On Linear Algebra

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Timothy Tarter
James Madison University
Department of Mathematics

Abstract

This lecture plan is intended for a four-day crash course in linear algebra, encapsulating ideas discussed at length in JMU's Math 300 and 434. It is not a replacement for an actual course in linear or advanced linear algebra, however, it provides exposure to topics from those courses. Day one covers vector spaces, bases, inner product spaces, orthogonality, orthonormality, and matrices as linear transformations.

Day One

What is a vector? Last week (in multivariable calculus) we said that a vector is a collection of points which have both direction and magnitude (distance from the origin). We noted that in $\mathbb{R}^3$, we could write any vector as

$$a\vec{i} + b\vec{j} + c\vec{k} = \langle a, b, c \rangle; \text{ for any } a, b, c \in \mathbb{R}. \tag{1}$$

This week, however, we will treat a vector as an element of a vector space. We will also write it a bit differently. Let $\vec{v} \in V$ with $dim(V) = n$. Then,

$$\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} \in V. \tag{2}$$

For now, assume $dim(V)$ to mean the length of a vector inside of V. We will formalize it soon.

Example 1. *Let* $v_1 = \begin{bmatrix} 2 \\ 3 \end{bmatrix} \in \mathbb{R}^2$. *We can see that,*

$$\begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 3 \end{bmatrix} = 2\begin{bmatrix} 1 \\ 0 \end{bmatrix} + 3\begin{bmatrix} 0 \\ 1 \end{bmatrix}. \tag{3}$$

In fact, for any $a, b \in \mathbb{R}$, we can actually find a similar result.

Example 2. *Let* $v_1 = \begin{bmatrix} a \\ b \end{bmatrix} \in \mathbb{R}^2$. *We can see that,*

$$\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} a \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ b \end{bmatrix} = a \begin{bmatrix} 1 \\ 0 \end{bmatrix} + b \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \tag{4}$$

As it turns out, the vectors

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} \text{ and } \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{5}$$

have a very nice property in $\mathbb{R}^2$, with an analog for $\mathbb{R}^n$.

Definition 1. *Let*

$$\vec{e_1} = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \vec{e_2} = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \ldots, \vec{e_n} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ n \end{bmatrix}. \tag{6}$$

We call $\{e_1, e_2, \ldots, e_n\}$ *the standard basis of* $\mathbb{R}^n$, *or any vector space* V *with* $dim(V) = n$.

Breaking some new terms down, what is a basis? We define the basis of a vector space V to be a set of “building blocks" for any vector in the vector space. We don’t want to have redundant building blocks and we don’t want to have too few building blocks to make a vector in V. As it turns out, there is a more mathematical way to phrase this argument.

Definition 2. *Let* $\beta = \{\vec{b_1}, \vec{b_2}, \ldots, \vec{b_n} \mid \vec{b_i} \in V\}$ *be a basis of* V. *Then,* β *has the following properties:*

- *(Linear Independence:) For the following equation to be true, each* $a_i \in \mathbb{R}$ *must be zero.*

$$a_1\vec{b_1} + a_2\vec{b_2} + \cdots + a_n\vec{b_n} = \vec{0} \tag{7}$$

- *(Spanning Set:) For any* $\vec{w} \in V$, *we can express* $\vec{w}$ *as a linear combination as the elements of* β*; rather, there exist* $a_i \in \mathbb{R}$ *such that*

$$\vec{w} = a_1\vec{b_1} + a_2\vec{b_2} + \cdots + a_n\vec{b_n}. \tag{8}$$

Definition 3. *For a vector space V, we define* $dim(V)$ *to be the number of vectors in its basis.*

The point? We have exactly as many building blocks as we need to make any vector in our vector space.

Motivating question: are bases unique? Answer: no.

Example 3. *Consider* $\mathbb{R}^2$ *with the basis*

$$\beta = \{ \begin{bmatrix} a \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ b \end{bmatrix} \}; \text{ for any } a, b \in \mathbb{R}. \tag{9}$$

Let $\vec{v} = \begin{bmatrix} c \\ d \end{bmatrix}$ *for some* $c, d \in \mathbb{R}$. *Then,*

$$\frac{c}{a}\begin{bmatrix} a \\ 0 \end{bmatrix} + \frac{d}{b}\begin{bmatrix} 0 \\ b \end{bmatrix} = \begin{bmatrix} \frac{c}{a}a \\ \frac{d}{b}b \end{bmatrix} = \begin{bmatrix} c \\ d \end{bmatrix} = \vec{v}. \tag{10}$$

So we can have funky bases for our vector spaces!

Example 4. *Show that* $\beta = \{\begin{bmatrix}1\\2\end{bmatrix}, \begin{bmatrix}3\\4\end{bmatrix}\}$ *isn't a basis by showing that it isn't linearly independent or spanning.*

Example 5. *Show that* $\beta = \{\begin{bmatrix}1\\2\\0\end{bmatrix}, \begin{bmatrix}-2\\1\\0\end{bmatrix}\}$ *isn't a basis by showing that it isn't spanning.*

Example 6. *Show that* $\beta = \{\begin{bmatrix}1\\2\\0\end{bmatrix}, \begin{bmatrix}-2\\1\\0\end{bmatrix}, \begin{bmatrix}0\\2\\1\end{bmatrix}, \begin{bmatrix}1\\2\\3\end{bmatrix}\}$ *isn't a basis by showing that it isn't linearly independent.*

Example 7. *Show that* $\beta = \{\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}, \begin{bmatrix}0\\0\\0\end{bmatrix}\}$ *isn't a basis by showing that it isn't spanning.*

So far, we've really been hand-waving about the notion of what a vector space is, in favor of computing examples of properties of vector spaces. For the sake of appeasing the standard approach to education in linear algebra, let's define the properties of a vector space now.

Definition 4. *A vector space* V *over a field* k *is a set endowed with vector addition (pointwise) and scalar multiplication, as one would expect from a field. More specifically, we have:*

- *(Vector Addition)* $\vec{v} + \vec{w} = \begin{bmatrix}v_1 + w_1\\v_2 + w_2\\\vdots\\v_n + w_n\end{bmatrix}$
- *(Associativity of +)* $\vec{v} + (\vec{w} + \vec{j}) = (\vec{v} + \vec{w}) + \vec{j}$
- *(Commutativity of +)* $\vec{v} + \vec{w} = \vec{w} + \vec{v}$
- *(Additive Identity)* $\vec{0} + \vec{v} = \vec{v}$
- *(Additive Inverse)* $\vec{v} - \vec{v} = \begin{bmatrix}v_1 + (-v_1)\\v_2 + (-v_2)\\\vdots\\v_n + (-v_n)\end{bmatrix} = \vec{0}$
- *(Scalar Multiplication)* $c \in \mathbb{R}$, $c\vec{v} = c\begin{bmatrix}v_1\\v_2\\\vdots\\v_n\end{bmatrix} = \begin{bmatrix}cv_1\\cv_2\\\vdots\\cv_n\end{bmatrix}$
- *(Associativity of *)* $c * (a * b)\vec{v} = (c * a) * b\vec{v}$
- *(Commutativity of *)* $c\vec{v} = \vec{v}c$
- *(Multiplicative Identity)* $1 * \vec{v} = \vec{v}$
- *(Multiplicative Inverse)* $\frac{1}{c}c\vec{v} = 1\vec{v} = \vec{v}$

- *(Distributive Property)* $a * (\vec{v} + \vec{w}) = a\vec{v} + a\vec{w}$.

Next question: are there any other properties which we might care about a vector having?

Definition 5. *We define the euclidean inner product of $\vec{v}, \vec{w} \in \mathbb{C}^n$ to be*

$$\langle \vec{v}, \vec{w} \rangle = \sum_{i=1}^{n} v_i \overline{w_i}, \tag{11}$$

where $\overline{w_i}$ is the conjugate of w_i, if $w_i \in \mathbb{C}$. If $w_i \in \mathbb{R}$, then $\overline{w_i} = w_i$.

We are familiar with the dot product from calculus III - and really, the euclidean inner product is the same idea. A lot of this is hopefully review from last week.

Definition 6. *We define the L^2 norm on $\mathbb{C}^n$ to be the distance between a point and the origin.*

$$||\vec{v}||_2 = \sqrt{\langle \vec{v}, \vec{v} \rangle} = \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2} \tag{12}$$

Definition 7. *If $||\vec{v}||_2 = 1$, we say that $\vec{v}$ is of unit length.*

Definition 8. *We say that two vectors, $\vec{v}, \vec{w} \in \mathbb{C}^n$ are orthogonal if*

$$\langle \vec{v}, \vec{w} \rangle = 0. \tag{13}$$

Definition 9. *We call two vectors $\vec{v}, \vec{w} \in \mathbb{C}^n$ orthonormal if they are both orthogonal and of unit length.*

Proposition 1. *We can make any vector $\vec{v} \in \mathbb{C}^n$ unit length by performing*

$$\vec{v}' = \frac{\vec{v}}{||\vec{v}||_2}. \tag{14}$$

Definition 10. *If a set of vectors is spanning for its vector space, and its contents are orthogonal (or orthonormal), it is a basis for its vector space. We call it an orthogonal or orthonormal basis respectively.*

Example 8. *Let*

$$\vec{v_1} = \begin{bmatrix} 1 \\ 3 \\ 2 \\ 0 \end{bmatrix}, \vec{v_2} = \begin{bmatrix} -3 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \vec{v_3} = \begin{bmatrix} 2 \\ -2 \\ 0 \\ 1 \end{bmatrix}, \vec{v_4} = \begin{bmatrix} -2 \\ 0 \\ 2 \\ 4 \end{bmatrix}. \tag{15}$$

Is it a basis? Orthogonal? Orthonormal?

Example 9. *Come up with a 3x3 orthogonal basis, and then figure out what you need to do to make it orthonormal.*

Proposition 2. *We can make any basis into an orthonormal basis using the Gram-Schmidt process. (I am omitting typing this out because it is long and there is pre-existing content on it, just google it).*

Okay - for those of you in the class who have heard of linear algebra, you're probably wondering, “where are all the matrices?" Isn't that what this class is about? You're right - linear algebra *is* about matrices, but the notion of a matrix is a bit more than just an array of numbers in a box. We've established that there exists this inner product thing that lets us get a scalar value out of two vectors which preserves the angle between the vectors and the magnitude of each vector. What if it actually had to also do with matrices?

Definition 11. *If $\vec{v_1}, \vec{v_2}, \ldots, \vec{v_n} \in V$ are vectors such that*

$$A = [\vec{v_1}\ \vec{v_2} \ldots \vec{v_n}], \tag{16}$$

we call A the matrix whose image is defined by the vectors $v_1 \ldots v_n$. A is a linear transformation from $V \to im(V)$.

Definition 12. *If A is an $m \times n$ matrix with row vectors $\vec{r_1} \ldots \vec{r_m}$ and $\vec{v} \in V$ is a vector of length n, then we define $A\vec{v}$ to be*

$$A\vec{v} = \begin{bmatrix} \langle \vec{r_1}\vec{v} \rangle \\ \langle \vec{r_2}\vec{v} \rangle \\ \vdots \\ \langle \vec{r_n}\vec{v} \rangle \end{bmatrix}. \tag{17}$$

Example 10. *Compute the following matrix-vector multiplication:*

$$\begin{bmatrix} 0 & -1 \\ -3 & 1 \\ 1 & -9 \end{bmatrix} \begin{bmatrix} 4 \\ -3 \end{bmatrix} = \tag{18}$$

What (visually) does this encode?

Definition 13. *If A is an $m \times n$ matrix with row vectors $\vec{r_1} \ldots \vec{r_m}$ and B is an $m \times l$ matrix with column vectors $\vec{c_1} \ldots \vec{c_l}$, we define AB to be*

$$AB = \begin{bmatrix} \langle \vec{r_1}\vec{c_1} \rangle & \ldots & \langle \vec{r_1}\vec{c_l} \rangle \\ \vdots & & \vdots \\ \langle \vec{r_m}\vec{c_1} \rangle & \ldots & \langle \vec{r_m}\vec{c_l} \rangle \end{bmatrix}. \tag{19}$$

Example 11. *Compute the following matrix multiplication:*

$$\begin{bmatrix} 3 & 2 \\ 4 & -1 \\ 0 & -9 \end{bmatrix} \begin{bmatrix} 1 & 4 \\ -1 & -2 \end{bmatrix} = \tag{20}$$

What (visually) does this encode?

Definition 14. *For a matrix A, we define A^T to be the transpose of A by making row 1 of A into column 1 of A^T, row 2 of A into column 2 of A^T, and so on.*

Example 12. *If*

$$A = \begin{bmatrix} -3 & 4 \\ -27 & 0 \end{bmatrix}, \tag{21}$$

compute A^T. What (visually) does this encode?

Definition 15. *We call a matrix 'diagonal' if all of its entries a_{ij}, $i \neq j$, are zero.*

Example 13.

$$A = \begin{bmatrix} 4 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -1 \end{bmatrix} \tag{22}$$

is a diagonal matrix.

Definition 16. *Earlier we defined the standard basis for a vector space V of dimension n by $[\vec{e_1}, \vec{e_2}, \ldots, \vec{e_n}]$. We call the matrix*

$$I = \begin{bmatrix} \vec{e_1} & \vec{e_2} & \ldots & \vec{e_n} \end{bmatrix} \tag{23}$$

the identity matrix. For any matrix A with n rows, $IA = A$ and for any matrix B with n columns $AI = A$.

Proposition 3. *We can easily tell if a list of vectors, $\beta = \{\vec{b_1} \ldots \vec{b_n}\}$ is an orthogonal or orthonormal basis by letting $A = [\vec{b_1} \ldots \vec{b_n}]$ and computing $A^T A$, and checking if it is diagonal (in which case, it is orthogonal). If $A^T A = I$, β is an orthonormal basis.*

Example 14. *Show that $\beta = \{ \begin{bmatrix} 1 \\ -2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix} \}$ is orthogonal, but not orthonormal. What would we have to do to make it orthonormal?*

Example 15. *Show that $\beta = \{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \\ 2 \end{bmatrix} \}$ is not an orthogonal basis. Is it a basis at all?*