RUBRIC:

Questions	Points	Score
Total		

Problem 1.3.1: Consider a one-dimensional rod, $0 \leq x \leq L$. Assume that the heat energy flowing out of the rod at $x = L$ is proportional to the temperature difference between the end temperature of the bar and the known external temperature. Derive (1.3.5); briefly, physically explain why $H > 0$.

If the heat energy flowing out of the rod (flux) is proportional to the temperature difference between the end temperature of the bar and the known external temperature, we really mean that, for some prescribed flux representing the known external temperature ($u_B(t)$), some constant K_0, and some proportionality constant H,

$$-K_0(L)\frac{\partial u}{\partial x}(L,t) = H[u(L,t) - u_B(t)].$$

Note, the LHS is negative because of the direction of the flow going from hot to cold. If H were less than or equal to zero, it would reverse the flow (and render the sign on LHS meaningless), so $H > 0$.

Problem 1.3.2: Two one-dimensional rods of different materials joined at $x = x_0$ are said to be in **perfect thermal contact** if the temperature is continuous at $x = x_0$:

$$u(x_{0-}, t) = u(x_{0+}, t)$$

and no heat energy is lost at $x = x_0$. What mathematical equation represents the latter condition at $x = x_0$? Under what special condition is $\frac{\partial u}{\partial x}$ continuous at $x = x_0$?

The mathematical equation which represents no loss of heat energy at $x = x_0$ is

$$-K_0^-(x_0)\frac{\partial u_1(x_0^-, t)}{\partial x} = -K_0^+(x_0)\frac{\partial u_2(x_0^+, t)}{\partial x} \tag{1}$$

where

$$u(x,t) = \begin{cases} u_1(x,t) \text{ for } 0 \leq x \leq x_0 \\ u_2(x,t) \text{ for } x_0 < x \leq L. \end{cases}$$

If $\frac{\partial u}{\partial x}$ is continuous at $x = x_0$, then $K_0^+(x_0) = K_0^-(x_0)$ implies that (1) becomes:

$$\frac{\partial u_1(x_0^-, t)}{\partial x} = \frac{\partial u_2(x_0^+, t)}{\partial x}.$$

Problem 1.4.1a,d, e-f: Determine the equilibrium temperature distribution for a one-dimensional rod with constant thermal properties with the following sources and boundary conditions:

(a) $Q = 0$, $u(0) = 0$, and $u(L) = T$.

The one dimensional heat equation is:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + Q = \frac{\partial^2 u}{\partial x^2} + 0 = \frac{\partial^2 u}{\partial x^2} \tag{2}$$

At equilibrium, (2) becomes

$$0 = \frac{\partial^2 u}{\partial x^2}.$$

Integrating twice with respect to x, we get:

$$0 = \frac{\partial u}{\partial x} + c_1$$

$$0 = u(x) + c_1 x + c_2 \Rightarrow u(x) = -c_1 x - c_2 \tag{3}$$

At this point, we insert our boundary conditions into (3).

$$u(0) = 0 = -c_1(0) - c_2 \Rightarrow c_2 = 0$$

$$u(L) = T = -c_1 L + 0 \Rightarrow c_1 = -\frac{T}{L}.$$

Thus, the equilibrium distribution of (2) according to our BCs is

$$u(x) = \frac{T}{L}x. \tag{4}$$

(d) $Q = 0$, $u(0) = T$, and $\frac{\partial u}{\partial x}(L) = \alpha$

Starting again at our equilibrium condition without a sink,

$$0 = \frac{\partial^2 u}{\partial x^2}.$$

Integrating twice with respect to x, we get:

$$0 = \frac{\partial u}{\partial x} + c_1 \tag{5}$$

$$0 = u(x) + c_1 x + c_2$$

$$u(x) = -c_1 x - c_2 \tag{6}$$

At this point, we insert our boundary conditions into (5), (6).

$$u(0) = T = -c_1(0) - c_2$$

$$c_2 = -T \tag{7}$$

and

$$\frac{\partial u}{\partial x}(L) = \alpha = -c_1$$

$$c_1 = -\alpha \tag{8}$$

derives (via (7) and (8)),

$$u(x) = \alpha x + T.$$

(e) $\frac{Q}{K_0} = 1$, $u(0) = T_1$, and $u(L) = T_2$

Now, we want to start at equilibrium again, but it's a little tougher since we have a constant sink.

$$0 = \frac{\partial^2 u}{\partial x^2} + 1 \tag{9}$$

Integrating (9) twice, we find

$$\begin{aligned} 0 &= \frac{\partial u}{\partial x} + x + c_1 \\ 0 &= u(x) + \frac{1}{2}x^2 + c_1 x + c_2 \\ u(x) &= -\frac{1}{2}x^2 - c_1 x - c_2. \end{aligned} \tag{10}$$

Plugging in our boundary values into (10), we find:

$$\begin{aligned} u(0) = T_1 &= -c_2 \\ c_2 &= -T_1 \end{aligned} \tag{11}$$

and

$$u(L) = T_2 = -\frac{1}{2}L^2 - c_1 L + T_1$$

$$\begin{aligned} c_1 L &= -\frac{1}{2}L^2 + T_1 - T_2 \\ c_1 &= -\frac{1}{2}L + \frac{T_1 - T_2}{L}. \end{aligned} \tag{12}$$

Finally, plugging (11) and (12) into (10), we get

$$\begin{aligned} u(x) &= -\frac{1}{2}x^2 - (-\frac{1}{2}L + \frac{T_1 - T_2}{L})x - (-T_1) \\ u(x) &= -\frac{1}{2}x^2 + (\frac{1}{2}L - \frac{T_1 - T_2}{L})x + T_1 \end{aligned} \tag{13}$$

(13) as the equilibrium temperature distribution.

(f) $\frac{Q}{K_0} = x^2$, $u(0) = T$, and $\frac{\partial u}{\partial x}(L) = 0$

Taking the same equilibrium setup as in (9), but with a new set of conditions, we have

$$0 = \frac{\partial^2 u}{\partial x^2} + x^2. \tag{14}$$

Integrating (14) twice, we find

$$0 = \frac{\partial u}{\partial x} + \frac{1}{3}x^3 + c_1 \tag{15}$$

$$0 = u(x) + \frac{1}{12}x^4 + c_1 x + c_2$$

$$u(x) = -\frac{1}{12}x^4 - c_1 x - c_2. \tag{16}$$

Now, we need to plug in our boundary conditions into (15) and (16).

$$u(0) = T = -c_2$$

$$c_2 = -T \tag{17}$$

and

$$\frac{\partial u}{\partial x} = -\frac{1}{3}x^3 - c_1$$

$$\frac{\partial u}{\partial x}(L) = 0 = -\frac{1}{3}L^3 - c_1$$

$$c_1 = -\frac{1}{3}L^3 \tag{18}$$

Therefore, plugging (17), (18) into (16), we find our equilibrium temperature distribution (19):

$$u(x) = -\frac{1}{12}x^4 + \frac{1}{3}L^3 x + T \tag{19}$$

Problem 1.4.2a-b: Consider the equilibrium temperature distribution for a uniform one-dimensional rod with sources $\frac{Q}{K_0} = x$ of thermal energy subject to the boundary conditions $u(0) = 0$ and $u(L) = 0$.

(a) Determine the heat energy generated per unit time inside the entire rod.

First, I'd like to recognize that the heat energy generated per unit time inside the rod is the flux balance over the length of the rod.

We start with

$$0 = \frac{\partial^2 u}{\partial x^2} + x. \tag{20}$$

Integrating (20) twice, we get

$$\begin{aligned} 0 &= \frac{\partial u}{\partial x} + \frac{1}{2}x^2 + c_1 \\ 0 &= u(x,0) + \frac{1}{6}x^3 + c_1 x + c_2 \\ u(x,0) &= -\frac{1}{6}x^3 - c_1 x - c_2. \end{aligned} \tag{21}$$

Plugging in our BCs, we find

$$\begin{aligned} u(0,0) &= 0 = -c_2 \\ c_2 &= 0 \end{aligned} \tag{22}$$

and

$$\begin{aligned} u(L) &= 0 = -\frac{1}{6}L^3 - c_1 L \\ c_1 L &= -\frac{1}{6}L^3 \\ c_1 &= -\frac{1}{6}L^2. \end{aligned} \tag{23}$$

Plugging in (22) and (23) to (21), we derive the equilibrium distribution, (24):

$$u(x,0) = -\frac{1}{6}x^3 + \frac{1}{6}L^2 x \tag{24}$$

Then to compute the flux balance between both ends, we want to differentiate (24) with respect to x and evaluate

$$\begin{aligned} \frac{\partial u}{\partial x}(L) - \frac{\partial u}{\partial x}(0) &= \left[-\frac{1}{2}x^2 + \frac{1}{6}L^2 \right]\Bigg|_0^L = \\ &= -\frac{1}{2}L^2 + \frac{1}{6}L^2 - [-\frac{1}{2}(0)^2 + \frac{1}{6}L^2] \\ &= -\frac{1}{2}L^2 \end{aligned} \tag{25}$$

So, the heat energy generated per unit time inside the rod is $-\frac{1}{2}L^2$.

(b) Determine the heat energy flowing out of the rod per unit time at $x = 0$ and at $x = L$ (remember, this is at equilibrium).

Accordingly, the heat energy flowing out of the rod per unit time at $x = 0$ and $x = L$ is

$$\frac{\partial u}{\partial x}(L) = -\frac{1}{2}L^2 + \frac{1}{6}L^2 = -\frac{1}{3}L^2$$

$$\frac{\partial u}{\partial x}(0) = \frac{1}{6}L^2.$$

Problem 1.4.3: Determine the equilibrium temperature distribution for a one-dimensional rod composed of two different materials in perfect thermal contact at $x = 1$. For $0 < x < 1$, there is one material ($c\rho - 1$, $K_0 = 1$) with a constant source ($Q = 1$), whereas for the other $1 < x < 2$, there are no sources ($Q = 0$, $c\rho = 2$, $K_0 = 2$ with $u(0) = 0 = u(2)$.

We have the equations for the whole rod at equilibrium as follows

$$\frac{\partial^2 u(x,0)}{\partial x^2} = \begin{cases} 0 = \frac{\partial^2 u}{\partial x^2} + 1 \text{ if } 0 < x < 1 \\ 0 = 2\frac{\partial^2 u}{\partial x^2} \text{ if } 1 \leq x < 2 \end{cases} .$$

From this, we derive

$$\frac{\partial^2 u(x,0)}{\partial x^2} = \begin{cases} \frac{\partial^2 u}{\partial x^2} = -1 \text{ if } 0 < x < 1 \\ 2\frac{\partial^2 u}{\partial x^2} = 0 \text{ if } 1 \leq x < 2 \end{cases} . \tag{26}$$

Integrating (26), we get

$$\frac{\partial u(x,0)}{\partial x} = \begin{cases} \frac{\partial u}{\partial x} = -x + c_1 \text{ if } 0 < x < 1 \\ \frac{\partial u}{\partial x} = \frac{1}{2}c_2 \text{ if } 1 \leq x < 2 \end{cases} . \tag{27}$$

Via (27), it holds that

$$\frac{(-x + c_1)}{2} = c_2 \tag{28}$$

at $x = 1$ (given perfect thermal contact). So, (28) becomes

$$\frac{-1 + c_1}{2} = c_2.$$

Then (27) becomes

$$\frac{\partial u(x,0)}{\partial x} = \begin{cases} \frac{\partial u}{\partial x} = -x + c_1 \text{ if } 0 < x < 1 \\ \frac{\partial u}{\partial x} = \frac{-1+c_1}{2} \text{ if } 1 \leq x < 2 \end{cases} . \tag{29}$$

Integrating (29), we get

$$u(x,0) = \begin{cases} u(x) = -\frac{1}{2}x^2 + c_1 x + c_3 \text{ if } 0 < x < 1 \\ u(x) = \frac{-1+c_1}{2}x + c_4 \text{ if } 1 \leq x < 2 \end{cases} . \tag{30}$$

Since $u(0) = 0 = u(2)$,

$$u(0) = c_3 = 0 \tag{31}$$

$$u(2) = \frac{-1 + c_1}{2}2 + c_4 = 0$$

$$c_4 = 1 - c_1. \tag{32}$$

Then, using (31) and (32) in (30), we get

$$u(x,0) = \begin{cases} u(x) = -\frac{1}{2}x^2 + c_1 x \text{ if } 0 < x < 1 \\ u(x) = \frac{-1+c_1}{2}x + 1 - 1c_1 \text{ if } 1 \leq x < 2 \end{cases} . \tag{33}$$

By perfect thermal contact, we have $u^-(1) = u^+(1)$. So,

$$-\frac{1}{2}(1)^2 + c_1(1) = \frac{-1 + c_1}{2}(1) + 1 - 1c_1$$

$$-\frac{1}{2} + c_1 = \frac{-1 + c_1}{2} + 1 - c_1$$

$$
\begin{aligned}
2c_1 &= \frac{-1+c_1}{2} + \frac{3}{2} \\
4c_1 &= -1 + c_1 + 3 \\
3c_1 &= 2 \\
c_1 &= \frac{2}{3}
\end{aligned} \tag{34}
$$

Finally, inserting (34) into (33),

$$
u(x,0) = \begin{cases} u(x) = -\frac{1}{2}x^2 + \frac{2}{3}x \text{ if } 0 < x < 1 \\ u(x) = -\frac{1}{6}x + 1 - \frac{2}{3} \text{ if } 1 \leq x < 2 \end{cases}.
$$

$$
u(x,0) = \begin{cases} u(x) = -\frac{1}{2}x^2 + \frac{2}{3}x \text{ if } 0 < x < 1 \\ u(x) = -\frac{1}{6}x + \frac{1}{3} \text{ if } 1 \leq x < 2 \end{cases}. \tag{35}
$$

Which is our final equilibrium distribution.

Problem 1.4.5: Consider a one-dimensional rod $0 \leq x \leq L$ of known length and constant thermal properties without sources or sinks. Suppose that the temperature is an unknown constant T at $x = L$. Determine T if we known (in the steady state) for the temperature and the heat flow at $x = 0$.

I can't lie, the last bit of this prompt is throwing me for a loop; I'm going to answer this question as if it reads "Determine T if we know the steady state distribution for the temperature, and the heat flow at $x = 0$." Thus,

$$\frac{\partial^2 u}{\partial x^2} = 0. \tag{36}$$

This implies that

$$u(x) = c_1 x + c_2 \tag{37}$$

$$\frac{\partial u}{\partial x} = c_1. \tag{38}$$

Then given

$$u(0) = T_0$$

$$u(0) = T_0 = c_1 0 + c_2$$

$$c_2 = T_0.$$

So if

$$u(L) = T$$

and then by (38), $u'(x)$ is constant,

$$T - T_0 = c_1(L - 0).$$

So

$$T = c_1 L + T_0.$$

Problem 1.4.7a-b: For the following problems, determine an equilibrium temperature distribution (if one exists). For what values of β are there solutions? Explain physically.

(a) $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + 1$, $u(x,0) = f(x)$, $\frac{\partial u}{\partial x}(0,t) = 1$, and $\frac{\partial u}{\partial x}(L,T) = \beta$

At equilibrium, we have

$$0 = \frac{\partial^2 u}{\partial x^2} + 1$$
$$-1 = \frac{\partial^2 u}{\partial x^2} \tag{39}$$

Integrating twice, we get

$$\frac{\partial u}{\partial x} = -x + c_1$$
$$u(x,t) = -\frac{1}{2}x^2 + c_1 x + c_2.$$

Using our first BC, $\frac{\partial u}{\partial x}(0,t) = 1$,

$$c_1 = 1. \tag{40}$$

Using our second condition,

$$\beta = -L + c_1 = 1 - L. \tag{41}$$

So for there to be solutions, β must equal $1 - L$. Physically, this equates to the idea that the length of the rod must be less than one. Using the trick from class (I'm still trying to intuit how it works), let E(t) denote total thermal energy:

$$\frac{E(t)}{c\rho} = \int_0^L u dx \tag{42}$$

So,

$$\frac{d}{dt}\frac{E(t)}{c\rho} = \frac{d}{dt}\int_0^L u dx = \int_0^L \frac{\partial u}{\partial t} dx. \tag{43}$$

But,

$$\int_0^L \frac{\partial u}{\partial t} dx = \int_0^L \frac{\partial^2 u}{\partial x^2} dx + \int_0^L 1 dx \tag{44}$$
$$\frac{1}{c\rho}\frac{dE}{dt} = \beta - 1 + L \tag{45}$$

However, since $\beta = 1 - L$, (45) becomes:

$$\frac{1}{c\rho}\frac{dE}{dt} = 0. \tag{46}$$

So, the change in total thermal energy is constant. Thus,

$$\int_0^L f(x) dx = \int_0^L -\frac{1}{2}x^2 + x + c_2 dx \tag{47}$$

$$= \left[-\frac{1}{6}x^3 + \frac{1}{2}x^2 + c_2 x \right]\Bigg|_0^L \tag{48}$$

$$= -\frac{1}{6}L^3 + \frac{1}{2}L^2 + c_2 L = \int_0^L f(x) dx \tag{49}$$

Then, solving (49) for c_2, we get:

$$c_2 = \frac{1}{L}\left[\frac{1}{6}L^3 - \frac{1}{2}L^2 + \int_0^L f(x)dx\right] \tag{50}$$

$$c_2 = \frac{1}{6}L^2 - \frac{1}{2}L + \frac{1}{L}\int_0^L f(x)dx \tag{51}$$

From (51), it follows that our steady state distribution is:

$$u(x,t) = -\frac{1}{2}x^2 + x + \frac{1}{6}L^2 - \frac{1}{2}L + \frac{1}{L}\int_0^L f(x)dx \tag{52}$$

(b) $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$, $u(x,0) = f(x)$, $\frac{\partial u}{\partial x}(0,t) = 1$, and $\frac{\partial u}{\partial x}(L,T) = \beta$

We start with our equilibrium equation,

$$0 = \frac{\partial^2 u}{\partial x^2}, \tag{53}$$

which becomes

$$0 = \frac{\partial u}{\partial x} + c_1 \tag{54}$$

$$\frac{\partial u}{\partial x} = -c_1 \tag{55}$$

$$0 = u(x) + c_1 x + c_2 \tag{56}$$

$$u(x) = -c_1 x + c_2. \tag{57}$$

From our BC's,

$$\frac{\partial u}{\partial x}(0,t) = 1 = -c_1 \tag{58}$$

$$\frac{\partial u}{\partial x}(L,T) = \beta = -c_1. \tag{59}$$

If $\beta = 1 = -c_1$, we can continue.

$$\frac{E(t)}{c\rho} = \int_0^L u dx \tag{60}$$

So,

$$\frac{d}{dt}\frac{E(t)}{c\rho} = \frac{d}{dt}\int_0^L u dx = \int_0^L \frac{\partial u}{\partial t}dx. \tag{61}$$

But,

$$\int_0^L \frac{\partial u}{\partial t}dx = \int_0^L \frac{\partial^2 u}{\partial x^2}dx \tag{62}$$

Plugging in our BCs,

$$\frac{1}{c\rho}\frac{dE}{dt} = \beta - 1 \tag{63}$$

However, since $\beta = 1$, (45) becomes:

$$\frac{1}{c\rho}\frac{dE}{dt} = 0. \tag{64}$$

So in this state ($\beta = 1 = -c_1$), we can derive a steady state distribution.

$$\int_0^L f(x)dx = \int_0^L -x + c_2 dx = \left[-\frac{1}{2}x^2 + c_2 x \right]\Bigg|_0^L \tag{65}$$

$$\int_0^L f(x)dx = -\frac{1}{2}L^2 + c_2 L \tag{66}$$

$$c_2 = \frac{1}{2}L + \frac{1}{L}\int_0^L f(x)dx. \tag{67}$$

Thus, our steady state solution is:

$$u(x,t) = -x + \frac{1}{2}L + \frac{1}{L}\int_0^L f(x) \tag{68}$$

Problem 1.4.10: Suppose $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + 4$, $u(x,0) = f(x)$, $\frac{\partial u}{\partial x}(0,t) = 5$, and $\frac{\partial u}{\partial x}(L,t) = 6$. Calculate the total thermal energy in the one-dimensional rod (as a function of time).

We know that total thermal energy is

$$E(t) = \int_0^L u(x,t)dx. \tag{69}$$

$$\frac{dE}{dt} = \frac{d}{dt}\int_0^L u(x,t)dx = \int_0^L \frac{\partial u}{\partial t}dx. \tag{70}$$

But by what we're given,

$$\int_0^L \frac{\partial u}{\partial t}dx = \int_0^L \frac{\partial^2 u}{\partial x^2} + 4dx \tag{71}$$

$$= \left[\frac{\partial u}{\partial x} + 4x\right]\Big|_0^L \tag{72}$$

$$= [6-5] + 4L. \tag{73}$$

Thus,

$$\frac{dE}{dt} = 4L + 1. \tag{74}$$

Integrating (74), we get

$$E(t) = (4L+1)t + c_1, \tag{75}$$

which at $t = 0, x = x$ has

$$E(0) = f(x) = c_1. \tag{76}$$

Over the whole domain of x though, we really want

$$c_1 = \int_0^L f(x)dx, \tag{77}$$

so that E is a function of time. Thus,

$$E(t) = (4L+1)t + \int_0^L f(x)dx \tag{78}$$

is our total thermal energy at any point in time.