Madison Institute for Mathematical Finance: Week One Definitions

September 8, 2025

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1. Calculus I (Differential Calculus)

Key Definitions

- **Limit:** $\lim_{x \to a} f(x) = L$ if values of $f(x)$ approach L as x gets arbitrarily close to a. *Foundation of calculus, formalizing "approach."*
- **Continuity:** f is continuous at a if $\lim_{x \to a} f(x) = f(a)$. *No jumps, holes, or breaks at the point.*
- **Derivative:** $f'(x) = \lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$. *Instantaneous rate of change of f at x.*
- **Differentiability:** f is differentiable at a if $f'(a)$ exists. *Differentiability $\implies$ continuity.*
- **Tangent Line:** $y = f(a) + f'(a)(x - a)$. *Local linear approximation of f.*

Differentiation Rules

- **Power Rule:** $\frac{d}{dx}(x^n) = nx^{n-1}$. Works for any real n, basic rule for polynomials.
- **Constant Rule:** $\frac{d}{dx}(c) = 0$. Constants do not change, so slope is zero.
- **Constant Multiple Rule:** $\frac{d}{dx}[cf(x)] = cf'(x)$. Constants factor out of derivatives.
- **Sum Rule:** $(f + g)' = f' + g'$. Differentiation distributes over addition.
- **Difference Rule:** $(f - g)' = f' - g'$. Same as sum rule, but with subtraction.
- **Product Rule:** $(fg)' = f'g + fg'$. Needed when multiplying functions.
- **Chain Rule:** $(f(g(x)))' = f'(g(x)) \cdot g'(x)$. Differentiates compositions, like peeling layers.
- **Exponential Rule:** $(e^x)' = e^x$. e^x is its own derivative.
- **Logarithm Rule:** $(\ln x)' = \frac{1}{x}$. Logarithms connect to reciprocals.
- **Trigonometric Rules:** $(\sin x)' = \cos x$, $(\cos x)' = -\sin x$, $(\tan x)' = \sec^2 x$. Derivatives of trig functions follow periodic patterns.

Theorems

- **Intermediate Value Theorem (IVT):** A continuous function on $[a, b]$ takes all intermediate values. Guarantees roots/solutions exist inside intervals.
- **Extreme Value Theorem (EVT):** A continuous function on a closed interval attains both a maximum and minimum. Ensures extrema exist.
- **Mean Value Theorem (MVT):** There exists $c \in (a, b)$ with $f'(c) = \frac{f(b)-f(a)}{b-a}$. Tangent slope equals average slope.

- **Rolle's Theorem:** If $f(a) = f(b)$, then some c has $f'(c) = 0$. Guarantees horizontal tangent.
- **L'Hôpital's Rule:** For $\frac{0}{0}$ or $\frac{\infty}{\infty}$ forms, $\lim \frac{f}{g} = \lim \frac{f'}{g'}$. Simplifies tricky limits.
- **FTC Part I:** $\frac{d}{dx} \int_a^x f(t)\,dt = f(x)$. Differentiation and integration are inverses.
- **FTC Part II:** $\int_a^b f(x)\,dx = F(b) - F(a)$. Definite integrals via antiderivatives.

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2. Calculus II (Integral Calculus and Series)

Key Definitions

- **Definite Integral:** Limit of Riemann sums, exact accumulation. Represents area, mass, work, etc.
- **Improper Integral:** Integral with infinite bounds or unbounded integrand, defined as a limit. May converge or diverge.
- **Series:** $\sum_{n=1}^{\infty} a_n$. Infinite sum defined via partial sums.
- **Convergence:** A series converges if partial sums approach a finite limit. Otherwise, it diverges.
- **Power Series:** $\sum c_n (x-a)^n$. Infinite polynomial expansion about a.
- **Taylor Series:** $\sum \frac{f^{(n)}(a)}{n!}(x-a)^n$. Approximates smooth functions locally.

Theorems

- **Integration Techniques:** Substitution, parts, trig, partial fractions. Essential methods for evaluating integrals.
- **Comparison Test:** Compare to known convergent/divergent series. Useful for improper integrals and series.
- **p-Test:** $\sum 1/n^p$ converges iff $p > 1$. Benchmark for convergence.
- **Alternating Series Test:** If $a_n \to 0$ and decreases, $\sum(-1)^n a_n$ converges. Explains convergence of alternating harmonic series.
- **Ratio Test:** If $\lim |a_{n+1}/a_n| < 1$, converges absolutely. Good for factorials/exponentials.
- **Root Test:** If $\lim \sqrt[n]{|a_n|} < 1$, converges absolutely. Useful for powers and exponentials.
- **Taylor Remainder:** $|R_n(x)| \leq \frac{M}{(n+1)!}|x-a|^{n+1}$. Bounds error in polynomial approximation.

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3. Calculus III (Multivariable Calculus)

Key Definitions

- **Partial Derivative:** Differentiate while holding other variables fixed. Measures sensitivity in one variable.
- **Gradient:** $\nabla f = (f_x, f_y, f_z)$. Points in direction of steepest increase.
- **Divergence:** $\vec{\nabla} f = \nabla f \cdot \mathbf{u}$. Rate of change in direction $\mathbf{u}$.
- **Tangent Plane:** $z = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$. Local linear approximation to surface.
- **Double/Triple Integrals:** Integrals over regions/volumes. Compute area, volume, mass, probability.

- **Line Integral:** $\int_C f(x,y)\,ds$ or $\int_C \mathbf{F}\cdot d\mathbf{r}$. Measures work or accumulation along curves.
- **Surface Integral:** $\iint_S \mathbf{F}\cdot d\mathbf{S}$. Measures flux (amount in or out) across surfaces.

Theorems

- **Clairaut's Theorem:** If second partials are continuous, $f_{xy} = f_{yx}$. Order of differentiation doesn't matter.
- **Multivariable Chain Rule:** Differentiates composite multivariable functions. Generalizes chain rule.
- **Second Derivative Test:** Classifies critical points using Hessian determinant. Distinguishes maxima, minima, saddle points.
- **Green's Theorem:** $\oint_C P\,dx + Q\,dy = \iint_D (Q_x - P_y)\,dA$. Links circulation around region to curl inside.
- **Divergence Theorem:** $\iint_{\partial V} F\cdot n\,dS = \iiint_V \nabla\cdot F\,dV$. Relates flux through boundary to divergence inside.
- **Stokes' Theorem:** $\iint_S (\nabla\times F)\cdot dS = \oint_{\partial S} F\cdot dr$. Generalizes Green's theorem to 3D surfaces.
- **FTC for Line Integrals:** If $F = \nabla f$, then $\int_C F\cdot dr = f(B) - f(A)$. Conservative fields give path-independent integrals.