Math 435 09/05/2025 Notes

September 5, 2025

Timothy Tarter
James Madison University
Department of Mathematics

Review

Definition 1. *A topological space is a pair $(X, \mathscr{U})$, (often (X, τ)), where X is a set, and $\mathscr{U}$ is a collection of subsets of X such that*

1. $\emptyset, X \in \mathscr{U}$
2. $\mathscr{U}$ *is closed under arbitrary union*
3. $\mathscr{U}$ *is closed under finite intersection.*

The elements of $\mathscr{U}$ are called open sets. Note: 'open set' here really just means that a set belongs to $\mathscr{U}$.

Today

- Examples of topologies

Example 1. *Let $X = \{a,\ b,\ c\}$. What topologies can we put on X?*

Topology 1 (the Indiscrete Topology)

$\{\emptyset, X\}$

Topology 2

$\{\emptyset, \{a, b, c\}, \{a, b\}, \{b\}\}$

NOT a Topology

$\{\emptyset, \{a, b, c\}, \{a, b\}, \{b, c\}\}$, since we don't have {b}.

Topology 3 (Discrete topology)

$\{\emptyset, \{a,b,c\}, \{a,b\}, \{b,c\}, \{a,c\}, \{a\}, \{b\}, \{c\}\} = \mathscr{P}(X) =$ all subsets of X.

Definition 2. *For any set X, the set $\{\emptyset, X\}$ is a topology called the indiscrete topology.*

Definition 3. *For any set X, $\mathscr{P}(X)$ is the discrete topology.*

Example 2. *The collection of all subsets of X such that the complement is either countable or all of X is a topology.*

Example 3. *Let X be a non-empty set and let $p \in X$. Define $\mathscr{U}$ to be the set of $\{U \subseteq X \mid p \in U\} \cup \{\emptyset\}$. This is a topology on X.*

Proof: $p \in X \subseteq X$ and $\emptyset \in \mathscr{U}$ satisfies our first point. Let $\bigcup A_\alpha$ be an arbitrary union of elements of U. Then $p \in A_p$ for some $A_p \subseteq \bigcup A_\alpha$ implies $p \in \bigcup A_\alpha \in \mathscr{U}$. For our final requirement, Let $\bigcap_{i=1}^n A_i$ be a finite intersection of elements of $\mathscr{U}$. Then $p \in A_i$ for any $i \in 1 \ldots n$ implies that $p \in \bigcap_{i=1}^n A_i \in \mathscr{U}$.

$\square$