Math 440 Exam 3 Practice Test

November 17, 2025

Timothy Tarter
James Madison University
Department of Mathematics

1 Given $f(x)$, graph $f(x)$, its Fourier series, and its periodic even / odd extensions.

2 Given $f(x)$, write down the Fourier series including coefficients, don't bother integrating anything.

3 Solve a PDE using method of eigenvalue expansion, justifying every differentiation.

4 State the criteria for term-by-term differentiation for even, odd, and full periodic extensions.

5 Prove term-by-term differentiation for Fourier series (sine, cosine, or full).

6 Show that even if f is continuous with f' piecewise smooth, f is not term-by-term differentiable unless $f(-L) = f(L)$.

7 Prove the following:

$$\int_{-L}^{L} e^{-i\frac{n\pi x}{L}} e^{i\frac{n\pi x}{L}} dx = \begin{cases} 0; \text{ if } m \neq n \\ 2L; \text{ if } m = n \end{cases}$$

1. Given $f(x)$, graph f, its Fourier series, and its periodic even/odd extensions

Let $f : (0, L) \to \mathbb{R}$ be given. Its $2L$–periodic extension is

$$\widetilde{f}(x + 2L) = \widetilde{f}(x), \qquad \widetilde{f}(x) = f(x) \text{ for } x \in (0, L).$$

The even and odd $2L$–periodic extensions f_e, f_o of f are

$$f_e(x) = \begin{cases} f(x), & x \in [0, L], \\ f(-x), & x \in [-L, 0), \end{cases} \qquad f_e(x + 2L) = f_e(x),$$

$$f_o(x) = \begin{cases} f(x), & x \in [0, L], \\ -f(-x), & x \in [-L, 0), \end{cases} \qquad f_o(x + 2L) = f_o(x).$$

To *graph*:

- Plot f on $(0, L)$.
- Reflect f evenly about $x = 0$ to get f_e on $(-L, L)$, then repeat every interval of length $2L$.
- Reflect f oddly about $x = 0$ to get f_o on $(-L, L)$, then repeat every interval of length $2L$.

If $\widetilde{f}$ is piecewise smooth on $[-L, L]$, its Fourier series is

$$\widetilde{f}(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} \big(a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L}\big),$$

and we can plot the partial sums

$$S_N(x) = \frac{a_0}{2} + \sum_{n=1}^{N} \big(a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L}\big)$$

on a few periods to see the approximation.

2. Fourier series formulas

For a $2L$–periodic, piecewise smooth function f we have:

Real (sine–cosine) series

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} \Big(a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L}\Big),$$

$$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos \frac{n\pi x}{L} \, dx, \qquad n \geq 0,$$

$$b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin \frac{n\pi x}{L} \, dx, \qquad n \geq 1.$$

Cosine series (even extension on $(0, L)$)

If f is given on $(0, L)$ and extended evenly, then

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{L},$$

with

$$a_n = \frac{2}{L} \int_0^L f(x) \cos \frac{n\pi x}{L}\, dx, \qquad n \geq 0.$$

Sine series (odd extension on $(0, L)$)

If f is given on $(0, L)$ and extended oddly, then

$$f(x) \sim \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L},$$

with

$$b_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L}\, dx, \qquad n \geq 1.$$

Complex form

For $2L$–periodic f,

$$f(x) \sim \sum_{n=-\infty}^{\infty} c_n e^{in\pi x/L}, \qquad c_n = \frac{1}{2L} \int_{-L}^{L} f(x) e^{-in\pi x/L}\, dx.$$

3. Example PDE via eigenvalue expansion (heat equation)

Consider the heat equation

$$u_t = \kappa u_{xx}, \quad 0 < x < L,\ t > 0,$$

with boundary and initial conditions

$$u(0, t) = 0, \quad u(L, t) = 0, \quad u(x, 0) = f(x).$$

Separation of variables

Seek $u(x, t) = X(x)T(t)$ with $X \not\equiv 0$, $T \not\equiv 0$. Then

$$X(x)T'(t) = \kappa X''(x)T(t)$$

and division by κXT gives

$$\frac{T'}{\kappa T} = \frac{X''}{X} = -\lambda.$$

Thus

$$X'' + \lambda X = 0, \qquad X(0) = 0,\ X(L) = 0,$$

and

$$T' + \kappa\lambda T = 0.$$

Spatial eigenvalue problem

Nontrivial solutions of $X'' + \lambda X = 0$ with $X(0) = X(L) = 0$ occur only for

$$\lambda_n = \left(\frac{n\pi}{L}\right)^2, \quad n = 1, 2, \dots,$$

with eigenfunctions

$$X_n(x) = \sin\frac{n\pi x}{L}.$$

Time factor

For each n,

$$T_n'(t) + \kappa\lambda_n T_n(t) = 0 \quad \Rightarrow \quad T_n(t) = e^{-\kappa\lambda_n t} = e^{-\kappa(n\pi/L)^2 t}.$$

Series solution

Form the eigenfunction expansion

$$u(x,t) = \sum_{n=1}^{\infty} b_n e^{-\kappa(n\pi/L)^2 t} \sin\frac{n\pi x}{L},$$

where the coefficients b_n are chosen to match $u(x,0) = f(x)$:

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L} \quad \Rightarrow \quad b_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx.$$

Justifying differentiations

Assume f is piecewise smooth so that the sine coefficients b_n decay at least like $1/n$. Then for any $t_0 > 0$ the factors $e^{-\kappa(n\pi/L)^2 t_0}$ decay exponentially in n, and

$$\sum_{n=1}^{\infty} \left| b_n e^{-\kappa(n\pi/L)^2 t_0} \right|$$

converges absolutely and uniformly in $x \in [0, L]$. Hence we may differentiate term-by-term:

$$u_t(x,t) = \sum_{n=1}^{\infty} -\kappa\left(\frac{n\pi}{L}\right)^2 b_n e^{-\kappa(n\pi/L)^2 t} \sin\frac{n\pi x}{L},$$

$$u_{xx}(x,t) = \sum_{n=1}^{\infty} -\left(\frac{n\pi}{L}\right)^2 b_n e^{-\kappa(n\pi/L)^2 t} \sin\frac{n\pi x}{L}.$$

Thus $u_t = \kappa u_{xx}$ for all $t > 0$, and u satisfies the PDE.

4. Criteria for term-by-term differentiation

Let f be $2L$–periodic and piecewise C^1 on $[-L, L]$.

Full Fourier series

If f is continuous on $\mathbb{R}$, $2L$–periodic, piecewise C^1 on $[-L, L]$, and f' is piecewise continuous and $2L$–periodic, then the Fourier series

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} \big(a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L}\big)$$

may be differentiated term-by-term for $x \in (-L, L)$, and the resulting series converges (at each point of continuity of f') to $f'(x)$. A necessary condition for f to be C^1 and $2L$–periodic is

$$f(-L) = f(L).$$

Cosine series (even extension)

Let $f : [0, L] \to \mathbb{R}$ be continuous, with f' piecewise continuous on $(0, L)$, and assume the even extension f_e is C^1 and $2L$–periodic. Equivalently,

$$f_e'(-L) = f_e'(L), \qquad f_e' \text{ is piecewise continuous.}$$

Then the cosine series of f may be differentiated term-by-term, and the result is the sine series of f' (the derivative of the even extension).

A necessary condition for f_e to be C^1 is

$$f'(0) = 0, \qquad f'(L) = 0.$$

Sine series (odd extension)

Let $f : [0, L] \to \mathbb{R}$ be continuous with f' piecewise continuous on $(0, L)$, and assume its odd extension f_o is C^1 and $2L$–periodic. Equivalently,

$$f(0) = 0, \qquad f(L) = 0, \qquad f_o' \text{ is piecewise continuous.}$$

Then the sine series of f may be differentiated term-by-term, and the result is the cosine series of f' (the derivative of the odd extension).

5. Proof of term-by-term differentiation (sketch)

Let f be $2L$–periodic, continuous, piecewise C^1, with f' piecewise continuous and $2L$–periodic, and $f(-L) = f(L)$.

The Fourier coefficients satisfy

$$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos \frac{n\pi x}{L}\, dx,$$
$$b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin \frac{n\pi x}{L}\, dx.$$

Integrating by parts (using periodicity and $f(-L) = f(L)$ so that boundary terms cancel) gives

$$\begin{aligned} a_n &= \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx \\ &= \frac{1}{L}\left[f(x)\,\frac{L}{n\pi}\sin\frac{n\pi x}{L}\right]_{-L}^{L} - \frac{1}{L}\int_{-L}^{L} f'(x)\,\frac{L}{n\pi}\sin\frac{n\pi x}{L}\,dx \\ &= -\frac{1}{n\pi}\int_{-L}^{L} f'(x)\sin\frac{n\pi x}{L}\,dx = -\frac{L}{n\pi}\,b_n', \end{aligned}$$

where

$$b_n' = \frac{1}{L}\int_{-L}^{L} f'(x)\sin\frac{n\pi x}{L}\,dx$$

is the n-th sine coefficient of f'. A similar computation shows

$$b_n = \frac{L}{n\pi}\,a_n',$$

where a_n' is the n-th cosine coefficient of f'.

Hence

$$\frac{d}{dx}\left(\frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n\cos\frac{n\pi x}{L} + b_n\sin\frac{n\pi x}{L})\right) = \sum_{n=1}^{\infty}\Big(-\frac{n\pi}{L}a_n\sin\frac{n\pi x}{L} + \frac{n\pi}{L}b_n\cos\frac{n\pi x}{L}\Big)$$

is exactly the Fourier series of f'.

Since f' is piecewise continuous and periodic, its Fourier series converges to f' at every point of continuity of f' (and to the midpoint of one-sided limits at discontinuities). Therefore the differentiated series converges to f' wherever f' is continuous. The sine and cosine cases follow by applying this result to the even or odd extension.

6. Necessity of $f(-L) = f(L)$

Suppose f is continuous, $2L$–periodic, f' is piecewise smooth, and the Fourier series of f is term-by-term differentiable, with derivative series converging to f'.

If the term-by-term derivative series converges uniformly on $[-L, L]$, then the resulting function must be $2L$–periodic and continuous. Hence f' must be continuous and $2L$–periodic, so f must be C^1 and $2L$–periodic (with continuous periodic derivative). A necessary condition for f to be C^1 and $2L$–periodic is

$$f(-L) = f(L),$$

since otherwise f has a jump discontinuity at $x = -L$ (equivalently $x = L$) and cannot have a continuous derivative there.

Equivalently: if $f(-L) \neq f(L)$, the $2L$–periodic extension has a jump at the endpoints. The Fourier series of f then converges at $x = \pm L$ to the midpoint of the jump,

$$\frac{f(L^-) + f(-L^+)}{2} = \frac{f(L) + f(-L)}{2},$$

and cannot represent a C^1 periodic function. Thus in this case there cannot exist a term-by-term derivative series converging to f' on $[-L, L]$.

Therefore, $f(-L) = f(L)$ is a necessary condition for term-by-term differentiability of the Fourier series.

7. Orthogonality integral

We want to prove

$$\int_{-L}^{L} e^{-i\frac{m\pi}{L}x} e^{i\frac{n\pi}{L}x}\,dx = \begin{cases} 0, & m \neq n, \\ 2L, & m = n. \end{cases}$$

Note

$$e^{-i\frac{m\pi}{L}x} e^{i\frac{n\pi}{L}x} = e^{i\frac{(n-m)\pi}{L}x}.$$

Case $m = n$

If $m = n$, the integrand is $e^0 = 1$, so

$$\int_{-L}^{L} e^{-i\frac{m\pi}{L}x} e^{i\frac{m\pi}{L}x}\,dx = \int_{-L}^{L} 1\,dx = 2L.$$

Case $m \neq n$

If $m \neq n$, write $k = n - m \neq 0$. Then

$$\begin{aligned} \int_{-L}^{L} e^{i\frac{(n-m)\pi}{L}x}\,dx = \int_{-L}^{L} e^{i\frac{k\pi}{L}x}\,dx &= \left[\frac{L}{ik\pi} e^{i\frac{k\pi}{L}x}\right]_{x=-L}^{x=L} \\ &= \frac{L}{ik\pi}\left(e^{ik\pi} - e^{-ik\pi}\right) = \frac{L}{ik\pi} \cdot 2i\sin(k\pi) \\ &= \frac{2L}{k\pi}\sin(k\pi) = 0, \end{aligned}$$

since $k \in \mathbb{Z} \setminus \{0\}$ implies $\sin(k\pi) = 0$.

This proves the desired orthogonality relation.