On Differential Equations

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Abstract

This lecture plan is intended for a four-day crash course in differential equations, encapsulating ideas discussed at length in JMU's Math 336. It is not a replacement for an actual course in differential equations, however, it provides exposure to topics from those courses. My notes are **heavily** aided by my notes from Math 336 with Dr. Alex Capaldi; there are some sections which are verbatim because I loved his class so much. If you ever get a chance to take his DiffEQ class, do it.

What is a Differential Equation?

Definition 1. *A Differential Equation (DE) is an equation relating rates of change with respect to one variable. The ones we will study will mostly be of the form:*

$$\frac{dy}{dx} = f(x, y) \tag{1}$$

and

$$\frac{d^2y}{dx^2} + \frac{dy}{dx} = f(x, y). \tag{2}$$

Example 1. *The equation*

$$\frac{dy}{dx} = y + 3x \tag{3}$$

can be solved as follows:

$$dy = (y + 3x)dx \tag{4}$$

$$\int 1dy = \int (y + 3x)dx \tag{5}$$

$$y + c_1 = yx + \frac{3}{2}x^2 + c_2 \tag{6}$$

$$y - yx = \frac{3}{2}x^2 + c_2 - c_1 \tag{7}$$

$$y(1 - x) = \frac{3}{2}x^2 + C \tag{8}$$

$$y = \frac{1}{1 - x}\left[\frac{3}{2}x^2 + C\right]. \tag{9}$$

At this point, we ask, what is C? To properly answer this, we should differentiate y with respect to X, and solve for C.

$$
\begin{aligned}
y' &= \frac{d}{dx}(3x^2)(2-2x)^{-1} + \frac{d}{dx}C(1-x)^{-1} && (10)\\
&= 6x(2-2x)^{-1} + (3x^2)(2)(2-2x)^{-2} - C(1-x)^{-2} && (11)\\
&= \frac{6x}{2-2x} + \frac{6x^2}{(2-2x)^2} - \frac{C}{(1-x)^2} && (12)
\end{aligned}
$$

But we actually know what y' is already:

$$y' = y + 3x \tag{13}$$

So we can set (12) equal to (13) and solve for C.

$$
\begin{aligned}
\frac{6x}{2(1-x)} + \frac{6x^2}{4(1-x)^2} - \frac{C}{(1-x)^2} &= y + 3x && (14)\\
3x(1-x) + \frac{3}{2}x^2 - C &= (y+3x)(1-x)^2 && (15)\\
C &= 3x(1-x) + \frac{3}{2}x^2 - (y+3x)(1-x)^2 && (16)
\end{aligned}
$$

Thus,

$$
\begin{aligned}
y &= \frac{1}{1-x}\left[\frac{3}{2}x^2 + 3x(1-x) + \frac{3}{2}x^2 - (y+3x)(1-x)^2\right] && (17)\\
&= \frac{3x^2}{(1-x)} + 3x - (y+3x)(1-x) && (18)\\
&= \frac{3x^2}{(1-x)} + 3x - (y)(1-x) - (3x)(1-x) && (19)\\
y + y(1-x) &= \frac{3x^2}{(1-x)} + 3x - (3x)(1-x) && (20)\\
y(1+1-x) &= \frac{3x^2}{(1-x)} + 3x^2 && (21)\\
y &= \frac{1}{2-x}\left[\frac{3x^2}{(1-x)} + 3x^2\right] && (22)
\end{aligned}
$$

Is our "closed form solution" to the equation $\frac{dy}{dx} = y + 3x$.

But what does it mean to solve a differential equation? In Example 1, we solved for y, since our rate of change was y'. Thus, we should be able to differentiate (22), plug in y and y' to (13), and have it be a true mathematical statement.

Initial Value Problems

Example 2. *Say we are given the DE,*

$$\frac{dy}{dx} = 3x^2 \tag{23}$$

By our previous solution, we can do

$$\int 1dy = \int 3x^2dx \tag{24}$$

$$y(x) = x^3 + C \tag{25}$$

Since if we were to differentiate this and plug it into (23), we wouldn't be able to find C, we need an ***initial condition****,* $y(x_0) = k$*. Consider* $y(0) = 1$*.*

$$y(0) = 1 = 0^3 + C \tag{26}$$

$$1 = C \tag{27}$$

Thus,

$$y(x) = x^3 + 1 \tag{28}$$

is our solution.

Group Work

Solve the following IVP's.

$$\frac{dy}{dx} = \frac{x+1}{x}; y(0) = 3 \tag{29}$$

$$\frac{dy}{dx} = \frac{x^2+3}{2} - 3; y(0) = 5 \tag{30}$$

$$\frac{dy}{dx} = \frac{-x+4}{7}; y(0) = -6. \tag{31}$$

Classifying DE's

Definition 2. *We call a DE*

- *Linear, if it is of the form*

$$a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_2(x)y'' + a_1(x)y' + a_0(x)y = f(x) \tag{32}$$

- *At equilibrium, if we set* y' *equal to zero*

$$\frac{dy}{dx} = f(x, y) = 0 \tag{33}$$

- *Separable, if it is of the form*

$$\frac{dy}{dx} = f(x)g(y) \tag{34}$$

- *Homogeneous, if it is of the form*

$$a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_2(x)y'' + a_1(x)y' + a_0(x)y = 0. \tag{35}$$

Group Work

Example 3. *Classify the following DE's:*

$$y'' + y'y = 0 \tag{36}$$
$$y'' + y = y' \tag{37}$$
$$y'' - y' + y = 3x + 2 \tag{38}$$
$$y' - y = x^2 y \tag{39}$$
$$y' = y + x \tag{40}$$

Theory of Linear DE's

Theorem 1. *Let y_1, y_2 be solutions to a linear differential equation. The following equation is also a solution, for any $k \in \mathbb{R}$.*

$$ky_1 + ky_2 \tag{41}$$

Theorem 2. *If $y_h(x)$ solves the linear homogeneous DE,*

$$y' + p(x)y = 0, \tag{42}$$

and $y_p(x)$ solves the linear non-homogeneous DE,

$$y' + p(x)y = f(x), \tag{43}$$

then $y = y_p(x) + y_h(x)$ solves (43) as well.

Expanding on theorem two: $y_h(x)$ is in the kernel of the function which solves (43). $y_p(x)$ is the image of that function. Accordingly, we call $y_h(x)$ the homogeneous solution and $y_p(x)$ the particular solution.

Theorem 3. *All first order homogeneous DE's have a solution of the form*

$$y = Ce^{-\lambda x} \tag{44}$$

Theorem 4. *All homogeneous DE's have a solution constructed of sums of exponential and trigonometric functions. This is called a Fourier series.*

General Solutions to Non-Homogeneous DE's (Method of Undetermined Coefficients)

Example 4. *Find all solutions to*

$$y' - y = x. \tag{45}$$

Solution 1. *Break (44) into its homogeneous form first and solve it:*

$$y' - y = 0 \tag{46}$$

$$\frac{dy}{dx} = y \tag{47}$$
$$\frac{dy}{y} = dx \tag{48}$$
$$\int \frac{1}{y} dy = \int dx \tag{49}$$
$$ln|y| = x + c \tag{50}$$
$$y = e^{x+c} \tag{51}$$
$$y = e^c e^x \tag{52}$$
$$y = Ce^x. \tag{53}$$

Next, make a "guess" for what y_p is such that

$$\frac{d}{dx}(Ce^x + y_p(x)) = x + y. \tag{54}$$

Since the RHS of (45) is a degree 1 polynomial, we will guess a degree 1 polynomial.

$$y_p = ax + b. \tag{55}$$

$$y_p' - y_p = a - (ax + b) = -ax + (a - b) \tag{56}$$

Setting (56) equal to RHS of (45), we find

$$-ax + (a - b) = 1x + 0. \tag{57}$$

This really splits into two equations,

$$-ax = x \tag{58}$$
$$a - b = 0. \tag{59}$$

Thus, $a = -1$ *and* $b = -1$ *gives*

$$y_p(x) = -x - 1 \tag{60}$$

and thus,

$$y = y_h + y_p = Ce^x - x - 1 \tag{61}$$

is the solution to (45).

This gives rise to a larger question: for non-homogeneous DE's, can we "guess" the particular solution based on the RHS of the function?

It turns out that we can, although it is important to remember that the non-homogeneous equations we are considering are still **linear**. For non-linear equations, there really may not be a closed-form solution, and often times, there isn't one.

Other important note: if the particular solution is already a part of the homogeneous solution, we multiply by x so that the two are linearly independent.

Forcing function $f(x)$ (RHS)	Typical guess for $y_p(x)$	Notes
Constant k	A	Try a constant. If it's a solution to homogeneous part, multiply by x.
Polynomial of degree n: $a_n x^n + \cdots + a_0$	$Ax^n + Bx^{n-1} + \cdots + K$ (same degree)	Same degree as $f(x)$. If overlap with homogeneous solution, multiply by x.
$e^{\alpha x}$	$Ae^{\alpha x}$	If $e^{\alpha x}$ solves homogeneous equation, multiply by x. If still overlap, multiply by x^2, etc.
Polynomial $\times e^{\alpha x}$	(Polynomial of same degree) $\times e^{\alpha x}$	Again, if overlap, multiply by powers of x.
$\sin(\beta x)$ or $\cos(\beta x)$	$A\cos(\beta x) + B\sin(\beta x)$	If overlap, multiply by x.
Polynomial $\times$ $\sin(\beta x)$ or $\cos(\beta x)$	(Polynomial of same degree) $\times [A\cos(\beta x) + B\sin(\beta x)]$	Multiply by x if needed.
$e^{\alpha x}\cos(\beta x)$ or $e^{\alpha x}\sin(\beta x)$	$e^{\alpha x}(A\cos(\beta x) + B\sin(\beta x))$	If overlap, multiply by x.
Sum of terms (e.g. $x^2 + 3e^{2x} + \sin x$)	Sum of corresponding guesses	Handle each forcing term separately.

Figure 1: Method of Undetermined Coefficients Cheat Sheet

Second Order Homogeneous DE's

To tackle problems of the form

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = 0, \tag{62}$$

we actually need a few things.

1. Preferably, $a_i(x)$ are all constant.
2. Take a guess that $y(x) = e^{\lambda x}$.

The resulting equation is called the characteristic equation (sounds familiar, right?). It's of the form,

$$a\lambda^2 + b\lambda + c. \tag{63}$$

Take its roots!

Distinct Real Roots

If $\lambda_1 \neq \lambda_2 \in \mathbb{R}$, then

$$y = C_1 e^{\lambda_1 x} + C_2 e^{\lambda_2 x} \tag{64}$$

Repeated Real Roots

If $\lambda_1 = \lambda_2 \in \mathbb{R}$, then

$$y = (C_1 + C_2 x)e^{\lambda x} \tag{65}$$

Complex Roots

If $\lambda_1 \neq \lambda_2 \in \mathbb{C}$ such that $\lambda = \alpha \pm \beta i$, then

$$y = e^{\alpha x}(C_1 cos(\beta x) + C_2 sin(\beta x)) \tag{66}$$

Example 5. *Consider the following equations (last three are group work):*

$$y'' + y = xsin(x) \tag{67}$$
$$y'' + y' = cos(x) \tag{68}$$
$$y'' + 4y = sin(2x) \tag{69}$$
$$y'' - 3y' + 2y = e^x \tag{70}$$

Solution 2. *For (62):*

Solve $y'' + y = x \sin x$.

(1) Homogeneous part: *Consider* $y_h'' + y_h = 0$.
Try $y_h = e^{rx}$. $\Rightarrow$ $r^2 e^{rx} + e^{rx} = (r^2+1)e^{rx} = 0$.
Since $e^{rx} \neq 0,\ r^2 + 1 = 0 \ \Rightarrow \ r = \pm i$.
Complex roots $0 \pm i \ \Rightarrow \ y_h(x) = C_1 \cos x + C_2 \sin x$.

(2) Particular solution guess: *RHS is* $x \sin x$, *a degree-*1 *polynomial* $\times \sin x$.
Normally guess $(Ax + B)\cos x + (Cx + D)\sin x$.
But since $\cos x, \sin x \in y_h$, *multiply by* x *to avoid overlap:*
$y_p = x\big((Ax + B)\cos x + (Cx + D)\sin x\big)$.

(3) Substitute and match: $y_p'' + y_p = -4A\,x \sin x + 4C\,x \cos x + (2A + 2D)\cos x + (-2B + 2C)\sin x$.
This must equal $x \sin x$.

Match coefficients: $\begin{cases} -4A = 1, \\ 4C = 0, \\ 2A + 2D = 0, \\ -2B + 2C = 0, \end{cases}$

(4) Particular solution: $y_p(x) = \frac{1}{4}\big(-x^2 \cos x + x \sin x\big)$.

(5) General solution: $\boxed{y(x) = C_1 \cos x + C_2 \sin x + \frac{1}{4}\big(-x^2 \cos x + x \sin x\big)\,.}$

The Laplace Transform

So far, we've covered a few ways to solve linear DE's. One way to solve more complicated DE's is called the Laplace transform - but it actually isn't only motivated by solving DE's. In signals processing, we often want to answer the question, "how similar is one function to another function?" For example, if we have a radio signal $f(t)$ and we want to know how similar it is to a sine way of frequency $\frac{\omega}{2\pi}$ over a symmetric domain $(-N, N)$, we look at:

$$\int_{-N}^{N} f(t) sin(\omega t) dt. \tag{71}$$

If f's frequency is very close to $\frac{\omega}{2\pi}$, the integrand will be very large from positive interference. Otherwise, many of the ups or downs will cancel out and it will be much smaller.

We can do the same with the most common function which shows up as a solution to DE's, $e^{\lambda t}$. To measure the similarity of $f(t)$ to e^{zt}, where $z \in \mathbb{C}$, we look at

$$\int_{-\infty}^{\infty} f(t)e^{-zt}dt. \tag{72}$$

If $f(t) = e^{zt}$, then the integrand is one (because the functions cancel out), and it integrates to infinity. If we split up $z = s + i\omega$ into its real an imaginary parts, we get two transforms:

$$\int_{-\infty}^{\infty} f(t)e^{-st}dt \text{ and } \int_{-\infty}^{\infty} f(t)e^{-i\omega t}dt. \tag{73}$$

The left-hand transform is the Laplace transform; the right-hand transform is the Fourier transform. We use Laplace for ODE's and Fourier for PDE's.

Definition 3. *The Laplace Transform of a function, $\mathcal{L}(f(t))$ is the function $F(s)$, given by:*

$$\mathcal{L}(f(t)) = F(s) = \int_{0}^{\infty} f(t)e^{-st}dt \tag{74}$$

for all s such that the integral converges.

In short, we take an equation from the **time** domain to the **frequency** domain.

Properties of the Laplace Transform

1. $\mathcal{L}(e^{at}) = \frac{1}{s-a}$ for $s > a$
2. $\mathcal{L}(1) = \frac{1}{s}$ for $s > 0$
3. $\mathcal{L}(f'(t)) = s\mathcal{L}(f(t)) - f(0)$; (note that we need an IC)
4. $\mathcal{L}(f''(t)) = s^2\mathcal{L}(f(t)) - sf(0) - f'(0)$
5. $\mathcal{L}(f(t) + g(t)) = \mathcal{L}(f(t)) + \mathcal{L}(g(t))s$
6. $\mathcal{L}(cf(t)) = c\mathcal{L}(f(t))$ for any scalar c.
7. $\mathcal{L}(t^n) = \frac{n!}{s^{n+1}}$
8. $\mathcal{L}(t^n e^{at}) = \frac{n!}{(s-a)^{n+1}}$
9. $\mathcal{L}(sin(bt)) = \frac{b}{s^2+b^2}$
10. $\mathcal{L}(cos(bt)) = \frac{s}{s^2+b^2}$
11. $\mathcal{L}(e^{at}sin(bt)) = \frac{b}{(s-a)^2+b^2}$
12. $\mathcal{L}(e^{at}cos(bt)) = \frac{s-a}{(s-a)^2+b^2}$

Solving IVPs with Laplace

The general method is:

1. Apply the Laplace transform to both sides
2. Solve the resulting equation for $F(s)$
3. Use Laplace transform tables to convert $F(s)$ back to an equation based on t.

Example 6. *Consider the homogeneous example,*

$$y'' + 3y' + 2y = 0, \qquad y(0) = 1, \quad y'(0) = 0.$$

Step 1. Take Laplace transforms.
Recall:

$$\mathcal{L}\{y''(t)\} = s^2Y(s) - sy(0) - y'(0), \quad \mathcal{L}\{y'(t)\} = sY(s) - y(0), \quad \mathcal{L}\{y(t)\} = Y(s).$$

Applying the Laplace transform to both sides:

$$\big(s^2Y(s) - sy(0) - y'(0)\big) + 3\big(sY(s) - y(0)\big) + 2Y(s) = 0.$$

Step 2. Substitute initial conditions.
With $y(0) = 1$, $y'(0) = 0$:

$$(s^2Y(s) - s) + (3sY(s) - 3) + 2Y(s) = 0.$$

Step 3. Collect terms.

$$(s^2 + 3s + 2)Y(s) = s + 3.$$

Hence

$$Y(s) = \frac{s+3}{s^2 + 3s + 2}.$$

Step 4. Factor and decompose.

$$s^2 + 3s + 2 = (s+1)(s+2),$$

so

$$Y(s) = \frac{s+3}{(s+1)(s+2)}.$$

Partial fractions:

$$\frac{s+3}{(s+1)(s+2)} = \frac{A}{s+1} + \frac{B}{s+2}.$$

Multiply through:

$$s + 3 = A(s + 2) + B(s + 1).$$

Choosing $s = -2$: $1 = -B \implies B = -1$.
Choosing $s = -1$: $2 = A \implies A = 2$. Thus,

$$Y(s) = \frac{2}{s+1} - \frac{1}{s+2}.$$

Step 5. Invert the transform.
Recall

$$\mathcal{L}^{-1}\left\{\frac{1}{s+a}\right\} = e^{-at}.$$

So

$$y(t) = 2e^{-t} - e^{-2t}.$$

Final Answer.

$$\boxed{y(t) = 2e^{-t} - e^{-2t}, \quad t \geq 0.}$$

That worked great! But does it work for the more painful non-homogeneous examples?

Example 7. *Consider the non-homogeneous example,*

$$y'' + 4y = 3\cos(t), \qquad y(0) = 0, \quad y'(0) = 0.$$

Step 1. Take Laplace transforms.

$$\mathcal{L}\{y''(t)\} = s^2Y(s) - sy(0) - y'(0), \quad \mathcal{L}\{y(t)\} = Y(s).$$

So

$$(s^2Y(s) - sy(0) - y'(0)) + 4Y(s) = 3 \cdot \frac{s}{s^2+1}.$$

Step 2. Substitute initial conditions.
With $y(0) = 0$, $y'(0) = 0$:

$$(s^2 + 4)Y(s) = \frac{3s}{s^2+1}.$$

Hence

$$Y(s) = \frac{3s}{(s^2+1)(s^2+4)}.$$

Step 3. Partial fraction decomposition.
We want

$$\frac{3s}{(s^2+1)(s^2+4)} = \frac{As+B}{s^2+1} + \frac{Cs+D}{s^2+4}.$$

Multiply through:

$$3s = (As + B)(s^2 + 4) + (Cs + D)(s^2 + 1).$$

Expanding:

$$3s = (A + C)s^3 + (B + D)s^2 + (4A + C)s + (4B + D).$$

Step 4. Match coefficients.
Comparing with $3s$ gives:

$$A + C = 0, \quad B + D = 0, \quad 4A + D = 3, \quad 4B + D = 0.$$

From this: $B = 0, D = 0, A = \frac{3}{4}, C = -\frac{3}{4}$. *Thus*

$$Y(s) = \frac{\frac{3}{4}s}{s^2 + 1} - \frac{\frac{3}{4}s}{s^2 + 4}.$$

Step 5. Inverse transform.
Recall

$$\mathcal{L}^{-1}\left\{\frac{s}{s^2 + a^2}\right\} = \cos(at).$$

So

$$y(t) = \tfrac{3}{4}\cos(t) - \tfrac{3}{4}\cos(2t).$$

Final Answer.

$$\boxed{y(t) = \tfrac{3}{4}\cos(t) - \tfrac{3}{4}\cos(2t), \quad t \geq 0.}$$

Side-note: I wish we had time to cover this thing called the heaviside, Kronecker Delta, and step function which let you artificially add impulses into the DE - essentially asking the question of “what if we have piecewise functions" or “what if something spikes?" It isn’t as applicable to finance, but it’s often used to stress-test DE models of real-world processes. That said, we’ll talk about them some in a different context during our week on Stochastic Differential Equations in week 8.

Systems of Equations & the Phase Plane

Consider

$$\frac{dx}{dt} = P(x, y) \tag{75}$$

$$\frac{dy}{dt} = Q(x, y). \tag{76}$$

We call this system “autonomous" because both equations are not dependent on time. If we set both (75) and (76) equal to zero, we get an equilibrium solution; moreover, based on factors that we will discuss soon, that equilibrium can either be

- Stable (attracts nearby solutions)
- Un-stable (repels nearby solutions in at least one direction)

Example 8. *Consider the system,*

$$\frac{dx}{dt} = y - sin(x) \tag{77}$$
$$\frac{dy}{dt} = x - y. \tag{78}$$

We can actually graph it on something called the ***phase plane****, as shown in Figure 2.*

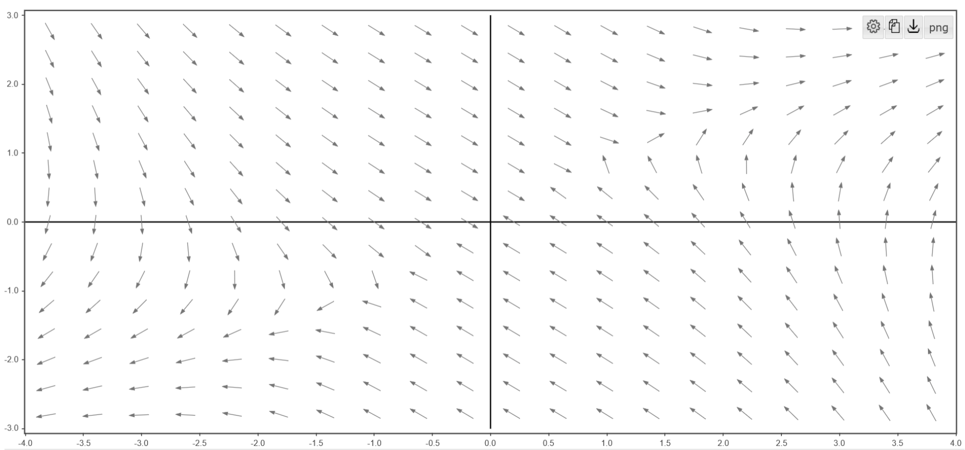

Figure 2: Phase Plane for Example 8

If we pick an initial condition at $t = 0$ for (x,y), we can actually see where it goes in real time. Consider the initial condition $x(0), y(0) = (-2, 1)$.

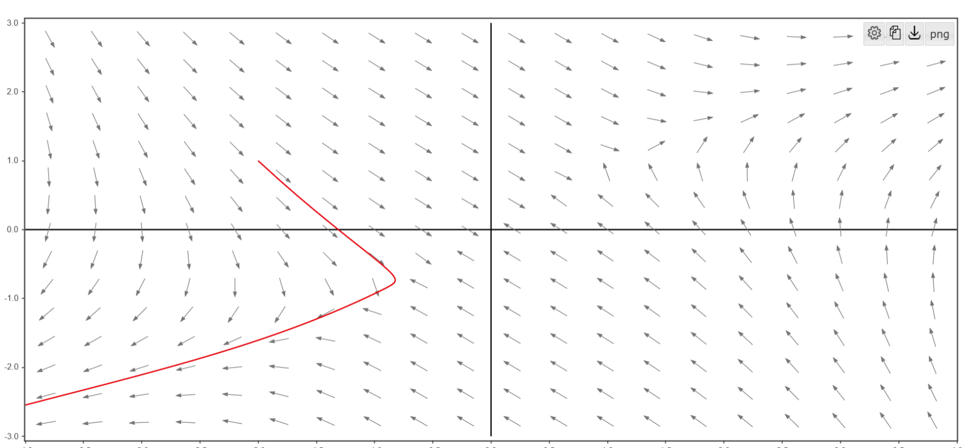

Figure 3: Initial Condition for Example 8

Pretty cool, right? It would probably be even cooler if we could figure out where all solutions converge to. As it turns out, we can do that by finding the equilibrium for each equation, then the equilibrium *solutions for both equations. The equation describing the equilibrium for one equation is called a* ***nullcline****, and it actually strictly determines the path of something in the phase plane. Let's find the nullclines and equilibria for*

$$\frac{dx}{dt} = y - sin(x) \tag{79}$$
$$\frac{dy}{dt} = x - y \tag{80}$$

We start by letting $\frac{dx}{dt} = 0,$ *and,* $0 = \frac{dy}{dt}$*. This yields our nullclines,*

$$y = sin(x) \tag{81}$$
$$y = x. \tag{82}$$

Thus, we have an equilibrium solution when

$$x = sin(x), \tag{83}$$

which only occurs at $x = 0$*. Since* $y = x$*, our equilibrium is* $(0, 0)$*. In a few minutes, we'll be able to determine the stability of this equilibrium mathematically, but for now, it is hopefully clear that it is unstable.*

Theorem 5. *All paths for a system of linear differential equations either converge to equilibrium attractors, or infinity.*

From this point onward, it will be helpful to only consider DE's of the form

$$\frac{dx}{dt} = ax + by + f(t) \tag{84}$$
$$\frac{dy}{dt} = cx + dy + g(t). \tag{85}$$

We can rewrite this in matrix form as

$$\begin{bmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} f(t) \\ g(t) \end{bmatrix} \tag{86}$$

or more succinctly,

$$\vec{x}' = A\vec{x} + f(\vec{t}). \tag{87}$$

If $f(\vec{t}) = 0$, the system is homogeneous.

Theory of Linear Systems of DE's

We first note that any order-n linear DE can be converted to a system of n equations.

Example 9. *Consider that we can represent*

$$x'' + 2x' - 3x = 0 \tag{88}$$

by letting

$$y_1 = x \tag{89}$$

and

$$y_2 = x'. \tag{90}$$

From this, we get

$$y_1' = y_2 \tag{91}$$

$$y_2' = x'' = -2x' + 3x = -2y_2 + 3y_1. \tag{92}$$

Thus, our system of first order linear DE's is

$$y_1' = 0y_1 + y_2 \tag{93}$$

$$y_2' = 3y_1 - y_2. \tag{94}$$

As an A matrix, this becomes

$$\vec{x}' = \begin{bmatrix} 0 & 1 \\ 3 & -2 \end{bmatrix} \vec{x}. \tag{95}$$

As it turns out, to ***solve*** *the system of linear DE's at any point in time, all we have to do is* ***compute the eigenvalues****, and use our method of undetermined coefficients to get a general solution! Recall our method of computing eigenvalues and eigenvectors:*

1. *Compute* $\det(A - \lambda I)$
2. *Solve the characteristic equation (yes, that one) for its roots (the eigenvalues)*
3. *Find a basis for* $\ker(A - \lambda_i I)$*, (the eigenvectors).*

If we follow this process, we get the following characteristic equation

$$(\lambda - 1)(\lambda + 3) \tag{96}$$

which yields $\lambda_1 = 1, \lambda_2 = -3$ *and the corresponding eigenvectors*

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}; \vec{v}_2 = \begin{bmatrix} 1 \\ -3 \end{bmatrix}. \tag{97}$$

Then the solution to our system of equations is

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = C_1 e^{1t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + C_2 e^{-3t} \begin{bmatrix} 1 \\ -3 \end{bmatrix}. \tag{98}$$

Since $x = y_1$*, the solution to our actual DE is*

$$x = C_1 e^t + C_2 e^{-3t}. \tag{99}$$

This is incredible! We can now compute the exact solution for systems of linear DE's,

Theorem 6. *If A has distinct, real eigenpairs,* $(\vec{v}_i, \lambda_i)$*, then the system of equations has a general solution:*

$$\vec{x} = \sum_{i=1}^{n} C_i e^{\lambda_i t} \vec{v}_i. \tag{100}$$

Moreover, the origin is either an attracting node, repelling node, or a saddle point, based on the following:

- $\lambda_i < 0$ *for all* $i \Rightarrow$ *attracting node*
- $\lambda_i > 0$ *for all* $i \Rightarrow$ *repelling node*
- λ_i *mixed* $\Rightarrow$ *saddle point.*

Finally, the distinct eigenvectors generalize the straight line solutions of the system.

The repelling node:

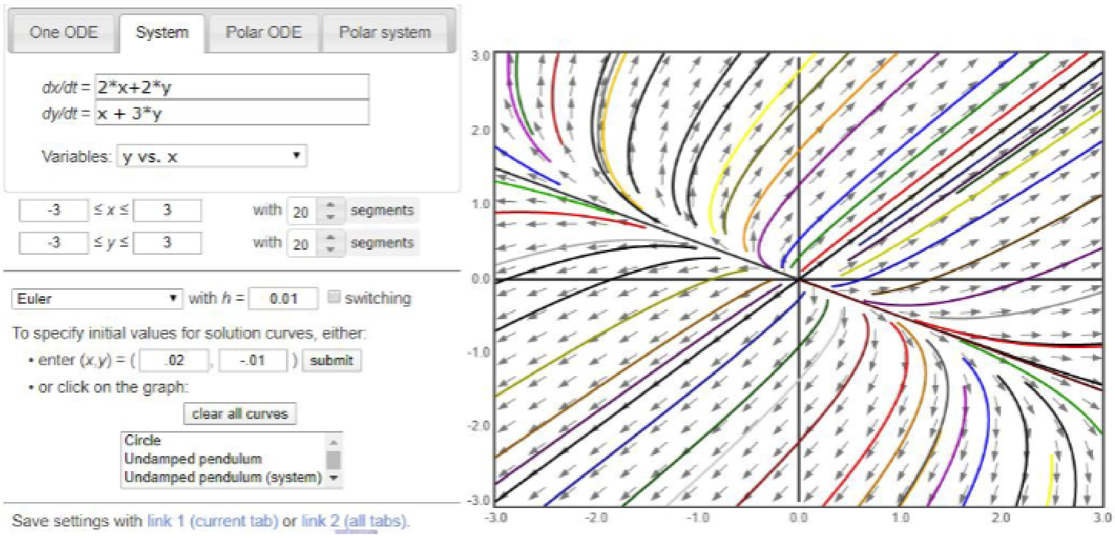

Figure 4: Repelling Node

The saddle point:

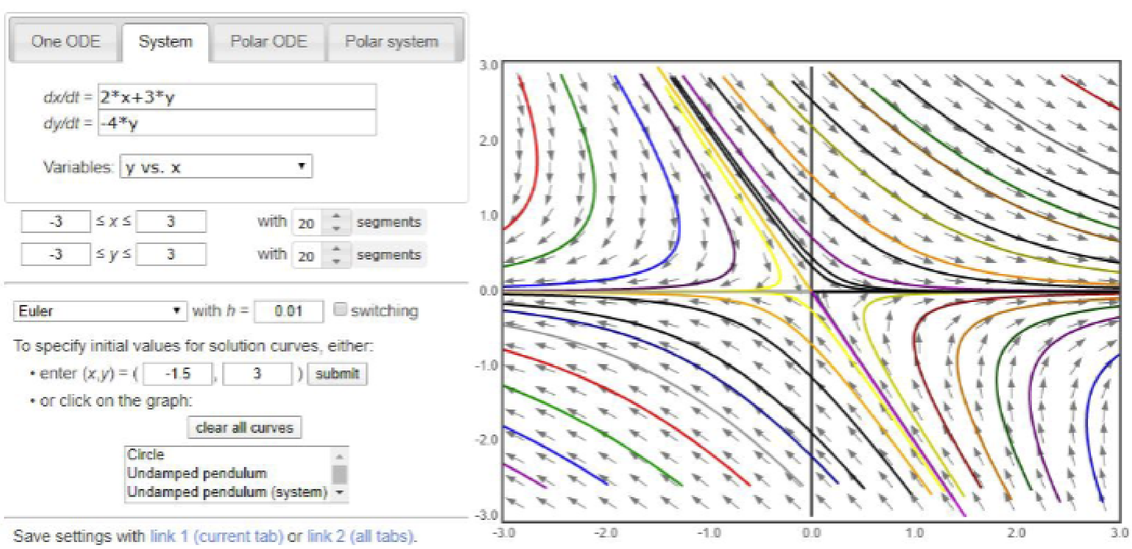

Figure 5: Saddle Point

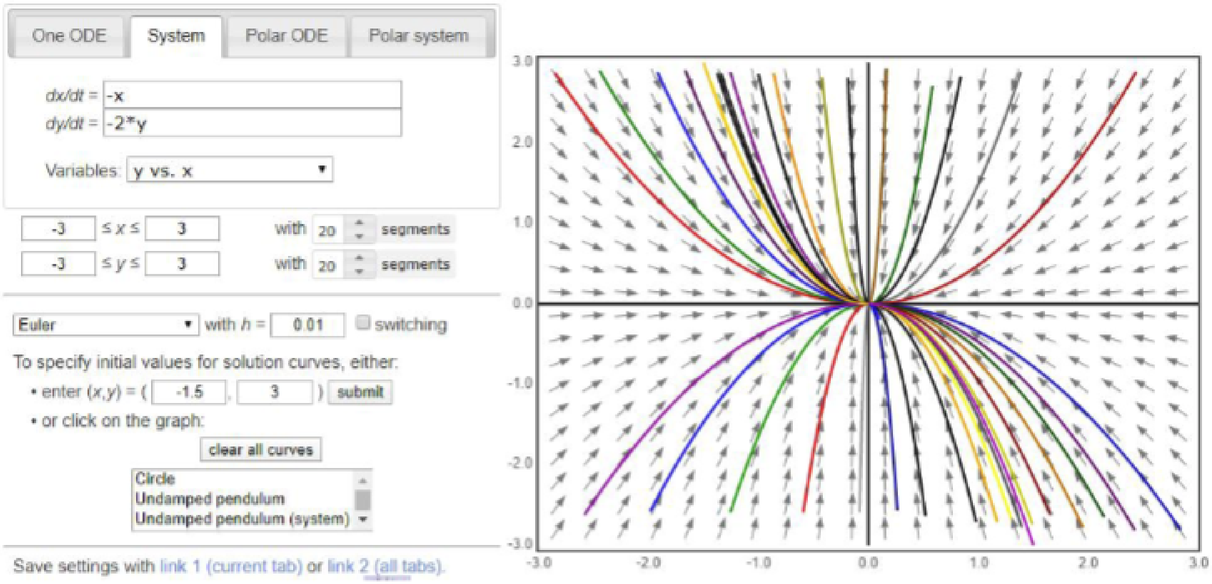

Figure 6: Attracting Node

This is all great, but what if we have complex eigenvalues?

Theorem 7. *If A has distinct, complex eigenpairs, $(\vec{v}_j, \lambda_j)$ with $\lambda_j = \alpha_j + \beta_j i$, then the system of equations has a general solution. It gets more complicated with dimensionality, so I'm going to write it on the board. The solution looks like a* ***spiral****. If $Re(\lambda) > 0$ then it is repelling from the center. If $Re(\lambda) < 0$ then it is attracted to the center. If $Re(\lambda) = 0$ then it is a "center."*

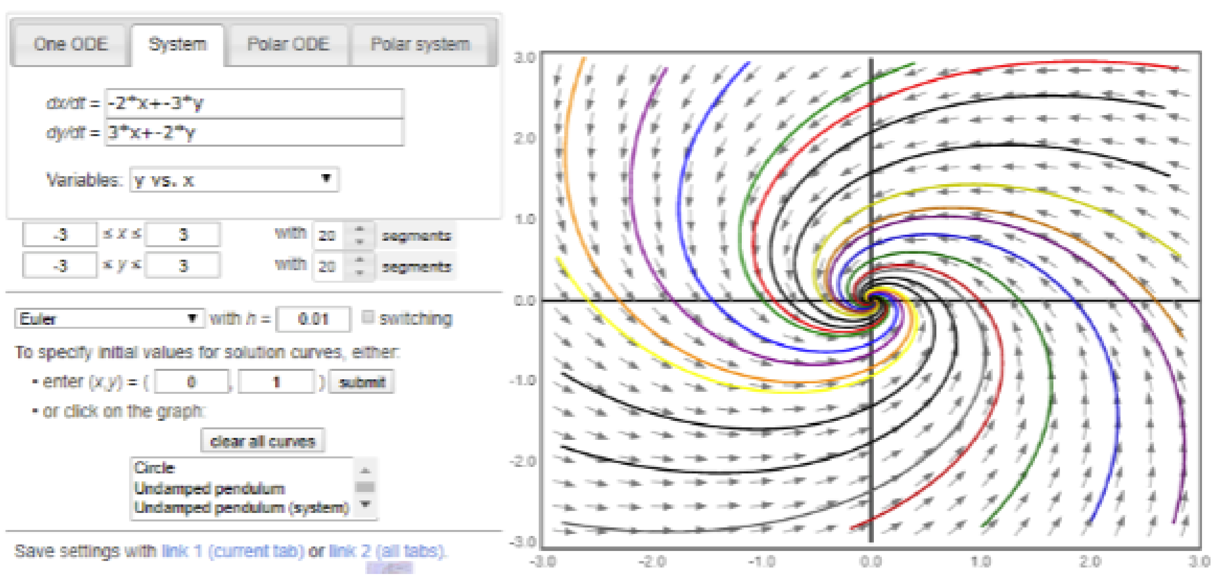

Figure 7: Attracting / Repelling Spiral

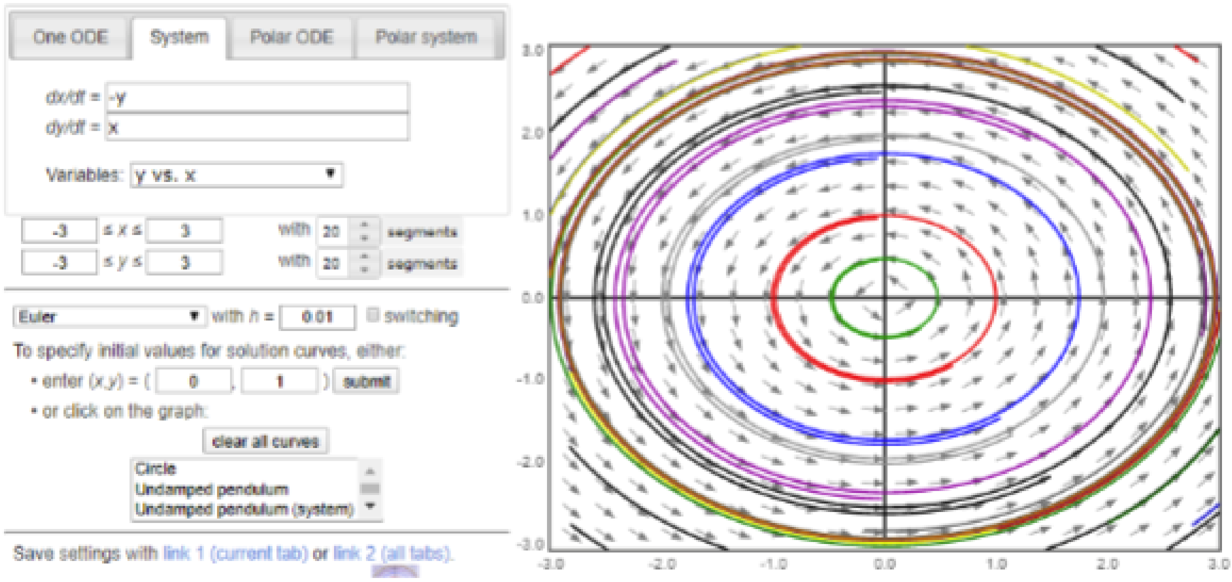

Figure 8: Center

There exists another case for repeated eigenvalues, but for the sake of time, I'm omitting it from the notes.

The Trace Determinant Plane

Definition 4. *The trace of a matrix, $Tr(A)$, is the sum of its diagonal entries.*

For 2×2 systems, the characteristic polynomial is

$$\lambda^2 - Tr(A)\lambda + Det(A) = 0. \tag{101}$$

Its solutions become

$$\lambda = \frac{T \pm \sqrt{T^2 - 4D}}{2}. \tag{102}$$

Thus, we can quickly characterize the types of solutions based on the trace and determinant of A. Let $\Delta = T^2 - 4D$. We can see that the value of Δ splits the trace-determinant plane apart. Then, based on the specific values of λ, we can figure out if the system is attracting, repelling, or degenerate.

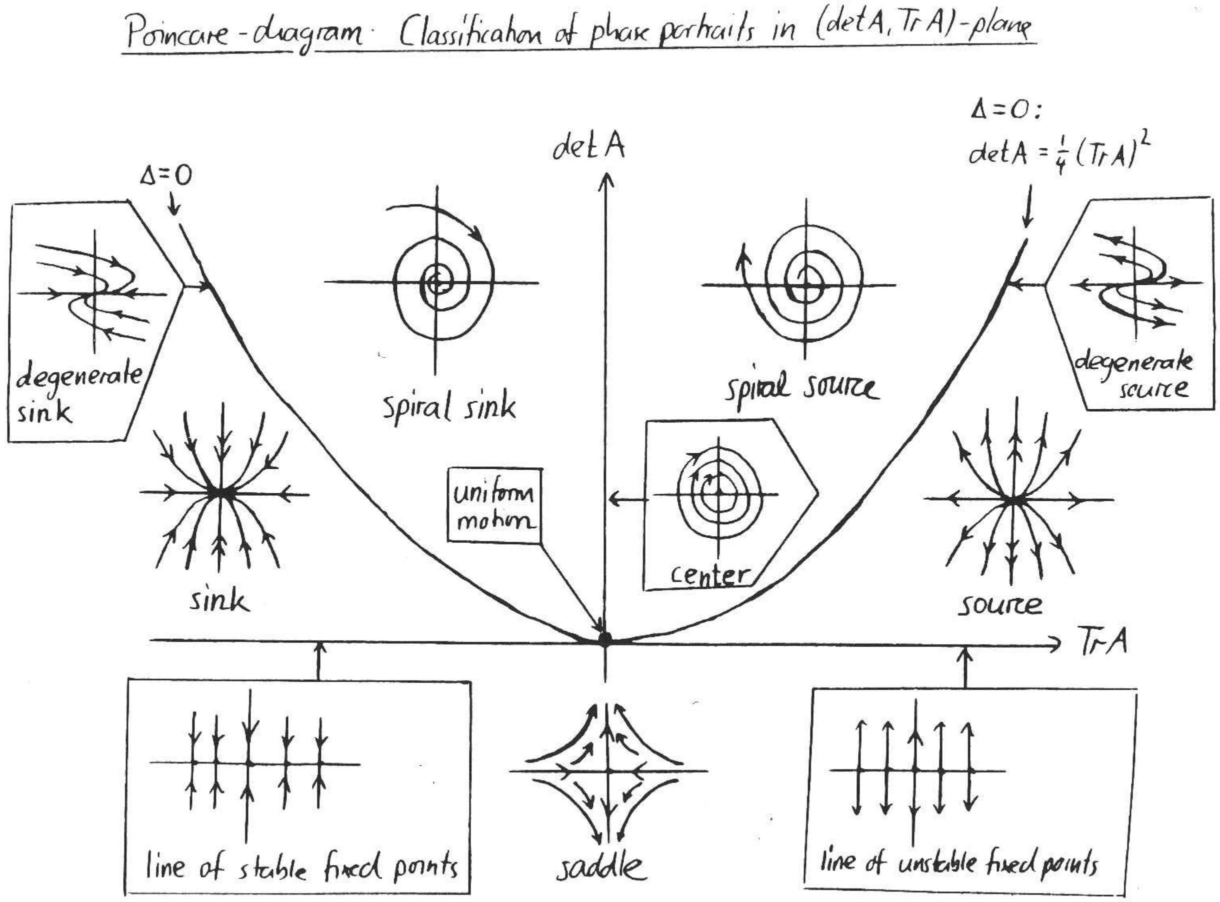

Figure 9: Trace Determinant Plane

Applications of DE's

Lotka-Volterra Equations

Mixing and Cooling Models

Harmonic Oscillator

Numerical Methods

Euler's Method

Monte-Carlo Simulation

Runge-Kutta