Math 360 Homework 1

September 8, 2025

Timothy Tarter
James Madison University
Department of Mathematics

Problem 1.3

Let $z_1, z_2 \in \mathbb{C}$. If $z_1 + z_2 \in \mathbb{R}$ and $z_1 z_2 \in \mathbb{R}$, show that either both $z_1, z_2 \in \mathbb{R}$ or $z_1 = \overline{z_2}$.

Proof: Let $z_1 = x_1 + y_1 i$, $z_2 = x_2 + y_2 i$. If $z_1 + z_2, z_1 z_2 \in \mathbb{R}$, then

$$x_1 + y_1 i + x_2 + y_2 i = (x_1 + x_2) + i(y_1 + y_2) = x_3 + 0i \tag{1}$$
$$(x_1 + y_1 i)(x_2 + y_2 i) = (x_1 x_2 - y_1 y_2) + i(x_1 y_2 + x_2 y_1) = x_4 + 0i. \tag{2}$$

Thus,

$$(x_1 + x_2) + i(y_1 + y_2) = x_3 + 0i \tag{3}$$
$$(x_1 x_2 - y_1 y_2) + i(x_1 y_2 + x_2 y_1) = x_4 + 0i. \tag{4}$$

This yields that,

$$(y_1 + y_2)i = 0i \tag{5}$$
$$y_1 = -y_2 \tag{6}$$

and

$$i(x_1 y_2 + x_2 y_1) = 0i \tag{7}$$
$$x_1 y_2 = -x_2 y_1. \tag{8}$$

Subbing in (6) to (8), we get:

$$(x_1 y_2) = -x_2(-y_2) = x_2 y_2 \tag{9}$$
$$x_1 y_2 = x_2 y_2 \tag{10}$$
$$x_1 = x_2. \tag{11}$$

Thus, either $x_1 = x_2$ and $y_1 = -y_2 \neq 0$ has $z_1 = \overline{z_2}$, or $x_1 = x_2$ and $y_1 = -y_2 = 0$, which means that $z_1, z_2 \in \mathbb{R}$.

$\square$

Problem 1.43

Show that the whole complex plane and the empty set are both open. Are they both closed?

Proposition: Both the whole complex plane and the empty set are open.

Let $z, w \in \mathbb{C}$. Let $\epsilon = d(z,w) = \left[(re(z) - re(w))^2 + (im(z) - im(w))^2\right]^{\frac{1}{2}}$. Then $B(z, \epsilon) \subseteq \mathbb{C}$. Since z, w are arbitrary in $\mathbb{C}$, $\mathbb{C}$ is open. For the empty set, it's vacuously true that it is open; for $\mathbb{C}$ to be endowed with a topology, the empty set needs to be open. That said, since we have no points in the empty set, all "open balls" are nonexistent, and are thus contained in the empty set. So $\emptyset$ is open.

Proposition: The whole complex plane and the empty set are closed.

By definition, a set is closed if its complement is open. Since $\mathbb{C}^c = \emptyset$, and $\emptyset$ is open, $\mathbb{C}$ is closed. Similarly for $\emptyset$, since $\emptyset^c = \emptyset$, which is open, $\emptyset$ is also closed.

$\square$

1.44: For each of the following point sets, find the interior points, exterior points, boundary points, and limit points.

Part A: $0 \leq Re(iz) \leq 3$

Well, if $z = x + iy \in \mathbb{C}$,

$$0 \leq Re(iz) \leq 3 \tag{12}$$
$$\iff \tag{13}$$
$$0 \leq Re(i(x+iy)) = Re(-y + ix) \leq 3 \tag{14}$$
$$0 \leq -y \leq 3 \tag{15}$$
$$0 \geq y \geq -3. \tag{16}$$

Thus, the set with points satisfying $0 \leq Re(iz) \leq 3$ is equal to $\{x + iy \mid 0 \geq y \geq -3\}$.

Interior Points

Its interior points are:

$$\{x + iy \mid 0 > y > -3\}. \tag{17}$$

Exterior Points

Its exterior points are:

$$\{x + iy \mid \{-\infty > y > -3\} \cup \{0 < y < \infty\}\}. \tag{18}$$

Boundary Points

Its boundary points are:

$$\{x + iy \mid y = 0 \text{ or } y = -3\} \tag{19}$$

Limit Points

Its limit points are:

$$\{x + iy \mid 0 \geq y \geq -3\} \tag{20}$$

since any deleted neighborhood of these points will be nonempty, including at the boundary.

Part B: $0 \leq Arg(z) < \frac{\pi}{4}$ and $|z| > 2$

This is exactly the set,

$$r > 2 \tag{21}$$

and

$$Re(z) \in (rcos(\frac{\pi}{4}), r) \tag{22}$$

$$Im(z) \in (0, rsin(\frac{\pi}{4})). \tag{23}$$

Thus,

$$x^2 + y^2 > 4 \tag{24}$$

$$\sqrt{x^2 + y^2}cos(\frac{\pi}{4}) < x < \sqrt{x^2 + y^2} \tag{25}$$

$$0 < y < \sqrt{x^2 + y^2}sin(\frac{\pi}{4}) \tag{26}$$

becomes

$$x^2 + y^2 > 4 \tag{27}$$

$$cos(\frac{\pi}{4}) < \frac{x}{x^2 + y^2} < 1 \tag{28}$$

$$0 < \frac{y}{\sqrt{x^2 + y^2}} < sin(\frac{\pi}{4}). \tag{29}$$

Interior Points

Its interior points are all points $z = (x + yi)$ satisfying

$$x^2 + y^2 > 4 \tag{30}$$

$$cos(\frac{\pi}{4}) < \frac{x}{x^2 + y^2} < 1 \tag{31}$$

$$0 < \frac{y}{\sqrt{x^2 + y^2}} < sin(\frac{\pi}{4}). \tag{32}$$

Exterior Points

Its exterior points are all points $z = (x + yi)$ satisfying

$$x^2 + y^2 \leq 4 \tag{33}$$

$$\frac{x}{x^2 + y^2} \leq cos(\frac{\pi}{4}) \text{ or } 1 \leq \frac{x}{x^2 + y^2} \tag{34}$$

$$\frac{y}{\sqrt{x^2 + y^2}} \leq 0 \text{ or } sin(\frac{\pi}{4}) \leq \frac{y}{\sqrt{x^2 + y^2}}. \tag{35}$$

Boundary Points

Its boundary points are all points $z = (x + yi)$ satisfying

$$x^2 + y^2 = 4 \tag{36}$$

$$cos(\frac{\pi}{4}) = \frac{x}{x^2 + y^2} \text{ or } \frac{x}{x^2 + y^2} = 1 \tag{37}$$

$$0 = \frac{y}{\sqrt{x^2 + y^2}} \text{ or } \frac{y}{\sqrt{x^2 + y^2}} = sin(\frac{\pi}{4}). \tag{38}$$

Limit Points

Its limit points are all points $z = (x + yi)$ satisfying

$$x^2 + y^2 \geq 4 \tag{39}$$

$$cos(\frac{\pi}{4}) \leq \frac{x}{x^2 + y^2} \leq 1 \tag{40}$$

$$0 \leq \frac{y}{\sqrt{x^2 + y^2}} \leq sin(\frac{\pi}{4}). \tag{41}$$