MIMF Lecture Week 2, Day 3: Orthonormality, Types of Matrices, Linear Functionals, Riesz Representation Theorem

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Orthonormality

Definition: A set of vectors $\{\mathbf{u}_1, \ldots, \mathbf{u}_m\} \subset \mathbb{R}^n$ is *orthonormal* if

$$\langle \mathbf{u}_i, \mathbf{u}_j \rangle = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

Example: Check whether $\{(1,0,1), (1,1,0), (0,1,1)\}$ is orthonormal.

$$\begin{aligned} \langle \mathbf{v}_1, \mathbf{v}_2 \rangle &= \underline{1 \cdot 1 + 0 \cdot 1 + 1 \cdot 0 = 1}. \\ \langle \mathbf{v}_1, \mathbf{v}_3 \rangle &= \underline{1 \cdot 0 + 0 \cdot 1 + 1 \cdot 1 = 1}. \\ \|\mathbf{v}_1\| &= \underline{\sqrt{1^2 + 0^2 + 1^2} = \sqrt{2}}. \end{aligned}$$

So the set is not orthonormal.

Applications:

- **Quant finance:** Principal component analysis (PCA) of covariance matrices uses orthonormal eigenvectors to identify uncorrelated risk factors.
- **Physics:** Quantum states form orthonormal bases in Hilbert spaces.
- **CS / ML:** Feature orthogonalization improves model interpretability.

Types of Matrices

Definitions:

- A matrix $A \in \mathrm{M}_{m\times n}(\mathbb{R})$ is *symmetric* if $\underline{A^T = A.}$
- A matrix $A \in \mathrm{M}_{m\times n}(\mathbb{F})$ is *Hermitian/self-adjoint* if $\underline{A^* = A.}$
- A matrix $A \in \mathrm{M}_{m\times n}(\mathbb{R})$ is *orthogonal* if $\underline{A^T A = I.}$
- A matrix $U \in \mathrm{M}_{m\times n}(\mathbb{F})$ is *unitary* if $\underline{U^* U = I.}$
- A matrix $U \in \mathrm{M}_{m\times n}(\mathbb{F})$ is *normal* if $\underline{UU^* = U^* U.}$
- A matrix $A \in \mathrm{M}_{m\times n}(\mathbb{F})$ is *positive semi-definite* if:

  1) $\underline{\langle A\mathbf{x}, \mathbf{x}\rangle \geq 0}$ for all $\mathbf{x} \in \mathbb{F}^n$; and,

  2) A is Hermitian/self-adjoint.

  What we care about: A positive semi-definite matrix has all <u>non-negative</u> eigenvalues.

- A matrix $A \in \mathrm{M}_{m\times n}(\mathbb{F})$ is *positive definite* if:

  1) $\underline{\langle A\mathbf{x}, \mathbf{x}\rangle > 0}$ for all $\mathbf{x} \in \mathbb{F}^n$; and,

  2) A is Hermitian/self-adjoint.

  A positive definite matrix has all <u>positive</u> eigenvalues.

Examples:

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$$

- A is Hermitian: <u>Yes</u>

- A has non-negative eigenvalues: <u>Yes</u>

- A is Positive semi-definite: <u>Yes</u>

- B is hermitian: No

- B is Positive semi-definite: No

- C: Left as an exercise to the reader :)

Applications:

- **Quant finance:** Covariance matrices (symmetric, PSD) underpin portfolio risk modeling.
- **Physics:** Hermitian operators correspond to observables with real eigenvalues.
- **CS:** Orthogonal/unitary matrices are used in QR decomposition and FFTs.

Linear Functionals and Riesz Representation

Definition: A *linear functional* is a linear map $L : \underline{\mathbb{F}^n} \to \underline{\mathbb{F}.}$
In English: A linear functional takes in something from a vector space and outputs a scalar in its underlying field.

Applications:

- **Quant finance:** Portfolio valuation and risk measures are linear functionals of asset payoffs. They take in vectors of information and output scalar risk measures.
- **Physics:** Fourier expansions use Riesz to represent functionals.
- **CS:** Duality in optimization and kernel methods rely on linear functionals.

Riesz Representation Theorem (RRT): Every linear functional L on $\mathbb{F}^n$ can be written as

$$L(\mathbf{x}) = \underline{\langle \mathbf{x}, \mathbf{y} \rangle} \quad \text{for a unique } \mathbf{y} \in \mathbb{F}^n.$$

In english: there exists a vector $\mathbf{y} \in \mathbb{F}^n$ such that the linear functional on $\mathbf{x}$, $L(\mathbf{x})$, can be written as the inner product of $\mathbf{x}$ with $\mathbf{y}$.

Example:

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}; \quad L(\mathbf{x}) = 2x_1 - 3x_2 + 5x_3$$

$$\text{Answer:} \quad \mathbf{y} = \begin{bmatrix} 2 \\ -3 \\ 5 \end{bmatrix}.$$

Test:

$$\mathbf{x} = \begin{bmatrix} 4 \\ 1 \\ 7 \end{bmatrix} \qquad L(\mathbf{x}) = 2(4) - 3(1) + 5(7) = 8 - 3 + 35 = 40$$

$$\mathbf{y} = \begin{bmatrix} 2 \\ -3 \\ 5 \end{bmatrix} \qquad \langle \mathbf{x}, \mathbf{y} \rangle = 4(2) + 1(-3) + 7(5) = 8 - 3 + 35 = 40.$$

What vector space does L pull from? $\underline{\mathbb{R}^3}$