Math 435 Homework 1

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Problem 1: Prove that a function $f : A \to B$ is surjective if and only if $g_1 \circ f = g_2 \circ f$ implies $g_1 = g_2$ for any set X and any two functions $g_1, g_2 : B \to X$.

(Direction 1) Let $f : A \to B$ such that $\forall b \in B$, $\exists a \in A$ with $f(a) = b$. We want to show that $g_1(f) = g_2(f)$ implies $g_1 = g_2$ for any $g_1, g_2 : B \to X$. Let $b \in B$ with

$$g_1(b) = g_2(b) \in X. \tag{1}$$

Since f is surjective, the fiber of b, $f^{-1}(b) = a$ for some $a \in A$. So, $g_1(f(a)) = g_2(f(b)), \forall a \in A$. But since f is surjective, B is the image of f, or $B = im(f)$. Notably B is the whole domain of g_1 and g_2. Then $g_1(B) = g_2(B)$. So $g_1 = g_2$.

(Direction 2) Let $g_1(f) = g_2(f)$ imply that $g_1(b) = g_2(b)$ for all $b \in B$. We want to show that $f : A \to B$ is surjective, either by showing that $\forall b \in B$, $\exists a \in A$ with $f(a) = b$ or $im(f) = B$ or there exists $\ell : B \to A$ with $f \circ \ell = id_B$.

Assume for contradiction that $im(f) \subseteq B$, but $B \not\subseteq im(f)$ (i.e., f is not surjective). Then let $x_0 \in B \setminus im(f)$ and $X = \{0, 1\}$. We define

$$g_1(x) = \begin{cases} 1 & \text{if } x = x_0 \\ 0 & \text{otherwise} \end{cases} \tag{2}$$

$$g_2(x) = 0. \tag{3}$$

Then for any $b \in im(f)$,

$$g_1(b) \neq g_2(b). \tag{4}$$

But since $x_0 \notin im(f)$,

$$g_1(im(f)) = g_2(im(f)) \to g_1 = g_2, \tag{5}$$

which is a contradiction. Thus, f is surjective.

$\square$

Problem 2: Given a set A, we define $id_A : A \to A$ by $id_A(a) = a$ for all $a \in A$.

Part 1: Given $h : A \to A$, prove that $h = id_A$ implies $h \circ f = f$ and $g \circ h = g$ for any functions $f : X \to A$ and $g : A \to Y$.

Let $h : A \to A$ as above, and $f : X \to A$. $Im(f) \subseteq A$, by definition. So $\forall a \in im(f)$, $h(a) = a$, which implies that, for any $x \in X$ with $f(x) = a \in A$, $h(f(x)) = h(a) = a$. Similarly, letting $g : A \to Y$, since the domain of g is A, $\forall a \in A$, $h(a) = a$ implies $g(h(a)) = g(a)$. Then $g(im(h)) = g(A)$.

□

Part 2: Given that $h : A \to A$, prove that $h = id_A$ if $h \circ f = f$ for any function $f : X \to A$.

Let $f : X \to A$ with $f(x) = a \in A$ for any $x \in X$. Then for any $x \in X$, $h(f(x)) = h(a) = id_A(a) = a = f(x)$. So $h \circ f = f$ for any f.

Problem 3: Given a function $f : A \to B$, we say that $g : B \to A$ is a left inverse for f if $g \circ f = id_A$. Analogously, we say that a function $h : B \to A$ is a right inverse for f if $f \circ h = id_B$.

Part 1: Without referencing elements explicitly, prove that if f has a left inverse, then f is injective.

Let $f : A \to B$ and $g : B \to A$ with $g \circ f = id_A$. We want to show that f is injective, i.e., that $f(b) = f(a)$ implies $b = a$ for any $a, b \in A$. So, let $a, b \in A$ with

$$f(b) = f(a) \tag{6}$$
$$a \neq b \tag{7}$$

But $g(f(x)) = x$ implies that

$$g(f(b)) = g(f(a)) \tag{8}$$
$$b = a \tag{9}$$

which is a contradiction. Then, f is injective.

□

Part 2: Show that if f has a right inverse, f is surjective.

Let $f : A \to B$ and $h : B \to A$ with $f \circ h = id_B$. We want to show that $\forall b \in B$, there exists $a \in A$ with $f(a) = b$. Assume for contradiction that there exists $b_0 \in B$ with no fiber in A. Then

$$f(h(b_0)) = b_0 \tag{10}$$

implies that $h(b_0) \in A$ is the fiber of b_0.

$\square$

Problem 4: Prove that every injection has a left inverse.

Let $f : A \to B$ with $f(a) = f(b)$ implies $a = b$. We want to show that there exists $g : B \to A$ with $g \circ f = id_A$. Let $im(f) = \{f(a) \in B \mid a \in A\}$. Notably we can define $h : B \to A$ as $g(f(a)) = a \in A$ because each element of $im(f) \subseteq B$ has exactly one preimage. If $b \notin im(f)$ and $g(b) = g(b_0)$ for some $b_0 \in im(f)$, then $b_0 = b$ and so b must be in $im(f)$. Thus, any injection has a left inverse.

$\square$