Math 248: Lab 7

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Written Portion

An object falling vertically through the air is subjected to viscous resistance a well as to the force of gravity. Assume that an object with mass m (in kilograms) is dropped from a height s_0 (in meters) and that the height of the object after t seconds is

$$s(t) = s_0 - \frac{mg}{k}t + \frac{m^2 g}{k^2}\left(1 - e^{-\frac{k}{m}t}\right),$$

where $g = 9.8$ meters/second^2 is the Earth's gravity and $k > 0$ represents the coefficient of air resistance (in kilograms/second).

Our goal for the rest of this lab is to solve for the time that this object hits the ground. Assume that it **DOES NOT** bounce.

1. **[3 pts]** This is a rootfinding problem. Assuming that $s_0 > 0$ **and without doing any math**, how many roots are possible? Why? (Seriously, don't do any math. Visualize the physical problem to answer this question.)

2. **[6 pts]** Does it make sense for any of the parameters (m, k, g) to be equal to zero? Justify for each parameter. (Again, don't do any math.)

3. **[15 pts]** Show that $s'(t) = 0$ **ONLY at** $t = 0$. Show lots of work and explain when needed. (Your response to part **2** will be useful here.)

4. **[6 pts]** Assume that you want to use Newton's method to solve this problem. What do your answers to parts **1** and **3** tell you about the likelihood of finding the root given any positive t_0 as an initial guess?

Part 1:

There should be one root! Since the $0 \leqslant t \leqslant T$, $s(0) = s_0$ and $\lim_{t\to\infty} s(t) = -\infty$. Then by the intermediate value theorem, we have at least one root. Moreover, since we don't have any quadratic terms (or higher) for t, we won't recross the x-axis.

Part 2:

If the ball has no mass, that makes no sense. That said, if we plug in $m = 0$, then $s(t) = s_0$, which does make sense - it just has no real world meaning. k can't be zero since k is in the denominator (no air resistance also doesn't make a lot of sense). Finally, $g = 9.8$ m/s^2 by assumption, so g is fixed to be non-zero. If g were zero, then $s(t) = s_0$, which makes sense.

Part 3:

$$\begin{aligned}
s'(t) &= -\frac{mg}{k} - \frac{m^2g}{k^2}\left(\frac{d}{dt}\frac{1}{e^{\frac{kt}{m}}}\right) && (1)\\
&= -\frac{mg}{k} - \frac{m^2g}{k^2}(-\frac{k}{m})e^{-\frac{kt}{m}} && (2)\\
&= -\frac{mg}{k} + \frac{mg}{k}e^{-\frac{kt}{m}} && (3)\\
&= \frac{mg}{k}\left(e^{-\frac{kt}{m}} - 1\right). && (4)
\end{aligned}$$

Since $mg \neq 0$ by part 2,

$$\begin{aligned}
0 &= e^{-\frac{kt}{m}} - 1 && (5)\\
1 &= e^{-\frac{kt}{m}} && (6)\\
ln(1) = 0 &= -\frac{kt}{m} && (7)
\end{aligned}$$

which implies that

$$kt = 0. \tag{8}$$

Since k cannot possibly be zero, t must be zero.

Part 4:

Since we have a unique root, and since the function is smooth, monotone, and concave down, this should work really nicely for Newton's method. The only iffy starting point is really $t = 0$ since that is the only point where $s'(t)$ is zero; since we are starting with any positive t_0, we aren't in any danger of using $t = 0$, so Newton's method should converge!