

# Math 440 Exam 2 Practice Test

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## 1 Describe / Explain:

### 1.1 Laplace's Equation (include well-posedness, uniqueness, and the mean value proposition).

Laplace's equation states that the sum of the first partials are equal to zero, i.e.,

$$0 = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} \quad (1)$$

Well-posedness: there exists a unique solution which depends continuously on non-homogeneous data.

Uniqueness: solutions to laplace's equation are unique.

Mean value proposition: the temperature at the origin is the average of the temperature at the boundary.

### 1.2 Linear Operators, provide an example of something which is a linear operator, and provide an example of something which is not a linear operator.

A linear operator,  $\mathcal{U} : V \rightarrow V$  where  $V$  is a vector space, has the property that for any  $\vec{x}, \vec{y} \in V$  and any scalar  $c$  in the field of  $V$ ,

1.  $\mathcal{U}(\vec{x} + \vec{y}) = \mathcal{U}(\vec{x}) + \mathcal{U}(\vec{y})$
2.  $\mathcal{U}(c\vec{x}) = c\mathcal{U}(\vec{x})$ .

Example:

$$L(u) = \frac{\partial}{\partial x} \left[ K_0(x) \frac{\partial u}{\partial x} \right] \quad (2)$$

Non-Example:

$$L(u) = \frac{\partial}{\partial x} \left[ K_0(u, x) \frac{\partial u}{\partial x} \right] \quad (3)$$

where  $K_0$  is a non-constant function.

## 2 Prove Orthogonality of Sines.

We want to show that

$$\int_0^L \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{m\pi x}{L}\right) dx = \begin{cases} 0; & \text{if } n \neq m \\ \frac{L}{2}; & \text{if } n = m. \end{cases} \quad (4)$$

$$\int_0^L \cos\left(\frac{n\pi x}{L}\right) \cos\left(\frac{m\pi x}{L}\right) dx = \begin{cases} 0; & \text{if } n \neq m \\ \frac{L}{2}; & \text{if } n = m \neq 0 \\ L; & \text{if } n = m = 0 \end{cases} \quad (5)$$

Starting off with (4), if  $n = m$ , then

$$\int_0^L \sin^2\left(\frac{n\pi x}{L}\right) dx = \int_0^L \frac{1}{2}(1 - \cos(2\frac{n\pi x}{L})) dx \quad (6)$$

$$= \frac{1}{2} \left[ x - \frac{L}{2n\pi} \sin\left(\frac{2n\pi x}{L}\right) \right] \Big|_0^L \quad (7)$$

$$= \frac{1}{2} [L - 0] = \frac{L}{2} \quad (8)$$

as desired. Alternatively, if  $n \neq m$ , then

$$\int_0^L \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{m\pi x}{L}\right) dx \quad (9)$$

$$= \int_0^L \frac{1}{2} [\cos\left(\frac{n\pi x}{L} - \frac{m\pi x}{L}\right) - \cos\left(\frac{n\pi x}{L} + \frac{m\pi x}{L}\right)] dx \quad (10)$$

$$= \frac{1}{2} \int_0^L \cos\left(\frac{\pi x(n-m)}{L}\right) - \cos\left(\frac{\pi x(n+m)}{L}\right) dx \quad (11)$$

$$= \frac{1}{2} \left[ \frac{L}{\pi(n-m)} \sin\left(\frac{\pi x(n-m)}{L}\right) - \frac{L}{\pi(n-m)} \sin\left(\frac{\pi x(n+m)}{L}\right) \right] \Big|_0^L \quad (12)$$

$$= 0 - 0 = 0 \quad (13)$$

as desired.

Moving to (5), if  $n \neq m$ , it works pretty much the same way as with sines. If  $n = m \neq 0$ , then

$$\int_0^L \cos^2\left(\frac{n\pi x}{L}\right) dx = \quad (14)$$

$$= \int_0^L \frac{1}{2}(1 + \sin\left(\frac{2n\pi x}{L}\right)) dx \quad (15)$$

$$= \frac{1}{2} \left[ x - \frac{L}{2n\pi} \cos\left(\frac{2n\pi x}{L}\right) \right] \Big|_0^L \quad (16)$$

$$= \frac{1}{2}[L + (-1 + 1)] \quad (17)$$

$$= \frac{L}{2} \quad (18)$$

as desired. Then finally, if  $n = m = 0$ ,

$$\int_0^L \cos^2(0) dx = \int_0^L 1 dx = L. \quad (19)$$

**3 Solve  $\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2}$  with certain BC's and IC's, addressing  $\lambda > 0, = 0, < 0$ .**

**3.1**  $u(0, t) = u(L, t) = 0$

Gives sine solutions.

**3.2**  $\frac{\partial u}{\partial x}(0, t) = 0 = \frac{\partial u}{\partial x}(L, t)$

Gives cosine solutions.

**3.3**  $u(0, t) = u(L, t) = 0$  and  $\frac{\partial u}{\partial x}(0, t) = 0 = \frac{\partial u}{\partial x}(L, t)$

Gives sine and cosine solutions.

**4 Solve  $\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2}$  at equilibrium in polar coordinates over the interval  $(-L, L)$ .**

We begin with the one-dimensional heat equation

$$u_t = \kappa u_{xx}, \quad x \in [-L, L],$$

and bend the wire so that its endpoints are in perfect thermal contact:

$$u(-L, t) = u(L, t), \quad u_x(-L, t) = u_x(L, t).$$

These are the *periodic boundary conditions*, corresponding physically to a wire bent into a circle of circumference  $2L$ .

At equilibrium, the temperature no longer depends on time, so

$$u_t = 0 \implies u_{xx} = 0.$$

Integrating twice gives

$$u_{\text{eq}}(x) = Ax + B.$$

The periodicity condition  $u_{\text{eq}}(-L) = u_{\text{eq}}(L)$  yields

$$A(-L) + B = A(L) + B \implies A = 0.$$

Hence

$$u_{\text{eq}}(x) = B = \text{constant.}$$

To determine  $B$ , note that the total heat (or average temperature) is conserved for periodic boundary conditions:

$$\frac{d}{dt} \int_{-L}^L u(x, t) dx = \kappa [u_x(x, t)]_{x=-L}^{x=L} = \kappa (u_x(L, t) - u_x(-L, t)) = 0.$$

Thus, the equilibrium constant equals the initial mean temperature:

$$u_{\text{eq}}(x) = \frac{1}{2L} \int_{-L}^L u_0(x) dx.$$

Therefore, the bent wire reaches a uniform equilibrium temperature equal to its initial average.

## 5 Solve Laplace's Equation on a rectangle, with LHS, Top, and Bottom = 0, RHS = $g_2(y)$ .

We want to start with Laplace's 2D equation,

$$0 = k \left[ \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right]. \quad (20)$$

If  $u(x, t) = X(x)Y(y)$ , then

$$0 = X''Y + XY'' \quad (21)$$

implies that

$$\frac{X''}{X} = \lambda = \frac{-Y''}{Y}. \quad (22)$$

Thus,

$$X'' - \lambda X = 0 \quad (23)$$

and

$$Y'' + \lambda Y = 0 \quad (24)$$

are our system of ODE's. Now we want to specify our boundary conditions.

This results in the following BCs:

- $u(0, y) = 0$
- $u(L, y) = g_2(y)$
- $u(x, 0) = 0$
- $u(x, H) = 0.$

Accordingly,

$$Y(y) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi y}{H}\right). \quad (25)$$

Then,

$$X'' - \lambda X = 0 \quad (26)$$

implies that

$$X(x) = A \sinh(\sqrt{\lambda}x) + B \cosh(\sqrt{\lambda}x). \quad (27)$$

Since

$$X(0) = B = 0, \quad (28)$$

then

$$X(x) = A \sinh\left(\frac{n\pi x}{H}\right). \quad (29)$$

But,

$$X(L) = g_2(y) = A \sinh\left(\frac{n\pi L}{H}\right) \quad (30)$$

$$A = \frac{g_2(y)}{\sinh\left(\frac{n\pi L}{H}\right)}. \quad (31)$$

So

$$u(x, y) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi y}{H}\right) \frac{g_2(y) \sinh\left(\frac{n\pi x}{H}\right)}{\sinh\left(\frac{n\pi L}{H}\right)} \quad (32)$$

where

$$A_n = \frac{2}{H} \int_0^H g_2(y) \sin\left(\frac{n\pi y}{H}\right) dy. \quad (33)$$

## 6 Extra Resources Which You Gotta Know or You Might Be Cooked:

Trig Identities:

$$\sin(a + b) = \sin(a)\cos(b) + \cos(a)\sin(b) \quad (34)$$

$$\cos(a + b) = \cos(a)\cos(b) - \sin(a)\sin(b) \quad (35)$$

$$\sin(a - b) = \sin(a)\cos(b) - \cos(a)\sin(b) \quad (36)$$

$$\cos(a - b) = \cos(a)\cos(b) + \sin(a)\sin(b) \quad (37)$$

If BC's are

- $u(0, t) = 0 = u(L, t)$  then

$$u(x, t) = \sum_{n=1}^{\infty} A_n e^{-kt(\frac{n\pi}{L})^2} \sin\left(\frac{n\pi x}{L}\right) \quad (38)$$

- $\frac{\partial u}{\partial x}(0, t) = 0 = \frac{\partial u}{\partial x}(L, t)$  then

$$u(x, t) = \sum_{n=0}^{\infty} B_n e^{-kt(\frac{n\pi}{L})^2} \cos\left(\frac{n\pi x}{L}\right) \quad (39)$$

where

$$A_n = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx \quad (40)$$

$$\text{and} \quad (41)$$

$$B_n = \frac{2}{L} \int_0^L f(x) \cos\left(\frac{n\pi x}{L}\right) dx. \quad (42)$$

Orthogonality of sines and cosines (formulas):

$$\int_0^L \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{m\pi x}{L}\right) dx = \begin{cases} 0; & \text{if } n \neq m \\ \frac{L}{2}; & \text{if } n = m. \end{cases} \quad (43)$$

$$\int_0^L \cos\left(\frac{n\pi x}{L}\right) \cos\left(\frac{m\pi x}{L}\right) dx = \begin{cases} 0; & \text{if } n \neq m \\ \frac{L}{2}; & \text{if } n = m \neq 0 \\ L; & \text{if } n = m = 0 \end{cases} \quad (44)$$