# Math 435 09/22/2025 Notes

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## Review

**Definition 1.** *A space $(X, \mathscr{U})$ is Hausdorff if given $x \neq y \in X$, there exists $U, V \in \mathscr{U}$ with $x \in U$ and $y \in U$ such that $U$ and $V$ are disjoint.*

## Today

- Quotient spaces
- Quotient spaces of Hausdorff spaces

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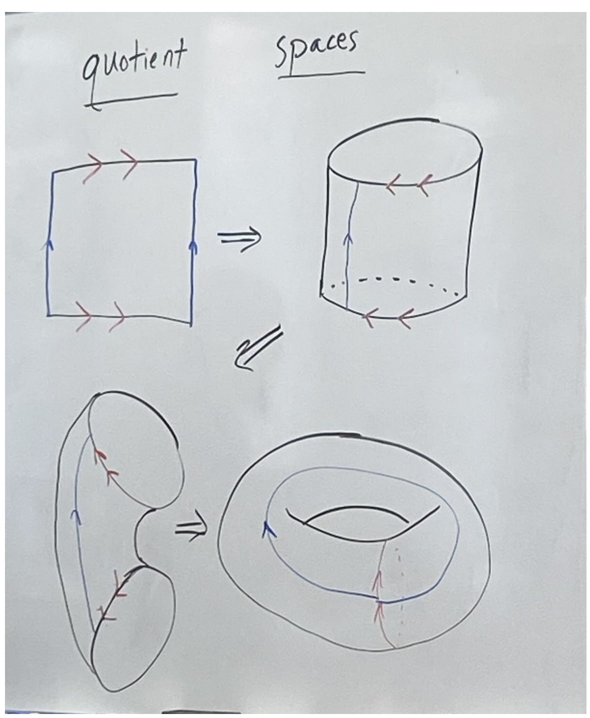


Figure 1: Mapping the fundamental strip to a Torus

**Definition 2.** *Let $X$ be a space and let $A$ be a set (not necessarily a subset of $X$). Let $p : X \to A$ be a surjective function. The quotient topology on $A$ has $U \subseteq A$ open iff $p^{-1}(A)$ is open in $X$, by definition. We call $p$ the quotient map of $X$ onto $A$.*

**Theorem 1.** *The quotient topology is a topology.*

**Proof:** Let $p : X \to A$ be surjective and let $\mathscr{U} = \{U \subseteq A \mid p^{-1}(U) \overset{\circ}{\subset} X\}$. $\emptyset \in \mathscr{U}$ since $p^{-1}(\emptyset) = \emptyset$. Moreover, $A \in \mathscr{U}$ since $p$ is surjective which implies $p^{-1}(A) = X$. (The rest of the proof is an exercise.)

$\square$

**Example 1.** *Let $X = \mathbb{R}$ under the standard topology, and let $A = \{a, b, c\}$. Then let*

$$p(x) = \begin{cases} a & \textit{if } x < 0 \\ b & \textit{if } x = 0 \\ c & \textit{if } x > 0 \end{cases} \quad . \tag{1}$$

*Thus, $p^{-1}(a) = (-\infty, 0)$, $p^{-1}(b) = \{0\}$, $p^{-1}(c) = (0, \infty)$. But $p^{-1}(b)$ isn't open in $\mathbb{R}$. So the topology on $\{a, b, c\}$ must be $\mathscr{U} = \{a, c, a \cup c, \emptyset, a \cup b \cup c\}$.*

**Definition 3.** *A partition of a set $A$ is a collection of disjoint, nonempty subsets that cover $A$.*

Often, for quotient spaces, we let A be a partition of X. Foreshadowing: if $X = [0, 1]$ with the subspace topology from $\mathbb{R}$, and we let our partition $X^*$ of A be $X^* = \{\{x\} | 0 < x < 1\} \cup \{0, 1\}$, we can make the Riemann sphere.