At several places, we have assumed that $\boxed{\frac{dy}{dt} = f(t,y)}$ has only one solution passing thr. one point in xy-plane.
[Recall geometric method].

This means that we assume that IVP

$$\text{IVP} \begin{cases} \frac{dy}{dt} = f(t,y) \\ y(t_0) = y_0 \end{cases}$$

has unique solution.

Aim:- To understand that when the above assumption is correct.

Existence and uniqueness theorems

- Cauchy-Peano's existence thm.
- Picard's existence and uniqueness thm.
- Picard's iteration method to approximate solution of IVP.

Main questions

- Does a solution exist?
- If a solution exists, is it unique?
- If solution exists uniquely, can we find it? or can we write it in form of elementary func?

To write thm. we need to define one notation

First see one definition

Let $f(t, y)$ be a real function defined on $D \subset \mathbb{R}^2$.
Then f is called Lipschitz continuous in its second variable (i.e. y) If the following condition holds

$$\boxed{\begin{aligned} &|f(t, y_1) - f(t, y_2)| \le K\, |y_1 - y_2| \\ &\qquad\qquad \forall\ (t, y_1), (t, y_2) \in D. \end{aligned}}$$

(Lip. condition)

Here K is called Lipschitz constant.

Notation

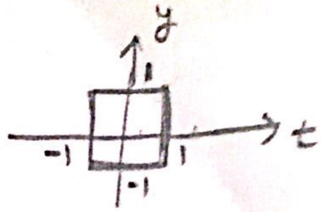
$Lip(D, K) = \{ f(t, y) : D \subset \mathbb{R}^2 \to \mathbb{R} :$ f is Lipschitz continuous in its second variable with Lipschitz constant $K \}$

Note: Lipschitz condition $\equiv$ $\dfrac{|f(t, y_1) - f(t, y_2)|}{|y_1 - y_2|} \le K$
$\forall\ (t, y_1), (t, y_2) \in D$.

$\boxed{\text{Test}}$ $\left|\dfrac{\partial f}{\partial y}\right| \le K$ in $D \Rightarrow f \in Lip(D, K)$ (Sufficient test only)

functions with bdd slope (wrt y).

Examples



① $D = \{(t,y) : |t| \leq 1,\ |y| \leq 1\}$

$f(t,y) = \sin y + t$

$\left|\frac{\partial f}{\partial y}\right| = |\cos y| \leq 1 \quad \forall\ (t,y) \in D$

Hence $f \in Lip(D,1)$.

② D is same as in Ex 1.

$f(t,y) = |y|$

Here $\frac{\partial f}{\partial y}$ does not exist at $(t,0)$

But try to verify the Lipschitz condition

$|f(t,y_1) - f(t,y_2)| = \big||y_1| - |y_2|\big| \leq |y_1 - y_2|$

Thus $f \in Lip(D,1)$

This example shows that our derivative test is sufficient and not necessary.

From sufficient test, we can not disprove the Lipschitz continuity

③ D is same as in Ex 1.

$f(t,y) = \sqrt{y} = y^{1/2}$

Check Lip. condition around t-axis

$$\frac{|f(t,y_1) - f(t,0)|}{|y_1 - 0|} = \frac{\sqrt{y_1}}{y_1} = \frac{1}{\sqrt{y_1}} \quad (y_1 > 0)$$

is not bdd if $y_1 \approx 0$.

Hence f is not Lipschitz continuous in D.

Cauchy - Peano's Thm. (W/P) (Existence thm)

Consider the following IVP

$$\frac{dy}{dt} = f(t, y)$$

$$y(t_0) = y_0$$

(Assumptions)

Let $D = \{(t, y) : |t - t_0| \leq a, \ |y - y_0| \leq b\}$

Let f be continuous and bounded on D,

Let $|f(t, y)| \leq L \ \forall \ (t, y) \in D$.

Let $h = \min.\{a, \frac{b}{L}\}$

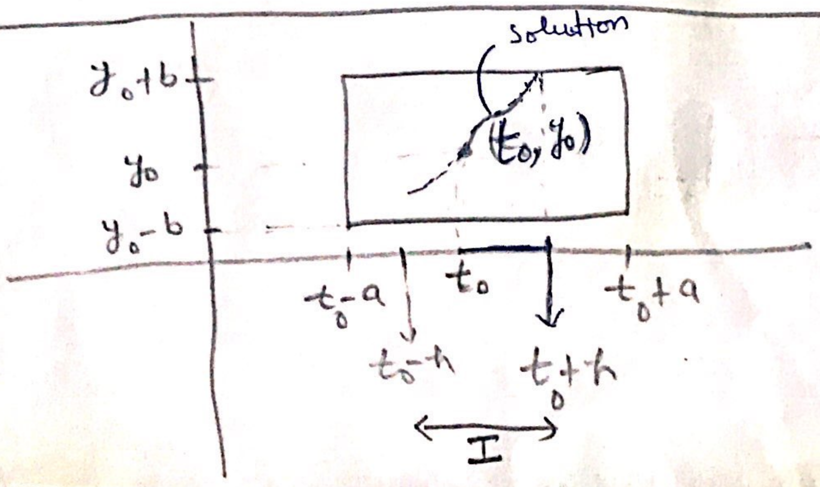
Let $I = [t_0 - h, t_0 + h]$

Then IVP has a solution on I.

Remarks

① Under assumptions, IVP has solution on I. But thm. does not stop the IVP to have a solution on larger interval $I = [t_0, t_0 + h^*]$; $h^* > h$.

② Assumptions set sufficient conditions, i.e. solution may exist if any/all assumptions do not hold.



In rough sense if $f(t, y)$ is continuous around (t_0, y_0) then solution exists

Picard's Thm. (Existence and uniqueness Thm.)

Consider to following IVP

$$\frac{dy}{dt} = f(t, y)$$

$$y(t_0) = y_0.$$

Let $D = \{(t, y) : |t - t_0| \le a,\ |y - y_0| \le b\}$.

Let f be continuous and bounded on D.

Let $|f(t, y)| \le L \quad \forall\ (t, y) \in D$.

Let $\boxed{f \in Lip.(D, K)}$ ← extra than Cauchy-Peano's Thm.

Let $h = \min\{a, \frac{b}{L}\}$

Let $I = [t_0 - h, t_0 + h]$

Then IVP has a solution on I.

moreover the solution is unique.

<u>Remarks</u> (i) Interval I may be extended.

(ii) Assumptions set sufficient conditions.

(iii) <u>Hint on proof</u>

$$IVP \equiv \boxed{y(t) = y_0 + \int_{t_0}^{t} f(s, y(s))\, ds} \to \text{Integral equation}$$

introduce iteration ($\boxed{\text{Picard's iterations}}$ to compute approximate solution of IVP <u>OR</u> integral equation)

$$\boxed{y_{i+1} = y_0 + \int_{t_0}^{t} f(s, y_i(s))\, ds \qquad i = 0, 1, 2, 3, \cdots}$$

<u>Take</u> $y_0(s) \equiv y_0$ and start computing $y_1, y_2, y_3, \cdots$. This sequence converges to $\boxed{y(t)}$.

Example Use picard's iterations to find approximate solution of following IVP

$$\frac{dy}{dt} = -y$$

$$y(0) = 1$$

Ist iteration

$$y_1(t) = y_0 + \int_{t_0}^{t} f(s, y_0(s))\, ds$$

given :-
$f(t, y) = -y$
$y_0 = 1$ (constant function)
$t_0 = 0$

$$= 1 + \int_0^t f(s, 1)\, ds = 1 + \int_0^t (-1)\, ds$$

$$= 1 + (-s)\Big|_0^t = 1 + (-t + 0) = 1 - t$$

IInd iteration

$$y_2(t) = y_0 + \int_{t_0}^{t} f(s, y_1(s))\, ds$$

$$= 1 + \int_0^t -(1-s)\, ds = 1 - \left(s - \frac{s^2}{2}\right)\Big|_0^t = 1 - t + \frac{t^2}{2}$$

IIIrd iteration

$$y_3(t) = 1 + \int_0^t -\left(1 - s + \frac{s^2}{2}\right) ds = 1 - \left(s - \frac{s^2}{2} + \frac{s^3}{3!}\right)\Big|_0^t$$

$$= 1 - t + \frac{t^2}{2} - \frac{t^3}{3!} + \cdots$$

$\vdots$

$$y_n(t) = \underbrace{1 - t + \frac{t^2}{2} - \frac{t^3}{3!} + \frac{t^4}{4!} - \cdots (-1)^n \frac{t^n}{n!}}$$

$\hookrightarrow e^{-t}$ as $n \to \infty$

[This means approximate solution.]

If you increase the number of iterations, you go more close to solution

Examples to learn about existence and uniqueness thms

① IVP $\frac{dy}{dt} = 4y^{3/4}$

$y(0) = 0$

See that the both of the following functions are solution

① $y(t) \equiv 0$

② $y(t) = t^4$

Verify that $f(t,y) = 4y^{3/4}$ is continuous around $(0,0)$

Hence Peano's thm ensures existence of solution.

Also verify that $f(t,y) = 4y^{3/4}$ is not Lipschitz around $(0,0)$

Thus Picard's thm is not applicable.

Remember!!

Since Assumptions of Picard's thm are not satisfied around $(0,0)$ so the solution is not unique

~~NOT~~ a CORRECT CLAIM

Think!! sufficient Conditions!

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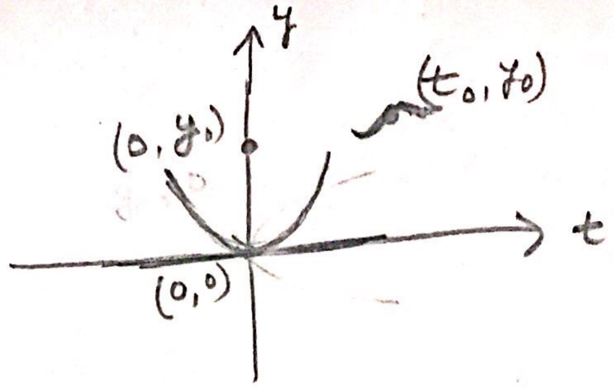
(2)

IVP $\begin{cases} \dfrac{dy}{dt} = \dfrac{2y}{t} \\ y(t_0) = y_0 \end{cases}$ $\longrightarrow$



If $t_0 \neq 0$ then $f(t,y) = \dfrac{2y}{t}$ is continuous around (t_0, y_0)

moreover $\left|\dfrac{\partial f}{\partial y}\right| = \left|\dfrac{2}{t}\right|$ is bdd.

Thus, we have unique solution.

Subcase-1

IVP $\begin{cases} \dfrac{dy}{dt} = \dfrac{2y}{t} \\ y(0) = 0 \end{cases}$

Check: each of the following is a solution

(i) $y(t) \equiv 0$

(ii) $y(t) = t^2$

Thus in this case, we have more than one solutions. But $f(t,y) = \dfrac{2y}{t}$ is not defined at $(0,0)$ and $\lim_{(t,y)\to(0,0)} f(t,y)$ does not exist [why; take two paths $y = t$ and $y = t^2$] $\longrightarrow$ $\boxed{\text{MA101 ?}}$

Hence solution exists around $(0,0)$ but peano's hypothesis are not satisfied. This gives clarification that hypothesis in Peano's thm are sufficient only.

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~~③ $\frac{dy}{dt} = \begin{cases} \frac{2y}{t} ; & t > 0 \\ 0 ; & t = 0 \end{cases}$ $y(0) = y_0$ and $y_0 \neq 0$. No solution.~~

Even If we take

IVP: $\frac{dy}{dt} = \begin{cases} \frac{2y}{t} ; & t > 0 \\ 0 ; & t \leq 0 \end{cases}$

$y(0) = 0$

Again see that both of the following functions are solutions around $(0,0)$

① $y(t) \equiv 0$

② $y(t) = t^2$

But function

$f(t,y) = \begin{cases} \frac{2y}{t} ; & t > 0 \\ 0 ; & t \leq 0 \end{cases}$ → This function is now well defined everywhere

is not continuous at $(0,0)$.

$\boxed{\text{MA101}}$

why; check

$\lim\limits_{(t,y) \to (0,0)} f(t,y)$ does not exist → why

along path $\left.\begin{array}{l} y = t \\ y = t^2 \end{array}\right\}$ different values.

$\boxed{\text{Subcase-2}}$

$\frac{dy}{dt} = \frac{2y}{t}$

$y(0) = y_0$; $(y_0 \neq 0)$

→ In this case, we do not have any solution.

see general solution of ODE $y = ct^2$ so Initial data $(0, y_0)$ where $y_0 \neq 0$ is not satisfied

Remember: $\boxed{\text{Lip continuity} \Rightarrow \text{Continuity}}$ $\boxed{\begin{array}{l} A \Rightarrow B \text{ then} \\ \neg B \Rightarrow \neg A \end{array}}$

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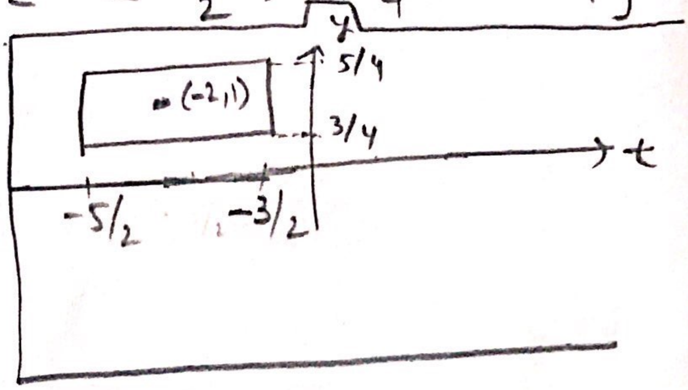
③

$$\frac{dy}{dt} = \frac{e^{(y^2-1)}}{1-t^2y^2}$$

$$y(-2) = 1$$

Take $D = \left\{(t,y) : |t+2| \le \frac{1}{2},\ |y-1| \le \frac{1}{4}\right\}$

$= \left\{(t,y) : -\frac{5}{2} \le t \le -\frac{3}{2},\ \frac{3}{4} \le y \le \frac{5}{4}\right\}$



In the D:

$\left.\begin{array}{l} t^2 \ge \frac{9}{4} \\ y^2 \ge \frac{9}{16} \end{array}\right\} \Rightarrow |1-t^2y^2| \ge \left|1-\frac{81}{64}\right| = \frac{17}{64} > \frac{1}{4}$

and

$\left|e^{(y^2-1)}\right| \le e^{\left(\frac{25}{16}-1\right)} = e^{\frac{9}{16}} < 3$

Thus $|f(t,y)| \le 3 \times 4 = 12$

moreover $\left|\frac{\partial f}{\partial y}\right| = \left|\frac{2y\, e^{(y^2-1)}[1-t^2y^2] + 2t^2y\, e^{(y^2-1)}}{(1-t^2y^2)^2}\right|$

$\le \left|\frac{2y\, e^{(y^2-1)}}{(1-t^2y^2)}\right| + \left|\frac{2t^2y\, e^{(y^2-1)}}{(1-t^2y^2)^2}\right|$

$\le \left|\frac{e^{y^2-1}}{1-t^2y^2}\right| |2y| + \left|\frac{e^{y^2-1}}{(1-t^2y^2)^2}\right| |2t^2y|$

$\le 12 \times 2 \times \frac{5}{4} + 3 \times 16 \times 2 \times \frac{25}{4} \times \frac{5}{4} = 30 + 750 = 780$

$h = \min\left\{a, \frac{b}{L}\right\} = \min\left\{\frac{1}{2}, \frac{1}{4 \times 12}\right\} = \frac{1}{48}$.

Thus Picard's Thm ensures that IVP has a unique solution on interval $\left[-2-\frac{1}{48}, -2+\frac{1}{48}\right]$. But I do't know any basic framework to find it.

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(4)

$$\frac{dy}{dt} = \sqrt{1+e^{-2t}}$$

It is a simple case of separation of variables

$$\boxed{y(t) = \int \sqrt{(1+e^{-2t})}\,dt + c}$$

→ can not express this integral in terms of some elementary functions.

Thus examples (3) and (4) explain the need of numerical methods. All numerical methods give approximate solutions.

Remember → Picard's iterations give approximate solution.

Is Picard's iteration technique a good numerical technique??

Other examples

$$\frac{dy}{dt} = e^{-\frac{t^2}{2}} \Rightarrow y(t) = \int e^{-t^2/2}\,dt + C$$

$$\frac{dy}{dt} = \frac{\sin t}{t} \Rightarrow y(t) = \int \frac{\sin t}{t}\,dt + C$$

(5)

IVP $\begin{cases} \dfrac{dy}{dt} = 1 + y^2 \\ y(0) = 0 \end{cases}$

Take $D = \{(t, y) : |t| \le 100, \ |y| \le 1\}$.

$$\left.\begin{aligned} |f| &= |1+y^2| \le 2 \\ \left|\frac{\partial f}{\partial y}\right| &= |2y| \le 2 \end{aligned}\right\} \Rightarrow h = \min\{a, \tfrac{b}{L}\} = \min\{100, \tfrac{1}{2}\} = \tfrac{1}{2}$$

Thus, Picard's Thm. ensures the existence of unique solution to IVP on interval

$[-\frac{1}{2}, \frac{1}{2}]$.

But, if you solve IVP by separable technique we obtain $y(t) = \tan(t)$ and it valid on $(-\frac{\pi}{2}, \frac{\pi}{2})$.

$\boxed{\text{See}}$ The interval $(-\frac{\pi}{2}, \frac{\pi}{2})$ is bigger that that predicted by the Picard's thm.

Thus, Picard's thm provides some inverval I for unique solution passing th. (t_0, y_0) but the thm. does not stop the problem to have a solution in bigger interval than I

Exercises:

① Find 4 Picard iterations for IVP

$$\frac{dy}{dt} = 2t(1-y) \quad ; \quad y(0) = 2.$$

moreover find exact solution of IVP. Is your solution unique around $(0,2)$. If yes, then verify that Picard iterative approximations converge to the unique solution of IVP.