Math 435 Homework 4

September 29, 2025

Timothy Tarter
James Madison University
Department of Mathematics

Problem 1:

Proposition 1. *(Square Lemma) Let $\mathcal{U}_1, \mathcal{U}_2$ be the topologies generated by bases $\mathcal{B}_1$ and $\mathcal{B}_2$ on X. If every element of $\mathcal{B}_1$ is a union of elements of $\mathcal{B}_2$, then $\mathcal{U}_1 \subseteq \mathcal{U}_2$.*

Prove: $\mathcal{U}_{A\times B}$ under the product topology coincides with $\mathcal{V}_{A\times B}$ under the subspace topology of $X \times Y$.

We want to show that $\mathcal{U}_{A\times B} \subseteq \mathcal{V}_{A\times B}$, that the product is contained in the subspace. It suffices by proposition 1 to show that every basis element of $\mathcal{U}_{A\times B}$ is a union of basis elements of $\mathcal{V}_{A\times B}$. Recall that any basis element U of $\mathcal{U}_{A\times B}$ can be written as

$$U = S \times T; \text{ where } S \text{ is open in } A, \text{ and } T \text{ is open in } B \tag{1}$$

Since S is open in A, then $S = A \cap P$ where P is open in X. Similarly, since T is open in B, $T = B \cap Q$ where Q is open in Y. Thus,

$$U = (A \cap P) \times (B \cap Q). \tag{2}$$

By DeMorgan's Laws,

$$U = (A \times B) \cap (P \times Q). \tag{3}$$

Notably, since $P \times Q$ is open in $X \times Y$, then U is the intersection of a set in $X \times Y$ with an open set in $\mathcal{U}_{A\times B}$, and is thus a basis element of $\mathcal{V}_{A\times B}$.

Going in the other direction we want to show that $\mathcal{V}_{A\times B} \subseteq \mathcal{U}_{A\times B}$. Again, by proposition 1, it suffices to show that every basis element of $\mathcal{V}_{A\times B}$ can be written as a union of the basis elements of $\mathcal{U}_{A\times B}$. Recall that any basis element V of $\mathcal{V}_{A\times B}$ can be written as

$$V = (A \times B) \cap (\bigcup_\alpha P_\alpha \times Q_\alpha) \tag{4}$$

where P_α is open in X and Q_α is open in Y. Then since by DeMorgan's Laws,

$$V = (A \cap \bigcup_\alpha P_\alpha) \times (B \cap \bigcup_\alpha Q_\alpha), \tag{5}$$

and since $\bigcup_\alpha P_\alpha$ is open in X, then $A \cap \bigcup_\alpha P_\alpha$ is open in A. Moreover, since $\bigcup_\alpha Q_\alpha$ is open in Y, $B \cap \bigcup_\alpha Q_\alpha$ is open in B. Then $(A \cap \bigcup_\alpha P_\alpha) \times (B \cap \bigcup_\alpha Q_\alpha)$ is open in $\mathcal{U}_{A \times B}$, and so the topologies coincide as desired.

□

Problem 2:

Definition 1. *A topology $\mathcal{U}$ over X is Hausdorff iff for any $x, y \in X, x \neq y$, there exists open sets U_x, U_y with $x \in U_x$ and $y \in U_y$ such that $U_x \cap U_y = \emptyset$.*

Let $(\mathcal{U}, X)$ be a Hausdorff space and let $A \subseteq X$. Show that the subspace topology, $\mathcal{V}$ on A is Hausdorff.

We want to show that for any $x, y \in A, x \neq y$, that there exist open sets $U_x, U_y \in \mathcal{V}$ with $x \in U_x$ and $y \in U_y$ with $U_x \cap U_y = \emptyset$. We note that an arbitrary set $V \in \mathcal{V}$ is of the form

$$V = \{A \cap U \mid U \in \mathcal{U}\}. \tag{6}$$

Since $x, y \in A \subseteq X$, then there exists disjoint open sets $U_x, U_y \in \mathcal{U}$ with $x \in U_x$ and $y \in U_y$ as desired. Then if we take

$$U'_x = U_x \cap A \tag{7}$$
$$U'_y = U_y \cap A, \tag{8}$$

we are guaranteed that $x \in U'_x$ and $y \in U'_y$, where $U'_x \cap U'_y = \emptyset$. The last thing we need to show is that U'_x and U'_y are actually in $\mathcal{V}$. This is quite simple since by (6), we know that the intersection of A with an open set in $\mathcal{U}$ is open in $\mathcal{V}$. Therefore, $\mathcal{V}$ is Hausdorff as desired.

□

Problem 3:

Let $(\mathcal{U}, X)$ be a topological space and let $A \subseteq X$ have the subspace topology.

Part A: If U is open in A and A is open in X, show that U is open in X.

Recall the definition of an open set U in A under the subspace topology from X. If U is such an open set in A, then $U = A \cap U'$ where U' is open in X. But by definition, $A \cap U'$ is open in X, since $A \subseteq X$ and $U' \subseteq X$ (intersections of open sets in X are open). So U is open in X.

□

Part B: If V is closed in A and A is closed in X, show that V is closed in X.

Since V is closed in A, there exists W which is open in A with $V = A \setminus W$. Since W is open in A, there exists U which is open in X such that $W = A \cap U$. Then, $V = A \setminus W = A \setminus (A \cap U)$. By set containment laws, $A \setminus (A \cap U) = A \cap (X \setminus U)$, but $X \setminus U$ is closed in X. Since intersections of closed sets in X are closed in X, and since both A and $X \setminus U$ are closed in X, $V = A \cap (X \setminus U)$ is closed in X, as desired.

$\square$

Problem 4:

Show that if $(\mathcal{U}, X)$ and $(\mathcal{V}, Y)$ are Hausdorff spaces, then so is their product space $(\text{Ł}, X \times Y)$.

Definition 2. *A topology $\mathcal{U}$ over X is Hausdorff iff for any $x, y \in X, x \neq y$, there exists open sets U_x, U_y with $x \in U_x$ and $y \in U_y$ such that $U_x \cap U_y = \emptyset$.*

Following from this definition, we know that if we have two distinct points, $u_1, u_2 \in X$ and $v_1, v_2 \in Y$, then there exist disjoint open sets in X and Y (respectively) containing each point. The exact containment is

- $u_1 \in U_1 \subseteq \mathcal{U}$
- $u_2 \in U_2 \subseteq \mathcal{U}$
- $v_1 \in V_1 \subseteq \mathcal{V}$
- $v_2 \in V_2 \subseteq \mathcal{V}$

with $U_1 \cap U_2 = \emptyset = V_1 \cap V_2$ and $u_1 \neq u_2, v_1 \neq v_2$. More importantly, if we pair $(u_1, v_1), (u_2, v_2) \in X \times Y$, if without loss of generality for construction, $u_1 = u_2$, and as long as $v_1 \neq v_2$, then $(u_1, v_1) \neq (u_2, v_2)$. Then since $V_1 \cap V_2 = \emptyset$, $U_1 \times V_1 \cap V_2 = U_2 \times V_1 \cap V_2 = \emptyset$. Since this is a generalization of the case where $u_1 \neq u_2$ and $v_1 \neq v_2$, as well as when $u_1 \neq u_2$ but $v_1 = v_2$, the proof is complete and $X \times Y$ is Hausdorff.

$\square$