Math 360 Homework 1

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Timothy Tarter
James Madison University
Department of Mathematics

Problem 1.3

Let $z_1, z_2 \in \mathbb{C}$. If $z_1 + z_2 \in \mathbb{R}$ and $z_1 z_2 \in \mathbb{R}$, show that either both $z_1, z_2 \in \mathbb{R}$ or $z_1 = \overline{z_2}$.

Proof: Let $z_1 = x_1 + y_1 i$, $z_2 = x_2 + y_2 i$. If $z_1 + z_2, z_1 z_2 \in \mathbb{R}$, then

$$x_1 + y_1 i + x_2 + y_2 i = (x_1 + x_2) + i(y_1 + y_2) = x_3 + 0i \tag{1}$$
$$(x_1 + y_1 i)(x_2 + y_2 i) = (x_1 x_2 - y_1 y_2) + i(x_1 y_2 + x_2 y_1) = x_4 + 0i. \tag{2}$$

Thus,

$$(x_1 + x_2) + i(y_1 + y_2) = x_3 + 0i \tag{3}$$
$$(x_1 x_2 - y_1 y_2) + i(x_1 y_2 + x_2 y_1) = x_4 + 0i. \tag{4}$$

This yields that,

$$(y_1 + y_2)i = 0i \tag{5}$$
$$y_1 = -y_2 \tag{6}$$

and

$$i(x_1 y_2 + x_2 y_1) = 0i \tag{7}$$
$$x_1 y_2 = -x_2 y_1. \tag{8}$$

Subbing in (6) to (8), we get:

$$(x_1 y_2) = -x_2(-y_2) = x_2 y_2 \tag{9}$$
$$x_1 y_2 = x_2 y_2 \tag{10}$$
$$x_1 = x_2. \tag{11}$$

Thus, either $x_1 = x_2$ and $y_1 = -y_2 \neq 0$ has $z_1 = \overline{z_2}$, or $x_1 = x_2$ and $y_1 = -y_2 = 0$, which means that $z_1, z_2 \in \mathbb{R}$.

$\square$

Problem 1.43

Show that the whole complex plane and the empty set are both open. Are they both closed?

Proposition: Both the whole complex plane and the empty set are open.

Let $z, w \in \mathbb{C}$. Let $\epsilon = d(z, w) = \left[(re(z) - re(w))^2 + (im(z) - im(w))^2\right]^{\frac{1}{2}}$. Then $B(z, \epsilon) \subseteq \mathbb{C}$. Since z, w are arbitrary in $\mathbb{C}$, $\mathbb{C}$ is open. For the empty set, it's vacuously true that it is open; for $\mathbb{C}$ to be endowed with a topology, the empty set needs to be open. That said, since we have no points in the empty set, all "open balls" are nonexistent, and are thus contained in the empty set. So $\emptyset$ is open.

Proposition: The whole complex plane and the empty set are closed.

By definition, a set is closed if its complement is open. Since $\mathbb{C}^c = \emptyset$, and $\emptyset$ is open, $\mathbb{C}$ is closed. Similarly for $\emptyset$, since $\emptyset^c = \emptyset$, which is open, $\emptyset$ is also closed.

$\square$