Math 435 08/25/2025 Notes

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Definition 1. *A metric space is a pair (X, d) where X is a set and* $d : X \times X \to \mathbb{R}$ *such that*

- $d(x, y) = 0$ *iff* $x = y$
- $d(x, y) = d(y, x)$ *for all* $x, y \in X$
- $d(x, z) \leq d(x, y) + d(y, z)$ *for all* $x, y, z \in X$.

Definition 2. *A pseudo-metric space is just (2) and (3) from above; a prime example is a Hausdorff space.*

Proposition 1. *Metrics are non-negative.*

Proof:

$$0 = d(x, x) \leq d(x, y) + d(y, x) = 2d(x, y) \tag{1}$$

implies

$$0 \leq d(x, y). \tag{2}$$

Definition 3. *The* L_2 *norm for Euclidean distance has* $d(x, y) = \left[\sum_{i=1}^{n} (x_i - y_i)^2 \right]^{\frac{1}{2}}$

Proposition 2. *The triangle inequality holds under the* L_2 *norm on* $\mathbb{R}$.

Proof: Define the polynomial,

$$(u_1 x + v_1)^2 + (u_2 x + v_2)^2 + \cdots + (u_n x + v_n)^2 = \tag{3}$$

$$= \sum_{i=1}^{n} \left[u_i^2 x^2 \right] + 2 \sum_{i=1}^{n} \left[u_i v_i x \right] + \sum_{i=1}^{n} \left[v_i^2 \right] = p(x) \tag{4}$$

Call the first summation term's coefficient A, the second B, and the third C. Notice that $p(x) \geq 0$ for all x, is quadratic, and has at most one real root. Thus, by the quadratic formula, the discriminant must be less than or equal to zero.

$$B^2 - 4AC \leq 0 \tag{5}$$

$\iff$

$$4\left[\sum_{i=1}^{n} u_i v_i\right]^2 - 4\sum_{i=1}^{n}\left[u_i^2\right]\sum_{i=1}^{n}\left[v_i^2\right] \leq 0. \tag{6}$$

Therefore,

$$\left[\sum_{i=1}^{n} u_i v_i\right]^2 \leq \sum_{i=1}^{n}\left[u_i^2\right]\sum_{i=1}^{n}\left[v_i^2\right]. \tag{7}$$

Switching to vector notation, (7) becomes

$$(\sum_{i=1}^{n} u_i v_i)^2 \leq (\sum_{i=1}^{n} u_i^2)(\sum_{i=1}^{n} v_i^2) \tag{8}$$

$$\left|\sum_{i=1}^{n} u_i v_i\right| \leq (\sum_{i=1}^{n} u_i^2)^{\frac{1}{2}}(\sum_{i=1}^{n} v_i^2)^{\frac{1}{2}} \tag{9}$$

$$\left|u_i \circ v_i\right| \leq ||\vec{u}||_2||\vec{v}||_2 \tag{10}$$

where $|| \circ ||_2$ is the L_2 norm on $\mathbb{R}^n$. Notably, (10) is the Cauchy-Schwarz inequality. Now, we want to show that

$$||\vec{u}||_2 + ||\vec{v}||_2 \geq ||\vec{u} + \vec{v}||_2. \tag{11}$$

So, let

$$||\vec{u} + \vec{v}||_2^2 = (\vec{u} + \vec{v}) \circ (\vec{u} + \vec{v}). \tag{12}$$

Then,

$$= ||\vec{u}||_2^2 + 2\vec{u} \circ \vec{v} + ||\vec{v}||_2^2 \tag{13}$$

$$\leq ||\vec{u}||_2^2 + 2|\vec{u} \circ \vec{v}| + ||\vec{v}||_2^2 \tag{14}$$

$$\leq ||\vec{u}||_2^2 + 2||\vec{u}||_2^2 * ||\vec{v}||_2^2 + ||\vec{v}||_2^2 \tag{15}$$

$$= (||\vec{u}||_2 + ||\vec{v}||_2)^2 \tag{16}$$

as desired.

$\square$