Math 360 Homework 3

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Problem 1: Find a value of z for which $1^z \neq 1$.

Note that we must not use the Principal log, otherwise $1^z \cong 1$. Thus, recall that

$$log(z) = log_e|z| + i(Arg(z) + 2k\pi) \tag{1}$$

and

$$z^c = exp(clog(z)). \tag{2}$$

Then

$$\begin{aligned}(1+0i)^c &= exp(c[log_e|1| + i(Arg(1+0i) + 2k\pi)]) && (3)\\ &= exp(ic(Arg(1+0i) + 2k\pi)) && (4)\\ &= e^{ic(0+2k\pi)}. && (5)\end{aligned}$$

Then

$$2cik\pi \neq 0. \tag{6}$$

So letting $c = a + bi$,

$$(a+bi)(2i\pi k) = -2b\pi k + i(2a\pi k). \tag{7}$$

(7) is nonzero for any nonzero b and $a \in \mathbb{Z}$. So any $z = a + bi$ where $a \in \mathbb{Z}$ and $b \neq 0 \subseteq \mathbb{R}$ will work.

Problem 2: Explain why $sin(z)$ and $cos(z)$ are unbounded as complex functions.

Recall that

$$cos(z) = \frac{e^{iz} + e^{-iz}}{2} = \frac{e^{-b+ai} + e^{b-ai}}{2} = \frac{1}{2}\left[e^{-b}(cos(a) + isin(a)) + e^{b}(cos(a) - isin(a))\right] \tag{8}$$

and

$$sin(z) = \frac{e^{iz} - e^{-iz}}{2i} = \frac{e^{-b+ai} - e^{b-ai}}{2i} = \frac{1}{2i}\left[e^{-b}(cos(a) + isin(a)) - e^{b}(cos(a) - isin(a))\right]. \tag{9}$$

Then we can take the limit as $b \to \pm\infty$ and get:

$$\lim_{b\to\infty} \frac{1}{2}\left[e^{-b}(cos(a)+isin(a))+e^{b}(cos(a)-isin(a))\right]=\frac{1}{2}e^{\infty}(cos(a)-isin(a))=\infty \quad (10)$$
$$\lim_{b\to\infty} \frac{1}{2}\left[e^{-b}(cos(a)+isin(a))+e^{b}(cos(a)-isin(a))\right]=\frac{1}{2}e^{\infty}(cos(a)+isin(a))=\infty \quad (11)$$
$$\lim_{b\to-\infty} \frac{1}{2i}\left[e^{-b}(cos(a)+isin(a))-e^{b}(cos(a)-isin(a))\right]=\frac{1}{2i}e^{\infty}(cos(a)+isin(a))=\infty \quad (12)$$
$$\lim_{b\to-\infty} \frac{1}{2i}\left[e^{-b}(cos(a)+isin(a))-e^{b}(cos(a)-isin(a))\right]=\frac{1}{2i}e^{\infty}(cos(a)-isin(a))=\infty. \quad (13)$$

So $sin(z)$ and $cos(z)$ are unbounded complex functions.

Problem 3: Prove that $cos(2z) = cos^2(z) - sin^2(z)$ holds for any $z \in \mathbb{C}$.

Recall equations (8) and (9) which say that

$$cos(z) = \frac{e^{iz}+e^{-iz}}{2} \quad (14)$$

and

$$sin(z) = \frac{e^{iz}-e^{-iz}}{2i}. \quad (15)$$

It follows then that

$$\begin{aligned} cos^2(z) &= \left[\frac{e^{iz}+e^{-iz}}{2}\right]^2 && (16)\\ &= \frac{1}{4}[e^{2iz}+e^{-2iz}+2e^{iz-iz}] && (17)\\ &= \frac{1}{4}[e^{2iz}+e^{-2iz}+2] && (18) \end{aligned}$$

and

$$\begin{aligned} sin^2(z) &= \left[\frac{e^{iz}-e^{-iz}}{2i}\right]^2 && (19)\\ &= -\frac{1}{4}[e^{2iz}+e^{-2iz}-2e^{iz-iz}] && (20)\\ &= -\frac{1}{4}[e^{2iz}+e^{-2iz}-2] && (21) \end{aligned}$$

Then,

$$\begin{aligned} cos^2(z)-sin^2(z) &= \frac{1}{4}[([e^{2iz}+e^{-2iz}+2])-(-[e^{2iz}+e^{-2iz}-2])] && (22)\\ &= \frac{1}{4}[2e^{2iz}+2e^{-2iz}+2-2] && (23)\\ &= \frac{e^{2iz}+e^{-2iz}}{2} && (24)\\ &= cos(2z) && (25) \end{aligned}$$

as desired.

$\square$

Problem 4: Find all values of z which satisfy $Log(z) = \frac{i\pi}{4}$.

Recall that

$$Log(z) = ln|z| + iArg(z). \tag{26}$$

Then

$$ln|z| = 0 \Rightarrow |z| = 1 \tag{27}$$

$$\text{and} \tag{28}$$

$$iArg(z) = i\frac{\pi}{4} \Rightarrow Arg(z) = \frac{\pi}{4}. \tag{29}$$

Notably, the only point in $\mathbb{C}$ with magnitude 1 and Argument $\frac{\pi}{4}$ is $\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i$. So $z = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i$.