

Math 440 Exam 2 Practice Test

October 14, 2025

Timothy Tarter
James Madison University
Department of Mathematics

1 Describe / Explain:

1.1 Laplace's Equation (include well-posedness, uniqueness, and the mean value proposition).

Laplace's equation states that the sum of the first partials are equal to zero, i.e.,

$$0 = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} \quad (1)$$

Well-posedness: there exists a unique solution which depends continuously on non-homogeneous data.

Uniqueness: solutions to laplace's equation are unique.

Mean value proposition: the temperature at the origin is the average of the temperature at the boundary.

1.2 Linear Operators, provide an example of something which is a linear operator, and provide an example of something which is not a linear operator.

A linear operator, $\mathcal{U} : V \rightarrow V$ where V is a vector space, has the property that for any $\vec{x}, \vec{y} \in V$ and any scalar c in the field of V ,

1. $\mathcal{U}(\vec{x} + \vec{y}) = \mathcal{U}(\vec{x}) + \mathcal{U}(\vec{y})$
2. $\mathcal{U}(c\vec{x}) = c\mathcal{U}(\vec{x})$.

Example:

$$L(u) = \frac{\partial}{\partial x} \left[K_0(x) \frac{\partial u}{\partial x} \right] \quad (2)$$

Non-Example:

$$L(u) = \frac{\partial}{\partial x} \left[K_0(u, x) \frac{\partial u}{\partial x} \right] \quad (3)$$

where K_0 is a non-constant function.

2 Prove Orthogonality of Sines.

We want to show that

$$\int_0^L \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{m\pi x}{L}\right) dx = \begin{cases} 0; & \text{if } n \neq m \\ \frac{L}{2}; & \text{if } n = m. \end{cases} \quad (4)$$

$$\int_0^L \cos\left(\frac{n\pi x}{L}\right) \cos\left(\frac{m\pi x}{L}\right) dx = \begin{cases} 0; & \text{if } n \neq m \\ \frac{L}{2}; & \text{if } n = m \neq 0 \\ L; & \text{if } n = m = 0 \end{cases} \quad (5)$$

Starting off with (4), if $n = m$, then

$$\int_0^L \sin^2\left(\frac{n\pi x}{L}\right) dx = \int_0^L \frac{1}{2} (1 - \cos(2\frac{n\pi x}{L})) dx \quad (6)$$

$$= \frac{1}{2} \left[x - \frac{L}{2n\pi} \sin\left(\frac{2n\pi x}{L}\right) \right] \Big|_0^L \quad (7)$$

$$= \frac{1}{2} [L - 0] = \frac{L}{2} \quad (8)$$

as desired. Alternatively, if $n \neq m$, then

$$\int_0^L \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{m\pi x}{L}\right) dx \quad (9)$$

$$= \int_0^L \frac{1}{2} [\cos\left(\frac{n\pi x}{L} - \frac{m\pi x}{L}\right) - \cos\left(\frac{n\pi x}{L} + \frac{m\pi x}{L}\right)] dx \quad (10)$$

$$= \frac{1}{2} \int_0^L \cos\left(\frac{\pi x(n-m)}{L}\right) - \cos\left(\frac{\pi x(n+m)}{L}\right) dx \quad (11)$$

$$= \frac{1}{2} \left[\frac{L}{\pi(n-m)} \sin\left(\frac{\pi x(n-m)}{L}\right) - \frac{L}{\pi(n+m)} \sin\left(\frac{\pi x(n+m)}{L}\right) \right] \Big|_0^L \quad (12)$$

$$= 0 - 0 = 0 \quad (13)$$

as desired.

Moving to (5), if $n \neq m$, it works pretty much the same way as with sines. If $n = m \neq 0$, then

$$\int_0^L \cos^2\left(\frac{n\pi x}{L}\right) dx = \quad (14)$$

$$= \int_0^L \frac{1}{2} (1 + \sin(\frac{2n\pi x}{L})) dx \quad (15)$$

$$= \frac{1}{2} \left[x - \frac{L}{2n\pi} \cos(\frac{2n\pi x}{L}) \right] \Big|_0^L \quad (16)$$

$$= \frac{1}{2} [L + (-1 + 1)] \quad (17)$$

$$= \frac{L}{2} \quad (18)$$

as desired. Then finally, if $n = m = 0$,

$$\int_0^L \cos^2(0) dx = \int_0^L 1 dx = L. \quad (19)$$

3 Solve $\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2}$ with certain BC's and IC's, addressing $\lambda > 0, = 0, < 0$.

3.1 $u(0, t) = u(L, t) = 0$

Gives sine solutions.

3.2 $\frac{\partial u}{\partial x}(0, t) = 0 = \frac{\partial u}{\partial x}(L, t)$

Gives cosine solutions.

3.3 $u(0, t) = u(L, t) = 0$ and $\frac{\partial u}{\partial x}(0, t) = 0 = \frac{\partial u}{\partial x}(L, t)$

Gives sine and cosine solutions.

4 Solve $\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2}$ at equilibrium in polar coordinates over the interval $(-L, L)$.

Note that the 1D heat equation can be expressed in polar coordinate with axial symmetry (meaning that $\frac{\partial^2 u}{\partial \theta^2} = 0$) as follows:

$$0 = \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial u}{\partial r} \right). \quad (20)$$

We also get the following BCs for free:

$$u(-L) = u(L) \quad (21)$$

$$\frac{\partial u}{\partial r}(-L) = \frac{\partial u}{\partial r}(L) \quad (22)$$

$$|u(0)| < \infty. \quad (23)$$

If we guess that $u(x, t) = g(t)\phi(x)$, then

$$\frac{1}{k} \frac{g'(t)}{g(t)} = -\lambda = \frac{\phi''(x)}{\phi(x)} \quad (24)$$

implies that

$$g(t) = e^{-k\lambda t} \quad (25)$$

and

$$\phi'' + \lambda\phi = 0. \quad (26)$$

If $\lambda > 0$, then $\phi(x) = A \cos(\sqrt{\lambda}x) + B \sin(\sqrt{\lambda}x)$. $\phi(-L) = \phi(L)$ implies,

$$A \cos(\sqrt{\lambda}L) + B \sin(\sqrt{\lambda}L) = A \cos(\sqrt{\lambda}L) - B \sin(\sqrt{\lambda}L) \quad (27)$$

$$2B \sin(\sqrt{\lambda}L) = 0 \quad (28)$$

$$\text{so} \quad (29)$$

$$\sqrt{\lambda}L = n\pi; n \in \mathbb{N} \quad (30)$$

$$\lambda = \left(\frac{n\pi}{L}\right)^2 \quad (31)$$

Then

$$\phi(x) = A_0 + \sum_{n=1}^{\infty} A_n e^{-k(\frac{n\pi}{L})^2 t} \cos\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} B_n e^{-k(\frac{n\pi}{L})^2 t} \sin\left(\frac{n\pi x}{L}\right) \quad (32)$$

where

$$A_n = \frac{2}{L} \int_{-L}^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx \quad (33)$$

$$\text{and} \quad (34)$$

$$B_n = \frac{2}{L} \int_{-L}^L f(x) \cos\left(\frac{n\pi x}{L}\right) dx. \quad (35)$$

5 Solve Laplace's Equation on a rectangle, with LHS, Top, and Bottom = 0, RHS = $g_2(y)$.

We want to start with Laplace's 2D equation,

$$0 = k \left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right]. \quad (36)$$

If $u(x, t) = X(x)Y(y)$, then

$$0 = X''Y + XY'' \quad (37)$$

implies that

$$\frac{X''}{X} = \lambda = \frac{-Y''}{Y}. \quad (38)$$

Thus,

$$X'' - \lambda X = 0 \quad (39)$$

and

$$Y'' + \lambda Y = 0 \quad (40)$$

are our system of ODE's. Now we want to specify our boundary conditions.

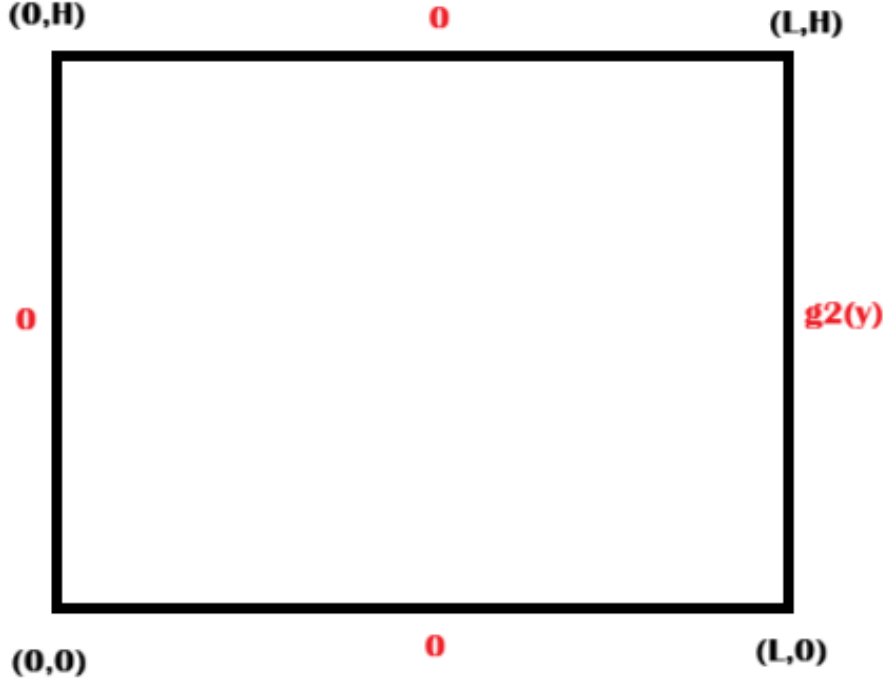


Figure 1: BCs for Problem 5

This results in the following BCs:

- $u(0, y) = 0$
- $u(L, y) = g_2(y)$
- $u(x, 0) = 0$
- $u(x, H) = 0$.

Accordingly,

$$Y(y) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi y}{H}\right). \quad (41)$$

Then,

$$X'' - \lambda X = 0 \quad (42)$$

implies that

$$X(x) = A \sinh(\sqrt{\lambda}x) + B \cosh(\sqrt{\lambda}x). \quad (43)$$

Since

$$X(0) = B = 0, \quad (44)$$

then

$$X(x) = A \sinh\left(\frac{n\pi x}{H}\right). \quad (45)$$

But,

$$X(L) = g_2(y) = A \sinh\left(\frac{n\pi L}{H}\right) \quad (46)$$

$$A = \frac{g_2(y)}{\sinh\left(\frac{n\pi L}{H}\right)}. \quad (47)$$

So

$$u(x, y) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi y}{H}\right) \frac{g_2(y) \sinh\left(\frac{n\pi x}{H}\right)}{\sinh\left(\frac{n\pi L}{H}\right)} \quad (48)$$

where

$$A_n = \frac{2}{H} \int_0^H g_2(y) \sin\left(\frac{n\pi y}{H}\right) dy. \quad (49)$$

6 Extra Resources Which You Gotta Know or You Might Be Cooked:

Trig Identities:

$$\sin(a + b) = \sin(a)\cos(b) + \cos(a)\sin(b) \quad (50)$$

$$\cos(a + b) = \cos(a)\cos(b) - \sin(a)\sin(b) \quad (51)$$

$$\sin(a - b) = \sin(a)\cos(b) - \cos(a)\sin(b) \quad (52)$$

$$\cos(a - b) = \cos(a)\cos(b) + \sin(a)\sin(b) \quad (53)$$

If BC's are

- $u(0, t) = 0 = u(L, t)$ then

$$u(x, t) = \sum_{n=1}^{\infty} A_n e^{-kt\left(\frac{n\pi}{L}\right)^2} \sin\left(\frac{n\pi x}{L}\right) \quad (54)$$

- $\frac{\partial u}{\partial x}(0, t) = 0 = \frac{\partial u}{\partial x}(L, t)$ then

$$u(x, t) = \sum_{n=0}^{\infty} B_n e^{-kt\left(\frac{n\pi}{L}\right)^2} \cos\left(\frac{n\pi x}{L}\right) \quad (55)$$

where

$$A_n = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx \quad (56)$$

$$\text{and} \quad (57)$$

$$B_n = \frac{2}{L} \int_0^L f(x) \cos\left(\frac{n\pi x}{L}\right) dx. \quad (58)$$

Orthogonality of sines and cosines (formulas):

$$\int_0^L \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{m\pi x}{L}\right) dx = \begin{cases} 0; & \text{if } n \neq m \\ \frac{L}{2}; & \text{if } n = m. \end{cases} \quad (59)$$

$$\int_0^L \cos\left(\frac{n\pi x}{L}\right) \cos\left(\frac{m\pi x}{L}\right) dx = \begin{cases} 0; & \text{if } n \neq m \\ \frac{L}{2}; & \text{if } n = m \neq 0 \\ L; & \text{if } n = m = 0 \end{cases} \quad (60)$$