A Faster Attack on LFSR Recurrences

An Alternative to Berlekamp-Massey for Complete Ciphertexts

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Overview

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- As it turns out, it repeats itself after 31 terms.

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$$(01111, x_{n+5} \equiv x_n + x_{n+2}) \implies 1001101001000010101110110001111$$

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- Start with two cases $n = 1$ and $n = 2$, then $x_1 = 0, x_2 = 1, x_3 = 1, x_4 = 0$.

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The solution to the system is $c_0 = 1$, $c_1 = 1$, thus we assume that the recurrence is $x_{n+2} \equiv x_n + x_{n+1}$, which **doesn't** fit the sixth digit, so make a new guess x_{n+3}.

Berlekamp-Massey

- The Gaussian Elimination Approach is computationally expensive.

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Let N be the length of the shortest recurrence, and M_N be the matrix containing the systems of equations for $[c_1, c_2, \ldots, c_n]^\top$, then $det(M_N) \equiv 1$ [mod 2], and for any $n > N$, $det(M_n) \equiv 0$ [mod 2][1].

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The first matrix with a determinant of one, with ε many zero determinants after the one, ***is*** *the coefficient matrix for the system of equations, equivalent to the recurrence polynomial.*

Main Results

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Once the bit recurrence string is obtained, recover the recurrence polynomial with BM.

Proof

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- Since $|\gamma|$ divides n by definition, our algorithm will identify it as a potential k. Since it is also a recurrence, it can be reconstructed into β by concatenating it with itself $\frac{n}{|\gamma|}$ times, and thus is contained in θ.

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- We want to show that $\gamma \in \theta$ and we want to show that there does not exist something smaller than γ in θ.
- Since $|\gamma|$ divides n by definition, our algorithm will identify it as a potential k. Since it is also a recurrence, it can be reconstructed into β by concatenating it with itself $\frac{n}{|\gamma|}$ times, and thus is contained in θ.
- To show that there is not a smaller element in θ than γ, we assume there exists $\eta \in \theta$ with $|\eta| < |\gamma|$. But if $\eta \in \theta$, then η can be reconstructed as a recurrence of β by concatenating itself $\frac{n}{|\eta|}$ times, and is therefore a smaller recurrence than γ, which we assumed to be the minimal recurrence. Therefore, there does not exist such an η, and $\min \theta = \gamma$.

Computational Complexity

Time Complexity

Algorithm	Complexity
THP	$O(n + log(n))$
BM	$O(n^2)$

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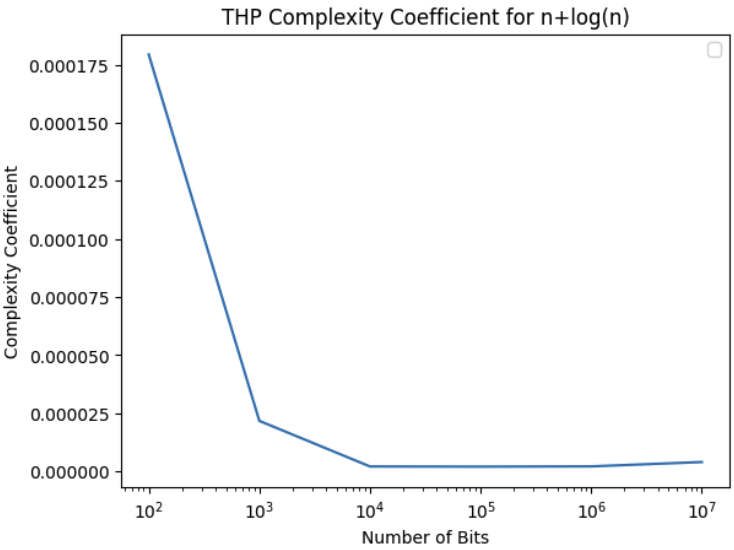
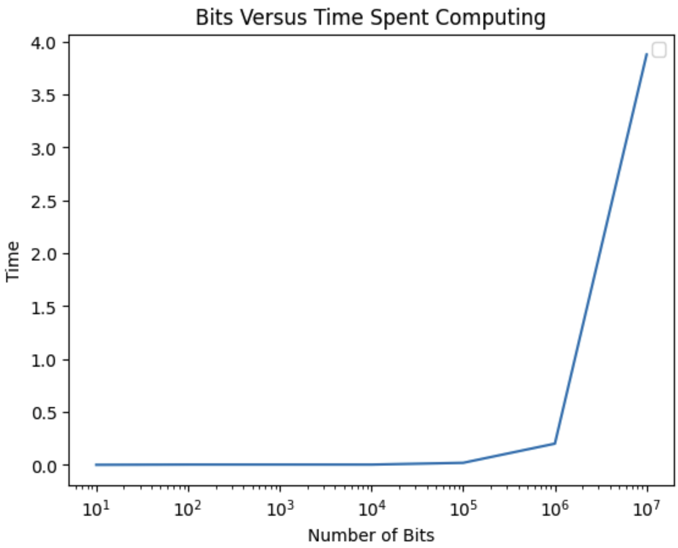
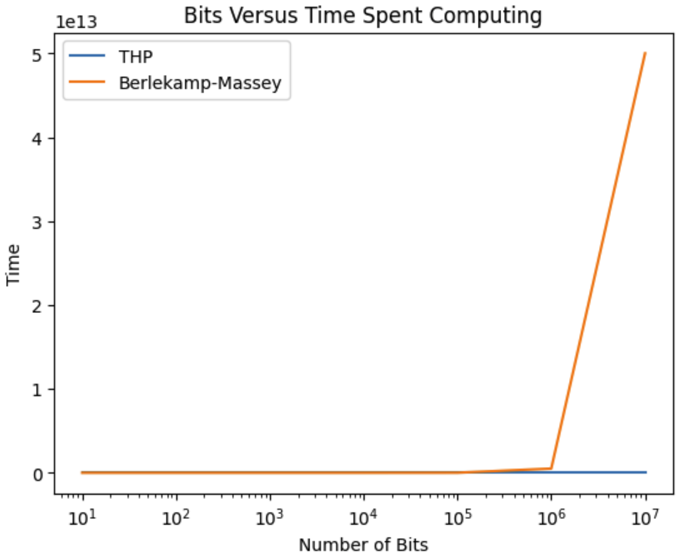
1. THP requires a full string whereas Berlekamp-Massey works for partial string as well.
2. Our time complexity prevails for large sequences, unfeasable for Berlekamp-Massey.


THP Complexity Coefficient for n+log(n)
Complexity Coefficient
0.000175
0.000150
0.000125
0.000100
0.000075
0.000050
0.000025
0.000000
10^2
10^3
10^4
10^5
10^6
10^7
Number of Bits


Bits Versus Time Spent Computing
4.0
3.5
3.0
2.5
2.0
1.5
1.0
0.5
0.0
Time
10^1
10^2
10^3
10^4
10^5
10^6
10^7
Number of Bits


Bits Versus Time Spent Computing
1e13
5
4
3
2
1
0
Time
THP
Berlekamp-Massey
10^1
10^2
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10^5
10^6
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GitHub Repository

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