

Math 435 10/29/2025 Notes

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Review

Definition 1. Let $f, g : X \rightarrow Y$ be continuous. Assume $[0, 1]$ has the subspace topology (from \mathbb{R}), and assume $X \times [0, 1]$ has the product topology. We say f and g are **homotopic**, and write $f \simeq g$ iff there exists a continuous function $H : X \times [0, 1] \rightarrow Y$ such that $H(x, 0) = f(x)$ and $H(x, 1) = g(x)$ for all $x \in X$. H is called a **homotopy**.

Definition 2. We will call a continuous function a map from now on.

Definition 3. Given spaces X and Y , the set of maps from X to Y is called $\text{hom}_\tau(X, Y)$.

Theorem 1. Homotopy of maps is an equivalence relation on $\text{hom}(X, Y)$.

Today

- Finishing the proof of theorem 1.
 - Homotopy Classes
 - Straight Line Homotopy
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Finishing up the proof of theorem 1:

1. $f \simeq f$ for any $f \in \text{hom}(X, Y)$.
2. $f \simeq g$ implies $g \simeq f$ for any $f, g \in \text{hom}(X, Y)$.
3. $f \simeq g$ and $g \simeq h$ implies $f \simeq h$ for any $f, g, h \in \text{hom}(X, Y)$.

Suppose $f \simeq g$ and $g \simeq h$ via H_1 and H_2 , respectively. Then we want to show that there exists an $G : X \times I \rightarrow Y$ such that $G(x, 0) = f(x)$ and $G(x, 1) = h(x)$. Let

$$G(x, t) = \begin{cases} H_1(x, 2t) & \text{where } 0 \leq t \leq \frac{1}{2} \\ H_2(x, 2t - 1) & \text{where } \frac{1}{2} \leq t \leq 1. \end{cases}$$

Notably, at $t = \frac{1}{2}$,

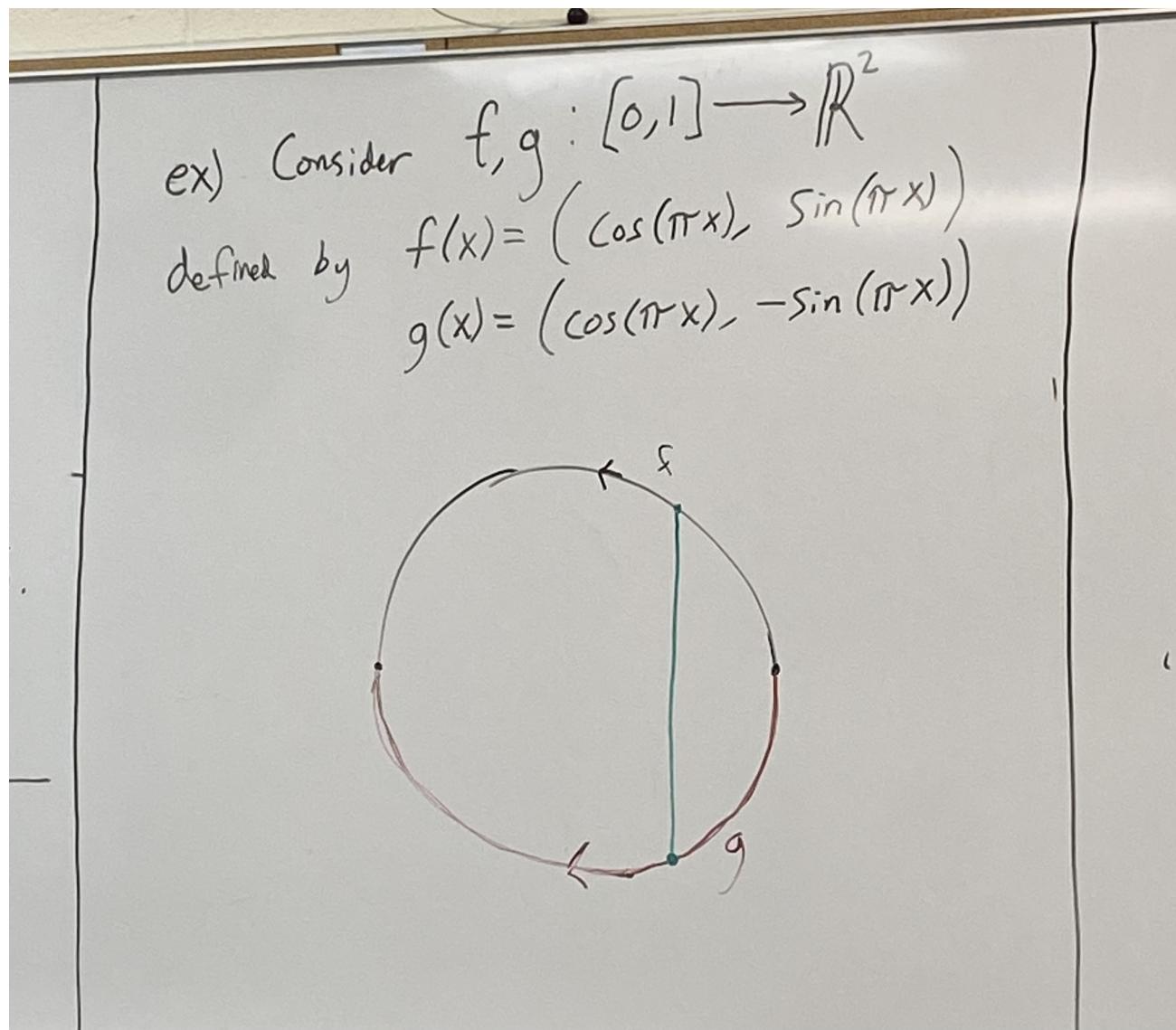
$$G(x, \frac{1}{2}) = H_1(x, 1) = g(x)$$

$$H_2(x, 0) = H_2(x, 0) = g(x).$$

This is continuous by the pasting lemma from WAY earlier in this course. \square

Definition 4. Since homotopy of maps is an equivalence relation on $\text{hom}(X, Y)$, we can consider the resulting equivalence classes $[f] = \{g \in \text{hom}(X, Y) | g \simeq f\}$. We call these **homotopy classes**. (We can define a group structure on this group of equivalence classes!)

Example 1. Consider the functions $f, g : I \rightarrow \mathbb{R}^2$ defined by $f(x) = (\cos(\pi x), \sin(\pi x))$ and $g(x) = (\cos(\pi x), -\sin(\pi x))$. The straight line homotopy takes a point on f and maps it straight down (or some affine transformation) to an associated point on g .



Theorem 2. If we have two vectors $v_1, v_2 \in \mathbb{R}^n$, we can define a clean homotopy parameterized by t as $f(t) = (1-t)v_1 + tv_2$. This is linear interpolation between v_1 and v_2 . This works for vector spaces, modules, and k -algebras.

Theorem 3. *The linear homotopy from $f \rightarrow g$ is $H(x, t) = (1 - t)f(x) + (t)g(x)$.*

In our example, this works out to be,

$$\begin{aligned} H(x, t) &= \langle (1 - t)\cos(\pi x), (1 - t)\sin(\pi x) \rangle + \langle t\cos(\pi x), -t\sin(\pi x) \rangle \\ &= \langle (1 - t)(\cos(\pi x) + t\cos(\pi x)), (1 - t)\sin(\pi x) - t\sin(\pi x) \rangle \\ &= \langle \cos(\pi x), (1 - 2t)\sin(\pi x) \rangle. \end{aligned}$$

Theorem 4. *Given $f, g : X \rightarrow Y \subseteq \mathbb{R}^n$, the linear homotopy $H(x, t) = (1 - t)f(x) + tg(x)$ is continuous if it is well defined (image is a subset of \mathbb{R}^n).*