Math 435 Homework 5

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Given a space $(X,\mathcal{U})$, a set A, and a surjective function $p\colon X \to A$, recall that the resulting quotient topology $\mathcal{U}_p$ on A has

$$U \subset A \text{ open in } A \iff p^{-1}(U) \subset X \text{ open in } X.$$

In other words, $\mathcal{U}_p = \{p^{-1}(U) \mid U \in \mathcal{U}\}$.

Problem 1:

Problem 1. If $\mathbb{R}$ has the standard topology, define $p\colon \mathbb{R} \to \{a,b,c,d,e\}$ by

$$p(x) = \begin{cases} a \text{ if } x > 2 \\ b \text{ if } x = 2 \\ c \text{ if } 0 \leq x < 2 \\ d \text{ if } -1 < x < 0 \\ e \text{ if } x \leq -1. \end{cases}$$

List the open sets in the quotient topology on $\{a,b,c,d,e\}$

- $p^{-1}(a) = \{(2,\infty)\}$ so a is open
- $p^{-1}(d) = \{(-1,0)\}$ so d is open

all other singleton elements of the quotient topology are not open because their preimages are not open in $\mathbb{R}$. Additionally, $\{c,d\}$, $\{a,d\}$, $\{a,c,d\}$, $\{d,e\}$, $\{a,d,e\}$, $\{c,d,e\}$, $\{a,c,d,e\}$, $\{a,b,c,d\}$, $\emptyset$, and $\{a,b,c,d,e\}$ are open because the union of their preimages are open in $\mathbb{R}$. (I kinda eye-balled it, it's not super hard to put together from the piecewise function. Hopefully I didn't need to list every individual preimage - if I did, I'm happy to revise this lol.)

Problem 2:

Problem 2. Given a space X, a set A, and a surjective function $p\colon X \to A$, it might be tempting to think that the quotient topology on A is precisely the collection of images of open sets in X. Prove that this is *not* true. That is, prove

$$\mathcal{U}_p \neq \{p(U) \mid U \text{ open in } X\}$$

in general. (Hint: the first counterexample I found was with $X = \{a, b, c\}$, equipped with the topology $\mathcal{U} = \{\varnothing, \{a\}, \{a, b\}, X\}$, and the set $A = \{1, 2\}$.)

Let $\mathcal{U}$ be defined as in the problem statement, and let $A = \{1, 2\}$. An easy counterexample is to let $P : X \to A$ be defined by

- $p(a) = 1$
- $p(b) = 2$
- $p(c) = 1.$

Then $im(p) = \{\emptyset, \{1\}, \{1, 2\}\}$ and the quotient topology on A is $\{\{1, 2\}, \emptyset\}$. Notably, it is missing $\{1\}$.

□

Problem 3:

Problem 3. Given a space X, a set A, and a surjective function $p\colon X \to A$, call a subset $U \subset X$ *special* (with respect to p) if U is open in X and $p^{-1}(p(U)) = U$. Prove that the quotient topology on A *is* precisely the collection of images of special subsets of X.

Recall that the quotient topology is the set of all subsets of A such that the inverse image of a subset is open. Let $U \subseteq X$ such that $p^{-1}(p(U)) = U$. Then $p(U) \subseteq A$ with $p^{-1}(p(U))$ open in X. So all the image of all special subsets is contained in the quotient topology. Conversely, let $B \subseteq A$ be open with $U \subseteq X$ such that $p(U) = B$. Then $p^{-1}(B) = p^{-1}(p(U)) = U$, which by definition is open in X. So the set of all open sets in the quotient topology is contained in the image of all special subsets. Thus, the quotient topology is exactly the collection of images of special subsets of X.

□

Problem 4:

Midterm grade: $24/25 = 96\%$. There were really only two small errors, both of which where only half of a point (in my opinion).