Math 360 Homework 5

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Problem 1:

Let $C = 2e^{i\theta} + 1$. Evaluate

$$\int_C \frac{e^z}{z-1} dz.$$

Recall Cauchy's integral formula, which states

$$f(z_0)2\pi i = \int_C \frac{f(z)}{z-z_0} dz.$$

Clearly, $f(z) = e^z$. So,

$$e^1 2\pi i = \int_C \frac{e^z}{z-1} dz.$$

Problem 2:

Evaluate

$$\sum_{k=1}^{\infty} \frac{(1+2i)^k}{5^k} = \sum_{k=1}^{\infty} \left(\frac{1+2i}{5}\right)^k.$$

Since

$$|\frac{1+2i}{5}| = \frac{1}{5}|1+2i| = \frac{1}{5}\sqrt{1^2+2^2} = \frac{\sqrt{5}}{5} < 1,$$

by properties of the Geometric series,

$$\sum_{k=1}^{\infty} \frac{(1+2i)^k}{5^k} = \frac{\frac{1+2i}{5}}{1 - \frac{1+2i}{5}}$$

since the series starts at $k = 1$. Since

$$1 - \frac{1}{5} - \frac{2}{5}i = \frac{4-2i}{5},$$

$$\frac{\frac{1+2i}{5}}{\frac{4-2i}{5}} = \frac{1+2i}{4-2i} = \frac{(1+2i)(4+2i)}{16+4} = \frac{1}{20}(4-4+10i) = \frac{i}{2}.$$

So

$$\sum_{k=1}^{\infty} \frac{(1+2i)^k}{5^k} = \frac{i}{2}.$$

Problem 3:

Can we use the ratio test to show that

$$\sum_{n=1}^{\infty} \frac{e^{in\theta}}{n^2}$$

is absolutely convergent?

Nope!

$$\lim_{n\to\infty} |\frac{a_{n+1}}{a_n}| = \lim_{n\to\infty} |\frac{e^{i\theta(n+1)}n^2}{(n+1)^2 e^{i\theta n}}| = \lim_{n\to\infty} |e^{i\theta}\frac{n^2}{(n+1)^2}|$$

By L'Hospital's rule,

$$= |e^{i\theta}| = 1.$$

Notably, for any θ, the ratio test must be inconclusive.

Problem 4:

Every convergent sequence, $\{a_n\} \to a$ is bounded.

Proof: Since for every $\epsilon > 0$, there exists an $N \in \mathbb{Z}$ such that for all $n \geq N$, $d(a, a_n) < \epsilon$, the last all but finitely many terms of the sequence are bounded in a ball around a. But then exactly finitely many terms are outside of this ball, so there exists at least one $n \in 1 \ldots N$ which satisfies

$$d(a_n, a) = \max\{d(a_m, a) | m \in 1 \ldots N\}.$$

Call this distance δ. Then for any $\epsilon > 0$,

$$\{a_n | n \in \mathbb{Z}\} \subseteq B(a, \delta + \epsilon).$$

□

Not every bounded sequence is convergent.

Example:

$$\{(-1)^n\}_{n\in\mathbb{Z}^{>0}}$$

is bounded but not convergent.

Problem 5:

Determine whether the following series converges:

$$\sum_{n=0}^{\infty} \frac{(4i)^n}{n!}$$

Ratio test!

$$\lim_{n\to\infty} |\frac{(4i)^{n+1}n!}{(n+1)!(4i)^n}| = \lim_{n\to\infty} |\frac{4i}{n+1}| = |4i| \lim_{n\to\infty} |\frac{1}{n+1}| = 0.$$

So the series converges.