Math 435 10/08/2025 Notes

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Timothy Tarter
James Madison University
Department of Mathematics

Review

Definition 1. *A map $f : X \to Y$ is a continuous function if inverse images of open sets of Y are open in X.*

Today

- An example of homeomorphic spaces
- Homeomorphism theorems
- Topological properties
- Hausdorffness is a topological property
- Connectedness

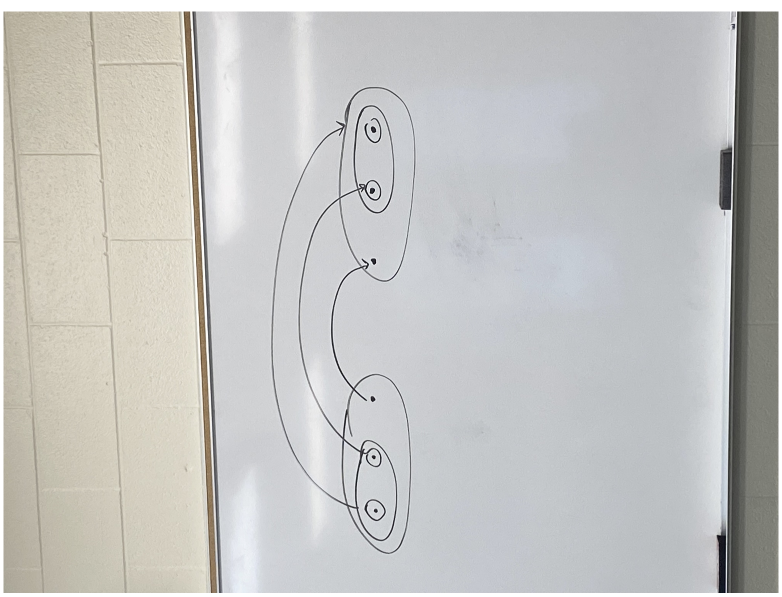

Figure 1: An example of homeomorphic spaces

Theorem 1. *Let $(X, \mathscr{U}_x)$ and $(Y, \mathscr{U}_y)$ be spaces. A continuous map $f : X \to Y$ is a homemorphism iff*

1. *$f : X \to Y$ is a bijection, and*
2. *$f : \mathscr{U}_x \to \mathscr{U}_y$ is a bijection*

Proof sketch: This is really saying that a homeomorphism is a bijection and that it maps open sets to open sets. Let $U \in \mathscr{U}_x$. For $f^{-1} : Y \to X$ to be continuous, $(f^{-1})^{-1}(U)$ must be open in Y for all open U in X. But $(f^{-1})^{-1} = f(U)$ because f is a bijection.

$\square$

Remark 1. *Consider the following:*

1. *The identity map on any set X is a homeomorphism.*
2. *If $f : X \to Y$ is a homeomorphism, then $f^{-1} : Y \to X$ is also a homemorphism.*
3. *If $f : X \to Y$ and $g : Y \to Z$ are homemorphisms then $g \circ f$ is a homeomorphism as well.*

Theorem 2. *Homeomorphism preserves Hausdorffness. I.e., if $f : X \to Y$ is a homeomorphism and X is Hausdorff, then Y must be Hausdorff.*

Proof: Let $x \neq y \in Y$. Then $f^{-1}(x) \neq f^{-1}(y)$ since f is a bijection, and $\exists U, V \subseteq X$ open with $U \cap V = \emptyset$, $f^{-1}(x) \in U$ and $f^{-1}(y) \in V$. Then $x \in f(U)$ and $y \in f(V)$. These are disjoint open sets since f is a bijection.

$\square$

Definition 2. *Any property of a space that is preserved by homeomorphism is called a topological property.*

Definition 3. *A space X is disconnected if there exists $U, V \subseteq X$ open with $U \cap V = \emptyset$ and $U \cup V = X$.*

Definition 4. *A space is connected iff it is not disconnected.*