Math 435 Homework 7

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Problem 1:

Let $X = \{*\}$ denote the space with one point and consider the constant functions $f, g : X \to \mathbb{R}^2 \setminus \{0\}$ defined by $f(*) = (1, 0)$ and $g(*) = (-1, 0)$. Prove that $f \simeq g$.

Consider the following homotopy:

$$H(x,t) = \begin{cases} \langle 1-2t, 2t \rangle; 0 \leq t \leq \frac{1}{2} \\ \langle 1-2t, -2t+2 \rangle; \frac{1}{2} \leq t \leq 1 \end{cases}$$

Since $H(x, 0) = (1, 0)$ and $H(x, 1) = (-1, 0)$, and since $H(x, t)$ is continuous at $t = \frac{1}{2}$, H is a valid homotopy for f and g.

□

Problem 2:

Given spaces X, Y, Z and continuous functions $f_1, f_2 : X \to Y$ and $g_1, g_2 : Y \to Z$, suppose $f_1 \simeq f_2$ and $g_1 \simeq g_2$. Prove that $g_1 \circ f_1 \simeq g_2 \circ f_2$.

Since $f_1 \simeq f_2$, there exists $H_1 : X \times I \to Y$ such that $H_1(x, 0) = f_1$ and $H_1(x, 1) = f_2$. Similarly, there exists $H_2 : Y \times I \to Z$ such that $H_2(y, 0) = g_1$ and $H_2(y, 1) = g_2$. Then the following homotopy gives $g_1 \circ f_1 \simeq g_2 \circ f_2$:

$$H'(x, y, t) = H_2(y, t) \circ H_1(x, t)$$

This holds since

$$H'(x, y, 0) = H_2 \circ H_1|_{t=0} = H_2 \circ f_1 = g_1 \circ f_1$$

$$H'(x, y, 1) = H_2 \circ H_1|_{t=1} = H_2 \circ f_2 = g_2 \circ f_2.$$

□

Problem 3:

Fix $n \geq 1$ and consider the unit ball $D = \{v \in \mathbb{R}^n; ||v||_2 \leq 1\} \subseteq \mathbb{R}^n$. Given a space X, prove that there is only one homotopy class of continuous functions from X to D.

Let $f, g : X \to D$ be continuous maps. Since D is a euclidean space, there exists a straight-line homotopy from f to g, by $H(x,t) = (1-t)f + (t)g$. Then by the triangle inequality,

$$||(1-t)f + tg||_2 = (1-t)||f||_2 + t||g||_2 \leq (1-t) * 1 + t * 1 = 1.$$

So, $H(x,t) \in D$ for all (x,t), and so H is a homotopy for any f and g. So there is only one homotopy class for continuous functions from X to D.

□

Problem 4:

A space X is contractible if the identity function $id_X : X \to X$ is homotopic to a constant function.

Part 1:

Suppose X and Y are homeomorphic. Prove that if X is contractible, then Y is contractible.

Since $X \simeq Y$, there exists $f : X \to Y$ and $g : Y \to X$ with $g \circ f \simeq id_X$ and $f \circ g \simeq id_Y$. We want to show that $id_Y \simeq$ a constant function. Really, this means we want to show that for an explicit f, g, $f \circ g$ is a constant function. Since X and Y are homeomorphic, let $f = g^{-1}$ and $g = f^{-1}$. Then $id_X \simeq g \circ f = g \circ g^{-1} = id_y$. But $id_y \simeq id_x \simeq$ a constant function, as desired. So Y is contractible.

□

Part 2:

Prove that $\mathbb{R}^n$ is contractible.

Let $h(x) = \vec{b} \in \mathbb{R}^n$, with $h : \mathbb{R}^n \to \mathbb{R}^n$. Then

$$H(\vec{x}, t) = (1-t)\vec{b} + t\vec{x}$$

is the straight-line homotopy which makes

$$H(\vec{x}, 0) = h(x)$$

and

$$H(\vec{x}, 1) = id_x(\vec{x}).$$

So $id_x \simeq h(x)$ for any constant function $h(x)$, as desired.

□