Problem 4.4.3(b): Consider a slightly damped vibrating string that satisfies

$$\rho_0 \frac{\partial^2 u}{\partial t^2} = T_0 \frac{\partial^2 u}{\partial x^2} - \beta \frac{\partial u}{\partial t}.$$

Determine the solution (by separation of variables) for a string with fixed ends and with initial conditions $u(x, 0) = f(x)$ and $\frac{\partial u}{\partial t}(x, 0) = g(x)$

First, assume that the solution is separable.

$$u(x, t) = T(t)X(x).$$

Then

$$\rho_0 T''X = T_0 X''T - \beta T'X$$

implies

$$\frac{\rho_0 T'' + \beta T'}{T_0 T} = -\lambda = \frac{X''}{X}.$$

First, to find $X(x)$, let $m = \pm\sqrt{\lambda}$. Then

$$X(x) = \sum_{n=1}^{\infty} a_n \sin(\frac{n\pi x}{L})$$

and, in the only non-trivial case, $\lambda > 0$,

$$\lambda = (\frac{n\pi}{L})^2.$$

Then to find $T(t)$, since

$$\rho_0 T'' + \beta T' + T_0 \lambda T = 0,$$

then

$$\rho_0 m^2 + \beta m + T_0 \lambda = 0$$

and we can solve for m with the quadratic formula,

$$m = \frac{-\beta \pm \sqrt{\beta^2 - 4\rho_0 T_0 \lambda}}{2\rho_0}.$$

Assume that the discriminant is less than zero and let $H = |\beta^2 - 4\rho_0 T_0 \lambda|$. Then,

$$m = \frac{-\beta \pm i\sqrt{H}}{2\rho_0}$$

implies

$$T(t) = e^{-\frac{\beta}{2\rho_0}t} \left(\cos(\frac{\sqrt{H}}{2\rho_0}t) + \sin(\frac{\sqrt{H}}{2\rho_0}t) \right).$$

Then,

$$\boxed{u(x, t) = \sum_{n=1}^{\infty} \sin(\frac{n\pi x}{L}) e^{-\frac{\beta}{2\rho_0}t} \left(A_n \cos(\frac{\sqrt{H}}{2\rho_0}t) + B_n \sin(\frac{\sqrt{H}}{2\rho_0}t) \right)}$$

where

$$\boxed{A_n = \frac{2}{L} \int_0^L f(x) \sin(\frac{n\pi x}{L}) dx.}$$

To get B_n, we differentiate u with respect to t, and apply boundary conditions as follows.

$$\frac{\partial u}{\partial t} = \sum_{n=1}^{\infty}\left(\frac{-\beta}{2\rho_0}e^{-\frac{\beta}{2\rho_0}t}\sin(\frac{n\pi x}{L})\right)\left(A_n\cos(\frac{\sqrt{H}}{2\rho_0}t)+B_n\sin(\frac{\sqrt{H}}{2\rho_0}t)\right)+\left(e^{-\frac{\beta}{2\rho_0}t}\sin(\frac{n\pi x}{L})\right)\left(\frac{\sqrt{H}}{2\rho_0}(B_n\cos(\frac{\sqrt{H}}{2\rho_0}t)-A_n\sin(\frac{\sqrt{H}}{2\rho_0}t))\right)$$

Evaluating this at $t=0$,

$$g(x) = \sum_{n=1}^{\infty}\left[-A_n\frac{\beta}{2\rho_0}\sin(\frac{n\pi x}{L})+B_n\frac{\sqrt{H}}{2\rho_0}\sin(\frac{n\pi x}{L})\right]$$

$$g(x) = \sum_{n=1}^{\infty}\left[(\frac{-\beta A_n+\sqrt{H}B_n}{2\rho_0})\sin(\frac{n\pi x}{L})\right]$$

gives

$$\frac{-\beta A_n+\sqrt{H}B_n}{2\rho_0} = \frac{2}{L}\int_0^L g(x)\sin(\frac{n\pi x}{L})dx$$

which yields

$$\boxed{B_n = \frac{4\rho_0}{L\sqrt{H}}\left(\int_0^L g(x)\sin(\frac{n\pi x}{L})dx+\beta A_n\right)}$$

Problem 4.4.7: If a vibrating string satisfying (4.4.1)-(4.4.3) is initially at rest ($g(x)=0$), show that

$$u(x,t) = \frac{1}{2}\left[F(x-ct)+F(x+ct)\right],$$

where $F(x)$ is the odd-periodic extension of $f(x)$.

The odd periodic extension of $f(x)$ is defined by

$$f(x)\sim F(x) = \sum_{n=1}^{\infty}B_n\sin(\frac{n\pi x}{L})$$

where

$$B_n = \frac{2}{L}\int_0^L f(x)\sin(\frac{n\pi x}{L})dx.$$

Accordingly,

$$F(x-ct) = \sum_{n=1}^{\infty}B_n\sin(\frac{n\pi x}{L}-\frac{n\pi ct}{L}) = \sum_{n=1}^{\infty}B_n\left[\sin(\frac{n\pi x}{L})\cos(\frac{n\pi ct}{L})-\cos(\frac{n\pi x}{L})\sin(\frac{n\pi ct}{L})\right]$$

and

$$F(x+ct) = \sum_{n=1}^{\infty}B_n\sin(\frac{n\pi x}{L}+\frac{n\pi ct}{L}) = \sum_{n=1}^{\infty}B_n\left[\sin(\frac{n\pi x}{L})\cos(\frac{n\pi ct}{L})+\cos(\frac{n\pi x}{L})\sin(\frac{n\pi ct}{L})\right]$$

Then,

$$\frac{1}{2}[F(x-ct)+F(x+ct)] =$$

$$= \frac{1}{2}\left[\sum_{n=1}^{\infty}B_n\left[\sin(\frac{n\pi x}{L})\cos(\frac{n\pi ct}{L})+\cos(\frac{n\pi x}{L})\sin(\frac{n\pi ct}{L})\right]+\sum_{n=1}^{\infty}B_n\left[\sin(\frac{n\pi x}{L})\cos(\frac{n\pi ct}{L})-\cos(\frac{n\pi x}{L})\sin(\frac{n\pi ct}{L})\right]\right]$$

$$= \sum_{n=1}^{\infty}B_n\sin(\frac{n\pi x}{L})\cos(\frac{n\pi ct}{L})$$

which is exactly $u(x,t)$ when $g(x)=0$.