Stress and Strain
Lecture 12

Engineering Mechanics - ME102

Courtesy: TMH

Stress & Strain: Axial Loading

- Suitability of a structure or machine may depend on the deformations in the structure as well as the stresses induced under loading. Statics analyses alone are not sufficient.

- Considering structures as deformable allows determination of *member forces and reactions which are statically indeterminate*.

- Determination of the stress distribution within a member also requires consideration of deformations in the member.

- Chapter 2 is concerned with deformation of a structural member under axial loading. Later chapters will deal with torsional and pure bending loads.

Normal Strain

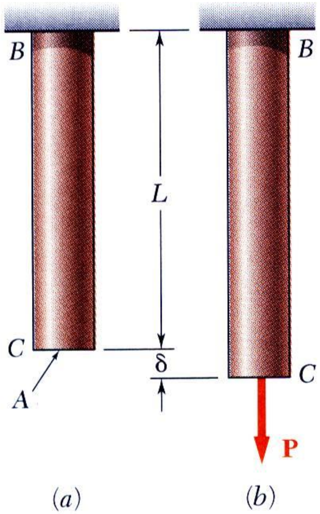


Fig. 2.1

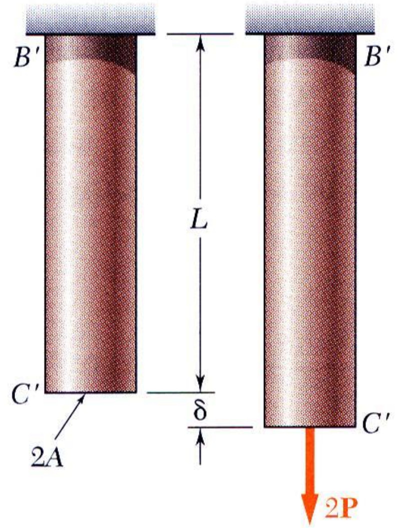


Fig. 2.3

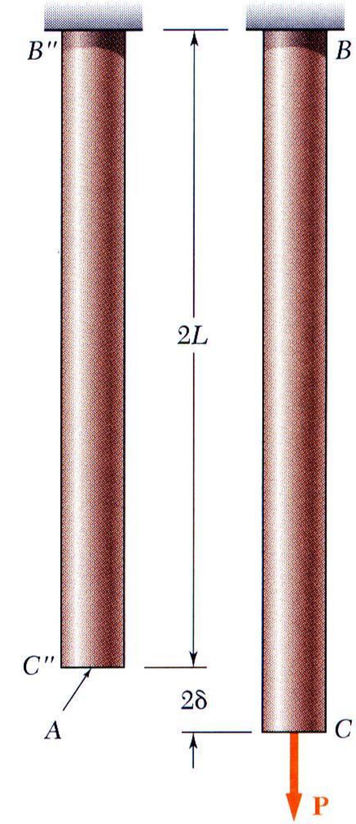


Fig. 2.4

$$\sigma = \frac{P}{A} = \text{stress}$$

$$\varepsilon = \frac{\delta}{L} = \text{normal strain}$$

$$\sigma = \frac{2P}{2A} = \frac{P}{A}$$

$$\varepsilon = \frac{\delta}{L}$$

$$\sigma = \frac{P}{A}$$

$$\varepsilon = \frac{2\delta}{2L} = \frac{\delta}{L}$$

Load-vs-displacement plot – can it be directly used for predicting deformation of another specimen of same material but different dimension?

Stress-Strain Test

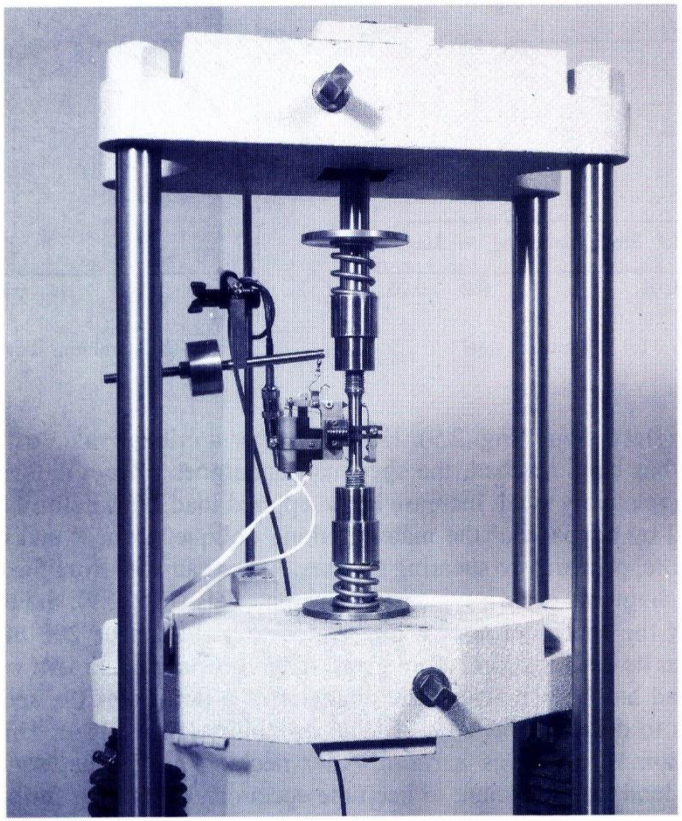


Fig. 2.7 This machine is used to test tensile test specimens, such as those shown in this chapter.

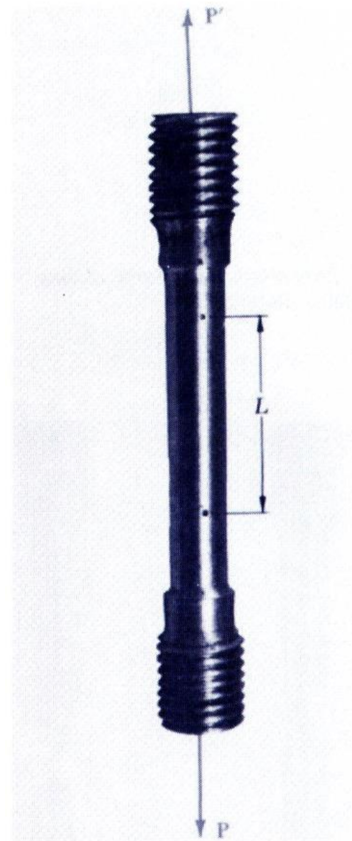
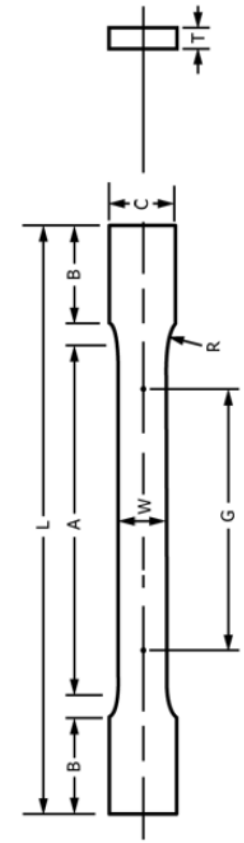


Fig. 2.8 Test specimen with tensile load.



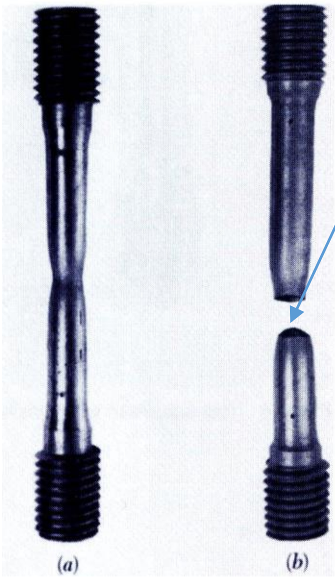
ASTM E8/E8M

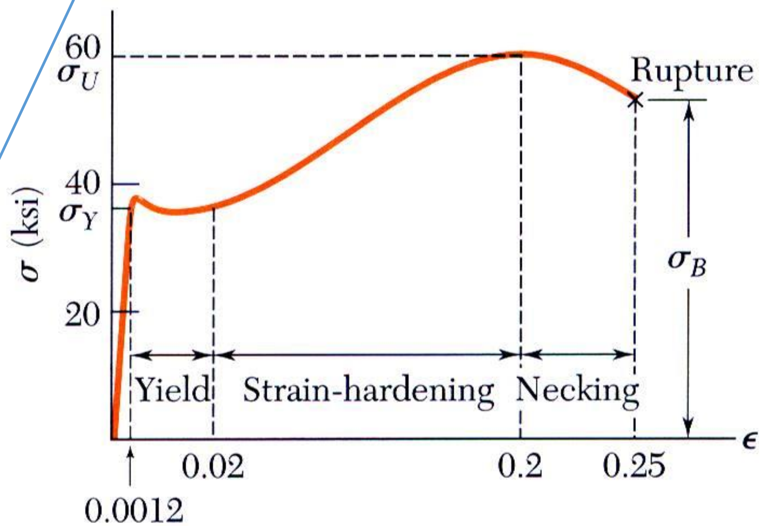
UTM – Universal Testing Machine, generally capable of different tests for materials (metal, non-metal, plastic, composites etc.) including tensile, compressive, cross tensile, lap shear, flexural – 3 point, 4 point bending etc.

ASTM (American Society for Testing and Materials) – ASTM or other standards to follow for sample dimensions, testing methods, testing parameters, analysis and many more for different materials.

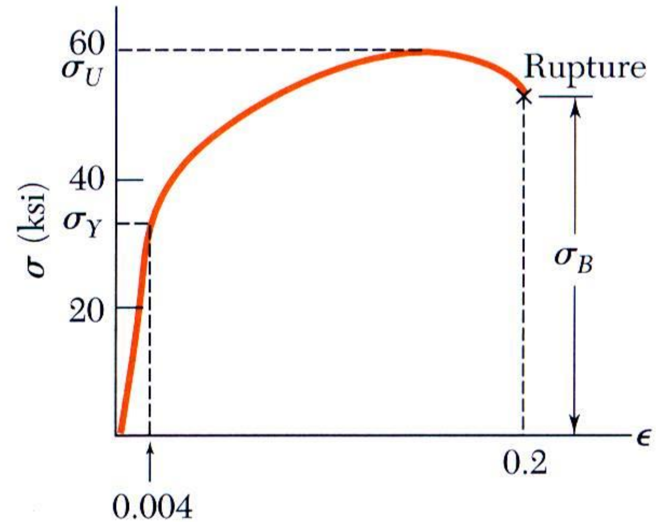
Stress-Strain Diagram: Ductile Materials

Cup and cone, 45°





(a) Low-carbon steel



(b) Aluminum alloy

σ_Y = Yield Strength
σ_U = Ultimate Strength
σ_B = Breaking Strength

Engineering stress, $\sigma = \frac{P}{A_0}$

Engineering strain, $\varepsilon = \frac{\delta}{L_0}$

True stress, $\sigma_{true} = \frac{P}{A}$

True strain, $\varepsilon_{true} = \ln\left(\frac{L}{L_0}\right)$

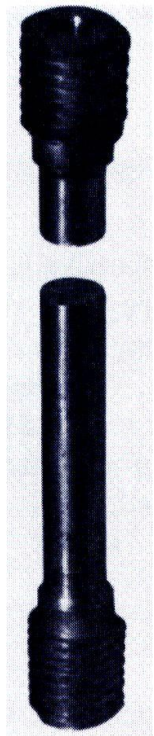
L_0 – initial length
L – instantaneous length
A_0 – initial c/s area
A - instantaneous area

Relation: $\sigma_{true} = \sigma(1+\varepsilon)$
$\varepsilon_{true} = \ln(1+\varepsilon)$

Courtesy: TMH

Stress-Strain Diagram: Brittle Materials

Brittle Material: Strain at rupture is much smaller



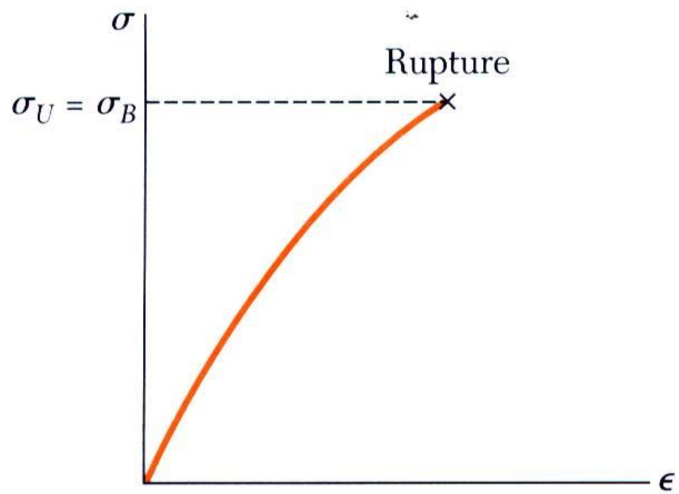


Fig. 2.11 Stress-strain diagram for a typical brittle material.

σ_Y = Yield Strength

σ_U = Ultimate Strength = σ_B = Breaking Strength

Hooke's Law: Modulus of Elasticity

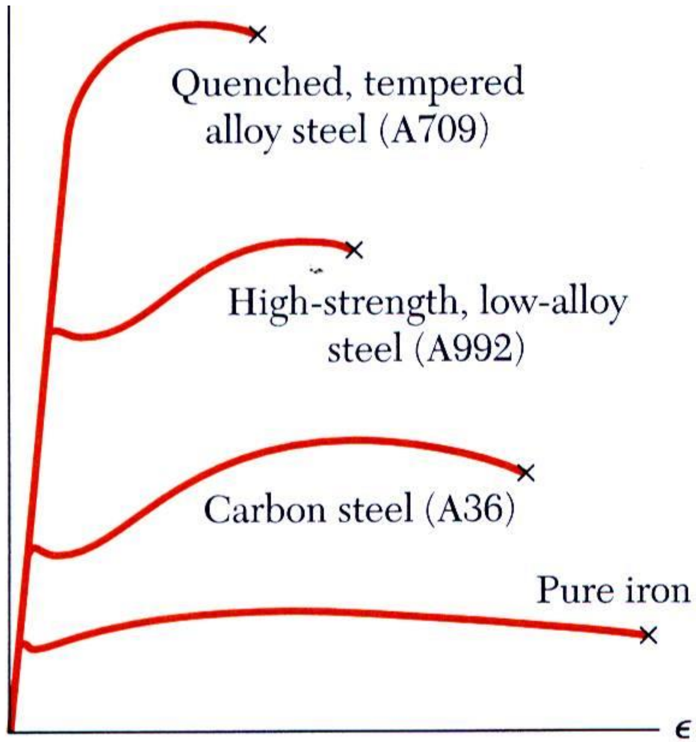


Fig. 2.16 Stress-strain diagrams for iron and different grades of steel.

- Below the yield stress

$$\sigma = E\varepsilon$$

E = Youngs Modulus or Modulus of Elasticity

- **Strength is affected by alloying, heat treating, and manufacturing process but stiffness (Modulus of Elasticity) is not.**

Elastic vs. Plastic Behavior

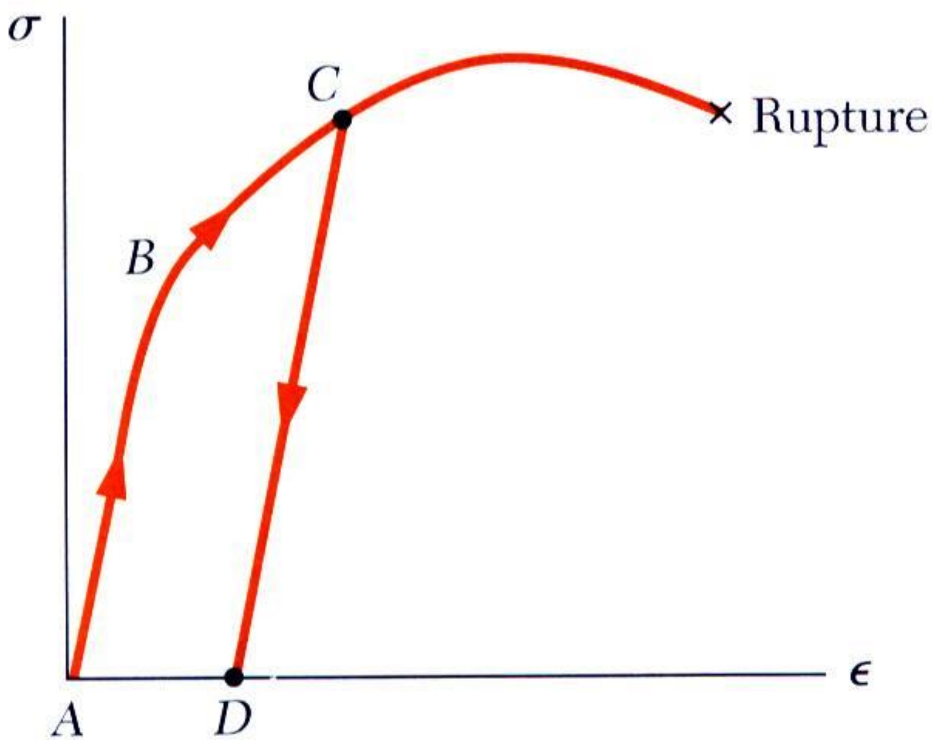


Fig. 2.18

- If the strain disappears when the stress is removed, the material is said to behave *elastically*.
- The largest stress for which this occurs is called the *elastic limit*.
- When the strain does not return to zero after the stress is removed, the material is said to behave *plastically*.

What will happen if again loaded (from D)?

Deformations Under Axial Loading

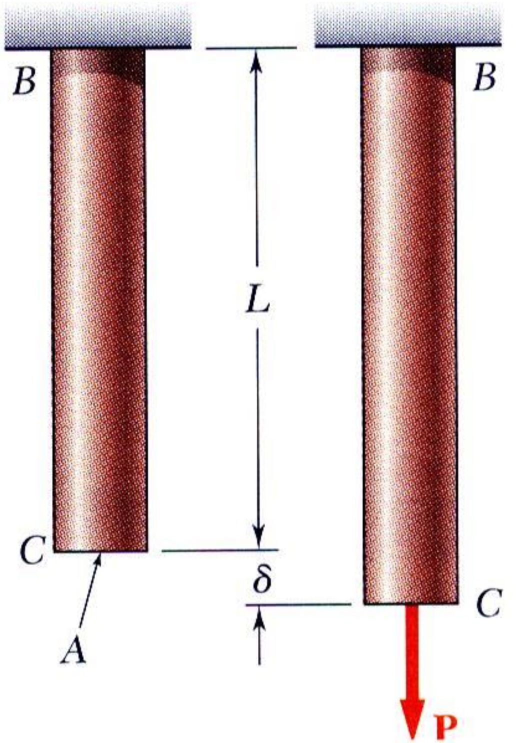


Fig. 2.22

- From Hooke's Law:

$$\sigma = E\varepsilon \qquad \varepsilon = \frac{\sigma}{E} = \frac{P}{AE}$$

- From the definition of strain:

$$\varepsilon = \frac{\delta}{L}$$

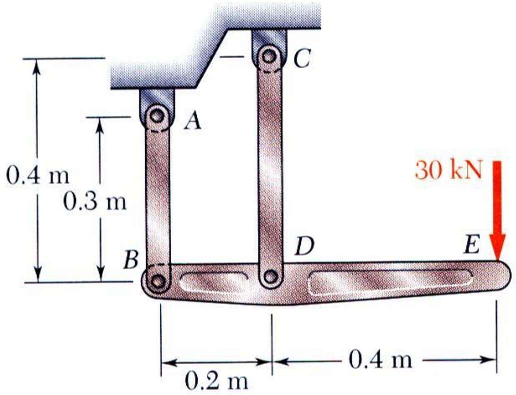
- Equating and solving for the deformation,

$$\delta = \frac{PL}{AE}$$

- With variations in loading, cross-section or material properties,

$$\delta = \sum_i \frac{P_i L_i}{A_i E_i}$$

Sample Problem 2.1



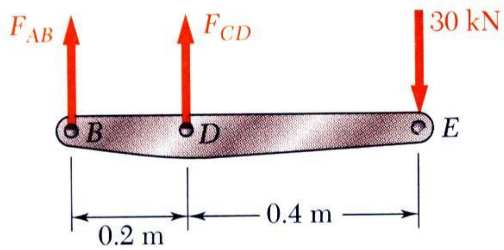
The rigid bar *BDE* is supported by two links *AB* and *CD*.

Link *AB* is made of aluminum (E = 70 GPa) and has a cross-sectional area of 500 mm^2. Link *CD* is made of steel (E = 200 GPa) and has a cross-sectional area of (600 mm^2).

For the 30-kN force shown, determine the deflection a) of *B*, b) of *D*, and c) of *E*.

SOLUTION:

Free body: Bar *BDE*



$$\sum M_B = 0$$

$$0 = -(30\,\text{kN} \times 0.6\,\text{m}) + F_{CD} \times 0.2\,\text{m}$$

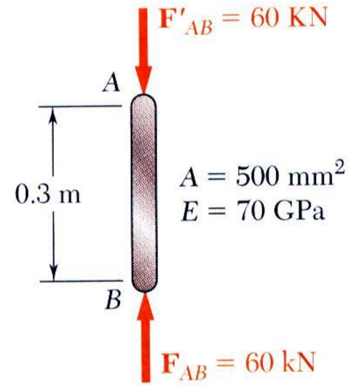
$$F_{CD} = +90\,\text{kN} \quad \textit{tension}$$

$$\sum \mathrm{M_D} = 0$$

$$0 = -(30\,\text{kN} \times 0.4\,\text{m}) - F_{AB} \times 0.2\,\text{m}$$

$$F_{AB} = -60\,\text{kN} \quad \textit{compression}$$

Displacement of *B*:



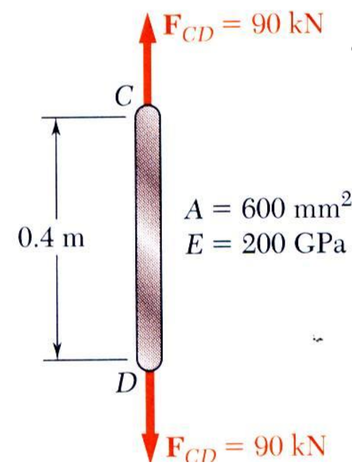
$$\delta_B = \frac{PL}{AE}$$

$$= \frac{\left(-60 \times 10^3\,\text{N}\right)(0.3\,\text{m})}{\left(500 \times 10^{-6}\,\text{m}^2\right)\left(70 \times 10^9\,\text{Pa}\right)}$$

$$= -514 \times 10^{-6}\,\text{m}$$

$$\boxed{\delta_B = 0.514\,\text{mm} \uparrow}$$

Displacement of *D*:

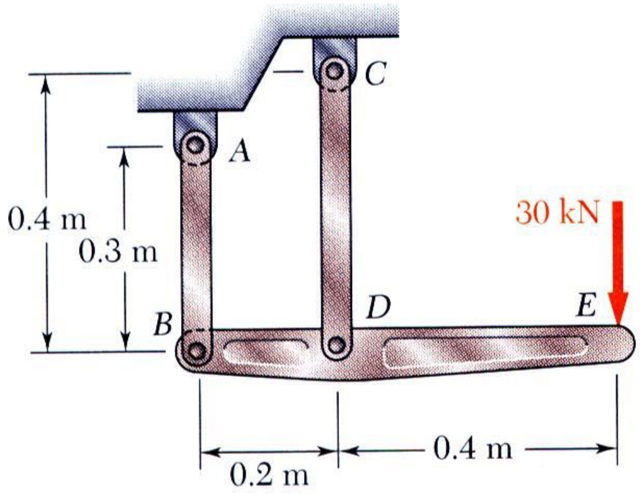


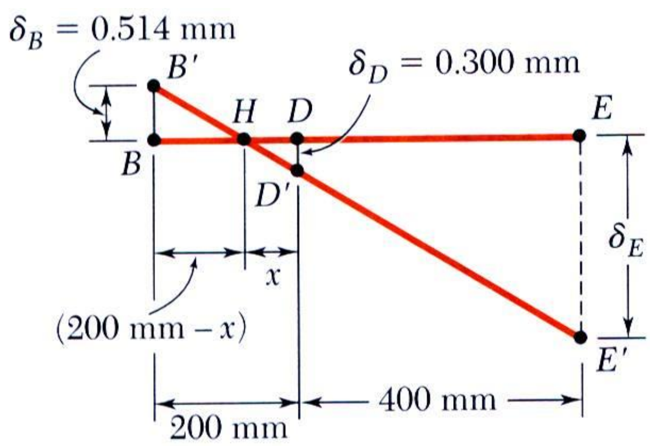
$$\delta_D = \frac{PL}{AE}$$

$$= \frac{\left(90 \times 10^3\,\text{N}\right)(0.4\,\text{m})}{\left(600 \times 10^{-6}\,\text{m}^2\right)\left(200 \times 10^9\,\text{Pa}\right)}$$

$$= 300 \times 10^{-6}\,\text{m}$$

$$\boxed{\delta_D = 0.300\,\text{mm} \downarrow}$$





Displacement of E:

$$\frac{BB'}{DD'} = \frac{BH}{HD}$$

$$\frac{0.514\text{ mm}}{0.300\text{ mm}} = \frac{(200\text{ mm}) - x}{x}$$

$$x = 73.7\text{ mm}$$

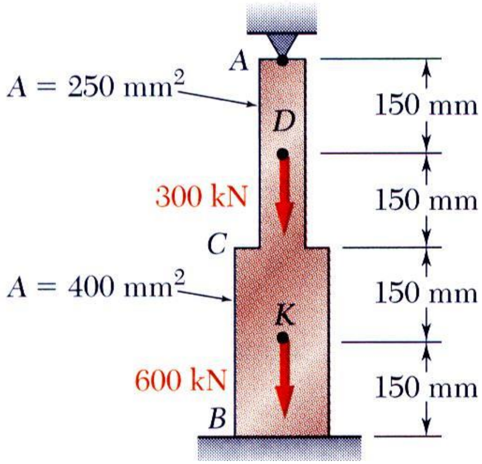
$$\frac{EE'}{DD'} = \frac{HE}{HD}$$

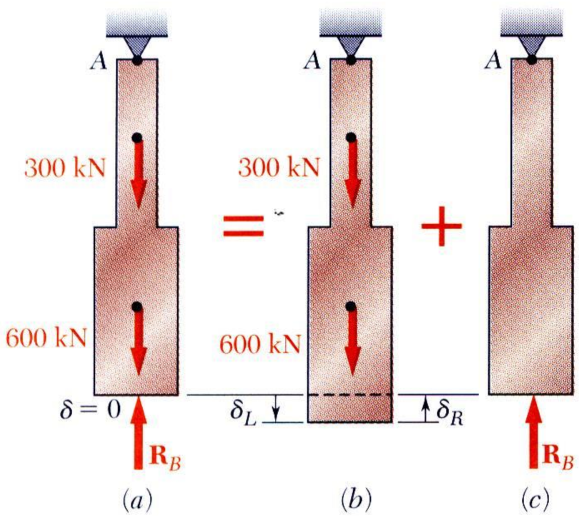
$$\frac{\delta_E}{0.300\text{ mm}} = \frac{(400 + 73.7)\text{mm}}{73.7\text{ mm}}$$

$$\delta_E = 1.928\text{ mm}$$

$$\boxed{\delta_E = 1.928\text{ mm} \downarrow}$$

Static Indeterminacy





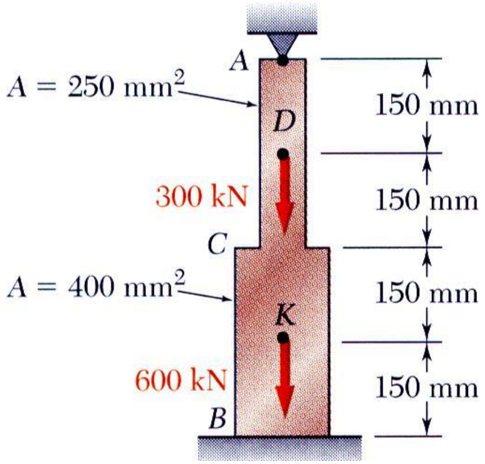
- Structures for which internal forces and reactions cannot be determined from statics alone are said to be *statically indeterminate*.
  - A structure will be statically indeterminate whenever it is held by more supports than are required to maintain its equilibrium.
  - Redundant reactions are replaced with unknown loads which along with the other loads must produce compatible deformations.
  - Deformations due to actual loads and redundant reactions are determined separately and then added or *superposed*.

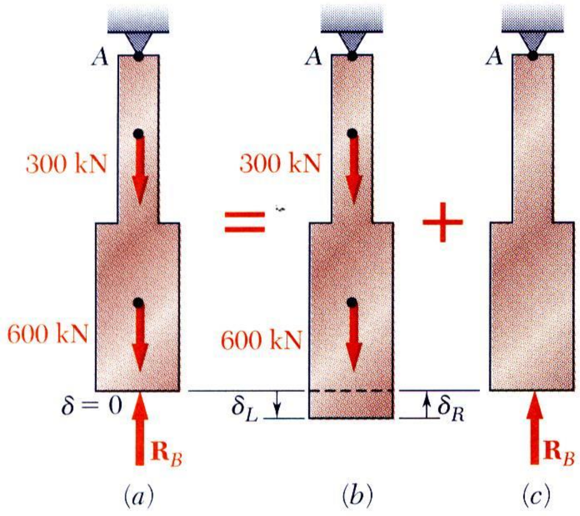
$$\delta = \delta_L + \delta_R = 0$$

Courtesy: TMH

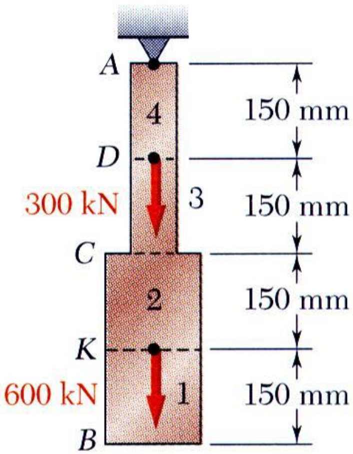
Example 2.04

Determine the reactions at *A* and *B* for the steel bar and loading shown, assuming a close fit at both supports before the loads are applied.





SOLUTION:



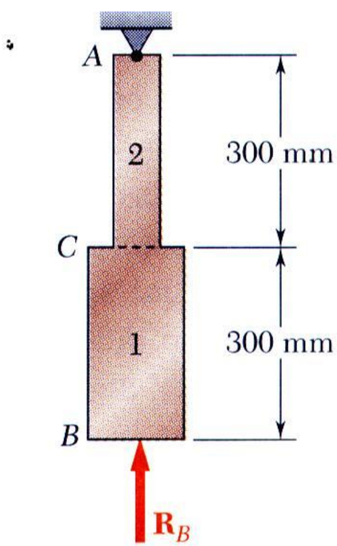
- Solve for the displacement at *B* due to the applied loads with the redundant constraint released,

$$P_1 = 0 \quad P_2 = P_3 = 600\times10^3\,\mathrm{N} \quad P_4 = 900\times10^3\,\mathrm{N}$$

$$A_1 = A_2 = 400\times10^{-6}\,\mathrm{m}^2 \quad A_3 = A_4 = 250\times10^{-6}\,\mathrm{m}^2$$

$$L_1 = L_2 = L_3 = L_4 = 0.150\,\mathrm{m}$$

$$\delta_{\mathrm{L}} = \sum_i \frac{P_i L_i}{A_i E_i} = \frac{1.125\times10^9}{E}$$



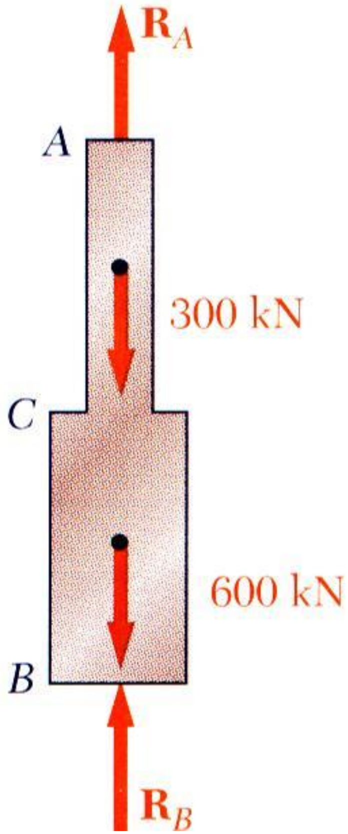
- Solve for the displacement at *B* due to the redundant constraint,

$$P_1 = P_2 = -R_B$$

$$A_1 = 400\times10^{-6}\,\mathrm{m}^2 \quad A_2 = 250\times10^{-6}\,\mathrm{m}^2$$

$$L_1 = L_2 = 0.300\,\mathrm{m}$$

$$\delta_R = \sum_i \frac{P_i L_i}{A_i E_i} = -\frac{\left(1.95\times10^3\right)R_B}{E}$$



- Require that the displacements due to the loads and due to the redundant reaction be compatible,

$$\delta = \delta_L + \delta_R = 0$$

$$\delta = \frac{1.125 \times 10^9}{E} - \frac{\left(1.95 \times 10^3\right) R_B}{E} = 0$$

$$R_B = 577 \times 10^3 \,\mathrm{N} = 577 \,\mathrm{kN}$$

- Find the reaction at *A* due to the loads and the reaction at *B*

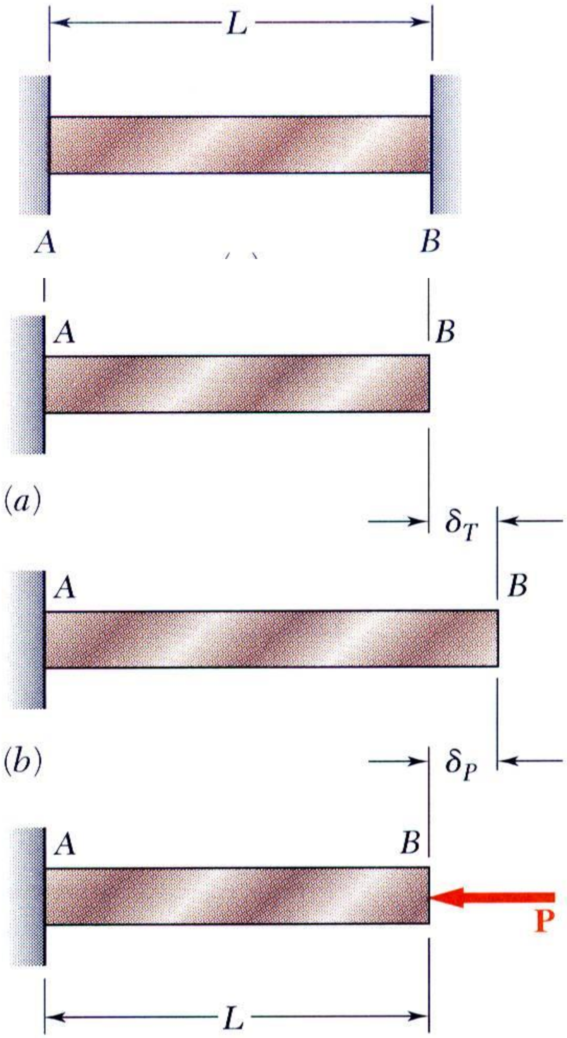
$$\sum F_y = 0 = R_A - 300\,\mathrm{kN} - 600\,\mathrm{kN} + 577\,\mathrm{kN}$$

$$R_A = 323\,\mathrm{kN}$$

$$\boxed{\begin{aligned} R_A &= 323\,\mathrm{kN} \\ R_B &= 577\,\mathrm{kN} \end{aligned}}$$

Can we find these reactions without using the concept of stress-strain?

Thermal Stresses



- A temperature change results in a change in length or *thermal strain*. There is no stress associated with the thermal strain unless the elongation is restrained by the supports.
- Treat the additional support as redundant and apply the principle of superposition.

$$\delta_T = \alpha(\Delta T)L \qquad \delta_P = \frac{PL}{AE}$$

$$\alpha = \text{thermal expansion coef.}$$

- The thermal deformation and the deformation from the redundant support must be compatible.

$$\delta = \delta_T + \delta_P = 0$$

$$\alpha(\Delta T)L + \frac{PL}{AE} = 0$$

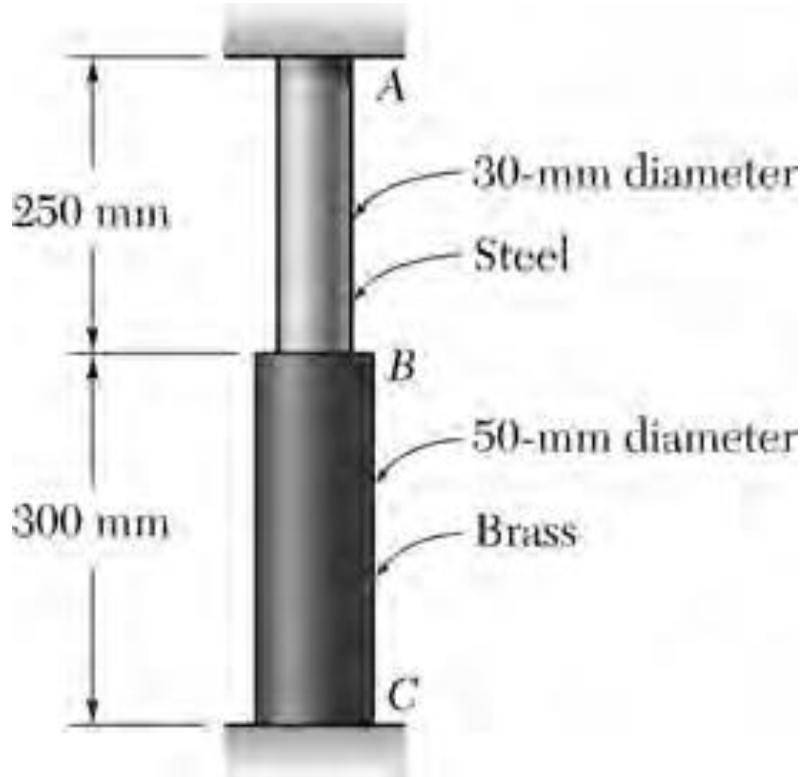
$$\delta = \delta_T + \delta_P = 0$$

$$P = -AE\alpha(\Delta T)$$

$$\sigma = \frac{P}{A} = -E\alpha(\Delta T)$$

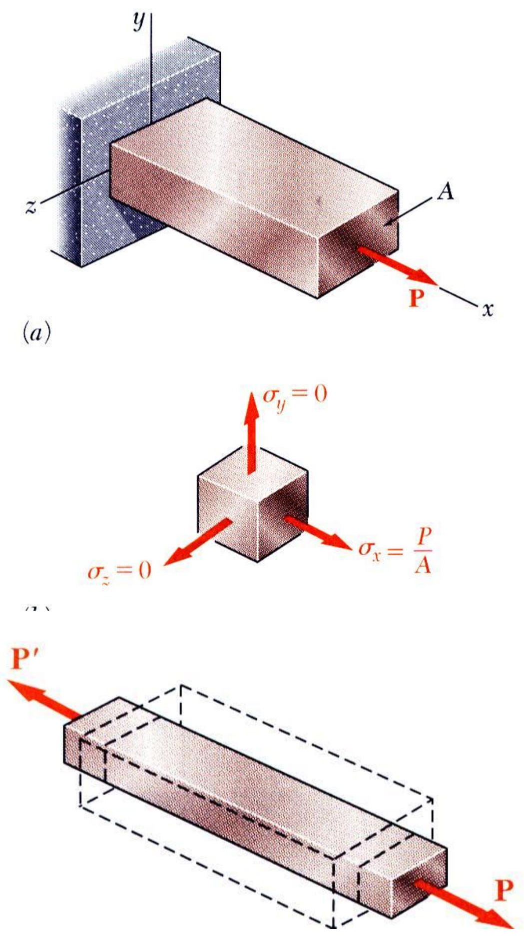
Problem

A rod consisting of two cylindrical portions AB and BC is restrained at both ends. Portion AB is made of steel ($E_s = 200$ GPa, $\alpha_s = 11.7\times10^{-6}/\,°\mathrm{C}$) and portion BC is made of brass ($E_b = 105$ GPa, $\alpha_b = 20.9\times10^{-6}/\,°\mathrm{C}$). Knowing that the rod is initially unstressed, determine the compressive force induced in ABC when there is a temperature rise of $50\,°\mathrm{C}$.



Courtesy: TMH

Poisson's Ratio



- For a slender bar subjected to axial loading:

$$\varepsilon_x = \frac{\sigma_x}{E} \qquad \sigma_y = \sigma_z = 0$$

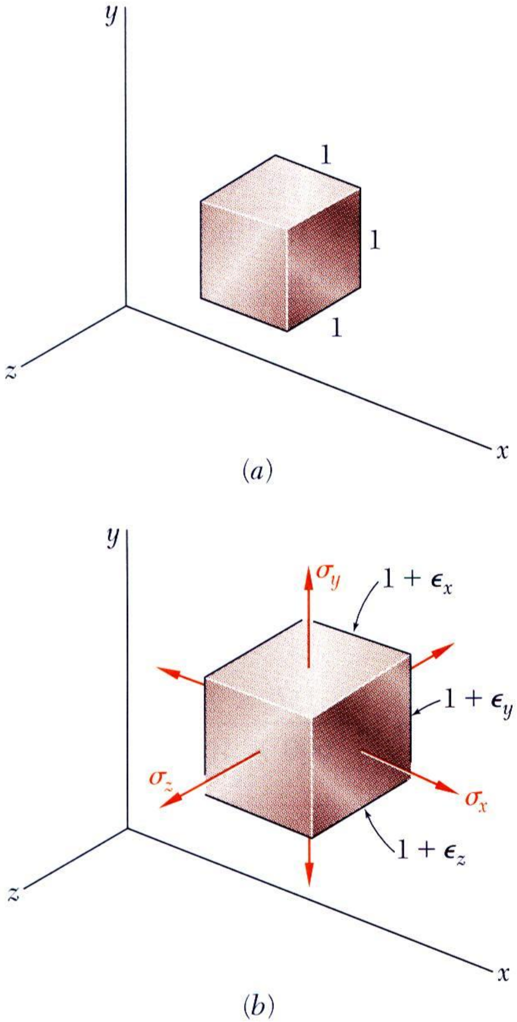
- The elongation in the x-direction is accompanied by a contraction in the other directions. Assuming that the material is isotropic (no directional dependence),

$$\varepsilon_y = \varepsilon_z \neq 0$$

- Poisson's ratio is defined as

$$\nu = \left| \frac{\text{lateral strain}}{\text{axial strain}} \right| = -\frac{\varepsilon_y}{\varepsilon_x} = -\frac{\varepsilon_z}{\varepsilon_x}$$

Generalized Hooke's Law



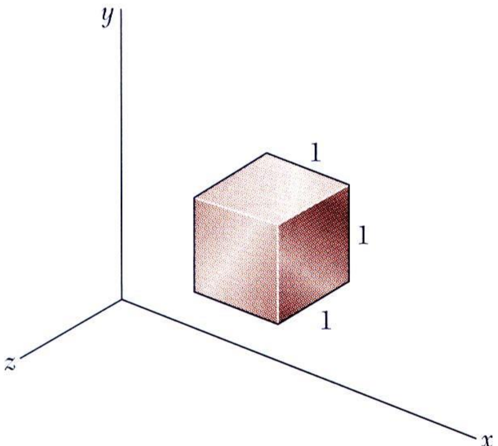
- For an element subjected to multi-axial loading, the normal strain components resulting from the stress components may be determined from the *principle of superposition*. This requires:
  1) strain is linearly related to stress
  2) deformations are small
- With these restrictions:

$$\varepsilon_x = +\frac{\sigma_x}{E} - \frac{\nu\sigma_y}{E} - \frac{\nu\sigma_z}{E}$$

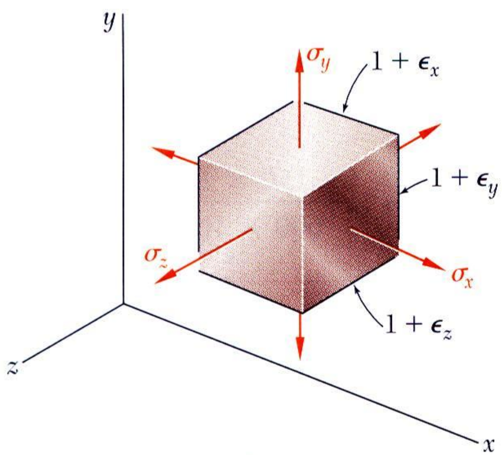
$$\varepsilon_y = -\frac{\nu\sigma_x}{E} + \frac{\sigma_y}{E} - \frac{\nu\sigma_z}{E}$$

$$\varepsilon_z = -\frac{\nu\sigma_x}{E} - \frac{\nu\sigma_y}{E} + \frac{\sigma_z}{E}$$

Dilatation: Bulk Modulus



(a)



(b)

- Relative to the unstressed state, the change in volume is

$$e = \left[(1+\varepsilon_x)(1+\varepsilon_y)(1+\varepsilon_z)\right] - 1 = \left[1+\varepsilon_x+\varepsilon_y+\varepsilon_z\right] - 1$$

$$= \varepsilon_x + \varepsilon_y + \varepsilon_z$$

$$= \frac{1-2\nu}{E}\left(\sigma_x + \sigma_y + \sigma_z\right)$$

$$= \text{dilatation (change in volume per unit volume)}$$

- For element subjected to uniform hydrostatic pressure,

$$e = -p\frac{3(1-2\nu)}{E} = -\frac{p}{k}$$

$$k = \frac{E}{3(1-2\nu)} = \text{bulk modulus}$$

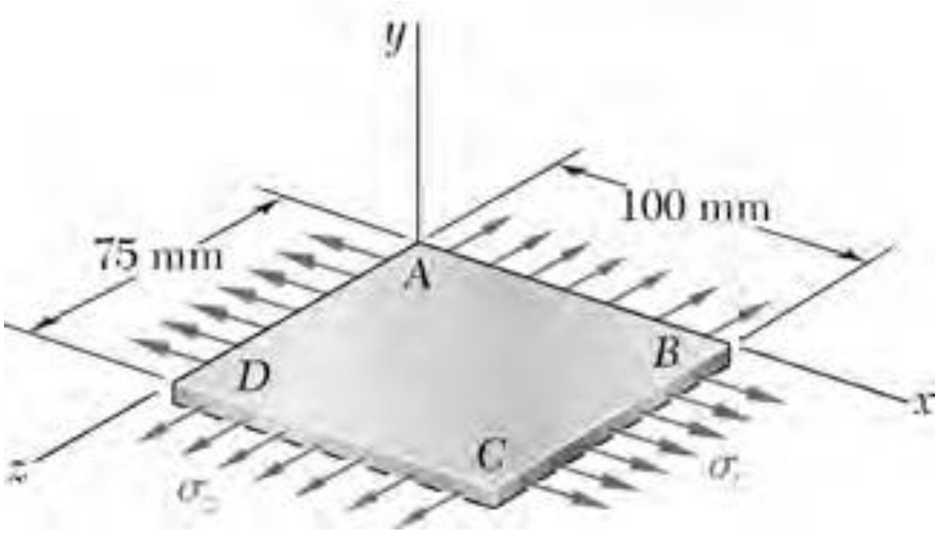
- Subjected to uniform pressure, dilatation must be negative, therefore

$$0 < \nu < \tfrac{1}{2}$$

Courtesy: TMH

Problem

A fabric used in air-inflated structures is subjected to a biaxial loading that results in normal stresses $\sigma_x = 120$ MPa and $\sigma_z = 160$ MPa. Knowing that the properties of the fabric can be approximated as $E = 87$ GPa and $\nu = 0.34$, determine the change in length of (*a*) side AB, (*b*) side BC, (*c*) diagonal AC.



Courtesy: TMH

Shearing Strain

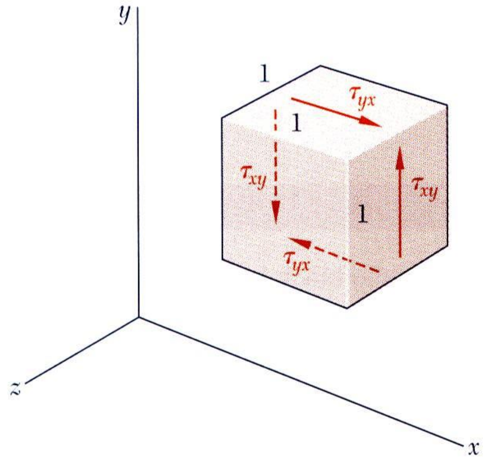


Fig. 2.46

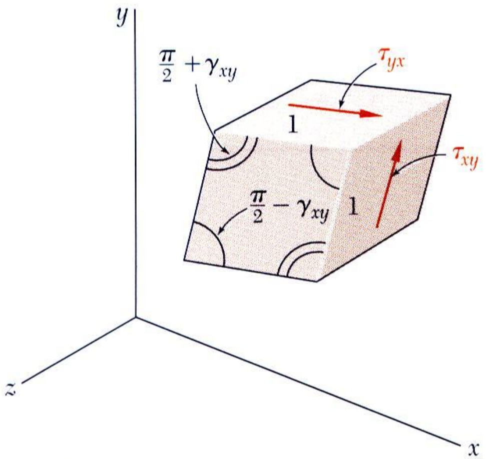


Fig. 2.47

- A cubic element subjected to a shear stress will deform into a rhomboid. The corresponding *shear* strain is quantified in terms of the change in angle between the sides,

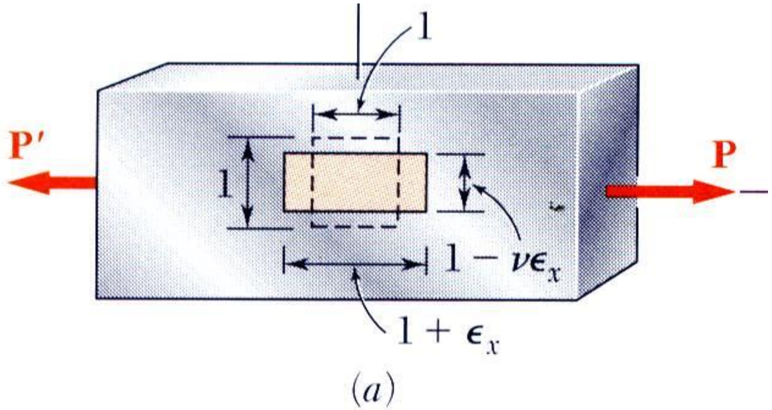
$$\tau_{xy} = f\left(\gamma_{xy}\right)$$

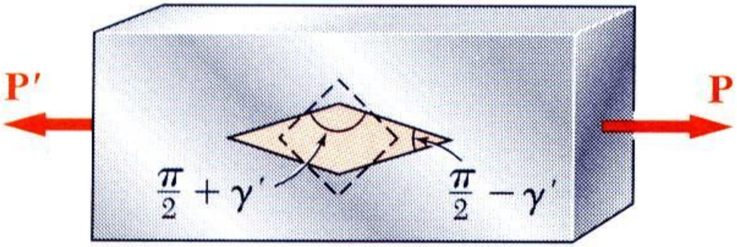
- A plot of shear stress vs. shear strain is similar the previous plots of normal stress vs. normal strain except that the **strength values are approximately half**. For small strains,

$$\tau_{xy} = G\gamma_{xy} \quad \tau_{yz} = G\gamma_{yz} \quad \tau_{zx} = G\gamma_{zx}$$

where G is the modulus of rigidity or shear modulus.

Relation Among E, ν, and G



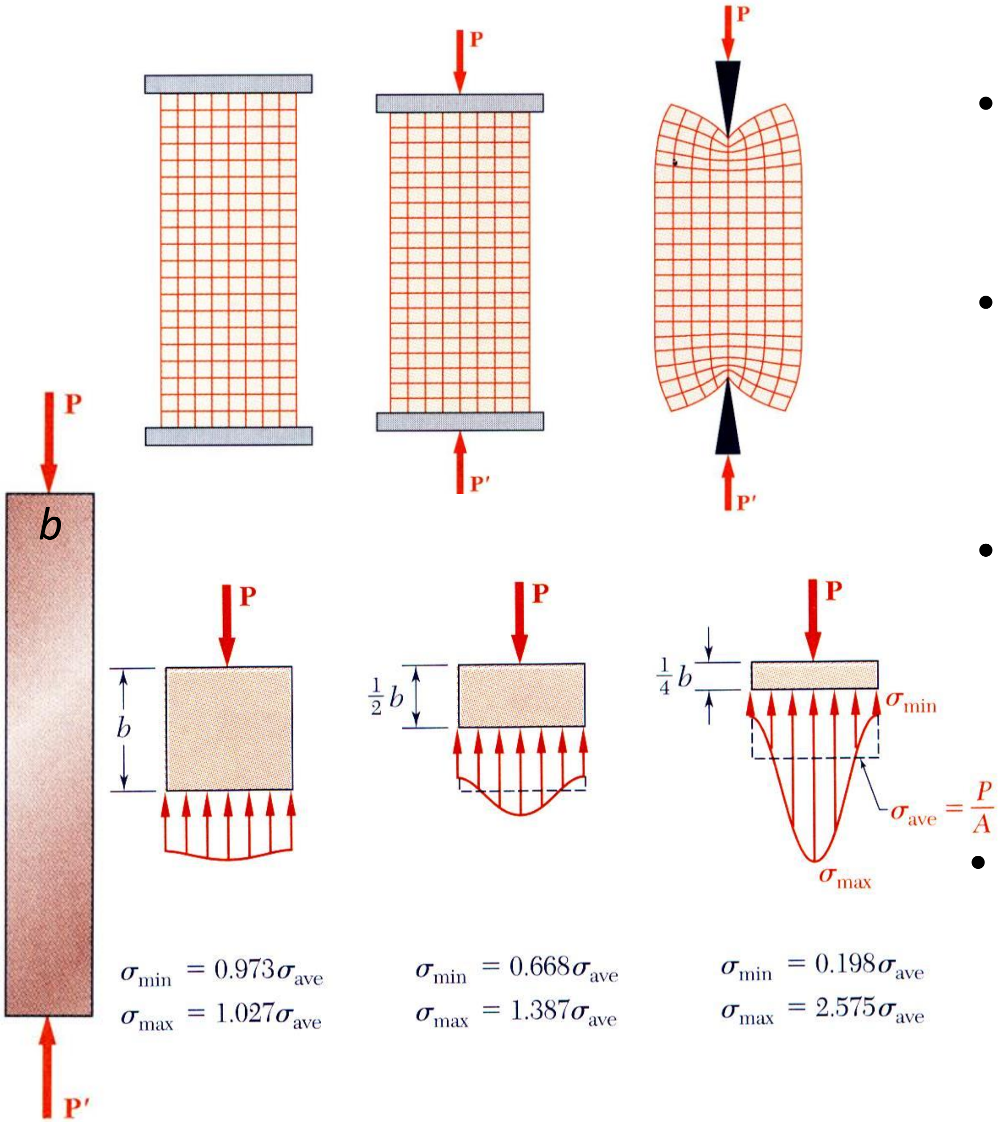


- An axially loaded slender bar will elongate in the axial direction and contract in the transverse directions.
- An initially cubic element oriented as in top figure will deform into a rectangular parallelepiped. The axial load produces a normal strain.
- If the cubic element is oriented as in the bottom figure, it will deform into a rhombus. Axial load also results in a shear strain.
- Components of normal and shear strain are related,

$$\frac{E}{2G} = (1 + \nu)$$

HW: Derive the relation between E, G, and nu.

Saint-Venant’s Principle



Estimated through FEM

- Loads transmitted through rigid plates result in uniform distribution of stress and strain.
- Concentrated loads result in large stresses in the vicinity of the load application point.
- Stress and strain distributions become uniform at a relatively short distance from the load application points.
- **Saint-Venant’s Principle:** Stress distribution may be assumed independent of the mode of load application except in the immediate vicinity of load application points.