Math 360 Notes

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Limits, Continuity, and Differentiability

Recall from Real Analysis that:

Definition 1. $\lim_{x\to z_0} f(z) = f(z_0)$ *iff* $\forall \epsilon > 0$, $\exists \delta > 0$ *such that* $d(z, z_0) < \delta$ *implies* $d(f(z) - f(z_0)) < \epsilon$. *Note,* $|\circ|$ *denotes the modulus of a complex number* z.

Suppose $\lim_{z\to z_0} f(z) = L$. Then $L = \alpha + i\beta$, $z_0 = x_0 + iy_0$, and

$$f(z) = u(x, y) + iv(x, y). \tag{1}$$

iff

$$\lim_{(x,y)\to(x_0,y_0)} u(x, y) = \alpha \tag{2}$$

$$\lim_{(x,y)\to(x_0,y_0)} v(x, y) = \beta. \tag{3}$$

It works out that all of our limit rules still hold, including L'Hopital's rule. Also, everything from limits carries over to continuity.

Theorem 1 (Extreme Value Theorem)**.** *Let* $f(z)$ *be a continuous function on a compact region* R*. Then* $f(z)$ *is bounded on* R*. In other words, there is an* $M \in R$ *such that* $|f(z)| < M$ *for any* $z \in R$*. (Continuous images of compact sets attain a maximum and minimum.)*

Differentiability

$$\frac{df}{dz} = \lim_{z\to z_0} \frac{f(z) - f(z_0)}{z - z_0} = \lim_{h\to 0} \frac{f(z + h) - f(z)}{h}. \tag{4}$$

It all works the same way.

Definition 2. *If* $f'(z)$ *exists at every point in some neighborhood of* z_0*, then* $f(z)$ *is said to be analytic* $\iff$ *holomorphic* $\iff$ *regular. If* $f(z)$ *is analytic at every point on a domain* D*, we say it is analytic on* D*.*

Remark: analytic is used in real analysis for function which can be expanded as a power series. In complex analysis, all differential functions can be expressed as a power series; in real analysis, this is not necessarily true.

Proposition 1. *$f(z) = \overline{z}$ (conjugation / reflection) is not differentiable.*

Proposition 2. $\frac{1}{i} = -i$

The Cauchy-Riemann Equations

If f is differentiable, then

$$f'(z) = u_x + iv_x \tag{5}$$

$$f'(z) = -iu_y + v_y \tag{6}$$

implies

$$u_x = v_y \tag{7}$$

$$u_y = -v_x. \tag{8}$$

Note, the converse of this statement is not true. If all of the partial derivatives are continuous, then f is analytic.