Math 435 10/13/2025 Notes

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Review

Today

- Compactness

Definition 1. *A collection of subsets, $\{C_\alpha \subseteq X\}_\alpha$ covers X if $\bigcup_\alpha C_\alpha = X$. It is an open cover if each C_α is open in X.*

Definition 2. *A collection $C' \subseteq C$ is called a subcover of X if it also covers X. We say that C admits a finite subcover if $\exists C' \subseteq C$ with $|C'| < \infty$.*

Definition 3. *A space X is compact if every open cover admits an open finite subcover.*

Example 1. $\mathbb{R}$ *is not compact.* $\{\ ,\ (-1,1),\ (0,2),\ (1,3),\ \}$ *covers* $\mathbb{R}$*. It contains no finite subcover.*

Definition 4. *Let X be a space and $A \subseteq X$ be a subset. A is compact in X if it is compact with the subspace topology.*

Lemma 1. *Let X be a space and let $A \subseteq X$ be a subset. Then A is compact in X iff every cover of A by sets open in X admits an open finite sub-cover.*

Theorem 1. *The continuous image of a compact set is compact.*

Remark 1. *Corollary: the quotient space of a compact space is compact.*

Theorem 2. *(Heine-Borel) Subsets of a euclidean space are compact iff they are closed and bounded.*

Theorem 3. *(1-point computation) Let X be a space and define*

$$Y = X \cup \{\infty\}. \tag{1}$$

Declare $U \subseteq Y$ to be open if

1. *$U \subseteq X$ open or if*
2. *$U = X \setminus$ some compact set.*

Then Y is compact.

Example 2. $\mathbb{R} \cup \{\infty\} \cong S^1$, *the 2d sphere.* $\mathbb{R}^2 \cup \{\infty\} \cong S^2$.