Math 435 09/08/2025 Notes

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Review

Definition 1. *A topological space is a pair $(X, \mathscr{U})$, (often (X, τ)), where X is a set, and $\mathscr{U}$ is a collection of subsets of X such that*

1. $\emptyset, X \in \mathscr{U}$
2. *$\mathscr{U}$ is closed under arbitrary union*
3. *$\mathscr{U}$ is closed under finite intersection.*

The elements of $\mathscr{U}$ are called open sets. Note: 'open set' here really just means that a set belongs to $\mathscr{U}$.

Today

- We can make a basis of a set
- The basis of a set induces a topology
- Every topology has a basis

Additional notes: our first exam is next Monday.

Definition 2. *Let $\mathscr{U}_1, \mathscr{U}_2$ be topologies on X. If $\mathscr{U}_1 \subseteq \mathscr{U}_2$ we say that $\mathscr{U}_2$ is finer than $\mathscr{U}_1$, and $\mathscr{U}_1$ is coarser than $\mathscr{U}_2$.*

Definition 3. *A collection $\mathscr{B}$ of subsets of a set X is called a basis if*

1. *The sets in $\mathscr{B}$ cover X, i.e., $\forall x \in X$ there exists $B \in \mathscr{B}$ with $x \in B$.*
2. *If $B_1, B_2 \in \mathscr{B}$ and if $x \in B_1 \cap B_2$, then there exists $B_3 \in \mathscr{B}$ such that $X \in B_3 \subseteq B_1 \cap B_2$.*

Proposition 1. *Given a basis $\mathscr{B}$ of X, the collection of sets obtained by taking arbitrary unions of elements of $\mathscr{B}$ is a topology. (This is called the topology generated by $\mathscr{B}$.)*

Proposition 2. *Intermediate Lemma: For any basis $\mathscr{B}$ on X, let $B_1 \dots B_n \in \mathscr{B}$, and if $x \in \bigcap_{i=1}^{n} B_i$, then there exists $B' \in \mathscr{B}$ with $x \in B' \subseteq \bigcap_{i=1}^{n} B_i$.*

Proof of Proposition 1: We claim the collection $\mathscr{U}$ of all unions of elements of $\mathscr{B}$ is a topology. We know that $X \in \mathscr{U}$ since we assume the sets in $\mathscr{B}$ cover X. We also know that $\emptyset \in \mathscr{U}$, since we can take arbitrary unions over the empty set. This satisfies requirement one.

For requirement two, we know that an arbitrary union of elements of $\mathscr{U}$ is again an arbitrary union of elements of $\mathscr{B}$.

For requirement three, we want to show that $\mathscr{U}$ is closed under finite intersection. Let $V = \bigcap_{i=1}^{n} U_i$ be the finite intersection of $U_i \in \mathscr{U}$. We want to show that $V = \bigcup_{i=1}^{k} B_i$, for elements of the basis $B_i \in \mathscr{B}$. If any $U_i = \emptyset$, then $V = \emptyset$. Otherwise, let each U_i be non-empty. So each $U_i = \bigcup B_i \in \mathscr{B}$. Now given $x \in V$, $x \in U_i$ for all i. Hence, for each i, there exists some $B_i \subseteq U_i$ with $x \in B_i$. Thus, $x \in \bigcap_{i=1}^{n} B_i$. By intermediate lemma, there exists B' such that $x \in B'_x$ with $x \in B'_x \subseteq \bigcap_{i=1}^{n} B_i \subseteq \bigcap_{i=1}^{n} U_i \subseteq V$. Since $x \in V$ was arbitrary, then we can find some subset $B'_x \subseteq V$. So $V = \bigcup_{x \in V} B'_x$, as desired.

□

Proposition 3. *Every topology has a basis.*

Proposition 4. *The collection of open balls in a metric space is a basis for the topology induced by the metric. (Called the metric topology).*

Definition 4. *If you can find a metric on a topology such that the topology induced by the metric is equivalent to the orignial topology, then that topology is metrizable.*

Example 1. *An example of this is the notion of a finite simplicial complex.*