Problem 1.5.10: Determine the equilibrium temperature distribution inside a circle ($r \leq r_0$) if the boundary is fixed at a temperature T_0.

Since the circle is axially symmetric so $u(r, \theta, z, t) = u(r, t)$. Moreover, since we're at equilibrium, our PDE is

$$0 = \frac{k}{r}\frac{\partial}{\partial r}(r\frac{\partial u}{\partial r}). \tag{1}$$

Since we're working over a division ring, we can multiply both sides by r without worry. So our PDE is

$$0 = k\frac{\partial}{\partial r}(r\frac{\partial u}{\partial r}). \tag{2}$$

Integrating, we get

$$\frac{\partial u}{\partial r} = \frac{c_1}{r} \tag{3}$$

and

$$u(r) = c_1 ln(r) + c_2. \tag{4}$$

In order for u to be less than infinity as $r \to 0$, we need $c_1 = 0$. Then

$$u(r) = c_2. \tag{5}$$

But

$$u(r_0) = T_0 = c_2 \tag{6}$$

so

$$\boxed{u(r) = T_0} \tag{7}$$

is our final temperature distribution.

Problem 1.5.11: Consider

$$\frac{\partial u}{\partial t} = \frac{k}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right), \quad a < r < b,$$

subject to $u(r,0) = f(r)$, $\frac{\partial u}{\partial r}(a,t) = \beta$, and $\frac{\partial u}{\partial r}(b,t) = 1$. For what value(s) of β does an equilibrium temperature distribution exist?

Since we are at equilibrium,

$$0 = k\frac{\partial}{\partial r}(r\frac{\partial u}{\partial r}). \tag{8}$$

Integrating, we get

$$c_1 = r\frac{\partial u}{\partial r}. \tag{9}$$

Using our BCs, we find that

$$\frac{\partial u}{\partial r}(a,t) = \beta = \frac{c_1}{r} \tag{10}$$

$$\Rightarrow c_1 = r\beta. \tag{11}$$

Plugging that in to (9), we find

$$\frac{\partial u}{\partial r} = \beta \tag{12}$$

But

$$\frac{\partial u}{\partial r}(b,t) = 1 = \beta \tag{13}$$

so there only exists an equilibrium distribution for $\beta = 1$.

Problem 2.2.1: Show that any linear combination of linear operators is a linear operator.

Let $\mathbb{L}, \mathbb{V}$ be arbitrary linear operators. Then let u, v be arbitrary elements in their respective vector space domains and c be a scalar in the field which the vector space is defined on. Then

- $\mathbb{L}(u+v) = \mathbb{L}(u) + \mathbb{L}(v)$
- $\mathbb{L}(cu) = c\mathbb{L}(u)$
- $\mathbb{V}(u+v) = \mathbb{V}(u) + \mathbb{V}(v)$
- $\mathbb{V}(cu) = c\mathbb{V}(u)$.

Let $\mathbb{W}(x) = \mathbb{L}(x) + \mathbb{V}(x)$. Then,

$$\begin{aligned}
\mathbb{W}(x+y) &= \mathbb{L}(x+y) + \mathbb{V}(x+y) && (14)\\
&= \mathbb{L}(x) + \mathbb{L}(y) + \mathbb{V}(x) + \mathbb{V}(y) && (15)\\
&= \mathbb{L}(x) + \mathbb{V}(x) + \mathbb{L}(y) + \mathbb{V}(y) && (16)\\
&= \mathbb{W}(x) + \mathbb{W}(y) && (17)
\end{aligned}$$

as desired. Similarly,

$$\begin{aligned}
\mathbb{W}(cx) &= \mathbb{L}(cx) + \mathbb{V}(cx) && (18)\\
&= c\mathbb{L}(x) + c\mathbb{V}(x) && (19)\\
&= c\mathbb{W}(x) && (20)
\end{aligned}$$

as desired.

$\square$

Problem 2.2.2: Show that

(a) $L(u) = \frac{\partial}{\partial x}\left[K_0(x)\frac{\partial u}{\partial x}\right]$ is a linear operator

Starting with the basics,

$$\begin{aligned}
\mathbb{L}(u+v) &= \frac{\partial}{\partial x}[K_0(x)\frac{\partial(u+v)}{\partial x}] && (21)\\
&= \frac{\partial}{\partial x}[K_0(x)[\frac{\partial u}{\partial x} + \frac{\partial v}{\partial x}]] && (22)\\
&= \frac{\partial}{\partial x}[K_0(x)\frac{\partial u}{\partial x} + K_0(x)\frac{\partial v}{\partial x}] && (23)\\
&= \frac{\partial}{\partial x}[K_0(x)\frac{\partial u}{\partial x}] + \frac{\partial}{\partial x}[K_0(x)\frac{\partial v}{\partial x}] && (24)\\
&= \mathbb{L}(u) + \mathbb{L}(v) && (25)
\end{aligned}$$

and

$$\begin{aligned}
\mathbb{L}(cu) &= \frac{\partial}{\partial x}[K_0(x)\frac{\partial cu}{\partial x}] && (26)\\
&= \frac{\partial}{\partial x}[cK_0(x)\frac{\partial u}{\partial x}] && (27)\\
&= c\frac{\partial}{\partial x}[K_0(x)\frac{\partial u}{\partial x}] && (28)\\
&= c\mathbb{L}(u) && (29)
\end{aligned}$$

as desired.

□

(b) and usually $L(u) = \frac{\partial}{\partial x}\left[K_0(x,u)\frac{\partial u}{\partial x}\right]$ is not a linear operator.

$$\mathbb{L}(u+v) = \frac{\partial}{\partial x}\left[K_0(x,u+v)\frac{\partial(u+v)}{\partial x}\right] \tag{30}$$

$$= \frac{\partial}{\partial x}\left[K_0(x,u+v)[\frac{\partial u}{\partial x} + \frac{\partial v}{\partial x}]\right] \tag{31}$$

$$= \frac{\partial}{\partial x}\left[K_0(x,u+v)\frac{\partial u}{\partial x} + K_0(x,u+v)\frac{\partial v}{\partial x}\right] \tag{32}$$

$$= \frac{\partial K_0(x,u+v)}{\partial x}\frac{\partial u}{\partial x} + K_0(x,u+v)\frac{\partial^2 u}{\partial x^2} + \frac{\partial K_0(x,u+v)}{\partial x}\frac{\partial v}{\partial x} + K_0(x,u+v)\frac{\partial^2 v}{\partial x^2} \tag{33}$$

$$= K_0(x,u+v)[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 v}{\partial x^2}] + \frac{\partial K_0(x,u+v)}{\partial x}[\frac{\partial u}{\partial x} + \frac{\partial v}{\partial x}] \tag{34}$$

which is not $\mathbb{L}(u) + \mathbb{L}(v)$.

□

Problem 2.2.3: Show that $\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} + Q(u,x,t)$ is linear if $Q = \alpha(x,t)u + \beta(x,t)$ and, in addition, homogeneous if $\beta(x,t) = 0$.

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} + \alpha(x,t)u + \beta(x,t) \tag{35}$$
$$\frac{\partial u}{\partial t} - k\frac{\partial^2 u}{\partial x^2} - \alpha(x,t)u = \beta(x,t) \tag{36}$$
$$\tag{37}$$

is linear. If $\beta = 0$, it's homogeneous, both by definition.

Problem 2.2.4: In this exercise we derive superposition principles for non homogeneous problems.

(a) Consider $L(u) = f$. If u_p is a particular solution, $L(u_p) = f$, and if u_1 and u_2 are homogeneous solutions, $L(u_i) = 0$, show that $u = u_p + c_1 u_1 + c_2 u_2$ is another particular solution.

$$
\begin{aligned}
L(u_p + c_1 u_1 + c_2 u_2) &= L(u_p) + c_1 L(u_1) + c_2 L(u_2) && (38)\\
&= f + 0 + 0 && (39)\\
&= f && (40)
\end{aligned}
$$

as desired.

□

(b) If $L(u) = f_1 + f_2$, where u_{pi} is a particular solution corresponding to f_i, what is a particular solution for $f_1 + f_2$?

$c_1u_1 + c_2u_2$ is such a particular solution since

$$L(c_1u_1 + c_2u_2) = c_1L(u_1) + c_2L(u_2) \tag{41}$$
$$= c_1f_1 + c_2f_2. \tag{42}$$

Problem 2.3.1 b,d: For the following partial differential equations, what ordinary differential equations are implied by the method of separation of variables?

(b) $\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} - v_0\frac{\partial u}{\partial x}$

Let

$$u(x,t) = X(x)T(t). \tag{43}$$

Then

$$\frac{\partial u}{\partial t} = X(x)T'(t) \tag{44}$$

and

$$\frac{\partial u}{\partial x} = X'(x)T(t) \tag{45}$$

$$\frac{\partial^2 u}{\partial x^2} = X''(x)T(t) \tag{46}$$

$$. \tag{47}$$

Plugging these equations in,

$$X(x)T'(t) = kX''(x)T(t) - v_0X'(x)T(t) \tag{48}$$

$$X(x)T'(t) = kT(t)[X''(x) - v_0X'(x)] \tag{49}$$

$$\frac{T'(t)}{kT(t)} = \lambda = \frac{X''(x) - v_0X'(x)}{X(x)} \tag{50}$$

we find that we get the system of equations,

$$T'(t) - \lambda kT(t) = 0 \tag{51}$$

$$X''(x) - v_0X'(x) - \lambda X(x) = 0. \tag{52}$$

(d) $\frac{\partial u}{\partial t} = \frac{k}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right)$ Let

$$u(r,t) = R(r)T(t). \tag{53}$$

Then

$$\frac{\partial u}{\partial t} = R(r)T'(t) \tag{54}$$
$$\frac{\partial u}{\partial r} = R'(r)T(t) \tag{55}$$
$$\frac{\partial^2 u}{\partial r^2} = R''(r)T(t). \tag{56}$$

Plugging these in,

$$R(r)T'(t) = \frac{k}{r^2}\frac{\partial}{\partial r}\left(r^2 R'(r)T(t)\right) \tag{57}$$
$$R(r)T'(t) = \frac{kT(t)}{r^2}\frac{\partial}{\partial r}\left(r^2 R'(r)\right) \tag{58}$$
$$\frac{T'(t)}{kT(t)} = \lambda = \frac{\frac{\partial}{\partial r}(r^2 R'(r))}{r^2 R(r)} \tag{59}$$

which gives us the system of equations,

$$T'(t) - \lambda k T(t) = 0 \tag{60}$$
$$\frac{\partial}{\partial r}(r^2 R'(r)) - \lambda r^2 R(r) = 0. \tag{61}$$

Problem 2.3.2 a,e: Consider the differential equation

$$\frac{d^2\phi}{dx^2} + \lambda\phi = 0,$$

where ϕ is a function of x only. Determine the eigenvalues λ (and corresponding eigenfunctions) if ϕ satisfies the following boundary conditions. Analyze three cases ($\lambda > 0$, $\lambda = 0$, and $\lambda < 0$). You may assume that the eigenvalues are real.

(a) $\phi(0) = 0$ and $\phi(\pi) = 0$

Let

$$\phi(x) = e^{rx}. \tag{62}$$

Then

$$\frac{\partial\phi}{\partial x^2} = r^2 e^{rx} \tag{63}$$

implies

$$r^2 e^{rx} + \lambda e^{rx} = 0 \tag{64}$$

which gives the characteristic equation

$$r^2 + \lambda = 0. \tag{65}$$

So our eigenvalues are $r = \pm\sqrt{-\lambda}$.

Case: $\lambda < 0$

If $\lambda < 0$, then $r = \pm\sqrt{|\lambda|}$ which gives us the solution

$$\phi(x) = Ae^{\sqrt{|\lambda|}x} + Be^{-\sqrt{|\lambda|}x}. \tag{66}$$

To solve for A and B, we use our BCs,

$$\phi(0) = A + B = 0 \Rightarrow A = -B \tag{67}$$

which implies

$$\phi(\pi) = 0 = A(e^{\sqrt{|\lambda|}\pi} - e^{-\sqrt{|\lambda|}\pi}). \tag{68}$$

Since $e^{\sqrt{|\lambda|}\pi} - e^{-\sqrt{|\lambda|}\pi}$ can't equal zero, then $A = 0 = B$. So

$$\phi(x) = 0 \iff \lambda < 0. \tag{69}$$

Case: $\lambda = 0$

If $\lambda = 0$, then $r = 0$ and

$$\frac{\partial^2\phi}{\partial x^2} = 0 \tag{70}$$

implies that

$$\phi(x) = Ax + B. \tag{71}$$

Since

$$\phi(0) = 0 = B \tag{72}$$

and

$$\phi(\pi) = 0 = A\pi, \tag{73}$$

then $A = 0 = B$ and

$$\phi(x) = 0 \iff \lambda = 0. \tag{74}$$

Case: $\lambda > 0$

If $\lambda > 0$, then $r = \pm i\sqrt{|\lambda|}$ which gives us the solution

$$\phi(x) = Acos(\sqrt{|\lambda|}x) + Bsin(\sqrt{|\lambda|}x). \tag{75}$$

Since

$$\phi(0) = 0 = A \tag{76}$$

and

$$\phi(\pi) = 0 = Bsin(\sqrt{|\lambda|}\pi), \tag{77}$$

to get a non-trivial solution, then

$$\sqrt{|\lambda|}\pi = n\pi; n \in 1, 2, \ldots \tag{78}$$

so, $\lambda = n^2$ where $n \in 1, 2, \ldots$. Thus,

$$\phi(x) = Bsin(n\pi); n \in 1, 2, \cdots \iff \lambda > 0. \tag{79}$$

(e) $\frac{d\phi}{dx}(0) = 0$ and $\phi(L) = 0$

Again guessing our solution and arriving at equation (65), we find that $r = \pm\sqrt{-\lambda}$.

Case: $\lambda < 0$

If $\lambda < 0$, then $r = \pm\sqrt{|\lambda|}$. Accordingly,

$$\phi(x) = Acosh(\sqrt{|\lambda|}x) + Bsinh(\sqrt{|\lambda|}x) \tag{80}$$

and

$$\frac{\partial\phi}{\partial x} = A\sqrt{|\lambda|}sinh(\sqrt{|\lambda|}x) + B\sqrt{|\lambda|}cosh(\sqrt{|\lambda|}x). \tag{81}$$

Then, since

$$\frac{\partial\phi}{\partial x}(0) = 0 = B\sqrt{|\lambda|} \Rightarrow B = 0 \tag{82}$$

and

$$\phi(L) = 0 = A\sqrt{|\lambda|}cosh(\sqrt{|\lambda|}L) \Rightarrow A = 0 \tag{83}$$

we get the trivial solution,

$$\phi(x) = 0 \iff \lambda < 0. \tag{84}$$

Case: $\lambda = 0$

If $\lambda = 0$, then $r = 0$, which implies that

$$\phi(x) = Ax + B \tag{85}$$

and

$$\frac{\partial\phi}{\partial x} = A. \tag{86}$$

Plugging in our BCs,

$$\frac{\partial\phi}{\partial x}(0) = 0 = A \tag{87}$$

and

$$\phi(L) = 0 = B \tag{88}$$

gives us the trivial solution,

$$\phi(x) = 0 \iff \lambda = 0. \tag{89}$$

Case: $\lambda > 0$

If $\lambda > 0$ then $r = \pm i\sqrt{|\lambda|}$. Accordingly, our general solution is

$$\phi(x) = Acos(\sqrt{|\lambda|}x) + Bsin(\sqrt{|\lambda|}x) \tag{90}$$

with

$$\frac{\partial\phi}{\partial x} = -A\sqrt{|\lambda|}sin(\sqrt{|\lambda|}x) + B\sqrt{|\lambda|}cos(\sqrt{|\lambda|}x). \tag{91}$$

Plugging in our BCs,

$$\frac{\partial \phi}{\partial x}(0) = B\sqrt{|\lambda|} = 0 \Rightarrow B = 0 \tag{92}$$

and

$$\phi(L) = 0 = Acos(\sqrt{|\lambda|}L). \tag{93}$$

In order to have a non-trivial solution,

$$\sqrt{|\lambda|}L = \frac{\pi}{2} + n\pi; n \in 1, 2, \ldots \tag{94}$$

implies

$$\lambda = \left(\frac{\pi(2n+1)}{2L}\right)^2. \tag{95}$$

So,

$$\phi(x) = Acos(\frac{\pi(2n+1)}{2L}x) \iff \lambda > 0 \tag{96}$$

is our solution.

Problem 2.3.3 a,c,d: Consider the heat equation

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2},$$

subject to the boundary conditions $u(0,t) = 0$ and $u(L,t) = 0$. Solve the initial value problem if the temperature is initially

(a) $u(x,0) = 6\sin\left(\frac{9\pi x}{L}\right)$

From stuff we did in class, we know that

$$u(x,t) = \sum_{n=1}^{\infty} a_n e^{-kt(\frac{n\pi}{L})^2} sin(\frac{n\pi x}{L}) \tag{97}$$

is the Fourier series for our solution given our BCs. Then

$$u(x,0) = 6sin(\frac{9\pi x}{L}) = \sum_{n=1}^{\infty} a_n sin(\frac{n\pi x}{L}). \tag{98}$$

By orthogonality of sines,

$$6sin(\frac{9\pi x}{L})sin(\frac{m\pi x}{L}) = \sum_{n=1}^{\infty} a_n sin(\frac{n\pi x}{L})sin(\frac{m\pi x}{L}) \tag{99}$$

implies

$$\int_0^L 6sin(\frac{9\pi x}{L})sin(\frac{m\pi x}{L})dx = \int_0^L \sum_{n=1}^{\infty} a_n sin(\frac{n\pi x}{L})sin(\frac{m\pi x}{L})dx \tag{100}$$

which becomes

$$\int_0^L 6sin(\frac{9\pi x}{L})sin(\frac{m\pi x}{L})dx = \sum_{n=1}^{\infty} a_n \int_0^L sin(\frac{n\pi x}{L})sin(\frac{m\pi x}{L})dx. \tag{101}$$

Since the terms of the infinite series on the right are zero iff $m \neq n$, then

$$\int_0^L 6sin(\frac{9\pi x}{L})sin(\frac{m\pi x}{L})dx = 0 + 0 + \cdots + a_m \int_0^L sin^2(\frac{m\pi x}{L})dx. \tag{102}$$

Using some trig identities, namely,

$$sin(a)sin(b) = \frac{1}{2}(cos(a-b) - cos(a+b)) \tag{103}$$

and

$$sin^2(x) = \frac{1-cos(2x)}{2}, \tag{104}$$

we get that (102) becomes

$$6\int_0^L \frac{1}{2}(cos(\frac{9\pi x}{L} - \frac{m\pi x}{L}) - cos(\frac{9\pi x}{L} + \frac{m\pi x}{L}))dx = a_m \int_0^L \frac{1-cos(2\frac{m\pi x}{L})}{2}dx \tag{105}$$

$$3\int_0^L cos(\frac{\pi x(9-m)}{L}) - cos(\frac{\pi x(9+m)}{L})dx = a_m \int_0^L \frac{1}{2} - \frac{cos(2\frac{m\pi x}{L})}{2}dx \tag{106}$$

$$3\left[\frac{L}{\pi(9-m)}sin(\frac{\pi x(9-m)}{L})) - \frac{L}{\pi(9+m)}sin(\frac{\pi x(9+m)}{L})\right]\Bigg|_0^L = a_m\left[\frac{1}{2}x - \frac{1}{2}\frac{L}{2m\pi}sin(\frac{2m\pi x}{L})\right]\Bigg|_0^L \tag{107}$$

$$3\left[\frac{L}{\pi(9-m)}sin(\frac{\pi x(9-m)}{L})) - \frac{L}{\pi(9+m)}sin(\frac{\pi x(9+m)}{L})\right]\Bigg|_0^L = a_m\frac{L}{2} \tag{108}$$

Notably, the LHS of (108) is zero unless $m = 9$. In this case, we revert back to (106), subbing in $m = 9$.

$$3\int_0^L cos(\frac{\pi x(0)}{L}) - cos(\frac{\pi x(18)}{L})dx = a_m\frac{L}{2} \tag{109}$$

$$3\int_0^L 1 - cos(\frac{18\pi x}{L})dx = a_m\frac{L}{2} \tag{110}$$

$$3\left[x - \frac{L}{18\pi}sin(\frac{18\pi x}{L})\right]\Bigg|_0^L = a_m\frac{L}{2} \tag{111}$$

$$3L = a_m\frac{L}{2} \tag{112}$$

$$6L = a_m L \tag{113}$$

$$a_m = 6. \tag{114}$$

Therefore,

$$\boxed{a_n = \begin{cases} 6 \text{ if n} = 9 \\ 0 \text{ otherwise} \end{cases}} \tag{115}$$

So,

$$\boxed{u(x,t) = 6e^{-kt(\frac{9\pi}{L})^2}sin(\frac{9\pi x}{L})} \tag{116}$$

(c) $u(x,0) = 2\cos\left(\frac{3\pi x}{L}\right)$

We know that

$$u(x,t) = \sum_{n=1}^{\infty} a_n e^{-kt(\frac{n\pi}{L})^2} sin(\frac{n\pi x}{L}) \tag{117}$$

is the Fourier series for our solution given our BCs. Then

$$u(x,0) = 2cos(\frac{3\pi x}{L}) = \sum_{n=1}^{\infty} a_n sin(\frac{n\pi x}{L}). \tag{118}$$

By orthogonality of sines, we know that

$$2sin(\frac{m\pi x}{L})cos(\frac{3\pi x}{L}) = \sum_{n=1}^{\infty} a_n sin(\frac{m\pi x}{L})sin(\frac{n\pi x}{L}) \tag{119}$$

implies

$$\int_0^L 2sin(\frac{m\pi x}{L})cos(\frac{3\pi x}{L})dx = \int_0^L \sum_{n=1}^{\infty} a_n sin(\frac{m\pi x}{L})sin(\frac{n\pi x}{L})dx \tag{120}$$

which becomes

$$\int_0^L 2sin(\frac{m\pi x}{L})cos(\frac{3\pi x}{L})dx = \sum_{n=1}^{\infty} a_n \int_0^L sin(\frac{m\pi x}{L})sin(\frac{n\pi x}{L})dx. \tag{121}$$

Since the terms of the infinite series on the right are zero iff $m \neq n$, let $m = n$. Then

$$\int_0^L 2sin(\frac{m\pi x}{L})cos(\frac{3\pi x}{L})dx = 0 + 0 + \cdots + a_m \int_0^L sin^2(\frac{m\pi x}{L})dx. \tag{122}$$

Invoking some trig identities, namely

$$sin(a+b) = sin(a)cos(b) + cos(a)sin(b) \tag{123}$$

$$sin(a-b) = sin(a)cos(b) - cos(a)sin(b) \tag{124}$$

$$\frac{1}{2}(sin(a+b) + sin(a-b)) = sin(a)cos(b) \tag{125}$$

we get

$$2\int_0^L \frac{1}{2}\left[sin(\frac{m\pi x}{L} + \frac{3\pi x}{L}) + sin(\frac{m\pi x}{L} - \frac{3\pi x}{L})\right]dx = a_m \int_0^L sin^2(\frac{m\pi x}{L})dx. \tag{126}$$

Since $sin^2(\theta) = \frac{1}{2}(1 - cos(2\theta))$,

$$\int_0^L sin(\frac{(m+3)\pi x}{L}) + sin(\frac{(m-3)\pi x}{L})dx = a_m \int_0^L \frac{1}{2}(1 - cos(\frac{2m\pi x}{L}))dx. \tag{127}$$

Integrating, we find that

$$\left[-\frac{L}{(m+3)\pi}cos(\frac{(m+3)\pi x}{L}) - \frac{L}{(m-3)\pi}cos(\frac{(m-3)\pi x}{L})\right]\Bigg|_0^L = \frac{a_m}{2}\left[x - \frac{L}{(m-3)\pi}sin(\frac{(m-3)\pi x}{L})\right]\Bigg|_0^L \tag{128}$$

$$[-\frac{L}{(m+3)\pi}cos((m+3)\pi) - (-\frac{L}{(m+3)\pi}cos(0))] + [-\frac{L}{(m-3)\pi}cos((m-3)\pi) - (-\frac{L}{(m-3)\pi}cos(0))] = \frac{La_m}{2} \tag{129}$$

$$-\frac{L}{(m+3)\pi}cos((m+3)\pi) + \frac{L}{(m+3)\pi} - \frac{L}{(m-3)\pi}cos((m-3)\pi) + \frac{L}{(m-3)\pi} = \frac{La_m}{2} \tag{130}$$

$$\frac{-L}{\pi}\left[\frac{1}{m+3}(cos((m+3)\pi) - 1) + \frac{1}{m-3}(cos((m+3)\pi) - 1)\right] = \frac{La_m}{2} \tag{131}$$

$$\frac{-2}{\pi}\left[\frac{1}{m+3}(cos((m+3)\pi) - 1) + \frac{1}{m-3}(cos((m+3)\pi) - 1)\right] = a_m. \tag{132}$$

From this, we can find the piecewise function,

$$\boxed{a_m = \begin{cases} 0; & \text{if m is odd} \\ \frac{2}{\pi}[\frac{1}{m+3} + \frac{1}{m-3}]; & \text{if m is even} \end{cases}} \tag{133}$$

which describes the coefficients of the Fourier series solution for the IVP.

(d) $u(x,0) = \begin{cases} 1 & 0 < x \leq L/2 \\ 2 & L/2 < x < L \end{cases}$

Starting off, we know that

$$\int_0^L u(x,0) sin(\frac{m\pi x}{L})dx = \sum_{n=1}^{\infty} a_n \int_0^L sin(\frac{m\pi x}{L}) sin(\frac{n\pi x}{L})dx \tag{134}$$

$$\int_0^{\frac{L}{2}} sin(\frac{m\pi x}{L})dx + \int_{\frac{L}{2}}^L 2sin(\frac{m\pi x}{L})dx = \sum_{n=1}^{\infty} a_n \int_0^L sin(\frac{m\pi x}{L}) sin(\frac{n\pi x}{L})dx. \tag{135}$$

Invoking orthogonality of sines,

$$\int_0^{\frac{L}{2}} sin(\frac{m\pi x}{L})dx + \int_{\frac{L}{2}}^L 2sin(\frac{m\pi x}{L})dx = a_m \int_0^L sin^2(\frac{m\pi x}{L})dx \tag{136}$$

$$\int_0^{\frac{L}{2}} sin(\frac{m\pi x}{L})dx + 2\int_{\frac{L}{2}}^L sin(\frac{m\pi x}{L})dx = \frac{1}{2}a_m \int_0^L 1 - cos(\frac{2m\pi x}{L})dx \tag{137}$$

$$2\int_0^{\frac{L}{2}} sin(\frac{m\pi x}{L})dx + 4\int_{\frac{L}{2}}^L sin(\frac{m\pi x}{L})dx = a_m \int_0^L 1 - cos(\frac{2m\pi x}{L})dx \tag{138}$$

$$2\left[-\frac{L}{m\pi} cos(\frac{m\pi x}{L})\Big|_0^{\frac{L}{2}} - 2\frac{L}{m\pi} cos(\frac{m\pi x}{L})\Big|_{\frac{L}{2}}^{L} \right] = a_m \frac{L}{2} \tag{139}$$

$$-\frac{4}{L}\left[\frac{L}{m\pi} cos(\frac{m\pi x}{L})\Big|_0^{\frac{L}{2}} + 2\frac{L}{m\pi} cos(\frac{m\pi x}{L})\Big|_{\frac{L}{2}}^{L} \right] = a_m \tag{140}$$

Evaluating this,

$$-\frac{4}{L}\left[\frac{L}{m\pi}[0-1] + 2\frac{L}{m\pi}[cos(m\pi) - 0] \right] = a_m \tag{141}$$

$$a_m = \frac{4}{m\pi} + \frac{8}{m\pi} cos(m\pi); m \in 1, 2, \ldots \tag{142}$$

we find that our final Fourier series solution is

$$u(x,t) = \sum_{n=1}^{\infty} a_n e^{-kt(\frac{n\pi}{L})^2} sin(\frac{n\pi x}{L}) \tag{143}$$

where

$$\boxed{a_n = \frac{4}{n\pi} + \frac{8}{n\pi} cos(n\pi); n \in 1, 2, \ldots} \tag{144}$$

Problem 2.3.4 a-b: Consider

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2},$$

subject to the boundary conditions $u(0,t) = 0$, $u(L,t) = 0$ and $u(x,0) = f(x)$.

(a) What is the total heat energy in the rod as a function of time?

Since total heat energy is defined by

$$\int_0^L c\rho u dx = c\rho \int_0^L \sum_{n=1}^{\infty} a_n e^{-kt(\frac{n\pi}{L})^2} sin(\frac{n\pi x}{L})dx \tag{145}$$

we can expand this out to

$$= \sum_{n=1}^{\infty} -c\rho a_n e^{-kt(\frac{n\pi}{L})^2} \left[(\frac{L}{n\pi} cos(\frac{n\pi x}{L})) \right]\Bigg|_0^L \tag{146}$$

$$\boxed{= \sum_{n=1}^{\infty} -c\rho a_n e^{-kt(\frac{n\pi}{L})^2} \left[\frac{L}{n\pi}((-1)^n - 1) \right]} \tag{147}$$

where

$$a_n = \frac{2}{L}\int_0^L f(x) sin(\frac{n\pi x}{L})dx. \tag{148}$$

(b) What is the flow of heat energy out to the rod at $x = 0$? at $x = L$?

Given part a,

$$\frac{\partial u}{\partial x} = \sum_{n=1}^{\infty} a_n e^{-kt(\frac{n\pi}{L})^2} \frac{n\pi}{L} cos(\frac{n\pi x}{L}). \tag{149}$$

Accordingly,

$$\phi(0,t) = -K_0 \frac{\partial u}{\partial x}(0,t) = -K_0 \sum_{n=1}^{\infty} a_n e^{-kt(\frac{n\pi}{L})^2} (\frac{n\pi}{L}) \tag{150}$$

$$\text{and} \tag{151}$$

$$\phi(L,t) = -K_0 \frac{\partial u}{\partial x}(0,t) = -K_0 \sum_{n=1}^{\infty} a_n e^{-kt(\frac{n\pi}{L})^2} (\frac{n\pi}{L})(-1)^n \tag{152}$$

as desired.