Math 435 11/03/2025 Notes

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Review

Today

- Paths, loops, and pointed spaces
- Degree Theory

Paths, loops, and pointed spaces

Now we will consider the homotopy between paths and loops.

Definition 1. *Let X be a space and let $x_0, x_1 \in X$. A path from x_0 to x_1 is a map from $f : [0,1] \to X$ such that $f(0) = x_0$ and $f(1) = x_1$.*

Definition 2. *A loop in X is a path from x_0 to x_0. We say that such a loop is based at x_0.*

Definition 3. *Given $a, b \in X$ and the paths γ_0, γ_1 from a to b, a path homotopy from γ_0 to γ_1 is a homotopy $H : [0,1] \times [0,1] \to X$ such that $H(0,t) = a$, $H(1,t) = b$, for all t.*

Definition 4. *A pointed / based space is a pair $(X, *)$ such that X is a topological space and $* \in X$. We'll refer to $*$ as the distinguished point. A map between pointed spaces (a pointed map) is $f : (X, *_x) \to (Y, *_y)$ is a map $f : X \to Y$ such that $f(*_x) = *_y$.*

Remark 1. *Let $(X, *_x)$ be a pointed space such that $\gamma : [0,1] \to X$ be a loop based at $*_x$. Note that we can identify γ with a pointed map $\gamma : (S^1, *_S) \to (X, *_x)$. Basically, loops are closed and the basically the same as traversing a circle.*

Degree Theory

Goal: understand homotopy classes of pointed maps from $(S^1, *) \to (S^1, *)$.

We will consider S^1 as the space $S^1 = \{(cos\theta, sin\theta) | \theta \in \mathbb{R}\}$ with the subspace topology. We'll refer to the point $(cos\theta, sin\theta)$ as θ, where $\theta_1 \sim \theta_2$ iff $\theta_1 = k2\pi + \theta_2$, $k \in \mathbb{Z}$. We'll take $(S^1, \theta = 0)$ to be pointed. We will also consider pointed maps $(S^1, 0) \to (S^1, 0)$.

Valid maps could be a straight line identity maps. But could we also have more complicated maps? We can view maps by $S' \times S'$: