# On Linear Algebra

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**Abstract**

This lecture plan is intended for a four-day crash course in linear algebra, encapsulating ideas discussed at length in JMU's Math 300 and 434. It is not a replacement for an actual course in linear or advanced linear algebra, however, it provides exposure to topics from those courses. Day two covers change of basis, Gaussian elimination for systems of equations, invertibility, matrices as linear transformations, the notion of the image of a linear transformation, the kernel of a linear transformation, an algorithm for finding a basis of both the image and kernel of a linear transformation, the determinant operator, eigenvectors and eigenvalues, how to find the eigenbasis of a matrix, and the spectral decomposition of a normal matrix.

## Day Two

Yesterday, we talked at length about the idea of a basis of a vector space. A key piece of why we wanted to formalize this notion was because we can represent any vector in $V$ as a linear combination of any basis of $V$. So what happens if we are given two bases of $V$, $\beta_1 = \{\vec{b_1} \dots \vec{b_n}\}$ and $\beta_2 = \{\vec{c}_1, \dots \vec{c}_n\}$, and a vector $\vec{v} \in V$ with respect to $\beta_1$,

$$\vec{v}_{\beta_1} = a_1\vec{b_1} + a_2\vec{b_2} + \cdots + a_n\vec{b_n}. \tag{1}$$

Is there a way to convert the RHS of (1) to some linear combination of basis vectors of $\beta_2$? There is!

$$\vec{v}_{\beta_2} = \begin{bmatrix} \langle a_1\vec{b_1}, c_1\rangle \\ \langle a_2\vec{b_2}, c_2\rangle \\ \vdots \\ \langle a_n\vec{b_n}, c_n\rangle \end{bmatrix} \tag{2}$$

We call this a "change of basis" for $V$. It follows that if we can convert a column vector from one basis to another, we can do the same thing for a matrix.

Consider a matrix whose columns form a basis for $V$. We know from our change of basis formulation that we can just as easily represent $V$ with any other basis. Let that other basis be the standard basis. So we can easily convert any matrix whose columns form a basis into the identity matrix,

$$I = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & & & \vdots & \\ 0 & 0 & 0 & \dots & 1 \end{bmatrix}. \tag{3}$$

In fact, the exact change of basis matrix which does this is called the inverse of $A$.

**Definition 1.** *We call $A$, an $n \times n$ matrix, invertible iff there exists a matrix $B$ with $BA = AB = I$. We often denote it $B = A^{-1}$. If there is only such a matrix $B$ with $BA = I$, then we call **B** the left inverse of A. The same notion follows if $AB = I$ but $BA \neq I$; we just call it the right inverse of A.*

Remark: A matrix is only a candidate for invertibility if it is square ($n \times n$).

**Proposition 1.** *A matrix is invertible iff its columns for a basis for V.*

It is at this point that we introduce an algorithm called **Gaussian elimination** for both determining whether a matrix can be inverted and then finding the inverse of a matrix. The rules are simple:

1. We can swap any two rows in a matrix
2. We can multiply any row by a scalar
3. We can add any row to another row.

The goal? Turn our matrix into the identity matrix.

**Definition 2.** *We define U to be an upper triangular matrix if $\{a_{ij} = 0$ for every $i < j\}$. A special case of this matrix is a diagonal matrix.*

**Example 1.**

$$U = \begin{bmatrix} 3 & 4 & 5 \\ 0 & -1 & 0 \\ 0 & 0 & 27 \end{bmatrix} \tag{4}$$

*is an upper triangular matrix.*

**Proposition 2.** *If we can use Gaussian elimination to turn $A$ into an upper triangular matrix, then $A$ is invertible and its columns are a basis for $V$.*

**Example 2.** *Let*

$$\beta = \{ \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -4 \\ 3 \end{bmatrix} \} \tag{5}$$

*be a set of vectors in $\mathbb{R}^3$. Use Gaussian elimination to show that*

$$A = \begin{bmatrix} 1 & -3 & 0 \\ 3 & 1 & -4 \\ 4 & 0 & 3 \end{bmatrix} \tag{6}$$

*is invertible, and $\beta$ is a basis.*

**Definition 3.** *We call the non-zero entries on the diagonal of a row-reduced matrix 'pivots'.*

**Proposition 3.** *If a column of A has a pivot in its row-reduced form, then its original column vector is a basis vector for* $im(A)$.

So all of this is great, but why did we say that we wanted to turn $A$ into $I$ with Gaussian elimination? How does any of this relate back to actually finding an inverse of A?

**Algorithm for Finding an Inverse of A:**

1. Make $[A|I]$ be the $n \times 2n$ matrix where we concatenate A with the Identity matrix.
2. Row reduce $[A|I]$ until $A = I$, applying all of the same transformations to $I$.
3. When $A = I$, the right-hand half of $[A|I]$ will be $A^{-1}$. Thus,

$$[A|I] \to [I|A^{-1}]. \tag{7}$$

**Example 3.** *Invert the following matrix:*

$$A = \begin{bmatrix} 3 & 2 & 0 \\ -1 & 6 & 2 \\ 2 & 0 & 1 \end{bmatrix}, \tag{8}$$

*then check that*

$$AA^{-1} = A^{-1}A = I. \tag{9}$$

Now, we have a way to change bases, invert matrices, check if a set of vectors is a basis for $V$, etc. Where are the applications?

**Example 4.** *Consider the system of equations,*

$$3x + 2y + 0z = -3 \tag{10}$$

$$-x + 6y + 2z = 4 \tag{11}$$

$$2x + 0y + z = 10. \tag{12}$$

*We could just as easily have written*

$$\begin{bmatrix} 3 & 2 & 0 \\ -1 & 6 & 2 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -3 \\ 4 \\ 10 \end{bmatrix}. \tag{13}$$

*I claim that, since we just found* $A^{-1}$ *in example* 3*, we can solve* (13) *as*

$$A\vec{x} = \vec{y} \tag{14}$$

*by computing*

$$A^{-1}A\vec{x} = A^{-1}\vec{y} \tag{15}$$

*or rather,*

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = A^{-1} \begin{bmatrix} -3 \\ 4 \\ 10 \end{bmatrix}. \tag{16}$$

**Group Work:**

**Example 5.** *Solve the following sets of equations using the augmented matrix method. Find $A^{-1}$ and use it to check your work.*

$$\begin{bmatrix} 1 & 2 & 3 \\ -1 & 4 & -5 \\ 0 & 7 & 1 \end{bmatrix} \vec{x} = \begin{bmatrix} 1 \\ 5 \\ 9 \end{bmatrix} \tag{17}$$

$$\begin{bmatrix} -7 & 1 & 3 \\ 0 & 2 & -12 \\ 1 & 3 & 1 \end{bmatrix} \vec{x} = \begin{bmatrix} 6 \\ -5 \\ 0 \end{bmatrix} \tag{18}$$

$$\begin{bmatrix} 0 & 4 & 0 \\ -1 & 3 & 17 \\ 6 & 2 & -1 \end{bmatrix} \vec{x} = \begin{bmatrix} -2 \\ 3 \\ 7 \end{bmatrix} \tag{19}$$

**Proposition 4.** *If $n = dim(V)$, and $A$ has $n$ pivots in its reduced row echelon form, the columns of $A$ are a basis for $V$.*

Following up on this idea, what about if we have less than $n$ pivots in the row-reduced form of $A$? Here, we want to formally define $im(A)$.

**Definition 4.** *Let A be a matrix, $A : V \to W$. We call $im(A) = \{A\vec{v} \in W \mid \vec{v} \in V\}$.*

We said last night that it was easy to have a function $f : A \to B$ with $im(f) = \{f(a)|a \in A\} \subseteq B$, but $im(f) \neq B$. (Ex. $f(x) = x^2$). So what do we consider $B \setminus im(f)$ to be?

**Definition 5.** *We denote rank(A) to be the number of pivots in the row-reduced form of A, and nullity(A) to be $dim(V)$ - the number of pivots in the row-reduced form of A.*

**Proposition 5.** *Rank-Nullity Theorem: $dim(V) = rank(A) + nullity(A)$.*

Since $B \setminus im(f)$ is all of the stuff that $A$ doesn't get sent to, it has to get sent somewhere. It actually ends up working out that everything in $A$ that doesn't get sent to a unique element of $B$ gets sent to the zero vector.

**Definition 6.** *Let A be a matrix, $A : V \to W$. We call $ker(A) = \{\vec{v} \in V \mid A\vec{v} = \vec{0} \in W\}$.*

It's kinda cool - if $A : V \to W$, then $\{A^{-1}(im(A))\} \cup \{ker(A)\} = V$. We can put our domain "back together" from our image! This gives us one more corollary about our idea of a basis.

**Proposition 6.** *$A$ is invertible / the columns of A are a basis for V / A is full-rank (has all of its pivots) iff $ker(A) = \emptyset = \{\}$. (For people who know the term, it also means that $A$ is an injective function - but don't worry about that yet).*

A question crops up now. How do we find a basis for $im(A)$ and $ker(A)$?

**Algorithm for Finding a Basis of $Im(A)$:**

1. Row reduce A to a diagonal matrix.
2. Circle the pivots.
3. Take the columns of the original matrix A corresponding to the columns with pivots in the row-reduced form of A. They form a basis for $Im(A)$.

**Algorithm for Finding a Basis of $Ker(A)$:**

1. Row reduce A to a diagonal matrix.
2. Circle the pivots.
3. Take an arbitrary vector, call it $\vec{w}$. For each i-th column without a pivot, take turns setting exactly one equal to one, and set all others to zero. Solve for what the entries need to be to satisfy
$$A\vec{w} = \vec{0}. \tag{20}$$
4. The set of those vectors are a basis for $ker(A)$.

Remark: if $A$ is non-square (either by virtue of not being linearly independent or not being spanning), we can still find solutions for
$$A\vec{x} = \vec{b}, \tag{21}$$
exactly by finding a basis for $Ker(A)$!

**Example 6.** *Find a basis for all solutions to*
$$\begin{bmatrix} 1 & 3 & 5 \\ -1 & 4 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix} \tag{22}$$

**Group Work:**

**Example 7.** *Find a basis for all solutions to the following examples.*
$$\begin{bmatrix} 0 & -1 & 2 \\ 9 & 7 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 0 \end{bmatrix} \tag{23}$$

$$\begin{bmatrix} -3 & 3 & 4 \\ -1 & 5 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \end{bmatrix} \tag{24}$$

$$\begin{bmatrix} 7 & 4 & 3 \\ 1 & -2 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -5 \\ 6 \end{bmatrix} \tag{25}$$

Running with this idea, I want to make a claim: for any matrix $A$ which is $n \times n$ (not necesarily full-rank), there exists a set of **eigenpairs**, $(\vec{v_i}, \lambda_i)$, such that

$$A\vec{v_i} = \lambda\vec{v_i}. \tag{26}$$

**Definition 7.** *We call $\vec{v_i}$ an eigenvector and $\lambda_i$ an eigenvalue.*

Remark: If $A$ is full rank, $n \times n$, there are $n$-many eigenpairs of A and the set of eigenvectors of $A$ form a basis for $V$ called an **eigenbasis** of $V$.

To actually compute the eigenpairs of $A$, we need one more concept called the determinant. Putting it simply,

**Definition 8.** *The determinant of a matrix is a scalar value representing the accumulated surface area between each vector. We compute it by row-reducing $A$ into its upper-triangular form, then taking the product of its diagonal entries. There are other methods for this, but this scales most easily.*

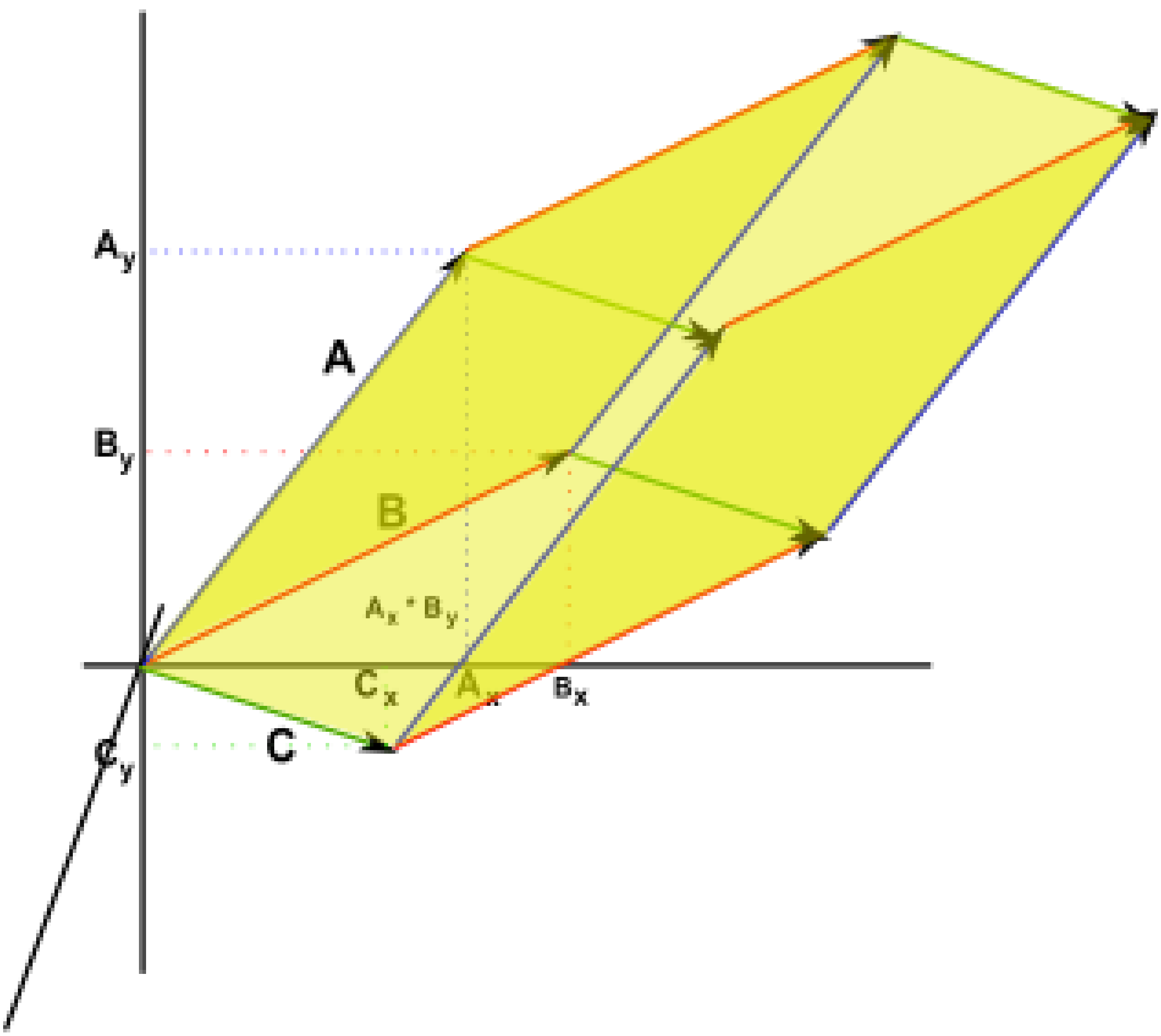


Figure 1: The Determinant in 3D

**Example 8.** *Let*

$$A = \begin{bmatrix} 3 & -1 & 0 \\ 1 & 3 & 2 \\ 2 & 0 & -1 \end{bmatrix}. \tag{27}$$

*Compute* $det(A)$.

**Proposition 7.** *A matrix A is invertible / is a basis for V / is full rank iff its determinant is non-zero iff each eigenpair is unique.*

**Algorithm for Finding Eigenpairs of A:**

1. Compute $A - \lambda I$, where $\lambda \in \mathbb{C}$ is a scalar.
2. Compute $det(A - \lambda I)$; this results in an equation called the **characteristic equation**.
3. Solve the characteristic equation for $\lambda$. Those are your eigenvalues.
4. Now, find a basis for the kernel of each $A - \lambda_j I$. These will be your eigenvectors.

**Example 9.** *Compute the eigenvalues and eigenvectors of*

$$A = \begin{bmatrix} 3 & -1 & 0 \\ 1 & 3 & 2 \\ 2 & 0 & -1 \end{bmatrix}. \tag{28}$$

**Proposition 8.** *For **any** triangular (upper or lower) matrix, its eigenvalues are its diagonal entries.*

**The Punchline: the Spectral Decomposition of A**

For any real-valued matrix $A$ such that $A^T A = AA^T$ (Nick will introduce a special word for this tomorrow), it has the following properties:

1. It is full-rank.
2. Its eigenvectors form a basis for $\mathbb{R}^n$. Call the matrix formed by its column vectors, $U$.
3. If $\lambda_1 \dots \lambda_n$ are the eigenvalues of A, then

$$A = U^T \text{diag}(\lambda_1 \dots \lambda_n) U. \tag{29}$$

(29) is called the Spectral Decomposition of A.

**Proposition 9.** *We can exponentiate any matrix which has a spectral decomposition as follows:*

$$A^k = U^T \textit{diag}(\lambda_1 \dots \lambda_n)^k U. \tag{30}$$

*Notably, since*

$$\textit{diag}(\lambda_1 \dots \lambda_n)^k = \textit{diag}(\lambda_1^k \dots \lambda_n^k), \tag{31}$$

*Then,*

$$A^k = U^T \textit{diag}(\lambda_1^k \dots \lambda_n^k) U. \tag{32}$$