Math 435 09/15/2025 Notes

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Review

Lemma 1. *Let $(X, \mathscr{U})$ be a topological space. If a collection $\mathscr{C} \subseteq \mathscr{U}$ satisfies the following,*

$$\forall U \in \mathscr{U} \text{ and } x \in U, \text{ there exists } C \in \mathscr{C} \text{ st. } x \in C \subseteq U, \tag{1}$$

then $\mathscr{C}$ is a basis that generates $\mathscr{U}$.

Today

- Subspace Topology

Definition 1. *Given topological spaces X and Y, the product topology on $X \times Y$ is generated by the basis*

$$\{U \times V \mid U \overset{\circ}{\subset} X, V \overset{\circ}{\subset} Y\}. \tag{2}$$

Theorem 1. *If $\mathscr{B}$ is a basis for a topology on X and $\mathscr{D}$ is a basis for a topology on Y, then*

$$\mathscr{C} = \{\mathscr{B} \times D \mid B \in \mathscr{B}, D \in \mathscr{D}\} \tag{3}$$

is a basis for the product topology on $X \times Y$.

Proof: (Use Lemma 1). Let W be an open set in $X \times Y$ and let $x \times y \in X \times Y$. <u>Punt to exercise.</u>

□

Definition 2. *Let $(X, \mathscr{U})$ be a topological space and let $Y \subseteq X$, (not necessarily open). The collection*

$$\mathscr{U}_Y = \{Y \cap U \mid U \in \mathscr{U}\} \tag{4}$$

*is a topology on Y. (This is called the **subspace topology**).*

Proof ($\mathscr{U}_Y$ is a Topology on Y): For part 1, since $\mathscr{U}$ is a topology, $\emptyset \in \mathscr{U}$ implies $Y \cap \emptyset = \emptyset \in \mathscr{U}_y$. Similarly, since $X \in \mathscr{U}$ then $Y = Y \cap X \in \mathscr{U}_Y$. Part 2 is satisfied by DeMorgan's laws: given an arbitrary collection of sets, $\{Y \cap U\}_\alpha$,we have $Y \cap U_\alpha = (Y \cap U)_\alpha$ where each $U_\alpha \overset{\circ}{\subset} X$. Then

$$\bigcup_\alpha (Y \cap U)\alpha = \bigcup_\alpha (Y \cap U_\alpha) = Y \cap (\bigcup_\alpha U_a \alpha). \tag{5}$$

Finally, for part 3, it works out similarly. <u>Punt to exercise.</u>

□

Lemma 2. *If $\mathscr{B}$ is a basis for the topology on X, then*

$$\{Y \cap B \mid B \in \mathscr{B}\} \tag{6}$$

is a basis for the subspace topology.

Proof: Given $U \overset{\circ}{\subset} X$ and $y \in Y \cap U$, we can choose $B \in \mathscr{B}$ such that $y \in B \subseteq U$, since $\mathscr{B}$ is a basis. Thus, $y \in Y \cap \mathscr{B} \subseteq Y \cap U$ and the claim follows by Lemma 1.

□

What about closed sets?

Definition 3. *Given a topological space, $(X, \mathscr{U})$, a set $V \subseteq X$ is closed if $V = X \setminus U$ for some $U \in \mathscr{U}$.*

Example 1. *Consider $Y = [0, 1] \cup (2, 3)$ with the subspace topology coming from the standard topology on the* ***ambient space****, $\mathbb{R}$. $[0, 1]$ is open in Y (but not in the standard topology on $\mathbb{R}$) since*

$$[0, 1] = Y \cap (-1, \frac{3}{2}) \overset{\circ}{\subset} \mathbb{R}. \tag{7}$$

Similarly, $(2, 3)$ is open in both Y and $\mathbb{R}$ since

$$(2, 3) = Y \cap (2, 3) \overset{\circ}{\subset} \mathbb{R} \tag{8}$$

is also open.

Notably, $[0, 1]$ and $(2, 3)$ are also both closed, since they are complements of each other ***in Y****.*