Math 435 Homework 3

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Problem 1:

Fix a set X, let $\mathfrak{U}_f$ denote the collection of subsets of X with finite complement, together with the empty set. That is,

$$\mathfrak{U}_f = \{X \setminus F \mid \text{ F is a finite subset of X}\} \cup \{\emptyset\} \tag{1}$$

Prove that $\mathfrak{U}_f$ is a topology on X. (This is the Zariski / finite complement topology).

Proof: We want to show the following things

1. $X, \emptyset \in \mathfrak{U}_f$
2. For any indexing set I, and $U_i \in \mathfrak{U}_f$, $\bigcup_{i \in I} U_i \in \mathfrak{U}_f$
3. For some $n \in \mathbb{Z}$, $\bigcap_{i=1}^{n} U_i = \mathfrak{U}_f$.

(Part 1) Let $F = \emptyset \subseteq X$. Then $X \setminus F = X$. So $X \in \mathfrak{U}_f$. Similarly, let $F = X$. Then $X \setminus X = \emptyset$. So $X, \emptyset \in \mathfrak{U}_f$.

(Part 2) Let I be some indexing set such that for every $i \in I$, $U_i \in \mathfrak{U}_f$ - or rather, U_i is the complement of a finite subset of X, F_x. So there exists $x \in U_j$ with $x \in X \setminus F_x \subseteq X$. Accordingly, call $U_j = X \setminus F_x$ Thus, $x \in X \setminus F_x = U_j \in \bigcup_{i=1}^{n} U_i$ implies $\bigcup_{i=1}^{n} U_i$ is the complement of the union of every F_{x_i}. By DeMorgan's laws,

$$\bigcup_{i=1}^{n} U_i = \bigcup_{i=1}^{n} X \setminus F_i = X \setminus \bigcap_{i=1}^{n} F_i. \tag{2}$$

But since $\bigcap_{i=1}^{n} F_i$ is a finite union of finite subsets of X, $\bigcap_{i=1}^{n} F_i$ is a finite subset of X Therefore, $\bigcup_{i=1}^{n} U_i = \bigcup_{i=1}^{n} X \setminus F_i \in \mathfrak{U}_f$.

(Part 3) Let $U_i \in \mathfrak{U}_f$ for every $i \in 1, 2, \ldots, n$. If any $U_j = \emptyset = X \setminus X$, then $\bigcap_{i=1}^{n} U_i = \emptyset \in \mathfrak{U}_f$. If each $U_j \neq \emptyset$, then $U_J = X \setminus F_j$ for some finite $F_j \subseteq X$. If each U_j is disjoint from the others, $j \in 1, \ldots n$, then $\bigcap_{i=1}^{n} U_i = \emptyset \in \mathfrak{U}_f$. Now, let $\bigcap_{i=1}^{n} U_i \neq \emptyset$. Then $\bigcap_{i=1}^{n} U_i = G \subseteq X$. So $G = X \setminus G^C$. Thus $G \in \mathfrak{U}_f$, as desired.

□

Problem 2:

Recall that the power set of X is called the discrete topology, $\mathcal{U}_d$ on X. Prove that a topology $\mathcal{U}$ is the discrete topology on X iff $\{x\} \in \mathcal{U}$ for all $x \in X$.

Proof: (Direction 1) Let $\mathcal{U} = \mathcal{U}_d = \mathcal{P}(X) = \{U \mid U \subseteq X\}$. We want to show that $\{x\} \in \mathcal{U}$ for all $x \in X$. Since $x \in X$, $\{x\} \subseteq X$. We defined $\mathcal{U}$ as the set of all subsets of X. So $\{x\} \in \mathcal{U}$, as desired.

(Direction 2) Now, assume that $\{x\} \in \mathcal{U}$ for all $x \in X$. We want to show that $\mathcal{U} = \mathcal{U}_d$, i.e., that $\mathcal{U}$ contains every subset of X. Since $\mathcal{U}$ is a topology, it is closed under arbitrary union. Thus, let $V \subseteq X$ be any subset of X. Then $V = \{x_1\} \cup \{x_2\} \cup \cdots \cup \{x_n\}$, and so $V \in \mathcal{U}$, as desired.

□

Problem 3:

We say two topologies $\mathcal{U}, \mathcal{U}'$ on a set X coincide if they are equal as sets, i.e., $\mathcal{U} \subseteq \mathcal{U}'$ and $\mathcal{U}' \subseteq \mathcal{U}$.

1. Give an example of a set X such that $\mathcal{U}_d$ coincides with the finite complement topology $\mathcal{U}_f$ on X.
2. Make and prove a conjecture about the class of sets for which the discrete topology and the finite complement topology coincide.

Part 1:

Let $X = \{\}$. Every finite subset of X is the empty set, whose complement is the empty set. So $\emptyset$ is the only subset, and every subset, of X. And $\mathcal{U}_f = \mathcal{U}_d$.

Part 2:

Proposition 1. *The discrete topology and the finite complement topology coincide on a set X iff X is a finite set.*

Proof: (Direction 1) Let X be any set, with $\mathcal{U}_f, \mathcal{U}_d$ on X such that $\mathcal{U}_f = \mathcal{U}_d$. We want to show that X is a finite set. Well, $U_d = \{U \mid U \subseteq X\}$ and $\mathcal{U}_f = \{X \setminus F \mid \text{F is a finite subset of X}\} \cup \{\emptyset\}$. So if U is a finite subset of X, then $U \in \mathcal{U}_d$. Assume X is non-finite. Then $X \setminus U$ is a non-finite set. But $X \setminus U \in \mathcal{U}_d$ implies $X \setminus U \in \mathcal{U}_f$. But $X \setminus U$ can't be in $\mathcal{U}_f$ because U is non-finite, which is a contradiction. Thus, X must be finite.

(Direction 2.1) Now, let X be a finite set and let $U \in \mathcal{U}_f$. We want to show that $U \in \mathcal{U}_d$, which implies that $\mathcal{U}_f \subseteq \mathcal{U}_d$. Since $U \in U_f$ and X is finite, there exists $F \subseteq X$ such that $U = X \setminus F$. Thus, $U \subseteq X$. So $U \in \mathcal{U}_f$ and $\mathcal{U}_f \subseteq \mathcal{U}_d$.

(Direction 2.2) Finally, let X be finite and $U \in \mathcal{U}_d$. We want to show that $U \in \mathcal{U}_f$. Since $U \in \mathcal{U}_d$, $U \subseteq X$. Since X is finite, U must be also be finite. Thus, let $F = X \setminus U$ implies $U = X \setminus F$. So $U \in \mathcal{U}_f$ implies $\mathcal{U}_d \subseteq \mathcal{U}_f$.

□

Problem 4:

Recall that the set of open balls in the plane $\mathbb{R}^2$ (with euclidean metric) is a basis that generates the standard topology τ on $\mathbb{R}^2$. Consider the set of open rectangles in the plane:

$$\mathcal{B} = \{(a,b)\times(c,d) \subseteq \mathbb{R}^2 \mid a < b \text{ and } c < d\}. \tag{3}$$

Part 1:

Prove that $\mathcal{B}$ is a basis for a topology τ' on $\mathbb{R}^2$. Recall:

Definition 1. *A collection $\mathcal{B}$ of subsets of a set X is called a basis if*

1. *The sets in $\mathcal{B}$ cover X, i.e., $\forall x \in X$ there exists $B \in \mathcal{B}$ with $x \in B$.*
2. *If $B_1, B_2 \in \mathcal{B}$ and if $x \in B_1 \cap B_2$, then there exists $B_3 \in \mathcal{B}$ such that $x \in B_3 \subseteq B_1 \cap B_2$.*

Proposition 2. *(From class) Let $\mathcal{U}_1, \mathcal{U}_2$ be the topologies generated by bases $\mathcal{B}_1$ and $\mathcal{B}_2$ on X. If every element of $\mathcal{B}_1$ is a union of elements of $\mathcal{B}_2$, then $\mathcal{U}_1 \subseteq \mathcal{U}_2$.*

Proof: (Part 1.1) We want to show that the sets in $\mathcal{B}$ cover $\mathbb{R}^2$. I.e., $\forall x \in X$ there exists $B \in \mathcal{B}$ with $x \in B$. Let $(\alpha,\beta) \in \mathbb{R}^2$. Then let $a_0 \in \mathbb{R}$ with $d(a_0,\alpha) = \epsilon > 0$ and $b_0 \in \mathbb{R}$ with $d(b_0,\beta) = \delta > 0$. Then $B(\alpha,\epsilon)\times B(\beta,\delta) \in \mathcal{B}$ is the exact element of $\mathcal{B}$ containing (α,β). Since α, β arbitrary, $\mathcal{B}$ covers X.

(Part 1.2) We want to show that if $B_1, B_2 \in \mathcal{B}$, and if $x \in B_1 \cap B_2$, then there exists $B_3 \in \mathcal{B}$ such that $x \in B_3 \subseteq B_1 \cap B_2$. Let $B_1, B_2 \in \mathcal{B}$ with $x = (a,b) \in B_1 \cap B_2$. We know that

$$B_1 = (a_1,b_1)\times(c_1,d_1) \tag{4}$$

$$B_2 = (a_2,b_2)\times(c_2,d_2). \tag{5}$$

Then,

$$B_1 \cap B_2 = (\max(a_1,a_2), \min(b_1,b_2)) \times (\max(c_1,c_2), \min(d_1,d_2)). \tag{6}$$

So for any (x,y) and (z,w) with $x,z \in (\max(a_1,a_2),\min(b_1,b_2))$ and $y,w \in (\max(c_1,c_2),\min(d_1,d_2))$, $(x,y)\times(z,w) \in B_3 \subseteq B_1 \cap B_2$, as desired.

□

Part 2:

Prove that τ coincides with τ'.

Proof: Let

$$\mathcal{B}_1 = \{B((x_0,y_0),\epsilon) \mid (x_0,y_0) \in \mathbb{R}^2, \epsilon > 0\} \tag{7}$$

$$\mathcal{B}_2 = \{(a,b)\times(c,d) \subseteq \mathbb{R}^2 \mid a < b \text{ and } c < d\}. \tag{8}$$

We want to show that every element of $\mathcal{B}_1$ is a union of the elements of $\mathcal{B}_2$, and vice versa.

Part 2.1 Let $B((x_0, y_0), \epsilon) \in \mathcal{B}_1$. We want to show that $B((x_0, y_0), \epsilon) = \bigcup_{i \in I}(a_i, b_i) \times (c_i, d_i)$, where $(a_i, b_i) \times (c_i, d_i) \in \mathcal{B}_2$. Let $(x, y) \in B((x_0, y_0), \epsilon)$. We want to find some δ such that the rectangle $(x - \delta, x + \delta) \times (y - \delta, y + \delta) \subseteq B((x_0, y_0), \epsilon)$. Let

$$\delta = \frac{\epsilon - \sqrt{(x - x_0)^2 + (y - y_0)^2}}{\sqrt{(x - x_0)^2} + \sqrt{(y - y_0)^2}} = \frac{\epsilon - \sqrt{(x - x_0)^2 + (y - y_0)^2}}{|x - x_0| + |y - y_0|}. \tag{9}$$

Then, since $(x, y) \in B((x_0, y_0), \epsilon)$,

$$\epsilon - \sqrt{(x - x_0)^2 + (y - y_0)^2} > 0. \tag{10}$$

Moreover, any point (x, y) can only be, at its furthest, at a corner. Thus, we need to divide by the (absolute value) distance between x and x_0, and y and y_0:

$$|x - x_0| + |y - y_0|. \tag{11}$$

Thus, for our choice of δ, we have that,

$$(x - \delta, x + \delta) \times (y - \delta, y + \delta) \subseteq B((x_0, y_0), \epsilon), \tag{12}$$

as desired.

Part 2.2 Let $(a_i, b_i) \times (c_i, d_i) \in \mathcal{B}_2$. We want to show that $(a_i, b_i) \times (c_i, d_i)$ can be written as a union of open balls. Let $x_0 \in (a_i, b_i)$ and $y_0 \in (c_i, d_i)$. Let $\epsilon = \min(|x_0 - a_i|, |x_0 - b_i|, |y_0 - c_i|, |y_0 - d_i|)$. Then $B((x_0, y_0), \epsilon) \in (a_i, b_i) \times (c_i, d_i)$ implies that $(a_i, b_i) \times (c_i, d_i) = \bigcup B((x_0, y_0), \epsilon)$ for every $(x_0, y_0) \in (a_i, b_i) \times (c_i, d_i)$, as desired.

Therefore, τ coincides with τ'.

$\square$