Math 435 11/19/2025 Notes

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Timothy Tarter
James Madison University
Department of Mathematics

Review

Definition 1. *A category $\mathcal{C}$ consists of two things,*

1. *A collection of objects, $Ob(\mathcal{C})$, and,*
2. *Morphisms between those objects, $Mor(\mathcal{C})$.*

These have the following properties.

- *Each morphism $f \in hom(\mathcal{C})$ has a **source** $src \in Ob(\mathcal{C})$ and a **target** $tgt \in Ob(\mathcal{C})$.*
- *There is a composition operation such that given objects $A, B, C \in Ob(\mathcal{C})$, $\circ : Mor(B, C) \times Mor(A, B) \to Mor(A, C)$ such that $\circ$ is associative.*
- *There exists an identity morphism, for all $A \in \mathcal{C}$, $\exists id_A \in Mor(A, A)$ with $f \circ id_A = f$ and $id_A \circ f = f$ for any $f \in Mor(A, B)$ and $id_A \circ g = g$ for any $g \in Mor(B, A)$.*

Today

- Category theory examples

Example 1. ***Set*** *has*

- *Ob(Set) contains sets*
- *Mor(Set) contains functions*

Example 2. ***Top***$_*$ *has*

- *Ob(Top$_*$) contains pointed / based topological spaces*
- *Mor(Top$_*$) contains pointed / based continuous functions*

Example 3. ***Grp*** *has*

- *Ob(Grp) contains groups*
- *Mor(Grp) contains group homomorphisms*

Example 4. *Consider: $f : A \to B$ is an epimorphism (surjective) iff $g_1 \circ f = g_2 \circ f$ implies $g_1 = g_2$ for any $g_1, g_2 : B \to X$.*

$$
\begin{tikzcd}
A \arrow[r, "f"] \arrow[rr, bend left, "g_1 \circ f"] \arrow[rr, bend right, "g_2 \circ f"'] & B \arrow[r, shift left, "g_2"] \arrow[r, shift right, "g_1"'] & X
\end{tikzcd}
$$

This holds universally for any category! So if we have any such diagram, in any category, f is an epimorphism.

Remark 1. *Principle of duality: if we reverse the arrows of the diagram in example 4, $f : B \to A$ is an monomorphism (injection). I.e., $f \circ g_1 = f \circ g_2$ implies $g_1 = g_2$.*

Example 5. *Unique factorization of the product:*

$$
\begin{tikzcd}
 & Z \arrow[d, dashed, "g!"] \arrow[ddl, "f_1"'] \arrow[ddr, "f_2"] & \\
 & X \times Y \arrow[dl, "\pi_x"] \arrow[dr, "\pi_y"'] & \\
X & & Y
\end{tikzcd}
$$