Dimension of the Sierpinski Triangle

Tarun Mistry | Mathematics

Imperial College London

Dimension and Fractals

Fractals

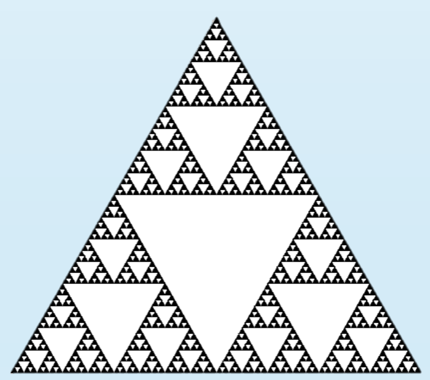
Fractals are shapes with non-integer dimensions that produce a geometric pattern that repeats at different scales (1) , with the potential to be self similar.

Dimension

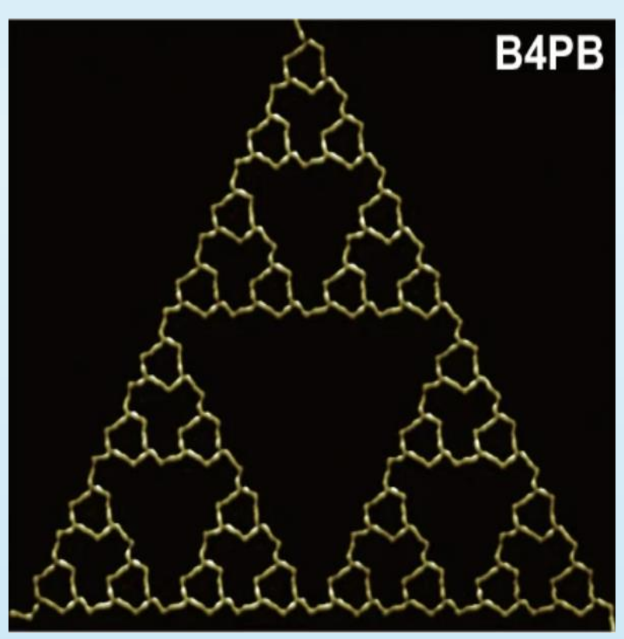
A dimension is a type of measurement such as length that varies between fractal types with higher dimensions represent a rougher shape. (1) Calculating these dimensions involves measuring and comparing the masses of a fractal from its original to its scaled forms.

What is the Sierpinski triangle?

This fractal is based around a triangle with four equal triangles within the original triangle. When the central triangle is removed, the new set of 3 triangles are then treated as if they were the original. This can be repeated infinitely within the finite space to create an infinite regression. (2)



Sierpinski Triangle in Chemical Engineering



The Sierpinski triangle has been modelled as chemical structures using iron and organic ' link molecules ' . The self assembling molecule, is the first order 5 fractal to have been synthesized. This molecule has potential to reduce cost and increase efficiency in the chemical engineering industry regarding regulation of redox reactions and their respective product synthesis. (3)

Lebesgue Measure

Want to show area tends to 0

Proof:

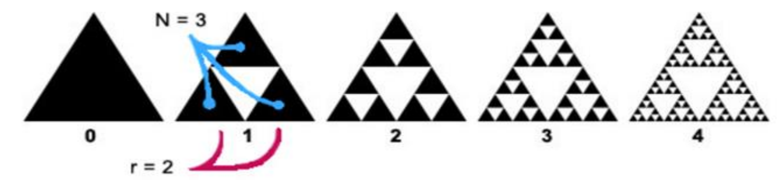
Suppose for contradiction that there is a point *Q* in the interior of the triangle. Then there is an inverted triangle centered at *Q,* in which, it is entirely contained in the initial triangle. This triangle contains a smaller triangle whose coordinates are multiples of $1/2^k$ for some *k*. But, if this triangle has not already been removed, it must have been holed in iteration $k + 1$, so it cannot be contained in the Sierpinski triangle, hence a contradiction. (4)

This shows dimension cannot be 2. We do not look at dimension of 0 or ∞ .

Lebesgue Covering dimension is restricted to integer solutions only.

Therefore 0< $dim_{Leb}(X)$ <2. The only integer solution for $dim_{Leb}(X)$ = 1.

Hausdorff Dimension



By scaling we can work out the dimension of the Sierpinski Triangle. The initial triangle of order 0 is in the top left. Following this we have the order 1 triangle which is made of 3 smaller identical triangles. Next, order 2 has 9 triangles. We see after each iteration the fractal has 3 times the previous iteration's triangles. N=3. The edge of each new triangle is half the previous hence r=2. So, using $N = r^D$. (2)

log(N) = D log(r)

$$D = \frac{\log(N)}{\log(r)}$$

$$D = \frac{\log(3)}{\log(2)} \approx 1.585$$

Box Counting

The Box counting dimension calculates the measure of an object as it scales. (6)

$$dim_B(X) = \lim_{\epsilon \to 0} \frac{\log(N(\epsilon))}{\log(1/\epsilon)}$$

We create a table with the data of our scale (ϵ) verses our number of boxes ($N(\epsilon)$) (6)

ϵ	$N(\epsilon)$
$\frac{1}{4}$	9
$\frac{1}{8}$	27
$\frac{1}{16}$	81

Here we see for scales in the form $1/2^k$ return $(N(\epsilon) \in \mathbb{N})$. Where $N(\epsilon)$ is in the form 3^k . So, using the equation assuming k is sufficiently large we have

$= dim_B(X)$

$= \frac{\log(3^k)}{\log(2^k)}$ (which by properties of logarithms)

$= \frac{k\log(3)}{k\log(2)} = \frac{\log(3)}{\log(2)} \approx 1.585$

Conclusions

We have explored and discussed a trilogy of methods to compute the dimension of the Sierpinski Triangle. Due to discrepancies between the methods only 2 of the 3 methods yield the same value of 1.585. Given the Lebesgue Covering Dimension restriction set, it can be concluded that the dimension of this fractal is 1.585. This fractal due to its nature has potentials in industrial settings, this could make way for the Sierpinski triangle to become part of our daily lives in areas such as in medicine.

References

1. Grant. Fractals are typically not self-similar [Internet]. Youtube; 2017 [cited 2022 Nov 6]. Available from: https://www.youtube.com/watch?v=gB9n2gHsHN4&t=604s
2. Móra P. ESTIMATE OF THE HAUSDORFF MEASURE OF THE SIERPINSKI TRIANGLE. Fractals. 2009 Jun;17(02):137–48.
3. Shang J, Wang Y, Chen M, Dai J, Zhou X, Kuttner J, et al. Assembling molecular Sierpiński triangle fractals. Nature Chem. 2015 May;7(5):389–93.
4. Semmes.S (2001). Some Novel Types of Fractal Geometry. Oxford Mathematical Monographs. Oxford University Press. p. 31. ISBN 0-19-850806-9. Zbl 0970.28001.
5. Complexity Explorer. Fractals and Scaling: Box counting the Sierpinski triangle [Internet]. 2017 [cited 2022 Dec 6]. Available from: https://www.youtube.com/watch?v=oUQ3C-6M3Ys
6. Khain, T. "FRACTALS AND DIMENSION." (2016). Available from: http://math.uchicago.edu/~may/REU2016/REUPapers/Khain.pdf