Math 435 09/24/2025 Notes

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Review

Definition 1. *Let* $(X, \mathscr{U}_x)$ *be a space and let* A *be a set. Given a surjective map* $p : X \to A$*, the quotient topology* $\mathscr{U}_p$ *on* A *is:*

$$A \supseteq U \in \mathscr{U}_p \iff p^{-1}(U) \in \mathscr{U}_x. \tag{1}$$

Today

- Examples of quotient spaces!

Example 1. *Let us have* $\mathbb{R}$ *under the standard topology. Let* $\mathbb{Z}$ *be our set. Let* $p(x) = x$ *if* $x \in \mathbb{Z}$ *and* $p(x) = n$ *if* n *is an odd integer and* $x \in (n-1, n+1)$*. (I.e.,* p *sends* $x \in \mathbb{R}$ *to the nearest odd integer if* $x \notin \mathbb{Z}$ *and* p *is the identity map if* $x \in \mathbb{Z}$*.) If* $n \in \mathbb{Z}$ *is odd, then* $p^{-1}(n) = (n-1, n+1)$ *which is open. If* $n \in \mathbb{Z}$ *is even, then* $p^{-1}(n) = \{n\}$ *which is closed in* $\mathbb{R}$*. Unions between odd integers are open, for example,* $p^{-1}(\{-1, 0, 1\}) = (-2, 2)$ *is open.*

Definition 2. *A collection* $\{U_\alpha\}_\alpha$ *of subsets* $U_\alpha \subset A$ *covers* A *if* $\bigcup_\alpha U_\alpha = A$*.*

Definition 3. *Let* A *be a set and let* B *be a collection of disjoint nonempty subsets of* A *that cover* A*.* B *is a partition of* A*.*

Observation: Given a space X and a partition X^* of X, there exists a natural surjection given by mapping each element to the unique set containing it.

Example 2. *Let* $X = \{a, b, c, d, e\}$*, with* $\mathscr{U} = \{\emptyset, \{a\}, \{a, b\}, \ldots, X\}$*. Let* $X^* = \{\{a, b\}, \{c, d, e\}\}$ *be a partition (not a topology) of* X*. What is the quotient topology* $\mathscr{U}_p$ *on* X^**? Let* $p : X \to X^*$ *with*

- $p(a), p(b) = \{a, b\} = A$
- $p(c), p(d), p(e) = \{c, d, e\} = B$

Then notice that $p^{-1}(A) = \{a, b\}$ *and* $p^{-1}(B) = \{c, d, e\}$ *and* $P^{-1}(\{A, B\}) = X$ *and* $p^{-1}(\emptyset) = \emptyset$*. Since* $\{c, d, e\}$ *is not open in* $\mathscr{U}_x$*, but* X *is, we find that the topology induced by* p *on* X^**,* $\mathscr{U}_p = \{\{A\}, \{A, B\}\}$*.*