~~Lecture~~-2

Revise the Lecture-1

We did exact ODEs of first order.

Given
$$M\,dx + N\,dy = 0 \quad \text{--- (1)} \qquad M \equiv M(x,y),\; N \equiv N(x,y)$$

Step 1: Convert ODE in form (1).

Step 2: Check $\boxed{\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}}$

Yes → Go to step 3

No → Can't apply this method to solve (1).

Step 3 In this step, we find a function $f \equiv f(x,y)$ such that

$$df = M\,dx + N\,dy \quad \text{--- (2)}$$

Thus solution (general solution) of (1) is

$$\boxed{f(x,y) = c} \quad \text{--- (3)}$$

where c is arbitrary real constant.

Think: If you have a continuous function $g(x)$ and $\boxed{dy = g\,dx}$, then ~~...~~ we can always find y by fundamental thm of single variable calculus $\boxed{\dfrac{dy}{dx} = g(x) \Rightarrow y = \int g(x)\,dx}$

→ Think about same question for (2) in two variables.

Now, see how can we find f in step 3.

Since, by definition, for $f \equiv f(x, y)$

$$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy$$

Now using (2), we obtain

$$\frac{\partial f}{\partial x} = M \quad \text{---} \quad (*)$$

and

$$\frac{\partial f}{\partial y} = N \quad \text{---} \quad (**)$$

two options.

by (*) obtain

$$f = \int M \, \partial x + \phi(y) \text{ --- } (\#)$$

$$\Rightarrow \frac{\partial f}{\partial y} = \frac{\partial}{\partial y} \int M \, \partial x + \frac{d\phi}{dy}$$

Now, using (**) obtain

$$N = \frac{\partial}{\partial y} \int M \, \partial x + \frac{d\phi}{dy}$$

$$\Rightarrow \frac{d\phi}{dy} = N - \frac{\partial}{\partial y} \int M \, \partial x$$

solve above for ϕ and put its value in (#) to obtain f.

by (**)

$$f = \int N \, \partial y + \phi(x) \text{ --- } (\#)$$

$$\Rightarrow \frac{\partial f}{\partial x} = \frac{\partial}{\partial x} \int N \, \partial y + \frac{d\phi}{dx}$$

Now, using (*) obtain

$$M = \frac{\partial}{\partial x} \int N \, \partial y + \frac{d\phi}{dx}$$

$$\Rightarrow \frac{d\phi}{dx} = M - \frac{\partial}{\partial x} \int N \, \partial y$$

Solve above for ϕ and put its value in (#) to obtain f.

Some Important facts

1. ODE 1 is called exact if $\exists$ f such that 2 holds.

2. General solution of 1 is 3. And 3 is an one parameter family of curves.

Here we read some definitions for any ODE of Ist order:

$$f\left(x, y, \frac{dy}{dx}\right) = 0 \quad \text{— } (1)$$

Then

(i) a real valued function $y = f(x)$ is called an **explicit solution** of 1 on some real interval I if

(a) $\frac{df}{dx}$ exists on I

(b) $f(x)$ and $\frac{df}{dx}$ satisfy 1 for all $x \in I$.

Moreover I is called Interval of validity for $f(x)$.

(ii) a relation $g(x, y) = 0$ is called an implicit solution of ① if it defines at least one explicit solution on an interval I.

(iii) an one-parameter familty $g(x, y, c) = 0$ is called general solution of ① if g satisfies ① for all real values of c.

(iv) a solution is called a particular solution to ① if it is obtained by finding value of c from general solution with the help of a given condition.

From above definition, Imp. Keywords to learn are

- Explicit solution ⎤
- Implicit solution ⎦ + Interval of validity.
- General solution
- particular solution

Examples to understand implicit / explicit solutions

Recall from basic mathematics

explicit functions $y = f(x)$ e.g. $y = x^2$

$y = 2\cos x + 3\sin x$

$y = e^x$

$\vdots$

implicit function $g(x, y) = 0$ e.g. $x^2 + y^2 = c^2$

$x^2 + y^3 + xy = 0$

$\vdots$

A solution in the form of explicit function is called explicit solution. Moreover an real interval I is said to be interval of validity if both solution and ODE are valid on I.

A solution in the form of implicit function is called implicit solution if it provides at least one explicit solution to ODE on some interval of validity

Let $g(x, y, c)$ be an one-parameter famility of curves. Here $c \in \mathbb{R}$ is the parameter. Then $g(x, y, c) = 0$ is called general solution if it solves ODE for each $c \in \mathbb{R}$.

For given ODE; first find general solution and then find a value of c with the help of a given condition. Thus we obtain particular solution.

ODE + condition at $x \in I$ = IVP

Example

$$\frac{dy}{dx} = -\frac{x}{y} \equiv x\,dx + y\,dy = 0 \quad \text{--- (1)}$$

See $g(x,y,c) = 0$ is general solution of (1)

when $g(x,y,c) = x^2 + y^2 + c$; $c \in \mathbb{R}$.

why; solve separable equation

$x\,dx + y\,dy = 0$

$\Rightarrow$ solution is

$\frac{x^2}{2} + \frac{y^2}{2} = c_1$

$\Rightarrow x^2 + y^2 = 2c_1 = c$; $c \in \mathbb{R}$

why; because g satisfies (1):

$x^2 + y^2 + c = 0$

$\Rightarrow 2x + 2y\frac{dy}{dx} = 0$

$\Rightarrow \frac{dy}{dx} = -\frac{x}{y}$

But if $c = k$ and $k > 0$ then

$x^2 + y^2 + k = 0$ is not an implicit solution

because for any $k > 0$

$y = \pm\sqrt{-k - x^2}$ is not a real function.

Again if $c = k$ and $k < 0$ then

$x^2 + y^2 - k = 0$ is an implicit solution

because it gives two explicit solution

$y = \pm\sqrt{k - x^2}$

and interval of validity for both is $(-\sqrt{k}, \sqrt{k})$

Solve IVP

ODE → $\frac{dy}{dx} = -\frac{x}{y}$

I/C → $y(0) = 6$

$\frac{dy}{dx} = -\frac{x}{y}$

$\equiv \quad x\,dx + y\,dy = 0$

It is separable. Thus solution is

$$\frac{x^2}{2} + \frac{y^2}{2} = c \quad \text{---} \quad (*)$$

put $x=0$ and $y=6$ in $(*)$ and obtain

$$\frac{36}{2} = c = 18$$

i.e.

$x^2 + y^2 - 36 = 0$ is a particular solution of given IVP.

If IVP is

$\frac{dy}{dx} = -\frac{x}{y}$

$y(0) = -6$

Then see again we have the same particular solution: $x^2+y^2-36=0$

We will see the exact meaning in next week: that same solution curve passes thr. $(0,6)$ and $(0,-6)$.

THINK. No harm to write general solution of given ODE as

$\boxed{x^2 + y^2 = c^2}$; $c \in \mathbb{R}$.

Let us see again Interval of validity for solution/ODE with the help of following examples

→ Find solution for the following IVP and determine the interval of validity for the solution.

$$\frac{dy}{dx} = 6y^2x \quad ; \quad y(1) = \frac{1}{25}$$

$$\frac{dy}{dx} = 6y^2x$$

$$\Rightarrow \quad \frac{dy}{y^2} = 6x\,dx \quad \text{(separable case)}$$

Thus solution is

$$-\frac{1}{y} = 6\frac{x^2}{2} + c = 3x^2 + c$$

$$\Rightarrow \quad y = -\frac{1}{3x^2+c} \quad \left[\begin{array}{c}\text{general solution} \\ \text{for ODE}\end{array}\right]$$

{ Now use I/C for solution to IVP, i.e. find particular solution

By using $y(1) = \frac{1}{25}$, we obtain

$$\frac{1}{25} = -\frac{1}{3+c} \Rightarrow 3+c = -25 \Rightarrow c = 28$$

Thus solution for IVP is

$$y = -\frac{1}{3x^2+28} = \frac{1}{28-3x^2}$$

[Here, we are lucky that it is already in explicit form.]

The solution is valid on following three intervals

$$\left(-\infty, -\sqrt{\frac{28}{3}}\right) ; \left(-\sqrt{\frac{28}{3}}, \sqrt{\frac{28}{3}}\right) ; \left(\sqrt{\frac{28}{3}}, \infty\right).$$

Since I/c lies here, therefore

Interval of validity for the solution to IVP is $\left(-\sqrt{\frac{28}{3}}, \sqrt{\frac{28}{3}}\right)$.

open interval.

Solve the following IVP

$$\frac{dy}{dx} = \frac{xy^3}{\sqrt{1+x^2}} \qquad y(0) = -1$$

$$\Rightarrow \frac{dy}{y^3} = \frac{x}{\sqrt{1+x^2}}\, dx$$

Separable ODE, thus general solution is

$$\boxed{-\frac{1}{2y^2} = \sqrt{1+x^2} + C}$$

$\Rightarrow$ put I/C $y(0)=-1$ to obtain C

$$-\frac{1}{2} = 1 + C \Rightarrow C = -\frac{3}{2}$$

i.e. solution to IVP is

$$\boxed{-\frac{1}{2y^2} = \sqrt{1+x^2} - \frac{3}{2}}$$

Suppose Interval of validity is also asked in the question, then remember Interval of validity is related with explicit solution, but above solution is in implicit form. For that simplify the solution and obtain

$$y = \pm \frac{1}{\sqrt{3 - 2\sqrt{1+x^2}}}$$

$\rightarrow$ From here you have to choose either +ive or -ive sign. I/C shows that -ive sign is valid.

Hence solution is

$$y = -\frac{1}{\sqrt{3 - 2\sqrt{1+x^2}}}$$

and it is valid when

$$3 - 2\sqrt{1+x^2} > 0 \qquad \boxed{\because\ 1+x^2 \geq 0}$$

$$\Rightarrow \quad -\frac{\sqrt{5}}{2} < x < \frac{\sqrt{5}}{2}$$

verify that initial x-value lies here.