RUBRIC:

Questions	Points	Score
Total		

Problem 1.2.1: Briefly explain the minus sign in conservation law (1.2.3) or (1.2.5) if $Q = 0$.

Since the flux enters at x and leaves at $x + \Delta x$, the average change over that window is

$$\phi(x + \Delta x, t) - \phi(x, t). \tag{1}$$

However, in the derivation of 1.2.3, to use the limit definition of a derivative, we take

$$\frac{\partial e}{\partial t} = \lim_{\Delta x \to 0} \frac{\phi(x, t) - \phi(x + \Delta x, t)}{\Delta x}, \tag{2}$$

which inverts the sign of the coefficient of $\frac{\partial \phi}{\partial x}$ to be -1. Functionally, this implies that the change in energy with respect to time is inversely related to the change in flux with respect to x. We explained this at the end of our first class by saying "the if the flux leaving the region we measure is greater than the flux entering the region we measure, thermal energy density is decreasing - and vice versa."

Problem 1.2.3: Derive the heat equation for a rod assuming constant thermal properties with variable cross-sectional area $A(x)$ assuming no sources by considering the total thermal energy between $x = a$ and $x = b$.

Consider a perfectly insulated region of a rod with no sources or sinks along its length.
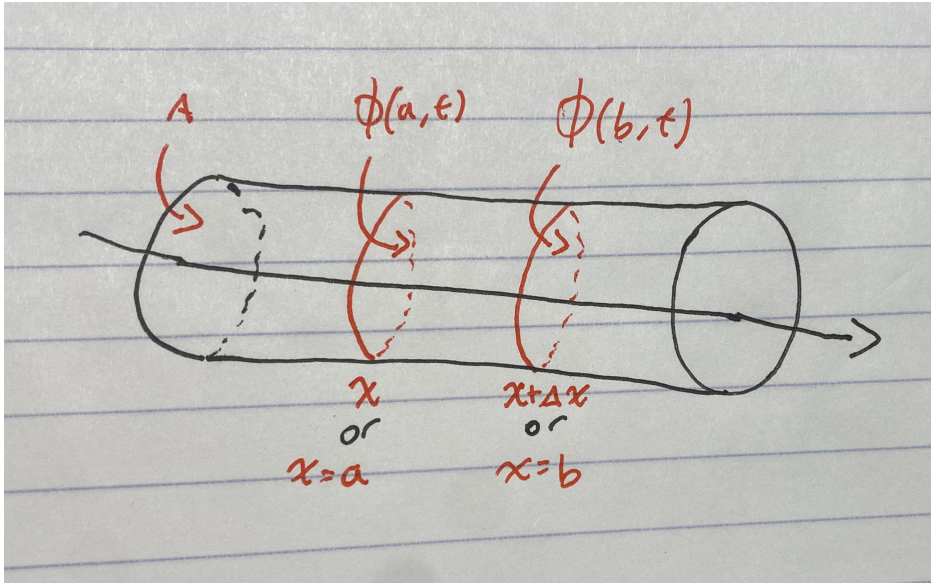

Figure 1: Diagram for Thermal Diffusion Equation Derivation

Let $e(x, t)$ denote the thermal energy density (units of energy/vol), and let $e(x, t) * A(x)\Delta x$ denote heat energy. Then let $\phi(x, t)$ denote the heat energy flux at point x (with units of energy/(time*area)). If we want to measure the change in the heat energy over the domain of $x = a$ to $x = b$, that means we really want

$$\frac{\partial}{\partial t} \int_a^b e(x, t) A(x) dx, \tag{3}$$

where A is a variable representing the area of the rod section at x. I claim that this is exactly equal to $\phi(a,t)A(x) - \phi(b,t)A$ if we have no sources or sinks, i.e.,

$$\frac{\partial}{\partial t}\int_a^b e(x,t)A(x)dx = \phi(a,t)A(a) - \phi(b,t)A(b). \tag{4}$$

$$\int_a^b \frac{\partial e(x,t)}{\partial t}A(x) + \frac{\partial A(x)}{\partial t}e(x,t)dx = \phi(a,t)A(a) - \phi(b,t)A(b). \tag{5}$$

Since A is a function of x, not t, we can zero out the second term of the integrand.

$$\int_a^b \frac{\partial e(x,t)}{\partial t}A(x)dx = \phi(a,t)A(a) - \phi(b,t)A(b). \tag{6}$$

$$\int_a^b \frac{\partial e(x,t)}{\partial t}A(x)dx = -\int_a^b \frac{\partial}{\partial x}\Big(\phi(x,t)A(x)\Big)dx \tag{7}$$

$$\int_a^b \frac{\partial e(x,t)}{\partial t}A(x)dx + \int_a^b \Big(\phi(x,t)A(x)\Big)dx = 0 \tag{8}$$

Therefore

$$\frac{\partial e(x,t)}{\partial t}A(x) + \frac{\partial}{\partial x}\Big(\phi(x,t)A(x)\Big) = 0. \tag{9}$$

We also know that, assuming constant thermal properties,

$$e(x,t) = c\rho u(x,t). \tag{10}$$

Thus,

$$\frac{\partial}{\partial t}(c\rho u(x,t))A(x) + \frac{\partial}{\partial x}\Big(\phi(x,t)A(x)\Big) = 0. \tag{11}$$

$$\frac{\partial u(x,t)}{\partial t}c\rho A(x) + \frac{\partial}{\partial x}\Big(\phi(x,t)A(x)\Big) = 0. \tag{12}$$

Moreover by Fourier's law, which states (using constant $K_0(x)$ due to constant thermal properties),

$$\phi(x,t) = -K_0\frac{\partial u(x,t)}{\partial x}, \tag{13}$$

we can sub out ϕ in (12) for

$$\frac{\partial u(x,t)}{\partial t}c\rho A(x) - \frac{\partial}{\partial x}\Big(K_0\frac{\partial u(x,t)}{\partial x}A(x)\Big) = 0. \tag{14}$$

This looks bad, but BOY does it get worse before it gets worse.

$$\frac{\partial u(x,t)}{\partial t}c\rho A(x) - K_0\Big(\frac{\partial A(x)}{\partial x}\frac{\partial u(x,t)}{\partial x} + \frac{\partial^2 u(x,t)}{\partial x^2}A(x)\Big) = 0. \tag{15}$$

Moving some stuff around, we derive our final equation (17).

$$\frac{\partial u(x,t)}{\partial t}c\rho A(x) = K_0\Big(\frac{\partial A(x)}{\partial x}\frac{\partial u(x,t)}{\partial x} + \frac{\partial^2 u(x,t)}{\partial x^2}A(x)\Big) \tag{16}$$

$$\frac{\partial u(x,t)}{\partial t} = \frac{K_0}{c\rho A(x)}\Big(\frac{\partial A(x)}{\partial x}\frac{\partial u(x,t)}{\partial x} + \frac{\partial^2 u(x,t)}{\partial x^2}A(x)\Big) \tag{17}$$

Problem 1.2.4: Derive the diffusion equation for a chemical pollutant.

(a) Consider the total amount of the chemical in a thin region between x and $x + \Delta x$.

We want to find the density of a chemical pollutant in a thin region between x and $x + \Delta x$. We can measure the pollutant density as $p(x,t)$ with units of $\frac{\text{mass}}{\text{volume}}$. Distributing $p(x,t)$ across our cross-sectional area times the length of the rod to get the total amount of pollutant, we need to multiply $p(x,t)$ by a $A\Delta x$ term to satisfy unit conversion. Next, we can measure the diffusive flux as $\phi(x,t)$ with units of $\frac{\text{mass}}{\text{area} * \text{time}}$. Let any source or sink be denoted as $Q(x,t)$ with units of $\frac{\text{mass}}{\text{volume}}$. Following from our intuition in class about heat diffusion, we want the rate of change of the pollutant density with respect to time to be equal to the change in flux with respect to x times the area of the region, plus the source / sink contribution times the area of the region. This is mathematically expressed as:

$$\frac{\partial p(x,t)}{\partial t} A\Delta x = -A(\phi(x,t) - \phi(x + \Delta x, t)) + Q(x,t)A\Delta x \tag{18}$$

Following our in-class derivation,

$$\frac{\partial p(x,t)}{\partial t} = -\frac{\phi(x,t) - \phi(x + \Delta x, t)}{\Delta x} + Q(x,t). \tag{19}$$

Taking the limit as $\Delta x \to 0$,

$$\lim_{\Delta x \to 0} \frac{\partial p(x,t)}{\partial t} = -\lim_{\Delta x \to 0} \frac{\phi(x,t) - \phi(x + \Delta x, t)}{\Delta x} + Q(x,t). \tag{20}$$

$$\frac{\partial p(x,t)}{\partial t} = -\frac{\partial \phi(x,t)}{\partial x} + Q(x,t). \tag{21}$$

We arrive at the final equation in terms of ϕ (21) for the total amount of the pollutant over our region. That said, we want to expand it with Fick's law so that it's functionally useful.

$$\frac{\partial p(x,t)}{\partial t} = -K_0 \frac{\partial^2 p(x,t)}{\partial x^2} + Q(x,t). \tag{22}$$

(b) Consider the total amount of the chemical between $x = a$ and $x = b$.

Same deal as above in (a). Kinda. We don't multiply by $A\Delta x$, but we do treat the change in heat energy with respect to time as

$$\frac{\partial}{\partial t} \int_a^b p(x,t)A\,dx. \tag{23}$$

At that point, the derivation is equivalent up to notational equivalence. The change in heat energy with respect to time should be equal to the change in flux over the region [a, b] times the area, plus the accumulation of the sink / source over the [a, b] region.

$$\frac{\partial}{\partial t} \int_a^b p(x,t)A\,dx = -A(\phi(b,t) - \phi(a,t)) + \int_a^b Q(x,t)A\,dx \tag{24}$$

Moving the $\frac{\partial}{\partial t}$ inside of the integral, and using FTC on the RHS, we get

$$\int_a^b \frac{\partial p(x,t)}{\partial t} A\,dx = -\int_a^b \frac{\partial \phi(x,t)}{\partial t} A\,dx + \int_a^b Q(x,t)A\,dx. \tag{25}$$

4

We can functionally eliminate A from all terms unless it is a variable term with respect to x. Because it wasn't specified, I'm assuming it to be constant.

$$\int_a^b \frac{\partial p(x,t)}{\partial t} dx = -\int_a^b \frac{\partial \phi(x,t)}{\partial t} dx + \int_a^b Q(x,t)dx. \tag{26}$$

Next, we want to move the RHS to the LHS.

$$\int_a^b \frac{\partial p(x,t)}{\partial t} dx + \int_a^b \frac{\partial \phi(x,t)}{\partial t} dx - \int_a^b Q(x,t)dx = 0 \tag{27}$$

$$\int_a^b \frac{\partial p(x,t)}{\partial t} + \frac{\partial \phi(x,t)}{\partial t} - Q(x,t)dx = 0 \tag{28}$$

From (27), we get our final equation in terms of ϕ, (29):

$$\frac{\partial p(x,t)}{\partial t} + \frac{\partial \phi(x,t)}{\partial t} - Q(x,t) = 0 \tag{29}$$

$$\frac{\partial p(x,t)}{\partial t} = -\frac{\partial \phi(x,t)}{\partial t} + Q(x,t) \tag{30}$$

Again converting (29) to a more useful form with Fick's law, we substitute in $\phi(x,t) = K_0 \frac{\partial p(x,t)}{\partial x}$.

$$\frac{\partial p(x,t)}{\partial t} = -K_0 \frac{\partial^2 p(x,t)}{\partial x^2} + Q(x,t). \tag{31}$$

Problem 1.2.5: Derive an equation for the concentration $u(x,t)$ of a chemical pollutant if the chemical is produced due to a chemical reaction at a rate of $\alpha u (\beta - u)$ per unit volume.

Treating $\alpha u(x,t)(\beta - u(x,t))$ as a source, and switching my notation from $u \to p$, plugging $\alpha u(x,t)(\beta - u(x,t)) = Q(x,t)$ to (29),

$$\frac{\partial p(x,t)}{\partial t} = -\frac{\partial \phi(x,t)}{\partial t} + \alpha p(x,t)(\beta - p(x,t)). \tag{32}$$

Because we want everything in terms of $\phi(x,t) = K_0 \frac{\partial p(x,t)}{\partial x}$, we will again utilize Fick's Law.

$$\frac{\partial p(x,t)}{\partial t} = -K_0 \frac{\partial^2 p(x,t)}{\partial x^2} + \alpha p(x,t)(\beta - p(x,t)) \tag{33}$$

Problem 1.2.8: If $u(x,t)$ is known, give an expression for the total thermal energy contained in a rod $(0 < x < L)$.

If E(t) is the total energy in the rod at time t, then,

$$E(t) = \int_0^L A(x)e(x,t)dx = \int_0^L A(x)u(x,t)\rho(x)c(x)dx. \tag{34}$$

Problem 1.2.9(a-b): Consider a thin, one-dimensional rod without sources of thermal energy whose lateral surface area is not insulated.

(a) Assume that the heat energy flowing out of the lateral sides per unit surface area per time is $w(x,t)$. Derive the partial differential equation for the temperature $u(x,t)$.

Since we don't have sources, $Q = 0$. Also assume $A, \rho, \& c$ to be constant. Let P be the perimeter of the rod. Then,

$$\frac{\partial}{\partial t} \int_a^b e(x,t) A dx = A(\phi(a,t) - \phi(b,t)) - \int_a^b w(x,t) P dx \tag{35}$$

$$\frac{\partial}{\partial t} \int_a^b e(x,t) A dx = -\int_a^b \frac{\partial \phi}{\partial x} A dx - \int_a^b w(x,t) P dx \tag{36}$$

Subbing out

$$e(x,t) = u(x,t) \rho c \tag{37}$$

for $e(x,t)$, we get

$$\frac{\partial}{\partial t} \int_a^b u(x,t) \rho c A dx = -\int_a^b \frac{\partial \phi}{\partial x} A dx - \int_a^b w(x,t) P dx \tag{38}$$

$$\int_a^b \frac{\partial u}{\partial t} \rho c A dx = -\int_a^b \frac{\partial \phi}{\partial x} A dx - \int_a^b w(x,t) P dx \tag{39}$$

$$\int_a^b \frac{\partial u}{\partial t} \rho c A dx + \int_a^b \frac{\partial \phi}{\partial x} A dx + \int_a^b w(x,t) P dx = 0 \tag{40}$$

Since the bounds of the integrands are arbitrary and exactly the same,

$$\int_a^b \frac{\partial u}{\partial t} \rho c A + \frac{\partial \phi}{\partial x} A + w(x,t) P dx = 0 \tag{41}$$

(39) implies that

$$\frac{\partial u}{\partial t} \rho c A + \frac{\partial \phi}{\partial x} A + w(x,t) P = 0 \tag{42}$$

Which yields our final equation (40) in terms of $\phi(x,t)$:

$$\frac{\partial u}{\partial t} = -\frac{1}{\rho c A} \left(\frac{\partial \phi}{\partial x} A + w(x,t) P \right) \tag{43}$$

To convert this to a more useful form, we use Fourier's law to sub out $\phi(x,t) = K_0 \frac{\partial u(x,t)}{\partial x}$

$$\frac{\partial u}{\partial t} = -\frac{1}{\rho c A} \left(K_0 \frac{\partial^2 u}{\partial x^2} A + w(x,t) P \right) \tag{44}$$

(b) Assume that $w(x,t)$ is proportional to the temperature difference between the rod $u(x,t)$ and a known outside temperature $\gamma(x,t)$ Derive

$$c\rho \frac{\partial u}{\partial t} = \frac{\partial}{\partial x} \left(K_0 \frac{\partial u}{\partial x} \right) - \frac{P}{A} \left[u(x,t) - \gamma(x,t) \right] h(x),$$

where $h(x)$ is a positive x-dependent proportionality, P is the lateral perimeter, and A is the cross-sectional area.

Ok, we're really close to this exact form with (44). Let's move some stuff around.

$$c\rho \frac{\partial u}{\partial t} = -\frac{1}{A} \left(K_0 \frac{\partial^2 u}{\partial x^2} A + w(x,t) P \right) \tag{45}$$

$$c\rho \frac{\partial u}{\partial t} = -K_0 \frac{\partial^2 u}{\partial x^2} - \frac{P}{A} w(x,t) \tag{46}$$

According to the assumption,

$$w(x,t) = [u(x,t) - \gamma(x,t)]h(x). \tag{47}$$

Then (46) becomes

$$c\rho \frac{\partial u}{\partial t} = -K_0 \frac{\partial^2 u}{\partial x^2} - \frac{P}{A}[u(x,t) - \gamma(x,t)]h(x), \tag{48}$$

as desired.