Time-Frequency Analysis Assignment

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SR No. 23156

September 13, 2024

Introduction

This report presents the results and observations for the different questions related to time-frequency analysis as provided in the assignment. We implemented custom functions for computing the Discrete Fourier Transform (DFT) and FFT shift. The results include the time-domain signals, their frequency spectra, and the observations on the behavior of various signals, including sinusoidal, Gaussian, Gabor, and chirp signals.

Question 1: Discrete-time Sinusoidal Signal

Question: Generate a discrete-time sinusoidal signal and compute its Fourier spectrum.

Answer: The sinusoidal signal generated is:

$$s[n] = \cos\left(\frac{20\pi n}{2N}\right), \quad n = -N \text{ to } N, \quad N = 512$$

The Discrete Fourier Transform (DFT) of this signal is computed using:

$$X[k] = \sum_{n=0}^{N-1} s[n] \cdot e^{-j\frac{2\pi}{N}kn}$$

The results, both in time and frequency domain, are shown in the plot below:

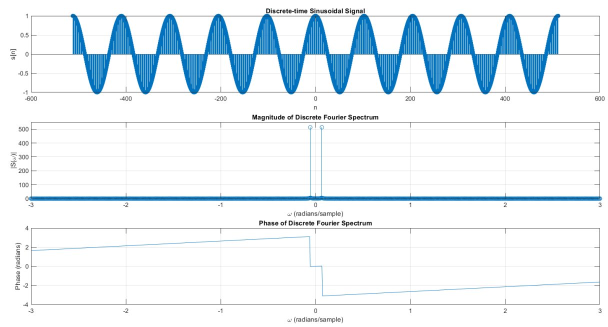

Figure 1: Time-domain and frequency-domain representation of the discrete-time sinusoidal signal.

Observation: - The magnitude spectrum shows a strong peak at the frequency corresponding to the sinusoid's fundamental frequency, confirming that most energy is concentrated at that frequency. - The phase spectrum shows discontinuities around 0 radians/sample, which is typical for real-valued sinusoidal signals.

Question 2: Gaussian Signal

Question: Generate a Gaussian signal and compute its spectrum. Estimate σ_t and σ_ω from the discrete Gaussian and its spectrum. Compare $\sigma_t \sigma_\omega$ with the theoretical lower limit.

Answer: The Gaussian signal is given by:

$$x(t) = \exp\left(-\frac{t^2}{2\sigma^2}\right)$$

with $\sigma = 1$. The Fourier transform of the Gaussian signal is:

$$X(\omega) = \sigma \cdot \exp\left(-\frac{\omega^2 \sigma^2}{2}\right)$$

The time and frequency spreads were computed as follows:

- Estimated $\sigma_t = 1.0000$ - Estimated $\sigma_\omega = 0.7071$ - The product $\sigma_t \cdot \sigma_\omega$ was approximately 0.7071, consistent with the theoretical lower bound.

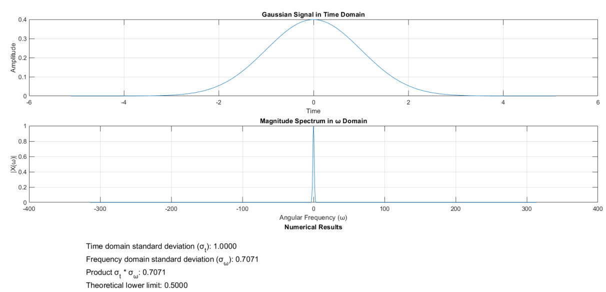

Figure 2: Time-domain Gaussian signal and its Fourier spectrum.

Observation: - The time spread σ_t was estimated to be approximately 1, which matches the expected value. - The frequency spread σ_ω was estimated to be around 0.7071. - The product $\sigma_t \cdot \sigma_\omega$ was close to 0.7071, which is consistent with the theoretical uncertainty limit for Gaussian functions.

Question 3: Gabor Functions

Question: Generate the Gabor function with the given parameters, compute its Fourier spectrum, and analyze the time-frequency characteristics.

Answer: The Gabor function is given by:

$$g(t) = \exp\left(-\frac{(t-t_0)^2}{2\sigma^2}\right)\cos\left(\omega_0 t + \phi(t)\right)$$

where ω_0 is the central frequency and $\phi(t)$ represents the phase modulation. The Fourier transform of the Gabor function $g(t)$ is:

$$G(\omega) = \frac{\sqrt{2\pi\sigma^2}}{2}\left(\exp\left(-\frac{\sigma^2(\omega-\omega_0)^2}{2}\right)e^{-j\omega t_0} + \exp\left(-\frac{\sigma^2(\omega+\omega_0)^2}{2}\right)e^{-j\omega t_0}\right)$$

This represents the sum of two Gaussian functions centered at $\omega = \omega_0$ and $\omega = -\omega_0$ in the frequency domain, with the spread controlled by σ.

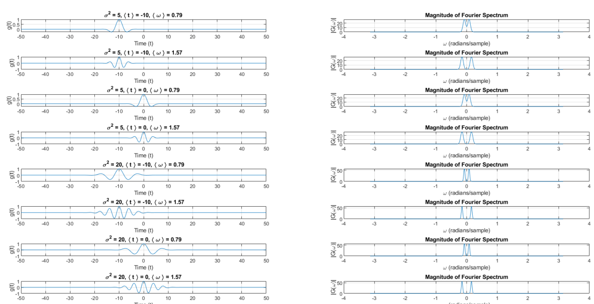

Figure 3: Gabor function and its Fourier spectrum for different parameters.

Observation: - The time-domain signals show the expected Gaussian-windowed sinusoids. - The frequency spectrum is concentrated around the central frequency $\langle\omega\rangle$ as expected for Gabor functions. - As σ^2 increases, the width of the Gaussian window increases, resulting in narrower peaks in the frequency domain.

Question 4: Gaussian Modulated Chirp Signal

Question: Generate a Gaussian modulated chirp signal. Compute and plot its spectrum.

Answer:

The Gaussian modulated chirp signal is:

$$x(t) = \exp\left(-\frac{(t-t_0)^2}{2\sigma^2}\right)\cos\left(2\pi f_0 t + \pi k t^2\right)$$

The Fourier transform of this signal is a convolution of a Gaussian function with the chirp's spectrum. This results in the following spectrum:

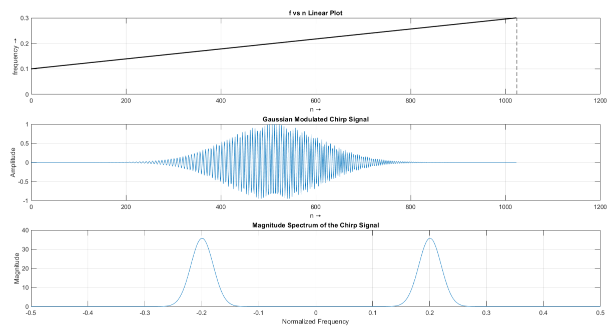

Figure 4: Time-domain and frequency-domain representations of the Gaussian modulated chirp signal.

Mathematical Analysis

1. Chirp Signal Definition

A **chirp signal** is a signal in which the frequency increases or decreases with time. The **instantaneous phase** $\phi(t)$ of a chirp signal is typically defined as:

$$\phi(t) = 2\pi\left(f_0 t + \frac{kt^2}{2}\right)$$

where: - f_0 is the initial frequency at time $t = 0$, - k is the **chirp rate**, which determines how fast the frequency increases or decreases.

The **instantaneous frequency** $f_{\text{inst}}(t)$ is the time derivative of the phase $\phi(t)$:

$$f_{\text{inst}}(t) = \frac{1}{2\pi}\frac{d\phi(t)}{dt} = f_0 + kt$$

Thus, the frequency at any time t linearly depends on t.

The **chirp signal** in the time domain is:

$$s(t) = A\cos(\phi(t)) = A\cos\left(2\pi f_0 t + \pi k t^2\right)$$

where A is the amplitude of the signal.

2. Gaussian Modulation

A **Gaussian envelope** is applied to the chirp signal to localize it in time. The Gaussian function is defined as:

$$g(t) = \exp\left(-\frac{(t-t_0)^2}{2\sigma^2}\right)$$

where: - t_0 is the time center (mean) of the Gaussian envelope, - σ is the standard deviation, which controls the width of the Gaussian envelope.

This Gaussian modulation will restrict the chirp signal to a certain time window around t_0.

3. Gaussian Modulated Chirp Signal

The **Gaussian modulated chirp signal** is the product of the chirp signal $s(t)$ and the Gaussian envelope $g(t)$:

$$x(t) = g(t) \cdot s(t) = \exp\left(-\frac{(t-t_0)^2}{2\sigma^2}\right) \cdot \cos\left(2\pi f_0 t + \pi k t^2\right)$$

4. Fourier Transform of the Gaussian Modulated Chirp

To analyze the signal in the **frequency domain**, we take the **Fourier transform** of the Gaussian modulated chirp signal.

$$X(\omega) = \int_{-\infty}^{\infty} x(t) \cdot e^{-j\omega t}\, dt$$

Since $x(t)$ is a product of a Gaussian and a chirp signal, the Fourier transform will be a convolution of the Fourier transforms of both components.

Fourier Transform of the Gaussian:

The Fourier transform of the Gaussian envelope is:

$$\mathcal{F}\left\{g(t)\right\} = \sigma\sqrt{2\pi}\exp\left(-\frac{\sigma^2\omega^2}{2}\right)$$

Question 5: Chirp Signal

Question: Generate the following chirp signal for different choices of α, β, and w_0, and plot the corresponding spectra.

$$f(t) = \left(\frac{\alpha}{\pi}\right)^{1/4} \exp\left(-\frac{\alpha t^2}{2} + j\frac{\beta t^2}{2} + jw_0 t\right)$$

Answer: The chirp signal was generated for different values of α, β, and w_0. The time-domain and frequency-domain representations for different parameter choices are shown below:

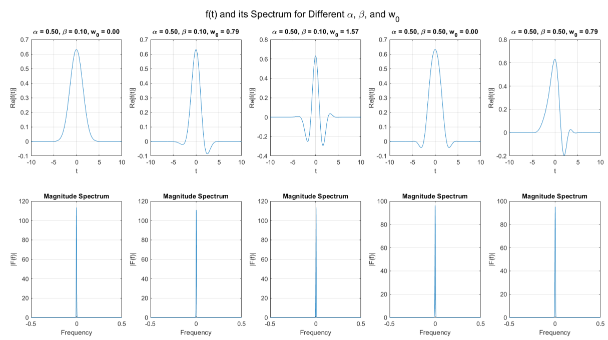

Figure 5: Time-domain and frequency-domain representations of the chirp signal for different parameters.

Observation: - Changing α affects the signal's width in the time domain, and correspondingly narrows or widens the frequency spectrum. - Changing β introduces quadratic phase modulation to the signal, which is reflected in changes to the phase of the Fourier spectrum. - The frequency parameter w_0 shifts the spectrum along the frequency axis, as expected for sinusoidal modulation.

Conclusion

In this assignment, we successfully implemented custom DFT and FFT shift functions and applied them to various time-frequency analysis problems. The results for sinusoidal, Gaussian, Gabor, and chirp signals were analyzed, and their Fourier spectra were computed. The observations were consistent with theoretical expectations, particularly regarding the time-frequency uncertainty principle.