MIMF Lecture Week 4, Day 2: Lagrange Interpolation

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Lagrange Interpolation

Motivation: Suppose we know values of a function $f(x)$ at several points:

$$(x_0, y_0), (x_1, y_1), \dots, (x_n, y_n).$$

We want to approximate $f(x)$ for other x values. One natural approach: find a polynomial $p(x)$ of degree $\leq n$ such that

$$p(x_i) = y_i \quad \text{for each } i = 0, 1, \dots, n.$$

This is called the *interpolation problem.*

Idea: To build $p(x)$, it is convenient to construct a set of special *basis polynomials* that "pick out" exactly one y_i at a time.

- We want $L_i(x_j) = 0$ if $j \neq i$, and $L_i(x_i) = 1$.
  - Look familiar? $L_i(x_j) = \delta_{ij}$ (Kronecker Delta!)
- Then $p(x) = \sum_{i=0}^{n} y_i L_i(x)$ will satisfy $p(x_j) = y_j$ for each j.

Construction of $L_i(x)$:

- Consider the product

$$L_i(x) = \underline{\hspace{3cm}}.$$

- Notice:
$$L_i(x_i) = \underline{\hspace{2em}}, \quad L_i(x_j) = \underline{\hspace{2em}} \quad (j \neq i).$$
- So $L_i(x)$ acts like a "selector" that turns on y_i at x_i and turns off all the others.

Formula:
$$p(x) = \sum_{i=0}^{n} y_i L_i(x).$$

Example: Interpolate the integer points $(0, -6), (3, 6), (6, 24)$.

$$L_0(x) = \underline{\hspace{20em}},$$

$$L_1(x) = \underline{\hspace{20em}},$$

$$L_2(x) = \underline{\hspace{20em}}.$$

So

$$= \underline{\hspace{12em}}.$$

Why it works:

- The $L_i(x)$ form a basis of polynomials that each "own" a single data point.
- By multiplying each $L_i(x)$ by y_i and summing, we guarantee interpolation at all x_j.
- The construction avoids solving systems of equations (as in other interpolation methods).

Applications:

- **Quant finance:** Interpolating yield curves and option implied volatility surfaces.
- **Physics:** Approximating physical quantities from discrete experimental data.
- **CS:** Numerical approximation, image resampling, and error-correcting codes.

Note: This method might not be the best for interpolating a high number of points. This is due to the fact that the Lagrange Interpolation method will, generally, create a polynomial of degree n when given n points to interpolate. (This can be a big killer of mathematical models!)