Math 440 Exam 2 Practice Test

October 14, 2025

Timothy Tarter
James Madison University
Department of Mathematics

1 Describe / Explain:

1.1 Laplace's Equation (include well-posedness, uniqueness, and the mean value proposition).

Laplace's equation states that the sum of the first partials are equal to zero, i.e.,

$$0 = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} \tag{1}$$

Well-posedness: there exists a unique solution which depends continuously on non-homogeneous data.

Uniqueness: solutions to laplace's equation are unique.

Mean value proposition: the temperature at the origin is the average of the temperature at the boundary.

1.2 Linear Operators, provide an example of something which is a linear operator, and provide an example of something which is not a linear operator.

A linear operator, $\mathscr{U} : V \to V$ where V is a vector space, has the property that for any $\vec{x}, \vec{y} \in V$ and any scalar c in the field of V,

1. $\mathscr{U}(\vec{x} + \vec{y}) = \mathscr{U}(\vec{x}) + \mathscr{U}(\vec{y})$
2. $\mathscr{U}(c\vec{x}) = c\mathscr{U}(\vec{x})$.

Example:

$$L(u) = \frac{\partial}{\partial x}\left[K_0(x)\frac{\partial u}{\partial x}\right] \tag{2}$$

Non-Example:

$$L(u) = \frac{\partial}{\partial x}\left[K_0(u,x)\frac{\partial u}{\partial x}\right] \tag{3}$$

where K_0 is a non-constant function.

2 Prove Orthogonality of Sines.

We want to show that

$$\int_0^L sin(\frac{n\pi x}{L})sin(\frac{m\pi x}{L}) = \begin{cases} 0;\text{ if } n \neq m \\ \frac{L}{2};\text{ if } n = m. \end{cases} \tag{4}$$

$$\int_0^L cos(\frac{n\pi x}{L})cos(\frac{m\pi x}{L}) = \begin{cases} 0\ ;\text{ if } n \neq m \\ \frac{L}{2};\text{ if } n = m \neq 0 \\ L\ ;\text{ if } n = m = 0 \end{cases} \tag{5}$$

Starting off with (4), if $n = m$, then

$$\int_0^L sin^2(\frac{n\pi x}{L})dx = \int_0^L \frac{1}{2}(1 - cos(2\frac{n\pi x}{L}))dx \tag{6}$$

$$= \frac{1}{2}\left[x - \frac{L}{2n\pi}sin(\frac{2n\pi x}{L})\right]\Big|_0^L \tag{7}$$

$$= \frac{1}{2}\left[L - 0\right] = \frac{L}{2} \tag{8}$$

as desired. Alternatively, if $n \neq m$, then

$$\int_0^L sin(\frac{n\pi x}{L})sin(\frac{m\pi x}{L})dx \tag{9}$$

$$= \int_0^L \frac{1}{2}[cos(\frac{n\pi x}{L} - \frac{m\pi x}{L}) - cos(\frac{n\pi x}{L} + \frac{m\pi x}{L})]dx \tag{10}$$

$$= \frac{1}{2}\int_0^L cos(\frac{\pi x(n-m)}{L}) - cos(\frac{\pi x(n+m)}{L})dx \tag{11}$$

$$= \frac{1}{2}[\frac{L}{\pi(n-m)}sin(\frac{\pi x(n-m)}{L}) - \frac{L}{\pi(n-m)}sin(\frac{\pi x(n-m)}{L})]\Big|_0^L \tag{12}$$

$$= 0 - 0 = 0 \tag{13}$$

as desired.

Moving to (5), if $n \neq m$, it works pretty much the same way as with sines. If $n = m \neq 0$, then

$$\int_0^L cos^2(\frac{n\pi x}{L})dx = \tag{14}$$
$$= \int_0^L \frac{1}{2}(1 + sin(\frac{2n\pi x}{L}))dx \tag{15}$$
$$= \frac{1}{2}[x - \frac{L}{2n\pi}cos(\frac{2n\pi x}{L})]\Big|_0^L \tag{16}$$
$$= \frac{1}{2}[L + (-1 + 1)] \tag{17}$$
$$= \frac{L}{2} \tag{18}$$

as desired. Then finally, if $n = m = 0$,

$$\int_0^L cos^2(0)dx = \int_0^L 1dx = L. \tag{19}$$

3 Solve $\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}$ with certain BC's and IC's, addressing $\lambda > 0, = 0, < 0$.

3.1 $u(0,t) = u(L,t) = 0$

Gives sine solutions.

3.2 $\frac{\partial u}{\partial x}(0,t) = 0 = \frac{\partial u}{\partial x}(L,t)$

Gives cosine solutions.

3.3 $u(0,t) = u(L,t) = 0$ and $\frac{\partial u}{\partial x}(0,t) = 0 = \frac{\partial u}{\partial x}(L,t)$

Gives sine and cosine solutions.

4 Solve $\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}$ at equilibrium in polar coordinates over the interval $(-L, L)$.

We begin with the one-dimensional heat equation

$$u_t = \kappa u_{xx}, \qquad x \in [-L, L],$$

and bend the wire so that its endpoints are in perfect thermal contact:

$$u(-L,t) = u(L,t), \qquad u_x(-L,t) = u_x(L,t).$$

These are the *periodic boundary conditions*, corresponding physically to a wire bent into a circle of circumference $2L$.

At equilibrium, the temperature no longer depends on time, so

$$u_t = 0 \quad \Longrightarrow \quad u_{xx} = 0.$$

Integrating twice gives

$$u_{\text{eq}}(x) = Ax + B.$$

The periodicity condition $u_{\text{eq}}(-L) = u_{\text{eq}}(L)$ yields

$$A(-L) + B = A(L) + B \quad \Longrightarrow \quad A = 0.$$

Hence

$$\boxed{u_{\text{eq}}(x) = B = \text{constant.}}$$

To determine B, note that the total heat (or average temperature) is conserved for periodic boundary conditions:

$$\frac{d}{dt}\int_{-L}^{L} u(x,t)\,dx = \kappa\,[u_x(x,t)]_{x=-L}^{x=L} = \kappa\,(u_x(L,t) - u_x(-L,t)) = 0.$$

Thus, the equilibrium constant equals the initial mean temperature:

$$\boxed{u_{\text{eq}}(x) = \frac{1}{2L}\int_{-L}^{L} u_0(x)\,dx.}$$

Therefore, the bent wire reaches a uniform equilibrium temperature equal to its initial average.

5 Solve Laplace's Equation on a rectangle, with LHS, Top, and Bottom $= 0$, RHS $= g_2(y)$.

We want to start with Laplace's 2D equation,

$$0 = k[\frac{\partial^2 u}{\partial x} + \frac{\partial^2 u}{\partial y}]. \tag{20}$$

If $u(x,t) = X(x)Y(y)$, then

$$0 = X''Y + XY'' \tag{21}$$

implies that

$$\frac{X''}{X} = \lambda = \frac{-Y''}{Y}. \tag{22}$$

Thus,

$$X'' - \lambda X = 0 \tag{23}$$

and

$$Y'' + \lambda Y = 0 \tag{24}$$

are our system of ODE's. Now we want to specify our boundary conditions.

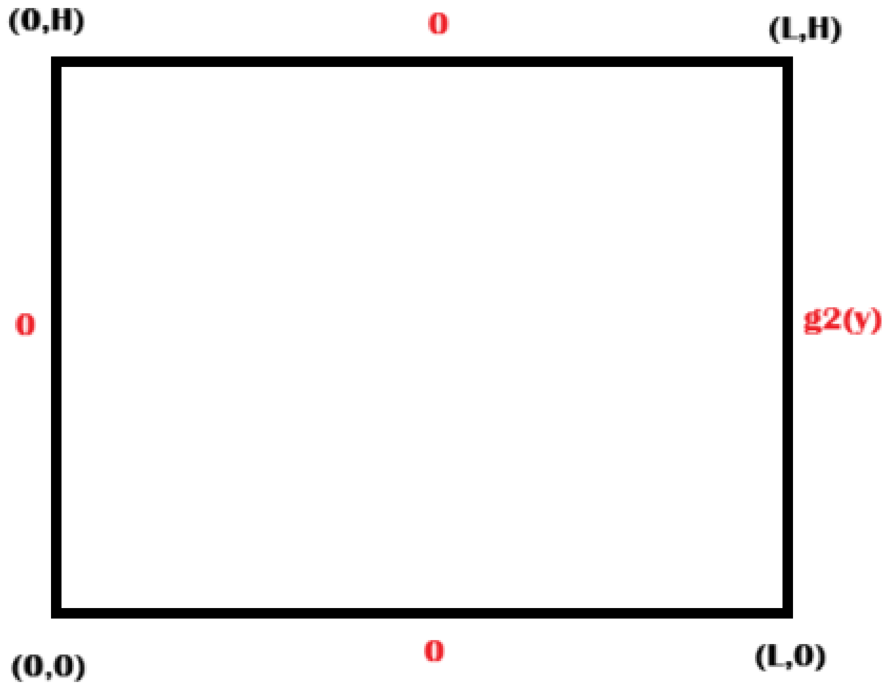

Figure 1: BCs for Problem 5

This results in the following BCs:

- $u(0, y) = 0$
- $u(L, y) = g_2(y)$
- $u(x, 0) = 0$
- $u(x, H) = 0.$

Accordingly,

$$Y(y) = \sum_{n=1}^{\infty} A_n sin(\frac{n\pi y}{H}). \tag{25}$$

Then,

$$X'' - \lambda X = 0 \tag{26}$$

implies that

$$X(x) = Asinh(\sqrt{\lambda}x) + Bcosh(\sqrt{\lambda}x). \tag{27}$$

Since

$$X(0) = B = 0, \tag{28}$$

then

$$X(x) = Asinh(\frac{n\pi x}{H}). \tag{29}$$

But,

$$X(L) = g_2(y) = Asinh(\frac{n\pi L}{H}) \tag{30}$$

$$A = \frac{g_2(y)}{sinh(\frac{n\pi L}{H})}. \tag{31}$$

So

$$u(x,y) = \sum_{n=1}^{\infty} A_n sin(\frac{n\pi y}{H}) \frac{g_2(y) sinh(\frac{n\pi x}{H})}{sinh(\frac{n\pi L}{H})} \tag{32}$$

where

$$A_n = \frac{2}{H} \int_0^H g_2(y) sin(\frac{n\pi y}{H}) dy. \tag{33}$$

6 Extra Resources Which You Gotta Know or You Might Be Cooked:

Trig Identities:

$$sin(a+b) = sin(a)cos(b) + cos(a)sin(b) \tag{34}$$

$$cos(a+b) = cos(a)cos(b) - sin(a)sin(b) \tag{35}$$

$$sin(a-b) = sin(a)cos(b) - cos(a)sin(b) \tag{36}$$

$$cos(a-b) = cos(a)cos(b) + sin(a)sin(b) \tag{37}$$

If BC's are

- $u(0,t) = 0 = u(L,t)$ then

$$u(x,t) = \sum_{n=1}^{\infty} A_n e^{-kt(\frac{n\pi}{L})^2} sin(\frac{n\pi x}{L}) \tag{38}$$

- $\frac{\partial u}{\partial x}(0,t) = 0 = \frac{\partial u}{\partial x}(L,t)$ then

$$u(x,t) = \sum_{n=0}^{\infty} B_n e^{-kt(\frac{n\pi}{L})^2} cos(\frac{n\pi x}{L}) \tag{39}$$

where

$$A_n = \frac{2}{L} \int_0^L f(x) sin(\frac{n\pi x}{L}) dx \tag{40}$$

$$\text{and} \tag{41}$$

$$B_n = \frac{2}{L} \int_0^L f(x) cos(\frac{n\pi x}{L}) dx. \tag{42}$$

Orthogonality of sines and cosines (formulas):

$$\int_0^L sin(\frac{n\pi x}{L})sin(\frac{m\pi x}{L}) = \begin{cases} 0; \text{ if } n \neq m \\ \frac{L}{2}; \text{ if } n = m. \end{cases} \tag{43}$$

$$\int_0^L cos(\frac{n\pi x}{L})cos(\frac{m\pi x}{L}) = \begin{cases} 0 \text{ ; if } n \neq m \\ \frac{L}{2}; \text{ if } n = m \neq 0 \\ L \text{ ; if } n = m = 0 \end{cases} \tag{44}$$