Proposal: Continuity in the Zariski Topology as a Bridge From Classical Topology to Algebraic Geometry

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1 Project Topic and Motivation

Briefly describe your proposed project topic. Why is this topic interesting or important in the context of topology? What aspect of topology will your project focus on?

I want to present what is essentially an exercise in the first chapter of Hartshorne's Algebraic Geometry. Specifically, I want to talk about functions which are or aren't continuous in the Zariski Topology. On the face of it, this may not seem wholly connected to Topology. That said, it is an interesting question because out of **"is $f(x)$ continuous?"**, we can develop the notion of an affine variety, the category of varieties, morphisms between varieties, regular functions, ideals of varieties, and rational (and birational) maps. Beyond the category of varieties, we are then left room to ask questions about the function field of varieties, local rings, and other interesting topics as time permits. Essentially, we can develop a followup question: **"how do varieties X, Y relate to each other, and what can $\mathcal{O}(X)$ tell us about $\mathcal{O}(Y)$ tell us about Y?" Furthermore, what is the relation between $\mathcal{O}(X), A(X),$ and $K(X)$?** This will lead us to our final question: **"when a function isn't continuous in the Zariski topology, what can we do to still classify its variety?"** (This is of course accomplished by the blowup of a point, or variety).

While this is clearly the setup of Algebraic Geometry, it is pertinent to our class because it shows the breadth of where Topology shows up! I feel like a lot of people don't know how versatile the tools we've built in this class are. Moreover, I'd really love to give people a pitch for what to study next, especially since I need people who know topology for the AG seminar that I'm starting. I will add: I am not trying to cover all (or really any) of modern AG in this presentation, so any references to sheaves, schemes, or particularly advanced category theory will be restricted to the open subset of things relevant to successfully motivating my research questions and getting an A on this project.

2 Goals and Guiding Questions

List the specific questions or goals you hope to address in your project. For example, what computations, proofs, visualizations, or explanations do you aim to produce?

Condensing the prose from section 1, the goal of my paper & presentation will be:

1. Is $f(x)$ continuous in the Zariski topology? (Explicit examples will be chosen)
2. How do varieties X, Y relate to each other, and what can $\mathcal{O}(X)$ tell us about $\mathcal{O}(Y)$ tell us about Y? Same for $K(X)$ and $K(Y)$. Furthermore, for a fixed variety X, what is the relation between $\mathcal{O}(X), A(X),$ and $K(X)$?
3. How do birational maps induce isomorphisms on open subsets, and how do local / global glueing theorems help us tie things back to topology?
4. When a function isn't continuous in the Zariski topology, what can we do to still classify its variety? How can we "fix" singularities?
5. Of the examples chosen for (1), which varieties are birational to each other? Are there multiple ways to show this?
6. Can we use restriction maps on open subsets to generalize continuity?

3 References and Resources

List potential textbooks, papers, websites, or other resources that you will consult. Include both required background material and any additional sources that may provide inspiration or guidance.

So, um, I have a few:

- Hartshorne's 'Algebraic Geometry' [HS1]
- Ravi Vakil's 'The Rising Sea: Foundations of Algebraic Geometry' [RV]
- Cox, Little, and O'Shea's 'Ideals, Varieties, and Algorithms' [CLO] .
- Herstein's 'Abstract Algebra' [HS2]
- Howie's 'Fields and Galois Theory' [HW]
- Munkres' 'Toplogy' [MK]
- Artin's 'Algebra' [AL]

Of these resources, we'll mostly use:

- [HS1] sections 1.1-1.6 for everything described in section (2) of this document
- [RV] chapter 1 (for category theory), chapter 2, 3 to say "...and hey sheaves & schemes are useful machinery for modern AG"... if I have time

- [CLO] chapters 1 and 2 to motivate some classical AG results
- [HS2] as a general reference for transitioning from group theory to ring theory
- [HW] as a general reference for ring and field theory
- [MK] because I would feel weird doing a topology project without citing Munkres
- [AL] chapter 10 because he has some nice prose on AG coming out of his explanation of ring theory.

4 Timeline and Next Steps

Outline a plan for how you will approach your project over the remainder of the semester, and how you plan to allocate time to research, writing, and preparation for the final presentation. If working in a group, briefly indicate how responsibilities will be shared.

Okay, I can't lie, if I hadn't already prepped a bunch of the preliminary stuff for the AG seminar I'm trying to run, this project would be basically impossible. As far as the presentation goes, I'd like to ask for a full 30 minute time-slot, just so I don't totally lose people by trying to go too fast. Will I still be going too fast? Almost certainly. But the goal is to pique their interest so that they can read more in my paper! Therefore, I think it will be okay.

I'm pretty self motivated to put this together; I really want to do my PhD in AG, so the more I can do in this next year to gain thorough understanding of the motivation, function, and form of AG through lenses I'm familiar with from my coursework, the better! Here's a rough map of what I want to accomplish and when.

- (11/14/2025) Get started on a first draft. Pick out examples of nice generators of varieties. Classic stuff like the twisted cubic, elliptic curves, Laurent series, etc.
- (11/21/2025) Computations of varieties completed, examples formalized, definitions for ring theory completed, and Hartshorne ch1 typed up the way I want to use it.
- (11/28/2025) Completed first draft. Research questions answered completely and rigorously. I should be able to teach out of my TeX document if needed.
- (12/01/2025) Revise my draft, start making slides in Beamer.
- (12/02/2025) Finish my slidedeck (draft), figure out what will be done purely on the board versus on slides. I like writing stuff down when I lecture, but I want to have enough images and viewable examples that there is motivating intuition.
- (12/03/2025) Start doing dry runs, revise slidedeck appropriately. Finalize my research paper.
- (12/04/2025) Submit a day early, celebrate by studying for my PDE's final.
- (12/05/2025) Wait for feedback from reviewer number 2!