Math 435 10/01/2025 Notes

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Review

Definition 1. *Given metric spaces (X, d_X) and (Y, d_y), a function $f : X \to Y$ is continuous at $x_0 \in X$ if for all $\epsilon > 0$ there exists $\delta > 0$ such that $d_x(x_0, x) < \delta$ then $d_y(f(x_0), f(x)) < \epsilon$.*

Equivalently,

Definition 2. *Let X and Y bet topological spaces. A map $f : X \to Y$ is topologically continuous if for $U \overset{\circ}{\subset} Y$, $f^{-1}(U) \overset{\circ}{\subset} X$. (Inverse images of open sets are open).*

Today

- Continuous functions and homeomorphisms.

Example 1. *Let $X = \{a, b, c, d\}$, $Y = \{1, 2, 3\}$ with $\mathscr{U}_x = \{\{a, b, c, d\}, \{\}, \{a, b\}, \{c\}, \{c, d\}, \{a, b, c\}\}$ and $\mathscr{U}_y = \{\{1, 2, 3\}, \{\}, \{1\}, \{2\}, \{1, 2\}\}$. Let $f : X \to Y$ be defined by*

- $f(a) = 2$
- $f(b) = 2$
- $f(c) = 1$
- $f(d) = 3$.

Additionally, let $g : X \to Y$ be defined by

- $g(a) = 1$
- $g(b) = 2$
- $g(c) = 2$
- $g(d) = 3$.

Claim: f is t-cont, g is not.

Proof (f is t-cont): $f^{-1}(\{1\}) = \{c\} \overset{\circ}{\subset} X$. $f^{-1}(\{2\}) = \{a,b\} \overset{\circ}{\subset} X$. $f^{-1}(\{1,2\}) = \{a,b,c\} \overset{\circ}{\subset} X$. $f^{-1}(Y) = X \overset{\circ}{\subset} X$. *So f is t-cont.*

Proof (g is not t-cont): $g^{-1}(\{2\}) = \{b,c\} \overset{\circ}{\not\subset} X$.

$\square$

Proposition 1. *Continuity depends on the function and the topologies!*

Proposition 2. *Any $f : X \to Y$ is continuous if the topology on X is the discrete topology (OR the indiscrete topology and f is surjective).*

Proposition 3. *A continuous map doesn't necessarily preserve the topological structure. A map which does is called a homeomorphism, which is a bijective continuous map $f : X \to Y$ such that if $U \overset{\circ}{\subset} X$, then $f(U) \overset{\circ}{\subset} Y$.*

Example 2. *$id : X \to X$ has $id^{-1}(U) = U$ for any open U in X, so it is continuous.*

Example 3. *$c : X \to Y$ is a constant map $c(x) = y_0$ is continuous.*

Theorem 1. *Let $Y \subseteq X$ with the subspace topology. Consider the inclusion $\iota : Y \to X$. It is continuous. The subspace topology is the coarsest (smallest) topology on Y such that ι is continuous.*

Theorem 2. *Given a space X, a set A, and a surjection $p : X \to A$, p is continuous with respect to the quotient topology. Moreover, it is the finest topology on A such that p is continuous.*