Math 435 08/27/2025 Notes

August 27, 2025

Timothy Tarter
James Madison University
Department of Mathematics

Definition 1. *We define the discrete metric to be*

$$d(x, y) = \begin{cases} 0, & \text{if } x = y, \\ 1, & \text{if } x \neq y. \end{cases} \tag{1}$$

.

Definition 2. *We define the taxicab metric on* $\mathbb{R}^n$ *to be*

$$d(x, y) = \sum_{i=1}^{n} |x_i - y_i| \tag{2}$$

Proposition 1. *The Taxicab Metric satisfies the triangle inequality.*

Proof:

$$d(x, z) = \sum_{i=1}^{n} |x_i - z_i| \tag{3}$$

$$= \sum_{i=1}^{n} |x_i - y_i + y_i - z_i| \leq \sum_{i=1}^{n} |x_i - y_i| + \sum_{i=1}^{n} |y_i - z_i| \tag{4}$$

$$= d(x, y) + d(y, z), \tag{5}$$

as desired.

□

Definition 3. *We define the Chebychev (Max) metric on* $\mathbb{R}^n$ *to be:*

$$d(x, y) = \max\{|x_i - y_i| : i \in 0 \ldots n\} \tag{6}$$

.

Definition 4. *We define the edit distance on strings of characters to be the number of pointwise differences between letters in the word. A special version of this is called the Hamming distance.*

Definition 5. *We define the Hamming distance on polynomials in* $\mathbb{Z}_2[x]$ *to be*

$$d(f,g) = \sum_{i=1}^{n} (f \oplus g))i \tag{7}$$

where $\oplus$ *denotes the 'XOR' operator.*

Definition 6. *We define the sup metric on the set of functions* $f : [a,b] \to \mathbb{R}$ *to be*

$$d(f,g) = \sup_{t \in [a,b]} |f(t) - g(t)| \tag{8}$$

.

Definition 7. *Given a metric space* (X,d)*, the open ball of radius* r *centered at* $x_0 \in X$ *is defined as*

$$B(x_0; r) = \{y \in X | d(x_0, y) < r\}. \tag{9}$$

Definition 8. *Given a metric space* (X,d)*, the closed ball of radius* r *centered at* $x_0 \in X$ *is defined as*

$$\overline{B(x_0; r)} = \{y \in X | d(x_0, y) \leq r\}. \tag{10}$$

Examples of Closed Balls Under These Metrics

L_2 Norm on $\mathbb{R}^2$, $\overline{B((0,0),1)}$:

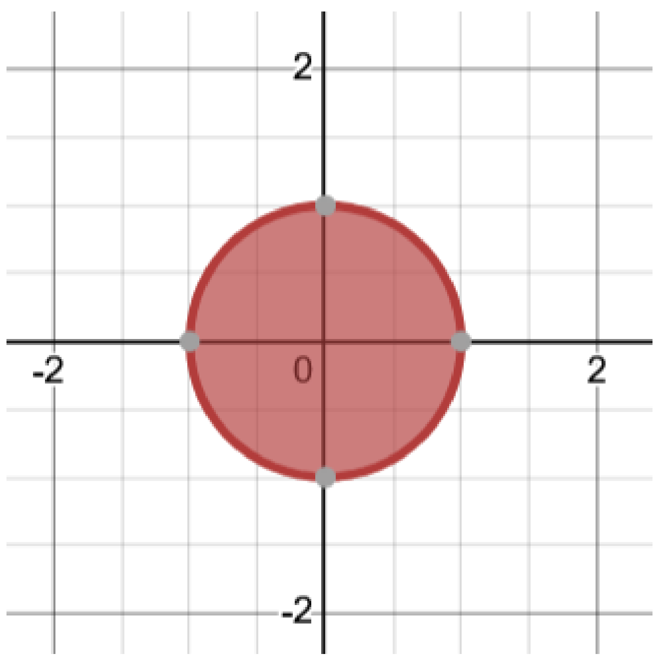

Figure 1: Euclidean Closed Ball

Taxicab Metric on $\mathbb{R}^2$, $\overline{B((0,0),1)}$:

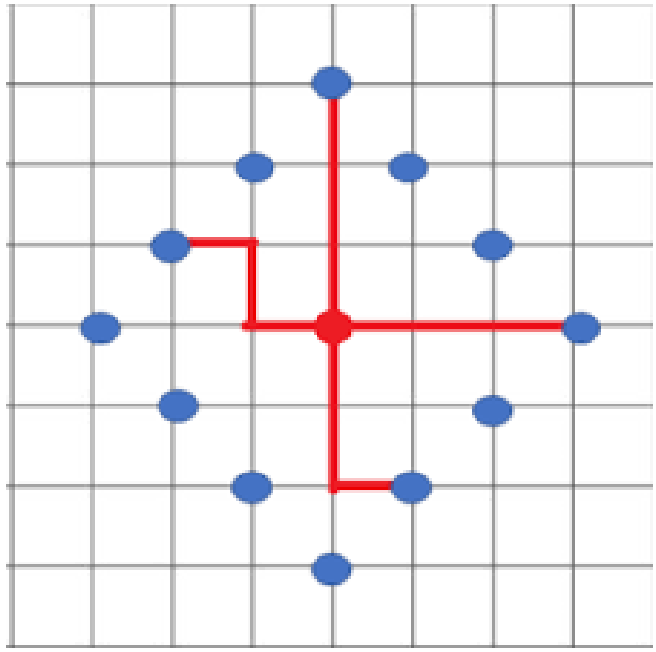

Figure 2: Taxicab Closed Ball

Max Metric on $\mathbb{R}^2$, $\overline{B((0,0),1)}$:

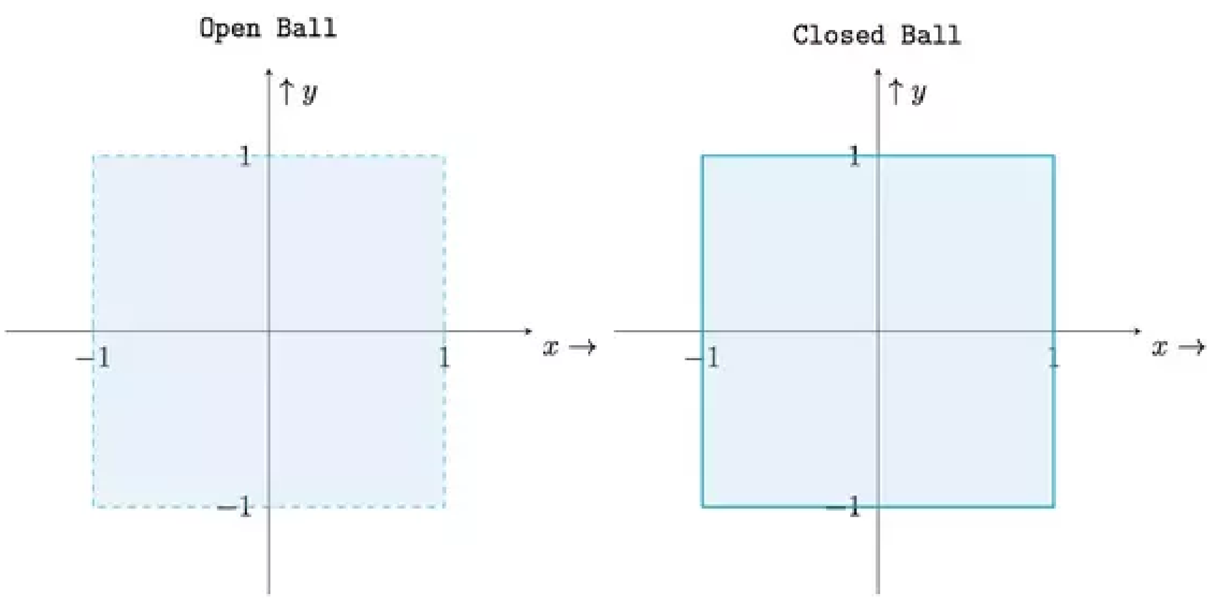

Figure 3: Chebychev Open & Closed Ball