Math 360 Notes

September 11, 2025

Timothy Tarter
James Madison University
Department of Mathematics

The Complex Exponential Function

Problem: the complex exponential function. Answer: Taylor series!

$$e^x = \sum_{i=0}^{\infty} \frac{x^i}{i!} \tag{1}$$

works for $x \in \mathbb{C}$!

$$z = re^{i\theta} = r\sum_{n=0}^{\infty} \frac{(i\theta)^n}{n!} \tag{2}$$

With a little calculus,

$$Re(e^{i\theta}) = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \cdots = cos\theta \tag{3}$$

$$Im(e^{i\theta} = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots = sin\theta. \tag{4}$$

So,

$$e^{i\theta} = cos\theta + isin\theta \tag{5}$$

. Let $\theta = \pi$. Then

$$e^{i\pi} = cos\pi + isin\pi = - \tag{6}$$

$$e^{i\pi} + 1 = 0. \tag{7}$$

So, the complex exponential function is periodic.

$$e^z = e^{x+yi} = e^{x+(y+2\pi)i} = e^{z+2\pi i}. \tag{8}$$

I.e., the exponential is **invariant** under vertical shifts by 2π and is no longer injective. So, no natural log function for the complex exponential.

Mapping the Fundamental Strip

We want to understand what the exponential function does to the horizontal strip:

$$\{a + bi \mid a \in \mathbb{R}, -\pi < b < \pi\} \tag{9}$$

Notice that $|e^z| = |e^{Re(z)}|$. So any vertical line segment of length 2π with fixed real part a will map onto the circle of radius $|e^a|$ centered at the origin. If $r > 0 \in \mathbb{R}$, pick $a = ln(r)$ to ensure a preimage. Then the exponential maps our horizontal strip to the entire complex plane except for zero.

Proposition 1. *Not injective implies no inverse.*

Consider

$$f(x, y) = v(x, y) + iu(x, y) \tag{10}$$

with

$$f(x, y) = (x + yi)^2 = (x^2 - y^2) + 2xyi \tag{11}$$

$$u(x, y) = x^2 - y^2 \tag{12}$$

$$v(x, y) = 2xy \tag{13}$$

So,

$$\frac{\partial u}{\partial x} = 2x = \frac{\partial v}{\partial y} \tag{14}$$

$$\frac{\partial u}{\partial y} = -2y = -\frac{\partial v}{\partial x} \tag{15}$$

is NOT a coincidence. (These are the Cauchy-Riemann Equations).

$f(Z) = iz$ rotates the plane by $\frac{\pi}{2}$. $f(x + iy) = i(x + iy) = -y + ix$.

When you multiply two complex numbers, you multiply the magnitudes and you add the arguments. (This will show up on tests).

How do affine (linear) functions behave? Any straight line in $\mathbb{C}$ is a combination of a rotation, magnification, and translation.