Math 435 09/12/2025 Notes

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Review

Definition 1. *A collection $\mathscr{B}$ of subsets of a set X is called a basis if*

1. *The sets in $\mathscr{B}$ cover X, i.e., $\forall x \in X$ there exists $B \in \mathscr{B}$ with $x \in B$.*
2. *If $B_1, B_2 \in \mathscr{B}$ and if $x \in B_1 \cap B_2$, then there exists $B_3 \in \mathscr{B}$ such that $X \in B_3 \subseteq B_1 \cap B_2$.*

Definition 2. *Let $\mathscr{U}_1, \mathscr{U}_2$ be topologies on X. If $\mathscr{U}_1 \subseteq \mathscr{U}_2$ we say that $\mathscr{U}_2$ is finer than $\mathscr{U}_1$, and $\mathscr{U}_1$ is coarser than $\mathscr{U}_2$.*

Today

- The product topology

Theorem 1. *Given a set X and the topology $\mathscr{U}$ generated by $\mathscr{B}$, $U \in \mathscr{U}$. I.e., U is an open set iff $\forall x \in U$, there exists $B_x \in \mathscr{B}$ such that $x \in B_x \subseteq U$.*

The Product Topology

Definition 3. *Let X, Y be topological spaces. The product topology on $X \times Y$ is the topology generated by the basis $\mathscr{B}$ consisting of all sets of the form $U \times V$ where U is open in the topology on X and V is open in the topology on Y.*

Check that $\mathscr{B}$ is a basis: (1) is trivial since we can let $U = X, V = Y$. Then X is open in X and Y is open in Y. So the product $X \times Y \in \mathscr{B}$, hence the basis covers X. (2) Let $(U_1 \times V_1) \cap (U_2 \times V_2) = (U_1 \cap U_2 \times V_1 \cap V_2)$. Notice that these intersections are open in their respective topologies, so this is an element of $\mathscr{B}$. So, given $x \in S = (U_1 \times V_1) \cap (U_2 \times V_2)$, we have $x \in (U_1 \cap U_2 \times V_1 \cap V_2) \subseteq S$. Thus (2) is satisfied.

$\square$

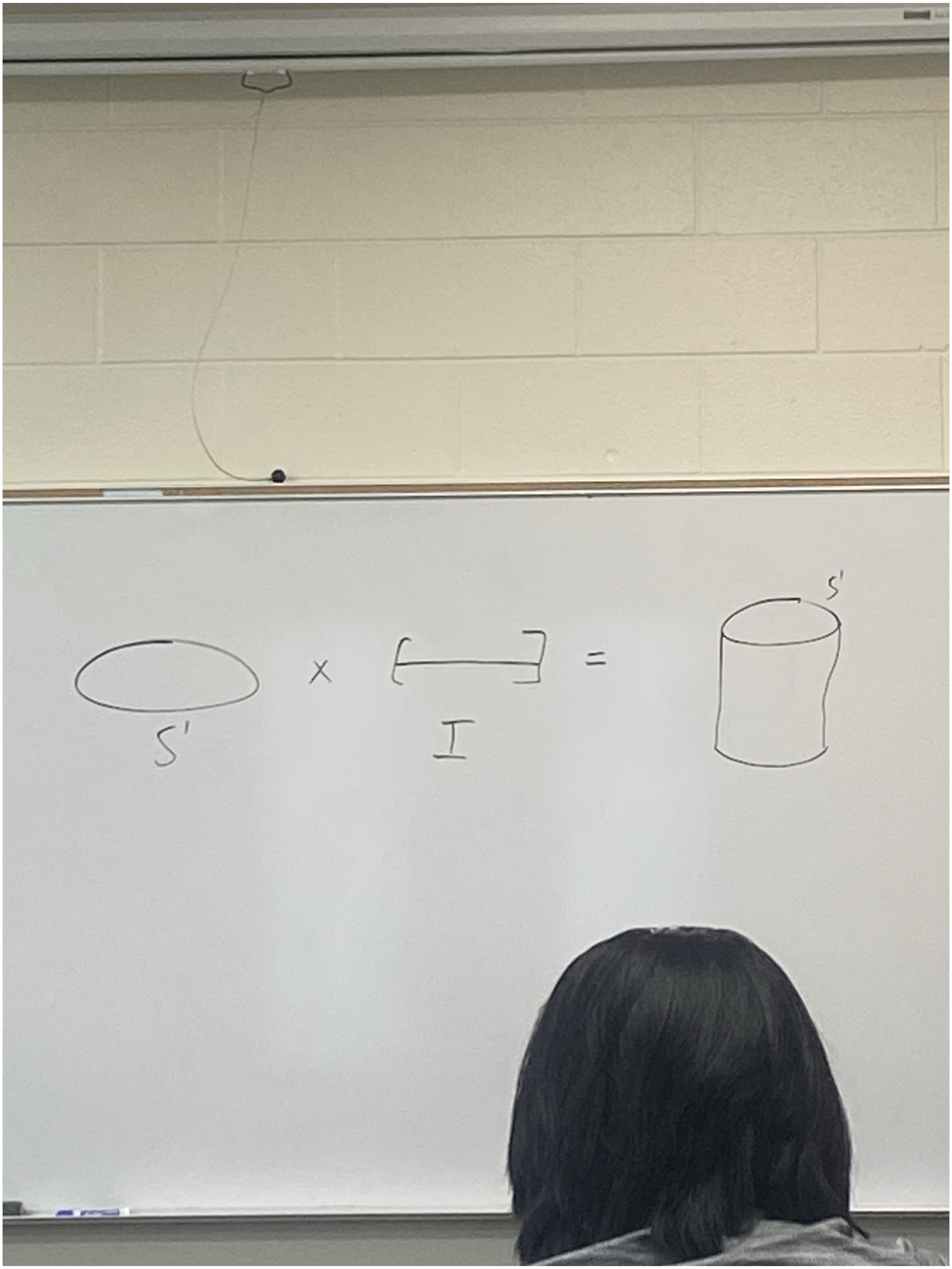

Figure 1: Example 1