Continuity in the Zariski Topology

A Bridge to Algebraic Geometry

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Motivating Questions

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1. For a fixed $f(x)$, how can we prove or disprove that $f(x)$ is continuous in the Zariski topology?
2. How do varieties X, Y relate to each other, and what can $\mathcal{O}(X)$ tell us about $\mathcal{O}(Y)$ tell us about Y? How about $K(X)$ and $K(Y)$?
3. Furthermore, for a fixed variety X, what is the relationship between $\mathcal{O}(X), A(X),$ and $K(X)$?
4. How do birational maps induce isomorphisms on open subsets, and how do local / global glueing theorems help us tie things back to topology?

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*Red entries are answered in the accompanied paper.

Getting a Grip on Affine n-Space

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The key difference: $\mathbb{A}^n$ doesn't preserve 0. In fact, affine space only preserves things invariant under linear transformation.

What Makes “Sense" in $\mathbb{A}^n$?

What Makes Sense	What Doesn’t Make Sense
Groups of rigid motions around a point	The origin - or any fixed point in $\mathbb{A}^n$
Ellipses	Circles
Parallelograms	Squares
Invariants of geometric symmetries	Angles between vectors and magnitudes of vectors

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$$Z(T) = \{p \in \mathbb{A}^n \; ; \; f_i(p) = 0 \text{ for all } f_i \in T\}$$

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Definition

$Y \subseteq \mathbb{A}^n$ is called an **algebraic set** if there exists a set of functions $T \subseteq k[x_1, \ldots, x_n]$ with $Y = Z(T)$.

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Additionally, it is somewhat intuitive that arbitrary unions of complements of algebraic sets are still complements of algebraic sets, and that finite intersections of complements of algebraic sets are also complements of algebraic sets.

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Algebraic geometry is (largely) the study and classification of varieties of many kinds (affine, quasi-affine, projective, and quasi-projective). We will develop certain tools (structural invariants) in this lecture and accompanied paper which can be used to classify varieties up to what is called birationality. The popular modern AG tools are called numerical invariants - usually cohomology - which (sort of) measure the genus of a simplicial complex. With topology as a frame of reference, this is a step up from the notion of homotopy classes of a space.

Examining Continuous Functions in $\mathbb{A}^1$

Remarks About Closed Sets in $\mathbb{A}^1$

Consider a function $f : \mathbb{A}^1 \to \mathbb{A}^1$. We know that $Z(f)$ *must* be a finite set by the Fundamental Theorem of Algebra.

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Conclusion: Closed sets are finite (or all of $\mathbb{A}^1$ or $\emptyset$).

As a corollary, we will also note that varieties in $\mathbb{A}^1$ are singleton sets.

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But $\mathbb{A}^m \setminus V = V^C$ is closed in $\mathbb{A}^m$. And $f^{-1}(V^C) = f^{-1}(\mathbb{A}^m \setminus V) = \mathbb{A}^n \setminus f^{-1}(V)$ is closed in $\mathbb{A}^n$.

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But $\mathbb{A}^m \setminus V = V^C$ is closed in $\mathbb{A}^m$. And $f^{-1}(V^C) = f^{-1}(\mathbb{A}^m \setminus V) = \mathbb{A}^n \setminus f^{-1}(V)$ is closed in $\mathbb{A}^n$. So inverse images of closed sets being closed imply topological continuity.

$\square$

Putting Things Together in $\mathbb{A}^1$

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Theorem

$f : \mathbb{A}^1 \to \mathbb{A}^1$ is continuous iff inverse images of finite sets are finite.

A Strict Classification of $\mathbb{A}^1$

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- In turn, this generalizes to a notion of glueability of continuous functions on intersection patches of open subsets.
- We can measure the failure of glueability of continuous data with sheaf cohomology.

Interested? Read the paper!

Github QR Code

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Additional resources used:

https://lekili.freemath.xyz/teaching/topics/Blowups.pdf

https://www.youtube.com/@HarpreetBedimath