Math 360 Homework 4

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Problem 1:

Evaluate

$$\oint_C \overline{z} dz$$

where C is the contour parameterized $z = t^2 + it$ and $0 \leq t \leq 2$.

Note,

$$dz = (2t + i)dt$$

and

$$\overline{z} = t^2 - it.$$

Accordingly,

$$\oint_C \overline{z} dz = \int_0^2 (t^2 - it)(2t + i)dt$$

Then,

$$\begin{aligned}
&= \int_0^2 2t^3 + t + i(t^2 - 2t)dt \\
&= \left[\frac{2}{4}t^4 + \frac{1}{2}t^2 - \frac{i}{3}t^3\right]\Bigg|_0^2 \\
&= \frac{16}{2} + \frac{4}{2} - \frac{8i}{3} \\
&= 10 - \frac{8i}{3}.
\end{aligned}$$

Problem 2:

Part A: Let C be the straight line joining 0 to $1 + i$.

Compute

$$\oint_C (z^2 - z)dz.$$

To parameterize C, $z = t(1+i)$, where $0 \le t \le 1$. Accordingly, $dz = (1+i)dt$. Then

$$\begin{aligned}(z^2 - z)dz &= (t^2(1+i)^2 - t(1+i))(1+i)dt \\ &= (t^2(2i) - t(1+i))(1+i)dt \\ &= (2it^2(1+i) - 2it)dt \\ &= (2it^2 - 2t^2 - 2it)dt \\ &= (-2t^2 + 2i(t^2 - t))dt.\end{aligned}$$

Therefore,

$$\begin{aligned}\int_0^1 (-2t^2 + 2it^2 - 2it)dt &= \left[-\frac{2}{3}t^3 + 2i(\frac{t^3}{3} - \frac{t^2}{2})\right]\Big|_0^1 \\ &= -\frac{2}{3} + 2i(\frac{1}{3} - \frac{1}{2}) \\ &= -\frac{2}{3} - 2i\frac{1}{6} \\ &= -\frac{2}{3} - i\frac{1}{3}\end{aligned}$$

$\square$

Part B: Let C be the straight line segment joining 0 to 1, followed by the straight line segment joining 1 to $1+i$.

Evaluate

$$\oint_{C_1} (z^2 - z)dz + \oint_{C_2} (z^2 - z)dz.$$

Notably for C_1, we can parameterize $z = t$ and $dz = dt$ for $0 \le t \le 1$, and for C_2, we can parameterize $z = 1 + it$ and $dz = idt$ for $0 \le t \le 1$. So,

$$\begin{aligned}\oint_{C_1} (z^2 - z)dz + \oint_{C_2} (z^2 - z)dz &= \int_0^1 (t^2 - t)dt + \int_0^1 ((1+it)^2 - (1+it))idt \\ &= \left[\frac{1}{3} - \frac{1}{2}\right] + \int_0^1 (-t - it^2)dt \\ &= . - \frac{1}{6} + \left[-\frac{1}{2} - \frac{i}{3}\right] = -\frac{2}{3} - i\frac{1}{3}.\end{aligned}$$

$\square$

Problem 3:

Let C be the circle parameterized by $2i + 2e^{it}$ with $0 \le t \le 2\pi$. Evaluate

$$\int_C \frac{z}{z^2 - 1} dz.$$

Since we are given the specific parameterization and since $dz = 2ie^{it}dt$,

$$\int_C \frac{z}{z^2 - 1} dz = \int_0^{2\pi} \frac{2i + 2e^{it}}{(2i + 2e^{it})^2 - 1} 2ie^{it}dt.$$

Let the denominator be $g(t)$. Then,

$$g(t) = (2i + 2e^{it})^2 - 1$$

and

$$g'(t) = 2(2i + 2e^{it})(2ie^{it}).$$

So

$$\int_0^{2\pi} \frac{2i + 2e^{it}}{(2i + 2e^{it})^2 - 1} 2ie^{it} dt = \int_0^{2\pi} \frac{1}{2} \frac{g'(t)}{g(t)} dt = \int_0^{2\pi} \frac{1}{2} \frac{d}{dt} \log g(t) dt.$$

$$= \frac{1}{2} \log g(t) \Big|_0^{2\pi} = \frac{1}{2}[\log((2i + 2e^{it})^2 - 1)] \Big|_0^{2\pi} = \frac{1}{2}[\log((2i + 2e^{2\pi i})^2 - 1) - \log((2i + 2)^2 - 1)]$$

$$= \frac{1}{2}[\log((2i + 2)^2 - 1) - \log((2i + 2)^2 - 1)] = \frac{1}{2}[0] = 0.$$

□

Problem 4:

Let C be the circle $|z| = 3$. Prove that

$$\left| \oint_C \frac{dz}{z + 1} \right| \leq 6\pi.$$

First we parameterize the circle,

$$z(t) = 3e^{it}, 0 \leq t \leq 2\pi.$$

Then

$$L = \int_0^{2\pi} |dz| = \int_0^{2\pi} 3dt = 6\pi.$$

Now we want to find $M = \sup |\frac{1}{1+z}|$ for $z \in C$. Notably, for $z \in C$, $|z| = 3$. By the triangle inequality,

$$|z + 1| \geq ||z| - |1|| = |3 - 1| = 2.$$

Therefore,

$$M \leq \frac{1}{2}.$$

By the ML inequality,

$$\left| \oint_C \frac{dz}{z + 1} \right| \leq \frac{1}{2} 6\pi \leq 6\pi.$$

□

Problem 5:

Let C be an ellipse centered at the origin with horizontal axis 4 and vertical axis 3. Evaluate

$$\oint_C \frac{1}{z} dz.$$

We can parameterize C with the circle of radius 4, $z = 4e^{it}$ for $0 \leq t \leq 2\pi$ by homotopy invariance since $\frac{1}{z}$ is analytic on $\mathbb{C} \setminus \{0\}$ between the circle of radius 4 and the ellipse described, which are mutually homotopic on $\mathbb{C} \setminus \{0\}$. Then,

$$\oint_C \frac{1}{z} dz = \oint_{|z|=4} \frac{dz}{z} = \int_0^{2\pi} \frac{1}{4e^{it}} 4ie^{it} dt = \int_0^{2\pi} i dt = 2\pi i.$$

$\square$