Math 440 Exam 2 Practice Test

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1 Describe / Explain:

1.1 Laplace's Equation (include well-posedness, uniqueness, and the mean value proposition).

Laplace's equation states that the sum of the first partials are equal to zero, i.e.,

$$0 = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} \tag{1}$$

Well-posedness: there exists a unique solution which depends continuously on non-homogeneous data.

Uniqueness: solutions to laplace's equation are unique.

Mean value proposition: the temperature at the origin is the average of the temperature at the boundary.

1.2 Linear Operators, provide an example of something which is a linear operator, and provide an example of something which is not a linear operator.

A linear operator, $\mathscr{U} : V \to V$ where V is a vector space, has the property that for any $\vec{x}, \vec{y} \in V$ and any scalar c in the field of V,

1. $\mathscr{U}(\vec{x} + \vec{y}) = \mathscr{U}(\vec{x}) + \mathscr{U}(\vec{y})$
2. $\mathscr{U}(c\vec{x}) = c\mathscr{U}(\vec{x})$.

Example:

$$L(u) = \frac{\partial}{\partial x}\left[K_0(x)\frac{\partial u}{\partial x}\right] \tag{2}$$

Non-Example:

$$L(u) = \frac{\partial}{\partial x}\left[K_0(u,x)\frac{\partial u}{\partial x}\right] \tag{3}$$

where K_0 is a non-constant function.

2 Prove Orthogonality of Sines.

We want to show that

$$\int_0^L sin(\frac{n\pi x}{L})sin(\frac{m\pi x}{L}) = \begin{cases} 0; \text{ if } n \neq m \\ \frac{L}{2}; \text{ if } n = m. \end{cases} \tag{4}$$

$$\int_0^L cos(\frac{n\pi x}{L})cos(\frac{m\pi x}{L}) = \begin{cases} 0 \text{ ; if } n \neq m \\ \frac{L}{2}; \text{ if } n = m \neq 0 \\ L \text{ ; if } n = m = 0 \end{cases} \tag{5}$$

Starting off with (4), if $n = m$, then

$$\int_0^L sin^2(\frac{n\pi x}{L})dx = \int_0^L \frac{1}{2}(1 - cos(2\frac{n\pi x}{L}))dx \tag{6}$$

$$= \frac{1}{2}\left[x - \frac{L}{2n\pi}sin(\frac{2n\pi x}{L})\right]\Big|_0^L \tag{7}$$

$$= \frac{1}{2}\left[L - 0\right] = \frac{L}{2} \tag{8}$$

as desired. Alternatively, if $n \neq m$, then

$$\int_0^L sin(\frac{n\pi x}{L})sin(\frac{m\pi x}{L})dx \tag{9}$$

$$= \int_0^L \frac{1}{2}[cos(\frac{n\pi x}{L} - \frac{m\pi x}{L}) - cos(\frac{n\pi x}{L} + \frac{m\pi x}{L})]dx \tag{10}$$

$$= \frac{1}{2}\int_0^L cos(\frac{\pi x(n-m)}{L}) - cos(\frac{\pi x(n+m)}{L})dx \tag{11}$$

$$= \frac{1}{2}[\frac{L}{\pi(n-m)}sin(\frac{\pi x(n-m)}{L}) - \frac{L}{\pi(n-m)}sin(\frac{\pi x(n-m)}{L})]\Big|_0^L \tag{12}$$

$$= 0 - 0 = 0 \tag{13}$$

as desired.

Moving to (5), if $n \neq m$, it works pretty much the same way as with sines. If $n = m \neq 0$, then

$$\int_0^L cos^2(\frac{n\pi x}{L})dx = \tag{14}$$
$$= \int_0^L \frac{1}{2}(1 + sin(\frac{2n\pi x}{L}))dx \tag{15}$$
$$= \frac{1}{2}[x - \frac{L}{2n\pi}cos(\frac{2n\pi x}{L})]\Big|_0^L \tag{16}$$
$$= \frac{1}{2}[L + (-1 + 1)] \tag{17}$$
$$= \frac{L}{2} \tag{18}$$

as desired. Then finally, if $n = m = 0$,

$$\int_0^L cos^2(0)dx = \int_0^L 1dx = L. \tag{19}$$

3 Solve $\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}$ with certain BC's and IC's, addressing $\lambda > 0, = 0, < 0$.

3.1 $u(0,t) = u(L,t) = 0$

Gives sine solutions.

3.2 $\frac{\partial u}{\partial x}(0,t) = 0 = \frac{\partial u}{\partial x}(L,t)$

Gives cosine solutions.

3.3 $u(0,t) = u(L,t) = 0$ and $\frac{\partial u}{\partial x}(0,t) = 0 = \frac{\partial u}{\partial x}(L,t)$

Gives sine and cosine solutions.

4 Solve $\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}$ at equilibrium in polar coordinates over the interval $(-L, L)$.

Note that the 1D heat equation can be expressed in polar coordinate with axial symmetry (meaning that $\frac{\partial^2 u}{\partial \theta^2} = 0$) as follows:

$$0 = \frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial u}{\partial r}). \tag{20}$$

We also get the following BCs for free:

$$u(-L) = u(L) \tag{21}$$
$$\frac{\partial u}{\partial r}(-L) = \frac{\partial u}{\partial r}(L) \tag{22}$$
$$|u(0)| < \infty. \tag{23}$$

If we guess that $u(x,t) = g(t)\phi(x)$, then

$$\frac{1}{k}\frac{g'(t)}{g(t)} = -\lambda = \frac{\phi''(x)}{\phi(x)} \tag{24}$$

implies that

$$g(t) = e^{-k\lambda t} \tag{25}$$

and

$$\phi'' + \lambda\phi = 0. \tag{26}$$

If $\lambda > 0$, then $\phi(x) = A\cos(\sqrt{\lambda}x) + B\sin(\sqrt{\lambda}x)$. $\phi(-L) = \phi(L)$ implies,

$$A\cos(\sqrt{\lambda}L) + B\sin(\sqrt{\lambda}L) = A\cos(\sqrt{\lambda}L) - B\sin(\sqrt{\lambda}L) \tag{27}$$

$$2B\sin(\sqrt{\lambda}L) = 0 \tag{28}$$

$$\text{so} \tag{29}$$

$$\sqrt{\lambda}L = n\pi; n \in \mathbb{N} \tag{30}$$

$$\lambda = (\frac{n\pi}{L})^2 \tag{31}$$

Then

$$\phi(x) = A_0 + \sum_{n=1}^{\infty} A_n e^{-k(\frac{n\pi}{L})^2 t}\cos(\frac{n\pi x}{L}) + \sum_{n=1}^{\infty} B_n e^{-k(\frac{n\pi}{L})^2 t}\sin(\frac{n\pi x}{L}) \tag{32}$$

where

$$A_n = \frac{2}{L}\int_{-L}^{L} f(x) sin(\frac{n\pi x}{L})dx \tag{33}$$

$$\text{and} \tag{34}$$

$$B_n = \frac{2}{L}\int_{-L}^{L} f(x) cos(\frac{n\pi x}{L})dx. \tag{35}$$

5 Solve Laplace's Equation on a rectangle, with LHS, Top, and Bottom $= 0$, RHS $= g_2(y)$.

6 Extra Resources Which You Gotta Know or You Might Be Cooked:

Trig Identities:

$$sin(a+b) = sin(a)cos(b) + cos(a)sin(b) \tag{36}$$

$$cos(a+b) = cos(a)cos(b) - sin(a)sin(b) \tag{37}$$

$$sin(a-b) = sin(a)cos(b) - cos(a)sin(b) \tag{38}$$

$$cos(a-b) = cos(a)cos(b) + sin(a)sin(b) \tag{39}$$

If BC's are

- $u(0,t) = 0 = u(L,t)$ then

$$u(x,t) = \sum_{n=1}^{\infty} A_n e^{-kt(\frac{n\pi}{L})^2} sin(\frac{n\pi x}{L}) \tag{40}$$

- $\frac{\partial u}{\partial x}(0,t) = 0 = \frac{\partial u}{\partial x}(L,t)$ then

$$u(x,t) = \sum_{n=1}^{\infty} B_n e^{-kt(\frac{n\pi}{L})^2} cos(\frac{n\pi x}{L}) \tag{41}$$

where

$$A_n = \frac{2}{L}\int_0^L f(x) sin(\frac{n\pi x}{L}) dx \tag{42}$$

$$\text{and} \tag{43}$$

$$B_n = \frac{2}{L}\int_0^L f(x) cos(\frac{n\pi x}{L}) dx. \tag{44}$$

Orthogonality of sines and cosines (formulas):

$$\int_0^L sin(\frac{n\pi x}{L}) sin(\frac{m\pi x}{L}) = \begin{cases} 0;\ \text{if } n \neq m \\ \frac{L}{2};\ \text{if } n = m. \end{cases} \tag{45}$$

$$\int_0^L cos(\frac{n\pi x}{L}) cos(\frac{m\pi x}{L}) = \begin{cases} 0\ ;\ \text{if } n \neq m \\ \frac{L}{2};\ \text{if } n = m \neq 0 \\ L\ ;\ \text{if } n = m = 0 \end{cases} \tag{46}$$