Math 435 11/12/2025 Notes

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Review

Today

- The fundamental group!

Definition 1. *Let f be a path in X that goes from x_0 to x_1 and let g be a path in X that goes from x_1 to x_2. The sum $f + g$ is the concatenation, which is a path from x_0 to x_2.*

Theorem 1. *The extension of $+$ to the set of path homotopy classes defined by $[f]+[g] := [f+g]$ is well-defined. ($[f]$ denotes the homotopy class of f.)*

To prove this we would show that if $[f] = [f']$ and $[g] = [g']$, then $[f + g] = [f' + g']$.

Caution: $f + g$ is defined only if f ends where g begins. To ensure that it is always defined, we will restrict our paths to be based loops, $(S', *_s) \to (X, *_x)$. This is important since any two of these start and end at $*_x$.

Definition 2. *We write $\pi_1(X, *_x)$ to denote the set of (path) homotopy classes of based loops from $(S', *_S) \to (X, *_x)$.*

Theorem 2. *The operation $+$ on $\pi_1(X, *_x)$ satisfies the following:*

1. *$[f] + ([g] + [h]) = ([f] + [g]) + [h]$ for every f, g, h.*
2. *Let O denote the pointed map, $0 : (S', *_s) \to (X, *_x)$ mapping all of S' to $*_x$. Then $[f] + [0] = [0] + [f] = [f]$. (Additive ID)*
3. *Let $\overline{f}$ be defined by $\overline{f}(\theta) = f(-\theta)$. Then $[\overline{f}] + [f] = [f] + [\overline{f}] = [0]$.*

Corollary: $(\pi_1(X, *), +)$ is a group! It is called the fundamental group of $(X, *)$.