Math 435 11/14/2025 Notes

November 14, 2025

Timothy Tarter
James Madison University
Department of Mathematics

Review

Today

- The fundamental group is an abelian group!

Theorem 1. $(\pi_1(S', *), +) \simeq (\mathbb{Z}, +)$.

Proof: Define $\varphi : \pi_1(S') \to \mathbb{Z}$ by $\varphi([f]) \to n$ where $n \in! \mathbb{Z}$ is such that $f \simeq C_n$ where $C_n(\theta) = n\theta$. We already saw that φ is a bijection. Now we need to check that φ is a homomorphism. Let $[f], [g] \in \pi_1(S')$ and let $n, m \in \mathbb{Z}$ such that $[f] = [C_n]$ and $[g] = [C_m]$. We need to prove that $\varphi([f] + [g]) = \varphi([f]) + \varphi([g])$. But $\varphi([C_n] + [C_m]) = \varphi([C_n + C_m])$. To summarize, we need to prove that $\varphi([C_n + C_m]) = n + m$. I.e., $[C_n + C_m] = [C_{n+m}]$. Equivalently, $C_n + C_m \simeq C_{n+m}$. This is true, but we'll omit the details here.

$\square$

Remark 1. *If $*_1$ and $*_2$ are two basepoints in S', then*

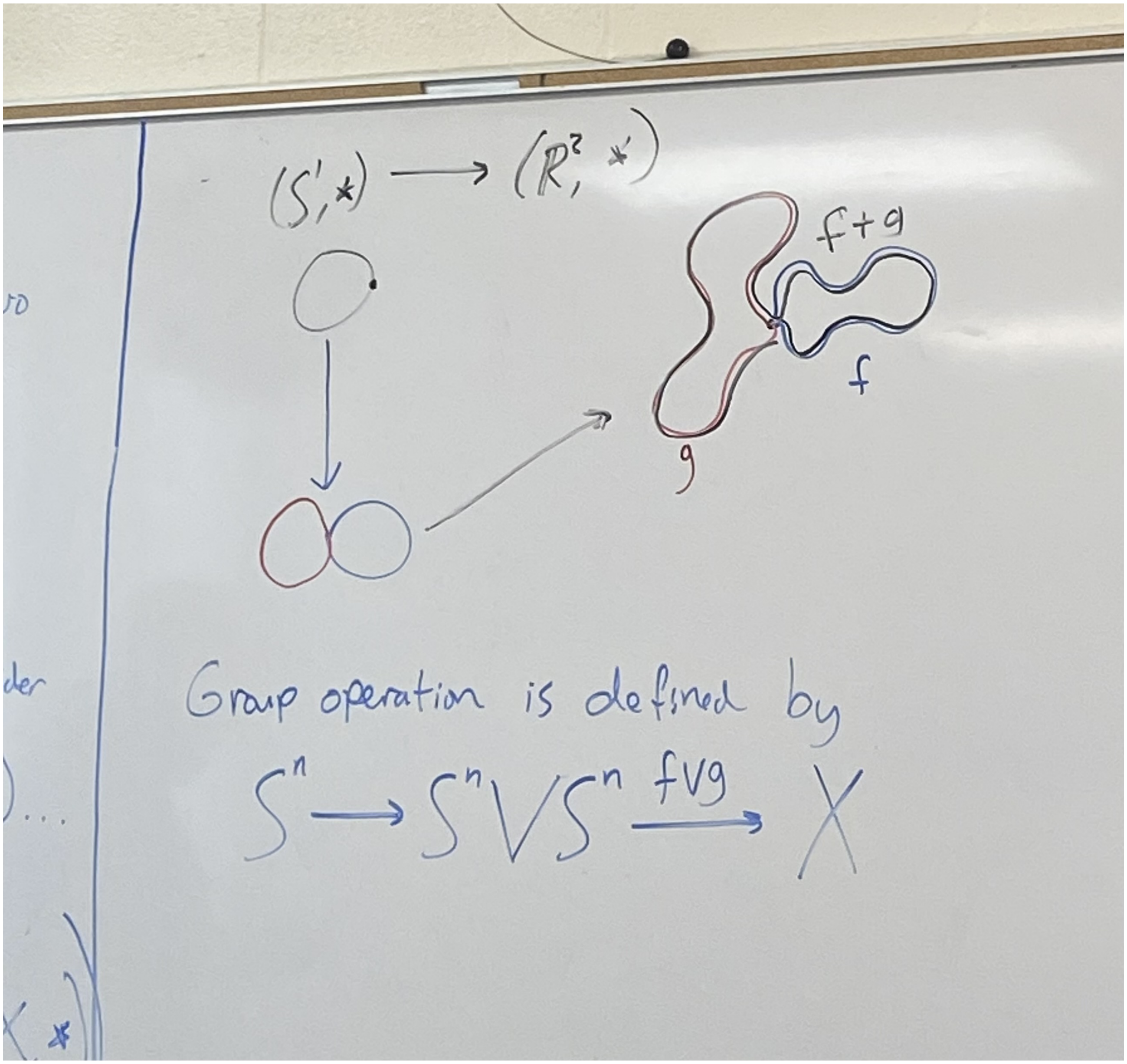
$$\pi_1(S', *_1) \simeq \pi_1(S', *_2).$$

More generally, we can consider $\pi_1(X, *)$ for any pointed space $(X, *)$. Even more generally, we can consider $\pi_n(X, *)$ where the elements of π_n are homotopy classes of pointed maps from $(S^n, *_{S^n}) \to (X, *)$.

Definition 1. *The group operation (wedge sum) is defined by*

$$S^n \to^{pinch} S^n \vee S^n \to^{f \vee g} X.$$

This is the same as taking two copies of S^n and glueing them at a point.



Theorem 2. *Given a group G, there exists a connected space $K(G,1)$ called an Eilenberg-MacLane space, or a classifying space, such that $\pi_1(K(G,1)) \cong G$.*

Example 1. *$G = \mathbb{Z}$, $K(\mathbb{Z},1) = S'$. What is $K(\mathbb{Z},2)$? It is infinite real projective space, $\mathbb{C}P^\infty$.*

Prepping category theory packages for next time:

$$\begin{tikzcd}
K \arrow[r, "f"] \arrow[dr, "h"'] & L \arrow[d, "g!"] \\
& \Delta K
\end{tikzcd}$$