Commutative Algebra & the Gr$\ddot{o}$bner Basis

Homework 1

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Problem 1: Let $G = \{g_1, g_2, \ldots, g_t\} \subseteq k[X]$ and let $0 \neq h \in k[X]$. Show that $\{g_1, \ldots, g_2\}$ is a Gröbner basis iff $\{hg_1, \ldots, hg_t\}$ is a Gröbner basis.

To prove this statement, we require two intermediary lemmas.

Lemma 1. *Let $0 \neq g, h \in k[X]$. Then if $\mathrm{lt}(g)$ denotes the leading term of the polynomial g, $\mathrm{lt}(hg) = \mathrm{lt}(h)\mathrm{lt}(g)$.*

Lemma 2. *Let I be an ideal of $k[X]$. Then if $LT(I)$ denotes the leading term ideal, $LT(hI) = \mathrm{lt}(h)LT(I)$.*

(Direction 1) Let $\{g_1, \ldots g_t\}$ be a Gröbner basis. Then by our lemmas,

$$\langle \mathrm{lt}(hg_1), \ldots, \mathrm{lt}(hg_t)\rangle = \langle \mathrm{lt}(h)\mathrm{lt}(g_1), \ldots, \mathrm{lt}(h)\mathrm{lt}(g_t)\rangle$$
$$= \mathrm{lt}(h)\langle lt(g_1), \ldots, \mathrm{lt}(g_t)\rangle = \mathrm{lt}(h)LT(G)$$

but since $\mathrm{lt}(h) \in k[X]$,

$$\mathrm{lt}(h)LT(G) = LT(G).$$

So $\{hg_1, \ldots, hg_t\}$ is a Gröbner basis.

(Direction 2) Now let $\{hg_1, \ldots, hg_t\}$ be a Gröbner basis. Then, by definition, if $I = \langle G\rangle$,

$$\langle \mathrm{lt}(hg_1), \ldots, \mathrm{lt}(hg_t)\rangle = LT(hG) = LT(hI).$$

By lemma 1, it follows that,

$$\langle \mathrm{lt}(hg_1), \ldots, \mathrm{lt}(hg_t)\rangle = \mathrm{lt}(h)LT(G) = LT(hG) = \mathrm{lt}(h)LT(I) = LT(hI).$$

So,

$$\text{lt}(h)LT(G) = \text{lt}(h)LT(I).$$

Notably, both sides are monomial ideals. We now introduce our final lemma, which completes this proof.

Lemma 3. *If $m \in k[X]$ is any monomial, and $mA = mB$ has both sides as monomial ideals, then $A = B$.*

(Proof of Lemma) Let $a \in A$. Then $ma \in mA = mB$, so $ma \in mB$ for some $b \in B$. Then for that b, $a = b \in B$ implies that for every $a \in A$, there exists a $b \in B$ such that $a = b$. Thus, $A \subseteq B$. Taking this direction as a "WLOG" statement, $B \subseteq A$ as well. Therefore, $A = B$.

As previously stated, since $\text{lt}(h)LT(G)$ and $\text{lt}(h)LT(I)$ are monomial ideals, by lemma 3,

$$LT(G) = LT(I)$$

and so, by definition, G is a Gröbner basis.

□

Remark 1. *Regarding lemma 3: I'm trying to keep a list of these minutiae so that I can work with ideals better, and this lemma seems like an important way to get around the whole* ***"we don't divide in this class"*** *thing.*

As a general question for Dr. Arnold - are there any other tricks like this (with ideals) that I should keep in mind from here on out? It seems like something I'll pick up more thoroughly in 431, but until then, I'd like to get better at working with ideals in proofs. I think that's where I'm making a lot of mistakes conceptually. Thanks!