Math 435 09/10/2025 Notes

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Review

Definition 1. *A collection $\mathscr{B}$ of subsets of a set X is called a basis if*

1. *The sets in $\mathscr{B}$ cover X, i.e., $\forall x \in X$ there exists $B \in \mathscr{B}$ with $x \in B$.*
2. *If $B_1, B_2 \in \mathscr{B}$ and if $x \in B_1 \cap B_2$, then there exists $B_3 \in \mathscr{B}$ such that $X \in B_3 \subseteq B_1 \cap B_2$.*

Definition 2. *Let $\mathscr{U}_1, \mathscr{U}_2$ be topologies on X. If $\mathscr{U}_1 \subseteq \mathscr{U}_2$ we say that $\mathscr{U}_2$ is finer than $\mathscr{U}_1$, and $\mathscr{U}_1$ is coarser than $\mathscr{U}_2$.*

Today

- Topology inclusion lemma
- Basis generation lemma.

Proposition 1. *Let $\mathscr{U}_1, \mathscr{U}_2$ be the topologies generated by bases $\mathscr{B}_1$ and $\mathscr{B}_2$ on X. If every element of $\mathscr{B}_1$ is a union of elements of $\mathscr{B}_2$, then $\mathscr{U}_1 \subseteq \mathscr{U}_2$.*

Proof: Let $U \in \mathscr{U}_1$. Then $U = \bigcup_{i=1}^n B_i$ with $B_i \in \mathscr{B}_1$. But each element of $\mathscr{B}_1$ is a union of elements of $\mathscr{B}_2$. So $U = \bigcup_{i=1}^n C_i$ where $C_i \in \mathscr{B}_2$, where $\mathscr{B}_2$ are the elements of $\mathscr{U}_2$. So $U \in \mathscr{U}_2$ implies $\mathscr{U}_1 \subset \mathscr{U}_2$.

$\square$

Lemma 1. *(How to get a basis from a topology) Let $(X, \mathscr{U})$ be a topological space. If $\mathscr{C} \subseteq \mathscr{U}$ such that for all $U \in \mathscr{U}$ and for all $x \in U$, there exists $C \in \mathscr{C}$ such that $x \in C \subseteq U$, then $\mathscr{C}$ is a basis for $\mathscr{U}$.*

Example 1. *Consider $(\mathbb{R}^2, \mathscr{U})$ where $\mathscr{U}$ is the **standard topology** (open sets look like sets without boundaries). Let $\mathscr{C}$ = collection of open balls. Then given $x \in U \subseteq \mathscr{U}$, there exists an open ball C such that $x \in C \subseteq U$.*

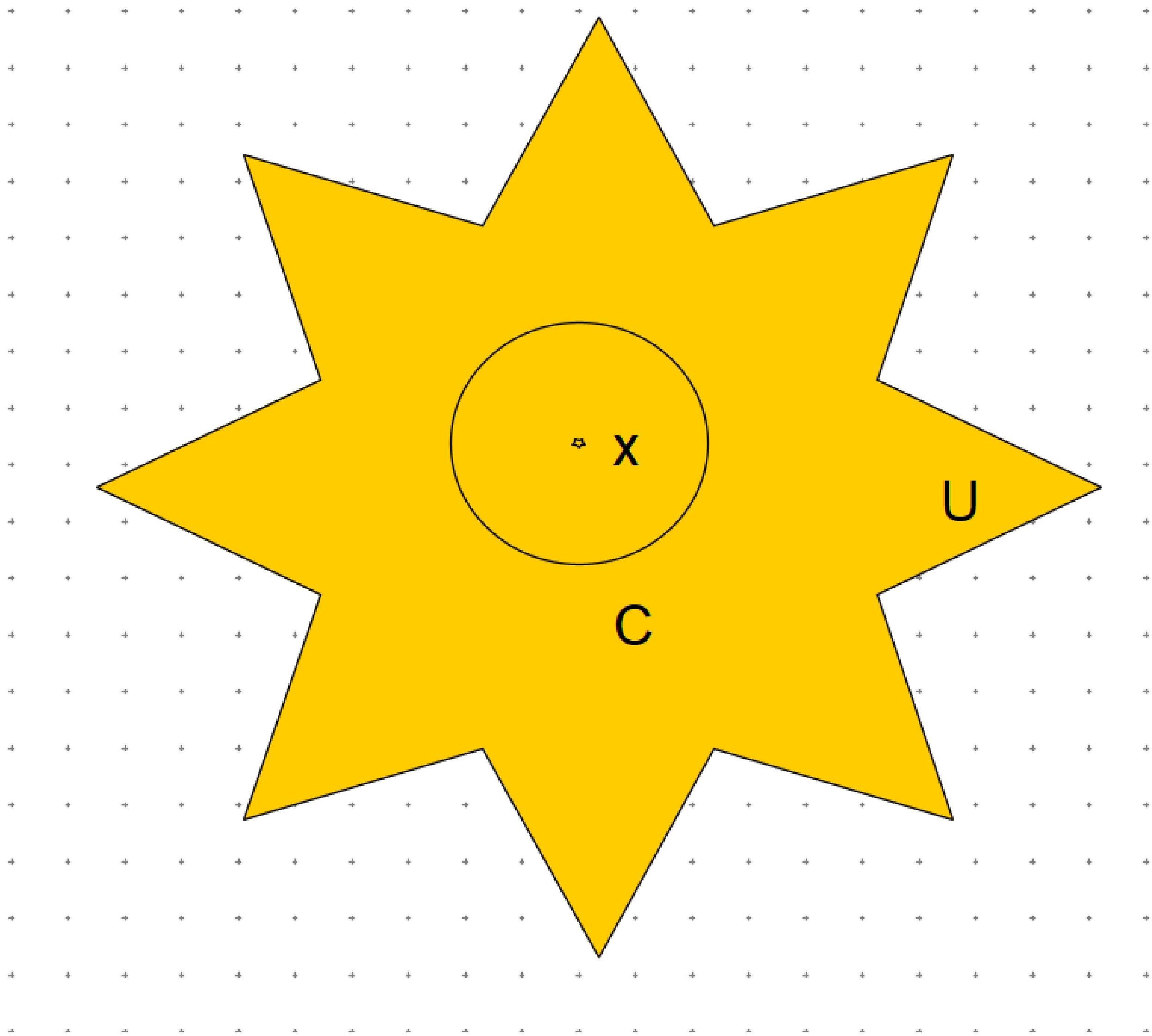

Figure 1: Example 1 Picture

Proof (Lemma 1): For the first part, we want to prove that $\mathscr{C}$ covers X. Then we want to prove the second axiom for showing that $\mathscr{C}$ is a basis. Finally, we want to show that it is a basis for the given topology, i.e., it actually generates $\mathscr{U}$.

For part 1, X itself is open in $\mathscr{U}$. So, given $x \in X$, there exists $c \in \mathscr{C}$ such that $x \in C \subseteq X$. This holds for all $x \in X$, so the sets in $\mathscr{C}$ cover X.

For part 2, suppose $x \in C_1 \cap C_2$ for some $C_1, C_2 \in \mathscr{C}$. Now, C_1, C_2 are open because they are elements of $\mathscr{C}$. Thus, $C_1 \cap C_2$ is open. Hence by proposition 1, there exists $C \in \mathscr{C}$ such that $x \in C \subseteq C_1 \cap C_2$.

Last but not least, to show that $\mathscr{C}$ generates $\mathscr{U}$, let $\mathscr{U}'$ denote the topology generated by $\mathscr{C}$. We want to show first that $\mathscr{U}' \subseteq \mathscr{U}$, then show that $\mathscr{U} \subseteq \mathscr{U}'$. We know that every element of $\mathscr{U}'$ is an arbitrary union of elements of $\mathscr{C}$. But, $\mathscr{C} \subseteq \mathscr{U}$. So every element of $\mathscr{U}'$ is also an element of $\mathscr{U}$. Thus, $\mathscr{U}' \subseteq \mathscr{U}$.

Now we want to show that $\mathscr{U} \subseteq \mathscr{U}'$. Let $U \in \mathscr{U}$. We want to show that U can be written as a union of elements of $\mathscr{C}$. Given $x \in U$, there exists $C_x \in \mathscr{C}$ such that $x \in C_x \subseteq U$. So $U = \bigcup_{x \in U} C_x$. Thus, $\mathscr{U}' = \mathscr{U}$.

$\square$

This lemma is **really useful** for showing that a set is a basis.