Math 435 09/03/2025 Notes

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Review

Definition 1. *$\mathscr{U}$ = collection of open sets in a given metric space (X, d).*

Definition 2. *An open ball $B(x_0; r)$ is open.*

Definition 3. *Every element of $\mathscr{U}$ is a union of open balls.*

Today

1. Continuity
2. Generalize topological spaces beyond metric spaces

Definition 4. *A NBHD of a point $x_0 \in X$ is any open set containing x_0.*

Definition 5. *$f : (X, d_x) \to (Y, d_y)$ is continuous at $x_0 \in X$ if $\forall \epsilon > 0$, $\exists \delta > 0$ with $d_x(x_0, x) < \delta$ implies $d_y(f(x_0), f(x)) < \epsilon$.*

Theorem 1. *$f : X \to Y$ is continuous iff for all $V \overset{\circ}{\subset} Y$, $f^{-1}(V) \overset{\circ}{\subset} X$. (Inverse images of open sets are open).*

Proof (Direction 1): Let $V \overset{\circ}{\subset} Y$ be open in Y. We want to show that $f^{-1}(V) \overset{\circ}{\subset} X$. I.e.,

$$\forall x_0 \in f^{-1}(V), \exists r > 0 \text{ with } B(x_0; r) \subseteq f^{-1}(V) \tag{1}$$

So, let

$$x_0 \in f^{-1}(V) \Rightarrow f(x_0) \in V. \tag{2}$$

By assumption, V is open, so $\exists \epsilon > 0$ with $B(f(x_0), \epsilon) \subseteq V$. We know f is continuous at x_0, so for $\epsilon > 0$, there exists $\delta > 0$ st.

$$d_x(x_0, x) < \delta \Rightarrow d_y(f(x_0), f(x)) < \epsilon. \tag{3}$$

Hence, $x \in B(x_0;\delta)$ implies that $f(x) \in B(f(x_0),\epsilon) \subseteq V$. Then $f(B(x_0,\delta)) \subseteq V$, i.e.,

$$B(x_0,\delta) \subseteq f^{-1}(V). \tag{4}$$

Since $x_0 \in f^{-1}(V)$ is arbitrary, we can find such an open ball for all $x \in f^{-1}(V)$. So, $f^{-1}(V) \overset{\circ}{\subset} X$, as desired.

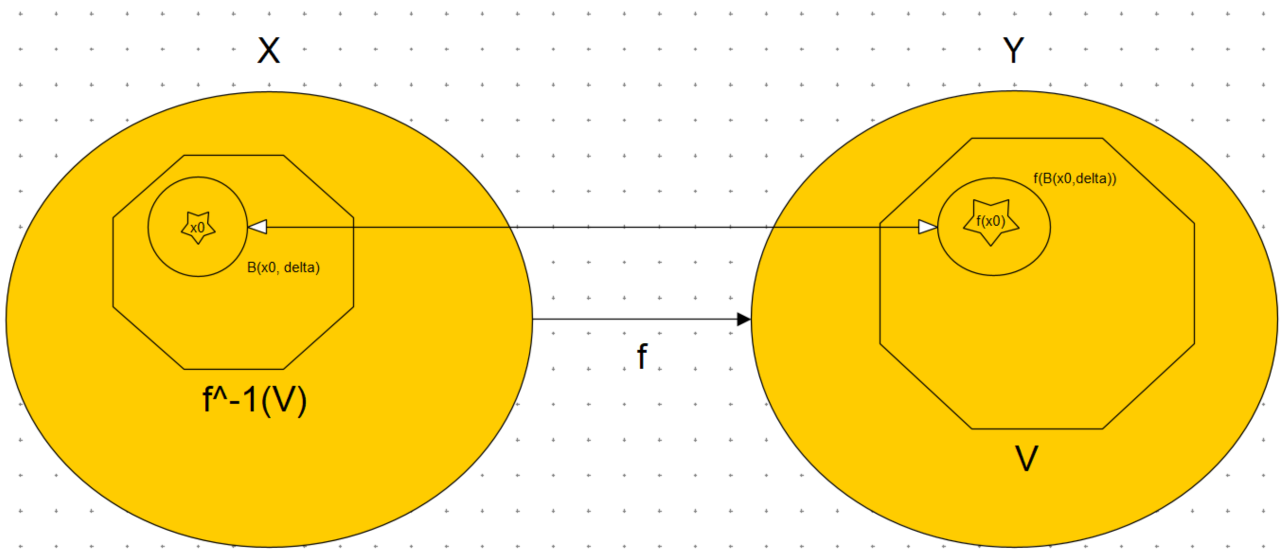

Figure 1: Topological Continuity Visualization (Dir 1)

(Direction 2): Fix $x_0 \in X$ and $\epsilon > 0$. Then

$$B(f(x_0);\epsilon) \overset{\circ}{\subset} Y \tag{5}$$

so,

$$U = f^{-1}(B(f(x_0),\epsilon)) \tag{6}$$

is open. Then there exists $\delta > 0$ with $B(x_0;\delta) \subseteq U$. Thus,

$$x \in B(x_0;\delta) \Rightarrow f(x_0) \in B(f(x_0);\epsilon), \tag{7}$$

which is exactly the epsilon-delta continuity criterion.

$\square$

Definition 6. *A topological space is a pair $(X, \mathscr{U})$, (often (X,τ)), where X is a set, and $\mathscr{U}$ is a collection of subsets of X such that*

1. *$\emptyset, X \in \mathscr{U}$*
2. *$\mathscr{U}$ is closed under arbitrary union*
3. *$\mathscr{U}$ is closed under finite intersection.*

The elements of $\mathscr{U}$ are called open sets. Note: 'open set' here really just means that a set belongs to $\mathscr{U}$.

Proposition 1. *Metrics induce a topology.*