

On Differential Equations

September 18, 2025

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Abstract

This lecture plan is intended for a four-day crash course in differential equations, encapsulating ideas discussed at length in JMU's Math 336. It is not a replacement for an actual course in differential equations, however, it provides exposure to topics from those courses. My notes are **heavily** aided by my notes from Math 336 with Dr. Alex Capaldi; there are some sections which are verbatim because I loved his class so much. If you ever get a chance to take his DiffEQ class, do it.

What is a Differential Equation?

Definition 1. A *Differential Equation (DE)* is an equation relating rates of change with respect to one variable. The ones we will study will mostly be of the form:

$$\frac{dy}{dx} = f(x, y) \quad (1)$$

and

$$\frac{d^2y}{dx^2} + \frac{dy}{dx} = f(x, y). \quad (2)$$

Example 1. The equation

$$\frac{dy}{dx} = y + 3x \quad (3)$$

can be solved as follows:

$$dy = (y + 3x)dx \quad (4)$$

$$\int 1dy = \int (y + 3x)dx \quad (5)$$

$$y + c_1 = yx + \frac{3}{2}x^2 + c_2 \quad (6)$$

$$y - yx = \frac{3}{2}x^2 + c_2 - c_1 \quad (7)$$

$$y(1 - x) = \frac{3}{2}x^2 + C \quad (8)$$

$$y = \frac{1}{1 - x} \left[\frac{3}{2}x^2 + C \right]. \quad (9)$$

At this point, we ask, what is C ? To properly answer this, we should differentiate y with respect to X , and solve for C .

$$y' = \frac{d}{dx}(3x^2)(2-2x)^{-1} + \frac{d}{dx}C(1-x)^{-1} \quad (10)$$

$$= 6x(2-2x)^{-1} + (3x^2)(2)(2-2x)^{-2} - C(1-x)^{-2} \quad (11)$$

$$= \frac{6x}{2-2x} + \frac{6x^2}{(2-2x)^2} - \frac{C}{(1-x)^2} \quad (12)$$

But we actually know what y' is already:

$$y' = y + 3x \quad (13)$$

So we can set (12) equal to (13) and solve for C .

$$\frac{6x}{2(1-x)} + \frac{6x^2}{4(1-x)^2} - \frac{C}{(1-x)^2} = y + 3x \quad (14)$$

$$3x(1-x) + \frac{3}{2}x^2 - C = (y + 3x)(1-x)^2 \quad (15)$$

$$C = 3x(1-x) + \frac{3}{2}x^2 - (y + 3x)(1-x)^2 \quad (16)$$

Thus,

$$y = \frac{1}{1-x} \left[\frac{3}{2}x^2 + 3x(1-x) + \frac{3}{2}x^2 - (y + 3x)(1-x)^2 \right] \quad (17)$$

$$= \frac{3x^2}{(1-x)} + 3x - (y + 3x)(1-x) \quad (18)$$

$$= \frac{3x^2}{(1-x)} + 3x - (y)(1-x) - (3x)(1-x) \quad (19)$$

$$y + y(1-x) = \frac{3x^2}{(1-x)} + 3x - (3x)(1-x) \quad (20)$$

$$y(1+1-x) = \frac{3x^2}{(1-x)} + 3x^2 \quad (21)$$

$$y = \frac{1}{2-x} \left[\frac{3x^2}{(1-x)} + 3x^2 \right] \quad (22)$$

Is our “closed form solution” to the equation $\frac{dy}{dx} = y + 3x$.

But what does it mean to solve a differential equation? In Example 1, we solved for y , since our rate of change was y' . Thus, we should be able to differentiate (22), plug in y and y' to (13), and have it be a true mathematical statement.

Initial Value Problems

Example 2. Say we are given the DE,

$$\frac{dy}{dx} = 3x^2 \quad (23)$$

By our previous solution, we can do

$$\int 1 dy = \int 3x^2 dx \quad (24)$$

$$y(x) = x^3 + C \quad (25)$$

Since if we were to differentiate this and plug it into (23), we wouldn't be able to find C , we need an **initial condition**, $y(x_0) = k$. Consider $y(0) = 1$.

$$y(0) = 1 = 0^3 + C \quad (26)$$

$$1 = C \quad (27)$$

Thus,

$$y(x) = x^3 + 1 \quad (28)$$

is our solution.

Group Work

Solve the following IVP's.

$$\frac{dy}{dx} = \frac{x+1}{x}; y(0) = 3 \quad (29)$$

$$\frac{dy}{dx} = \frac{x^2+3}{2} - 3; y(0) = 5 \quad (30)$$

$$\frac{dy}{dx} = \frac{-x+4}{7}; y(0) = -6. \quad (31)$$

Classifying DE's

Definition 2. We call a DE

- *Linear*, if it is of the form

$$a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_2(x)y'' + a_1(x)y' + a_0(x)y = f(x) \quad (32)$$

- *At equilibrium*, if we set y' equal to zero

$$\frac{dy}{dx} = f(x, y) = 0 \quad (33)$$

- *Separable*, if it is of the form

$$\frac{dy}{dx} = f(x)g(y) \quad (34)$$

- *Homogeneous*, if it is of the form

$$a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_2(x)y'' + a_1(x)y' + a_0(x)y = 0. \quad (35)$$

Group Work

Example 3. Classify the following DE's:

$$y'' + y'y = 0 \quad (36)$$

$$y'' + y = y' \quad (37)$$

$$y'' - y' + y = 3x + 2 \quad (38)$$

$$y' - y = x^2y \quad (39)$$

$$y' = y + x \quad (40)$$

Theory of Linear DE's

Theorem 1. Let y_1, y_2 be solutions to a linear differential equation. The following equation is also a solution, for any $k \in \mathbb{R}$.

$$ky_1 + ky_2 \quad (41)$$

Theorem 2. If $y_h(x)$ solves the linear homogeneous DE,

$$y' + p(x)y = 0, \quad (42)$$

and $y_p(x)$ solves the linear non-homogeneous DE,

$$y' + p(x)y = f(x), \quad (43)$$

then $y = y_p(x) + y_h(x)$ solves (43) as well.

Expanding on theorem two: $y_h(x)$ is in the kernel of the function which solves (43). $y_p(x)$ is the image of that function. Accordingly, we call $y_h(x)$ the homogeneous solution and $y_p(x)$ the particular solution.

Theorem 3. All first order homogeneous DE's have a solution of the form

$$y = Ce^{-\lambda x} \quad (44)$$

Theorem 4. All homogeneous DE's have a solution constructed of sums of exponential and trigonometric functions. This is called a Fourier series.

General Solutions to Non-Homogeneous DE's (Method of Undetermined Coefficients)

Example 4. Find all solutions to

$$y' - y = x. \quad (45)$$

Solution 1. Break (44) into its homogeneous form first and solve it:

$$y' - y = 0 \quad (46)$$

$$\frac{dy}{dx} = y \quad (47)$$

$$\frac{dy}{y} = dx \quad (48)$$

$$\int \frac{1}{y} dy = \int dx \quad (49)$$

$$\ln|y| = x + c \quad (50)$$

$$y = e^{x+c} \quad (51)$$

$$y = e^c e^x \quad (52)$$

$$y = C e^x. \quad (53)$$

Next, make a “guess” for what y_p is such that

$$\frac{d}{dx}(C e^x + y_p(x)) = x + y. \quad (54)$$

Since the RHS of (45) is a degree 1 polynomial, we will guess a degree 1 polynomial.

$$y_p = ax + b. \quad (55)$$

$$y_p' - y_p = a - (ax + b) = -ax + (a - b) \quad (56)$$

Setting (56) equal to RHS of (45), we find

$$-ax + (a - b) = 1x + 0. \quad (57)$$

This really splits into two equations,

$$-ax = x \quad (58)$$

$$a - b = 0. \quad (59)$$

Thus, $a = -1$ and $b = -1$ gives

$$y_p(x) = -x - 1 \quad (60)$$

and thus,

$$y = y_h + y_p = C e^x - x - 1 \quad (61)$$

is the solution to (45).

This gives rise to a larger question: for non-homogeneous DE's, can we “guess” the particular solution based on the RHS of the function?

It turns out that we can, although it is important to remember that the non-homogeneous equations we are considering are still **linear**. For non-linear equations, there really may not be a closed-form solution, and often times, there isn't one.

Other important note: if the particular solution is already a part of the homogeneous solution, we multiply by x so that the two are linearly independent.

Forcing function $f(x)$ (RHS)	Typical guess for $y_p(x)$	Notes
Constant k	A	Try a constant. If it's a solution to homogeneous part, multiply by x .
Polynomial of degree n : $a_n x^n + \dots + a_0$	$Ax^n + Bx^{n-1} + \dots + K$ (same degree)	Same degree as $f(x)$. If overlap with homogeneous solution, multiply by x .
$e^{\alpha x}$	$Ae^{\alpha x}$	If $e^{\alpha x}$ solves homogeneous equation, multiply by x . If still overlap, multiply by x^2 , etc.
Polynomial $\times e^{\alpha x}$	(Polynomial of same degree) $\times e^{\alpha x}$	Again, if overlap, multiply by powers of x .
$\sin(\beta x)$ or $\cos(\beta x)$	$A \cos(\beta x) + B \sin(\beta x)$	If overlap, multiply by x .
Polynomial $\times \sin(\beta x)$ or $\cos(\beta x)$	(Polynomial of same degree) $\times [A \cos(\beta x) + B \sin(\beta x)]$	Multiply by x if needed.
$e^{\alpha x} \cos(\beta x)$ or $e^{\alpha x} \sin(\beta x)$	$e^{\alpha x} (A \cos(\beta x) + B \sin(\beta x))$	If overlap, multiply by x .
Sum of terms (e.g. $x^2 + 3e^{2x} + \sin x$)	Sum of corresponding guesses	Handle each forcing term separately.

Figure 1: Method of Undetermined Coefficients Cheat Sheet

Second Order Homogeneous DE's

To tackle problems of the form

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = 0, \quad (62)$$

we actually need a few things.

1. Preferably, $a_i(x)$ are all constant.
2. Take a guess that $y(x) = e^{\lambda x}$.

The resulting equation is called the characteristic equation (sounds familiar, right?). It's of the form,

$$a\lambda^2 + b\lambda + c. \quad (63)$$

Take its roots!

Distinct Real Roots

If $\lambda_1 \neq \lambda_2 \in \mathbb{R}$, then

$$y = C_1 e^{\lambda_1 x} + C_2 e^{\lambda_2 x} \quad (64)$$

Repeated Real Roots

If $\lambda_1 = \lambda_2 \in \mathbb{R}$, then

$$y = (C_1 + C_2 x) e^{\lambda x} \quad (65)$$

Complex Roots

If $\lambda_1 \neq \lambda_2 \in \mathbb{C}$ such that $\lambda = \alpha \pm \beta i$, then

$$y = e^{\alpha x}(C_1 \cos(\beta x) + C_2 \sin(\beta x)) \quad (66)$$

Example 5. Consider the following equations (last three are group work):

$$y'' + y = x \sin(x) \quad (67)$$

$$y'' + y' = \cos(x) \quad (68)$$

$$y'' + 4y = \sin(2x) \quad (69)$$

$$y'' - 3y' + 2y = e^x \quad (70)$$

Solution 2. For (62):

Solve $y'' + y = x \sin x$.

(1) Homogeneous part: Consider $y_h'' + y_h = 0$.

Try $y_h = e^{rx}$. $\Rightarrow r^2 e^{rx} + e^{rx} = (r^2 + 1)e^{rx} = 0$.

Since $e^{rx} \neq 0$, $r^2 + 1 = 0 \Rightarrow r = \pm i$.

Complex roots $0 \pm i \Rightarrow y_h(x) = C_1 \cos x + C_2 \sin x$.

(2) Particular solution guess: RHS is $x \sin x$, a degree-1 polynomial $\times \sin x$.

Normally guess $(Ax + B) \cos x + (Cx + D) \sin x$.

But since $\cos x, \sin x \in y_h$, multiply by x to avoid overlap:

$y_p = x((Ax + B) \cos x + (Cx + D) \sin x)$.

(3) Substitute and match: $y_p'' + y_p = -4Ax \sin x + 4Cx \cos x + (2A + 2D) \cos x + (-2B + 2C) \sin x$.

This must equal $x \sin x$.

$$\text{Match coefficients: } \begin{cases} -4A = 1, \\ 4C = 0, \\ 2A + 2D = 0, \\ -2B + 2C = 0, \end{cases}$$

(4) Particular solution: $y_p(x) = \frac{1}{4}(-x^2 \cos x + x \sin x)$.

(5) General solution: $y(x) = C_1 \cos x + C_2 \sin x + \frac{1}{4}(-x^2 \cos x + x \sin x)$.

The Laplace Transform

So far, we've covered a few ways to solve linear DE's. One way to solve more complicated DE's is called the Laplace transform - but it actually isn't only motivated by solving DE's. In signals processing, we often want to answer the question, "how similar is one function to another function?" For example, if we have a radio signal $f(t)$ and we want to know how similar it is to a sine wave of frequency $\frac{\omega}{2\pi}$ over a symmetric domain $(-N, N)$, we look at:

$$\int_{-N}^N f(t) \sin(\omega t) dt. \quad (71)$$

If f 's frequency is very close to $\frac{\omega}{2\pi}$, the integrand will be very large from positive interference. Otherwise, many of the ups or downs will cancel out and it will be much smaller.

We can do the same with the most common function which shows up as a solution to DE's, $e^{\lambda t}$. To measure the similarity of $f(t)$ to e^{zt} , where $z \in \mathbb{C}$, we look at

$$\int_{-\infty}^{\infty} f(t)e^{-zt} dt. \quad (72)$$

If $f(t) = e^{zt}$, then the integrand is one (because the functions cancel out), and it integrates to infinity. If we split up $z = s + i\omega$ into its real and imaginary parts, we get two transforms:

$$\int_{-\infty}^{\infty} f(t)e^{-st} dt \text{ and } \int_{-\infty}^{\infty} f(t)e^{-i\omega t} dt. \quad (73)$$

The left-hand transform is the Laplace transform; the right-hand transform is the Fourier transform. We use Laplace for ODE's and Fourier for PDE's.

Definition 3. The Laplace Transform of a function, $\mathcal{L}(f(t))$ is the function $F(s)$, given by:

$$\mathcal{L}(f(t)) = F(s) = \int_0^{\infty} f(t)e^{-st} dt \quad (74)$$

for all s such that the integral converges.

In short, we take an equation from the **time** domain to the **frequency** domain.

Properties of the Laplace Transform

1. $\mathcal{L}(e^{at}) = \frac{1}{s-a}$ for $s > a$
2. $\mathcal{L}(1) = \frac{1}{s}$ for $s > 0$
3. $\mathcal{L}(f'(t)) = s\mathcal{L}(f(t)) - f(0)$; (note that we need an IC)
4. $\mathcal{L}(f''(t)) = s^2\mathcal{L}(f(t)) - sf(0) - f'(0)$
5. $\mathcal{L}(f(t) + g(t)) = \mathcal{L}(f(t)) + \mathcal{L}(g(t))$
6. $\mathcal{L}(cf(t)) = c\mathcal{L}(f(t))$ for any scalar c .
7. $\mathcal{L}(t^n) = \frac{n!}{s^{n+1}}$
8. $\mathcal{L}(t^n e^{at}) = \frac{n!}{(s-a)^{n+1}}$
9. $\mathcal{L}(\sin(bt)) = \frac{b}{s^2+b^2}$
10. $\mathcal{L}(\cos(bt)) = \frac{s}{s^2+b^2}$
11. $\mathcal{L}(e^{at}\sin(bt)) = \frac{b}{(s-a)^2+b^2}$
12. $\mathcal{L}(e^{at}\cos(bt)) = \frac{s-a}{(s-a)^2+b^2}$

Solving IVPs with Laplace

The general method is:

1. Apply the Laplace transform to both sides
2. Solve the resulting equation for $F(s)$
3. Use Laplace transform tables to convert $F(s)$ back to an equation based on t .

Example 6. Consider the homogeneous example,

$$y'' + 3y' + 2y = 0, \quad y(0) = 1, \quad y'(0) = 0.$$

Step 1. Take Laplace transforms.

Recall:

$$\mathcal{L}\{y''(t)\} = s^2Y(s) - sy(0) - y'(0), \quad \mathcal{L}\{y'(t)\} = sY(s) - y(0), \quad \mathcal{L}\{y(t)\} = Y(s).$$

Applying the Laplace transform to both sides:

$$(s^2Y(s) - sy(0) - y'(0)) + 3(sY(s) - y(0)) + 2Y(s) = 0.$$

Step 2. Substitute initial conditions.

With $y(0) = 1$, $y'(0) = 0$:

$$(s^2Y(s) - s) + (3sY(s) - 3) + 2Y(s) = 0.$$

Step 3. Collect terms.

$$(s^2 + 3s + 2)Y(s) = s + 3.$$

Hence

$$Y(s) = \frac{s + 3}{s^2 + 3s + 2}.$$

Step 4. Factor and decompose.

$$s^2 + 3s + 2 = (s + 1)(s + 2),$$

so

$$Y(s) = \frac{s + 3}{(s + 1)(s + 2)}.$$

Partial fractions:

$$\frac{s + 3}{(s + 1)(s + 2)} = \frac{A}{s + 1} + \frac{B}{s + 2}.$$

Multiply through:

$$s + 3 = A(s + 2) + B(s + 1).$$

Choosing $s = -2$: $1 = -B \implies B = -1$.

Choosing $s = -1$: $2 = A \implies A = 2$. Thus,

$$Y(s) = \frac{2}{s + 1} - \frac{1}{s + 2}.$$

Step 5. Invert the transform.

Recall

$$\mathcal{L}^{-1}\left\{\frac{1}{s + a}\right\} = e^{-at}.$$

So

$$y(t) = 2e^{-t} - e^{-2t}.$$

Final Answer.

$$\boxed{y(t) = 2e^{-t} - e^{-2t}, \quad t \geq 0.}$$

That worked great! But does it work for the more painful non-homogeneous examples?

Example 7. Consider the non-homogeneous example,

$$y'' + 4y = 3\cos(t), \quad y(0) = 0, \quad y'(0) = 0.$$

Step 1. Take Laplace transforms.

$$\mathcal{L}\{y''(t)\} = s^2Y(s) - sy(0) - y'(0), \quad \mathcal{L}\{y(t)\} = Y(s).$$

So

$$(s^2Y(s) - sy(0) - y'(0)) + 4Y(s) = 3 \cdot \frac{s}{s^2 + 1}.$$

Step 2. Substitute initial conditions.

With $y(0) = 0$, $y'(0) = 0$:

$$(s^2 + 4)Y(s) = \frac{3s}{s^2 + 1}.$$

Hence

$$Y(s) = \frac{3s}{(s^2 + 1)(s^2 + 4)}.$$

Step 3. Partial fraction decomposition.

We want

$$\frac{3s}{(s^2 + 1)(s^2 + 4)} = \frac{As + B}{s^2 + 1} + \frac{Cs + D}{s^2 + 4}.$$

Multiply through:

$$3s = (As + B)(s^2 + 4) + (Cs + D)(s^2 + 1).$$

Expanding:

$$3s = (A + C)s^3 + (B + D)s^2 + (4A + C)s + (4B + D).$$

Step 4. Match coefficients.

Comparing with $3s$ gives:

$$A + C = 0, \quad B + D = 0, \quad 4A + C = 3, \quad 4B + D = 0.$$

From this: $B = 0, D = 0, A = \frac{3}{4}, C = -\frac{3}{4}$. Thus

$$Y(s) = \frac{\frac{3}{4}s}{s^2 + 1} - \frac{\frac{3}{4}s}{s^2 + 4}.$$

Step 5. Inverse transform.

Recall

$$\mathcal{L}^{-1}\left\{\frac{s}{s^2 + a^2}\right\} = \cos(at).$$

So

$$y(t) = \frac{3}{4} \cos(t) - \frac{3}{4} \cos(2t).$$

Final Answer.

$$\boxed{y(t) = \frac{3}{4} \cos(t) - \frac{3}{4} \cos(2t), \quad t \geq 0.}$$

Side-note: I wish we had time to cover this thing called the heaviside, Kronecker Delta, and step function which let you artificially add impulses into the DE - essentially asking the question of “what if we have piecewise functions” or “what if something spikes?” It isn’t as applicable to finance, but it’s often used to stress-test DE models of real-world processes. That said, we’ll talk about them some in a different context during our week on Stochastic Differential Equations in week 8.

Systems of Equations & the Phase Plane

Consider

$$\frac{dx}{dt} = P(x, y) \tag{75}$$

$$\frac{dy}{dt} = Q(x, y). \tag{76}$$

We call this system “autonomous” because both equations are not dependent on time. If we set both (75) and (76) equal to zero, we get an equilibrium solution; moreover, based on factors that we will discuss soon, that equilibrium can either be

- Stable (attracts nearby solutions)
- Un-stable (repels nearby solutions in at least one direction)

Example 8. Consider the system,

$$\frac{dx}{dt} = y - \sin(x) \quad (77)$$

$$\frac{dy}{dt} = x - y. \quad (78)$$

We can actually graph it on something called the **phase plane**, as shown in Figure 2.

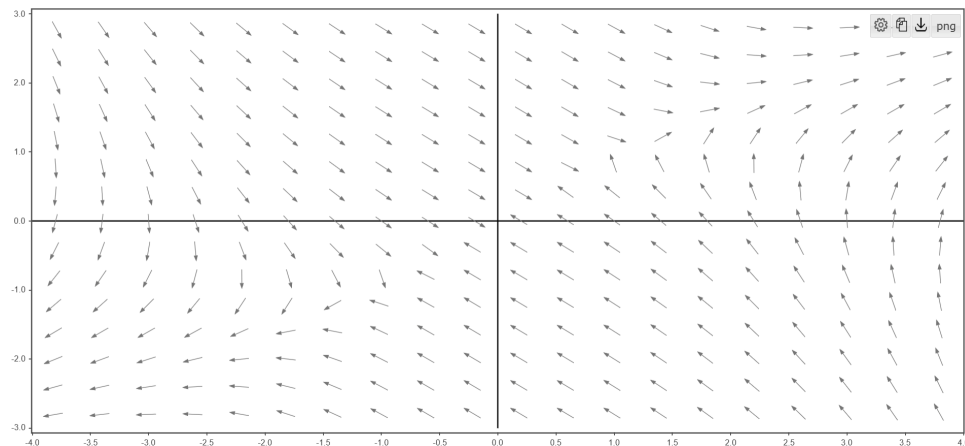


Figure 2: Phase Plane for Example 8

If we pick an initial condition at $t = 0$ for (x,y) , we can actually see where it goes in real time. Consider the initial condition $x(0), y(0) = (-2, 1)$.

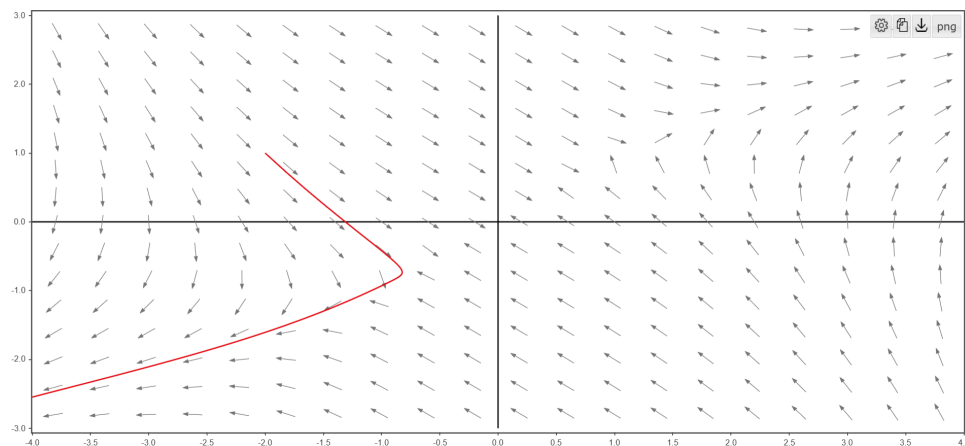


Figure 3: Initial Condition for Example 8

Pretty cool, right? It would probably be even cooler if we could figure out where all solutions converge to. As it turns out, we can do that by finding the equilibrium for each equation, then the equilibrium

solutions for both equations. The equation describing the equilibrium for one equation is called a **nullcline**, and it actually strictly determines the path of something in the phase plane. Let's find the nullclines and equilibria for

$$\frac{dx}{dt} = y - \sin(x) \quad (79)$$

$$\frac{dy}{dt} = x - y \quad (80)$$

We start by letting $\frac{dx}{dt} = 0$, and, $0 = \frac{dy}{dt}$. This yields our nullclines,

$$y = \sin(x) \quad (81)$$

$$y = x. \quad (82)$$

Thus, we have an equilibrium solution when

$$x = \sin(x), \quad (83)$$

which only occurs at $x = 0$. Since $y = x$, our equilibrium is $(0, 0)$. In a few minutes, we'll be able to determine the stability of this equilibrium mathematically, but for now, it is hopefully clear that it is unstable.

Theorem 5. All paths for a system of linear differential equations either converge to equilibrium attractors, or infinity.

From this point onward, it will be helpful to only consider DE's of the form

$$\frac{dx}{dt} = ax + by + f(t) \quad (84)$$

$$\frac{dy}{dt} = cx + dy + g(t). \quad (85)$$

We can rewrite this in matrix form as

$$\begin{bmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} f(t) \\ g(t) \end{bmatrix} \quad (86)$$

or more succinctly,

$$\vec{x}' = A\vec{x} + \vec{f}(t). \quad (87)$$

If $\vec{f}(t) = 0$, the system is homogeneous.

Theory of Linear Systems of DE's

We first note that any order- n linear DE can be converted to a system of n equations.

Example 9. Consider that we can represent

$$x'' + 2x' - 3x = 0 \quad (88)$$

by letting

$$y_1 = x \quad (89)$$

and

$$y_2 = x'. \quad (90)$$

From this, we get

$$y_1' = y_2 \quad (91)$$

$$y_2' = x'' = -2x' + 3x = -2y_2 + 3y_1. \quad (92)$$

Thus, our system of first order linear DE's is

$$y_1' = 0y_1 + y_2 \quad (93)$$

$$y_2' = 3y_1 - 2y_2. \quad (94)$$

As an A matrix, this becomes

$$\vec{x}' = \begin{bmatrix} 0 & 1 \\ 3 & -2 \end{bmatrix} \vec{x}. \quad (95)$$

As it turns out, to **solve** the system of linear DE's at any point in time, all we have to do is **compute the eigenvalues**, and use our method of undetermined coefficients to get a general solution! Recall our method of computing eigenvalues and eigenvectors:

1. Compute $\det(A - \lambda I)$
2. Solve the characteristic equation (yes, that one) for its roots (the eigenvalues)
3. Find a basis for $\ker(A - \lambda_i I)$, (the eigenvectors).

If we follow this process, we get the following characteristic equation

$$(\lambda - 1)(\lambda + 3) \quad (96)$$

which yields $\lambda_1 = 1$, $\lambda_2 = -3$ and the corresponding eigenvectors

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}; \vec{v}_2 = \begin{bmatrix} 1 \\ -3 \end{bmatrix}. \quad (97)$$

Then the solution to our system of equations is

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = C_1 e^{1t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + C_2 e^{-3t} \begin{bmatrix} 1 \\ -3 \end{bmatrix}. \quad (98)$$

Since $x = y_1$, the solution to our actual DE is

$$x = C_1 e^t + C_2 e^{-3t}. \quad (99)$$

This is incredible! We can now compute the exact solution for systems of linear DE's,

Theorem 6. If A has distinct, real eigenpairs, (\vec{v}_i, λ_i) , then the system of equations has a general solution:

$$\vec{x} = \sum_{i=1}^n C_i e^{\lambda_i t} \vec{v}_i. \quad (100)$$

Moreover, the origin is either an attracting node, repelling node, or a saddle point, based on the following:

- $\lambda_i < 0$ for all $i \Rightarrow$ attracting node
- $\lambda_i > 0$ for all $i \Rightarrow$ repelling node
- λ_i mixed \Rightarrow saddle point.

Finally, the distinct eigenvectors generalize the straight line solutions of the system.

The repelling node:

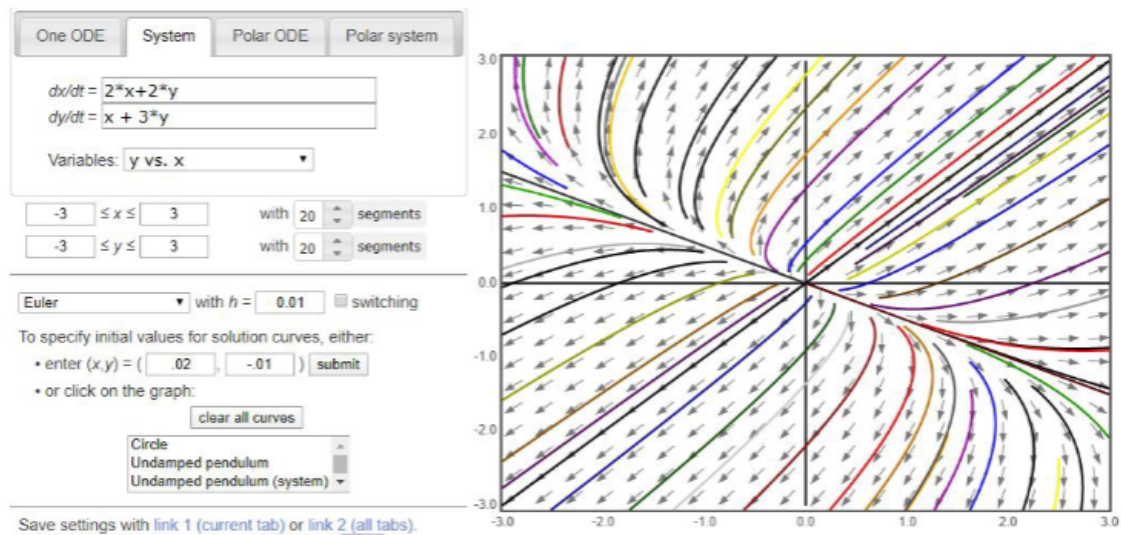


Figure 4: Repelling Node

The saddle point:

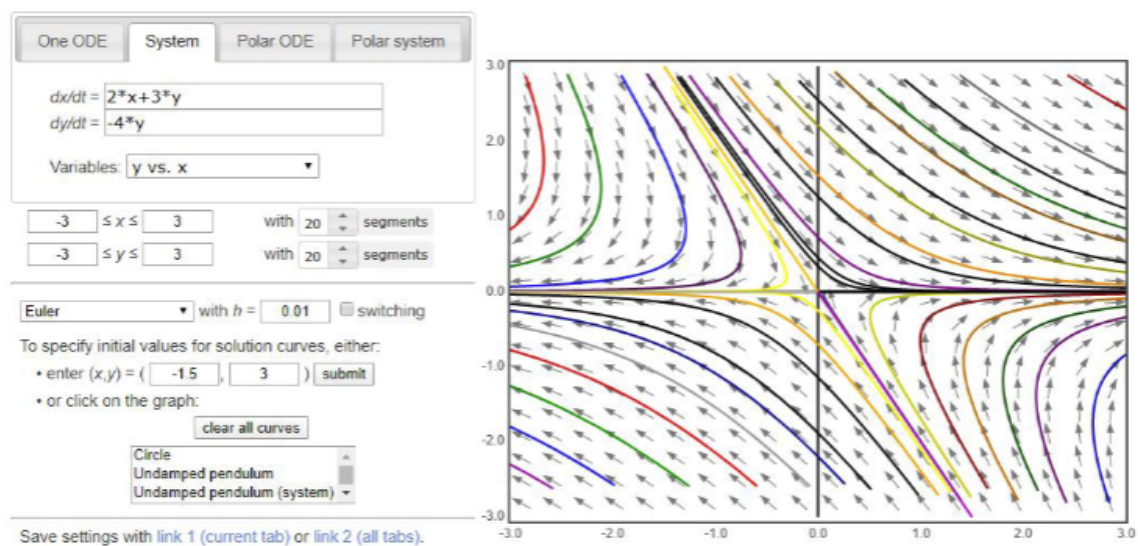


Figure 5: Saddle Point

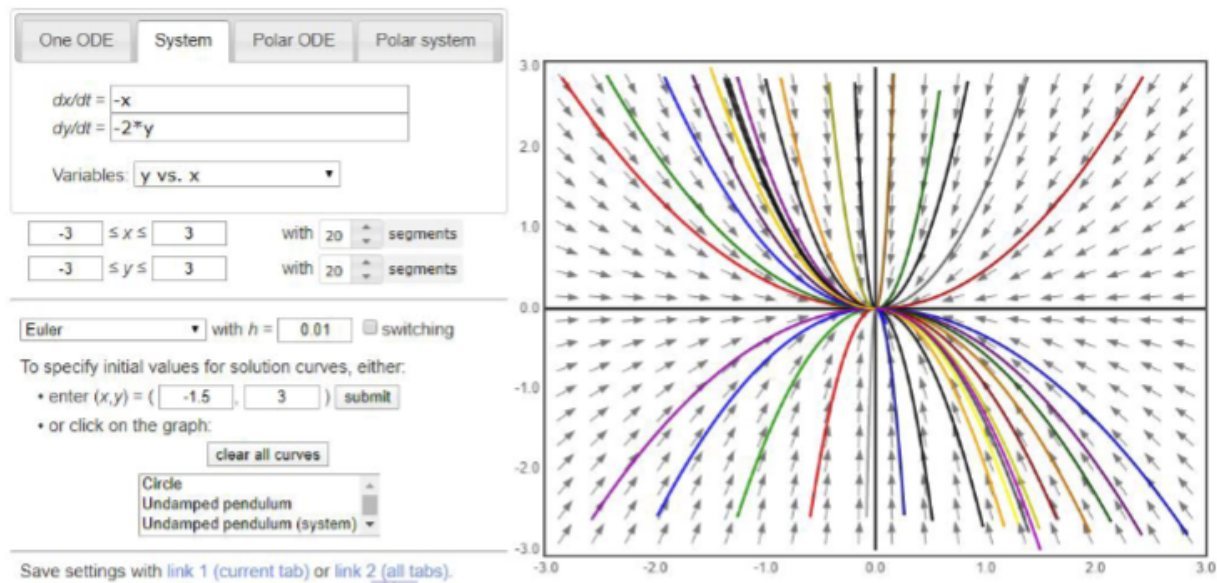


Figure 6: Attracting Node

This is all great, but what if we have complex eigenvalues?

Theorem 7. If A has distinct, complex eigenpairs, (\vec{v}_j, λ_j) with $\lambda_j = \alpha_j + \beta_j i$, then the system of equations has a general solution. It gets more complicated with dimensionality, so I'm going to write it on the board. The solution looks like a **spiral**. If $\text{Re}(\lambda) > 0$ then it is repelling from the center. If $\text{Re}(\lambda) < 0$ then it is attracted to the center. If $\text{Re}(\lambda) = 0$ then it is a "center."

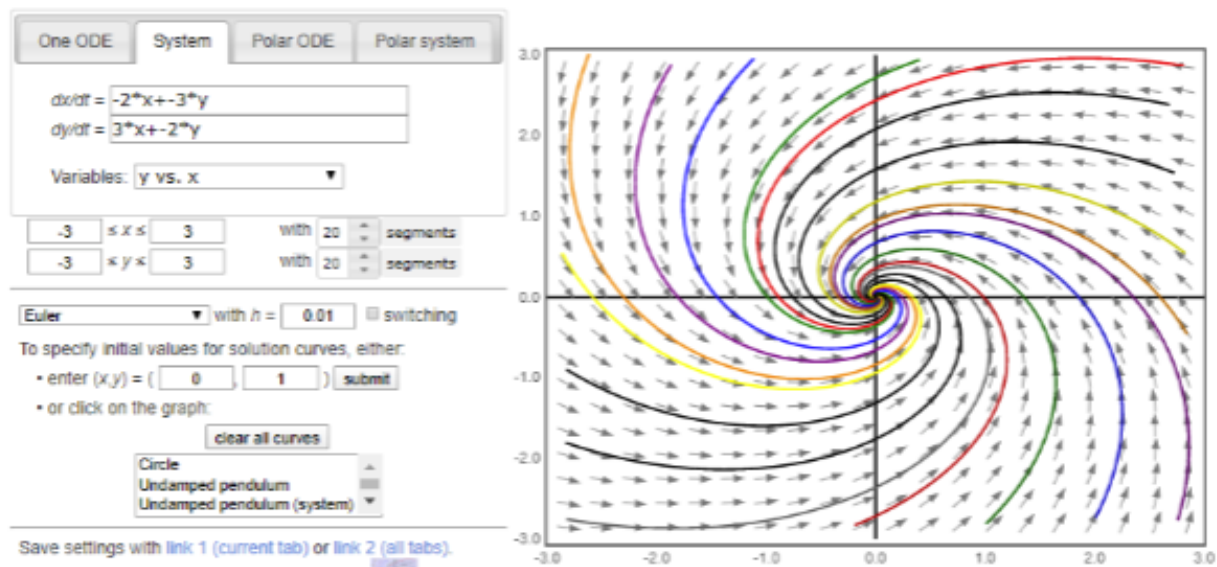


Figure 7: Attracting / Repelling Spiral

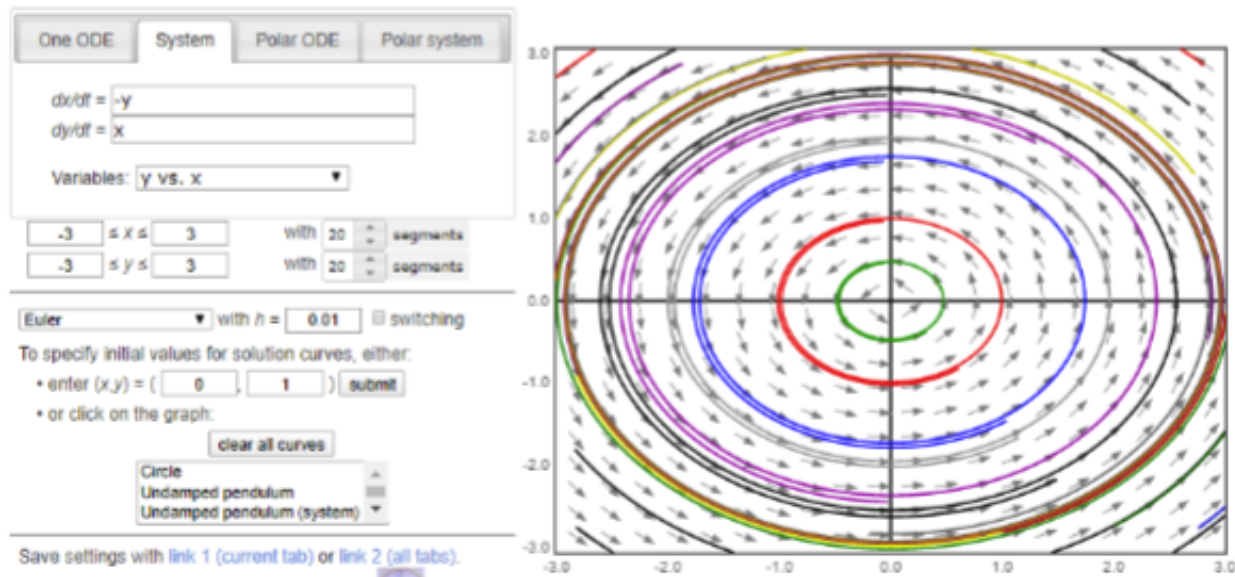


Figure 8: Center

There exists another case for repeated eigenvalues, but for the sake of time, I'm omitting it from the notes.

The Trace Determinant Plane

Definition 4. The trace of a matrix, $Tr(A)$, is the sum of its diagonal entries.

For 2×2 systems, the characteristic polynomial is

$$\lambda^2 - Tr(A)\lambda + Det(A) = 0. \quad (101)$$

Its solutions become

$$\lambda = \frac{-T \pm \sqrt{T^2 - 4D}}{2}. \quad (102)$$

Thus, we can quickly characterize the types of solutions based on the trace and determinant of A. Let $\Delta = T^2 - 4D$. We can see that the value of Δ splits the trace-determinant plane apart. Then, based on the specific values of λ , we can figure out if the system is attracting, repelling, or degenerate.

Poincaré - diagram: Classification of phase portraits in $(\det A, \text{Tr } A)$ -plane

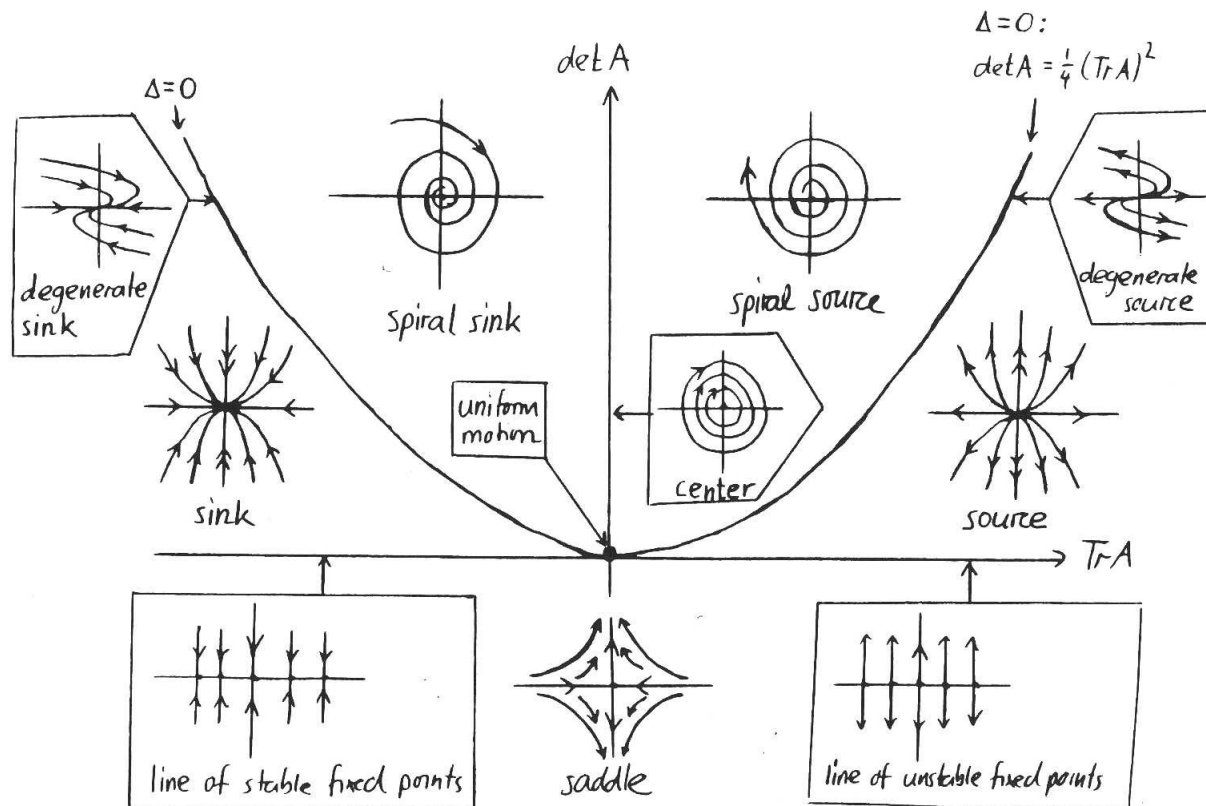


Figure 9: Trace Determinant Plane

Numerical Methods

Euler's Method for Numerical Integration

Consider the IVP,

$$\frac{dy}{dx} = y - x^2 + 1; y(0) = 0.5. \quad (103)$$

We could solve it the way we've spent the 18 pages solving DE's, but what if there's an easier way to approximate the y position that we care about?

Fix $h = 0.2$. Then iterate through, $n = 1, \dots, 5$.

$$y_{n+1} = y_n + h(y'(x_n, y_n)) \quad (104)$$

From this process, we can build out a table.

Euler table ($h = 0.2$)

n	x_n	y_n	$f(x_n, y_n) = y_n - x_n^2 + 1$	$y_{n+1} = y_n + h f$
0	0.0	0.500000	1.500000	0.800000
1	0.2	0.800000	1.760000	1.152000
2	0.4	1.152000	1.992000	1.550400
3	0.6	1.550400	2.190400	1.988480
4	0.8	1.988480	2.348480	2.458176

Figure 10: Euler Table

How can we code this up and graph it?

General Algorithm:

1. Pick what end value (x_f), you want the region to end at such that it is (x, x_f) .
2. Pick your granularity = n .
3. $dx = h = \frac{x_f - x}{n}$.
4. Recursively compute

$$f(x_n, y_n) = DE \quad (105)$$

$$y_{n+1} = y_n + h f \quad (106)$$

until you reach your granularity.

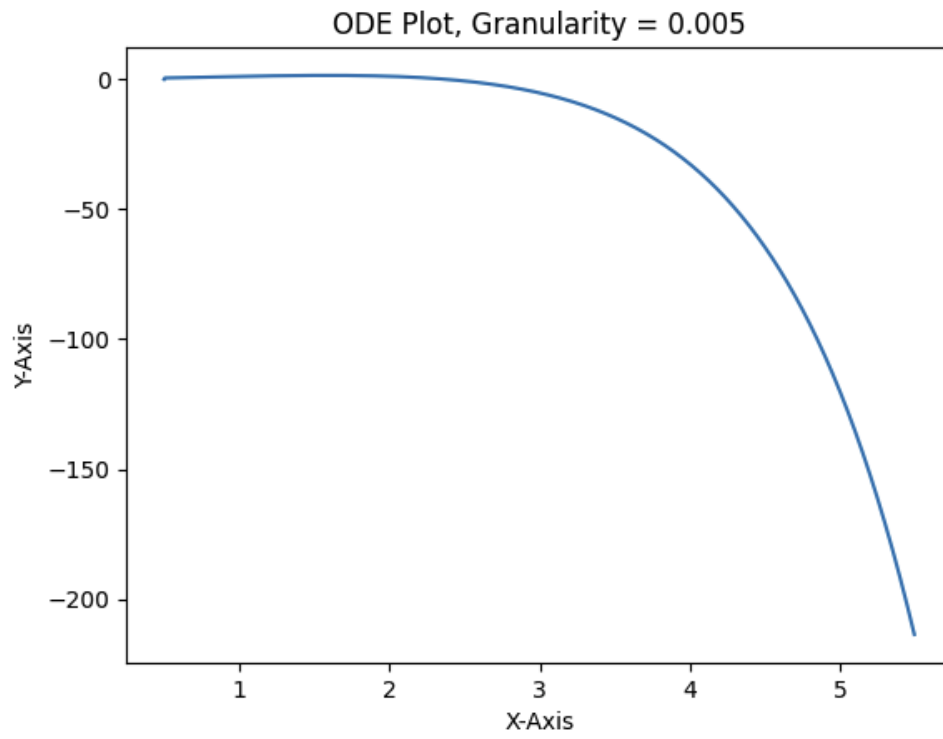


Figure 11: Highest Granularity

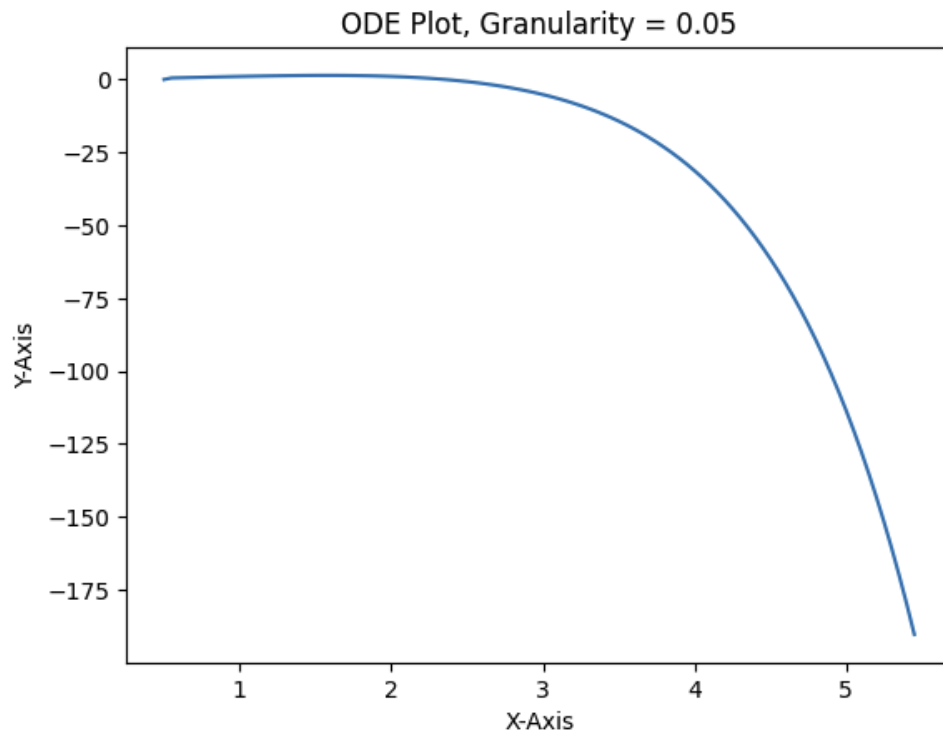


Figure 12: Medium Granularity

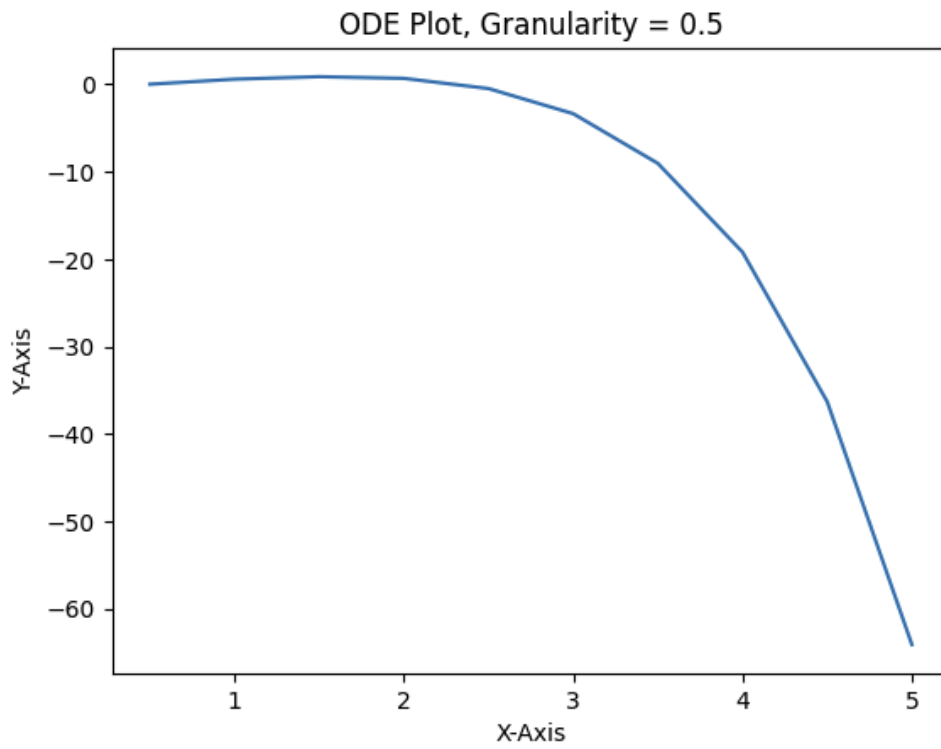


Figure 13: Low Granularity

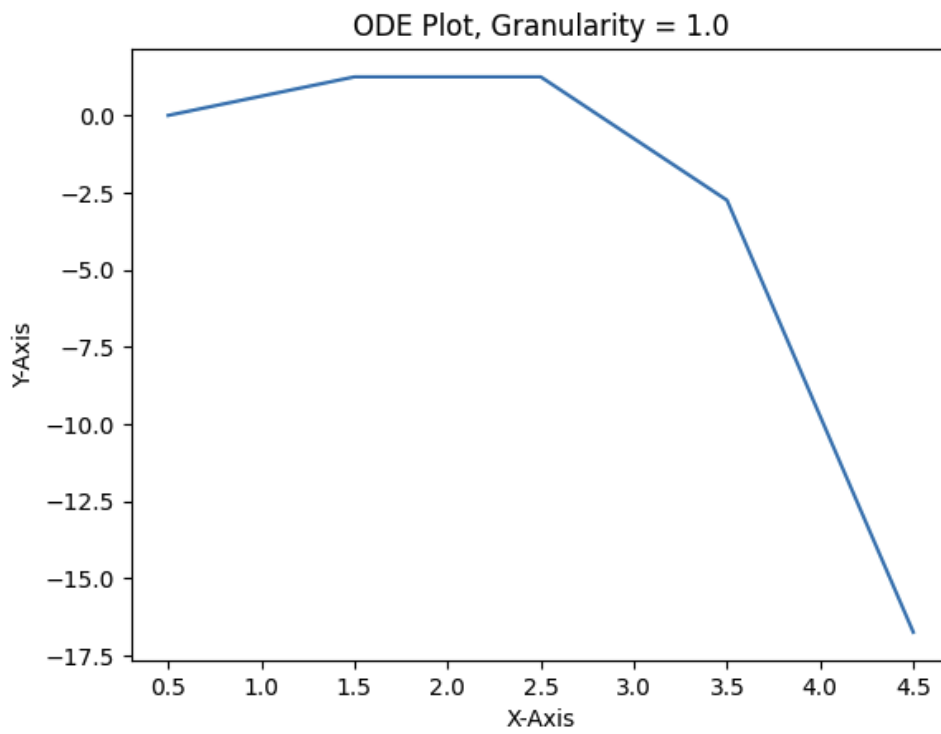


Figure 14: Lowest Granularity

Monte-Carlo Simulation

Note that if we have a DE,

$$\frac{dy}{dx} = P(x, y), \quad (107)$$

we really want to integrate both sides to get

$$\int \frac{dy}{dx} = \int p(x, y) dx \quad (108)$$

$$y(x) = \dots \quad (109)$$

Previously, when we talked about Riemann integration, we assumed that the dx was uniform across our partition $[a, b]$.

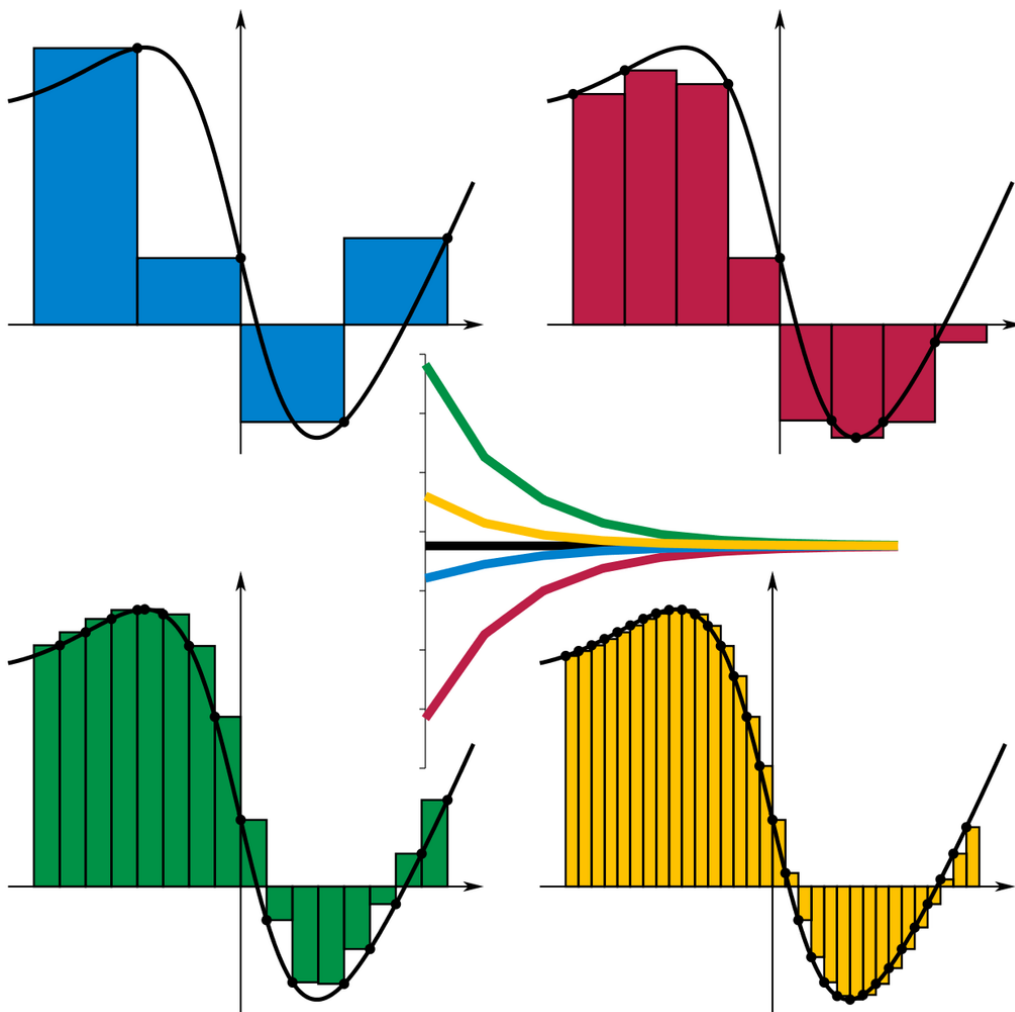


Figure 15: Riemann Integration

If you recall from lecture 1, we discussed the Riemann-Stieltjes integral by which our partition of $[a, b]$ was determined by $\alpha(x) : \mathbb{R} \rightarrow \mathbb{R}$. What if we let $\alpha(x)$ be a random sample of our partition of $[a, b]$? This is called **integration by simulation**. How do we implement this algorithmically in practice?

General Algorithm:

1. Pick a distribution, typically the uniform distribution over $[a, b]$, which says that $p(x = k) = \frac{1}{b-a}$.
2. Sample randomly to determine x , and order the samples from least to greatest.
3. Determine $dx = \{h_i | h_i = dx_i - dx_{i-1}\}$.
4. Use Euler's Method with this set of dx entries (as h).

Note: in the limit, Monte-Carlo simulation (with the uniform distribution) becomes the limit of Euler's method in the limit, which becomes Riemann Integration. The reason why this is a more useful formulation is because the Riemann-Stieltjes Integral is a version of the Ito Integral, which will be the foundation of SDE's later on in this course. Monte-Carlo simulation is a numerical evaluation of the RS integral for an α specified by a probability distribution. We'll see this as a specific example in week 5 for the expected value of a distribution; (this is the equivalent of an SDE with all components constant except $\alpha(x)$).

Runge-Kutta

Next week!

Applications of DE's

Lotka-Volterra Equations

The Lotka-Volterra model is a system of nonlinear differential equations that was developed to explicitly model predator-prey population dynamics for mathematical biology. While it is a very simple model, it has shown the test of time as a robust framework for modeling ecosystems.

The assumptions that the model make are as follows (GeeksForGeeks, 2024):^[11]

- The population of prey increases either linearly or exponentially in the absence of predators.
- The availability of prey is the only factor limiting the predator's population expansion.
- There is no age structure, constant reproduction, or atomic homogeneity between predator and prey.
- The rate of encounters between predators and prey is directly correlated with the rate of predation.
- The predator has a density-independent, constant mortality rate.

Parameters

The model takes into account the following parameters:^[11]

- P = prey population
- Q = predator population
- a = birth rate of prey
- b = predation rate (the rate at which the predators kill the prey)
- δ = efficiency rate (the rate at which eating prey creates new predators)
- n = natural death rate of predators (the rate at which predators will die if there are no prey to eat)

Model

The formalization of the system of equations follows the rates of change of both the populations. For the prey model, we get:

$$\frac{dP}{dt} = P(a - bQ) \quad (110)$$

Likewise, for the predator model, we get:

$$\frac{dQ}{dt} = Q(-n + \delta P) \quad (111)$$

Special Case: Affili Et. Al. Predator-Prey Model

In 2024, Affili Et. Al. published a monograph titled, “A New Lotka-Volterra Model of Competition with Strategic Aggression: Civil Wars When Strategy Comes Into Play”^[16] under Springer, a book which made significant developments on systems with shared resources - such as publicity and public belief (for our use case). The model doesn’t deviate too heavily from the classic Lotka-Volterra model, but does introduce the novel idea of a variational “strategy” parameter. They do an extremely thorough job of proving the extensive dynamics of their model, and we will reuse many of their results, theorems, and lemmas.

Parameters

The factors that the ‘Civil War’ model takes into account are:

- a = the portion of the population fighting = level of aggression = strategy
- p = the second group’s fitness to resources
- $c_i = \frac{\text{endured}}{\text{inflicted}}$ kills for population ‘i’ (note: $c_1 = \frac{1}{c_2}$)
- u = the population density of the first population = $\frac{pop_1}{pop_1 + pop_2}$
- v = the population density of the second population = $\frac{pop_2}{pop_1 + pop_2}$

Model

The system of differential equations is as follows:

$$\frac{du}{dt} = u(1 - u - v) - acu \quad (112)$$

$$\frac{dv}{dt} = pv(1 - u - v) - au \quad (113)$$

Model Discussion & Important Results

A big element worth noting about this model is that everything is measured in density - so the phase plane is constrained to $[0, 1] \times [0, 1] \subseteq \mathbb{R}^2$. A great deal of the monograph is devoted to partitioning this solution space into finite and non-finite time convergence solutions - namely, for any smooth curve μ that enters and exits $G = [0, 1] \times [0, 1] \subseteq \mathbb{R}^2$ exactly once, we can partition G into two sets, β (the set of infinite time solutions) and ε (the set of finite time solutions). Additionally, we note that μ is a stable solution.

G is Partitioned into Finite and Non-Finite Time Solutions by a Smooth Curve, μ :

It is proved in the monograph that certain trajectories cannot enter or exit through μ based on their starting position. Formally, let $S = \{(u, v) \in [0, 1] \times [0, 1] \mid v < g(u)\}$, where $g: [0, 1] \rightarrow [0, +\infty)$ is a monotone increasing, continuous function such that there exists a finite (possibly empty) set $Z \subset [0, 1]$ for which $g \in C^1([0, 1] \setminus Z)$, and in addition, the limits

$$\lim_{u \rightarrow z^\pm} g'(u)$$

exist for all $z \in Z$ (possibly distinct and possibly equal to $+\infty$). Let $N(\hat{u}, g(\hat{u}))$ be the set of outward unit normal vectors to ∂S at $(\hat{u}, g(\hat{u}))$, defined by

$$v = \left(-\frac{g'(\hat{u})}{\sqrt{1 + (g'(\hat{u}))^2}}, \frac{1}{\sqrt{1 + (g'(\hat{u}))^2}} \right). \quad (114)$$

Let $\{I_k\}_{k \in A}$, where A is the set of admissible strategies, and $\{k_j\}_{j \in C}$, where C is a finite subset of \mathbb{N} , be two finite collections of disjoint intervals in $[0, 1]$, the I_k being open and the K_j being closed (possibly coinciding with singletons) in the topology induced by $[0, 1]$, such that

$$\bigcup_{k \in A} I_k \cup \bigcup_{j \in C} K_j = [0, 1]. \quad (115)$$

Suppose that for all $\hat{u} \in \bigcup_{k \in A} I_k$ it holds that

$$\min_{v \in N(\hat{u}, g(\hat{u}))} (F(\hat{u}, g(\hat{u})), G(\hat{u}, g(\hat{u}))) \cdot v > 0 \quad (116)$$

and that for all $\hat{u} \in \bigcup_{j \in C} K_j$ it holds that

$$\min_{v \in N(\hat{u}, g(\hat{u}))} (F(\hat{u}, g(\hat{u})), G(\hat{u}, g(\hat{u}))) \cdot v = 0. \quad (117)$$

Then no trajectory enters S through a point of the set

$$\partial S \cap \{(u, v) : u \in [0, 1], v = g(u)\}. \quad (118)$$

The proof of this is in **no way trivial**, and can be found on pages 32 - 36 of the monograph. That said, this proof is extremely important - this implies that non-finite time solutions never become finite time solutions.

Similarly, it is shown (in a much shorter proof no less) that no finite-time solutions ever become non-finite time solutions, and thus, we have a complete partition of $G = [0, 1] \times [0, 1]$.

G is Partitioned into Four Basins of Attraction, and No Trajectory Exits A_3 if $ac \neq 1$:

It is also proved that there exist four unique basins of attraction, A_1, A_2, A_3 , and A_4 , which are divided by the curves

$$\frac{dv}{dt} = 0 \quad (119)$$

which is given by

$$\sigma(v) = 1 - \frac{pv^2 + a}{pv + a} \quad (120)$$

and

$$\frac{du}{dt} = 0 \quad (121)$$

The formalization of these partitions are as follows:

- $A_1 = \{(u, v) \in [0, 1] \times [0, 1] \mid \frac{du}{dt} \leq 0, \frac{dv}{dt} \geq 0\}$
- $A_2 = \{(u, v) \in [0, 1] \times [0, 1] \mid \frac{du}{dt} \leq 0, \frac{dv}{dt} \leq 0\}$
- $A_3 = \{(u, v) \in [0, 1] \times [0, 1] \mid \frac{du}{dt} \geq 0, \frac{dv}{dt} \leq 0\}$
- $A_4 = \{(u, v) \in [0, 1] \times [0, 1] \mid \frac{du}{dt} \geq 0, \frac{dv}{dt} \geq 0\}$

It is then proved that if $ac \neq 1$, that no trajectory exits A_3 - a very fruitful and interesting proof, especially for analyzing systems efficiently.

If $ac = 1$ and the Initial Condition Lies on μ , its ω -Limit is $\{(0, 0)\}$:

Affili Et. Al. defines the ω -limit, given $(u_0, v_0) \in [0, 1] \times [0, 1]$, as:

$$\omega(u_0, v_0) = \{(x, y) \in \mathbb{R}^2 : \phi_{(u_0, v_0)}(t) \in [0, 1] \times [0, 1],$$

for all $t \geq 0$,

and there exists $\{t_i\}_{i \in \mathbb{N}}$ with $t_i \rightarrow -\infty$,

$$\text{and } \lim_{i \rightarrow +\infty} \phi_{(u_0, v_0)}(t_i) = (x, y)\}$$

In short, if $ac = 1$ and the initial condition lies on μ , then the system will go to $(0,0)$ in its limit. (Note: $\phi_{(u_0, v_0)}(t)$ is the trajectory in the phase plane.)

Equilibria Based on Values of ac : If $ac \in (0, 1)$:

- $(0, 0)$ is a source
- $(0, 1)$ is a sink
- $(\frac{1-ac}{1+pc}pc, \frac{1-ac}{1+pc})$ is a saddle

If $ac > 1$:

- $(0, 0)$ is a saddle
- $(0, 1)$ is a sink

If $ac = 1$:

- $(0, 0)$ has a positive eigenvalue and a null eigenvalue
- $(0, 1)$ is a sink

How Fitness to Resources (p) Affects Strategy:

If $p = 1$:

- Strategy doesn't affect the outcome, just the time spent at war.

If $p < 1$ (the first force is more efficient):

- If $u + v \leq 1$, then the resource market isn't over-saturated, and the first population should pick a small a . This is the same as not being very aggressive.
- If $u + v > 1$, the first population should pick a very large a . This is the same as being very aggressive.

If $p > 1$ (the second force is more efficient):

- If $u + v \leq 1$, pick a very large value of a , as a moderate or small a value will cause a coexistence cycle.
- If $u + v > 1$, pick $a = 0$ until the resource market is no longer over-saturated, then pick a very large value of a .

Mixing and Cooling Models

Harmonic Oscillator