

Math 360 Homework 5

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1 Find all of the zeros of $f(z) = \sin z$ and determine their orders.

$$\begin{aligned}\sin(z) &= \sum_{n=0}^{\infty} z^{2n+1} \frac{(-1)^n}{(2n+1)!} = z - \frac{z^3}{5!} + \frac{z^5}{7!} - \dots \\ &= z \left(1 - \frac{z^2}{5!} + \frac{z^4}{7!} \right)\end{aligned}$$

so $\sin(z)$ has a zero of order 1 at $z = 0$. Since $n\pi$ is also a zero of $\sin(z)$, each $n\pi$ (with $n \in \mathbb{Z}$) is a zero of order one.

2 Consider the function $f(z) = \frac{e^{-iz}}{z^2+1}$. Find all of the singularities of this function and evaluate the residue at each of them.

$$\frac{1}{z^2+1}e^{-iz} = \frac{1}{z^2+1} \sum_{n=0}^{\infty} \frac{(-iz)^n}{n!} = \frac{1}{z^2+1} \left(1 - iz - \frac{z^2}{2!} + \frac{iz^3}{3!} + \dots \right)$$

Since $z^2 + 1$ has zeroes at $\pm i$, we have simple poles at $z = \pm i$. Using the fact that, for simple poles,

$$\text{Res}(f, z_0) = \lim_{z \rightarrow z_0} (z - z_0)f(z)$$

we can start by computing $z_0 = i$.

$$\text{Res}(f, i) = \lim_{z \rightarrow i} (z - i)f(z) = \lim_{z \rightarrow i} (z - i) \frac{e^{-iz}}{z^2 + 1} = \lim_{z \rightarrow i} (z - i) \frac{e^{-iz}}{(z - i)(z + i)} = \lim_{z \rightarrow i} \frac{e^{-iz}}{z + i} = \frac{e}{2i}$$

Then, computing $z_0 = -i$,

$$\text{Res}(f, -i) = \lim_{z \rightarrow -i} (z + i)f(z) = \lim_{z \rightarrow -i} (z + i) \frac{e^{-iz}}{z^2 + 1} = \lim_{z \rightarrow -i} (z + i) \frac{e^{-iz}}{(z - i)(z + i)} = \lim_{z \rightarrow -i} \frac{e^{-iz}}{z - i} = -\frac{e^{-1}}{2i}.$$

So the residues are $\frac{e}{2i}$ and $-\frac{1}{2ei}$.

- 3** Let C be the circle centered at the point $z = i$. Evaluate the contour integral of the above function.

$$\int_C \frac{e^{-iz}}{z^2 + 1} dz = 2\pi i \operatorname{Res}(f(z), i) = 2\pi i * \frac{e}{2i} = \pi e.$$

- 4** Let C be the unit circle. Evaluate the following integral.

$$\int_C \frac{z^2 + 2z + 5}{z^2 + 2iz - 3z - 6i} dz = \int_C \frac{z^2 + 2z + 5}{(z - 3)(z + 2i)}$$

Accordingly, we have simple poles at $z = 3, -2i$. Since C is the unit circle, we have no poles inside C , so by Cauchy's integral theorem

$$\int_C \frac{z^2 + 2z + 5}{(z - 3)(z + 2i)} dz = 0$$