Math 360 Homework 5

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1 Find all of the zeros of $f(z) = \sin z$ and determine their orders.

$$\sin(z) = \sum_{n=0}^{\infty} z^{2n+1} \frac{(-1)^n}{(2n+1)!} = z - \frac{z^3}{5!} + \frac{z^5}{7!} - \dots$$
$$= z\left(1 - \frac{z^2}{5!} + \frac{z^4}{7!}\right)$$

so $\sin(z)$ has a zero of order 1 at $z = 0$. Since $n\pi$ is also a zero of sin(z), each $n\pi$ (with $n \in \mathbb{Z}$) is a zero of order one.

2 Consider the function $f(z) = \frac{e^{-iz}}{z^2+1}$. Find all of the singularities of this function and evaluate the residue at each of them.

$$\frac{1}{z^2+1} e^{-iz} = \frac{1}{z^2+1} \sum_{n=0}^{\infty} \frac{(-iz)^n}{n!} = \frac{1}{z^2+1}\left(1 - iz - \frac{z^2}{2!} + \frac{iz^3}{3!} + \dots\right)$$

Since $z^2 + 1$ has zeroes at $\pm i$, we have simple poles at $z = \pm i$. Using the fact that, for simple poles,

$$\operatorname{Res}(f, z_0) = \lim_{z \to z_0} (z - z_0) f(z)$$

we can start by computing $z_0 = i$.

$$\operatorname{Res}(f, i) = \lim_{z \to i} (z - i) f(z) = \lim_{z \to i} (z - i) \frac{e^{-iz}}{z^2+1} = \lim_{z \to i} (z - i) \frac{e^{-iz}}{(z-i)(z+i)} = \lim_{z \to i} \frac{e^{-iz}}{z+i} = \frac{e}{2i}$$

Then, computing $z_0 = -i$,

$$\operatorname{Res}(f, -i) = \lim_{z \to -i} (z+i) f(z) = \lim_{z \to -i} (z+i) \frac{e^{-iz}}{z^2+1} = \lim_{z \to -i} (z+i) \frac{e^{-iz}}{(z-i)(z+i)} = \lim_{z \to -i} \frac{e^{-iz}}{z-i} = -\frac{e^{-1}}{2i}.$$

So the residues are $\frac{e}{2i}$ and $-\frac{1}{2ei}$.

3 Let C be the circle centered at the point $z = i$. Evaluate the contour integral of the above function.

Assuming that C doesn't contain $-i$, just by how this is worded,

$$\int_C \frac{e^{-iz}}{z^2+1} dz = 2\pi i \text{ Res}(f(z), i) = 2\pi i * \frac{e}{2i} = \pi e.$$

4 Let C be the unit circle. Evaluate the following integral.

$$\int_C \frac{z^2+2z+5}{z^2+2iz-3z-6i} dz = \int_C \frac{z^2+2z+5}{(z-3)(z+2i)}$$

Accordingly, we have simple poles at $z = 3, -2i$. Since C is the unit circle, we have no poles inside C, so by Cauchy's integral theorem

$$\int_C \frac{z^2+2z+5}{(z-3)(z+2i)} dz = 0$$