Math 435 10/27/2025 Notes

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Review

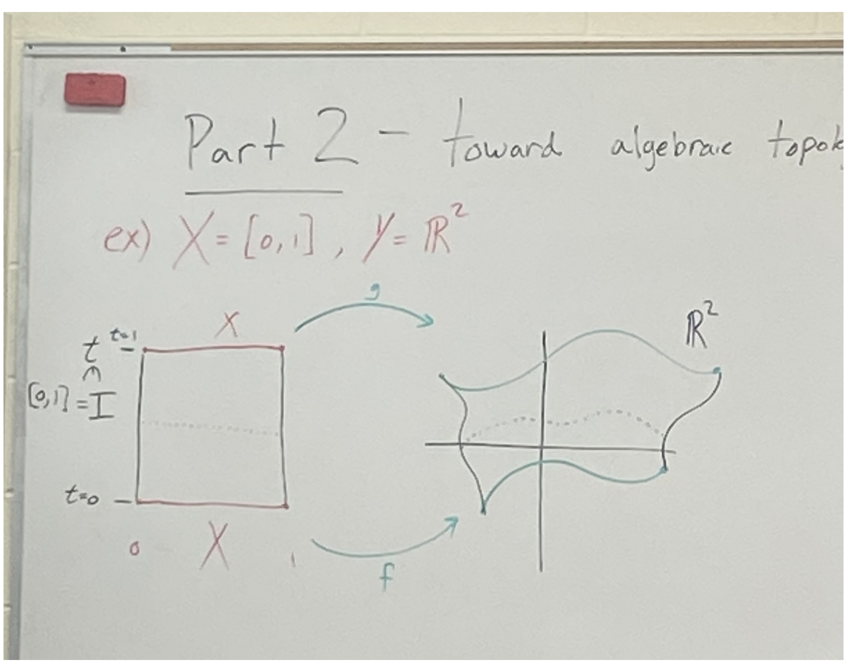
Today

- Starting Algebraic Topology!
- Homotopy

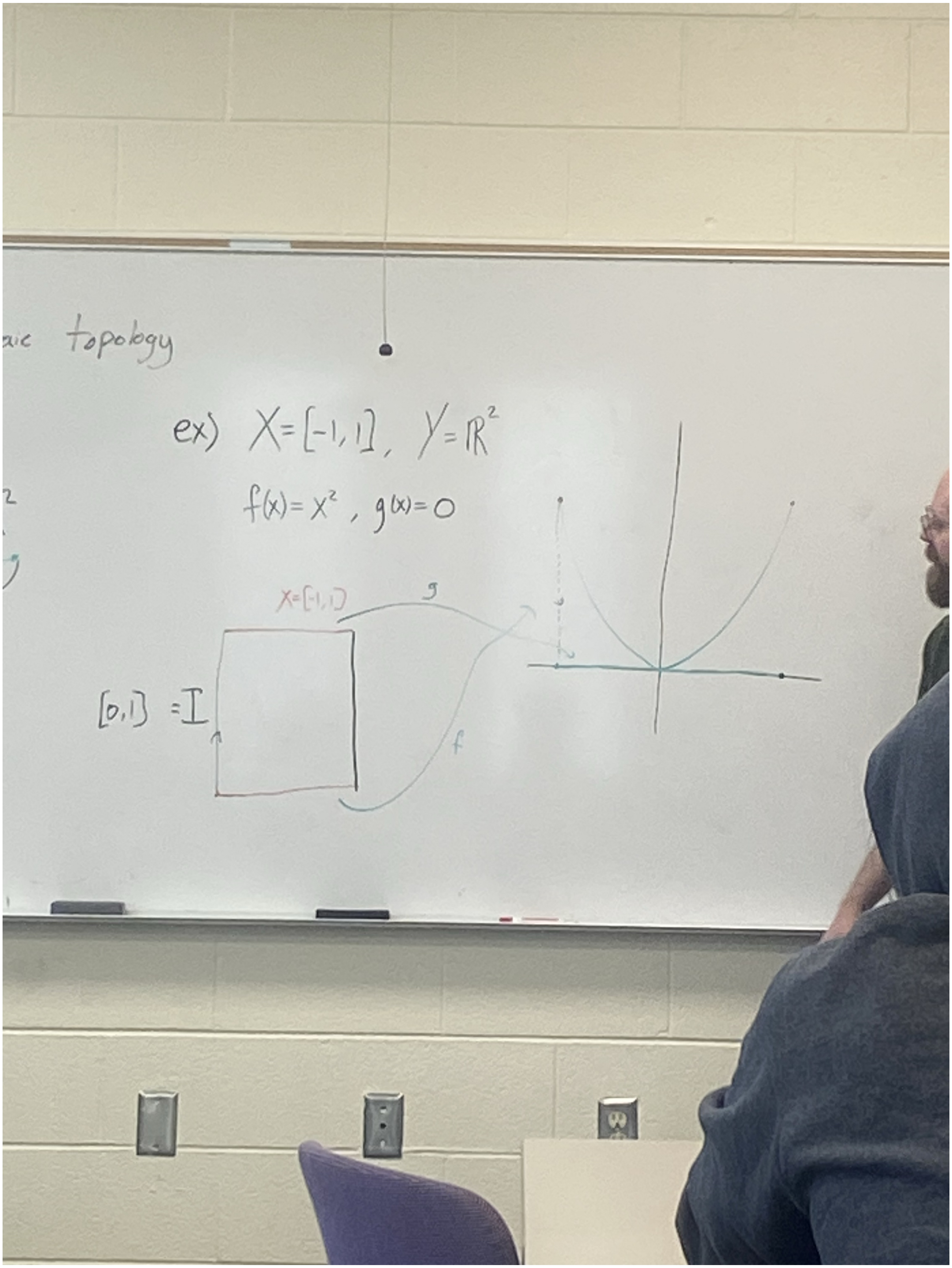
Homotopy

Definition 1. *Let $f, g : X \to Y$ be continuous. Assume $[0,1]$ has the subspace topology (from $\mathbb{R}$), and assume $X \times [0,1]$ has the product topology. We say f and g are* ***homotopic****, and write $f \simeq g$ iff there exists a continuous function $H : X \times [0,1] \to Y$ such that $H(x,0) = f(x)$ and $H(x,1) = g(x)$ for all $x \in X$. H is called a* ***homotopy****.*

Example 1. *Let $X = [0,1], Y = \mathbb{R}^2$.*



Example 2. *Let* $X = [-1, 1], Y = \mathbb{R}^2$*. Let* $f(x) = x^2$ *and* $g(x) = 0$*. This is called the straight line homotopy, and it implies that all polynomials are homotopic.*



Definition 2. *An equivalence relation on a set A is a subset $R \subseteq A \times A$ such that:*

1. *(Reflexive)* $(a, a) \in R$, $\forall a \in A$
2. *(Symmetric)* $(a, b) \in R$ *implies that* $(b, a) \in R$, $\forall a, b \in A$
3. *(Transitive)* $(a, b), (b, c) \in R$ *implies* $(a, c) \in R$, $\forall a, b, c \in A$.

We usually write $a \sim b$ to mean $(a, b) \in R$.

Definition 3. *Let $C(X, Y) = \{$continuous maps from X to $Y\}$.*

Theorem 1. $\simeq$ *(homotopy) is an equivalence relation on the set of continuous maps $X \to Y$.*

Proof:

1. Let $f \in C(X, Y)$. Is $f \simeq f$? Yes, via the homotopy, $H : X \times I \to Y$, where $H(x, t) = f(x)$.
2. Let $f, g \in C(X, Y)$ such that $f \simeq g$ via $H : X \times I \to Y$. We now want to show that there exists an $G : X \times I \to Y$. This works for $G(x, t) = H(x, 1 - t)$. Notice that $G(x, 0) = H(x, 1 - 0) = H(x, 1) = g(x)$ and $G(x, 1) = H(x, 1 - 1) = H(x, 0) = f(x)$, as desired, for any $x \in X$.
3. Transitivity: Exercise.