Department of Mathematics
Indian Institute of Technology Patna
B.Tech. II year (Autumn Semester: 2020-21)

Tutorial Sheet-2: MA201 (Complex Analysis)

1. Find the limit of each of the following functions at given points:
   (i) $f(z) = \dfrac{xy}{x^2+y^2} + 2xyi;$ when $z \to 0$,
   (ii) $f(z) = \dfrac{xy^3}{x^3+y^3} + \dfrac{x^8}{y^2+1}i;$ when $z \to 0$,
   (iii) $f(z) = \dfrac{3z^4 - 2z^3 + 8z^2 - 2z + 5}{z-i};$ when $z \to i$,
   (iv) $f(z) = \dfrac{z^3+8}{z^4+4z^2+16};$ when $z \to 2e^{\pi i/3}$,
   (v) $f(z) = \frac{z-\bar{z}}{z+\bar{z}}$; when $z \to 1+i$,
   (vi) $f(z) = (|z|^2 - i\bar{z})$; when $z \to 1-i$.

2. If $\lim_{z\to z_0} f(z) = l$, show that $\lim_{z\to z_0} |f(z)| = |l|$.

3. Let $f(z) = \frac{Re(z)}{|z|}$, $z \neq 0$ and $f(0) = 1$. Is $f(z)$ continuous at the origin?

4. Let $f(z) = \sqrt{z} = \sqrt{|z|}e^{iArg(z)/2}$, where $z \neq 0$. Show that $f(z)$ is discontinuous at each point along the negative real axis.

5. Test the continuity of the following functions:
   (i) $f(z) = z^2$ (ii) $f(z) = \cot z$ (iii) $f(z) = \dfrac{\tanh z}{z^2+1}$.

6. Show that the function is continuous at the given point z_0:
   (i) $f(z) = \frac{Re(z)}{z+iz} - 2z^2;$ $z_0 = e^{i\pi/4}$,
   (ii) $f(z) = \frac{z^3-1}{z^2+z+1}$, $|z| \neq 1$ and $f(z) = \frac{-1+i\sqrt{3}}{2}$, $|z| = 1;$ $z_0 = \frac{1+i\sqrt{3}}{2}$.

7. Establish the $C-R$ equations in polar coordinates.

8. Derive the Laplace equation in polar form.

9. Show that the function $f(z) = \begin{cases} \dfrac{x^3(1+i) - y^3(1-i)}{x^2+y^2}, & z \neq 0, \\ 0, & z = 0 \end{cases}$ is not differentiable at the origin (0, 0); although, the $C-R$ equations are satisfied there.

10. Show that the function $f(z) = \dfrac{|z^2 - \bar{z}^2|^{1/2}}{2}$ is not differentiable at the origin (0, 0). Although, the $C-R$ equations are satisfied there.

11. Show that for the function $f(z) = \begin{cases} \dfrac{\bar{z}^2}{z}, & z \neq 0 \\ 0, & z = 0 \end{cases}$, $C-R$ equations are satisfied at the origin (0, 0). Also, check whether or not $f'(0)$ exists?

12. Prove that $f(z) = xy + yi$ is everywhere continuous but not analytic.

13. Prove that for the function $f(z) = x^3 + i(1-y)^3$, we can write $f'(z) = u_x + iv_x = 3x^2$, only when $z = i$.

14. Check the continuity and analyticity of the function $f(z) = z\bar{z}$ everywhere.

15. Prove that the following functions are nowhere differentiable:
   (i) $f(z) = |z|$ (ii) $f(z) = Re(z)$ (iii) $f(z) = Im(z)$ (iv) $f(z) = \bar{z}$
   (v) $f(z) = z - \bar{z}$ (vi) $f(z) = 2x + ixy^2$ (vii) $f(z) = e^x e^{-iy}$.

16. Prove that in each of the following cases, U is a harmonic function. Further, find a function V such that $f(z) = U + Vi$ is analytic.

1. $U = e^x(x \cos y - y \sin y)$,
2. $U = 4xy - x^3 + 3xy^2$,
3. $U = \sin x \cosh y + 2 \cos x \sinh y + x^2 - y^2 + 4xy$,
4. $U = x^3 - 3xy^2 - 3x^2 - 3y^2 + 1$.

17. Prove that an analytic function with constant modulus is constant.

18. Show that if $u(x, y)$ and $v(x, y)$ are harmonic functions in a domain D, then the function $f(z) = (u_y - v_x) + i(u_x + v_y)$ is analytic in D.

19. Verify the validity of the statement: "If the function $f(z) = u(x, y) + iv(x, y)$ is analytic at a point z, then necessarily the function $f(z) = v(x, y) - iu(x, y)$ is analytic at z."