Math 435 Homework 6

October 19, 2025

Timothy Tarter
James Madison University
Department of Mathematics

Problem 1.1: Let $f : \mathbb{R} \to \mathbb{R}$ be $f(x) = \dfrac{x-1}{2}$, where the domain $(\mathbb{R}, \mathcal{U})$ has the standard topology and the codomain $(\mathbb{R}, \mathcal{U}_f)$ has the finite complement topology $\mathcal{U}_f = \{\emptyset\} \cup \{\mathbb{R} \setminus F : F \subset \mathbb{R} \text{ finite}\}$. Show that f is continuous.

We want to show that for every $V \in \mathcal{U}_f$, $f^{-1}(B) \in \mathcal{U}$. Notably, if $V = \emptyset$, then $f^{-1}(V) = \emptyset \in \mathcal{U}$. If V is nonempty, then $V = \mathbb{R} \setminus F$ for some finite $F \subseteq \mathbb{R}$. It is worth noting that any such F is a union of singleton sets, which are closed in the standard topology on $\mathbb{R}$. Accordingly,

$$f^{-1}(V) = f^{-1}(\mathbb{R} \setminus F) = \mathbb{R} \setminus f^{-1}(F). \tag{1}$$

We want to then show that $f^{-1}(F)$ is finite (a union of singleton sets), and thus closed in the standard topology on $\mathbb{R}$ (and that $\mathbb{R} \setminus f^{-1}(F)$ is open). To find f^{-1}, it suffices to solve

$$f^{-1}(x) = a(\frac{x-1}{2}) + b = x. \tag{2}$$

$$\frac{a}{2}x = x \text{ and } \frac{-a}{2} + b = 0 \tag{3}$$

$$\text{accordingly, } a = 2 \text{ and } b = 1. \tag{4}$$

So,

$$f^{-1}(x) = 2x + 1. \tag{5}$$

If F is a finite set, then $f^{-1}(F)$ is a finite set, since $\mathbb{R}$ is translationally invariant for affine functions. Thus, $\mathbb{R} \setminus f^{-1}(F)$ is open and f is continuous.

□

Problem 1.2: Let $f : \mathbb{R} \to \mathbb{R}$ be $f(x) = \dfrac{x-1}{2}$. Give the domain $(\mathbb{R}, \mathcal{U})$ the standard topology and the codomain $(\mathbb{R}, \mathcal{V})$ the lower-limit topology, whose basis is $\mathcal{B} = \{[a, b) : a < b,\ a, b \in \mathbb{R}\}$. Show that f is not continuous.

It suffices to find an open set $B \in \mathcal{B}$ such that $f^{-1}(B) \notin \mathcal{U}$. Take $B = [0, 1)$. Then

$$f^{-1}([0,1)) = \{x \in \mathbb{R} : 0 \leq \tfrac{x-1}{2} < 1\} = [1,3). \tag{6}$$

But $[1,3)$ is not open in the standard topology $\mathcal{U}$ since it contains its boundary. Hence $f^{-1}[0,1) \notin \mathcal{U}$, so f is not continuous.

$\square$

Problem 2: Given a space X with a topology $\mathcal{U}$, let $Y \subseteq X$ be a subset of X. We want to show that the subspace topology, $\mathcal{V} = \{U \cap Y | U \in \mathcal{U}\}$, on Y is the smallest (coarsest) topology on Y for which the inclusion $i : Y \to X$ is continuous.

Note that $i : Y \to X$ is defined by $i(y) = y \in X$ for every $y \in Y$.

Let i be as defined above, from the subspace topology $\mathcal{V}$ to $\mathcal{U}$. We want to show first that i is continuous. $i^{-1} : X \to Y$ is the map, for $U \in \mathcal{U}$, $i^{-1}(U) = U \cap Y$. Thus, $i^{-1}(U)$ is open in Y by definition.

Now, if i is continuous then every open set U in $\mathcal{U}$ has a preimage which is open in $\mathcal{V}$ by definition:

$$i^{-1}(U) = U \cap Y \in \mathcal{V}. \tag{7}$$

Thus, $U \cap Y$ is open in any topology on Y for every $U \in \mathcal{U}$. Accordingly, it must be minimal.

$\square$

Problem 3: Given a space X with a topology $\mathcal{U}$, a set A, and a surjective function $p : X \to A$, prove that the quotient topology τ on A is the finest topology on A for which p is continuous.

Let $(X, \mathcal{U})$ be a topological space with $p : X \to A$ a surjection and define the quotient topology τ by

$$\tau = \{U \subseteq A | p^{-1}(U) \in \mathcal{U}\}. \tag{8}$$

If $U \in \tau$, then by definition, $p^{-1}(U) \in \mathcal{U}$. Therefore, p is continuous. Let S be any topology on A such that p is continuous. Then for every $U \in S$, continuity gives $p^{-1}(U) \in \mathcal{U}$. Therefore, $U \in \tau$ implies that $S \subseteq \tau$. So τ is the finest topology on A which makes p continuous.

$\square$

Problem 4: Let X and Y be spaces. Define the set $(X \times Y)^* = \{X \times \{y\} | y \in Y\}$. Let $p : X \times Y \to (X \times Y)^*$ be the map defined by $(x, y) \to X \times \{y\}$. Prove that $(X \times Y)^*$ equipped with the quotient topology is homeomorphic to Y.

We want to show

1. There exists a continuous bijection $h : (X \times Y)^* \to Y$ by $h(X \times \{y\}) = \{y\}$
2. That h^{-1} is also a continuous bijection.

To show that h is continuous, let U be open in Y. Then $h^{-1}(U) = \{X \times \{y\} | y \in U\}$, which is open in $(X \times Y)^*$ by construction. So h is continuous. Similarly, to show that h^{-1} is continuous, let $U = X \times B$ be open in $(X \times Y)^*$, where B is any open set in Y. Then $h(U) = h(X \times B) = B$, which is open in Y by construction. So both h and h^{-1} are continuous maps. To show that h is injective and surjective, we want to show that h and h^{-1} are mutual inverses. This point is simply by construction;

$$h(X \times \{y\}) = y \tag{9}$$

$$\text{and} \tag{10}$$

$$h^{-1}(y) = X \times \{y\} \tag{11}$$

thus, h is a homeomorphism between $(X \times Y)^*$ and Y.

$\square$