The Algebraic Geometry of Sine and Cosine Roses (& Some Origami)

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Abstract

For Valentines day this past year, I coded up some sine roses in MATLAB, and folded them into origami for my girlfriend. She thought it was neat, and asked to learn some of the math behind why it worked! Recently, I've been studying Algebraic Geometry with Dr. Beth Arnold and Algebraic Topology with Dr. Jonathan Bush, and I've gained some more insight on *why* my little trick worked. I decided to write this as a way to show a more casual math-enjoyer (like my girlfriend) some of the really deep and beautiful mathematical context behind sine and cosine roses - and their origami representations! I can't promise that it's simple, but it is interesting if nothing else. (I've also been told that it is quite romantic, so feel free to use the tricks included in this paper!)

Sine and Cosine Roses

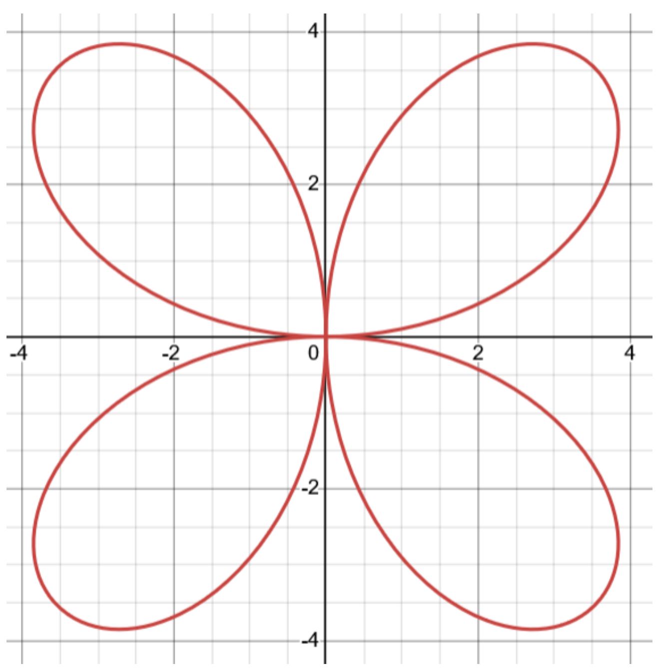

Figure 1: This is a Sine Rose! Specifically, $r = 5sin(2\theta)$

Why r and θ? Where are x and y?

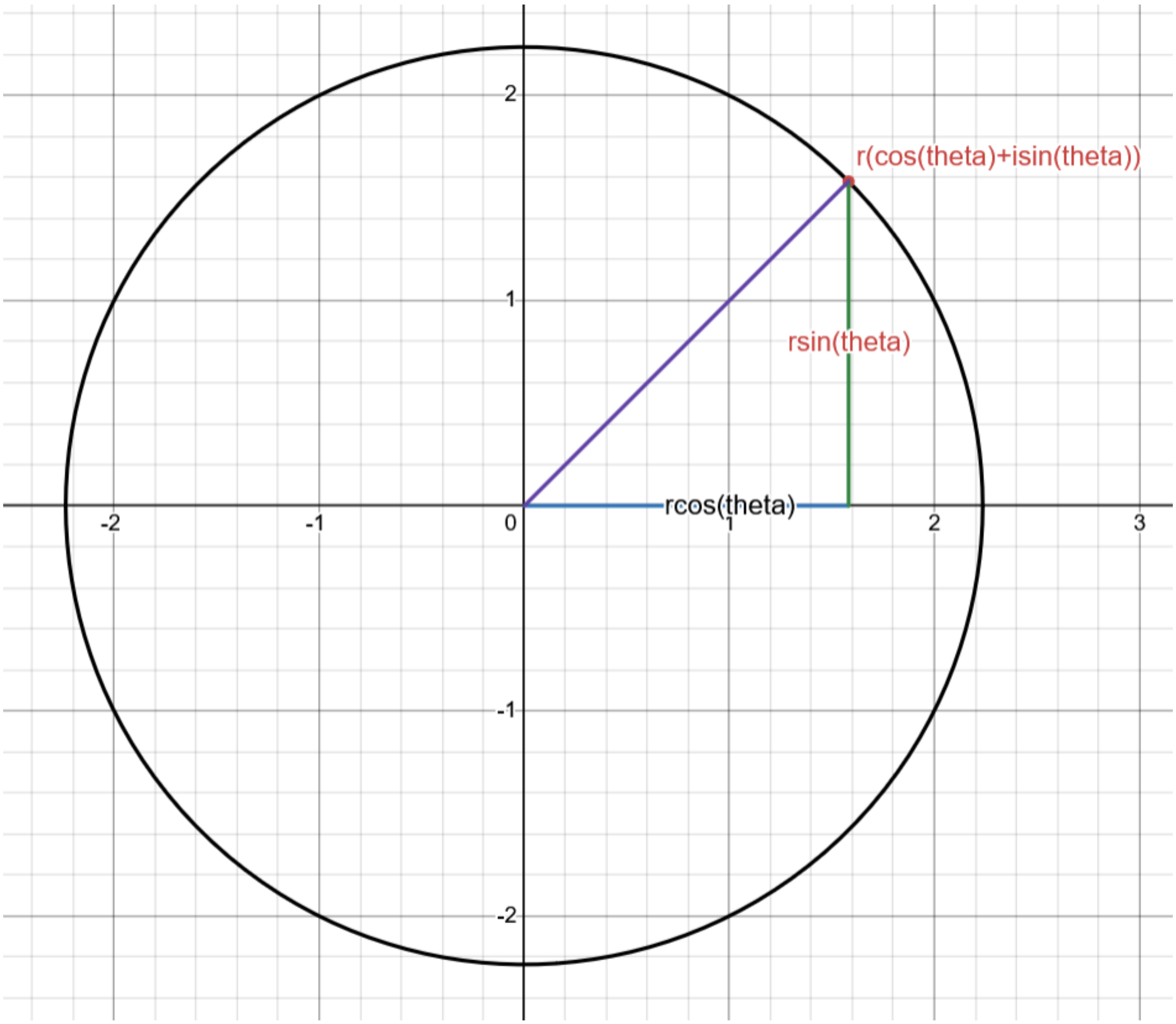

Figure 2: The Unit Circle, and $x + iy = r(cos(\theta) + isin(\theta))$

We can parameterize this as follows:

$$\begin{aligned} r^2 &= x^2 + y^2 \\ x &= rcos(\theta) \\ y &= rsin(\theta) \\ \theta &\in (-\pi, \pi]. \end{aligned}$$

Proposition 1. *We can make the sine rose* $r = 5sin(2\theta)$ *with the equation,*

$$\boxed{\frac{1}{5}(x^2 + y^2)^3 - 20x^2y^2 = 0.} \tag{1}$$

Proof: Using our little parameterization, we can sub out all of our x and y symbols.

$$\frac{1}{5}(r^2)^3 - 20x^2y^2 = 0$$

Sub in $x^2 + y^2 = r^2$

$$\frac{1}{5}(r^2)^3 - 20(rcos^2(\theta))^2y^2 = 0$$

Sub in $x = rcos(\theta)$ and $y = rsin(\theta)$

$$\frac{1}{5}(r^2)^3 - 20(rcos^2(\theta))^2(rsin^2(\theta))^2 = 0$$

Simplify and expand

$$\frac{1}{5}r^6 - 20r^2cos^4(\theta)r^2sin^4(\theta) = 0$$

Collect like terms

$$\frac{1}{5}r^6 - 20r^4cos^4(\theta)sin^4(\theta) = 0$$

Multiply by 5 and divide by r^4

$$r^2 - 100cos^4(\theta)sin^4(\theta) = 0$$

Move stuff to RHS

$$r^2 = 100cos^4(\theta)sin^4(\theta) =$$

Square root everything

$$r = 10cos^2(\theta)sin^2(\theta) = 5 * 2cos^2(\theta)sin^2(\theta)$$

Use $2cos^2(\theta)sin^2(\theta) = sin(2\theta)$

$$r = 5 * sin(2\theta) \text{ as desired.}$$

□

Is there another way to parameterize this sine rose?

Yes!

Proposition 2. *We can parameterize*

$$\frac{1}{5}(x^2 + y^2)^3 - 20x^2y^2 = 0.$$

$$\Longleftrightarrow$$

$$r = 5sin(2\theta)$$

as

$$\boxed{(x(t), y(t)) = \big(10\sin t\ \cos^2 t,\ 10\sin^2 t\ \cos t\big), \quad t \in [0, 2\pi)}$$

Proof: We start with the implicit curve

$$\frac{1}{5}(x^2 + y^2)^3 - 20x^2y^2 = 0.$$

Passing to polar coordinates via $x = r\cos\theta$, $y = r\sin\theta$, we have

$$x^2 + y^2 = r^2, \qquad x^2y^2 = r^4\cos^2\theta\ \sin^2\theta.$$

Substituting,

$$\frac{1}{5}(r^2)^3 - 20\big(r^4\cos^2\theta\,\sin^2\theta\big) = 0 \quad\Longrightarrow\quad \frac{1}{5}r^6 - 20r^4\cos^2\theta\,\sin^2\theta = 0.$$

Factoring r^4 and ignoring the trivial solution $r = 0$,

$$\frac{1}{5}r^2 = 20\cos^2\theta\,\sin^2\theta.$$

Using $\sin(2\theta) = 2\sin\theta\cos\theta$ so that $\cos^2\theta\,\sin^2\theta = \frac{1}{4}\sin^2(2\theta)$, we get

$$\frac{1}{5}r^2 = 5\sin^2(2\theta) \quad\Longrightarrow\quad r^2 = 25\sin^2(2\theta).$$

Thus any choice $r = \pm\,5\sin(2\theta)$ traces the same locus (changing the sign of r only shifts θ by π). We therefore pick the simple representative

$$r(\theta) = 5\sin(2\theta).$$

Setting $t = \theta$, the Cartesian parameterization is

$$x(t) = r(t)\cos t = 5\sin(2t)\cos t = 10\sin t\,\cos^2 t, \tag{2}$$

$$y(t) = r(t)\sin t = 5\sin(2t)\sin t = 10\sin^2 t\,\cos t, \tag{3}$$

which parameterizes the curve for $t \in [0, 2\pi)$.

$$\boxed{(x(t), y(t)) = \big(10\sin t\,\cos^2 t,\ 10\sin^2 t\,\cos t\big), \quad t \in [0, 2\pi),\ \text{as desired.}}$$

So, we have three ways to parameterize a sine rose... why do we care?

Functionally, it's easier to draw things on a computer if it's a function of one variable. But as it turns out, it makes folding a piece of paper into a sine rose easier too! Moreover, there's some deep algebraic geometry and algebraic topology going on behind the scenes to make it happen.

Algebra, Topology, and Origami

Affine Varieties, Ideals, and Grobner Basis

In Algebraic Geometry, one of the things we like to study is called an affine variety. A loose definition is as follows: an affine variety generated by some polynomials is the set of points which make all of the polynomials zero simultaneously. More technically, we define it as:

Definition 1. *The affine variety generated by a set of polynomials $f_1, \ldots, f_s \in k[x]$, for some field k of characteristic zero, is*

$$\mathbb{V}(f_1, \ldots, f_s) = \{p \in k^n \mid f_i(p) = 0\,\textit{for all } i\ \in 1, \ldots, s\} \tag{4}$$

Affine varieties are special because they tie together polynomial equations by their shared roots. For now, suffice it to say that this is a property of polynomial functions which we care about for our application. Moving forwards, we have two more concepts that we need to tie the idea of parameterization, affine varieties, cosine roses, and origami back together.

In Abstract Algebra, we define a polynomial ring over a field k to be the set of polynomials (like one uses in calculus and high-school algebra) with coefficients in k. The theory of polynomial rings goes quite deep (see Galois Theory for example), however, all we really need right now is the notion of a basis for such a structure.

Definition 2. *For a polynomial ring, $k[x_1, x_2, \ldots, x_n]$, we say that the basis of $k[x_1, x_2, \ldots, x_n]$ is the set of all permutations of $[x_1, x_2, \ldots, x_n]$ multiplied together. More specifically, $k[x_1, x_2, \ldots, x_n]$ is the set of all polynomials with coefficients in k and independent variables*

$$x_1, x_2, \ldots, x_n, x_1x_2, x_1x_3, \ldots x_1x_n, x_2x_3, \ldots, x_2x_n, etc. \tag{5}$$

This definition is somewhat unsatisfactory because it does not abstract very easily, at least how we've written it here. That said, it is nice because we can clearly see that for any two polynomials in $k[x_1, x_2, \ldots, x_n]$, they either have some factors in common, or they don't. (Note, factoring means the same thing it did in high-school algebra; we want to split $f(x_1, x_2, \ldots, x_n)$ into $f(x_1, x_2, \ldots, x_n) = (x_1 + a_1)(x_1 + a_2) \ldots (x_1 + a_n) + (x_2 + b_1)(x_2 + b_2) \ldots (x_2 + b_m) + \ldots$, if possible.) We call this "linear factoring." If f is a non-linear polynomial, i.e., we have any non-zero terms which are two of our basis elements multiplied by each other, it gets significantly more complicated to factor a polynomial linearly. We are also not guaranteed that f factors linearly over k, unless $k = \mathbb{C}$. In the same sense that we have prime counting numbers, we have prime polynomials over k - we just call them irreducible!

All of this is to the point of deriving what is called a **quotient ring of** $k[x_1 \ldots x_n]$**.** If we collect a bunch of polynomials with certain properties, we can create what is called an **ideal of** $k[x_1 \ldots x_n]$ **generated by** $f_1 \ldots f_s$**.** The specific properties are as follows:

Definition 3. *We call*

$$I = \langle f_1 \ldots f_s \rangle = \{\sum_{i=1}^{s} h_i f_i \mid h_i \in k[x_1 \ldots x_n]\} \tag{6}$$

the ideal generated by $f_1 \ldots f_s$ if for any $g, h \in I$ and any $\ell \in k[x_1 \ldots x_n]$,

1. $g + h \in I$
2. $g\ell \in I$.

The significance of ideals of polynomial rings is that we can "quotient" $k[x_1 \ldots x_n]$ by I, as follows.

Definition 4. *We define*

$$k[x_1 \ldots x_n]/\langle f_1 \ldots f_s \rangle = \{g + \langle f_1 \ldots f_s \rangle \mid g \in k[x_1 \ldots x_n]\} \tag{7}$$

as the distinct polynomials in $k[x_1 \ldots x_n]$ which are NOT in $\langle f_1 \ldots f_s \rangle$.

One such way we can work out if a polynomial (of one variable) is in $k[x]/I$ is actually to just do long division! If it has a remainder of zero, then it is in I, if it doesn't, then it becomes its remainder in $k[x]/I$. This is a non-trivial thing, because it actually turns out that the remainder polynomial is the **greatest common divisor** of the $f_1 \ldots f_s$ with g.

It also turns out that if an ideal is generated by many polynomials, some of them may be redundant! We can find out, again, by taking the $gcd(f_1 \ldots f_s)$. For polynomials of one variable, we call this the Gröbner basis of I.

Definition 5. *More generally, the Gröbner basis of an ideal is the minimum set of polynomials needed to express all of the information in I. It is traditionally computed using Buchbaker's algorithm.*

To tie everything back to sine and cosine roses, we need one last idea. What is the ideal of a variety? Since a variety is a set of common zeroes of a function, and an ideal is the set of linear combinations of some polynomials, we want an intuitive definition that connects the structure of ideals to the computation of functions in varieties. Thus,

Definition 6. *For a variety $V = \mathbb{V}(f_1 \ldots f_s)$, we define the ideal of V to be:*

$$I(V) = \{f \in k[x_1 \ldots x_n] \mid f(v) = 0 \textit{ for every } v \in V\}. \tag{8}$$

So, in summary: we have many parameterizations of our sine rose. All of them share common zeroes, provided we use all of the necessary equations for each parameterization.

Proposition 3. $\mathbb{V}(r - 5sin(2\theta)) = \mathbb{V}(\frac{1}{5}(x^2 + y^2)^3 - 20x^2y^2) = \mathbb{V}(10sin(t)cos^2(t), 10sin^2(t)cos(t))$

The proof is actually that we can make a variable substitution to turn each equation into the other equations, which we've already done!

$\square$

Why is this important? Because...

Proposition 4. $I(\mathbb{V}(r-5sin(2\theta))) = I(\mathbb{V}(\frac{1}{5}(x^2+y^2)^3-20x^2y^2)) = I(\mathbb{V}(10sin(t)cos^2(t), 10sin^2(t)cos(t)))$

Namely, the actual $I(V)$ here is the Gröbner basis for any of these equations! I'm omitting the proof though, because like most mathematicians, I am beautifully lazy.

Connecting I(V) to Topology

I want to preface this section by saying that I am not a topologist, and that I really only have an undergraduate understanding of the field at best. With that disclaimer out of the way, to most laypeople, Topology can be best explained by saying that it (sorta) largely deals with spaces. Algebraic topology looks at the structure of simplicial complexes and affine schemes (which behave kinda like our quotient rings - I'm handwaving heavily here though) as a spacial issue regarding the number of holes in an object (genus). In addition to the number of holes in an object, if we have an *orientation for the sides of an object*, we can take it and turn it into another object.

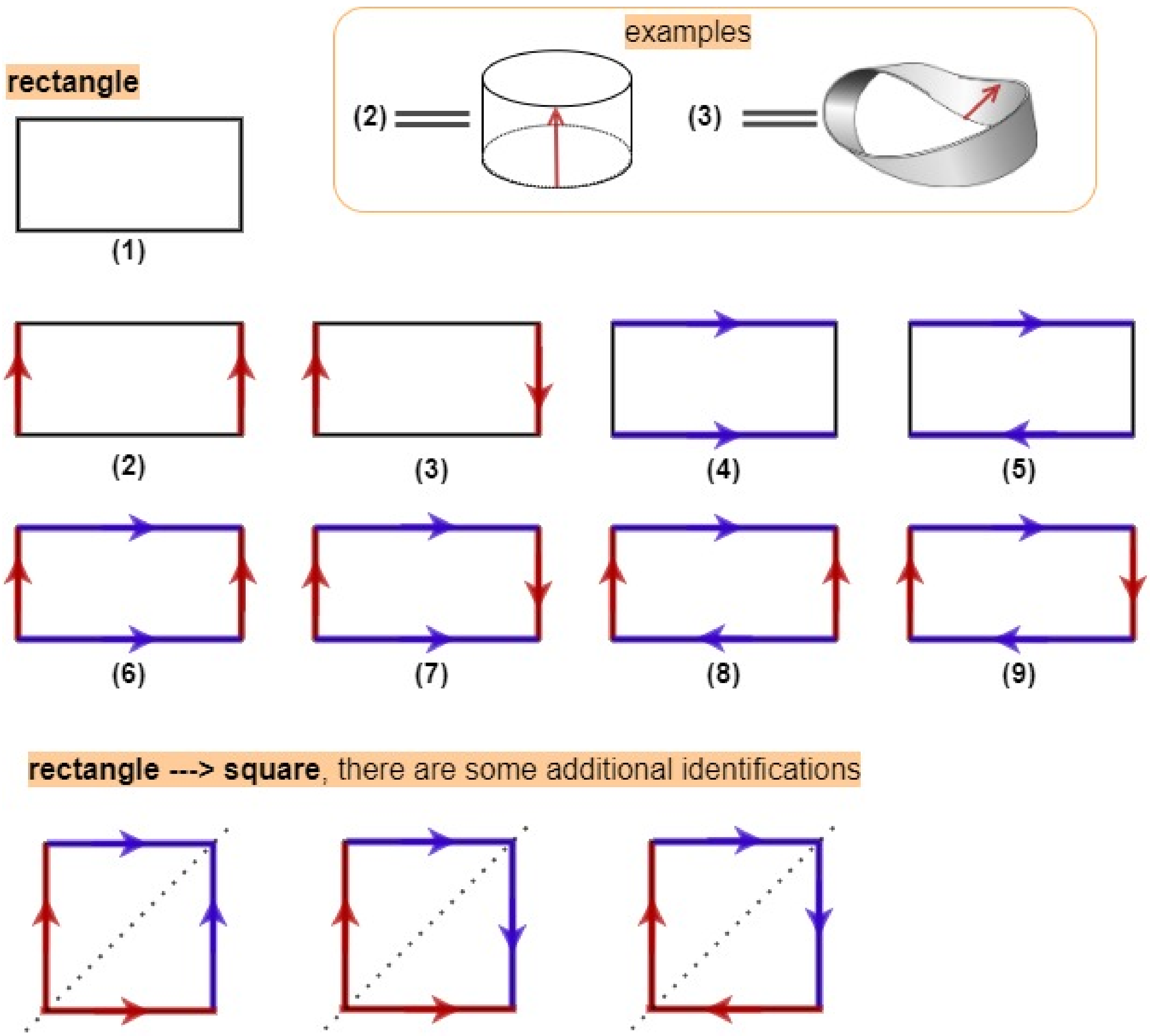

Figure 3: Oriented Plane ⇒ Möbius Strip

Our example of the Mobius strip actually represents itself as a quotient ring (which is a type of scheme),

$$M \simeq (\mathbb{R} \times [0, 1])/\mathbb{Z}. \tag{9}$$

(Sidenote, the Mobius strip is strictly a globally un-orientable object, however, we can orient its sides when we glue them together locally.)

This begs the question, if we let

$$V = \mathbb{V}(\frac{1}{5}(x^2 + y^2)^3 - 20x^2y^2), \tag{10}$$

what affine scheme is $k[x, y]/I(V)$? Moreover, is it constructed in such a way that we can fold it into an origami-like structure? (Notably, this second question is the same as asking if it has any holes in it).

As a matter-of-fact, $k[x, y]/I(V)$ is diffeomorphic (isomorphism / structure equivalence between smooth manifolds) to a closed curve on a sphere, which has no holes in it (genus zero). This means that there exists an origami folding pattern which takes a flat sheet of paper, and without cutting or altering it, folds it into a paper rose with four-fold symmetry. (I will loosely sketch the proof in the paragraph below, but one can intuit that this is true from the fact that origami roses with four-fold symmetry have existed for a long time before this paper).

Proof Sketch: Since we have a rational parameterization in one variable for our sine rose, there exists a dominant rational map from the Riemann sphere $\mathbb{P}^1_k$ to the projective closure of $\frac{1}{5}(x^2 + y^2)^3 - 20x^2y^2 = 0$, which is $(x^2 + y^2)^3 - 100x^2y^2 = 0$. This dominant rational map is injective to any Zariski-open subset of the projective closure, and so any closed curve of the projective closure is diffemorphic to the Riemann sphere.

$\square$

Conclusion

In short, we've mathematically proven that one can mathematically construct an origami rose with four-fold symmetry, without cutting or breaking a piece of paper. Since this paper really couldn't be complete without an origami instruction on how to make such a four-fold-symmetric rose, here is an instructional I found online which was similar to my construction for Valentines day.

https://www.instructables.com/Origami-Rose-in-Bloom-Part-1/

Happy folding!