Continuity in the Zariski Topology as a Bridge From Topological Continuity to Sheaf Glueability in Affine Algebraic Geometry

December 10, 2025

Math 435 Final Project
Timothy Tarter, Undergraduate Student
James Madison University
Department of Mathematics

Abstract

In this paper, we develop the notion of continuous maps in the Zariski topology as a motivation for classical Algebraic Geometry. Along the way, we establish the machinery of modern Algebraic Geometry in order to more effectively answer the following questions:

1. For a fixed $f(x)$, how can we prove or disprove that $f(x)$ is continuous in the Zariski topology?
2. How do varieties X, Y relate to each other, and what can $\mathcal{O}(X)$ tell us about $\mathcal{O}(Y)$ tell us about Y? How about $K(X)$ and $K(Y)$?
3. Furthermore, for a fixed variety X, what is the relationship between $\mathcal{O}(X), A(X),$ and $K(X)$?
4. How do birational maps induce isomorphisms on open subsets, and how do local / global glueing theorems help us tie things back to topology?
5. Can we use restriction maps on open subsets to generalize continuity?
6. How can we measure failures of glueability with the cohomology of sheaves? What is the topological intuition for this?

By the end of this paper, we hope to inspire the reader to see the purpose (and genius) in employing sheaves and cohomology to solve more abstract versions of these problems.

Contents

1 A Reference Regarding Ring Theory

In this paper, we will largely deal with polynomial rings and their residue fields, among other algebraic structures. While the following is no replacement for full courses in Abstract Algebra I & II, it is meant to be a helpful guide for intuition and reference while reading this paper.

Definition 1. *A ring R is a set with two binary operations, $+ : R \times R \to R$ and $* : R \times R \to R$ on it, satisfying the following axioms:*

1. (additive closure) for all $a, b \in R$, $a + b \in R$
2. (additive identity) $0 \in R$ has, for all $a \in R$, $a + 0 = 0 + a = a$
3. (additive inverse) if $a \in R$, there exists $-a \in R$ with $a + (-a) = -a + a = 0$
4. (associativity of $+$) for all $a, b, c \in R$, $a + (b + c) = (a + b) + c$
5. (commutativity of $+$) for all $a, b \in R$, $a + b = b + a$
6. (multiplicative closure) for all $a, b \in R$, $a * b \in R$
7. (associativity of $*$) for all $a, b, c \in R$, $a * (b * c) = (a * b) * c$
8. (bilinear distributive property) for all $a, b, c \in R$, $a(b + c) = ab + ac$ and $(a + b)c = ac + bc$
9. (multiplicative identity) $1 \in R$ where for all $a \in R$, $1 * a = a * 1 = a$

Definition 2. *When R has multiplicative inverses for all nonzero elements, we call R a **division ring**.*

Definition 3. *When R has commutative multiplication, i.e., for all $a, b \in R$, $a * b = b * a$, we call R a **commutative ring**. In this paper, we will assume that any ring worth noting is commutative unless otherwise specified.*

Definition 4. *If R is a commutative division ring, we call it a **field**.*

Definition 5. *A **zero divisor** is a non-zero element z of a ring R such that there exists an element $x \in R$ with $xz = 0$.*

Definition 6. *An **integral domain** is a ring which has no zero divisors.*

Example 1. $\mathbb{Z}_6$ *is not an integral domain since* $3 * 2$ *[mod 6]* $\equiv 0$.

Example 2. $\mathbb{Z}_7$ *is an integral domain since* $\gcd(7, n) = 1$ *for every* $n < 7$.

Theorem 1. $\mathbb{Z}_p$ *is an integral domain (and actually a field) if p is a prime integer.*

Definition 7. *$I \subseteq R$ is an **ideal** if it satisfies the following axioms:*

1. *(additive identity)* $0 \in I$
2. *(additive closure) if* $a, b \in I$, $a + b \in I$
3. *(sticky multiplication) if* $a \in R$ *and* $b \in I$, *then* $ab \in I$.

Definition 8. *If $\{f_1, \ldots, f_n\} = F \subseteq R$ is a list of elements of R, we say that $\langle F \rangle$ is the ideal generated by F.*

$$\langle F \rangle = \{\sum_{i=1}^{n} h_i f_i \, : \, h_i \in R, f_i \in F\}$$

Definition 9. *If an ideal $I \subseteq R$ can be generated by one element, I is called a* ***principal ideal****.*

Definition 10. *If all ideals of a ring R are principally generated, R is called a* ***Principal Ideal Domain****, or* ***PID****.*

Theorem 2. *Integral domain $\Leftarrow$ PID $\Leftarrow$ Field*

Definition 11. *Let $X = \{x_0, \ldots, x_n\}$ be a list of independent variables and let k be an algebraically closed field. We call $k[X]$ the* ***polynomial ring*** *with coefficients in k and variables in X.*

Definition 12. *Let $\mathfrak{m} \subseteq R$ be a proper ideal. If there does not exist a proper ideal I such that $\mathfrak{m} \subseteq I \subseteq R$ and $\mathfrak{m} \neq I$, then $\mathfrak{m}$ is called a* ***maximal ideal*** *of R.*

Definition 13. *Let $I \subseteq R$ be an ideal of R. We define the* ***quotient ring*** *to be,*

$$R/I = \{a + I \ ; \, a \in R\}.$$

Theorem 3. *$R/\mathfrak{m}$ is called the* ***residue field of R****. In particular, for any polynomial ring $A = k[X]$, $A/\mathfrak{m} \cong k$ has k as its residue field.*

Definition 14. *$I \subseteq R$ is a prime ideal iff*

- *I is a proper subset of R, and,*
- *if for any $a, b \in R$, and $ab \in I$, then $a \in I$ or $b \in I$.*

In practice, it is common that we mean that a prime ideal of a polynomial ring is one generated by irreducible polynomials. With a little bit of work, it becomes clear that these definitions are biconditionally equivalent.

Theorem 4. *If $I \subseteq R$ is a prime ideal of R, then R/I is an integral domain. If I is maximal, R/I is a field.*

Example 3. $\mathbb{R}[x]/\langle x^2 + 1 \rangle \cong \mathbb{C}$.

Definition 15. *The ideal generated by a subset $Y \subseteq k^n$ is as follows*

$$I(Y) = \{f \in k[X] \ ; \ f(p) = 0 \text{ for all } p \in Y\}$$

Obviously, I am omitting a great deal of Abstract Algebra II here. However, this should not be a great impediment to the reader as this section is merely a light synopsis of complicated material. For deeper understanding, please consult [AR] .

2 Continuity in the Zariski Topology

2.1 Making Sense of Affine Space

To successfully motivate the notion of continuous functions, we need to define a few things:

1. An affine n-space over a field k
2. Zero sets of polynomials as algebraic sets
3. The Zariski Topology
4. Irreducible open subsets
5. The affine variety generated by a set of polynomials
6. The definition of a continuous map.

Definition 16. *We define the **affine n-space**, $\mathbb{A}^n$ over k to be the set of n-tuples with coordinates in k^n. Note, this is not the same as the coordinate ring of $k[X]$.*

The notion of affine n-space clearly draws some intuition from vector-space like intuition, but there are some stark differences between vector spaces and affine spaces. The main property which exemplifies this fact is that affine spaces don't preserve anything which varies under linear transformations. As an example, in affine n-space,

- circles are identified with arbitrary ellipses, and,
- squares are identified with arbitrary parallelograms.

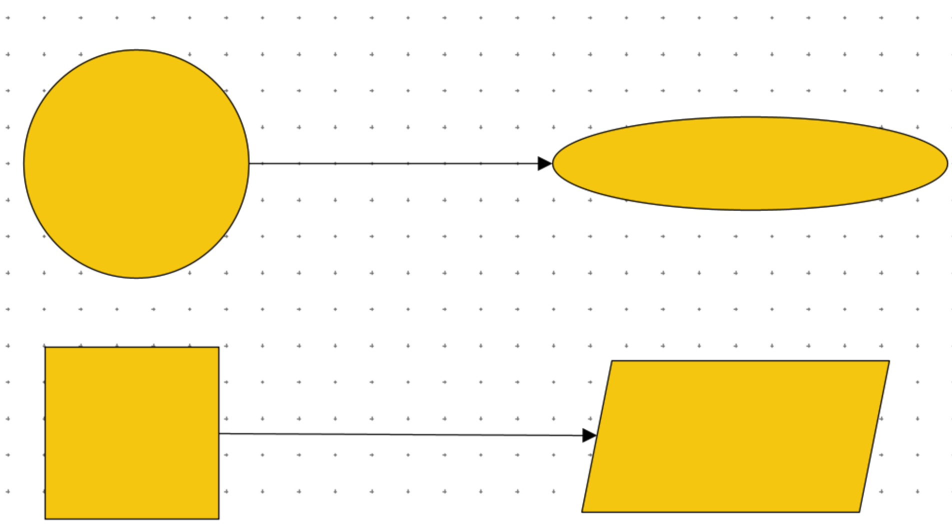

Figure 1: Circles and squares aren't preserved structures

We can distill this further into a list of "things which don't make sense" in affine space:

1. angles
2. magnitudes

3. zero.

In place of these traditional geometric constructions, we are left with other things which "make sense." Namely this list contains, ellipses, parallelograms, geometric linear invariants (like symmetries), and so on. Having laid a foundational intuition for affine space, we next seek to analyze functions in affine space by examining their zero sets as geometric constructions.

Definition 17. *Let $\{f_1, \dots f_s\} = F \subseteq A = k[X]$. The zero set of these polynomials is defined as follows:*

$$Z(F) = \{p \in \mathbb{A}^n \ ; \ f_i(p) = 0 \textit{ for all } i\}$$

In conjunction with this definition, we present the following as a means of regarding subsets of $\mathbb{A}^n$ as their own geometric objects:

Definition 18. *$Y \subseteq \mathbb{A}^n$ is called an algebraic set if there exist polynomials $F = \{f_1 \dots f_S\} \subseteq k[X]$ with*

$$Y = Z(F).$$

From the notion of algebraic sets, there is induced a natural topology which allows us to probe topological data in affine space.

Definition 19. *Open subsets in the Zariski Topology on an affine n-space $\mathbb{A}^n$ are subsets whose complements are algebraic sets. Accordingly, closed sets are algebraic sets.*

As an additional structure on the Zariski topology (but really any topology), we are interested in defining the notion of irreducibility. While this may seem disconnected from geometry at first, I encourage the reader to consider other types of irreducible algebraic structures, such as,

- prime numbers,
- polynomials,
- subspaces,
- etc.

Definition 20. *If τ is a topology on a set X, and $U \subseteq X$ is closed, U is irreducible iff there do not exist $V_1, V_2 \subseteq U$, which are also closed, with $V_1 \cup V_2 = U$.*

2.2 Introducing Varieties

Within the scope of algebraic geometry, the notion of an irreducible closed set generalizes itself in the following manner.

Definition 21. *$Y \subseteq \mathbb{A}^n$ is called an affine variety if there exist polynomials $\{f_1, \dots f_s\} = F \subseteq A$ such that $Y = Z(F)$ is irreducible under the induced topology. Respectively, a quasi-affine variety is an open subset of an affine variety.*

Largely, algebraic geometry is the study of these varieties. There are many flavors (affine, projective, quasi-affine, and quasi-projective), but in this paper, we will only really discuss affine and quasi-affine varieties. For now, we will abandon the notion of varieties to discuss continuity on the affine line, although, we will later tie continuity back in to the broader topic of varieties and their algebraic invariants.

Definition 22. *Let X, Y be topological spaces. $f : X \to Y$ is a continuous map iff for all $U \overset{\circ}{\subset} Y$, $f^{-1}(U) \overset{\circ}{\subset} X$.*

In order to compute whether a function is continuous on the affine line, $\mathbb{A}^1$, we must examine what open sets look like under the image of a function $f : \mathbb{C} \to \mathbb{C}$. **The emphasis here is on the fact that $\mathbb{C}$ is the affine line $\mathbb{A}^1$, otherwise, all of the following will change.** Additionally, we will focus on complex algebraic geometry as a grounding mechanism for computability.

We know that open subsets of $\mathbb{C}$ must be the complement of the zero set of a list of polynomials. Accordingly the closed sets must be

1. all of $\mathbb{C}$
2. all finite subsets of $\mathbb{C}$.

So open sets are complements of finite sets. Then by definition (20), we know that for $f : \mathbb{C} \to \mathbb{C}$ to be continuous, the inverse image of the complement of any finite set must be the complement of a finite set. Expressed mathematically, we have

Lemma 1. *$f : \mathbb{C}_1 \to \mathbb{C}_2$ is continuous iff for all $U \overset{\circ}{\subset} \mathbb{C}_2$ where there exists $V = U^C$ in $\mathbb{C}_2$ such that $|V| = n \in \mathbb{Z}$, $f^{-1}(U)$ has a $W \in \mathbb{C}_1$ where $f^{-1}(U) = W^c$.*

Equivalently, we have another theorem:

Theorem 5. *$f : \mathbb{C}_1 \to \mathbb{C}_2$ is continuous if the inverse image of every closed set in $\mathbb{C}_2$ is closed in $\mathbb{C}_1$.*

Recall now that the closed sets in the Zariski topology on $\mathbb{C}$ are either finite sets or all of $\mathbb{C}$. Then we can induce a stronger criterion still for continuity:

Theorem 6. *$f : \mathbb{C}_1 \to \mathbb{C}_2$ is continuous iff every non-constant fiber (pre-image) from $\mathbb{C}_2$ is closed in $\mathbb{C}_1$.*

2.3 A Set of Classification Theorems for $\mathbb{A}^1$

What types of functions can we immediately classify as continuous maps out of this definition?

1. Non-constant polynomial maps
2. Regular functions

3. Rational functions which correspond to regular functions.

Let's prove it!

Definition 23. *A rational function is an element of the field of fractions of a polynomial ring $k[X]$, and, for $f, g \in k[X]$ is of the form*

$$\frac{f(X)}{g(X)}.$$

Definition 24. *$f \in k[X]$ is (loosely - we will refine this definition later) called a regular function if there exists $h, g \in k[X]$ with $g \neq 0$ such that*

$$f(X) = \frac{h(X)}{g(X)}.$$

Note: to align this definition with the one I provide later on, we are taking the quasi-affine variety to be the whole space.

Theorem 7. *Non-constant polynomial maps are continuous from $\mathbb{C}$ to $\mathbb{C}$ in the Zariski topology.*

Proof: Let $f \in \mathbb{C}[X]$ be a non-constant polynomial viewed by $z \mapsto f(z)$ and let $U \overset{\circ}{\subset} C_2$ have $V \subseteq \mathbb{C}_2$ be such that $U = V^C$ and V is finite. We want to show that $f^{-1}(U) \overset{\circ}{\subset} \mathbb{C}_1$. Well, since

$$f^{-1}(U) = f^{-1}(V^C) = f^{-1}(\mathbb{C} \setminus V) = \mathbb{C} \setminus f^{-1}(V),$$

as long as $f^{-1}(V)$ is closed, f is continuous. Now, since V is closed, we have two cases:

1. $V = \mathbb{C}$
2. $V = \{a_1, \ldots, a_n\} \subseteq \mathbb{C}$.

In case 1, $f^{-1}(V) = f^{-1}(\mathbb{C}) = \mathbb{C}$, and is so closed.

In case 2,

$$f^{-1}(V) = f^{-1}(\{a_1, \ldots, a_n\}) = \bigcup_{i=1}^{n} f^{-1}(a_i).$$

So as long as each $f^{-1}(a_i)$ is finite, our claim is proven. Well, for $a_i \in V$, there exists $z \in \mathbb{C}_1$ with $f(z) = a_i$. So $f(z) - a_i = 0$ means that $f - a_i$ vanishes on $f(z) - a_i$ (which in $\mathbb{A}^1$ is an equivalent definition for an algebraic set). By the fundamental theorem of algebra, $\deg(f)$ is finite, and so $f(z)$ has finitely many such cases, fixing $f^{-1}(a_i)$ to also be finite. Thus, we have proven our claim.

$\square$

Theorem 8. *Non-constant regular functions are continuous from $\mathbb{C}$ to $\mathbb{C}$ in the Zariski topology.*

Proof: $f(X)$ is defined to be a non-constant polynomial map, so by theorem (7), it is continuous. I highlight this case for the properties which regular functions provide later on, less for their importance now.

$\square$

Theorem 9. *Rational functions aren't necessarily continuous, and the ones that are correspond to regular functions.*

Proof: Let
$$f(x) = \frac{g(x)}{h(x)}$$
with $h(x) \neq 0$. If we divide by zero somewhere, there isn't a preimage,
$$f^{-1}(\frac{g(x)}{0}) = f^{-1}(\pm\infty).$$
Accordingly, if we can end up with a division by zero, $f(x)$ isn't continuous. Notably, $\pm\infty$ isn't even an element of $\mathbb{A}^1$!

But, if $f(x)$ is a regular function, we have that $h(x) \neq 0$ for all $x \in \mathbb{C}$. By theorem 8, we know that f is so continuous. Therefore, if a rational function is continuous, it can be represented by a regular function.

$\square$

This should rightfully cause people to ask: doesn't this mean that, given our setup of regular functions, $h(x)|g(x)$?

Yes, it does! So the rational representations of a regular function $f(x)$ are really just
$$f(x) = f(x)\frac{g(x)}{g(x)} = \frac{h(x)}{g(x)}.$$

2.4 Interesting Examples of Discontinuous Functions

Now that we've stated these important facts about regular functions and polynomial maps in $\mathbb{C} \to \mathbb{C}$ (or equivalently in $\mathbb{A}^1$), let's consider some transcendental examples in $\mathbb{A}^1$.

Example 4. *(The Exponential Function) Claim: e^x is discontinuous in the Zariski topology.*

One would think that e^x is continuous, since it has no obvious zeroes. That said, we can identify certain properties of e^x in $\mathbb{C}$, provided we borrow a tool from complex analysis: the complex Log and the notion of branch cuts. Here's the setup: we want to say that the set of all non-constant fibers from the image of e^x are finite in $\mathbb{C}$. That means, for each of our singleton sets $\{a\} \subseteq \mathbb{C}$, $f^{-1}(\{a\}) = \{z \in \mathbb{C} \ ; \ z = e^a\}$. Accordingly, this is the equivalent of taking $Log_e(e^x)$.

For reasons beyond explanation here, the complex logarithm is defined as,
$$Log(z) = \ln|z| + 2k\pi i.$$
The "principal" branch of the Log function only examines $\ln|z|$, but those aren't all of the complex numbers which satisfy $Log(e^z) = a$. So accordingly, the pre-image of every closed (finite) point in $im(e^z) = \{a \in \mathbb{C} \ ; \ a = z + 2k\pi i\}$. However, that isn't finite, and it isn't all of $\mathbb{C}$. So e^z isn't continuous in the Zariski topology.

$\square$

Example 5. *(The Sine and Cosine Functions) This is a fairly easy example which motivates the idea of branches. The principal branch of* $\sin(\theta)$ *is where we define* $\sin(\theta)$ *exclusively on* $[0, 2\pi)$. *Every other branch is defined on some integer multiple of* $[0, 2\pi)$, *and so the preimage of any singleton set in im(sin*(θ)*) is infinite. The same holds for* $\cos(\theta)$.

$\square$

3 Just Enough Algebraic Geometry to Be Dangerous

The title of this section is a tribute to [RV] , but also serves as an adequate warning to the reader. Algebraic Geometry (in my experience) is a field which is aided heavily by **every other field of mathematics** and it is very easy to get bogged down by the rigor and abstraction. Just like [RV] says in his preface: I urge the reader of this paper to take time to really digest the abstract concepts in this section and to always attempt to find classical motivations for definitions and theorems included here.

Anyways. Now that we have established certain functions which are continuous on the complex affine line, let's consider general (potentially non-continuous) functions in higher dimensions of affine space, and their varieties. What relationships can we build between algebraic invariants of varieties and how can we tie it back to our topological intuition?

DISCLAIMER: all of the results below assume that we are working in an affine or quasi-affine variety unless *explicitly stated otherwise*. Many results stated in (3) do not hold for projective varieties.

3.1 Dimensions of Spaces

First, we need to construct some things about the dimension of varieties, as well as the dimension of ideals and polynomial rings. Initially, it is helpful to place bounds on chains within our topological spaces. We achieve this in the following way:

Definition 25. *A topological space X is called a Notherian topology if it satisfies the descending chain condition for closed subsets: for any sequence $Y_1 \supseteq Y_2 \supseteq \dots$ of closed subsets, there is an integer r such that $Y_r = Y_{r+1} = \dots$.*

As a result of this Notherian construction, we get a nicely behaved notion of irreducibility which makes varieties a richer topic to study.

Definition 26. *In a Notherian topological space X, every nonempty closed subset Y can be expressed as a finite union $Y = Y_1 \cup \cdots \cup Y_r$ of irreducible closed subsets Y_i. If we require that $Y_i \not\supseteq Y_j$ for $i \neq j$, then the Y_i are uniquely determined. They are called the irreducible components of Y.*

It easily follows that,

Theorem 10. *Any algebraic set in $\mathbb{A}^n$ can be expressed uniquely as a union of varieties, no one containing another.*

In a dual manner, one might ask how we can quantify the dimension of a ring. At first, these constructions might seem disconnected, but it will become immediately evident just how tethered these two concepts are.

Definition 27. *In a ring A, the height of a prime ideal $\mathfrak{p}$ is the supremum of all integers n such that there exists a chain $\mathfrak{p}_0 \subseteq \mathfrak{p}_1 \subseteq \cdots \subseteq \mathfrak{p}_n = \mathfrak{p}$ of distinct prime ideals. We define the dimension of A to be the supremum of the heights of all prime ideals.*

Recall from earlier that we highlighted the idea of irreducibility, tying together irreducible polynomials with prime numbers and irreducible subspaces. We will now introduce a combinatorial proposition linking the dimension of algebraic sets to their coordinate rings. In a major way, this proposition sets up the role of irreducibility of ideals, varieties, and topological spaces. We will realize this result soon.

Proposition 1. *If Y is an affine algebraic set, then the dimension of Y is equal to the dimension of its affine coordinate ring $A(Y)$.*

Proof: If Y is an affine algebraic set in $\mathbb{A}^n$, then the closed irreducible subsets of Y correspond to prime ideals of A containing $I(Y)$, the ideal generated by Y (see definition 15). These in turn correspond to prime ideals of $A(Y)$. Hence $\dim Y$ is the length of the longest chain of prime ideals in $A(Y)$, which is its dimension.

□

A resulting proposition is as follows:

Proposition 2. *A variety Y in $\mathbb{A}^n$ has dimension $n-1$ iff it is the zero set, $Z(f)$ of a single non-constant irreducible polynomial in $A = k[X]$.*

It isn't intrinsically obvious why this is true; but there is a deep geometric reason why this is forced to be true. Attacking the forward dimension, if we let Y be a variety in $\mathbb{A}^n$ with dimension $n-1$, we can consider a few assumptions for contradiction.

1. What if $Y = Z(\{f_1 \dots f_s\}$, i.e., Y is the zero set of multiple polynomials?
2. What if $Y = Z(f)$, but f isn't irreducible?
3. What if $Y = Z(f)$, but f is constant?

Tackling (1), we know that since Y is irreducible, $I(Y)$ is a prime ideal. Moreover, since A is a notherian factorization domain, the height of prime ideals are 1. So $I(Y)$ is generated by a single polynomial.

Towards (2), if f isn't irreducible, then, f is some product of polynomials $g_1 \dots g_r \in k[X]$,

$$f = \prod_{i=1}^{r} g_i.$$

Then it follows that

$$Z(f) = \bigcup_{i=1}^{r} Z(g_i).$$

So Y isn't irreducible.

Finally, for (3), if f is constant, its zero set either has dimension 0 or dimension n. This is trivial.

The converse direction is a bit easier. If a polynomial is non-constant and irreducible, its ideal is prime, and has height of $n-1$. Then by proposition 1, our claim is proven.

Having now sketched why proposition 2 is true, we can tie it back to our notion of continuity on the affine line. We know that non-constant polynomials on the affine line are continuous. The subset of these polynomials which are irreducible then have dimension 0 varieties. Notably, any reducible polynomial (on the affine line) will yield an algebraic set, not a variety.

The next invariant we will define will answer a question about what other functions might have shared properties on open subsets of these varieties (and general varieties). This may not seem terribly important at first, but the idea of glueability of sections on open subsets provides one of the primary construction of modern algebraic geometry: presheaves and sheaves. In order to provide this construction, we must first define the category of varieties. Recall our prior definition of regular functions:

3.2 The Category of Varieties

Definition 28. *Let Y be a quasi-affine variety in $\mathbb{A}^n$. A function $f : Y \to k$ is regular at a point $P \in Y$ if there is an open neighborhood U with $P \in U \subseteq Y$, and polynomials $g, h \in A = k[X]$, such that h is nowhere zero on U and $f = g/h$ on U. f is regular on Y if it is regular at every point of Y.*

Definition 29. *The category of varieties, $\mathfrak{Var}$, has the following characteristics:*

- *Elements of $Ob(\mathbf{V})$ are varieties*
- *Morphisms of $\mathbf{V}$ are continuous maps $\varphi : X \to Y$ such that for every open set $V \overset{\circ}{\subset} Y$ and for every regular function $f : V \to k$, the function $f \circ \varphi : \varphi^{-1}(V) \to k$ is regular.*

The notion of a morphism in $\mathfrak{Var}$ isn't implicitly clear from the definition. It may be helpful to examine the diagram below and "chase" objects around for a bit to gain intuition on how varieties interact.

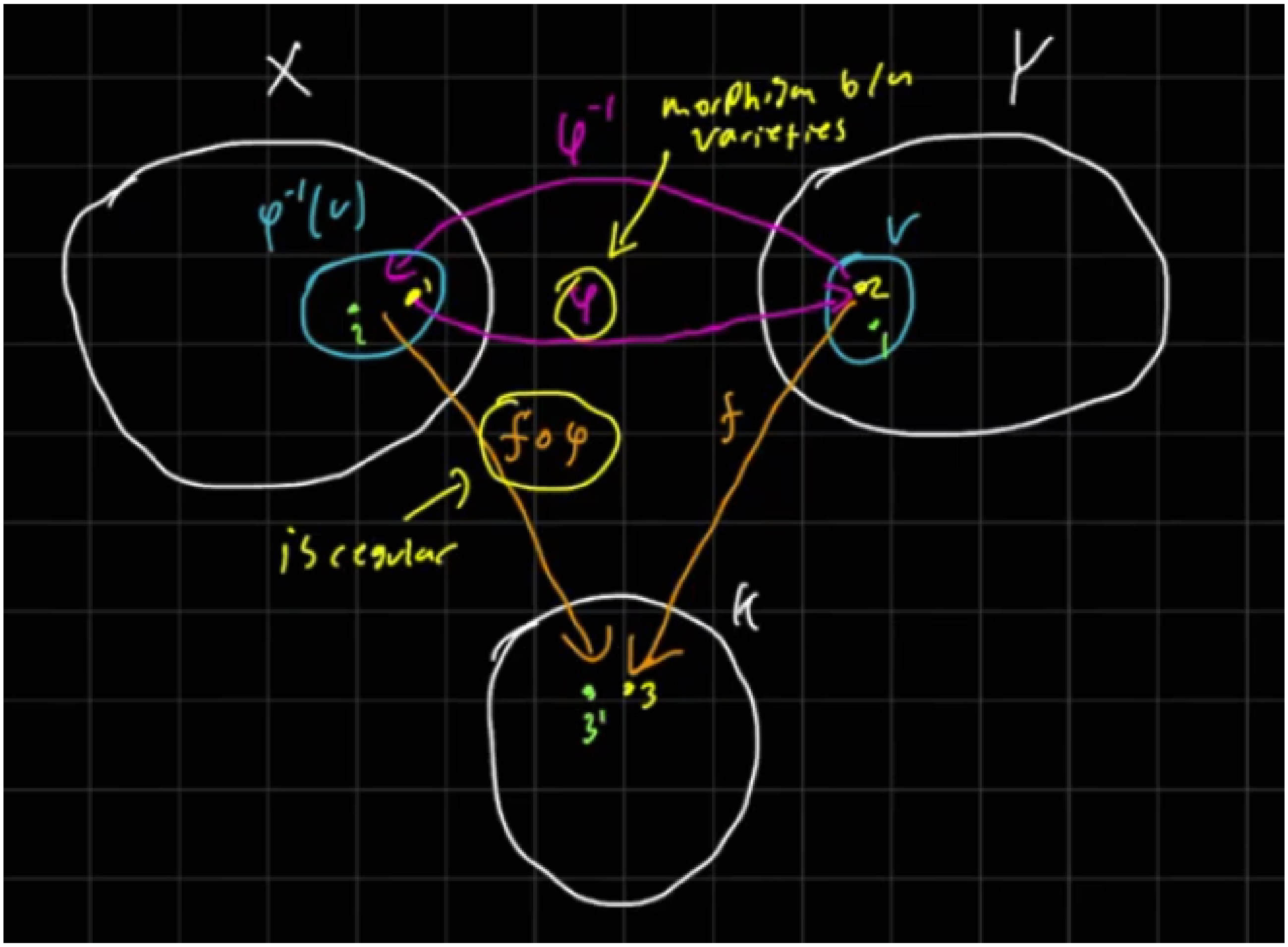

Figure 2: A potentially unhelpful universal diagram about morphisms of varieties

Definition 30. *An isomorphism of varieties is a morphism which admits an inverse morphism. Note that an isomorphism must be both a bijection and bicontinuous, but that a bicontinuous bijection might not be isomorphism.*

As an example, consider the following. Let k be an algebraically closed field of characteristic p. We define the Frobenius morphism to be $\varphi : \mathbb{A}^1 \to \mathbb{A}^1$ by $\varphi(t) = t^p$ [mod p]. φ is bicontinuous and bijective, but is only an isomorphism if p is prime / k is a perfect field.

3.3 Certain Algebraic Invariants

Now that we have scoped out the category of varieties (to a small degree), we can move towards our first few algebraic invariants.

Definition 31. *Let Y be a variety (of any kind). We call $\mathcal{O}(Y)$ the ring of regular functions on Y.*

$\mathcal{O}(Y)$ is a particularly interesting invariant because we can put a finer invariant on it by localizing it at a point (or multiplicative subset) in Y. It is beyond the scope of the questions that this paper intends to answer to give a universal definition of localization. For now, it suffices to say that if $S \subseteq Y$ is a multiplicative set, then the localization of Y at S is

$$S^{-1}Y = \{\frac{y}{s}\ ;\ y \in Y, s \in S\}.$$

There are two flavors of localizations, both of which adhere to the same universal commutative diagram, but present themselves in opposite ways. The flavor which we have presented (localizing at a point rather than localizing at an ideal) is what we will typically mean by localization, at least in this paper. Anyways - onto local rings as a special inclusion of $\mathcal{O}(Y)$:

Definition 32. *If $P \in Y$ is a point, then we define the local ring of P to be $\mathcal{O}_p$, the set of equivalence classes of all pairs $\langle U, f\rangle$ where $P \in U \overset{\circ}{\subset} Y$, f is regular on U, and $\langle U, f\rangle \sim \langle V, g\rangle$ iff $f = g$ on $U \cap V$. Notably, the maximal ideal $\mathfrak{m}$ of a local ring is the set of elements which vanish at P. Additionally, $\mathcal{O}_p/\mathfrak{m} \cong k$.*

This is a crucial definition for having algebraic objects to work with - we've eluded to a 1-1 correspondence between affine varieties and $O(Y)$, and for each variety Y, we've also hinted at a 1-1 correspondence between the prime ideals of $A = k[X]$ and varieties. However, local rings are truly our first example of what we will later call the "stalk of a sheaf." If we can build a notion of glueability of these "stalks," we can use local data to deduce global behavior of varieties. Naturally, sheaves are a great deal of the modern technology which we employ in algebraic geometry.

As a direct result of this motivation, we define the function field of a variety as follows.

Definition 33. *If Y is a variety, we define the function field $K(Y)$ to be the set of equivalence classes of all pairs $\langle U, f\rangle$ where $U \overset{\circ}{\subset} Y$, f is regular on U, and $\langle U, f\rangle \sim \langle V, g\rangle$ iff $f = g$ on $U \cap V$. The key difference between the $K(Y)$ and $\mathcal{O}_P$ is that $\mathcal{O}_P$ is defined on open sets which contain P, and $K(Y)$ is on all open sets.*

It is worth noting that while $K(Y)$ is not a sheaf, it is closely related to the notion of the structure sheaf on Y (denoted $\mathcal{O}_Y$), which is the collection of local rings under the same equivalence relation.

Proposition 3. *$K(Y)$ is a field.*

Rough justification: Addition and multiplication are well defined, and we can get multiplicative inverses by letting $\langle U, f\rangle^{-1} = \langle U - Z(f), \frac{1}{f}\rangle$. The rest falls into place by construction.

It is now worth stating a formal inclusion map between these invariants.

Proposition 4. *There is a natural injection from $\mathcal{O}(Y) \hookrightarrow \mathcal{O}_P \hookrightarrow K(Y)$, which yields $\mathcal{O}(Y)$ and $\mathcal{O}_P$ as subrings of $K(Y)$.*

Why is this "natural?" Well, $\mathcal{O}(Y)$ is the strictest form of $\mathcal{O}_p$ - the open subset is the entire set, so we have the smallest number of functions. Then $\mathcal{O}_p$ only requires that functions agree in the neighborhood of p, so we will naturally have more (or potentially an equal number of) functions in this ring. Then $K(Y)$ is just the ring regular functions which are gluable (note, there is no glueing actually happening) on intersections of open sets. So $K(Y)$ is the largest of the three rings, and naturally includes $\mathcal{O}(Y)$ and $\mathcal{O}_P$.

Our next remark has to do with why we've been calling $\mathcal{O}(Y)$, $\mathcal{O}_p$, and $K(Y)$ "invariants."

Remark 1. *If we have two varieties which are isomorphic, $X \cong Y$, then $\mathcal{O}(Y) \cong \mathcal{O}(X)$, $\mathcal{O}_p(Y) \cong \mathcal{O}_q(X)$ where p and q have a 1-1 correspondence, and $K(X) \cong K(Y)$. These are called invariants of the variety Y up to isomorphism (in the category of varieties).*

As a note, we've barely used the "affine-ness" of our varieties so far. In fact, most of what we've established will also hold for projective varieties (if the reader is so inclined to learn about them on their own). That said, our next result is highly dependent on the structure of an affine variety.

Theorem 11. *Let Y be an affine variety with affine coordinate ring $A(Y)$. Then*

- $\mathcal{O}(Y) \cong A(Y)$
- $K(Y) \cong$ *field of fractions of* $A(Y)$.

As a reversal of remark 1, it is worth noting the following.

Lemma 2. *Two varieties are isomorphic iff their affine coordinate rings are isomorphic as k-algebras.*

Having now discussed a string of correspondences and relations between structures, we want to establish a specific correspondence between prime ideals of $A = k[X]$ and varieties in $\mathbb{A}^n$ which falls out of the above set of constructions.

Theorem 12. *Prime ideals in $k[X]$ correspond to irreducible algebraic sets (varieties) in $\mathbb{A}^n$.*

Thus far, we have discussed invariants of varieties, correspondences between them, and how the dimension of certain invariants forces properties of both varieties and the continuity of polynomials which generate them. As a motivation for local rings, we loosely discussed the notion of sheaves and how the construction of a sheaf on a structure gives us a way to capture the glueability of sections of the underlying structure on intersections of open sets. While this concept is incredibly important, we have so far only focused on functions (polynomials) which generate varieties - not the relationship which glueing induces on morphisms. Let us remedy this.

Lemma 3 (Morphism Glueing Lemma)**.** *Let X and Y be varieties, let φ and ψ be two morphisms from X to Y and suppose there is a nonempty open subset $U \subseteq X$ such that $\varphi|_U = \psi|_U$. Then $\varphi = \psi$.*

The proof of this lemma is a bit beyond the scope of this paper, so I will refer the reader to [HS1] section 1.4.

3.4 The Category of Varieties with (Dominant) Rational Maps

When we consider morphisms between varieties (in $\mathfrak{Var}$), they must be defined everywhere. This is a problem, as it is often "too rigid" of a definition to be useful. In response to this issue, we want to modify our previous notion of morphisms in $\mathfrak{Var}$ to meet the following intuitional guidelines:

Object	Intuition
Rational map	a continuous map where it makes sense to have one
Birational map	an isomorphism where it makes sense to have one
Birational equivalence	two varieties have the same function field

Table 1: Motivation for $\mathfrak{Dvar}$

Progressing on to formality:

Definition 34. *Let X, Y be varieties. A rational map $\varphi : X \to Y$ is an equivalence class of pairs $\langle U, \varphi_U \rangle$ where U is a nonempty open subset of X and φ_U is a morphism of U to Y, and where $\langle U, \varphi_U \rangle \equiv \langle V, \varphi_V \rangle$ iff $\varphi_U = \varphi_V$ on $U \cap V$.*

In essence, this allows us pair open subsets with morphisms in a glueable way. If the image of φ is dense in Y, i.e., $\overline{\mathrm{im}(\varphi)} = Y$, then φ is called a **dominant rational map**. It is worth noting that **Lemma 3**, our morphism glueing lemma, implies that the relation on elements of rational maps is an equivalence relation. Also, rational maps are not generally a map from X to Y. Rather, they are a collection of glueable morphisms from X to Y. This becomes a natural construction for putting structure sheaves on varieties.

Definition 35. *We will call $\mathfrak{Dvar}$ the category of varieties with dominant rational maps as morphisms. It is clear that dominant rational maps can be composed, and thus satisfy the requirements to be morphisms in $\mathfrak{Dvar}$.*

3.5 Birational Maps

An isomorphism in $\mathfrak{Dvar}$ is called a birational map. As provided in Table 1, we want an isomorphism to mean "an isomorphism minus (maybe) finitely many points which have weird behavior." We can formalize this as:

Definition 36. *A rational map $\varphi : X \to Y$ is a **birational map** if φ admits an inverse.*

This may seem like an uninteresting definition at first, however, certain birational maps are particularly interesting because they allow us to "fix" varieties generated by discontinuous functions.

3.5.1 Blowing Up Varieties

It isn't often that laypeople get to see that mathematicians have a sense of humor. Personally, I think that naming (arguably) the most important example of a birational map the "blowup of a variety" is one of the few cases where the clouds part and light shines through. I mean seriously - I get to tell people that I blow a variety of things up at work! They must think I'm so cool.

In all seriousness though, the blowing up of a variety is the most important example of birationality for this paper. Although we have not yet dealt with projective varieties, the formal definition of a blowup requires them. In an attempt to avoid a complete explanation of projective varieties, I will try to handwave the construction of $\mathrm{Bl}(Y)$ with a picture.

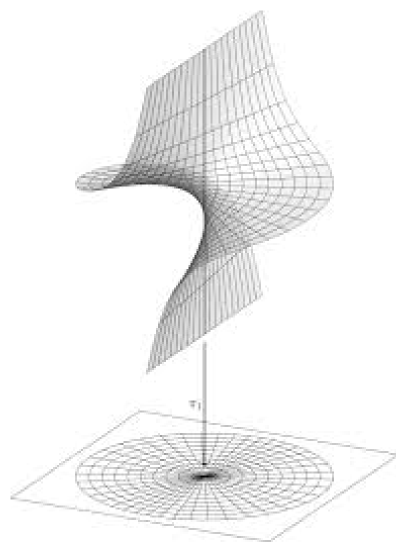

Figure 3: The blowup of a complex manifold

Consider a function $f : \mathbb{C} \to \mathbb{C}$ with a singularity at the origin (maybe a node where a curve traces over itself at 0). If we replace 0 with a bundle of all lines going through 0, we can identify the curves entering zero by the tangent line which they attach themselves to. As a result, around zero, the blowup of $\mathbb{C}$ looks like figure 3. Everywhere else, it is just $\mathbb{C}$.

Formally, we can define the blowup at a point as follows, although it is really beyond the scope of this paper.

Definition 37. *If Y is a closed subvariety of $\mathbb{A}^n$ passing through 0, then we define the blowing up of Y at 0 to be $\widetilde{Y} = (\varphi^{-1}(Y - 0))^-$ where φ is the birational map $\varphi : Y \to \mathbb{A}^n$ and $()^-$ denotes the Zariski closure of the preimage of $Y - 0$. To blow up any other point $P \in \mathbb{A}^n$, make a linear change of coordinates sending $P \to 0$.*

A more common definition of the blowup of a variety is

$$Bl_0(\mathbb{A}^n) = \{(x, [\ell]) \ ; \ x \in \ell \subseteq \mathbb{A}^n\} \subseteq \mathbb{A}^n \times \mathbb{P}^{n-1}.$$

3.5.2 The Larger Scope of Birational Equivalence

We say that two varieties X, Y are **birational** if there exists a birational map between them. The key idea of why we care is that we are really admitting the following:

1. $K(X) \cong K(Y)$ as k-algebras, i.e., the rational functions (which are implicitly defined on dense open subsets) on both varieties are the same, but not necessarily the regular functions.

2. Geometrically, two **curves** X and Y are isomorphic in $\mathfrak{Var}$, minus finitely many points.

   - In higher dimensional varieties, instead of removing points, we remove general divisors (eg, closed subsets of codimension 1), not just points.

Note the parallels between our requirements for continuous functions on the affine line and this last point regarding birational equivalence!

3.6 Generalizing Continuity and Discontinuity With the Morphism Glueing Lemma

Armed with the example of the blowup, we have come to see that what really matters (with regards to continuity) is whether or not regular functions glue nicely. Really, continuity in $\mathfrak{Var}$ is a rigid concept, and birationality is our way of loosening the rigidity of $\mathfrak{Var}$ by expanding it to $\mathfrak{Dvar}$. Moreover, the glueing lemma guarantees uniqueness of birationality by restricting a morphism to an open dense subset of a variety. Accordingly, we can generalize continuity by restricting morphisms to large open subsets that behave nicely, and glueing together morphisms on subsets (when possible) to obtain global behavior - even if a variety isn't globally well behaved.

4 Sheaves, Complexes, and Cohomology

While this is a beautiful construction, one might very well ask the question: what happens when glueability fails? Better yet, how can we measure those failures?

As a bottom line up front, it isn't easy. We have to establish a *massive* amount of theory *very* quickly. Since this paper is intended for those with a background in topology, after we present the structure of sheaves, we are going to motivate homological algebra as an extension of homotopy. However, algebraic geometry takes what is arguably a much less intuitive approach (but is universally accurate) to defining homology and cohomology groups via derived functors. So, we will also cover that approach. When it comes down to actually computing the cohomology of sheaves, we **do not** have the space to present anything more than a basic construction. I can tell you what it does, but we're going to have to omit the 'how.' At any rate, let's get into it!

4.1 Formalizing Pre-Sheaves and Sheaves

So far, we have alluded to the fact that sheaves on varieties are systems of local data that assign sections to pairs of open subsets and restriction maps, with glueing occurring on intersections of open subsets. The essential piece that makes a sheaf special is that local data *must* glue. We will now formalize this notion by defining presheaves which record local data and restriction maps but impose no gluing constraints, followed by the glueing condition that turns a presheaf into a sheaf.

Definition 38. *Let X be a topological space (for the purpose of this paper, X is a variety under the Zariski topology). A **presheaf** $\mathscr{F}$ of abelian groups on X consists of:*

- *for every open set $U \subseteq X$, an abelian group $\mathcal{F}(U)$, called the **sections** of $\mathcal{F}$ over U, and*

- *for every inclusion of open sets $V \subseteq U$, a* restriction map

$$\rho_{UV} : \mathcal{F}(U) \to \mathcal{F}(V), \qquad s \mapsto s|_V,$$

such that:

1. *(Identity): For all open sets U and all $s \in \mathcal{F}(U)$,*

$$s|_U = s.$$

2. *(Transitivity): If $W \subseteq V \subseteq U$, then for all $s \in \mathcal{F}(U)$,*

$$(s|_V)|_W = s|_W.$$

If each $\mathcal{F}(U)$ is a ring (or respectively module, set) and the restriction maps are ring (or respectively module, set) homomorphisms, then $\mathcal{F}$ is called a presheaf of rings (or respectively modules, sets). Additionally, depending on the context, we will interchange $\mathcal{F}(U)$ with

1. $\Gamma(U, \mathscr{F})$, or,
2. $H^0(U, \mathscr{F})$.

There is no need to worry further about the meaning of this notation yet.

Although you may not think so, you are already familiar with presheaves. For example, each of the following is a presheaf:

- functions on open subsets
- regular functions, $\mathcal{O}_Y$
- vector fields
- differential forms
- solutions to PDE's on open sets
- etc...

Each of these examples encodes local data, but doesn't tell you how to glue them together. Queue, sheaves:

Definition 39. *A presheaf $\mathcal{F}$ on a topological space X is called a **sheaf** if it satisfies the following two axioms for every open set $U \subseteq X$ and every open cover $\{U_i\}_{i \in I}$ of U:*

1. *Locality (Uniqueness): If $s, t \in \mathcal{F}(U)$ are two sections such that*

$$s|_{U_i} = t|_{U_i} \qquad \textit{for all } i \in I,$$

   then

$$s = t.$$

2. *Gluing (Existence): If for each $i \in I$ we are given a section $s_i \in \mathcal{F}(U_i)$ such that the sections agree on overlaps:*

$$s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j} \qquad \text{for all } i, j \in I,$$

then there exists a section $s \in \mathcal{F}(U)$ such that

$$s|_{U_i} = s_i \qquad \text{for all } i \in I.$$

This section s is unique by the Locality axiom.

Some examples of sheaves are as follows.

Example 6. *For a variety X, the **structure sheaf** $\mathcal{O}_X$ is defined by*

$$\mathcal{O}_X(U) = \{f : U \to k \mid f \text{ is regular on } U\}$$

for every open set $U \subseteq X$, with restriction maps given by ordinary restriction of functions. Since regular functions that agree on overlaps glue uniquely, $\mathcal{O}_X$ is a sheaf.

Example 7. *Fix a point $p \in X$ and an abelian group A. The **skyscraper sheaf** at p with value A is the sheaf $\mathcal{F}$ defined by*

$$\mathcal{F}(U) = \begin{cases} A & p \in U, \\ 0 & p \notin U, \end{cases}$$

with restriction maps as the identity on A (when both open sets contain p) and the zero map otherwise. This is a sheaf because sections over any cover glue uniquely.

There's a really great YouTube video explaining some examples of (pre)sheaves for those who want more depth on this topic.

https://www.youtube.com/watch?v=j7YmI5Prmnk

Now that we are clear on what sheaves are, we can start thinking about homotopy as a motivation for homology, and eventually cohomology of sheaves.

4.2 Homotopy as a Motivator for Homology

Recall that the fundamental group π_n of a pointed space X tells us how many n-dimensional holes there are in X. This is accomplished by contracting loops into equivalence classes. However, as $|\pi_1(X)|$ grows, it becomes a non-abelian group. As this is an undesirable trait, the topological intuition for connecting homotopy to homology is that, in degree 1, homology is (loosely) the abelianization of the fundamental group. More broadly, homology can be viewed as a linearized shadow of homotopy that forgets higher-order nonlinearity.

This is not, in any way, the full story here. It is just an initial motivation for why those with a topology background may be interested in homology groups. While we lose some data from the fundamental group (functorily speaking), homology is often easier to compute, and still holds a lot of data about the underlying topological space.

While this transition is one approach to viewing homology, there is a drier (but more accurate) approach to constructing homology groups.

4.3 Short Exact Sequences, (Co)Homology 101, and Derived Functors

So far, we've seen that the intuition for homology groups come from homotopy theory. However, homology (and cohomology) is an intrinsically algebraic phenomenon which arises from the failure of certain functors to preserve exactness. To set up an algebraic approach to homology, we need (short) exact sequences:

Definition 40. *A complex at B of modules (or sheaves, groups, rings, etc) is a sequence*

$$\dots \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow \dots$$

where $g \circ f = 0$, and is ***exact*** *at B if $\ker g = imf$. Moreover, a short exact sequence is a complex which is exact at A, B, C of the form,*

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

From this presentation of short exact sequences, we can finally define the homology group at B.

Definition 41. *If we have a complex at B, the homology at B,*

$$H(B) = \ker g / imf.$$

We typically call the elements of $\ker g$ cycles and the elements of imf boundaries. So the homology at B is the cycles mod the boundaries. Notably, a sequence is exact at B iff $H(B) = 0$.

Categorically speaking, the cohomology at B is obtained by applying a contravariant functor (essentially an arrow reversion which turns chains into cochains) to the sequence, then computing the homology at B in the new cochain complex.

The key idea here is that homology shows us what gets left behind when we move on in the sequence from B, and cohomology detects failures of exactness after we apply a contravariant functor to the sequence. This failure of exactness on the cohomology of sheaves is exactly the failure of glueability of presheaves.

Expanding on this idea of functor failure of exactness: if we want to relate short exact sequences to each other, we do so functorily. A functor is called **left exact** if it preserves the exactness of the first / left two elements of every short exact sequence (note that this means it preserves injections and kernels); similarly a functor is **right exact** if it preserves the exactness of the last / right two elements of every short exact sequence (note that this means it preserves surjections and cokernels). Most naturally occurring functors fail to be exact in one direction.

4.3.1 Cohomology as a Result of Right Derived Functors

As we begin to think of cohomology as a failure of exactness, we want a functorial language that tells us everything we need to know. This language is exactly that of derived functors.

To set up derived functors, we need two preliminaries, injective and projective resolutions. I'll be honest, I don't have a great intuition for these yet. The best explanation that I can give is that injective / projective resolutions force left / right exact functors to behave exactly in their image. Eg., every left / right exact functor becomes fully exact when restricted to injective / projective objects (again, respectively).

Oh, also: as a last-minute convention, consider an abelian category to be something like the category of sheaves, varieties, A-modules, etc. It's really important, but not worth explaining here since we are focused on sheaf cohomology.

Definition 42. *Let $\mathcal{A}$ be an abelian category. A **resolution** of an object $A \in \mathcal{A}$ is an exact complex*

$$\cdots \longrightarrow R^{-2} \longrightarrow R^{-1} \longrightarrow R^0 \longrightarrow A \longrightarrow 0$$

together with a morphism $R^0 \to A$ such that the sequence is exact at every R^i and at A.

*If each R^i belongs to the class of injective (or projective, respectively) objects, the resolution is called an **injective resolution** (or **projective resolution** respectively).*

With the notion of an injective resolution established, we can now define left and right derived functors as ways to measure failure of exactness of a functor, otherwise known as cohomology.

Definition 43. *Let $\mathcal{A}$ and $\mathcal{B}$ be abelian categories, and let $F : \mathcal{A} \to \mathcal{B}$ be a left exact functor. Assume that every object of $\mathcal{A}$ admits an injective resolution.*

For an object $A \in \mathcal{A}$, choose an injective resolution

$$0 \longrightarrow A \longrightarrow I^0 \longrightarrow I^1 \longrightarrow I^2 \longrightarrow \cdots .$$

Applying F to this resolution gives a complex

$$0 \longrightarrow F(I^0) \longrightarrow F(I^1) \longrightarrow F(I^2) \longrightarrow \cdots .$$

*The i**-th right derived functor** of F is defined to be the i-th cohomology group of this complex, where $I^\bullet = I^0 \longrightarrow I^1 \longrightarrow I^2 \longrightarrow \cdots$, i.e.,*

$$R^i F(A) = H^i(F(I^\bullet)) .$$

*Different choices of injective resolution give canonically isomorphic results, so R^iF is well-defined up to unique isomorphism. The sequence $\{R^iF\}_{i \geq 0}$ is called the **right derived functor** of F.*

The definition of the left derived functor of a right exact functor is almost identical to the one above.

Definition 44. *Let $\mathcal{A}$ and $\mathcal{B}$ be abelian categories, and let $G : \mathcal{A} \to \mathcal{B}$ be a right exact functor. Assume that every object of $\mathcal{A}$ admits a projective resolution.*

For an object $A \in \mathcal{A}$, choose a projective resolution

$$\cdots \longrightarrow P_2 \longrightarrow P_1 \longrightarrow P_0 \longrightarrow A \longrightarrow 0.$$

Applying G to this resolution gives a chain complex

$$\cdots \longrightarrow G(P_2) \longrightarrow G(P_1) \longrightarrow G(P_0) \longrightarrow G(A) \longrightarrow 0.$$

*The i**-th left derived functor** of G is defined to be the i-th homology group of this complex:*

$$L_iG(A) = H_i(G(P_\bullet)).$$

*The sequence $\{L_iG\}_{i\geq 0}$ is called the **left derived functor** of G.*

The key takeway here is that right derived functors tell us the cohomology data of the complex and that left derived functors tell us the homology data. We are naturally interested in the right-derived interpretation.

4.4 Much Ado About Cohomology

So far, we've constructed a beautiful cathedral of derived functor abstraction which lets us construct (co)homology on any abelian category. While this is a truly non-trivial fabrication, in practice, it is simultaneously like

1. watering a potted plant with a fire-hose, and,
2. setting one's house on fire to prove that water can douse a flame.

The ethos of this approach is that we can take our fancy modern cohomology machinery in any direction and let it run. For the purpose of this paper, that direction is the cohomology of sheaves, which measures the failure of glueability of sections. Making this explicit:

Object:	Right Derived Functors	Cohomology of Sheaves
What It Studies:	Failure of Right Exactness	Failure of Glueability

Table 2: Keeping track of why we care

4.4.1 The Cohomology of Sheaves

To introduce the cohomology of sheaves, we need to introduce the functor which induces it.

Definition 45 (Global Sections Functor)**.** *Let X be a topological space and let $\mathfrak{sh}(X)$ denote the category of sheaves of abelian groups on X. The* global sections functor

$$\Gamma(X, -) : \mathfrak{sh}(X) \longrightarrow \mathbf{Ab}$$

is defined on objects by

$$\Gamma(X, \mathcal{F}) = \mathcal{F}(X),$$

the abelian group of sections of the sheaf $\mathcal{F}$ over the whole space X.

For a morphism of sheaves $\varphi : \mathcal{F} \to \mathcal{G}$, the induced map

$$\Gamma(X, \varphi) : \Gamma(X, \mathcal{F}) \longrightarrow \Gamma(X, \mathcal{G})$$

is given by applying φ on global sections:

$$\Gamma(X, \varphi)(s) = \varphi_X(s).$$

Thus $\Gamma(X, -)$ extracts the global data of a sheaf, assigning to each sheaf its group of global sections.

Notably, $\Gamma(X, -)$ is left exact on all types of sheaves and thus preserves kernels and injections, but is not right exact for some sheaves (specifically sheaves on non-affine varieties and non-quasi-coherent sheaves on affine varieties), which would preserve cokernels and surjections. This is due to the fact that a surjective morphism of sheaves need not induce a surjection on global sections of projective varieties. (This is admittedly both entirely outside of the scope of the paper, as well as my comfort zone for answering detailed questions.) Since this paper deals strictly with affine geometry, $\Gamma(X, -)$ is exact, but this detail is worth noting, especially for those who have read this far into the paper.

Here's the big definition, the cohomology of sheaves:

Definition 46. *We obtain the **cohomology of a sheaf** $\mathcal{F}$ by taking the right derived functors of the global section functor,*

$$H^i(X, \mathcal{F}) = R^i\Gamma(X, \mathcal{F}).$$

Within this framework, the 0-th cohomology group recovers the global sections themselves, and higher cohomology groups measure how and why gluing fails. This intuition reconciles the notation immediately following definition 38 above.

As a final remark on sheaf cohomology: computing this is a *bear*. The nice part of sheaf cohomology is that

1. it always exists, and,
2. it is the universal way to extend the global sections functor to a cohomological functor.

The downside is that we need a more practical cohomology theory to compute sheaf cohomology in practice: the Cech cohomology. This theory has its own downsides, although we will not discuss it further here.

5 Conclusion

In this paper, we have

- thoroughly defined affine space, the Zariski topology, and the category of varieties
- proved that the only continuous functions on the affine line (under the Zariski topology) are non-constant polynomials
- shown how algebraic invariants like $\mathcal{O}(Y)$, $\mathcal{O}_p$, and $K(Y)$ can provide us with data about a variety Y
- constructed a natural relationship between those invariants
- connected topological continuity with glueability of regular functions on open subsets
- used birationality and restriction maps on open subsets of varieties to generalize continuity in the broader scope of algebraic geometry
- introduced sheaves as an extension of the notion of glueability
- provided a survey of sheaf cohomology as a method for measuring the failure of glueability of sections on varieties.

Ultimately, everything we have developed is a refinement of our initial idea of continuity in the Zariski topology. Regular functions are the algebraic analog of continuous functions on open sets, and the structure sheaf $\mathcal{O}_Y$ completes the idea that continuity is determined locally and glues uniquely. Birational maps and rational functions extend this by allowing continuity on dense open subsets instead of requiring continuity on the entire space. Sheaf cohomology then completes the picture by measuring the precise ways in which local "continuous" data fails to globalize. Thus, from affine space to sheaves to cohomology, this entire narrative is a skyscraper built on the premise of Zariski continuity.

6 Acknowledgments

I would like to thank Dr. Bush for assigning such a neat end of semester project and for letting me go off the AG deep end, even though this was technically written for a topology class. Additionally, I would like to thank Dr. Field for suggesting the initial topic, and for being such a kind and encouraging mentor to me.

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