Math 435 08/22/2025 Notes

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Definition 1. *$f : A \to B$ is injective iff $f(x_1) = f(x_2)$ implies $x_1 = x_2$ for all $x_1, x_2 \in A$.*

Definition 2. *$f : A \to B$ i s surjective iff for all $y \in B, \exists x \in A$ with $y = f(x)$, or $Im(f) = B$.*

Proposition 1. *A function $f : A \to B$ is injective iff $f \circ g_1 = f \circ g_2$ implies $g_1 = g_2$ for all sets X and functioning $g_1, g_2 : X \to A$.*

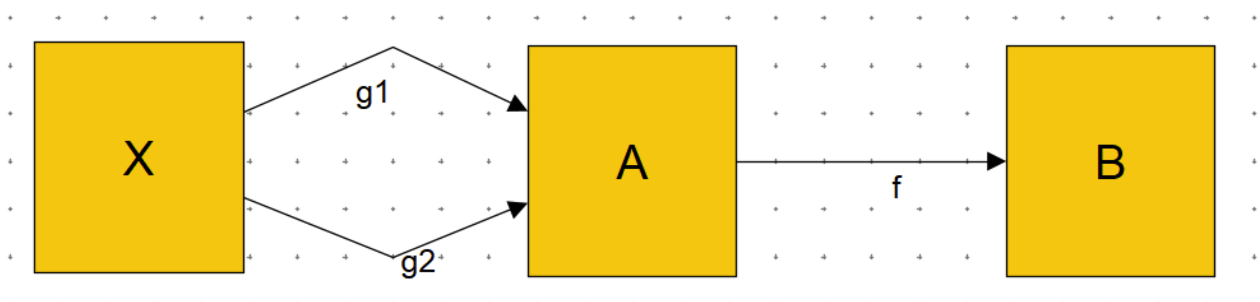

Figure 1: Left Cancellation for Injectivity

Proof:

(Direction 1) Let X, g_1, g_2 be given as above. We must show that $g_1 = g_2$, i.e. $g_1(x) = g_2(x)$ $\forall x \in X$. Well,

$$f(g_1(x)) = f \circ g_1(x) = f \circ g_2(x) = f(g_2(x)). \tag{1}$$

But, f is injective, so $g_1(x) = g_2(x)$.

(Direction 2) Assume $f \circ g_1(x) = f \circ g_2(x) \Rightarrow g_1 = g_2$. Let $a_1, a_2 \in A$ such that $f(a_1) = f(a_2)$. Then we need to show that $a_1 = a_2$. Let X be a set with one element, $X = \{x\}$. Define $g_1(x) = a_1$ and $g_2(x) = a_2$. Then,

$$f \circ g_1(x) = f(a_1) = f(a_2) = f \circ g_2(x). \tag{2}$$

So,

$$a_1 = g_1(x) = g_2(x) = a_2. \tag{3}$$

$\square$

Proposition 1 is significant because we can now characterize injectivity without referencing set elements!

Proposition 2. *A function $f : A \to B$ is surjective iff $g_1 \circ f = g_2 \circ f$ implies $g_1 = g_2$ for all X, $g_1, g_2 : B \to X$.*

Proof is exercise.

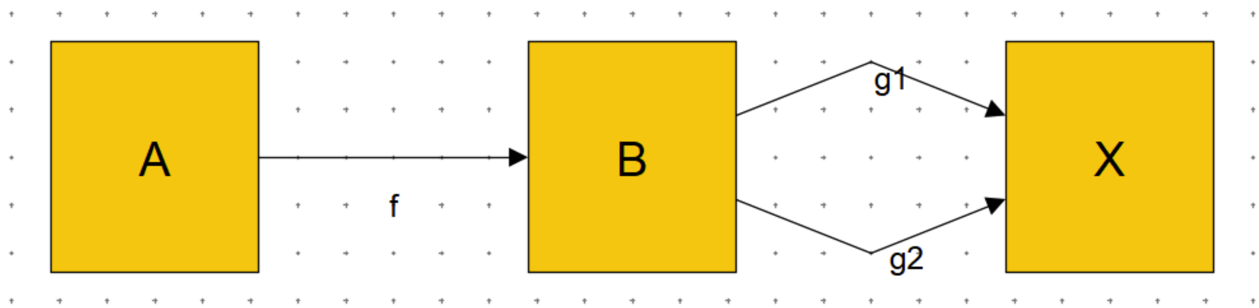

Figure 2: Right Cancellation for Surjectivity

Definition 3. *Given a set A, the identity function, $id_A : A \to A$ is defined by $id_A(a) = a$ for all $a \in A$.*

Definition 4. *Given $f : A \to B$, a function $g : B \to A$ is a right inverse if $f \circ g = id_B$.*

Definition 5. *Given $f : A \to B$, a function $g : B \to A$ is a left inverse if $g \circ f = id_A$.*

Proposition 3. *Every injective function has a left inverse.*

Proof is exercise.

Proposition 4. *Every surjective function has a right inverse iff ZFC.*

Proof:

Definition 6. ***Axiom of Choice (ZFC):*** *Given any set X whose elements are non-empty sets, there exists a choice function $c : X \to A$ such that $c(A) \in A$, for all $A \in X$.*

Let $f : A \to B$ be surjective. We want to show that there exits $g : B \to A$ with $f \circ g = id_B$. By surjectivity, for every $b \in B$, the fiber $f^{-1}(b)$ is non-empty. By ZFC $\exists$ c such that for all $b \in B$,

$$c(f^{-1}(b) \in f^{-1}(b). \tag{4}$$

Well, $f(c(f^{-1}(b))) = b$. Define $g(b) = c(f^{-1}(b))$ for all $b \in B$. Then $f \circ g(b) = b$, and g is a right inverse.

$\square$