Math 435 08/29/2025 Notes

August 29, 2025

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Definition 1. *$Y \subseteq X$ is open if for all $x \in Y$, there exists some $r > 0$ such that $B(x; r) \subseteq Y$.*

Proposition 1. *An open ball is an open set.*

Proof. Fix a metric space X, a point $a \in X$, and some radius $r > 0$. We want to show that $B(a; r)$ is open in X. Let $x \in B(a; r)$. Then $d(x, a) < r$, by definition. Define $\epsilon = r - d(x, a)$. Then we claim $B(x; \epsilon) \subseteq B(a; r)$. Given $y \in B(x; \epsilon)$, $d(y, x) < \epsilon$. Then,

$$d(y, a) \leq d(y, x) + d(x, a) < \epsilon + d(x, a) = (r - d(x, a)) + d(x, a) = r. \tag{1}$$

So,

$$d(y, a) < r. \tag{2}$$

Hence, $y \in B(a; r)$, which proves our claim that $B(x; \epsilon) \subseteq B(a; r)$.

□

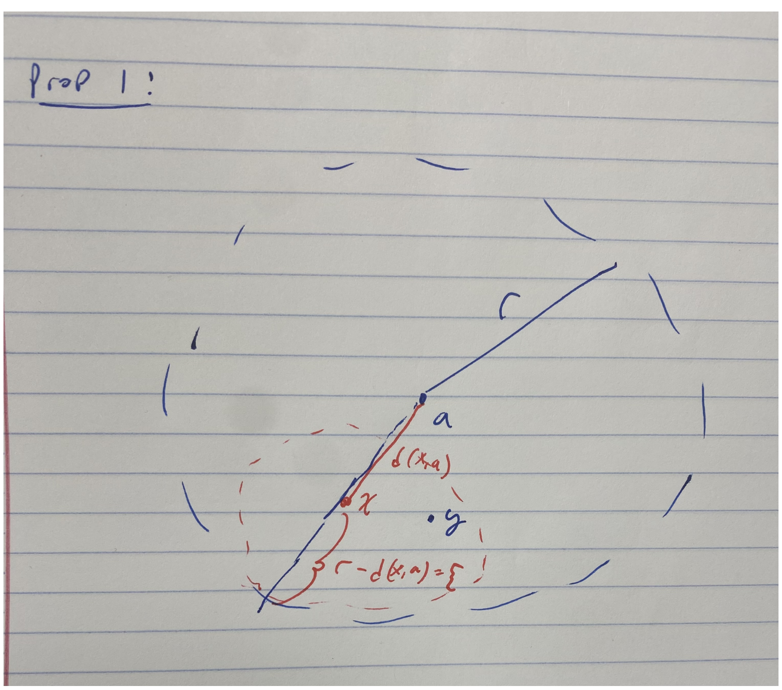

Figure 1: Reference for Proposition 1

Proposition 2. *Let (X,d) be a metric space. Let $\mathscr{U}$ denote the collection of open subsets of X. Then*

1. *$\emptyset, X \in \mathscr{U}$.*
2. *Arbitrary unions of elements of $\mathscr{U}$ belong to $\mathscr{U}$.*
3. *Finite intersections of elements of $\mathscr{U}$ belong to $\mathscr{U}$.*

We call $\mathscr{U}$ a topological space.

Proof. (1) is vacuously open. Given $x \in X$, $B(x;r) \subseteq X$ by definition. (2) If $\{U_i\}_{i\in I} \subseteq \mathscr{U}$, let $x \in \bigcup_{i\in I} U_i$. By definition, $x \in U_j$ for some $j \in I$. Since U_j is open, there exists $r > 0$ such that $B(x;r) \subseteq U_j$. Well, if $x \in B(x;r) \subseteq U_j$, then $B(x;r) \subseteq \bigcup_{i\in I} U_i$. Since x is arbitrary, $\bigcup_{i\in I} U_i$ is open. (3) proof is the last problem on Homework 2.

□

Example 1. *Why only finite intersections? In $\mathbb{R}$, consider $B(0,1) \supset B(0,\frac{1}{2}) \supset B(0,\frac{1}{3}) \supset \dots$.*

$$\bigcap_{n=1}^{\infty} B(0;\frac{1}{n}) = \{0\}. \tag{3}$$

Notably $\{0\}$ is a closed set in $\mathbb{R}$. Thus, in sets with LUB property, we need explicitly finitely many intersections.

Proposition 3. *Every open set is a union of open balls.*

Proof. Let U be open. Given $x \in U$, there exists $r_x > 0$ with $B(x;r_x) \subseteq U$. Hence, $U = \bigcup_{x\in U} B(x;r_x)$, omitting the trivial set containment proof. For visual reference, consider the open cover of an open set.

□

Definition 2. *Let (X,d) be a metric space. Given $x \in X$, a neighborhood (nbhd) of x is any open set containing x.*

Definition 3. *A function $f : (X,d_x) \to (Y,d_y)$ is continuous at $x_0 \in X$ if $\forall\epsilon > 0$ there exists $\delta > 0$ such that $d_x(x_0,x) < \delta$ implies that $0 < d_y(f(x_0),f(x)) < \epsilon$.*

Proposition 4. *A function $f : (X,d_x) \to (Y,d_y)$ is continuous if for any $U \overset{\circ}{\subset} Y$, its fiber, $f^{-1}(U)$, is open in X. Rather, $f^{-1}(U) \overset{\circ}{\subset} X$.*