Commutative Algebra, Gr$\ddot{o}$bner Basis, and Algebraic Geometry, Week One Notes

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Reference Texts: Ideals, Varieties, and Algorithms (Cox, Little, O'Shea), An Introduction to Gr$\ddot{o}$bner Basis (Adams & Loustaunau), and Algebraic Geometry (Hartshorne)

§1.1 **Polynomials and Affine Spaces**

Definition 1. *A monomial in $x_1, \ldots, x_n$ is a product*

$$x^\alpha = x_1^{\alpha_1} * x_2^{\alpha_2} * \cdots * x_n^{\alpha_n} \tag{1}$$

and $|\alpha| = \sum_{i \in I} \alpha_i$ is the total degree of x^α.

Definition 2. *A polynomial in $x_1, \ldots, x_n$ with coefficients in a field K is*

$$f = \sum_\alpha a_\alpha x^\alpha; \qquad a_\alpha \in K \tag{2}$$

Definition 3. *The n-dimensional affine space over a field K is*

$$k^n = \{(a_1, \ldots, a_n) | a_1, \ldots, a_n \in K\} \tag{3}$$

Proposition 1. *Let K be a field of characteristic zero, $f \in K[x_1, \ldots x_n]$. Then $f = 0$ if and only if $f : k^n \to K$ is the zero function.*

Proposition 2. *Let K be as above, and $f, g \in K[x_1, \ldots, x_n]$. Then $f = g$ if and only if $f : k^n \to K$ and $g : k^n \to K$ are the same function.*

§1.2 **Affine Varieties**

Definition 4. *Let K be a field and $f_1, \ldots, f_s \in K[x_1, \ldots, x_n]$. Then*

$$\mathbb{V}(f_1, \ldots, f_s) = \{(a_1, \ldots, a_n) \in k^n | f_i(a_1, \ldots, a_n) = 0, \forall i \in \{1, \ldots, s\}\} \tag{4}$$

is the affine variety defined for $f_1, \ldots, f_s$ over K.

Proposition 3. *Affine varieties are closed under finite union and intersection.*

Assigned Problems

- (7)
- (11)
- (12)

§1.3 Parameterizations of Affine Varieties

Proposition 4. *The unit circle, $x^2 + y^2 = 1$ is parameterized by the equations*

$$x = \frac{1 - t^2}{1 + t^2} \tag{5}$$

$$y = \frac{2t}{1 + t^2}, \tag{6}$$

but doesn't contain (-1,0) due to geometric construction.

Definition 5. *Given $V = \mathbb{V}(f_1, \dots, f_s) \subseteq k^n$, we define a rational parametric representation of V to consist of the rational functions*

$$r_1 \dots r_n \in K[t_1 \dots t_m] \tag{7}$$

such that the points

$$x_1 = r_1(t_1, \dots, t_m) \tag{8}$$

$$\downarrow \tag{9}$$

$$x_n = r_n(t_1, \dots, t_m) \tag{10}$$

lie in V.

Definition 6. *We often refer to $\mathbb{V}(f_1, \dots, f_s)$ as an implicit representation of V.*

Proposition 5. *Not every affine variety has a rational parametric representation. If an affine variety does have a rational parametric representation, we call it unirational.*

Proposition 6. *Given a parametric representation of an affine variety, we can always find an implicit representation.*

Example 1. Elimination Theory: *Given*

$$x = 1 + t \tag{11}$$

$$y = 1 + t^2 \tag{12}$$

we get

$$t = x - 1 \tag{13}$$

$$y = 1 + (x - 1)^2 \tag{14}$$

$$y = x^2 - 2x + 2 \tag{15}$$

So, $\mathbb{V}(-y + x^2 - 2x + 2)$ is the implicit representation of $\mathbb{V}(1 + t, 1 + t^2)$.

Example 2. *To parameterize $x^2+y^2=1$ (the unit circle) geometrically, we select (-1,0) as our fixed point, and then target the top half of the circle with*

$$1+t^2 \tag{16}$$

and the bottom half of the circle with

$$1-t^2. \tag{17}$$

If you draw it on a piece of paper, it makes sense.

Example 3. *We want to parameterize $\mathbb{V}(y-x^2, z-x^3)$. Let*

$$x=t \tag{18}$$

in

$$y-x^2=z-x^3=0. \tag{19}$$

Then,

$$\begin{align} x &= t \tag{20}\\ y &= t^2 \tag{21}\\ z &= t^3 \tag{22} \end{align}$$

implies that

$$\begin{align} \vec{r(t)} &= \langle t, t^2, t^3 \rangle \tag{23}\\ \vec{r'(t)} &= \langle 1, 2t, 3t^2 \rangle. \tag{24} \end{align}$$

Using a bit of multivariable calculus, we know that the tangent line is:

$$\begin{align} &= \vec{r}(t) + u\vec{r'}(t) \tag{25}\\ &= \langle t, t^2, t^3 \rangle + u\langle 1, 2t, 3t^2 \rangle \tag{26}\\ &= \langle t+u, 2tu+t^2, t^3+3t^2u \rangle. \tag{27} \end{align}$$

Relaxing t, we find that

$$\begin{align} x &= t+u \tag{28}\\ y &= t^2+2tu \tag{29}\\ z &= t^3+3t^2u, \tag{30} \end{align}$$

where t tells us where we are on the curve, and u tells us where we are on the tangent line.

Definition 7. *The Bezier cubic is defined by*

$$\begin{align} x &= (1-t)^3x_0 + 3t(1-t)^2x_1 + 3t^2(1-t)x_2 + t^3x_3 \tag{31}\\ y &= (1-t)^3y_0 + 3t(1-t)^2y_1 + 3t^2(1-t)y_2 + t^3y_3 \tag{32} \end{align}$$

for $0 \le t \le 1$. Note,

$$\begin{align} (x(0), y(0)) &= (x_0, y_0) \tag{33}\\ (x(1), y(1)) &= (x_3, y_3) \tag{34}\\ (x'(0), y'(0)) &= 3(x_1-x_0, y_1-y_0) \tag{35}\\ (x'(1), y'(1)) &= 3(x_3-x_2, y_3-y_2) \tag{36}\\ & \tag{37} \end{align}$$

defines the control polygon, which the Cubic always lies inside.

Assigned Problems

- (1)
- (3)
- (4)
- (9)
- (14)
- (15)

§1.4 **Ideals**

Definition 8. *$I \subseteq K[x_1 \dots x_n]$ is an ideal if,*

1. *$0 \in I$*
2. *If $f, g \in I$, $f + g \in I$*
3. *If $f \in I, h \in K[x_1 \dots x_n]$ implies $hf \in I$.*

Definition 9. *The ideal generated by $f_1 \dots f_s \in K[x_1 \dots x_n]$ is defined as*

$$\langle f_1 \dots f_s \rangle = \{\sum_{i=1}^{s} h_i f_i | h_1 \dots h_s \in K[x_1 \dots x_n]\}. \tag{38}$$

Note that, for $f_i \in \langle f_1 \dots f_s \rangle$, $f_i = 0$ and $\sum f_i = 0$.

Proposition 7. *If $f_1 \dots f_s$ and $g_1 \dots g_s$ are bases of the same ideal in $K[x_1 \dots x_n]$ so that $\langle f_1 \dots f_s \rangle = \langle g_1 \dots g_s \rangle$ implies $\mathbb{V}(f_1 \dots f_s) = \mathbb{V}(g_1 \dots g_s)$.*

Definition 10. *Let $V \subseteq k^n$. Then,*

$$I(\mathbb{V}) = \{f \in K[x_1 \dots x_n] | f(a_1 \dots a_n) = 0, \forall (a_1 \dots a_n) \in \mathbb{V}\} \tag{39}$$

Proposition 8. *$I(\mathbb{V})$ is an ideal.*

Proof: $0 \in I(\mathbb{V})$ since the zero polynomial vanishes in k^n, and thus, in $\mathbb{V}$. Let $f, g \in I(\mathbb{V})$ and $h \in K[x_1 \dots x_n]$. Let $(a_1 \dots a_n) \in \mathbb{V}$. Then,

$$f(a_1 \dots a_n) + g(a_1 \dots a_n) = 0 + 0 = 0 \tag{40}$$
$$h(a_1 \dots a_n) f(a_1 \dots a_n) = h(a_1 \dots a_n) * 0 = 0. \tag{41}$$

Thus, $I(\mathbb{V})$ is an ideal.

$\square$

Example 4. *Consider $V = \{0, 0\} \in k^2$; then its ideal, $I(\{(0, 0)\})$, consists of all of the polynomials which vanish at the origin. We claim that*

$$I(\{(0, 0)\}) = \langle x, y \rangle. \tag{42}$$

The proof is an exercise.

Example 5. *Consider* $V = k^n \in k^n$*; then its ideal,* $I(k^n)$*, consists of all of the polynomials which vanish everywhere in* k^n*. If* K *has characteristic zero, we claim that*

$$I(k^n) = \{0\}. \tag{43}$$

The proof is an exercise.

Proposition 9. *(This one is mine and makes a nice boilerplate).* $I(\mathbb{V}(f_1 \dots f_s)) = \langle f_1, \dots, f_s \rangle$ *if we can write any polynomial* $f \in k[x_1 \dots x_n]$ *as*

$$f = r + \sum_{i=1}^{s} h_i f_i, \tag{44}$$

letting $h_i \in K$ *and* $r \in K[x_1 \dots x_n]$*, and show that* $r = 0$ *using the parameterization of* f*.*

Example 6. *We claim that* $I(\mathbb{V}(y - x^2, z - x^3)) = \langle y - x^2, z - x^3 \rangle$*. Since* $y - x^2, z - x^3 \in I$*,*

$$h_1(y - x^2) + h_2(z - x^3) \in I. \tag{45}$$

Thus, $\langle y - x^2, z - x^3 \rangle \subseteq I$*. To prove that* $r = 0$*, we use the parameterization of the twisted cubic,* (t, t^2, t^3)*. Since* f *vanishes on* $\mathbb{V}$*, we obtain*

$$0 = f(t, t^2, t^3) = 0 + 0 + r(t). \tag{46}$$

Thus, set equality holds.

Proposition 10. $\langle f_1 \dots f_s \rangle \subseteq I(\mathbb{V}(f_1 \dots f_s))$*.*

Proposition 11. *Let* V *and* W *be affine varieties in* k^n*. Then:*

- $V \subseteq W$ *iff* $I(V) \supseteq I(W)$
- $V = W$ *iff* $I(V) = I(W)$*.*

Three key questions:

1. (Ideal Description) Can every ideal be finitely generated by polynomials in $K[x_1 \dots x_n]$? **Yes, by Hilbert.**
2. (Ideal Membership) If $f_1 \dots f_s \in K[x_1 \dots x_n]$, is there an algorithm to decide whether a given $f \in K[x_1 \dots x_n]$ lies in $\langle f_1 \dots f_s \rangle$?
3. (Nullstellnsatz) Given $f_1 \dots f_s \in K[x_1 \dots x_n]$, what is the exact relation between $\langle f_1 \dots f_s \rangle$ and $I(\mathbb{V}(f_1 \dots f_s))$?

Assigned Problems

- (2)
- (3)
- (5)
- (7)
- (8)
- (9)