Math 360 Homework 2

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Problem 2.30a: Find an analytic function $f(z)$ whose real part $u(x,y) = y^3 - 3x^2y$, $f(0+i) = 1+i$.

Taking partials we get,

$$u_x = -6xy \tag{1}$$
$$u_y = 3y^2 - 3x^2. \tag{2}$$

Using the CR equations,

$$u_x = v_y \tag{3}$$
$$v_x = -u_y \tag{4}$$

we know that

$$v_y = -6xy \tag{5}$$
$$v_x = -3y^2 + 3x^2 \tag{6}$$

so

$$-6\int xydy = -3xy^2 + f(x) \tag{7}$$
$$3\int -y^2 + x^2dx = -3y^2x + x^3 + f(y). \tag{8}$$

Then

$$f(x) = x^3 \tag{9}$$
$$f(y) = 0. \tag{10}$$

So,

$$v(x,y) = -3xy^2 + x^3 + C \tag{11}$$

is the general analytic function whose imaginary part is a harmonic conjugate for $u(x, y)$. To solve for C, we plug in

$$f(0,1) = 1 + i = 1 + i(C). \tag{12}$$

So, $C = 1$

$$f(z) = y^3 - 3x^2y + i(-3xy^2 + x^3 + 1) \tag{13}$$

works out to be the analytic function,

$$f(z) = (y^3 - 3x^2y) + i(-3xy^2 + x^3 + 1). \tag{14}$$

Problem 2.32a: Determine the values of the parameters appearing in the following function such that the function becomes analytic. $f(z) = (x^2 + axy + by^2) + i(cx^2 + dxy + y^2)$

For $f(z)$ to be analytic, it needs to satisfy the CR equations everywhere (I'm assuming that analytic = entire here). Taking partials, we get

$$\begin{aligned} u_x &= 2x + ay & (15)\\ u_y &= 2by + ax & (16)\\ v_x &= 2cx + dy & (17)\\ v_y &= 2y + dx. & (18) \end{aligned}$$

Then by CR,

$$\begin{aligned} 2x + ay &= 2y + dx & (19)\\ &\text{and} & (20)\\ -2by - ax &= 2cx + dy. & (21) \end{aligned}$$

Separating these equations by independent variables, we derive

$$\begin{aligned} 2x &= dx & (22)\\ ay &= 2y & (23)\\ -2by &= dy & (24)\\ -ax &= 2cx & (25) \end{aligned}$$

Dividing out the independent variables x and y, we get

$$\begin{aligned} d &= 2 & (26)\\ a &= 2 & (27)\\ -2b = 2 \Rightarrow b &= -1 & (28)\\ -2 = 2c \Rightarrow c &= -1. & (29) \end{aligned}$$

Therefore, the function which is analytic is:

$$f(x) = (x^2 + 2xy - y^2) + i(-x^2 + 2xy + y^2) \tag{30}$$

□

Problem 2.34: If $u(x, y)$ is a harmonic function, determine whether or not u^2 is also harmonic.

If u is harmonic, it satisfies

$$u_{xx} + u_{yy} = 0. \tag{31}$$

Then for u^2 to be harmonic,

$$(u^2)_{xx} + (u^2)_{yy} = 0. \tag{32}$$

Differentiating each expression, we get

$$(u^2)_{xx} = (2u * u_x)_x = 2(u_x^2 + u * u_{xx}) \tag{33}$$

$$(u^2)_{yy} = (2u * u_y)_y = 2(u_y^2 + u * u_{yy}) \tag{34}$$

Then, by (31), we should satisfy

$$2(u_x^2 + u * u_{xx}) + 2(u_y^2 + u * u_{yy}) = 0 \tag{35}$$

$$\text{however, if we expand this, we get} \tag{36}$$

$$u_x^2 + u_y^2 + u(u_{xx} + u_{yy}) = 0 \tag{37}$$

$$u_x^2 + u_y^2 + u * 0 = 0 \tag{38}$$

$$u_x^2 + u_y^2 = 0 \tag{39}$$

which only holds if u is a constant function. Therefore, any non-constant harmonic function u does not satisfy $\Delta(u^2) = 0$.

□