Number Representation

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Number System :: The Basics

- We are accustomed to using the so-called decimal number system
 - □ Ten digits :: 0,1,2,3,4,5,6,7,8,9
 - Every digit position has a weight which is a power of 10
 - □ Base or radix is 10

Example:

$$234 = 2 \times 10^{2} + 3 \times 10^{1} + 4 \times 10^{0}$$

 $250.67 = 2 \times 10^{2} + 5 \times 10^{1} + 0 \times 10^{0} + 6 \times 10^{-1} + 7 \times 10^{-2}$

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Binary Number System

- Two digits:
 - □ 0 and 1
 - Every digit position has a weight which is a power of 2
 - □ Base or radix is 2
- Example:

$$110 = 1 \times 2^{2} + 1 \times 2^{1} + 0 \times 2^{0}$$

$$101.01 = 1 \times 2^{2} + 0 \times 2^{1} + 1 \times 2^{0} + 0 \times 2^{-1} + 1 \times 2^{-2}$$

Decimal Numbers:

- **10** Symbols {0,1,2,3,4,5,6,7,8,9}, Base or Radix is 10
- 136.25 = 1 × 10² + 3 × 10¹ + 6 × 10⁰ + 2 × 10⁻¹ + 3 × 10⁻²

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Binary Numbers:

- **❖** 2 Symbols {0,1}, Base or Radix is 2
- **♦** $101.01 = 1 \times 2^2 + 0 \times 2^1 + 1 \times 2^0 + 0 \times 2^{-1} + 1 \times 2^{-2}$

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Octal Numbers:

- **8** Symbols {0,1,2,3,4,5,6,7}, Base or Radix is 8
- $621.03 = 6 \times 8^2 + 2 \times 8^1 + 1 \times 8^0 + 0 \times 8^{-1} + 3 \times 8^{-2}$

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Hexadecimal Numbers:

- **16** Symbols {0,1,2,3,4,5,6,7,8,9,A,B,C,D,E,F}, Base is 16
- **♦** $6AF.3C = 6 \times 16^2 + 10 \times 16^1 + 15 \times 16^0 + 3 \times 16^{-1} + 12 \times 16^{-2}$

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Binary-to-Decimal Conversion

- Each digit position of a binary number has a weight
 - □ Some power of 2
- A binary number:

$$B = b_{n-1} b_{n-2} \dots b_1 b_0 \cdot b_{-1} b_{-2} \dots b_{-m}$$

Corresponding value in decimal:

$$D = \sum_{i=-m}^{n-1} b_i 2^i$$

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Examples

101011 →
$$1x2^5 + 0x2^4 + 1x2^3 + 0x2^2 + 1x2^1 + 1x2^0$$

= 43
 $(101011)_2 = (43)_{10}$
.0101 → $0x2^{-1} + 1x2^{-2} + 0x2^{-3} + 1x2^{-4}$
= .3125
 $(.0101)_2 = (.3125)_{10}$
101.11 → $1x2^2 + 0x2^1 + 1x2^0 + 1x2^{-1} + 1x2^{-2}$
= 5.75
 $(101.11)_2 = (5.75)_{10}$

Decimal to Binary: Integer Part

- Consider the integer and fractional parts separately.
- For the integer part:
 - Repeatedly divide the given number by 2, and go on accumulating the remainders, until the number becomes zero.
 - Arrange the remainders in reverse order.

	Base	Numb	Rem	
I	2	89		1 4
ı	2	44	1	
	2	22	0	
	2	11	0	
	2	5	1	
	2	2	1	
	2	1	0	
		0	1	

$$(89)_{10} = (1011001)_2$$

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$(89)_{10} = (10)$	$011001)_2$
--------------------	-------------

2	66	
	33	0
2	16	1
2	8	0
2	4	0
2	2	0
2	1	0
	0	1

$$(66)_{10} = (1000010)_2$$

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---------------------------	--

2	66	
2	33	0
2	16	1
2	8	0
2	4	0
2	2	0
2	1	0
	0	1

$$(66)_{10} = (1000010)_2$$

2	239	
2	119	1
2	59	1
2	29	1
2	14	1
2	7	0
2	3	1
2	1	1
	0	1

$$(239)_{10} = (11101111)_2$$

Decimal to Binary: Fraction Part

- Repeatedly multiply the given fraction by 2.
 - Accumulate the integer part (0 or 1).
 - If the integer part is 1, chop it off.
- •Arrange the integer parts in the order they are obtained.

Example: 0.634

```
.634 \times 2 = 1.268
```

$$.268 \times 2 = 0.536$$

$$.536 \times 2 = 1.072$$

$$.072 \times 2 = 0.144$$

$$.144 \times 2 = 0.288$$

•

•

$$(.634)_{10} = (.10100.....)_2$$

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 $(.634)_{10} = (.10100.....)_2$

Example: 0.0625

 $.0625 \times 2 = 0.125$

 $.1250 \times 2 = 0.250$

 $.2500 \times 2 = 0.500$

 $.5000 \times 2 = 1.000$

 $(.0625)_{10} = (.0001)_2$

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 $(.0625)_{10} = (.0001)_2$

$$(37)_{10} = (100101)_2$$

$$(.0625)_{10} = (.0001)_2$$

 $(37.0625)_{10} = (100101.0001)_2$



Hexadecimal Number System

- A compact way of representing binary numbers
- 16 different symbols (radix = 16)

```
0 \rightarrow 0000 \quad 8 \rightarrow 1000
1 \rightarrow 0001 \quad 9 \rightarrow 1001
2 \rightarrow 0010 \quad A \rightarrow 1010
3 \rightarrow 0011 \quad B \rightarrow 1011
4 \rightarrow 0100 \quad C \rightarrow 1100
5 \rightarrow 0101 \quad D \rightarrow 1101
6 \rightarrow 0110 \quad E \rightarrow 1110
7 \rightarrow 0111 \quad F \rightarrow 1111
```

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Binary-to-Hexadecimal Conversion

- For the integer part,
 - □ Scan the binary number from right to left
 - Translate each group of four bits into the corresponding hexadecimal digit
 - Add leading zeros if necessary
- For the fractional part,
 - □ Scan the binary number from left to right
 - Translate each group of four bits into the corresponding hexadecimal digit
 - Add trailing zeros if necessary



Example

- 1. $(1011 \ 0100 \ 0011)_2 = (B43)_{16}$
- 2. $(10 \ 1010 \ 0001)_2 = (2A1)_{16}$
- 3. $(.1000 \ 010)_2 = (.84)_{16}$
- 4. $(101 \cdot 0101 \cdot 111)_2 = (5.5E)_{16}$

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Hexadecimal-to-Binary Conversion

 Translate every hexadecimal digit into its 4-bit binary equivalent

Examples:

```
(3A5)_{16} = (0011 \ 1010 \ 0101)_2

(12.3D)_{16} = (0001 \ 0010 \ .0011 \ 1101)_2

(1.8)_{16} = (0001 \ .1000)_2
```

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Unsigned Binary Numbers

An n-bit binary number

$$B = b_{n-1}b_{n-2}....b_2b_1b_0$$

- 2ⁿ distinct combinations are possible, 0 to 2ⁿ-1.
- For example, for n = 3, there are 8 distinct combinations
 - □ 000, 001, 010, 011, 100, 101, 110, 111
- Range of numbers that can be represented

$$n=8 \rightarrow 0 \text{ to } 2^8-1 (255)$$

$$n=16 \rightarrow 0 \text{ to } 2^{16}-1 (65535)$$

$$n=32 \rightarrow 0$$
 to $2^{32}-1$ (4294967295)



Signed Integer Representation

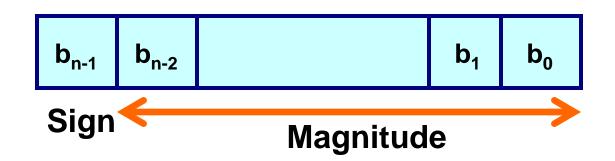
- Many of the numerical data items that are used in a program are signed (positive or negative)
 - □ Question:: How to represent sign?

- Three possible approaches:
 - □ Sign-magnitude representation
 - One's complement representation
 - □ Two's complement representation



Sign-magnitude Representation

- For an n-bit number representation
 - □ The most significant bit (MSB) indicates sign
 - $0 \rightarrow positive$
 - 1 → negative
 - ☐ The remaining n-1 bits represent magnitude



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Contd.

Range of numbers that can be represented:

```
Maximum :: +(2^{n-1}-1)
Minimum :: -(2^{n-1}-1)
```

A problem:

Two different representations of zero

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One's Complement Representation

- Basic idea:
 - Positive numbers are represented exactly as in sign-magnitude form
 - Negative numbers are represented in 1's complement form
- How to compute the 1's complement of a number?
 - □ Complement every bit of the number (1→0 and 0→1)
 - ☐ MSB will indicate the sign of the number
 - $0 \rightarrow positive$
 - 1 → negative

Example :: n=4

$$00000 \rightarrow +0$$
 $10000 \rightarrow -7$
 $00010 \rightarrow +1$
 $10010 \rightarrow -6$
 $0010 \rightarrow +2$
 $1010 \rightarrow -5$
 $0011 \rightarrow +3$
 $1011 \rightarrow -4$
 $0100 \rightarrow +4$
 $1100 \rightarrow -3$
 $0101 \rightarrow +5$
 $1101 \rightarrow -2$
 $0110 \rightarrow +6$
 $1110 \rightarrow -1$
 $0111 \rightarrow +7$
 $1111 \rightarrow -0$

To find the representation of, say, -4, first note that

$$+4 = 0100$$

-4 = 1's complement of 0100 = 1011

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Contd.

Range of numbers that can be represented:

```
Maximum :: +(2^{n-1}-1)
Minimum :: -(2^{n-1}-1)
```

A problem:

Two different representations of zero.

```
+0 → 0 000....0
-0 → 1 111....1
```

- Advantage of 1's complement representation
 - □ Subtraction can be done using addition
 - □ Leads to substantial saving in circuitry

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Two's Complement Representation

- Basic idea:
 - Positive numbers are represented exactly as in sign-magnitude form
 - Negative numbers are represented in 2's complement form
- How to compute the 2's complement of a number?
 - □ Complement every bit of the number (1→0 and 0→1), and then add one to the resulting number
 - ☐ MSB will indicate the sign of the number
 - $0 \rightarrow positive$
 - 1 → negative

Example: n=4

$$0000 \rightarrow +0$$

$$0001 \rightarrow +1$$

$$0010 \rightarrow +2$$

$$0011 \rightarrow +3$$

$$0100 \rightarrow +4$$

$$0101 \to +5$$

$$0110 \rightarrow +6$$

$$0111 \rightarrow +7$$

$$1010 \rightarrow -6$$

$$1111 \rightarrow -1$$

To find the representation of, say, -4, first note that

$$+4 = 0100$$

$$-4 = 2$$
's complement of $0100 = 1011+1 = 1100$

Rule: Value = - msb*2⁽ⁿ⁻¹⁾ + [unsigned value of rest]

Example:
$$0110 = 0 + 6 = 6$$
 $1110 = -8 + 6 = -2$

$$1110 = -8 + 6 = -2$$



Contd.

Range of numbers that can be represented:

```
Maximum :: +(2^{n-1}-1)
```

Minimum :: -2^{n-1}

- Advantage:
 - Unique representation of zero
 - Subtraction can be done using addition
 - Leads to substantial saving in circuitry
- Almost all computers today use the 2's complement representation for storing negative numbers

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Contd.

- In C
 - □ short int
 - 16 bits \rightarrow + (2¹⁵-1) to -2¹⁵
 - □ int or long int
 - 32 bits \rightarrow + (2³¹-1) to -2³¹
 - □ long long int
 - 64 bits \rightarrow + (2⁶³-1) to -2⁶³



Adding Binary Numbers

■ Basic Rules:

- $\Box 0 + 0 = 0$
- $\Box 0 + 1 = 1$
- $\Box 1 + 0 = 1$
- \Box 1+1=0 (carry 1)

Example:

01101001

00110100

10011101

Subtraction Using Addition :: 1's Complement

- How to compute A B?
 - \square Compute the 1's complement of B (say, B₁).
 - \square Compute R = A + B₁
 - ☐ If the carry obtained after addition is '1'
 - Add the carry back to R (called end-around carry)
 - That is, R = R + 1
 - The result is a positive number

Else

 The result is negative, and is in 1's complement form



Example 1 :: 6-2

1's complement of 2 = 1101

Assume 4-bit representations

Since there is a carry, it is added back to the result

The result is positive



Example 2 :: 3 – 5

1's complement of 5 = 1010

3 :: 0011 A

-5 :: 1010 B₁



-2

Assume 4-bit representations

Since there is no carry, the result is negative

1101 is the 1's complement of 0010, that is, it represents –2

Subtraction Using Addition :: 2's Complement

- How to compute A B?
 - \square Compute the 2's complement of B (say, B₂)
 - \square Compute R = A + B₂
 - ☐ If the carry obtained after addition is '1'
 - Ignore the carry
 - The result is a positive number

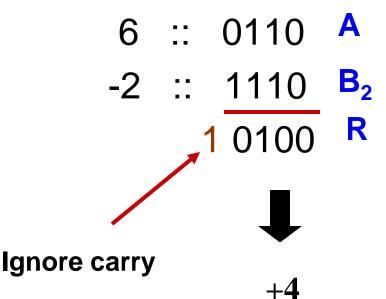
Else

The result is negative, and is in 2's complement form



Example 1 :: 6-2

2's complement of
$$2 = 1101 + 1 = 1110$$



Assume 4-bit representations

Presence of carry indicates that the result is positive

No need to add the endaround carry like in 1's complement



Example 2 :: 3-5

2's complement of 5 = 1010 + 1 = 1011

3 :: 0011 A

-5 :: 1011 B₂

1110 R



-2

Assume 4-bit representations

Since there is no carry, the result is negative

1110 is the 2's complement of 0010, that is, it represents –2

2's complement arithmetic: More Examples

- Example 1: 18-11 = ?
- 18 is represented as 00010010
- 11 is represented as 00001011
 - 1's complement of 11 is 11110100
 - 2's complement of 11 is 11110101
- Add 18 to 2's complement of 11

00010010

+ 11110101

00000111 (with a carry of 1 which is ignored)

00000111 is 7



- Example 2: 7 9 = ?
- 7 is represented as 00000111
- 9 is represented as 00001001
 - 1's complement of 9 is 11110110
 - 2's complement of 9 is 11110111
 - Add 7 to 2's complement of 9

00000111

+ 11110111

11111110 (with a carry of 0 which is ignored)

11111110 is -2

Adding two +ve (-ve) numbers should not produce a -ve (+ve) number. If it does, overflow (underflow) occurs

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Another equivalent condition: carry in and carry out from Most Significant Bit (MSB) differ.

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```
(64) 01000000
(4) 00000100
-----
(68) 01000100
```

carry (out)(in) 0 0

Adding two +ve (-ve) numbers should not produce a -ve (+ve) number. If it does, overflow (underflow) occurs

Another equivalent condition: carry in and carry out from Most Significant Bit (MSB) differ.

(64) 01000000

(4) 00000100

(68) 01000100

carry (out)(in) 0 0 (64) 01000000

(96) 01100000

(-96) 10100000

carry out in 0 1

differ:

overflow



Floating-point Numbers

- The representations discussed so far applies only to integers
 - □ Cannot represent numbers with fractional parts
- We can assume a decimal point before a signed number
 - In that case, pure fractions (without integer parts)
 can be represented
- We can also assume the decimal point somewhere in between
 - □ This lacks flexibility
 - Very large and very small numbers cannot be represented

Representation of Floating-Point Numbers

A floating-point number F is represented by a doublet <M,E>:

```
F = M \times B^{E}
```

- B → exponent base (usually 2)
- M → mantissa
- E → exponent
- M is usually represented in 2's complement form, with an implied binary point before it
- For example,

```
In decimal, 0.235 \times 10^6
In binary, 0.101011 \times 2^{0110}
```

Example :: 32-bit representation



- ☐ M represents a 2's complement fraction
 - 1 > M > -1
- ☐ E represents the exponent (in 2's complement form)
 - 127 > E > -128
- Points to note:
 - □ The number of significant digits depends on the number of bits in M
 - 6 significant digits for 24-bit mantissa
 - □ The range of the number depends on the number of bits in E
 - \bullet 10³⁸ to 10⁻³⁸ for 8-bit exponent.



A Warning

- The representation for floating-point numbers as shown is just for illustration
- The actual representation is a little more complex
- Example: IEEE 754 Floating Point format

IEEE 754 Floating-Point Format (Single Precision)

S	E (Exponent)	M (Mantissa)
(31)	(30 23)	(22 0)

S: Sign (0 is +ve, 1 is -ve)

E: Exponent (More bits gives a higher range)

M: Mantissa (More bits means higher precision)

[8 bytes are used for double precision]

Value of a Normal Number:

$$(-1)^{S} \times (1.0 + 0.M) \times 2^{(E - 127)}$$

An example

S	E (Exponent)	M (Mantissa)
(31)	(30 23)	(22 0)

1	10001100	11011000000000000000000

Value of a Normal Number:

=
$$(-1)^{S} \times (1.0 + 0.M) \times 2^{(E-127)}$$

= $(-1)^{1} \times (1.0 + 0.1101100) \times 2^{(10001100 - 1111111)}$

Representing 0.3

S	E (Exponent)	M (Mantissa)
(31)	(30 23)	(22 0)

0.3 (decimal)

= 0.01001001001001001001001...

 $= 1.00100100100100100100100100... \times 2^{-2}$

= $1.00100100100100100100100100... \times 2^{125-127}$

 $= (-1)^{S} \times (1.0 + 0.M) \times 2^{(E-127)}$

0 01111101 00100100100100100100

What are the largest and smallest numbers that can be represented in this scheme?

Representing 0

S	E (Exponent)	M (Mantissa)
(31)	(30 23)	(22 0)
0	0000000	000000000000000000000000000000000000000
1	0000000	000000000000000000000000000000000000000

Representing Inf (∞)

0	11111111	000000000000000000000000000000000000000
1	11111111	000000000000000000000000000000000000000

Representing NaN (Not a Number)

0	11111111	Non zero
1	11111111	Non zero

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Representation of Characters

- Many applications have to deal with non-numerical data.
 - Characters and strings
 - □ There must be a standard mechanism to represent alphanumeric and other characters in memory
- Three standards in use:
 - Extended Binary Coded Decimal Interchange Code (EBCDIC)
 - Used in older IBM machines
 - American Standard Code for Information Interchange (ASCII)
 - Most widely used today
 - □ UNICODE
 - Used to represent all international characters.
 - Used by Java



ASCII Code

- Each individual character is numerically encoded into a unique 7-bit binary code
 - □ A total of 2⁷ or 128 different characters
 - □ A character is normally encoded in a byte (8 bits), with the MSB not been used.
- The binary encoding of the characters follow a regular ordering
 - □ Digits are ordered consecutively in their proper numerical sequence (0 to 9)
 - □ Letters (uppercase and lowercase) are arranged consecutively in their proper alphabetic order

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Some Common ASCII Codes

```
'A' :: 41 (H) 65 (D)
'B' :: 42 (H) 66 (D)
'Z' :: 5A (H) 90 (D)
'a' :: 61 (H) 97 (D)
'b' :: 62 (H) 98 (D)
'z' :: 7A (H) 122 (D)
```

```
'0' :: 30 (H) 48 (D)
'1' :: 31 (H) 49 (D)
'9' :: 39 (H) 57 (D)
'(' :: 28 (H) 40 (D)
'+' :: 2B (H) 43 (D)
": 3F (H) 63 (D)
'\n' :: 0A (H) 10 (D)
'\0' :: 00 (H) 00 (D)
```



Character Strings

 Two ways of representing a sequence of characters in memory



The first location contains the number of characters in the string, followed by the actual characters



The characters follow one another, and is terminated by a special delimiter



String Representation in C

- In C, the second approach is used
 - □ The '\0' character is used as the string delimiter
- Example:

"Hello"





- A null string "" occupies one byte in memory.
 - □ Only the '\0' character