Math 435 Homework 2

September 6, 2025

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Problem 1: Prove that the discrete metric on a set X is a distance metric.

Recall that the discrete metric is defined by

$$d(x,y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y \end{cases} \tag{1}$$

We want to show, for any $x, y, z \in X$, that:

1. $d(x,y) = 0$ iff $x = y$
2. $d(x,y) = d(y,x)$
3. $d(x,z) \leq d(x,y) + d(y,z)$.

(1) $d(x,y) = 0$ iff $x = y$

(Direction 1) Let $d(x,y) = 0$. Then $x = y$, by definition. (Direction 2) Let $x = y$. Then $d(x,y) = 0$, also by definition.

(2) $d(x,y) = d(y,x)$

Let $x = y$. Then $d(x,y) = 0 = d(y,x)$ since $x = y = x$. Alternatively, let $x \neq y$. Then $d(x,y) = 1 = d(y,x)$ since the points are distinct. Thus, d is symmetric.

(3) $d(x,z) \leq d(x,y) + d(y,z)$

Let $x = z$. Then $d(x,z) = 0$. If $y = x = z$,then

$$d(x,z) = 0 = d(x,y) + d(y,z) = 0 + 0 = 0. \tag{2}$$

If $y \neq x = z$, then

$$d(x,z) = 0 < d(x,y) + d(y,z) = 1 + 1 = 2. \tag{3}$$

Alternatively, if $x \neq z$, then $d(x,z) = 1$. If, WLOG, $y \neq x$ and $y = z$ then $d(x,y) = 1$ implies that

$$d(x,z) = 1 = d(x,y) + d(x,z) = 1 + 0 = 1. \tag{4}$$

If $y \neq x$ and $y \neq z$, then $d(x,y) = d(z,y) = 1$ implies

$$d(x,z) = 1 < d(x,y) + d(y,z) = 1 + 1. \tag{5}$$

Thus, d is a distance metric on any set X.

□

Problem 2: Consider the following function, d. Show that it is a metric on $\mathbb{R}^2 \times \mathbb{R}^2$.

Let $(x_i, y_i) \in \mathbb{R}^2$. We define $d : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ as:

$$d((x_1,y_1),(x_2,y_2)) = \begin{cases} 0 & \text{if } (x_1,y_1) = (x_2,y_2) \\ |y_1| + |x_2 - x_1| + |y_2| & \text{if } (x_1,y_1) \neq (x_2,y_2) \end{cases} \tag{6}$$

We want to show, for any $x, y, z \in X$, that:

1. $d((x_1,y_1),(x_2,y_2)) = 0$ iff $(x_1,y_1) = (x_2,y_2)$
2. $d((x_1,y_1),(x_2,y_2)) = d((x_2,y_2),(x_1,y_1))$
3. $d((x_1,y_1),(x_3,y_3)) \leq d((x_1,y_1),(x_2,y_2)) + d((x_2,y_2),(x_3,y_3))$.

(1) $d((x_1,y_1),(x_2,y_2)) = 0$ iff $(x_1,y_1) = (x_2,y_2)$

This is by definition, as in problem 1.

(2) $d((x_1,y_1),(x_2,y_2)) = d((x_2,y_2),(x_1,y_1))$

The trivial case is if $(x_1,y_1) = (x_2,y_2)$; alternatively if $(x_1,y_1) \neq (x_2,y_2)$, then since addition is commutative, and since $|x_1 - x_2| = |x_2 - x_1|$,

$$|y_1| + |x_2 - x_1| + |y_2| = |y_2| + |x_1 - x_2| + |y_1|. \tag{7}$$

Thus d is symmetric.

(3) $d((x_1,y_1),(x_3,y_3)) \leq d((x_1,y_1),(x_2,y_2)) + d((x_2,y_2),(x_3,y_3))$

Again, the trivial case is where all points are the same. Then $0 = 0 + 0$. Similarly, if $x_1 = x_3$ and $y_1 = y_3$, then $0 \leq \ldots$ implies that the triangle inequality holds. Finally, if all of the points are distinct, then

$$
\begin{aligned}
d((x_1,y_1),(x_3,y_3)) - d((x_1,y_1),(x_2,y_2)) - d((x_2,y_2),(x_3,y_3)) &= && (8)\\
= |y_1| + |x_3 - x_1| + |y_3| - (|y_1| + |x_2 - x_1| + |y_2|) - (|y_2| + |x_3 - x_2| + |y_3|) &= && (9)\\
= |y_1| + |x_3 - x_1| + |y_3| - |y_1| - |x_2 - x_1| - |y_2| - |y_2| - |x_3 - x_2| - |y_3| &= && (10)\\
= |\cancel{y_1}| + |x_3 - x_1| + |\cancel{y_3}| - |\cancel{y_1}| - |x_2 - x_1| - |y_2| - |y_2| - |x_3 - x_2| - |\cancel{y_3}| &= && (11)\\
|x_3 - x_1| - |x_2 - x_1| - |y_2| - |y_2| - |x_3 - x_2| &= && (12)\\
|x_3 - x_1| - |x_2 - x_1| - 2|y_2| - |x_3 - x_2| & && (13)
\end{aligned}
$$

If it were true that $|x_3 - x_1| - |x_2 - x_1| - 2|y_2| - |x_3 - x_2| > 0$, then our metric would not satisfy the triangle equality. I.e., it would be true that:

$$|x_3 - x_1| > |x_2 - x_1| + 2|y_2| + |x_3 - x_2| \tag{14}$$

However, in $\mathbb{R}$ with the absolute value metric,

$$|x_3 - x_1| \leq |x_3 - x_2| + |x_2 - x_1|. \tag{15}$$

So,

$$|x_3 - x_1| \leq |x_2 - x_1| + 2|y_2| + |x_3 - x_2| \tag{16}$$

and the triangle inequality holds for our metric.

$\square$

Problem 3:

Given a pair of metric spaces (X_1, d_1) and (X_2, d_2), and two points $x = (x_1, x_2)$ and $y = (y_1, y_2)$ in $X_1 \times X_2$, we can define the metric

$$d_\infty(x,y) = \max\{d_1(x_1,y_1), d_2(x_2,y_2)\}. \tag{17}$$

Show that this is indeed a distance metric.

(1) $d_\infty(x,y) = 0$ iff $x = y$

(Direction 1:) Let $d_\infty(x,y) = 0$. Then $\max\{d_1(x_1,y_1), d_2(x_2,y_2)\} = 0$. Since d_1, d_2 are metrics on X_1, X_2, then $x_1 = y_1$ and $x_2 = y_2$ by the very property we seek to prove for d_∞.

(Direction 2:) Let $x = y$. Then by the above mentioned property of d_1 and d_2, $d_1(x,y) = d_2(x,y) = 0$. So $\max\{0,0\} = 0$.

(2) $d_\infty(x,y) = d_\infty(y,x)$

The max function is symmetric, so $\max(a,b) = \max(b,a)$ implies d_∞ is symmetric as well.

(3) $d_\infty(x,z) \leq d_\infty(x,y) + d_\infty(y,z)$

We have

$$d_\infty(x,z) - d_\infty(x,y) - d_\infty(y,z) = \tag{18}$$
$$\max\{d_1(x_1,z_1), d_2(x_2,z_2)\} - \max\{d_1(x_1,y_1), d_2(x_2,y_2)\} - \max\{d_1(y_1,z_1), d_2(y_2,z_2)\} \tag{19}$$

Since d_1, d_2 are distance metrics by assumption, we know that they satisfy the triangle inequality. Thus, we can consider $d_1(x_1,z_1) - d_1(x_1,y_1) - d_1(y_1,z_1) \leq 0$ and $d_2(x_2,z_2) - d_2(x_2,y_2) - d_2(y_2,z_2) \leq 0$ to be true.

This leads us to ask a fundamental question: if WLOG $\max\{d_1(x_1,z_1), d_2(x_2,z_2)\} = d_1(x_1,z_1)$, would $d_2(x_2,y_2) > d_1(x_1,y_1)$ or $d_2(y_2,z_2) > d_1(y_1,z_1)$ even be possible? If we assume that it could be, we actually end up contradicting the assumption that $\max\{d_1(x_1,z_1), d_2(x_2,z_2)\} = d_1(x_1,z_1)$, since d_1, d_2 have the triangle inequality

$$d_1(x_1,z_1) \leq d_1(x_1,y_1) + d_1(y_1,z_1), \tag{20}$$
$$d_2(x_2,z_2) \leq d_2(x_2,y_2) + d_2(y_2,z_2), \tag{21}$$

implies

$$\max\{d_1(x_1,y_1) + d_1(y_1,z_1), d_2(x_2,z_2)\} = \tag{22}$$
$$\max\{d_1(x_1,y_1) + d_1(y_1,z_1), d_2(x_2,y_2) + d_2(y_2,z_2)\} = \tag{23}$$
$$d_1(x_1,y_1) + d_1(y_1,z_1). \tag{24}$$

Thus, we have covered all possible cases, and d_∞ satisfies the triangle inequality, and is a metric.

$\square$

Problem 4: Given a metric space (X,d), let $U_1 \ldots U_n \subseteq X$ be a finite collection of open sets. Prove that the intersection $\bigcap_{i=1}^n U_i$ is open.

Proof: Let $x_0 \in \bigcap_{i=1}^n U_i$. Then $x_0 \in U_j$, for every $j \in 1 \ldots n$. But U_j is open by assumption, so there exists some $\epsilon_j > 0$ with $B(x_0;\epsilon_j) \subseteq U_j$, for every U_j. Let $\epsilon = \min\{\epsilon_1 \ldots \epsilon_n\}$. Then $B(x_0;\epsilon) \subseteq \bigcap_{i=1}^n U_i$, and $\bigcap_{i=1}^n U_i$ is open, as desired.

$\square$