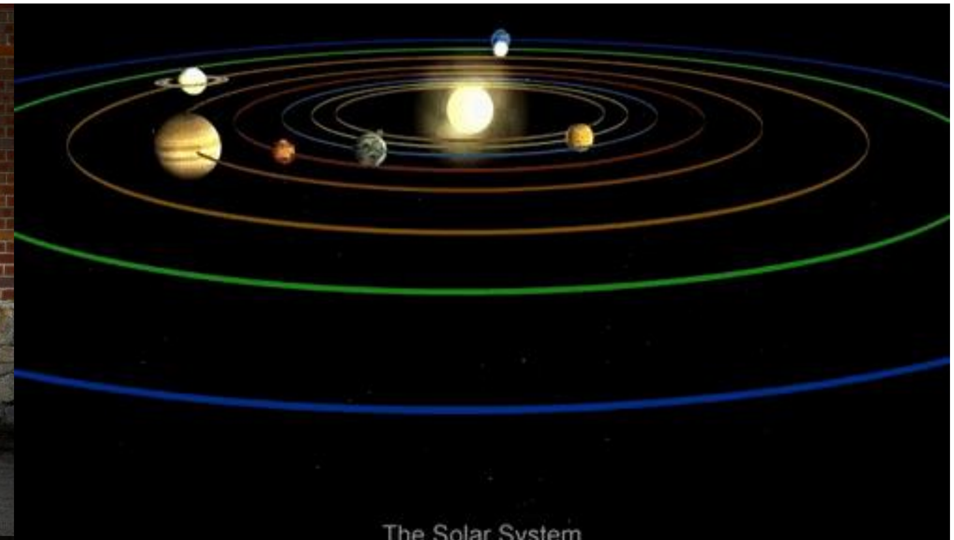
Work , Energy and Conservation laws



GIFs:GiPHY, Gfycat

Recap

$$W_{ba} = \int_a^b \vec{F} \cdot d\vec{r}$$

Evaluation of this integral depends on knowing what path the particle actually follows

Constrained motion

Conservative forces

Recap….

$$V(\vec{r}) - V(\vec{r}_0) = -\int_{r_0}^{r} \vec{F}(\vec{r}\,') \cdot d\vec{r}\,'$$

If V(r) is path independent,

$$dV(\vec{r}) = -\vec{F}(\vec{r}) \cdot d\vec{r}$$

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$$\vec{F}\left(\vec{r}\right) = -\nabla V(\vec{r})$$

Curl of F is $\vec{\nabla} \times \vec{F}\left(\vec{r}\right) = -\left[\vec{\nabla} \times \nabla V(\vec{r})\right] = 0$

Concept of equilibrium

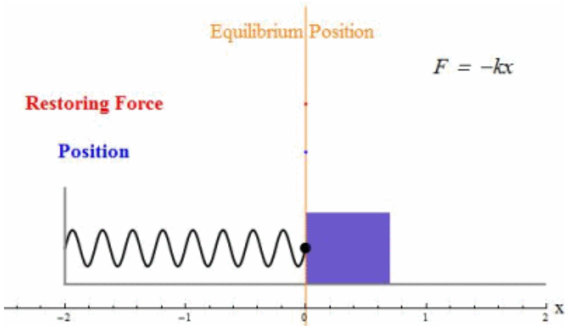
$\vec{F}\left(\vec{r}\right) = -\nabla V(\vec{r})$ is useful for visualizing the stability of a system

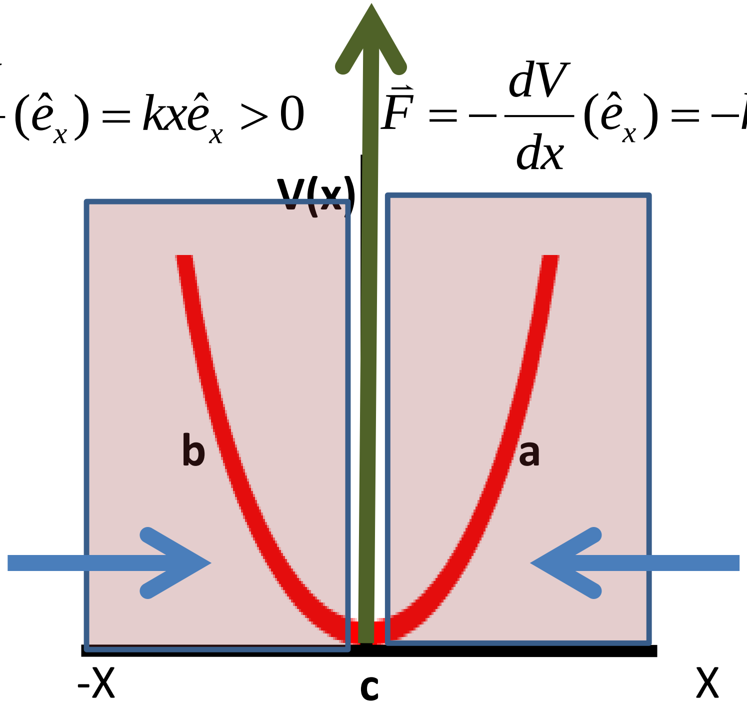
Examples

1-D Harmonic Oscillator

$$\vec{F} = -\frac{dV}{dx}(\hat{e}_x) = 0$$

$$\vec{F} = -\frac{dV}{dx}(\hat{e}_x) = kx\hat{e}_x > 0 \qquad \vec{F} = -\frac{dV}{dx}(\hat{e}_x) = -kx\hat{e}_x < 0$$

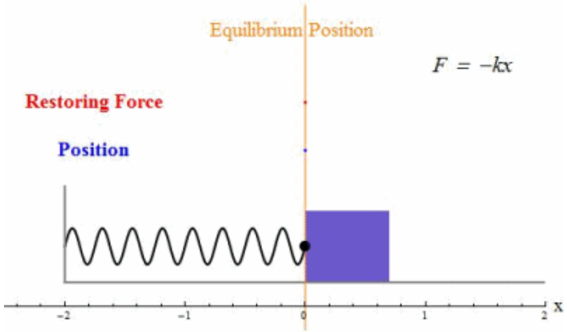




$$V(x) = \frac{1}{2}kx^2$$

$\frac{dV}{dx} = 0$ → Equilibrium of the system

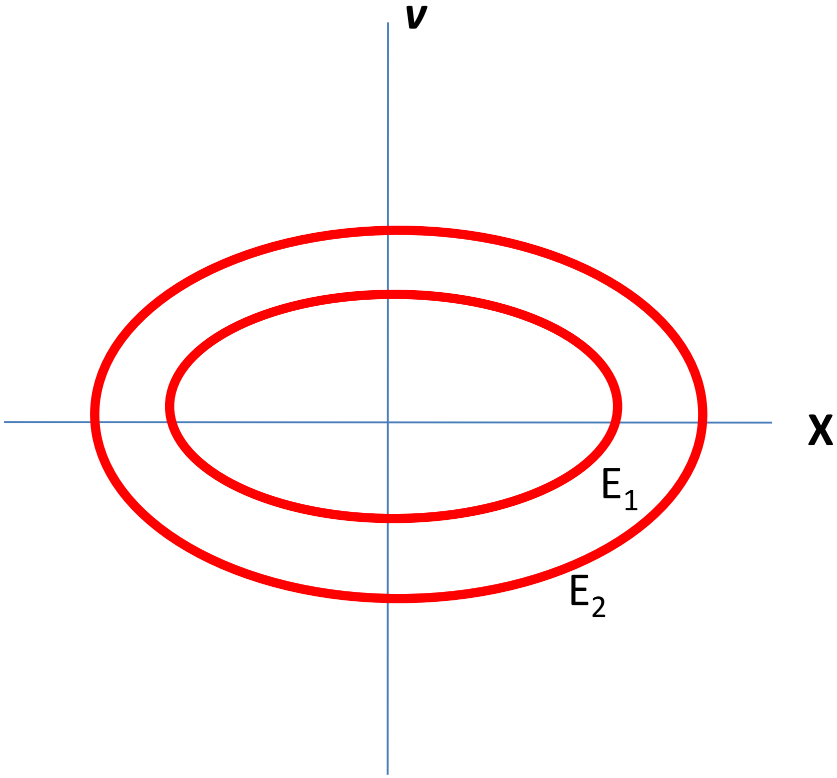
Velocity Vs. Position plot



$$E = \frac{1}{2}kx^2 + \frac{1}{2}mv^2$$

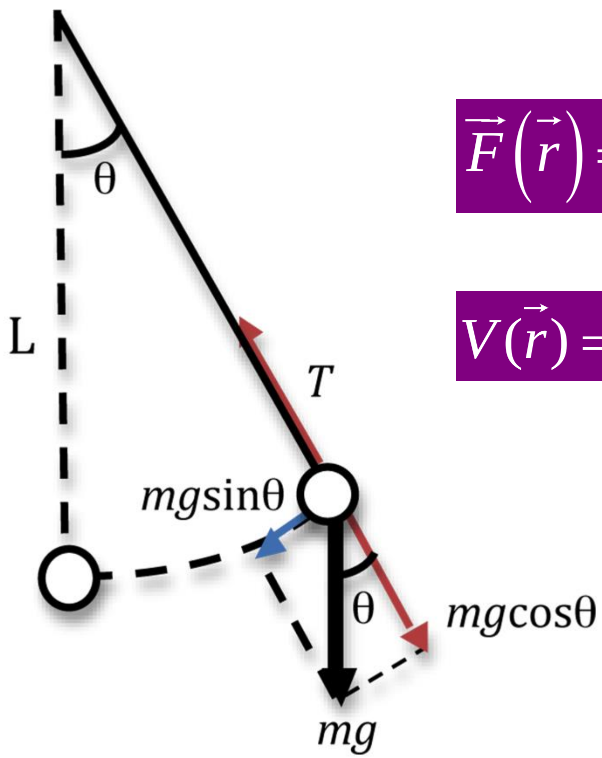
Plot for energy = E_1

$E_2 > E_1$



Concept of equilibrium

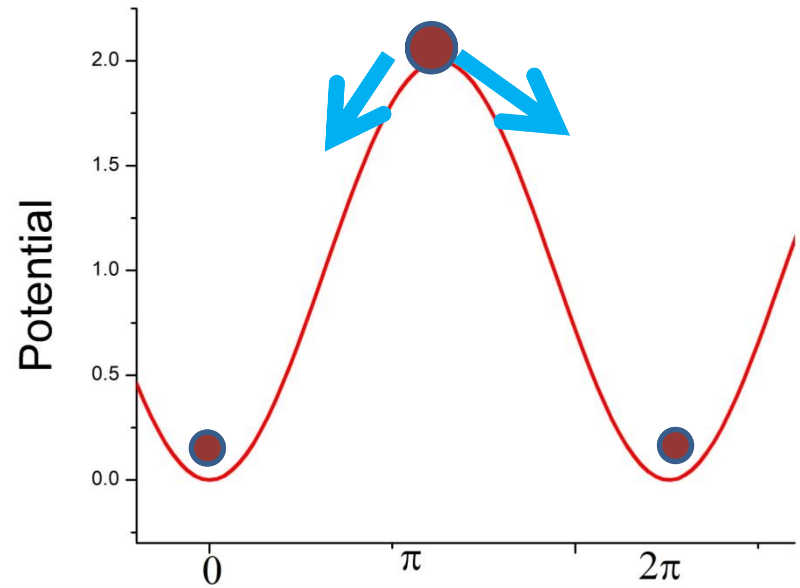
Simple Pendulum

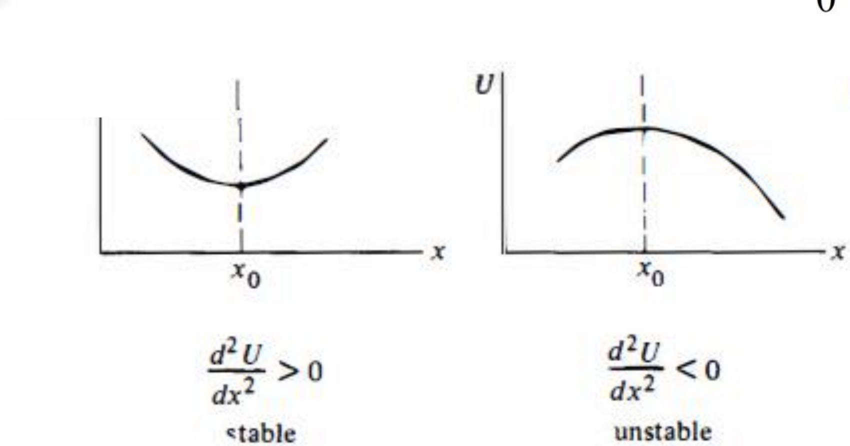


$$\vec{F}\left(\vec{r}\right) = -mg\sin\theta\hat{e}_\theta$$

$$V(\vec{r}) = mgl(1-\cos\theta)$$

Unstable equilibrium





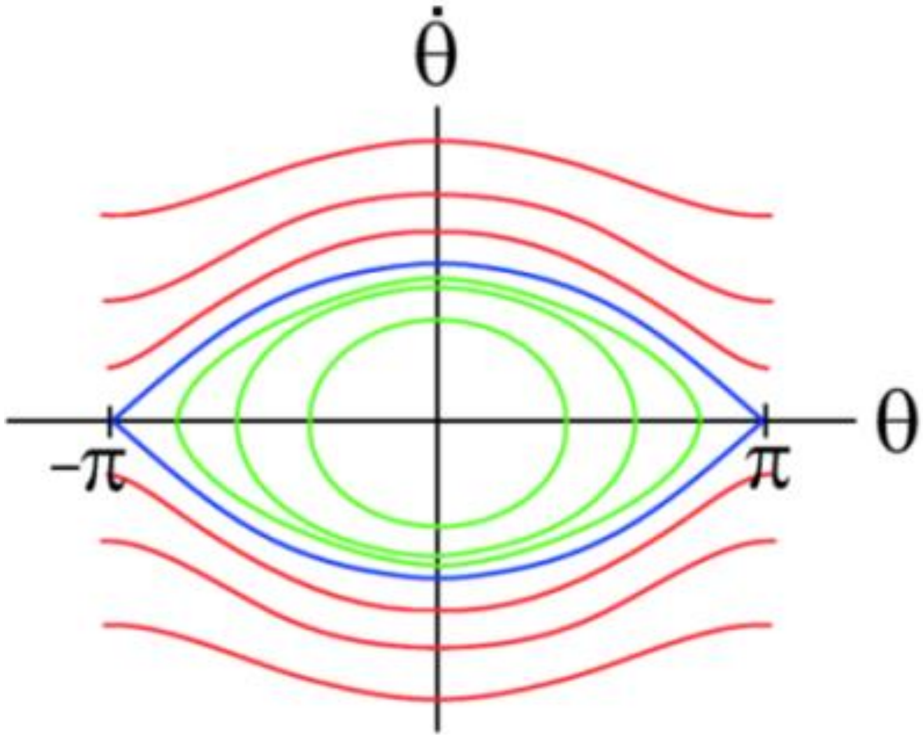
Plot $\dot{\theta}$ v/s θ ?

$$E = \frac{1}{2} ml^2 \dot{\theta}^2 + mgl(1 - \cos\theta)$$

$$E = \frac{1}{2} ml^2 \dot{\theta}^2 + mgl(1 - (1 - \frac{\theta^2}{2!} + ...)$$

$$E = \frac{1}{2} ml^2 \dot{\theta}^2 + mgl \frac{\theta^2}{2!}$$

$$\frac{\dot{\theta}^2}{2E/ml^2} + \frac{\theta^2}{2E/mgl} = 1$$

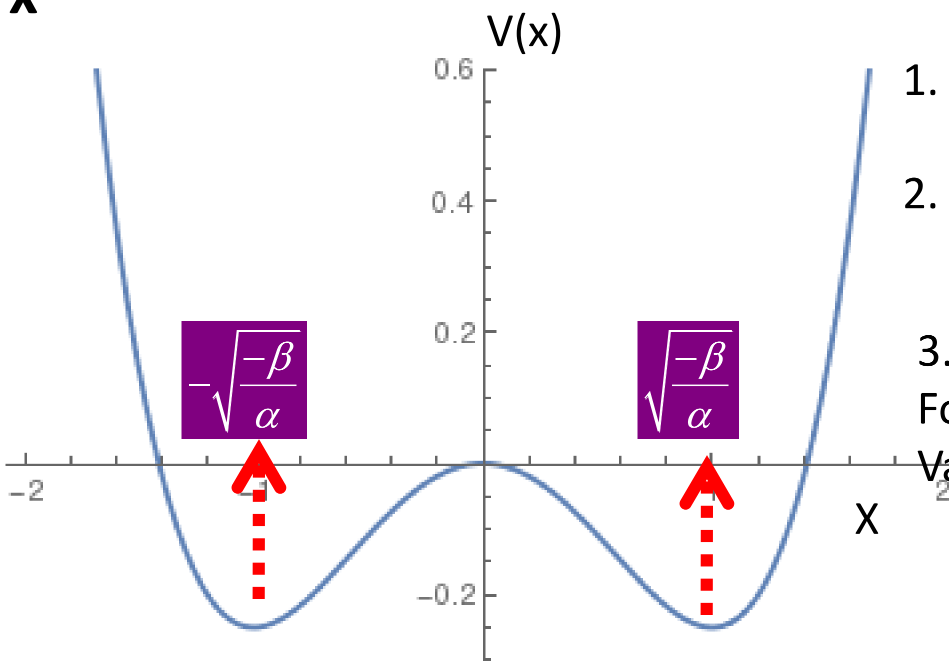


Concept of equilibrium

Duffing Oscillator is an example of a periodically forced oscillator with a nonlinear elasticity

$$V(x)=\frac{1}{2}\beta x^2+\frac{1}{4}\alpha x^4,\quad \beta<0$$

Plot V(x) v/s x



1. Maxima at x =0

Minima at $-\sqrt{\frac{-\beta}{\alpha}}$

and $\sqrt{\frac{-\beta}{\alpha}}$

2. Zero crossing at 3 points, x = 0, $-\sqrt{\frac{-2\beta}{\alpha}}$ $\sqrt{\frac{-2\beta}{\alpha}}$

To Plot the graph

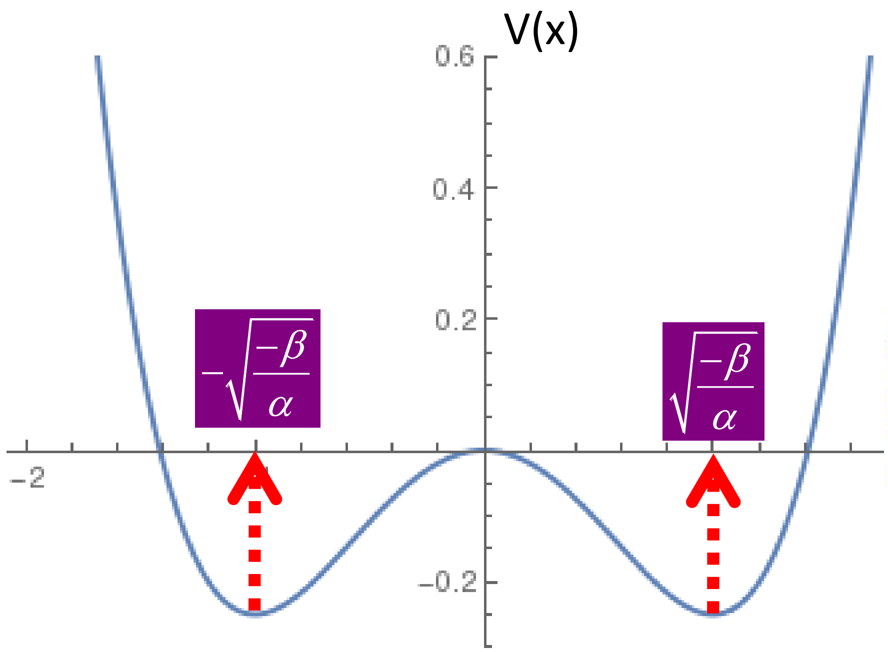
1. Find Maxima and Minima
2. Find the zero crossing points
3. Imagine the function For smaller and larger Values of x

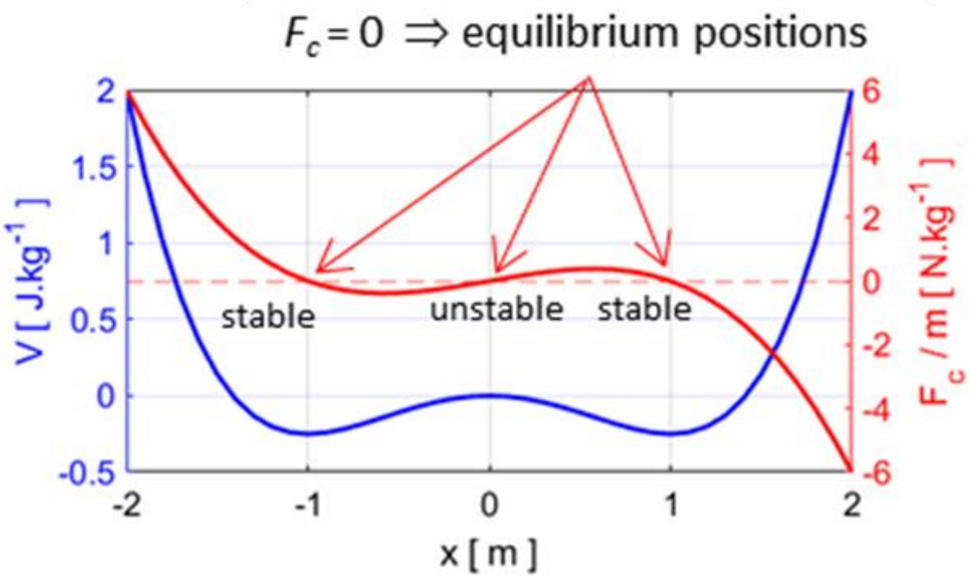
Concept of equilibrium

Duffing Oscillator

$$V(x) = \frac{1}{2}\beta x^2 + \frac{1}{4}\alpha x^4, \quad \beta < 0$$

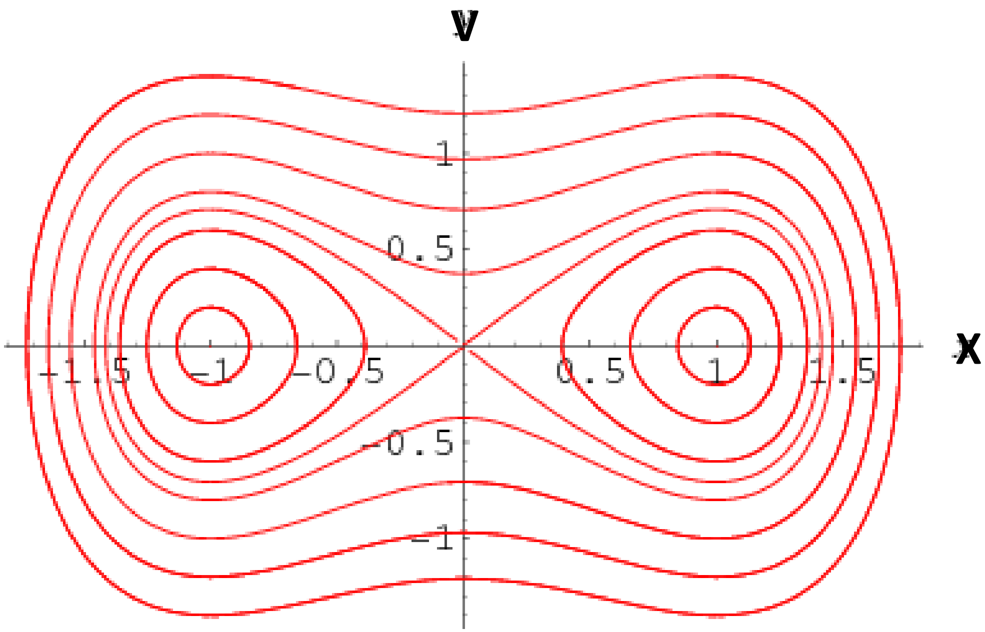
Component of Force $F(x) = -\frac{dV(x)}{dx} = -\beta x - \alpha x^3$

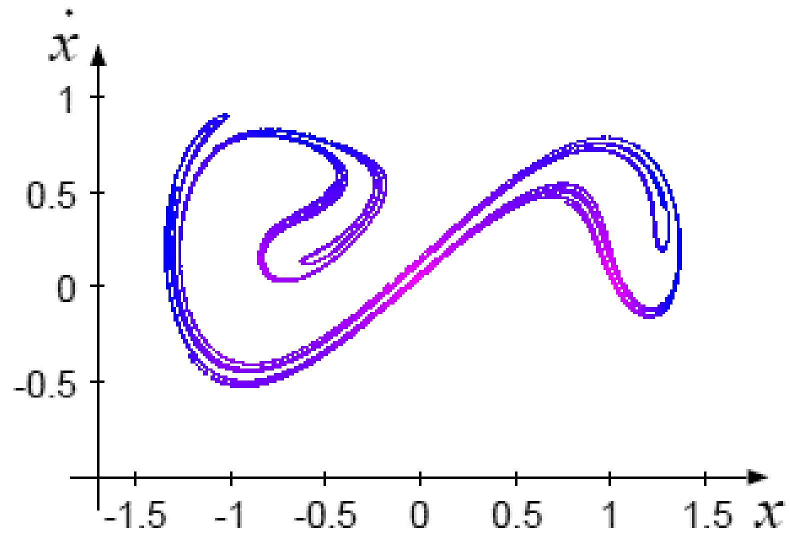




Velocity Vs. Position plot

$$E = \frac{1}{2}\beta x^2 + \frac{1}{4}\alpha x^4 + \frac{1}{2}mv^2$$



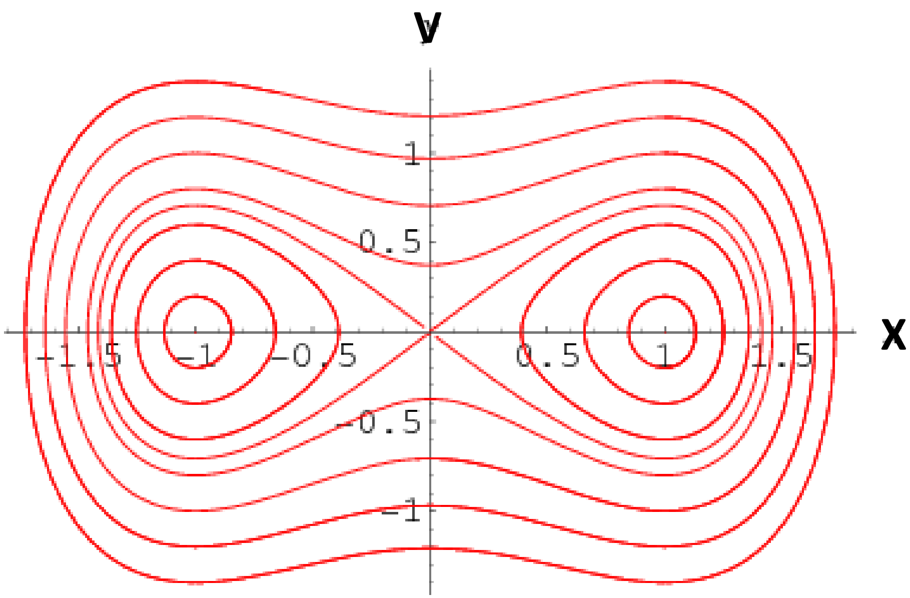


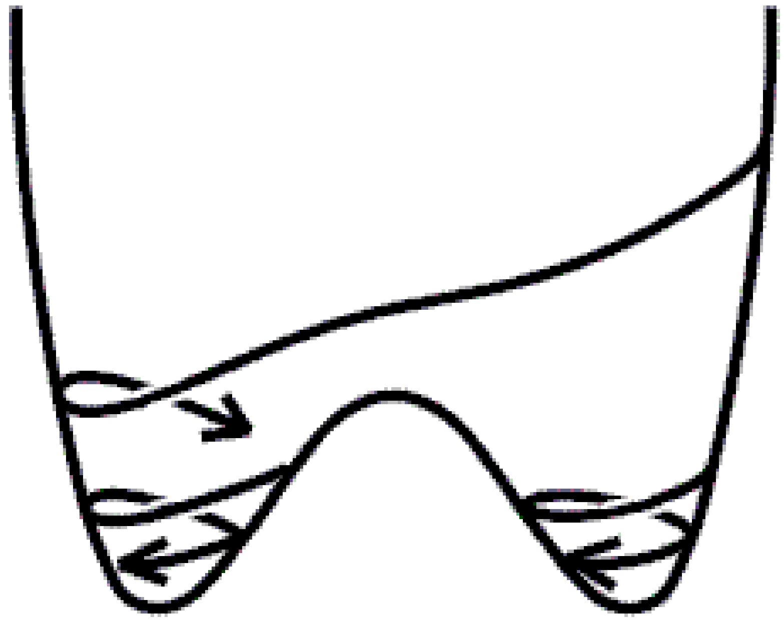
Periodic change of the chaotic attractor of the Duffing oscillator for α=1 , β=−1, and ω=1

Takashi Kanamaru (2008), Scholarpedia, 3(3):6327.

Velocity Vs. Position plot

$$E = \frac{1}{2}\beta x^2 + \frac{1}{4}\alpha x^4 + \frac{1}{2}mv^2$$

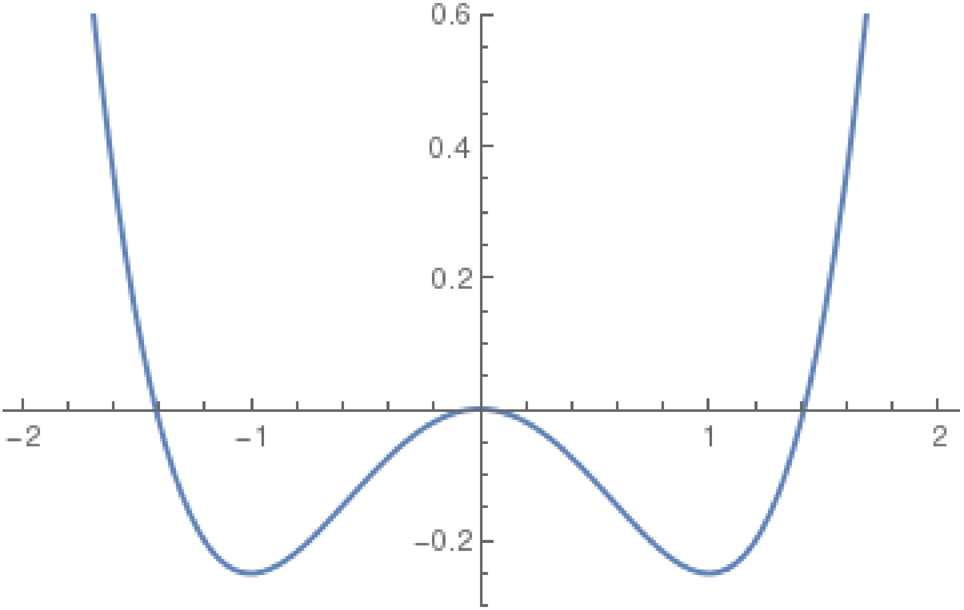


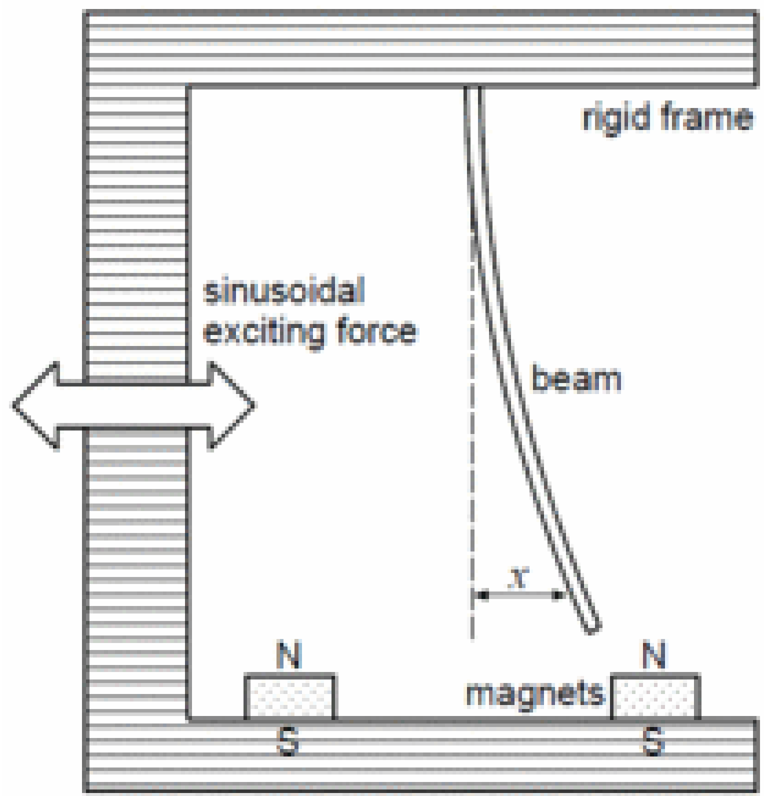


Takashi Kanamaru (2008), Scholarpedia, 3(3):6327.

Physical Example

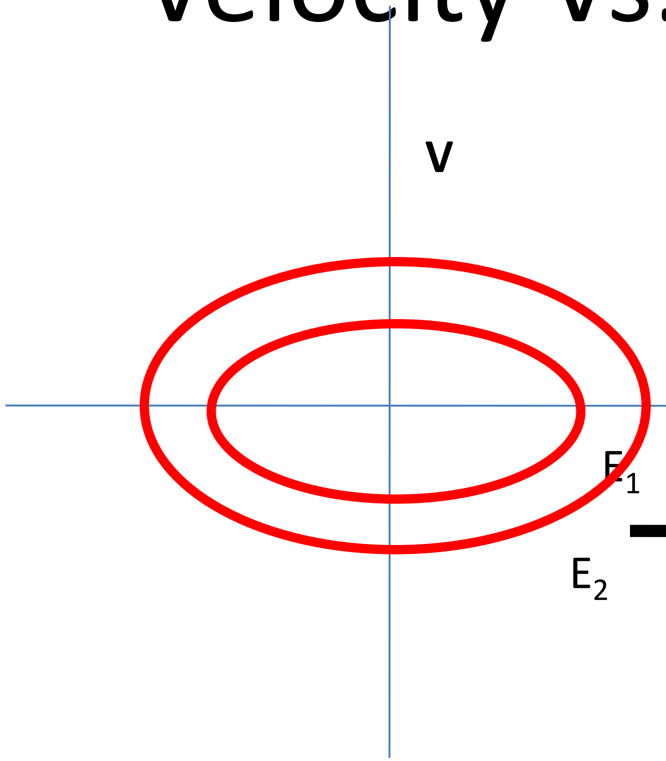
Duffing oscillator: a model of a periodically forced steel beam which is deflected toward the two magnets.

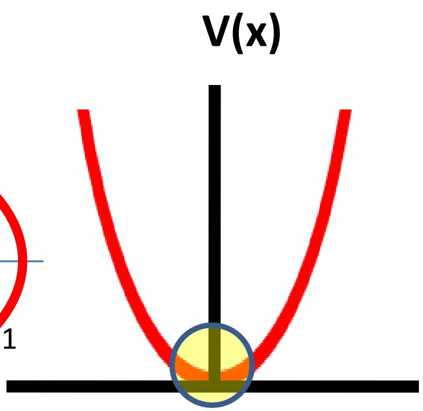


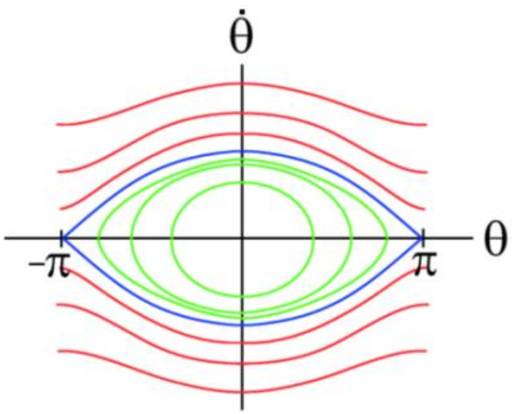


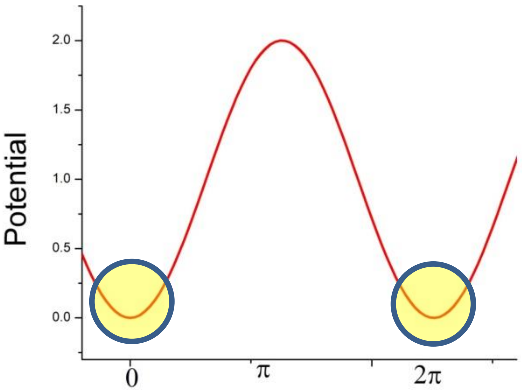
Moon and Holmes, 1979; Guckenheimer and Holmes, 1983; Ott, 2002

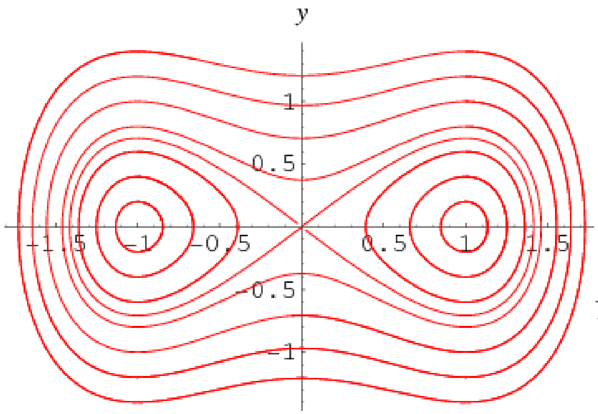
Velocity Vs. Position plot (Recap)

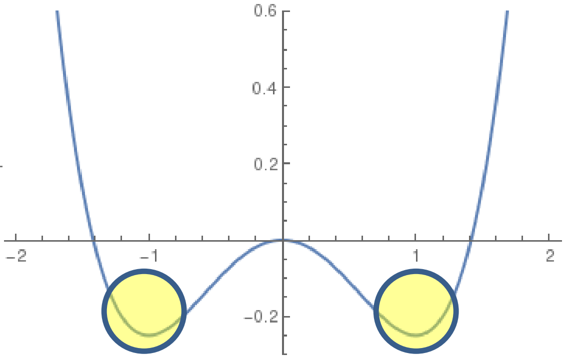




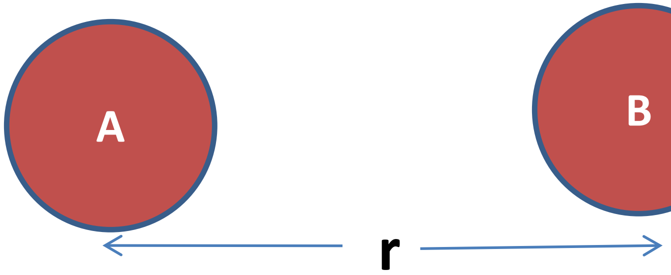








Interatomic potential

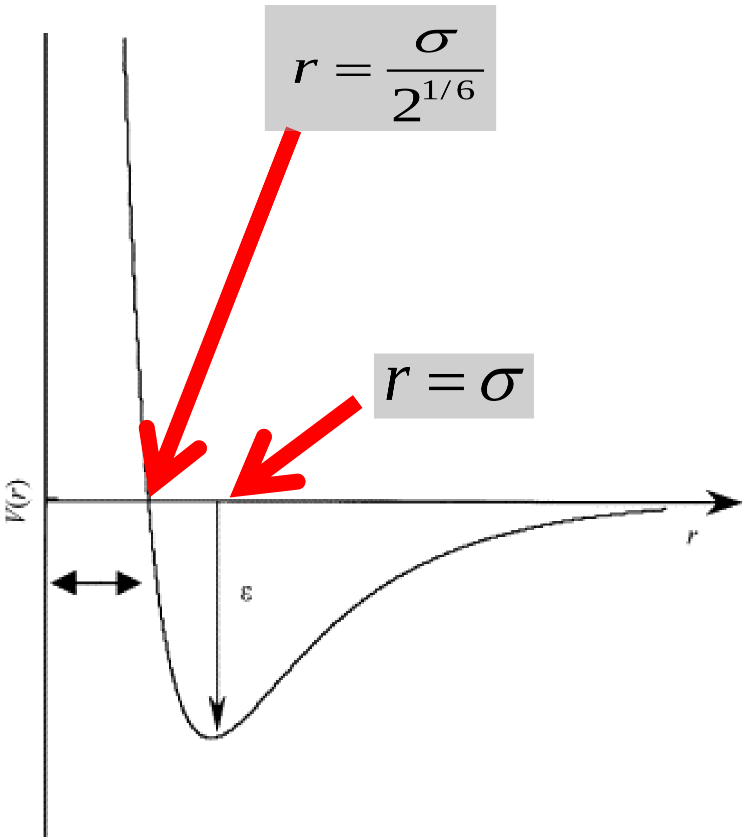


Lennard Jones Potential

$$U_{LJ}(r) = \varepsilon\left[\left(\frac{\sigma}{r}\right)^{12} - 2\left(\frac{\sigma}{r}\right)^{6}\right]$$

$$\varepsilon > 0$$

Interatomic potential



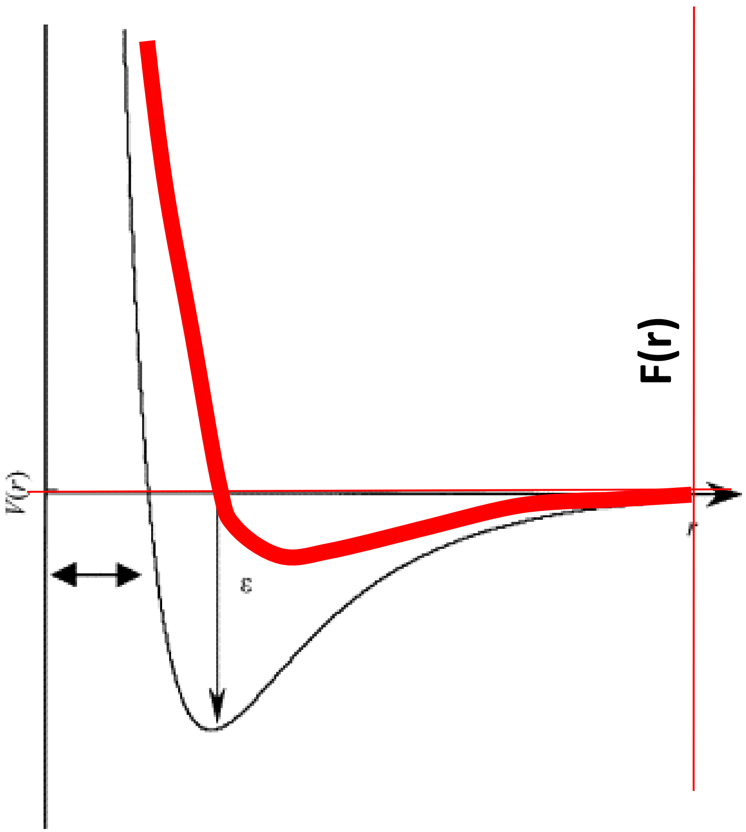
Lennard Jones Potential

$$U_{LJ}(r) = \varepsilon\left[\left(\frac{\sigma}{r}\right)^{12} - 2\left(\frac{\sigma}{r}\right)^{6}\right]$$

Zeros: $r = \infty,\ r = \frac{\sigma}{2^{1/6}}$

Minimum: $r = \sigma$

Interatomic force



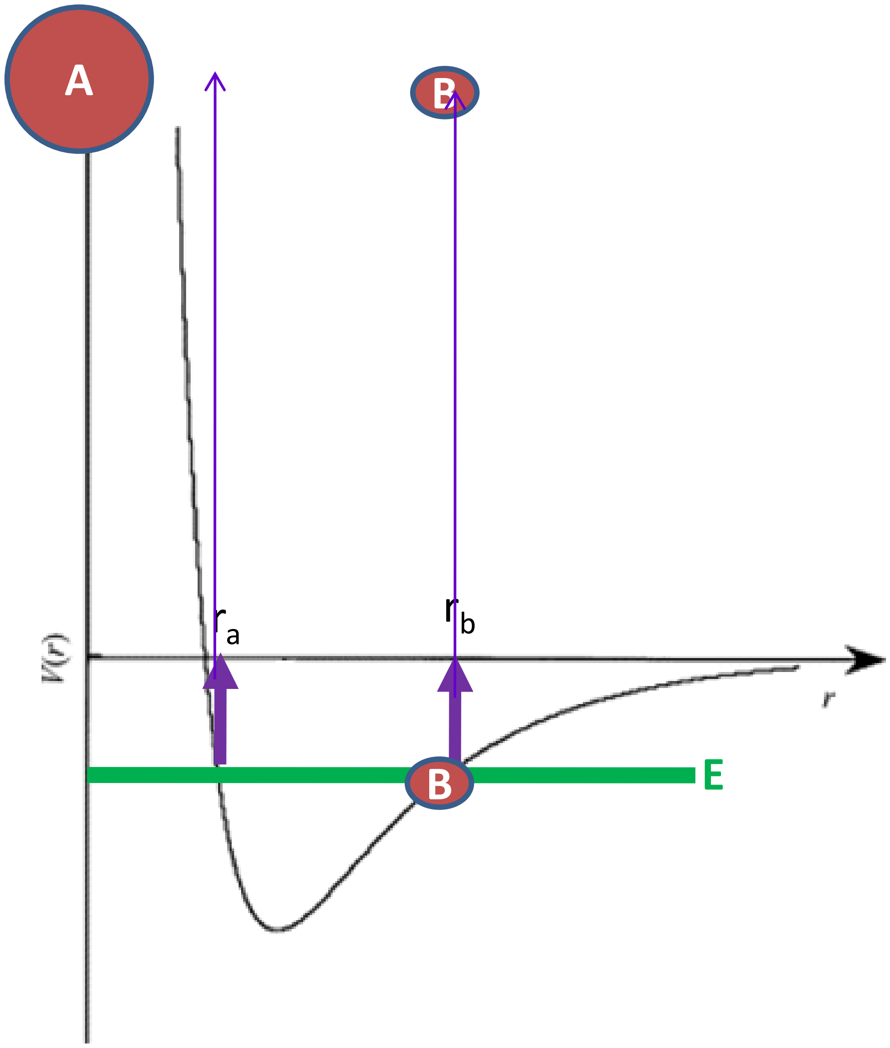
Lennard Jones Potential

$$U_{LJ}(r) = \varepsilon\left[\left(\frac{\sigma}{r}\right)^{12} - 2\left(\frac{\sigma}{r}\right)^{6}\right]$$

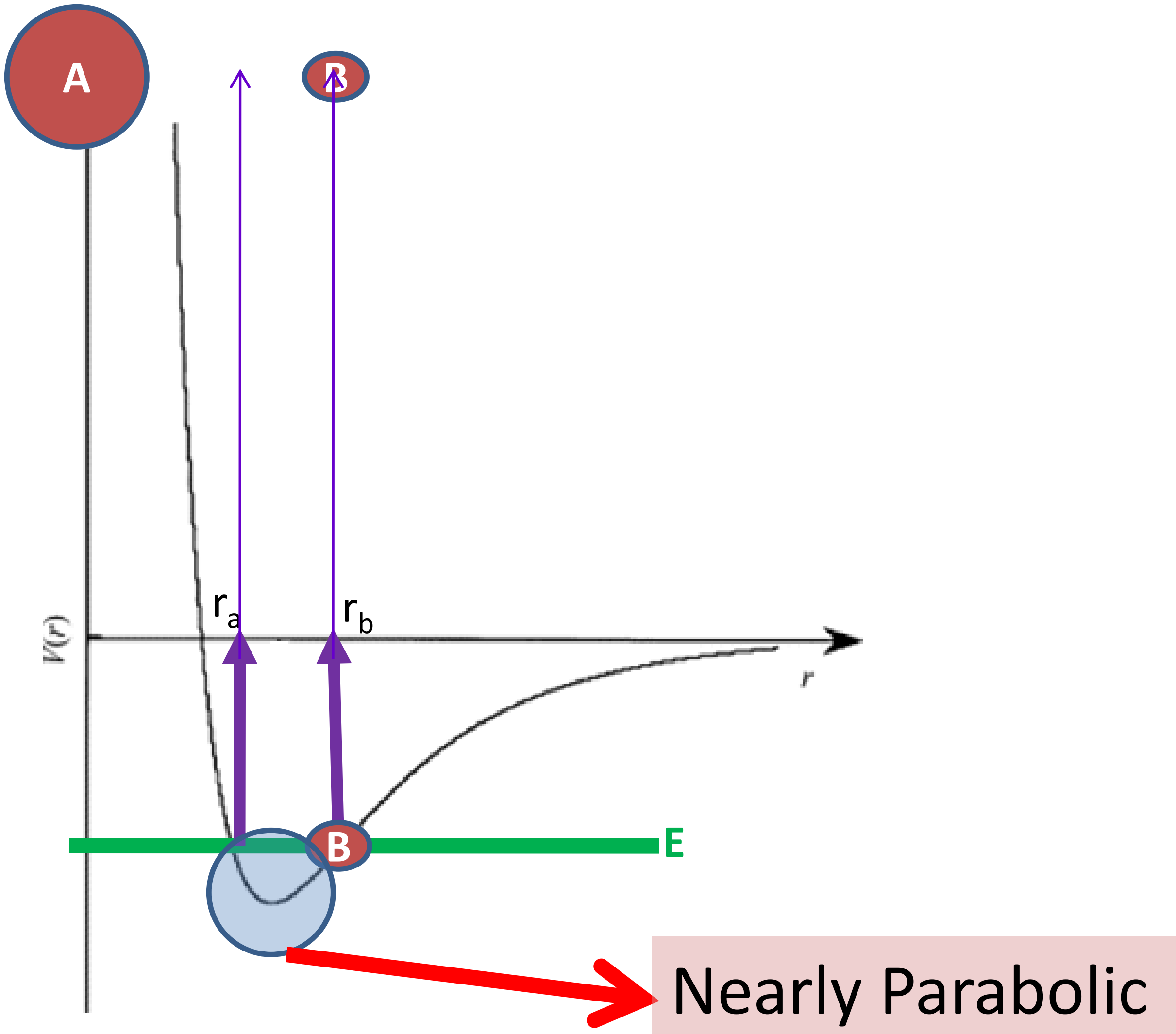
Lennard Jones Force

$$\vec{F} = 12\varepsilon\left[\frac{\sigma^{12}}{r^{13}} - \frac{\sigma^{6}}{r^{7}}\right]\hat{e}_r$$

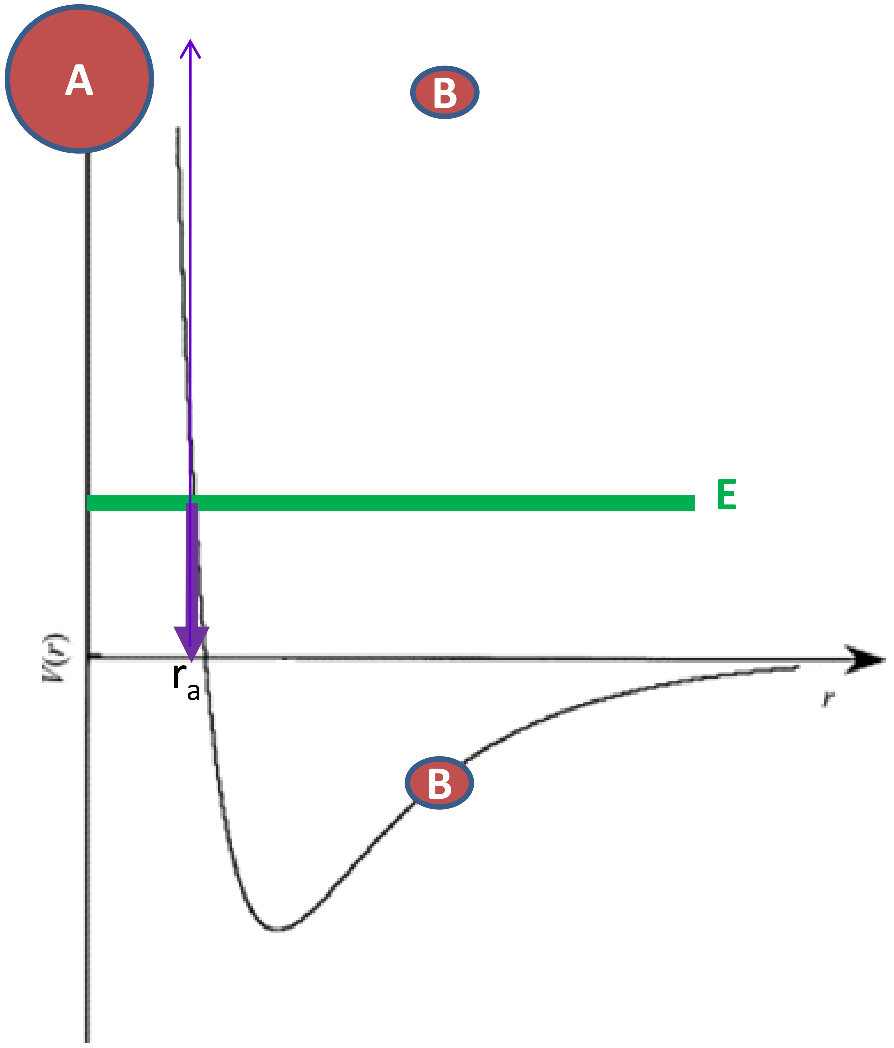
Interatomic potential



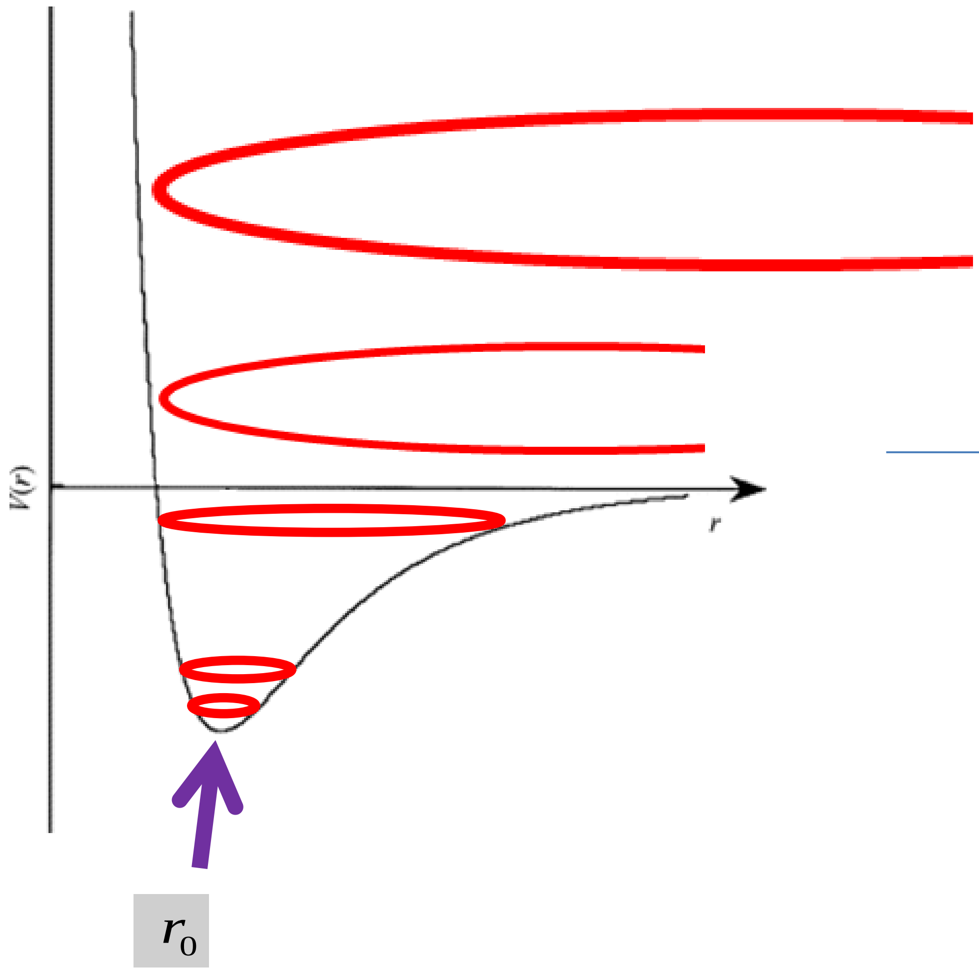
Interatomic potential

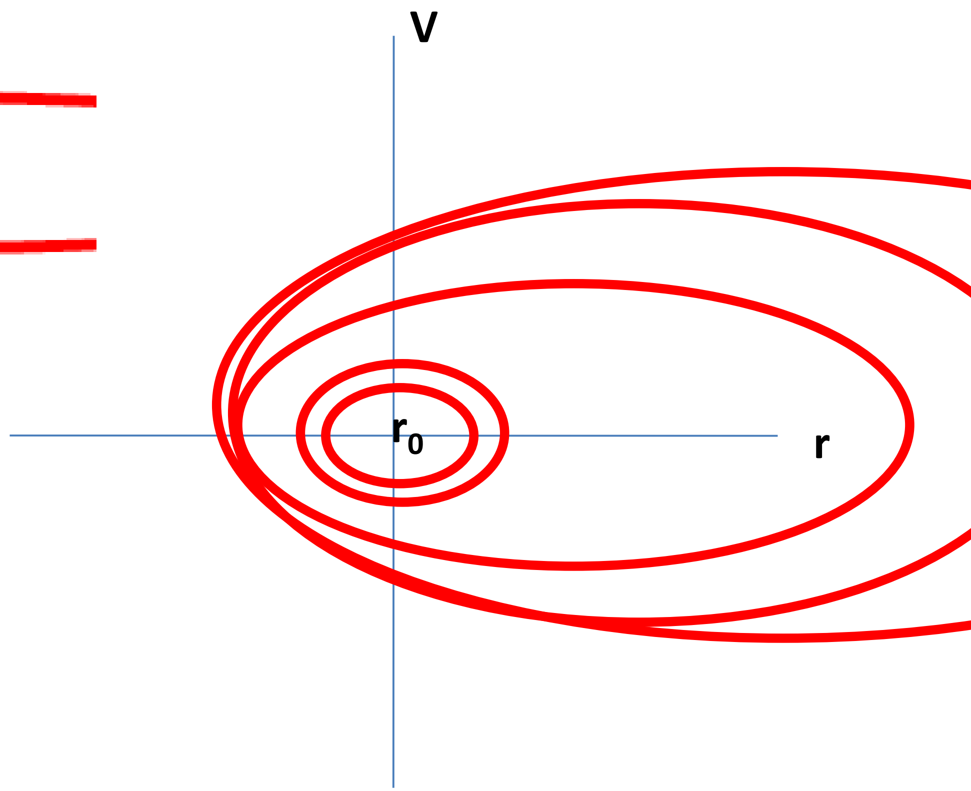


Interatomic potential

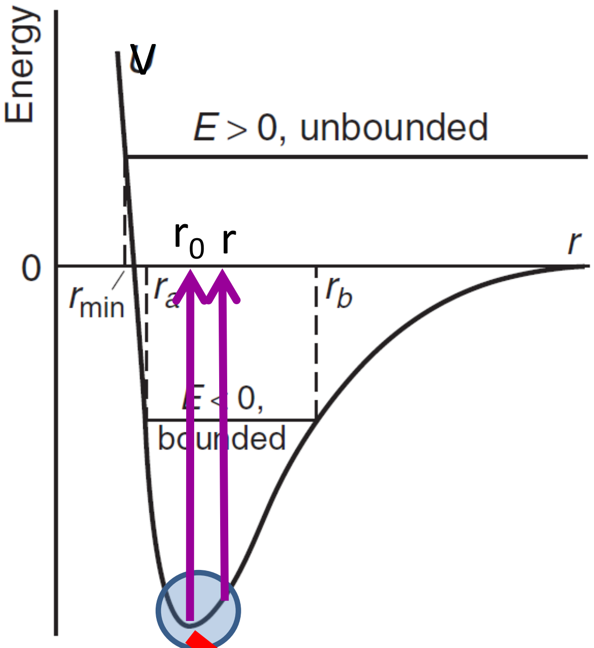


Velocity Vs. Position plot





Small Oscillations



$$V(r) = V(r_0) + \cancel{(r - r_0)\frac{dV}{dr}\bigg|_{r_0}} + \frac{1}{2}(r - r_0)^2 \frac{d^2V}{dr^2}\bigg|_{r_0} + \cdots$$

$$V(r) = V(r_0) + \frac{1}{2}(r - r_0)^2 \frac{d^2V}{dr^2}\bigg|_{r_0}$$

$$V(r) = \text{Constant} + \frac{1}{2}kx^2$$

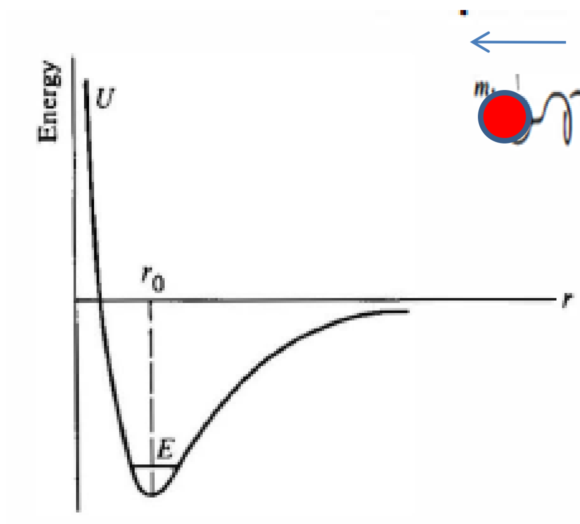
Nearly Parabolic

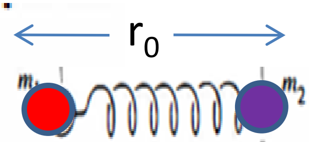
Harmonic Oscillator

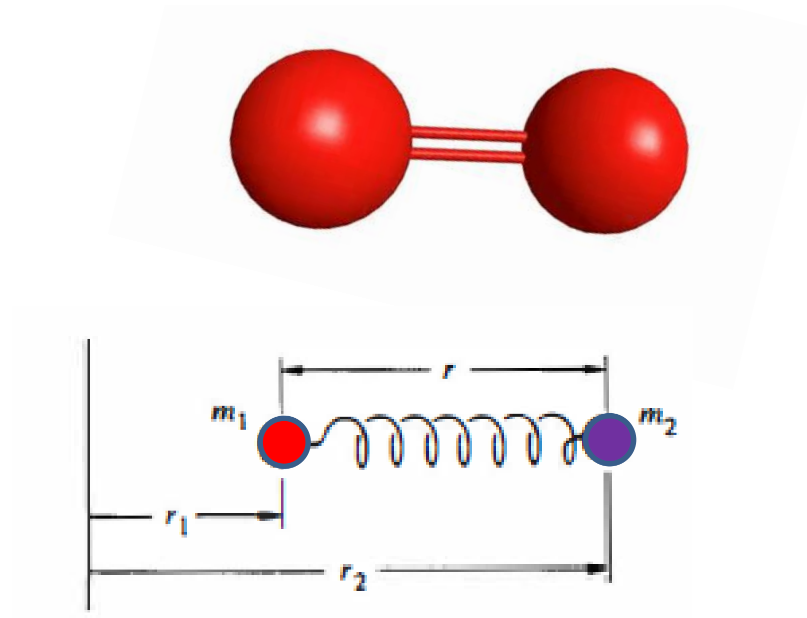
$$k = \omega^2 m = \frac{d^2V}{dr^2}\bigg|_{r_0}$$

Molecular vibrations

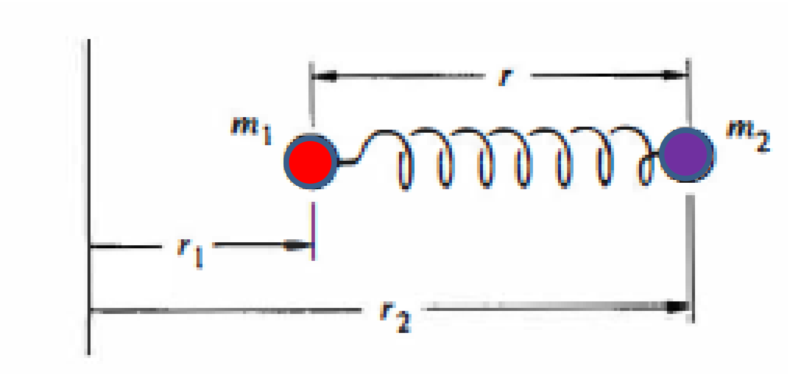
How to find the vibration frequency of diatomic molecule which is bound with very low energy such that their separation is almost close to equilibrium distance r_0?







Equation of motion



Provided, r_0 is the equilibrium distance,

$$m_1 \ddot{r}_1 = k(r - r_0)$$

$$m_2 \ddot{r}_2 = -k(r - r_0),$$

$$\ddot{r}_2 - \ddot{r}_1 = \ddot{r} = -k\left(\frac{1}{m_1} + \frac{1}{m_2}\right)(r - r_0)$$

Equation of motion

$$\ddot{r}_2 - \ddot{r}_1 = \ddot{r} = -k\left(\frac{1}{m_1} + \frac{1}{m_2}\right)(r - r_0)$$

or

$$\ddot{r} = -\frac{k}{\mu}(r - r_0),$$

$$\mu = m_1 m_2/(m_1 + m_2)$$

By Comparing with Harmonic Oscillator, we get the frequency of vibrations as

$$\omega = \sqrt{\frac{k}{\mu}}$$
$$= \sqrt{\left.\frac{d^2U}{dr^2}\right|_{r_0} \frac{1}{\mu}}.$$

Recap....

$$\boldsymbol{V(\vec{r}) - V(\vec{r}_0) = -\int_{r_0}^{r} \vec{F}(\vec{r}\,') \cdot d\vec{r}\,'}$$

If V(r) is path independent,

$$\boldsymbol{dV(\vec{r}) = -\vec{F}(\vec{r}) \cdot d\vec{r}}$$

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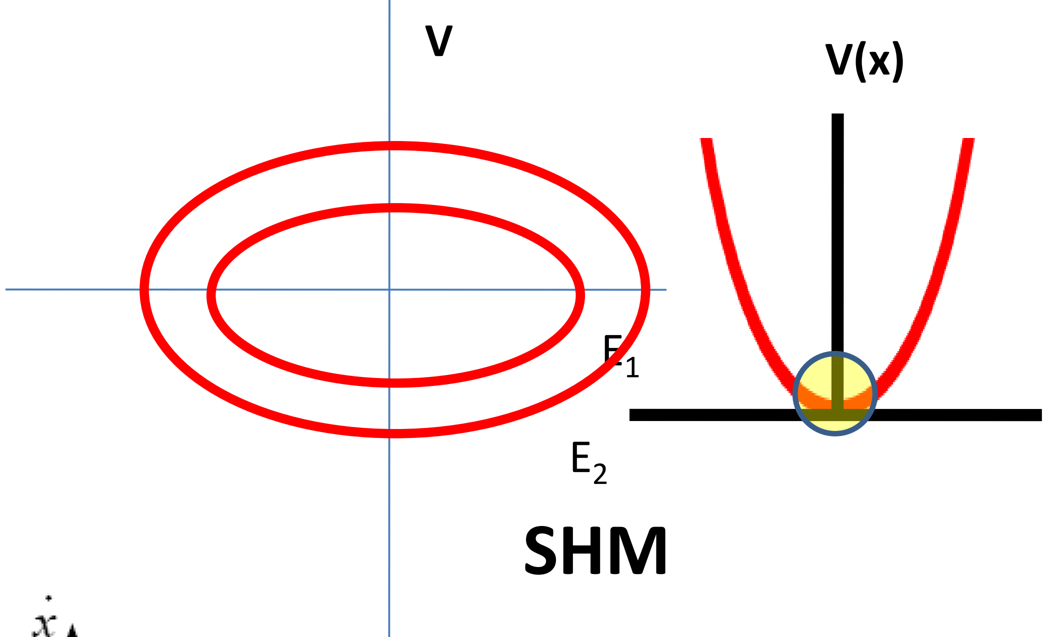
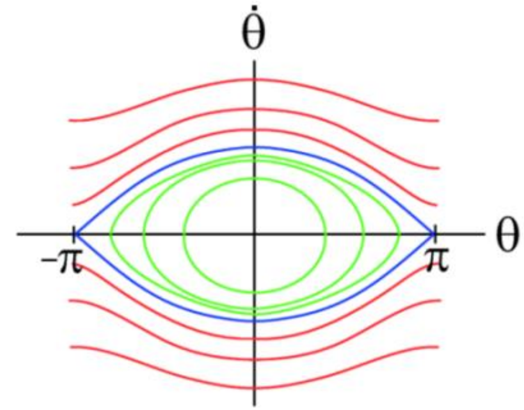
$$\vec{F}\left(\vec{r}\right) = -\nabla V(\vec{r})$$

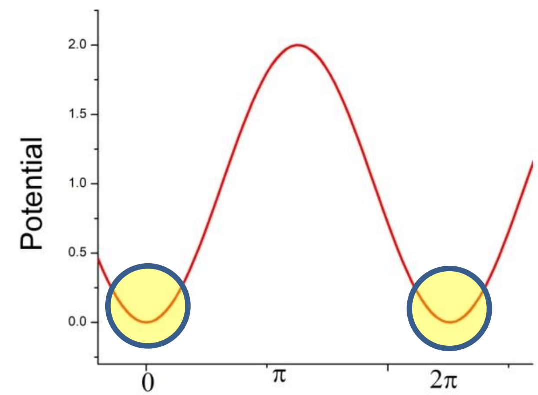
Curl of F is $\vec{\nabla} \times \vec{F}\left(\vec{r}\right) = -\left[\vec{\nabla} \times \nabla V(\vec{r})\right]$

Concept of equilibrium (3 examples)

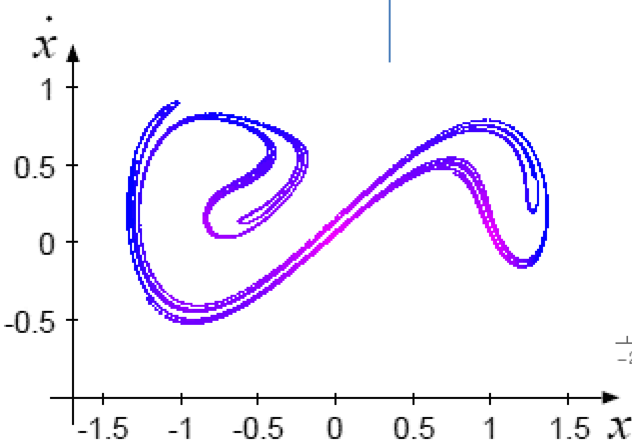


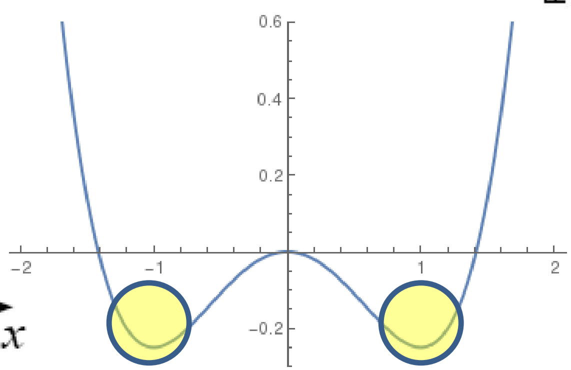
SHM





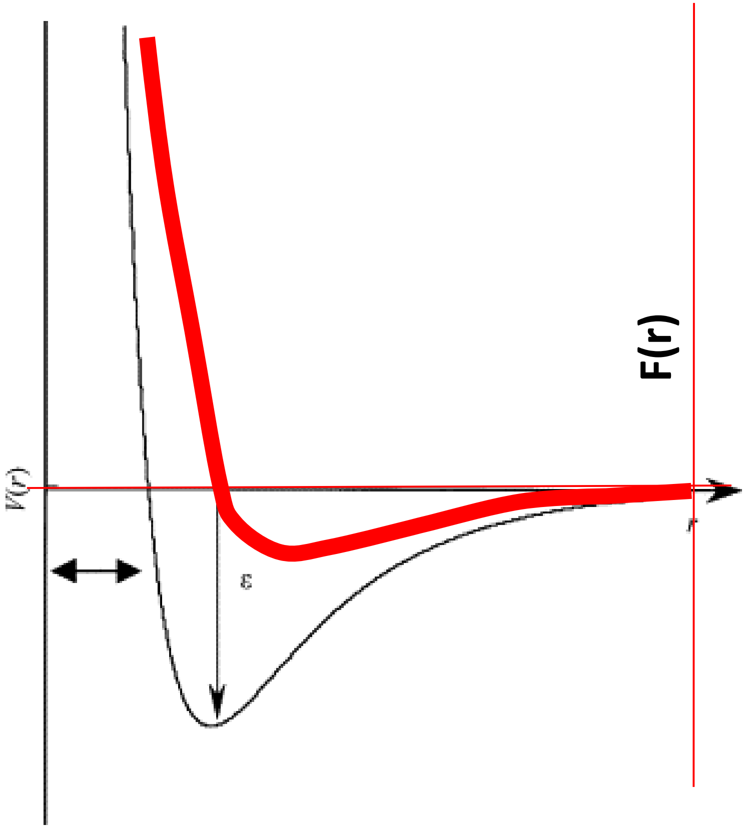
Simple pendulum





Duffing oscillator

Interatomic force



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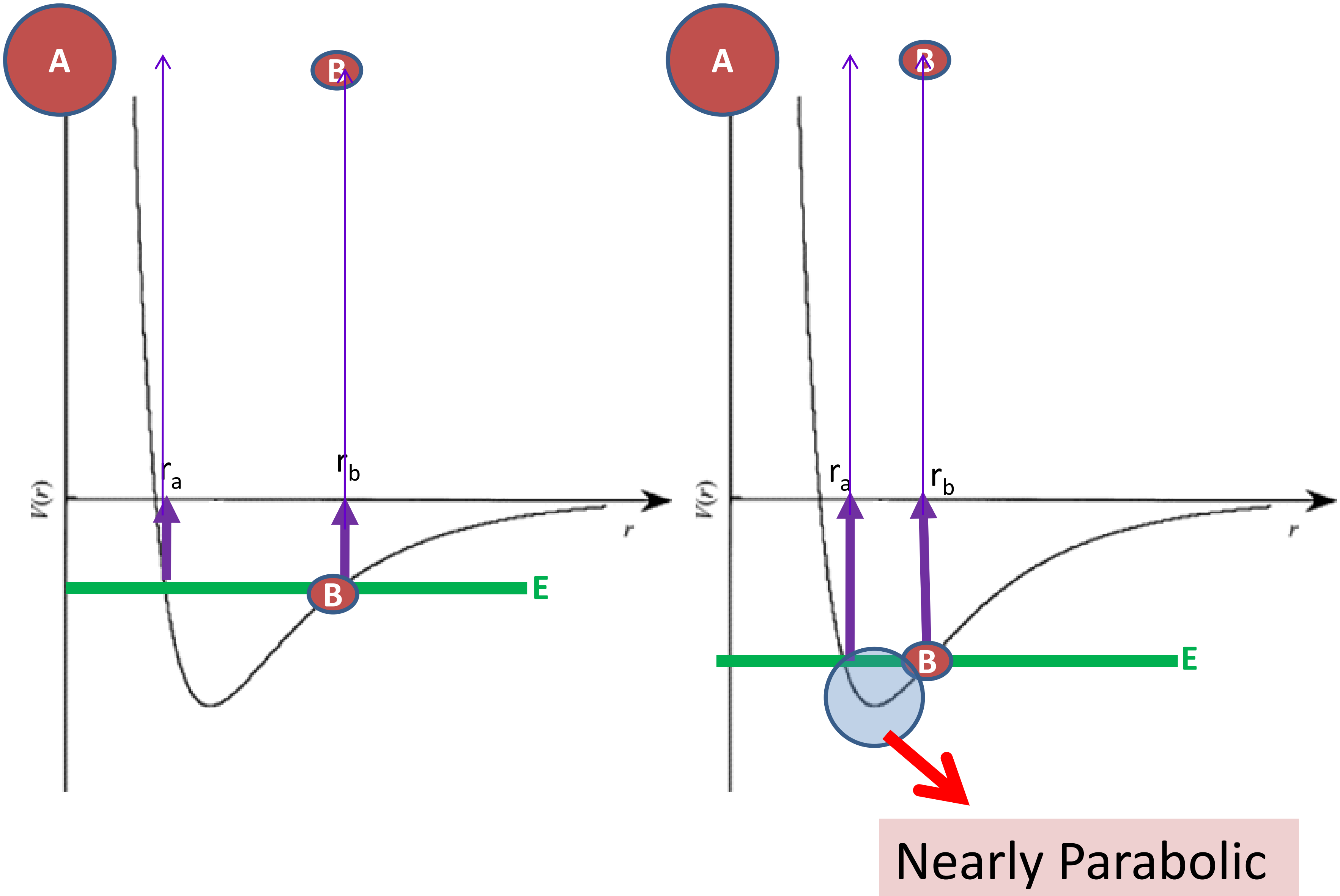

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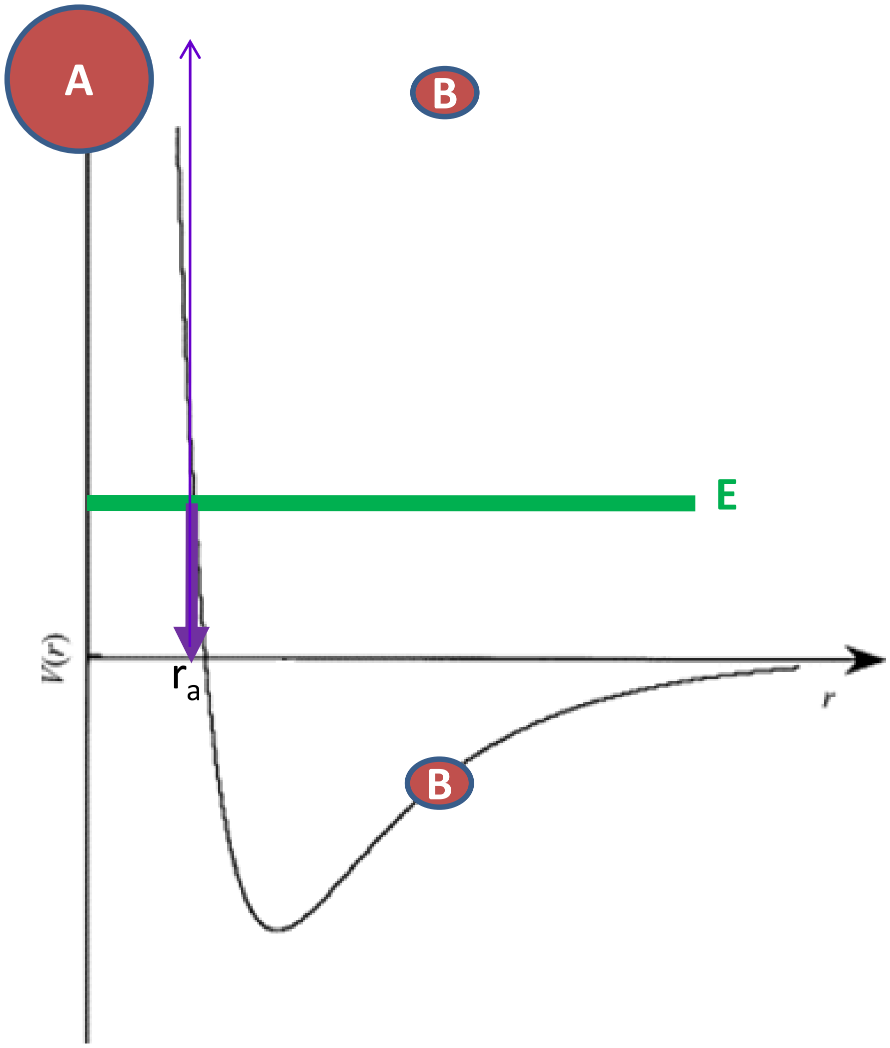
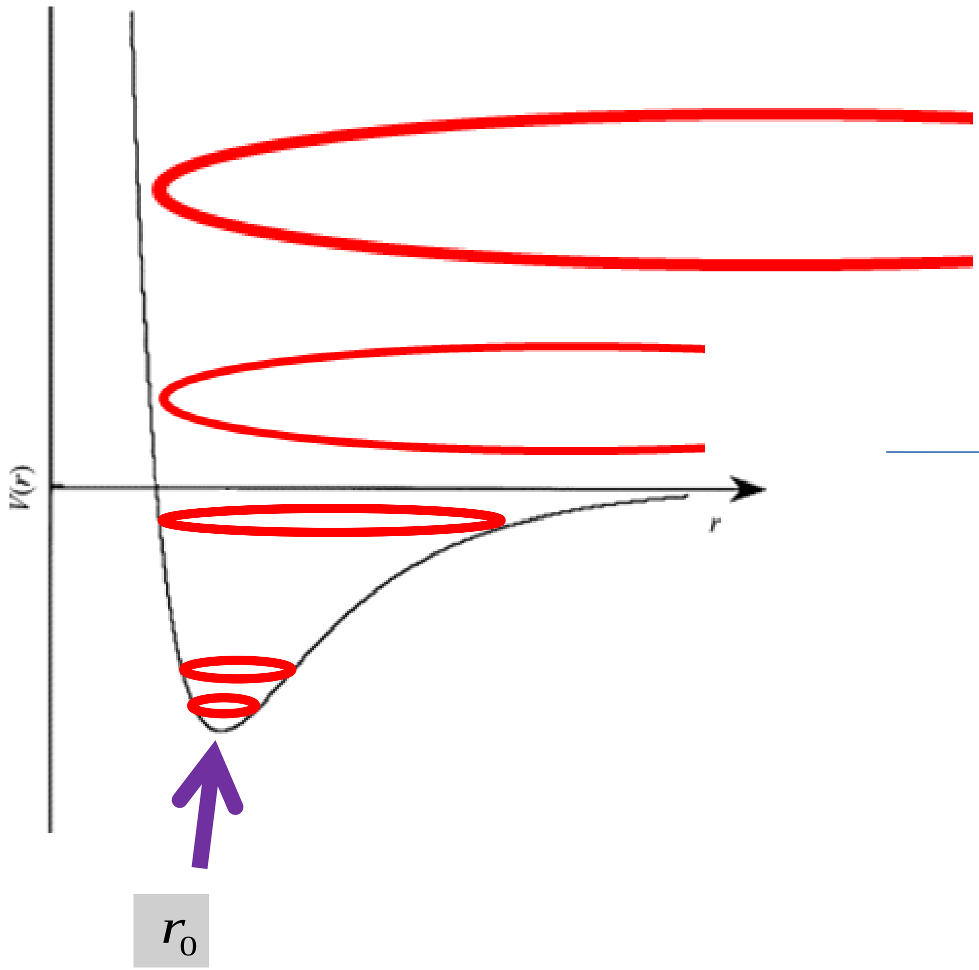
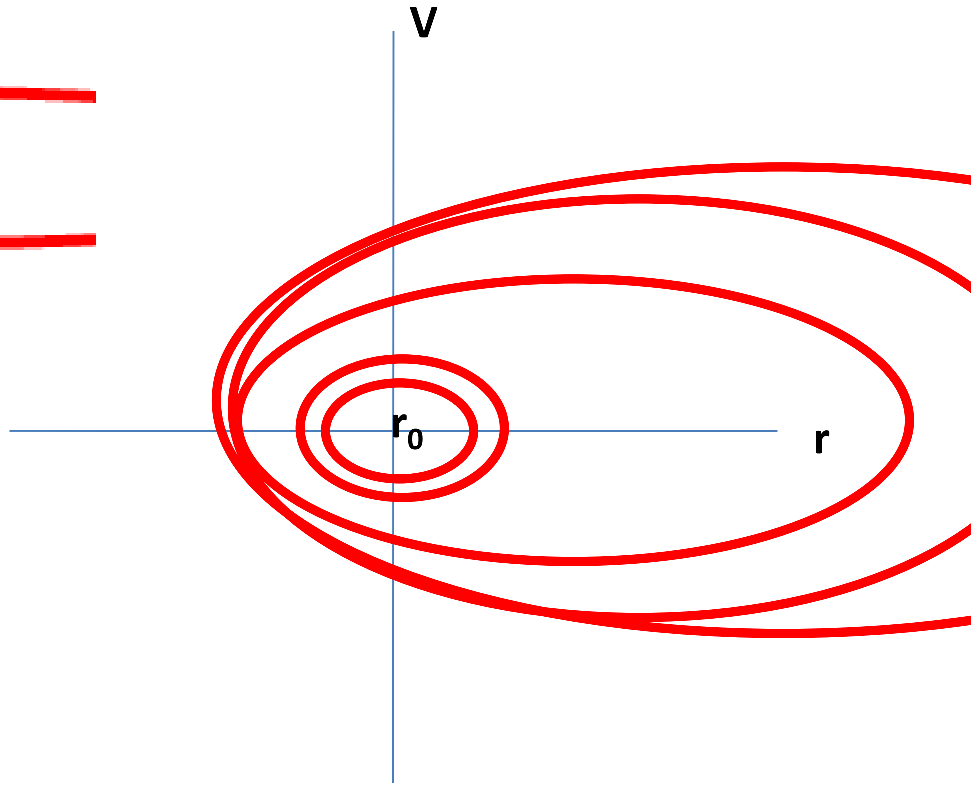
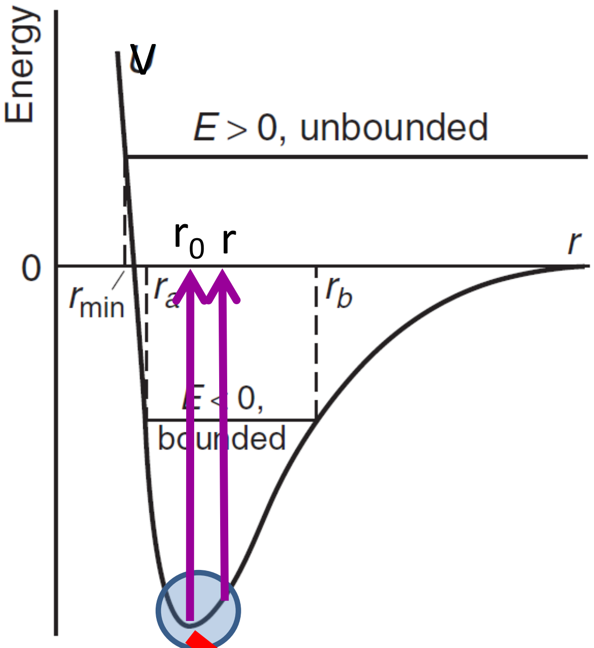


Velocity Vs. Position plot





Small Oscillations



Nearly Parabolic

$$V(r) = V(r_0) + (r - r_0)\cancel{\left.\frac{dV}{dr}\right|_{r_0}} + \frac{1}{2}(r - r_0)^2 \left.\frac{d^2V}{dr^2}\right|_{r_0} + \cdots$$

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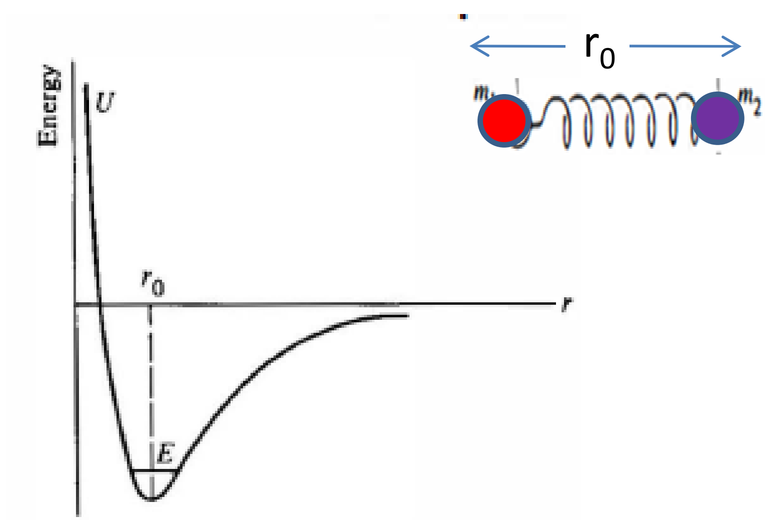
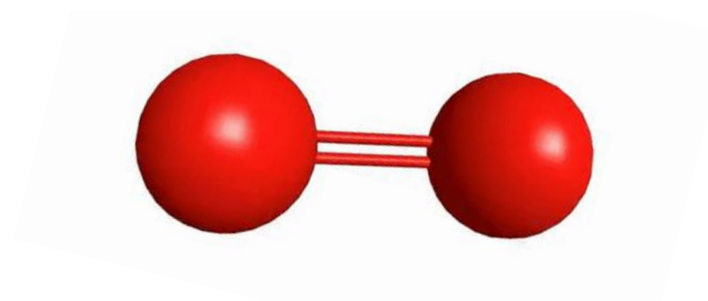
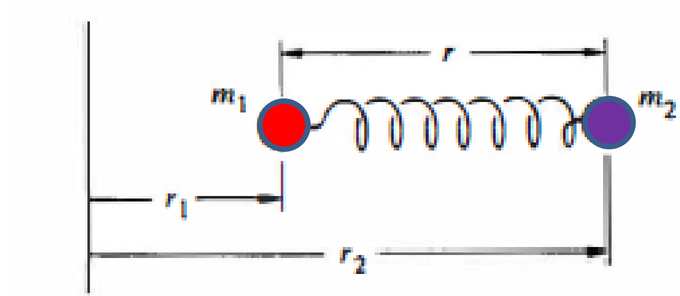
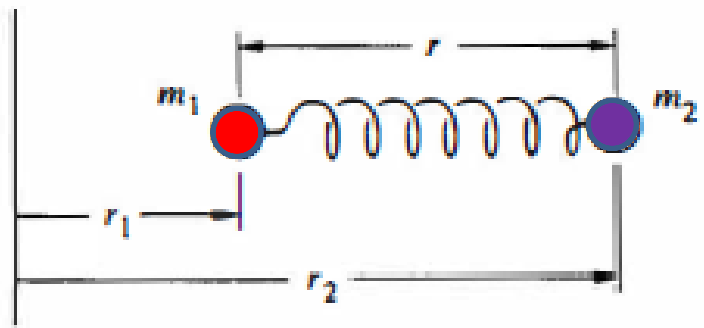






Equation of motion



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$$\ddot{r}_2 - \ddot{r}_1 = \ddot{r} = -k\left(\frac{1}{m_1} + \frac{1}{m_2}\right)(r - r_0)$$

Equation of motion

$$\ddot{r}_2 - \ddot{r}_1 = \ddot{r} = -k\left(\frac{1}{m_1} + \frac{1}{m_2}\right)(r - r_0)$$

or

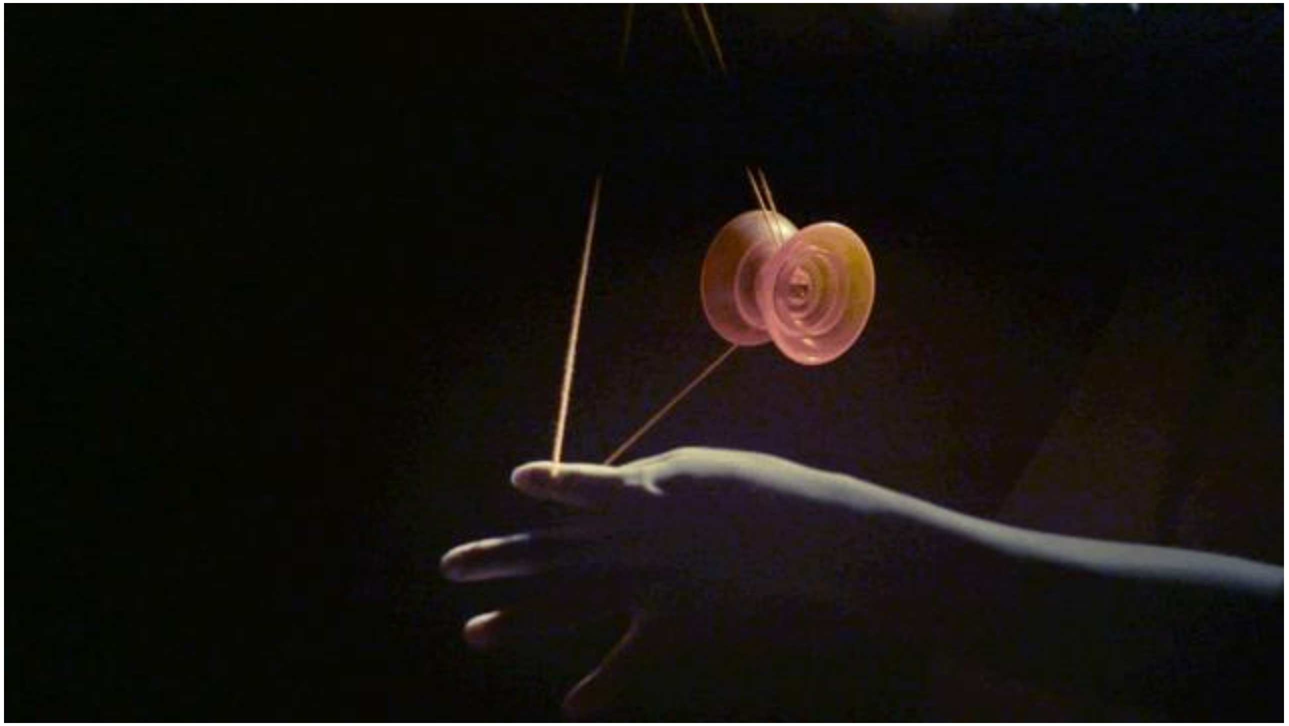
$$\ddot{r} = -\frac{k}{\mu}(r - r_0),$$

$$\mu = m_1 m_2/(m_1 + m_2)$$

By Comparing with Harmonic Oscillator, we get the frequency of vibrations as

$$\omega = \sqrt{\frac{k}{\mu}} = \sqrt{\left.\frac{d^2U}{dr^2}\right|_{r_0} \frac{1}{\mu}}.$$

RIGID BODY IN MOTION



Rotation Vs Translation

Translation

Mass

Linear momentum

Force

Rotation

Moment of Inertia

Angular momentum

Torque

Dynamics of Rotation [Recap]

$$\vec{\tau} = I\vec{\alpha}$$

$$\vec{F} = m\vec{a}$$

$\vec{\tau}$ = External Torque

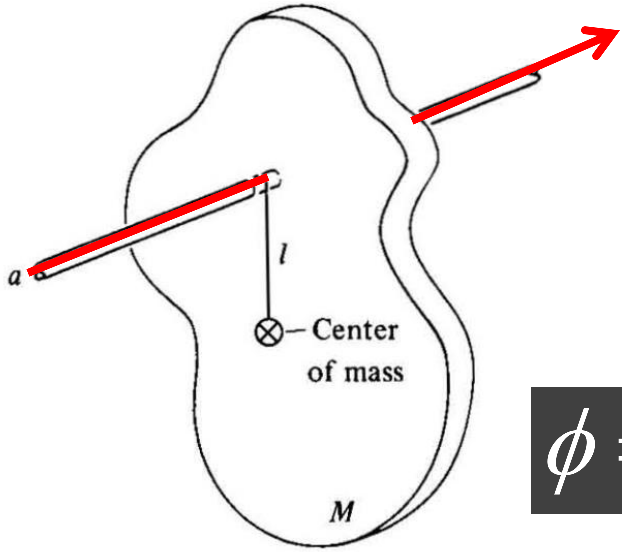
I=Moment of Inertia

α=Angular acceleration

Rotational Kinetic Energy

$$KE = \frac{1}{2}I\omega^2$$

Rigid body Pendulum [Recap]

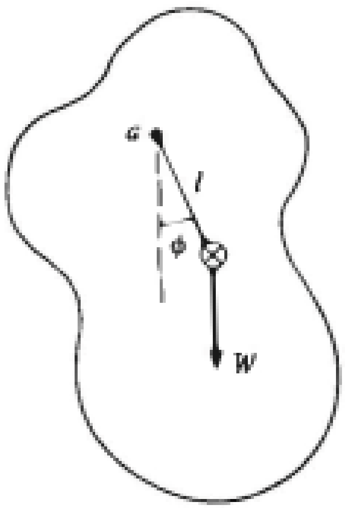


$$-lW\sin\varphi = I_a\ddot{\varphi}$$

$$I_a\ddot{\varphi} + lMg\varphi = 0$$

$$\omega = \sqrt{\frac{Mlg}{I_a}}$$

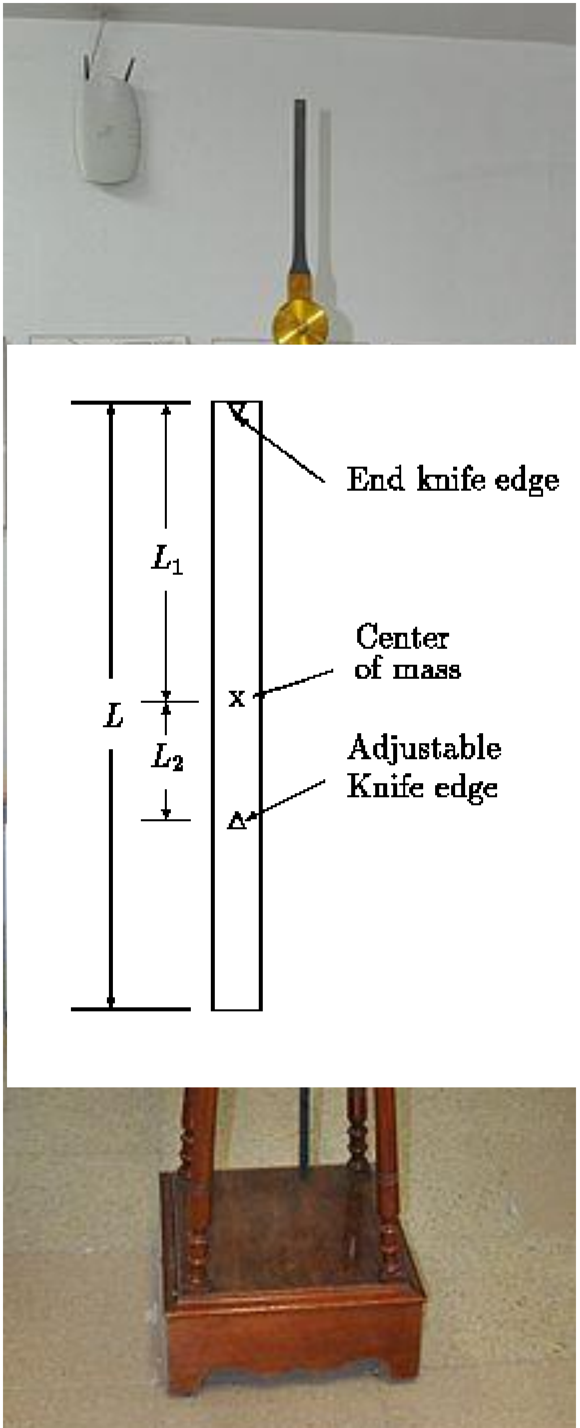
$$\phi = A\sin\omega t + B\cos\omega t$$



Radius of gyration $k = \sqrt{\frac{I_0}{M}}$, where I_0 is the moment of inertia about its center of mass.

Parallel axis theorem $I_a = I_0 + Ml^2$

$$\omega = \sqrt{\frac{gl}{k^2 + l^2}}$$



Kater’s pendulum

$$\omega = \sqrt{\frac{gl}{k^2 + l^2}}$$

$$T_A = 2\pi\sqrt{\frac{k^2 + {L_1}^2}{gL_1}}$$

$$T_B = 2\pi\sqrt{\frac{k^2 + {L_2}^2}{gL_2}}$$

The periods are made identical by adjusting L_1 and L_2

$$k^2 = L_1 L_2$$

$$g = 4\pi^2\sqrt{\frac{L_1 + L_2}{T^2}}$$

The only geometrical quantity needed is the distance between the knife edges.

The position of the center of mass need not be known.

Work Energy Theorem

$$m\int_{x_a}^{x_b} \frac{dv}{dt}\,dx = \int_{x_a}^{x_b} F(x)\,dx$$

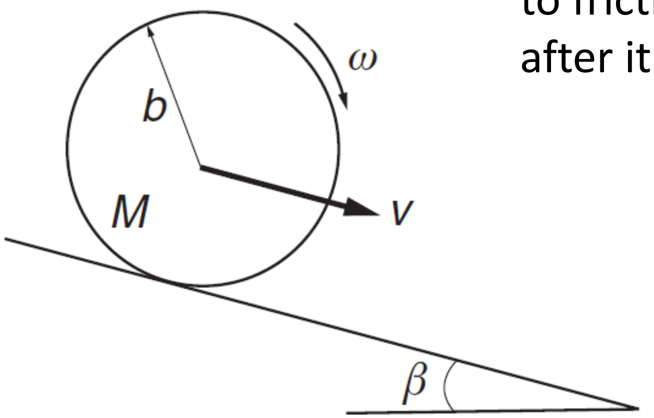
$$\frac{1}{2}m{v_b}^2 - \frac{1}{2}m{v_a}^2 = \int_{x_a}^{x_b} F(x)\,dx$$

Work Energy Theorem for rigid body motion

$$\frac{1}{2}I{\omega_b}^2 - \frac{1}{2}I{\omega_a}^2 = \int_{\theta_a}^{\theta_b} \tau\,d\theta$$

$$W_{ba} = K_b - K_a$$

Drum rolling down a plane (Rotation and translation)



If the drum starts from rest and rolls without slipping (due to frictional force **f**), find the speed of its center of mass, V, after it has descended a height h.

$$\int_a^b \vec{F}\cdot d\vec{r} = \frac{1}{2}MV_b^2 - \frac{1}{2}MV_a^2$$

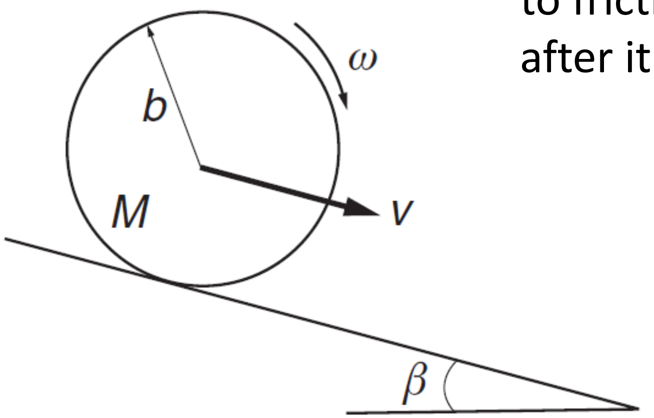
Translation

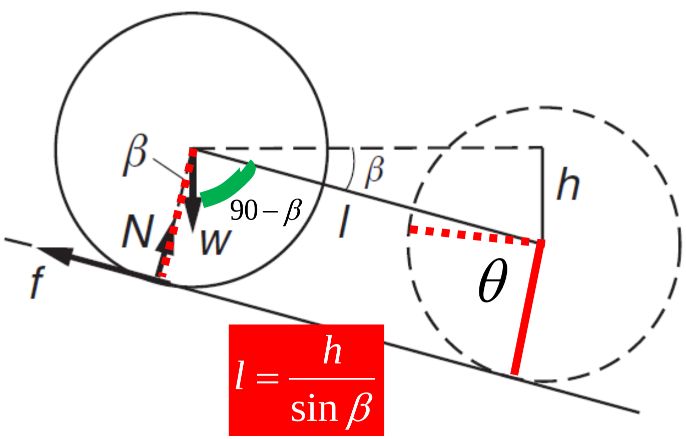
$90-\beta$

$$\left(W\sin\beta - f\right)l = \frac{1}{2}MV^2$$

Drum rolling down a plane (Rotation and translation)

If the drum starts from rest and rolls without slipping (due to frictional force **f**), find the speed of its center of mass, V, after it has descended a height h.





$$l = \frac{h}{\sin\beta}$$

Rotation

$$\frac{1}{2}I\omega_b{}^2 - \frac{1}{2}I\omega_a{}^2 = \int_0^\theta \tau\, d\theta$$

$$\tau = bf$$

$$fb\theta = \frac{1}{2}I\omega^2$$

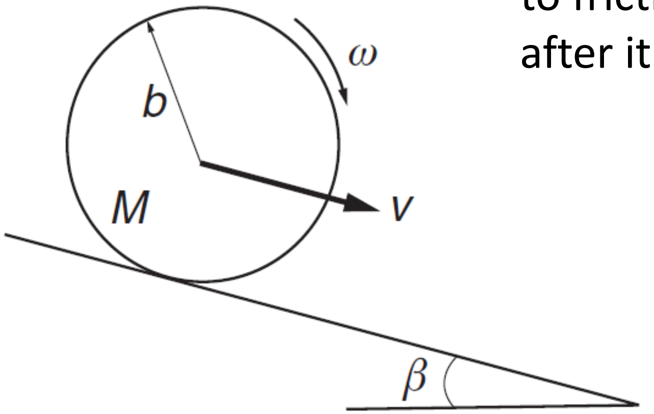
For rolling without slipping bθ=*l*

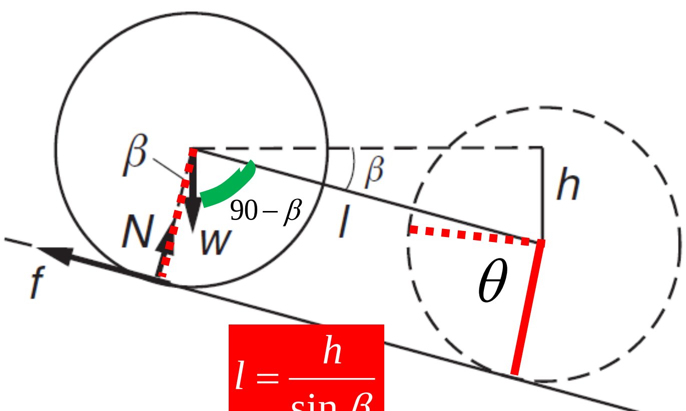
$$fl = \frac{1}{2}I\frac{V^2}{b^2}$$

$$\omega = V/b$$

Drum rolling down a plane (Rotation and translation)

If the drum starts from rest and rolls without slipping (due to frictional force **f**), find the speed of its center of mass, V, after it has descended a height h.





$$l = \frac{h}{\sin\beta}$$

Rotation and translation

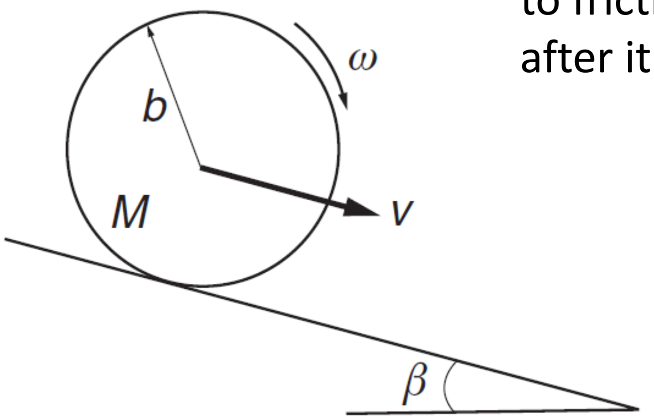
$$\left(W\sin\beta - f\right)l = \frac{1}{2}MV^2$$

$$fl = \frac{1}{2}I\frac{V^2}{b^2}$$

Substituting for $\sin\beta$ and fl

$$V = \sqrt{\frac{4gh}{3}}$$

Drum rolling down a plane (Rotation and translation)

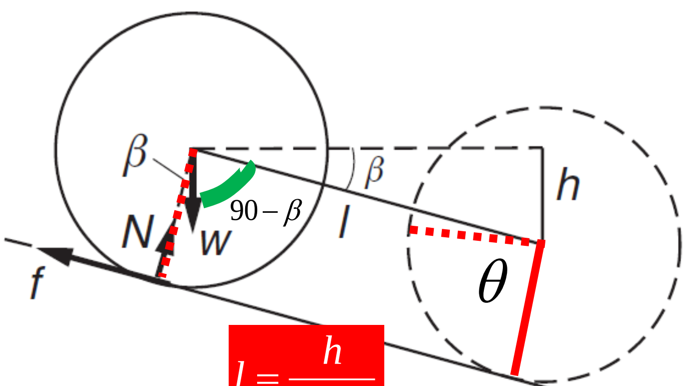


If the drum starts from rest and rolls without slipping (due to frictional force **f**), find the speed of its center of mass, V, after it has descended a height h.

Rotation and translation

$$\left(W \sin\beta - f\right) l = \frac{1}{2} M V^2$$

$$fl = \frac{1}{2} I \omega^2$$



$$l = \frac{h}{\sin\beta}$$

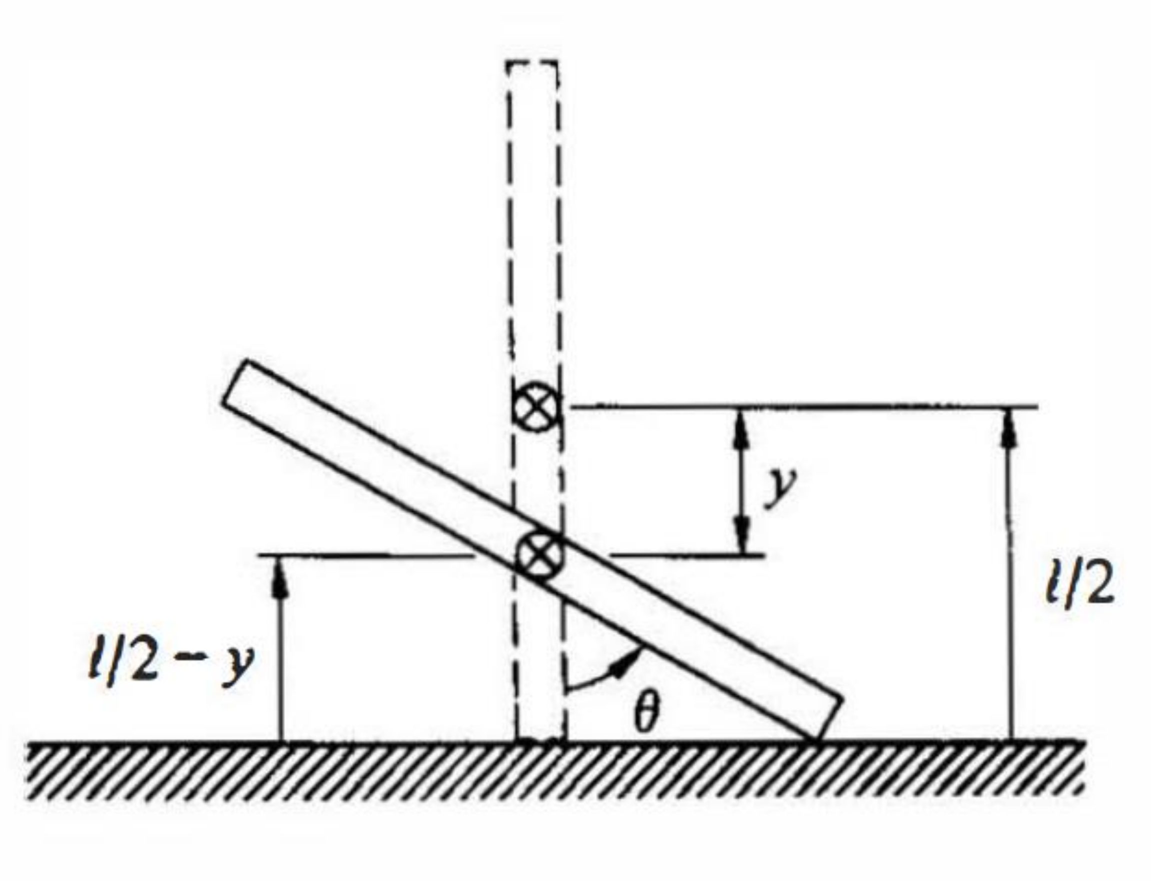
Frictional force is not dissipative!

Friction decreases the translational energy by an amount ‘fl’ and increases the rotational energy by the same amount.

Friction transforms mechanical energy from one mode to another.

The falling stick (Rotation and translation)

A stick of length l and mass M, initially upright on a frictionless table, starts falling. The problem is to find the speed of the center of mass as a function of position. Assume that center of mass fall straight down.



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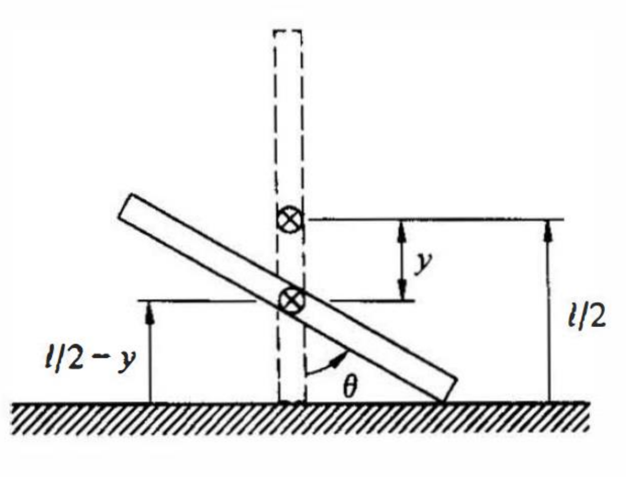


$$E = \frac{Mgl}{2}$$

$$E = \frac{1}{2} I \dot{\theta}^2 + \frac{1}{2} M \dot{y}^2 + Mg\left(\frac{l}{2} - y\right)$$

$$y = \frac{l}{2}(1 - \cos\theta)$$

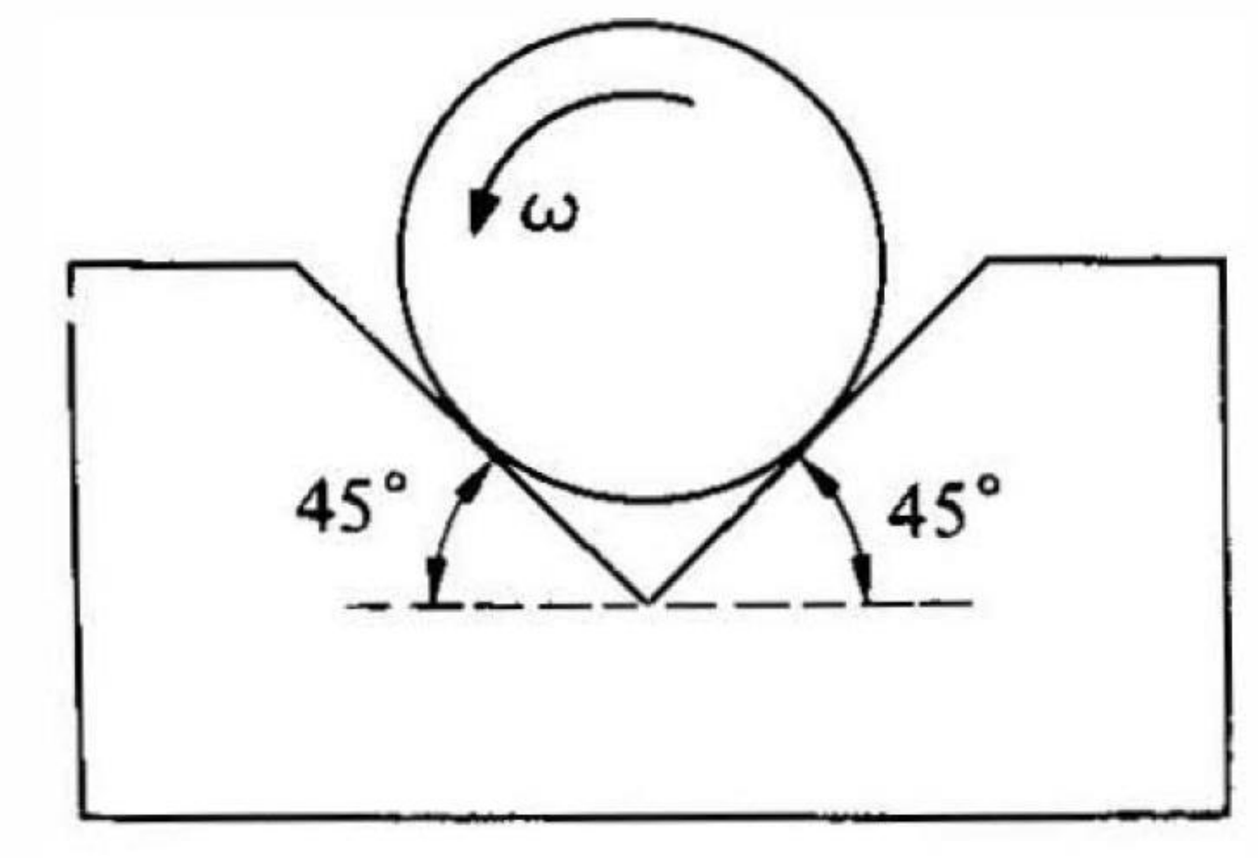
$$\dot{y} = \frac{l}{2}(\sin\theta)\dot{\theta}$$

$$\dot{\theta} = \frac{2}{l\sin\theta}\dot{y}$$

$$\dot{y} = \left[\frac{6gy\sin^2\theta}{3\sin^2\theta + 1}\right]^{1/2}$$

The cylinder in a groove (Rotation)

A cylinder of mass M and radius R is rotated in a uniform V groove with constant angular velocity ω. The coefficient of friction between the cylinder and each surface is μ. What torque must be applied to the cylinder to keep it rotating.



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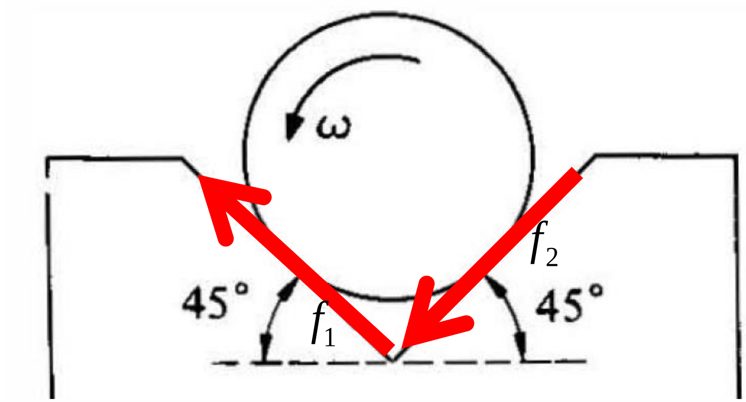
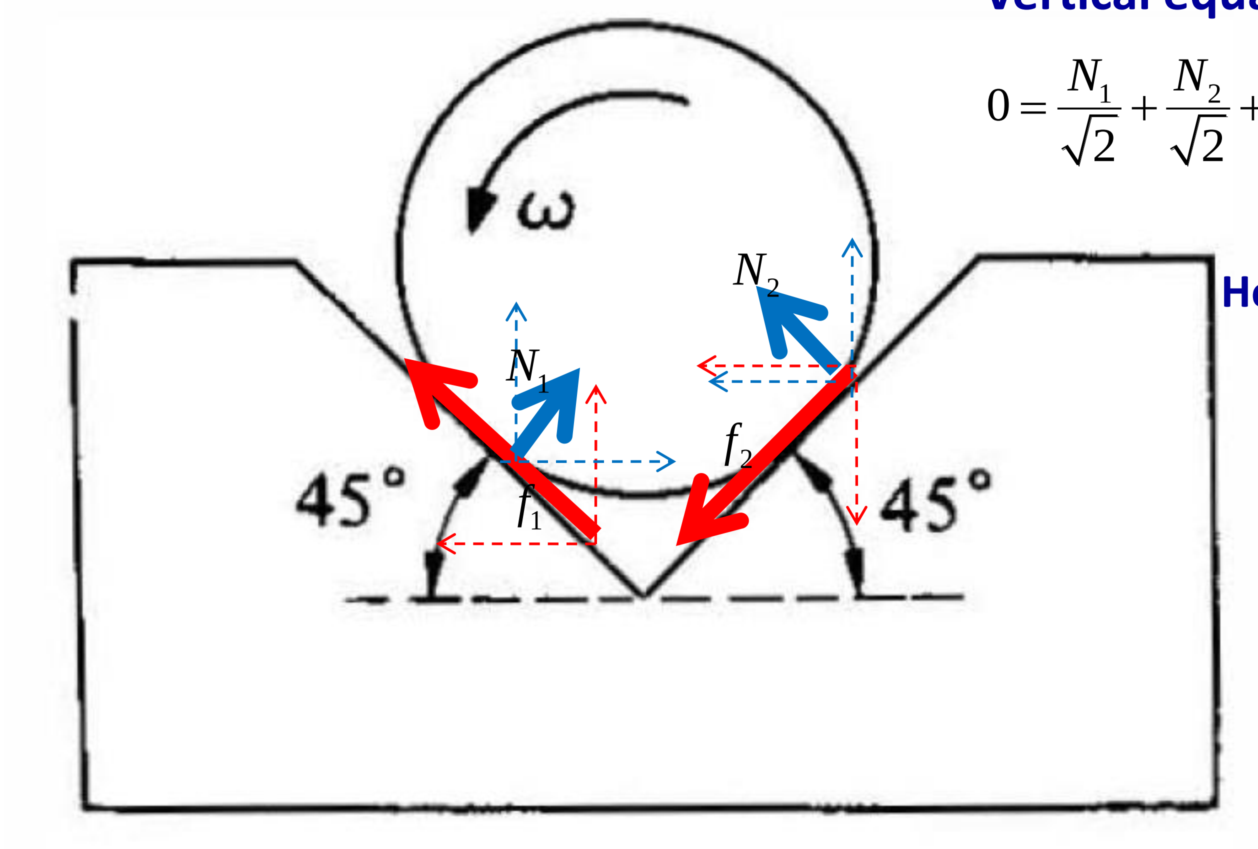




Minimum torque required= $\tau_{f1} + \tau_{f2}$

Vertical equation of motion:

$$0 = \frac{N_1}{\sqrt{2}} + \frac{N_2}{\sqrt{2}} + \frac{f_1}{\sqrt{2}} - \frac{f_2}{\sqrt{2}} - Mg$$

Horizontal equation of motion:

$$0 = \frac{N_1}{\sqrt{2}} - \frac{N_2}{\sqrt{2}} - \frac{f_1}{\sqrt{2}} - \frac{f_2}{\sqrt{2}}$$

The cylinder in a groove (Rotation)

Vertical equation of motion:

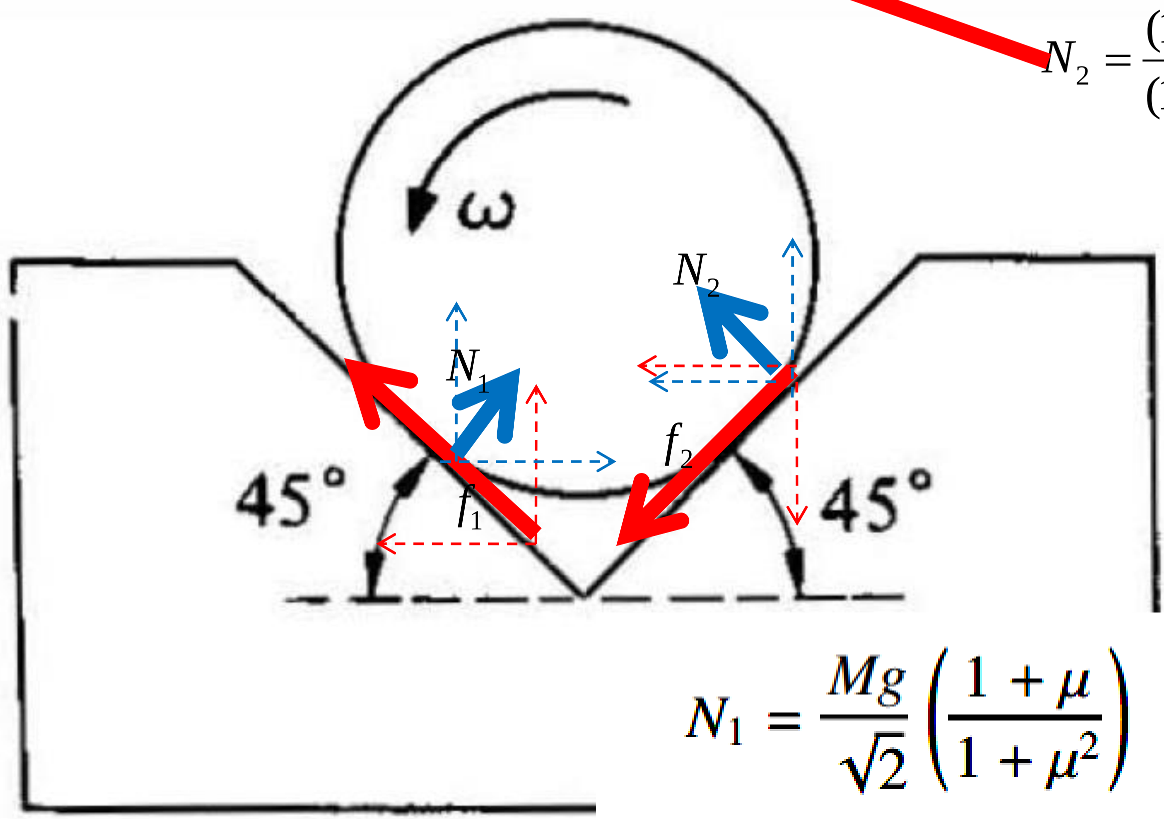
$$0 = \frac{N_1}{\sqrt{2}} + \frac{N_2}{\sqrt{2}} + \frac{f_1}{\sqrt{2}} - \frac{f_2}{\sqrt{2}} - Mg$$

Horizontal equation of motion:

$$0 = \frac{N_1}{\sqrt{2}} - \frac{N_2}{\sqrt{2}} - \frac{f_1}{\sqrt{2}} - \frac{f_2}{\sqrt{2}}$$

Using $f = \mu N$

$$N_2 = \frac{(1-\mu)}{(1+\mu)} N_1$$



$$N_1 = \frac{Mg}{\sqrt{2}}\left(\frac{1+\mu}{1+\mu^2}\right) \qquad N_2 = \frac{Mg}{\sqrt{2}}\left(\frac{1-\mu}{1+\mu^2}\right)$$

The cylinder in a groove (Rotation)

$$N_1 = \frac{Mg}{\sqrt{2}}\left(\frac{1+\mu}{1+\mu^2}\right) \qquad N_2 = \frac{Mg}{\sqrt{2}}\left(\frac{1-\mu}{1+\mu^2}\right)$$

Minimum torque required= $\tau_{f1} + \tau_{f2} = (f_1 + f_2)R$

$$= \mu(N_1 + N_2)R$$

$$\sqrt{2}Mg\left(\frac{\mu}{1+\mu^2}\right)R$$

RIGID BODY IN MOTION

Dynamics of Rotation [Recap]

$$\vec{\tau} = I\vec{\alpha}$$

$$\vec{F} = m\vec{a}$$

$\vec{\tau}$ = External Torque

I=Moment of Inertia

α=Angular acceleration

Work Energy Theorem for Rotational motion

$$\frac{1}{2}I\omega_b{}^2 - \frac{1}{2}I\omega_a{}^2 = \int_{\theta_a}^{\theta_b} \tau d\theta$$

$$W_{ba} = K_b - K_a$$

Rotational Kinetic Energy

$$KE = \frac{1}{2}I\omega^2$$

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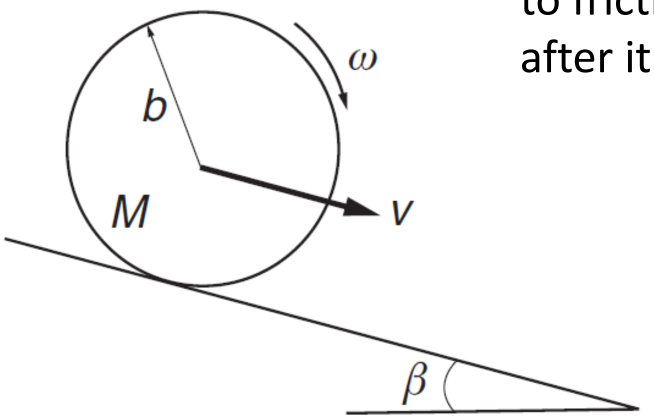
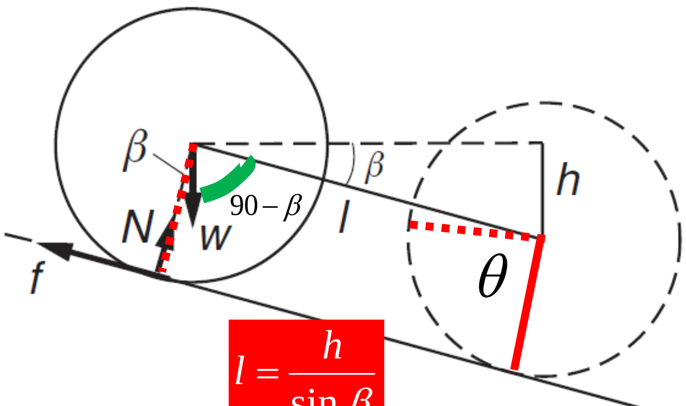


$$V = \sqrt{\frac{4gh}{3}}$$

$$\left(W \sin\beta - f\right) l = \frac{1}{2} M V^2$$



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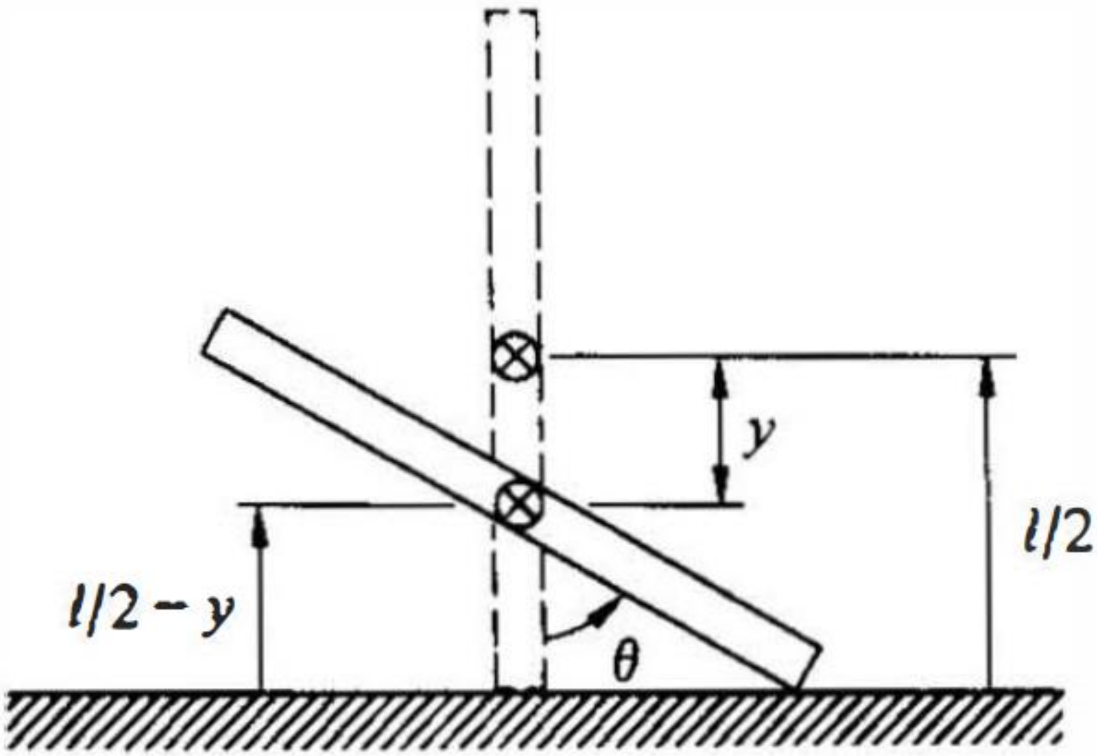
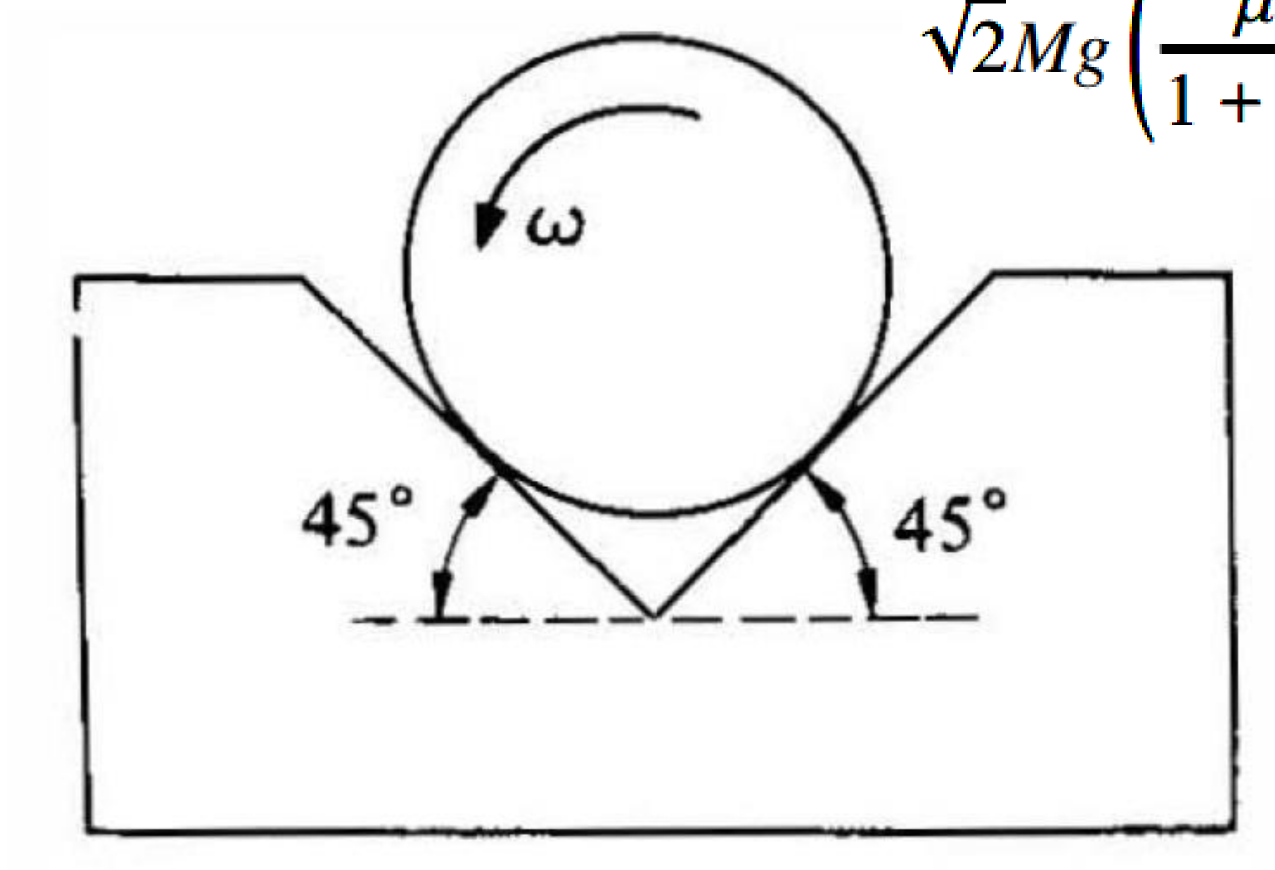


$$\dot{y} = \left[\frac{6gy\sin^2\theta}{3\sin^2\theta + 1}\right]^{1/2}$$

The cylinder in a groove (Rotation)

A cylinder of mass M and radius R is rotated in a uniform V groove with constant angular velocity ω. The coefficient of friction between the cylinder and each surface is μ. What torque must be applied to the cylinder to keep it rotating.

$$\sqrt{2}Mg\left(\frac{\mu}{1+\mu^2}\right)R$$



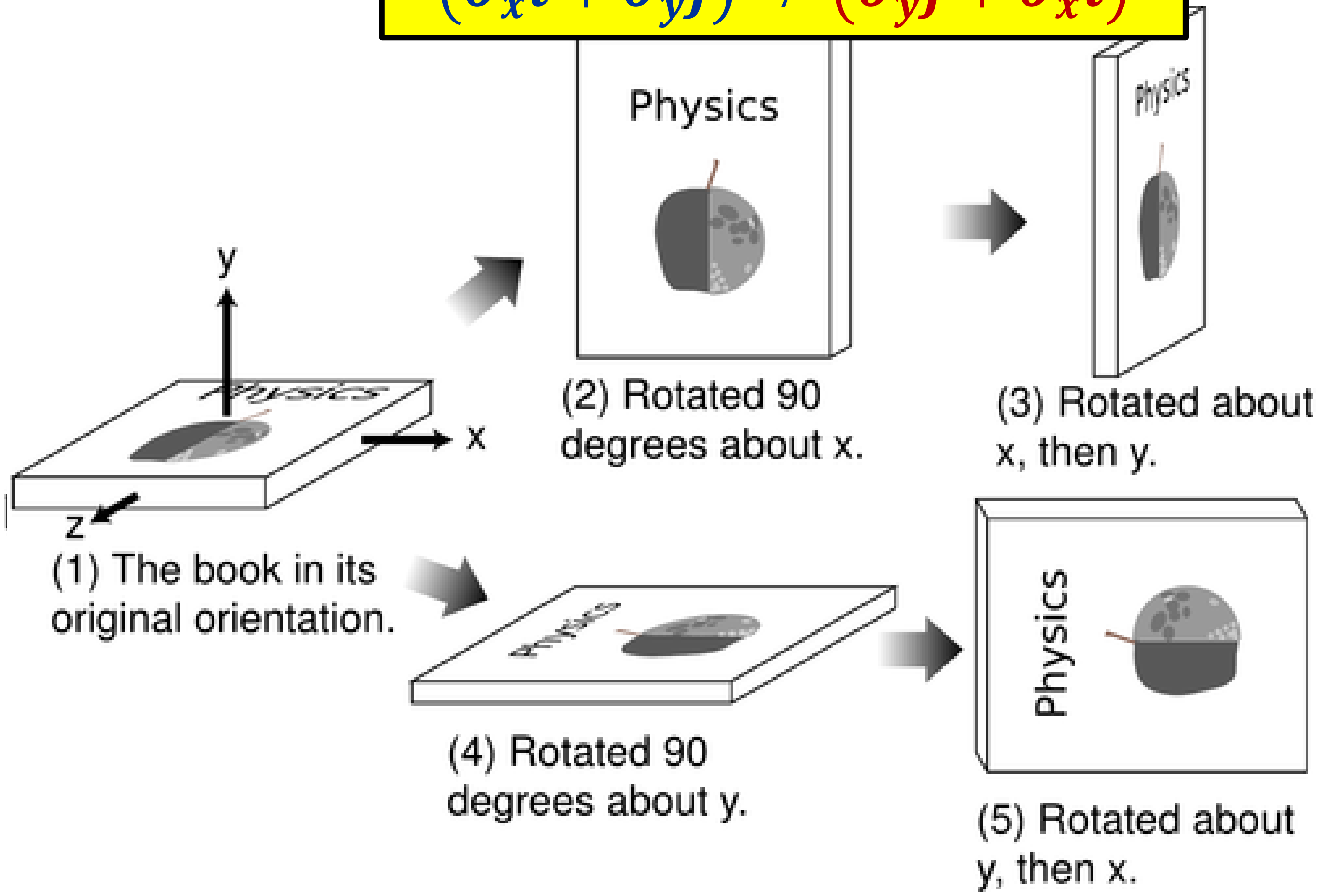
Vector nature **of angular velocity** and **angular momentum**

Can we specify the angular orientation of the body by a vector?

$$\vec{\boldsymbol{\theta}} = \left(\boldsymbol{\theta}_x \hat{\boldsymbol{\imath}} + \boldsymbol{\theta}_y \hat{\boldsymbol{\jmath}} + \boldsymbol{\theta}_z \widehat{\boldsymbol{k}}\right)?$$

Vector nature **of angular velocity** and **angular momentum**

$$\left(\boldsymbol{\theta_x}\hat{\boldsymbol{\imath}} + \boldsymbol{\theta_y}\hat{\boldsymbol{\jmath}}\right) \neq \left(\boldsymbol{\theta_y}\hat{\boldsymbol{\jmath}} + \boldsymbol{\theta_x}\hat{\boldsymbol{\imath}}\right)$$



$$\left(\boldsymbol{\theta_x}\hat{\boldsymbol{\imath}} + \boldsymbol{\theta_y}\hat{\boldsymbol{\jmath}}\right)$$

$$\left(\boldsymbol{\theta_y}\hat{\boldsymbol{\jmath}} + \boldsymbol{\theta_x}\hat{\boldsymbol{\imath}}\right)$$

Vector nature **of angular velocity** and **angular momentum**

$$\boxed{\left(\boldsymbol{\theta}_x\hat{\boldsymbol{\imath}} + \boldsymbol{\theta}_y\hat{\boldsymbol{\jmath}}\right) \neq \left(\boldsymbol{\theta}_y\hat{\boldsymbol{\jmath}} + \boldsymbol{\theta}_x\hat{\boldsymbol{\imath}}\right)}$$

Can we specify the angular orientation of the body by a vector?

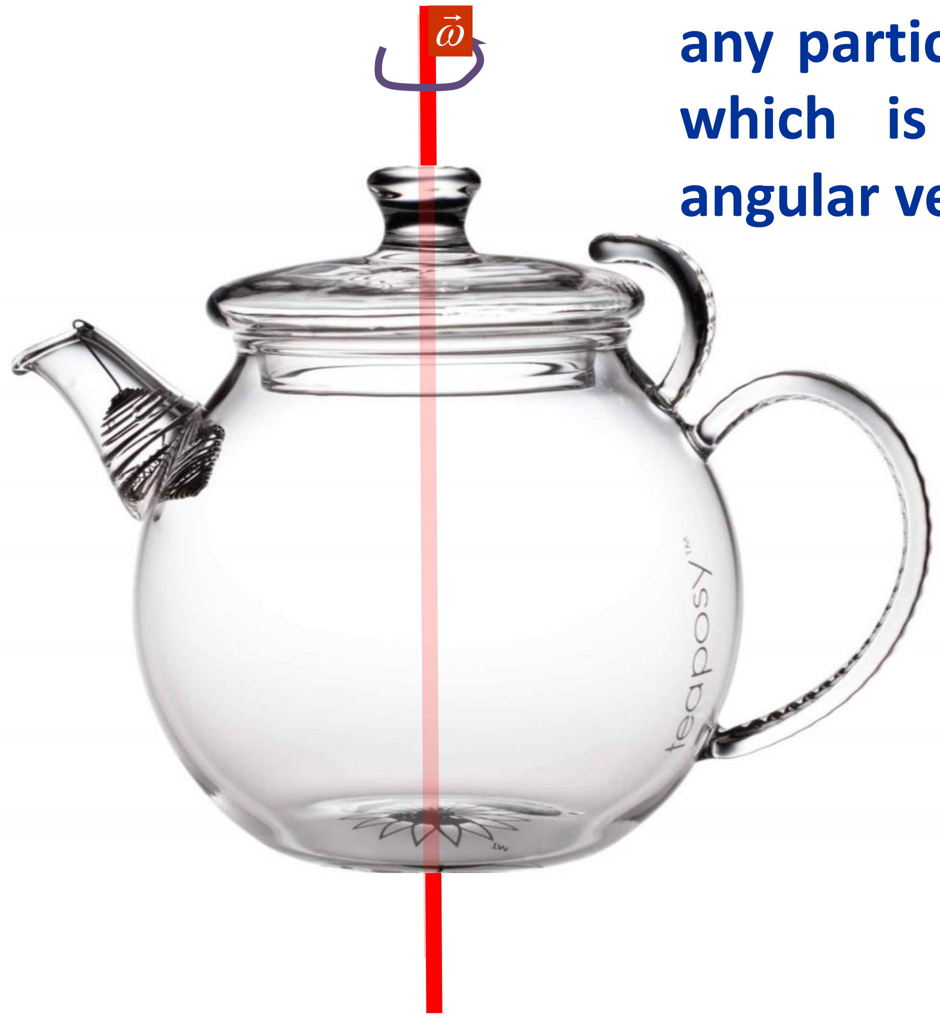
$$\vec{\boldsymbol{\theta}} = \left(\boldsymbol{\theta}_x\hat{\boldsymbol{\imath}} + \boldsymbol{\theta}_y\hat{\boldsymbol{\jmath}} + \boldsymbol{\theta}_z\widehat{\boldsymbol{k}}\right)?$$

Answer is NO

Can we specify the angular velocity of the body by a vector? Yes.

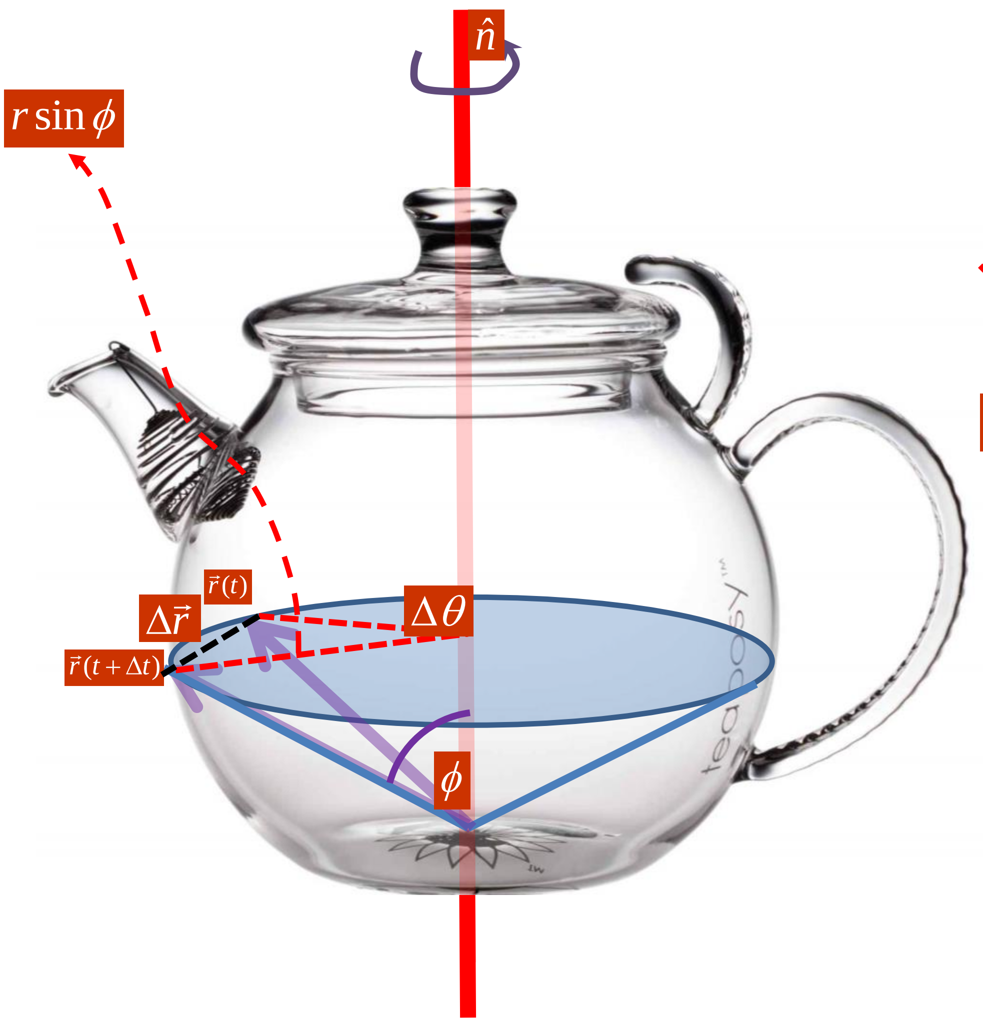
$$\vec{\boldsymbol{\omega}} = \left(\boldsymbol{\omega}_x\hat{\boldsymbol{\imath}} + \boldsymbol{\omega}_y\hat{\boldsymbol{\jmath}} + \boldsymbol{\omega}_z\widehat{\boldsymbol{k}}\right)$$

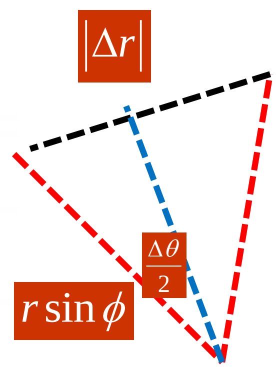
Vector nature **of angular velocity** and **angular momentum**



What is the velocity of any particle in tea kittle, which is rotating with angular velocity ω?

Vector nature **of angular velocity** and **angular momentum**





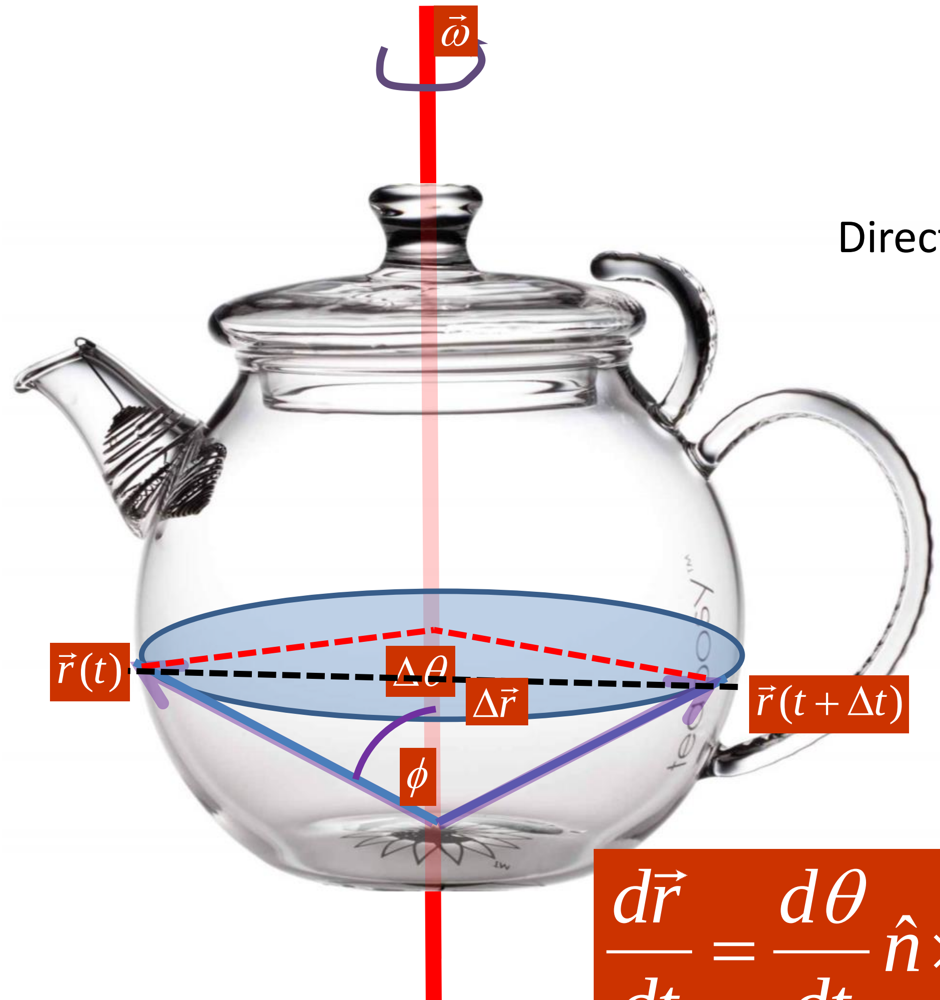
$$|\Delta r| = 2r\sin\phi\sin\frac{\Delta\theta}{2}$$

For infinitesimal rotation,

$$|\Delta r| = r\sin\phi\,\Delta\theta$$

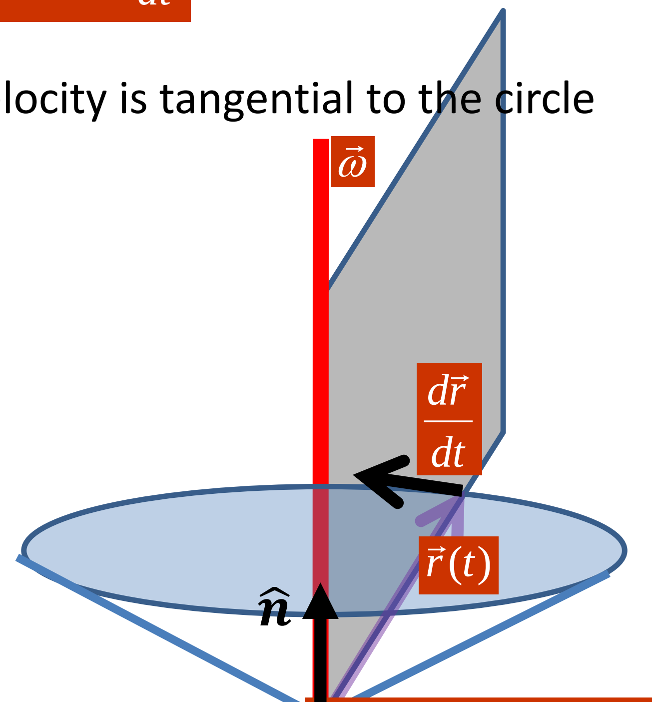
$$\left|\frac{dr}{dt}\right| = r\sin\phi\,\frac{d\theta}{dt}$$

Vector nature **of angular velocity** and **angular momentum**



$$\left|\frac{dr}{dt}\right| = r\sin\phi\ \frac{d\theta}{dt}$$

Direction of velocity is tangential to the circle



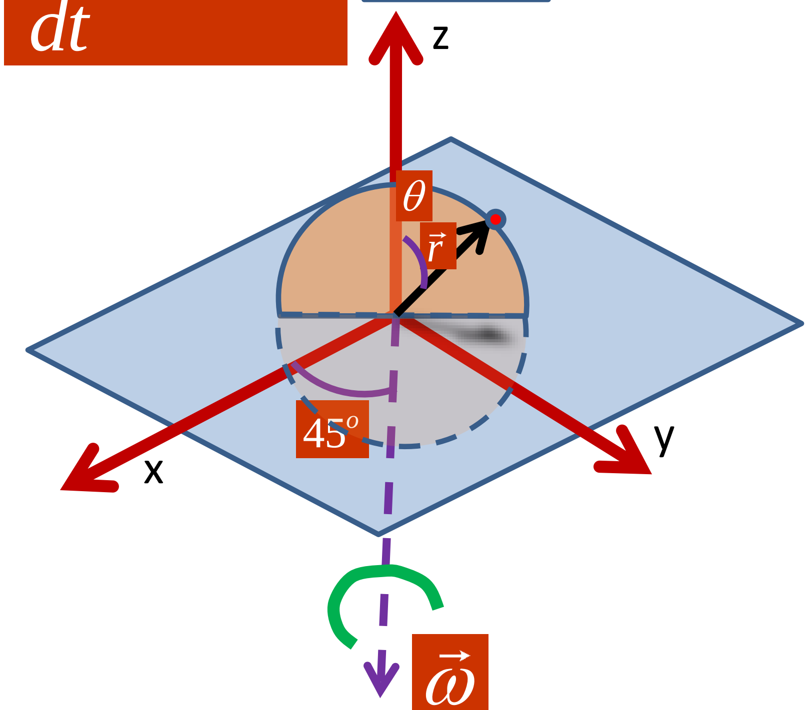
$$\frac{d\vec{r}}{dt} = \frac{d\theta}{dt}\hat{n}\times\vec{r} = \vec{\omega}\times\vec{r}$$

$$\frac{d\vec{A}}{dt} = \vec{\omega}\times\vec{A}$$

Find the velocity of a particle rotating in a vertical plane

$$\frac{d\vec{r}}{dt} = \vec{\omega}\times\vec{r}$$

?



$$z = r\cos\theta$$

$$x = -\frac{r\sin\theta}{\sqrt{2}}$$

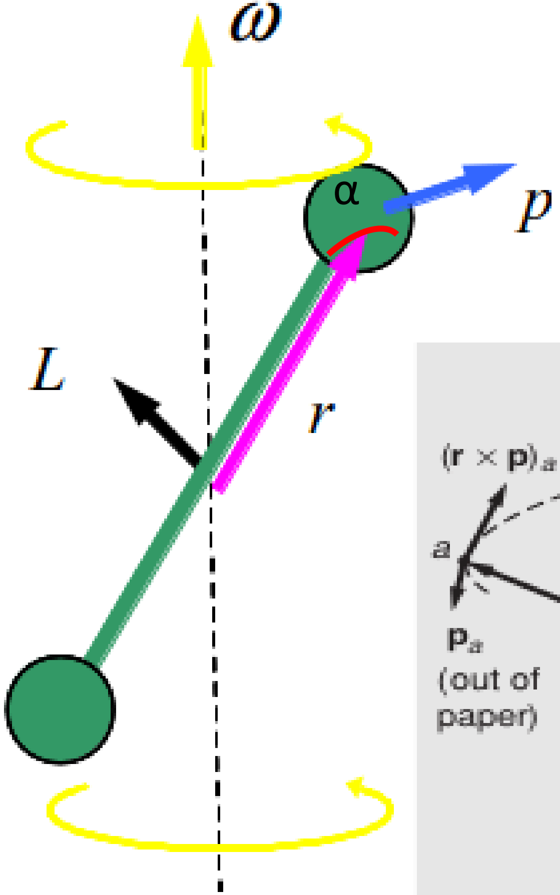
$$y = \frac{r\sin\theta}{\sqrt{2}}$$

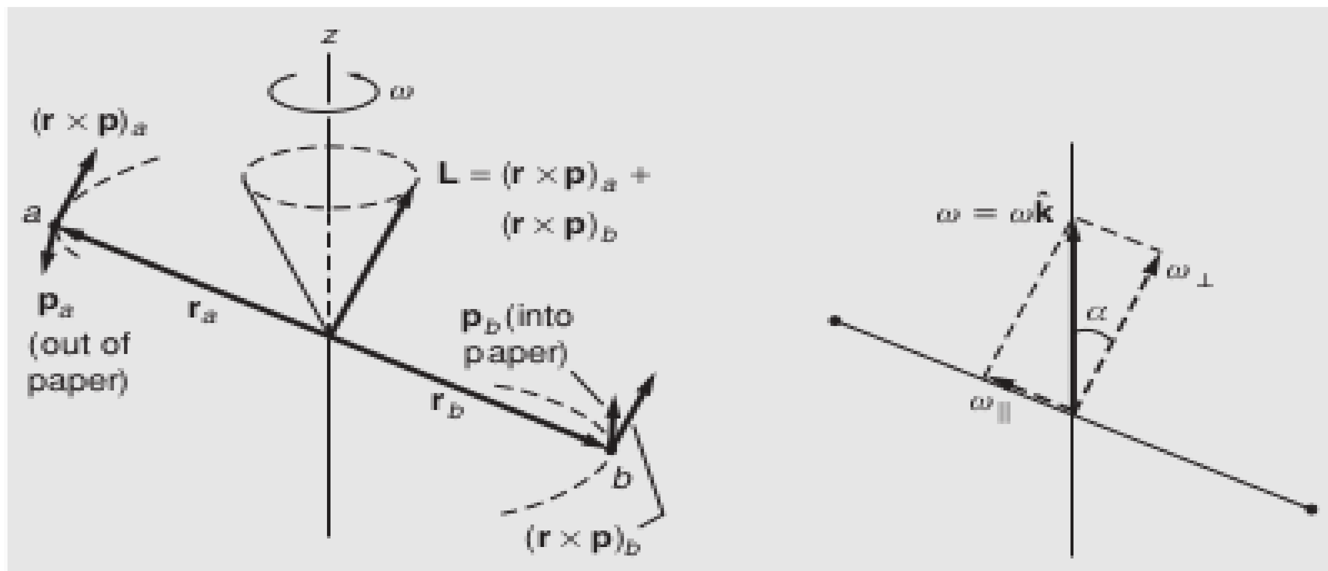
$$\vec{r} = r\left(-\frac{\sin\theta}{\sqrt{2}}\hat{e}_x + \frac{\sin\theta}{\sqrt{2}}\hat{e}_y + \cos\theta\hat{e}_z\right)$$

$$\vec{\omega} = \frac{\omega}{\sqrt{2}}\hat{e}_x + \frac{\omega}{\sqrt{2}}\hat{e}_y$$

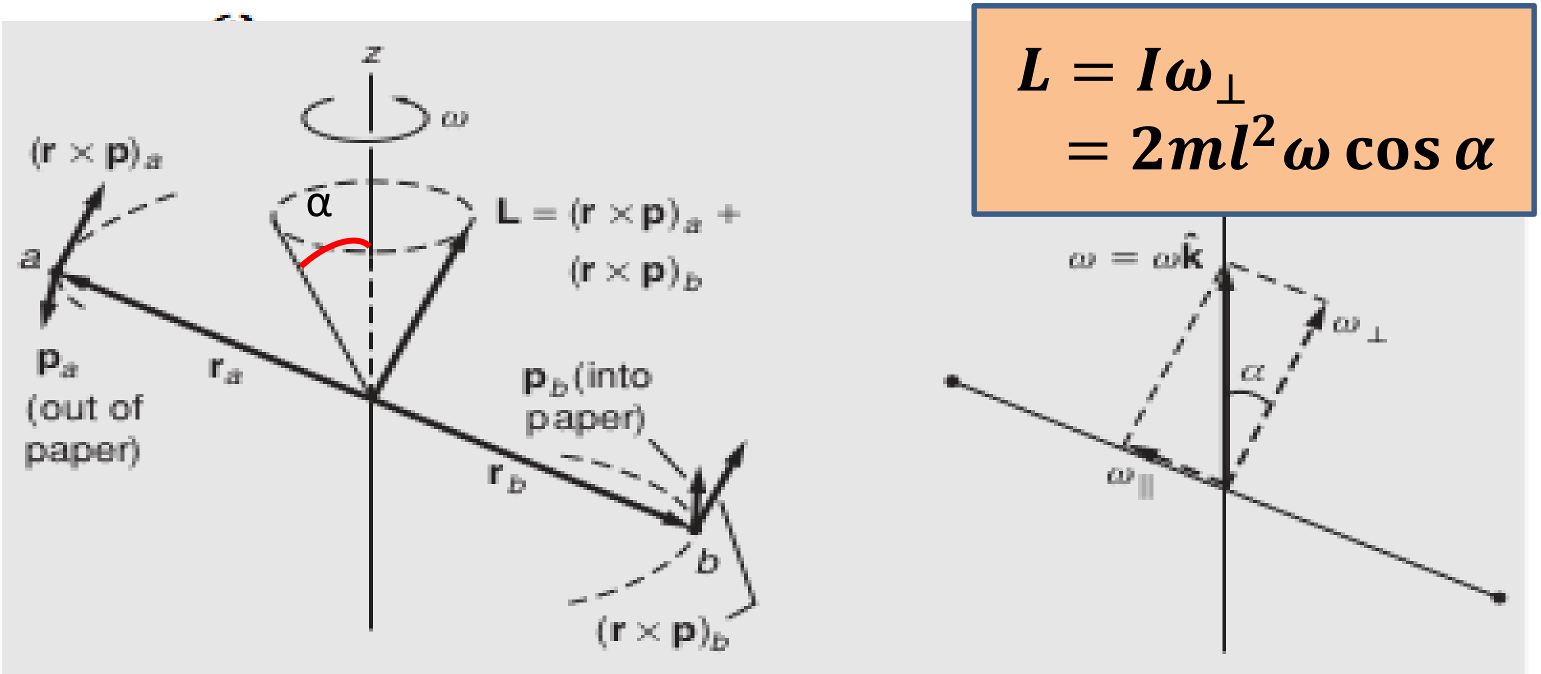
$$\frac{d\vec{r}}{dt} = \vec{\omega}\times\vec{r} = \omega r\left(-\frac{\cos\theta}{\sqrt{2}}\hat{e}_x + \frac{\cos\theta}{\sqrt{2}}\hat{e}_y - \sin\theta\hat{e}_z\right)$$

Angular Momentum of a Rotating Skew Rod





Angular Momentum of a Rotating Skew Rod

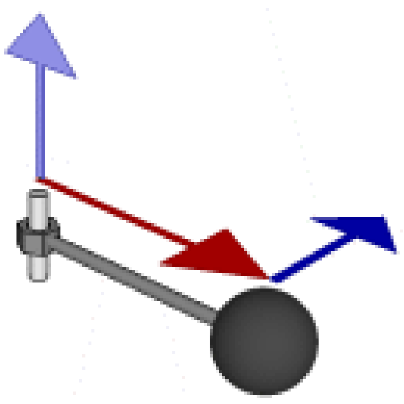


$$L = I\omega_\perp$$
$$= 2ml^2\omega\cos\alpha$$

$$\vec{L} = \sum(\vec{r}_i \times \vec{p}_i)$$

$$p = ml\omega\cos\alpha$$
$$r = l$$
$$\therefore L = 2m\omega l^2 \cos\alpha$$

$\tau = \mathbf{r} \times \mathbf{F}$
$\mathbf{L} = \mathbf{r} \times \mathbf{p}$



Moment of Inertia Matrix

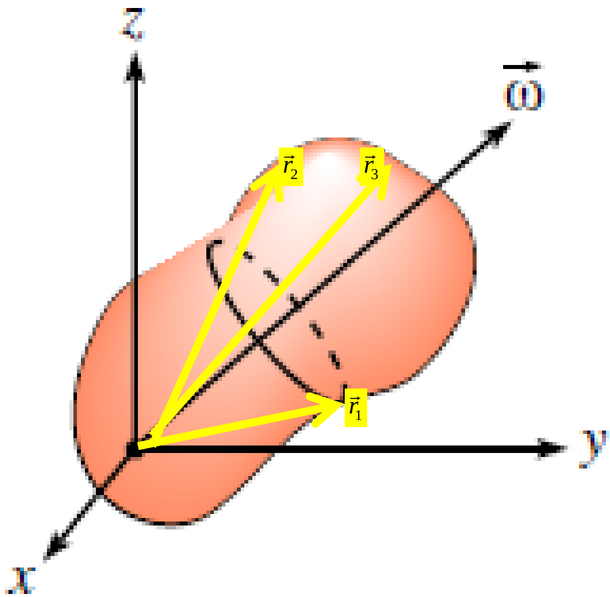
$$\vec{v} = \frac{d\vec{r}}{dt} = \vec{\omega} \times \vec{r}$$

$$\vec{L} = \vec{r} \times m\vec{v}$$

$$\vec{L} = \boxed{[I]}[\vec{\omega}]$$

$$\vec{\tau} = [I][\vec{\alpha}]$$

Angular momentum of a rigid body



$$\vec{L} = \vec{r} \times \vec{p}$$

$$\vec{p} = m\frac{d\vec{r}}{dt} = m\left(\vec{\omega} \times \vec{r}\right)$$

$$\vec{L} = \sum_{j=1}^{N}\left[\vec{r}_j \times m_j\left(\vec{\omega} \times \vec{r}_j\right)\right]$$

$$\vec{\omega} = \omega_x \hat{e}_x + \omega_y \hat{e}_y + \omega_z \hat{e}_z$$

$$\vec{\omega} \times \vec{r}_j = \begin{vmatrix} \hat{e}_x & \hat{e}_y & \hat{e}_z \\ \omega_x & \omega_y & \omega_z \\ x_j & y_j & z_j \end{vmatrix}$$

$$\vec{r}_j \times m_j\left(\vec{\omega} \times \vec{r}_j\right)?$$

Angular momentum of a rigid body

$$\vec{L} = \sum_{j=1}^{N} \left[\vec{r}_j \times m_j \left(\vec{\omega} \times \vec{r}_j \right) \right]$$

$$L_x = \sum_{j=1}^{N} \left[m_j \left(y_j^2 + z_j^2 \right) \omega_x - m_j x_j y_j \omega_y - m_j x_j z_j \omega_z \right]$$

Let us introduce moment of Inertias

$$I_{xx} = \sum m_j \left(y_j^2 + z_j^2 \right)$$

$$I_{xy} = -\sum m_j x_j y_j$$

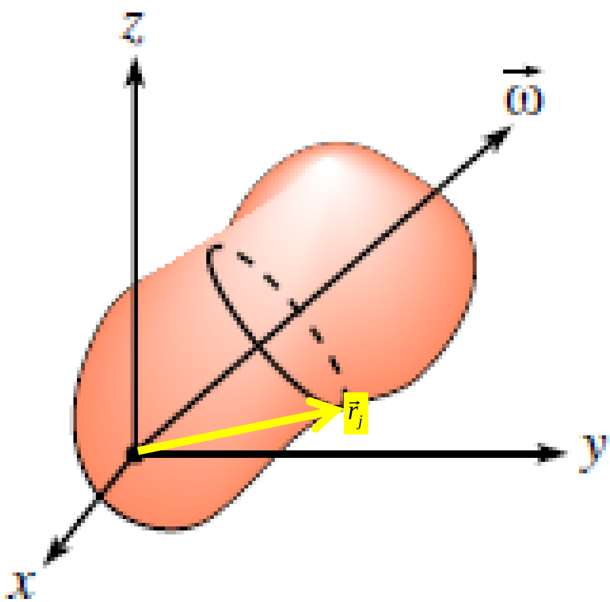
$$I_{xz} = -\sum m_j x_j z_j$$

Moment of inertia

Product of inertia

$$L_x = I_{xx} \omega_x + I_{xy} \omega_y + I_{xz} \omega_z$$

Angular momentum of a rigid body



$$L_x = I_{xx}\omega_x + I_{xy}\omega_y + I_{xz}\omega_z$$

$$L_y = I_{yx}\omega_x + I_{yy}\omega_y + I_{yz}\omega_z$$

$$L_z = I_{zx}\omega_x + I_{zy}\omega_y + I_{zz}\omega_z$$

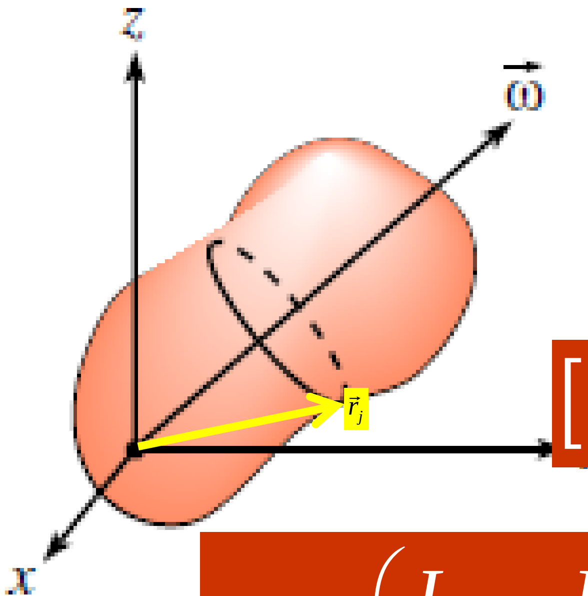
Different from what you learned!

Matrix Equation

$$\begin{pmatrix} L_x \\ L_y \\ L_z \end{pmatrix} = \begin{pmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{pmatrix} \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix}$$

$$[L] = [I][\omega]$$

Angular momentum of a rigid body



$$\begin{pmatrix} L_x \\ L_y \\ L_z \end{pmatrix} = \begin{pmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{pmatrix} \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix}$$

$$[L] = [I][\omega]$$ **Equivalent to** $$\vec{p} = m\vec{v}$$ **?**

$$[I] = \begin{pmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{pmatrix}$$

Moment of Inertia Matrix

Moment of Inertia Matrix

$$L_x = I_{xx}\omega_x + I_{xy}\omega_y + I_{xz}\omega_z$$

$$L_y = I_{yx}\omega_x + I_{yy}\omega_y + I_{yz}\omega_z$$

$$L_z = I_{zx}\omega_x + I_{zy}\omega_y + I_{zz}\omega_z$$

The fact that *I* is a matrix means that $\vec{L}$ and $\vec{\omega}$ do not necessarily point in the same direction.