END SEMESTER

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Que 1: $f_{X,Y}(x,y) = e^{-y}$, $0 < x < y < \infty$

Find

1) Correlation coefficient between X and Y.
2) Conditional Variance V(X|

Solution:

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dy$$

$$= \int_{x}^{\infty} e^{-y}\,dy = \left[-e^{-y}\right]_x^{\infty}$$

$$\boxed{f_X(x) = e^{-x}} \quad \text{——— (1)}$$

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx$$

$$= \int_{0}^{x} e^{-y}\,dy = \left[e^{-y}x\right]_0^{y}$$

$$\boxed{f_Y(y) = ye^{-y}} \quad \text{——— (2)}$$

$$E[X] = \int_{-\infty}^{\infty} x\, f_X(x)\,dx = \int_0^{\infty} xe^{-x}\,dx$$

$$= \left[xe^{-x}(-1) - e^{-x}\right]_0^{\infty} = \left[-xe^{-x} - e^{-x}\right]_0^{\infty}$$

$$= 1$$

$$\boxed{E[X] = 1} \quad \text{——— (3)}$$

$$E[Y] = \int_{-\infty}^{\infty} y\, f_Y(y)\,dy = \int_0^{\infty} y^2 e^{-y}\,dy$$

$$= \left[-y^2e^{-y} - 2y\cdot e^{-y} - 2e^{-y}\right]_0^{\infty} = 2$$

$$\boxed{E[Y] = 2} \quad \text{——— (4)}$$

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$$E[XY] = \int_{-\infty}^{\infty} x \cdot y \, f_{XY}(x,y)\, dx\, dy$$

$$= \int_{0}^{\infty} x \int_{x}^{\infty} y e^{-y} dy\, dx$$

$$E[XY] = \int_{0}^{\infty} x \cdot \left[-ye^{-y} - e^{-y}\right]_{x}^{\infty} dx$$

$$\Rightarrow E[XY] = \int_{0}^{\infty} x \cdot (xe^{-x} + e^{-x})\, dx$$

$$= \int_{0}^{\infty} e^{-x}(x^2 + x)\, dx$$

$$E[XY] = \left[-x^2 e^{-x} + 3(-xe^{-x} - e^{-x})\right]_{0}^{\infty}$$ (Applying integration by parts twice)

$$= \left[\left(-e^{-x}(x^2 + 2x + 2)\right) + \left(-e^{-x}(x+1)\right)\right]_{0}^{\infty}$$

$$= 2 + 1$$

$$\boxed{E[XY] = 3} \quad \text{——— } (3)$$

We know; Covariance

$$Cov(X,Y) = E[XY] - E[X] \cdot E[Y]$$

Substituting the values.

$$Cov(X,Y) = 3 - 1 \cdot 2$$

$$= 1$$

$$\boxed{Cov(X,Y) = 1}$$

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$$E[X^2] = \int_0^\infty x^2 e^{-x}\,dx = \left[-x^2e^{-x} - 2xe^{-x} - 2e^{-x}\right]_0^\infty$$

$$\Rightarrow \boxed{E[X^2] = 2} \text{ ——— } (6)$$

$$E[Y^2] = \int_0^\infty y^2 \cdot ye^{-y}\,dy = \left[-y^3e^{-y}\right]_0^\infty + 3\left[y^2(-e^{-y}) - \int 2y(-e^{-y})dy\right]$$

$$= -e^{-y}y^3 - 3y^2e^{-y} + 6\left[y(-e^{-y}) - e^{-y}\right]$$

$$= -e^{-y}\left[y^3 + 3y^2 + 6y + 6\right]_0^\infty$$

$$\Rightarrow \boxed{E[Y^2] = 6} \text{ ——— } (7).$$

$$V(X) = E[X^2] - [E[X]]^2$$

$$= 2 - (1)^2 = 2 - 1$$

$$= 1$$

$$\boxed{V(X) = 1} \qquad \Rightarrow \sigma_x = \sqrt{V(x)} = \sqrt{1} = 1$$

$$\boxed{\sigma_x = 1}$$

$$V(Y) = E[Y^2] - [E[Y]]^2$$

$$= 6 - 2^2$$

$$= 4$$

$$\boxed{V(Y) = 2} \qquad \Rightarrow \sigma_y = \sqrt{V(y)} = \sqrt{2}$$

$$\boxed{\sigma_y = \sqrt{2}}$$

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As $Cov(X,Y) = 1$, $\sigma_x = 1$, $\sigma_y = \sqrt{2}$

$$\rho(x,y) = \frac{Cov(X,Y)}{\sigma_x \cdot \sigma_y}$$

$$= \frac{1}{1 \cdot \sqrt{2}} = \frac{1}{\sqrt{2}}$$

Correlation Coefficient

$$\boxed{\rho(X,Y) = \frac{1}{\sqrt{2}}}$$

To find conditional variance. $V(X|Y)$

$$V(X|Y) = E[X^2|Y] - (E[X|Y])^2$$

$$f(x|y) = \frac{f(x,y)}{f_Y(y)} = \frac{e^{-y}}{y e^{-y}} = \frac{1}{y}.$$

$$E(X|Y) = \int_0^y x \, f_{X|Y}(x|y) \, dx$$

$$= \int_0^y x \, \frac{1}{y} \, dx$$

$$= \left. \frac{x^2}{2y} \right|_0^y$$

$$\boxed{E[X|Y] = \frac{y}{2}}$$

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$$E[X^2|Y] = \int_0^y x^2 f_{X|Y}(x|y)\,dx$$

$$= \int_0^y x^2\left(\frac{1}{y}\right)dx$$

$$= \left.\frac{x^3}{3y}\right|_0^y = \frac{y^2}{3}$$

$$\boxed{E[X^2|Y] = \frac{y^2}{3}}$$

$$V(X|Y) = E[X^2|Y] - E[X|Y]^2$$

Substituting the values

$$V(X|Y) = \frac{y^2}{3} - \left(\frac{y}{2}\right)^2$$

$$\boxed{V(X|Y) = \frac{y^2}{12}} \rightarrow \text{Conditional Variance}$$

Answers for Solution 1

$f_{X,Y}(x,y) = e^{-y}$

$f_X(x) = e^{-x}$

$f_Y(y) = ye^{-y}$

$E[X] = 1$

$E[Y] = 2$

$E[XY] = 3$

$Cov(X,Y) = 1$

$E[X^2] = 2$

$E[Y^2] = 6$

$V(X) = 1$

$\sigma_x = 1$

$V(Y) = 2$

$\sigma_y = \sqrt{2}$

$E[X|Y] = \frac{y}{2}$

$E[X^2|Y] = \frac{y^2}{3}$

$\Rightarrow$ Correlation Coefficient

$\rho(X,Y) = \frac{1}{\sqrt{2}}$

$\Rightarrow$ Conditional Variance

$V(X|Y) = \frac{y^2}{12}$

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Que 2: $f_{XY}(x,y) = A e^{-\frac{1}{2}(x^2 - 2xy + 4y^2)}$ $\quad -\infty < x, y < \infty$

$A \rightarrow$ appropriate constant.

Find

1) all parameters of bivariate normal
2) A.

Solution: (X, Y) has a bivariate normal distribution.

$$\Rightarrow f_{XY}(x,y) = \frac{1 \cdot e^{\frac{-1}{2(1-\rho^2)}\left\{\left(\frac{x-\mu_1}{\sigma_1}\right)^2 + \left(\frac{y-\mu_2}{\sigma_2}\right)^2 - 2\rho\left(\frac{x-\mu_1}{\sigma_1}\right)\left(\frac{y-\mu_2}{\sigma_2}\right)\right\}}}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}}.$$

Comparing it with the given equation:

$$x^2 = \left(\frac{x-\mu_1}{\sigma_1}\right)^2\left(\frac{1}{1-\rho^2}\right) \quad ; \mu_1 = 0$$ (as in the given eq there are no terms of x)

$$x^2 = \frac{x^2}{\sigma_1^2(1-\rho^2)}$$

$$\boxed{\sigma_1^2(1-\rho^2) = 1} \quad \text{——— (1)}$$

$$4y^2 = \frac{(y-\mu_2)^2}{\sigma_2^2(1-\rho^2)} \quad ; \mu_2 = 0$$ (as in the given eq. there are no terms of y)

$$4 = \frac{1}{\sigma_2^2(1-\rho^2)}$$

$$\boxed{\sigma_2^2(1-\rho^2) = \frac{1}{4}} \quad \text{——— (2)}$$

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$$xy = \rho\left(\frac{x}{\sigma_1}\right)\left(\frac{y}{\sigma_2}\right)\left(\frac{1}{1-\rho^2}\right)$$

$$\boxed{\frac{\sigma_1\sigma_2(1-\rho^2)}{\rho} = 1} \quad \text{———} \quad (3).$$

from (1), (2) and (3),

$$\sigma_1^2(1-\rho^2) = 1$$

$$\sigma_2^2(1-\rho^2) = \frac{1}{4}$$

$$\frac{\sigma_1\sigma_2(1-\rho^2)}{\rho} = 1$$

$$(1) \times (2) \Rightarrow \sigma_1^2\sigma_2^2(1-\rho^2) = \frac{1}{4} \longrightarrow (4)$$

$$((3))^2 = \frac{\sigma_1^2\sigma_2^2(1-\rho^2)^2}{\rho^2} = 1 \quad \text{———} \quad (5)$$

Substituting.

$$\frac{1}{4\rho^2} = 1 \Rightarrow \rho^2 = \frac{1}{4} \Rightarrow \rho = \pm\frac{1}{2}$$

but.

$$(1) \div (2) \Rightarrow \frac{\sigma_1}{\sigma_2} = \frac{1}{2}$$

$$(1) \div (3) \Rightarrow \frac{\sigma_2}{\sigma_1\rho} = \frac{1}{2\rho} = 1$$

$$\Rightarrow \boxed{\rho = +\frac{1}{2}}$$

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Substituting ρ in ①

$\boxed{\sigma_1 = \frac{2}{\sqrt{3}}} \Rightarrow \boxed{\sigma_1^2 = \frac{4}{3}}$

Substituting ρ in ②

$\boxed{\sigma_2 = \frac{1}{\sqrt{3}}} \Rightarrow \boxed{\sigma_2^2 = \frac{1}{3}}$

$$\Rightarrow A = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} = \frac{1}{2\pi \frac{2}{\sqrt{3}} \times \frac{1}{\sqrt{3}} \times \sqrt{\frac{3}{4}}}$$

$$= \frac{1}{\frac{2\pi \times \cancel{2}}{\cancel{2}\sqrt{3}}}$$

$$\boxed{A = \frac{\sqrt{3}}{2\pi}}$$

Answers for ques 2:

1) $\mu_1 = 0$

$\mu_2 = 0$

$\rho = \frac{1}{2}$

$\sigma_1 = \frac{2}{\sqrt{3}} \Rightarrow \sigma_1^2 = \frac{4}{3}$

$\sigma_2 = \frac{1}{\sqrt{3}} \Rightarrow \sigma_2^2 = \frac{1}{3}$

2) $A = \frac{\sqrt{3}}{2\pi}$

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Que 3: Given that no. of cells arriving at a certain center is a Poisson Process $(X(t))$ with $\lambda = 2$.

The probability mass function is:

$$f(K) = P(X=k) = \frac{e^{-\lambda}\lambda^{k}}{k!} \quad ; k = 0, 1, 2, \ldots$$

Ans: As $\lambda = 2$

$$f(k) = P(X=k) = \frac{e^{-2}2^{k}}{k!}$$

i) no cell arrives at center during a 9 minute period.

$P(X=0)$

$$P(X(9)=0) = \frac{e^{-2\times 9}(2\times 9)^{0}}{0!} = \frac{e^{-18}(1)}{1}$$

$$\boxed{P(X=0) = e^{-18}}$$

ii) more than 3 cells ~~pass~~ arrive at center during $\frac{3}{2}$ minute period

$$P(X(\tfrac{3}{2})>3) = 1 - P(X \le 3)$$

$$= 1 - \sum_{i=0}^{3}\left(\frac{e^{-3}(3)^{i}}{i!}\right)$$

$$= 1 - e^{-3}\left(1 + \frac{3}{1} + \frac{9}{2} + \frac{27}{6}\right)$$

$$= 1 - 13e^{-3}$$

$$\boxed{P(X>3) = 1 - 13e^{-3}}$$

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Answers for que 3

$P(x=0) = e^{-18}$

$P(X>3) = 1-13e^{-3}$

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