16. Roots of the equation $z^n = 1$ are $1, \omega, \omega^2, \ldots \omega^{n-1}$ where $\omega = \cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}$

sum of the p th power of these roots

$$1^p + (\omega)^p + (\omega^2)^p + \cdots + (\omega^{n-1})^p$$

<u>Case-I</u> If p is not a multiple of n, $\omega^p \neq 1$

$$1^p + \omega^p + (\omega^p)^2 + \cdots + (\omega^p)^{n-1}$$

$$= \frac{1-(\omega^p)^n}{1-\omega^p} \quad \left[\because 1+z+z^2+\cdots+z^{n-1} = \frac{1-z^n}{1-z}\right]$$

$$= \frac{1-(\omega^n)^p}{1-\omega^p} = \frac{1-(1)^p}{1-\omega^p} = 0$$

<u>Case-II</u> If p is a multiple of n, say $p = nm$ where m is an integer

$$\omega^p = \omega^{nm} = (\omega^n)^m = 1^m = 1$$

$$1 + \omega^p + (\omega^p)^2 + \cdots + (\omega^p)^{n-1}$$

$$= 1 + 1 + 1 + \cdots + 1 \quad (\text{upto } n \text{ terms})$$

$$= n$$

17. $z^n - 1 = (z-1)(z-z_1)\cdots(z-z_{n-1}) = (z-1)\prod_{j=1}^{n-1}(z-z_j)$

again $z^n - 1 = (z-1)(1+z+z^2+\cdots+z^{n-1}) = (z-1)\sum_{j=0}^{n-1} z^j$

Let $f(z) = \prod_{j=1}^{n-1}(z-z_j)$, $g(z) = \sum_{j=0}^{n-1} z^j$

$z \neq 1$ then $f(z) = g(z)$.

Since both $f(z)$ and $g(z)$ are continuous then

$$f(1) = g(1)$$

i.e $(1-z_1)(1-z_2)\cdots(1-z_{n-1}) = 1+1+\cdots+1 = n$

(18) Consider the poly. $f(z) = (1-z)^n - 1$

Now, ω is a root of $f(z)$ iff $(1-\omega)^n = 1$

i.e $1-\omega$ is an nth root of unity.

$\therefore$ $f(z)$ has exactly n roots $z_0, z_1, \dots, z_{n-1}$ and

$$1 - z_k = e^{2\pi k i/n}$$

consider $\alpha_k = \frac{\pi k}{n}$

We have

$$z_k = 1 - e^{2i\alpha_k} = e^{i\alpha_k}\left(e^{-i\alpha_k} - e^{i\alpha_k}\right)$$

$$= -2i\, e^{i\alpha_k} \frac{e^{i\alpha_k} - e^{i\alpha_k}}{2i}$$

$$= -2i\, e^{i\alpha_k} \sin\alpha_k$$

Productof the non zero roots of $f(z)$ is

$$z_1 z_2 z_3 \cdots z_{n-1} = \left(-2i e^{\alpha_1 i} \sin\alpha_1\right)\left(-2i e^{\alpha_2 i} \sin\alpha_2\right) \cdots \left(-2i e^{\alpha_{n-1} i} \sin\alpha_{n-1}\right)$$

$$= (-1)^{n-1} 2^{n-1} i^{n-1} e^{i(\alpha_1+\alpha_2+\cdots+\alpha_{n-1})} \sin\alpha_1 \sin\alpha_2 \cdots \sin\alpha_{n-1}$$

We have

$$\alpha_1 + \alpha_2 + \cdots + \alpha_{n-1} = \frac{\pi}{n} + \frac{2\pi}{n} + \cdots + \frac{(n-1)\pi}{n}$$

$$= \frac{\pi}{n}(1+2+\cdots+n-1) = \frac{\pi}{n}\frac{(n-1)n}{2} = \frac{(n-1)\pi}{2}$$

$$e^{i(\alpha_1+\alpha_2+\cdots+\alpha_{n-1})} = e^{\frac{\pi}{2} i (n-1)} = \left(e^{\frac{\pi}{2} i}\right)^{n-1} = i^{n-1}$$

$$(-1)^{n-1} 2^{n-1} i^{n-1} e^{i(\alpha_1+\cdots+\alpha_{n-1})} = (-1)^{n-1} 2^{n-1} i^{n-1} i^{n-1}$$

$$= (-1)^{n-1} 2^{n-1} (-1)^{n-1}$$

$$= 2^{n-1}$$

and

$z_1 \cdot z_2 \cdots z_{n-1} = 2^{n-1} (\sin\alpha_1)(\sin\alpha_2) \cdots (\sin\alpha_{n-1})$

$= 2^{n-1} \left(\sin\frac{\pi}{n}\right) \sin\left(\frac{2\pi}{n}\right) \cdots \left(\sin\frac{(n-1)\pi}{n}\right)$ — (1)

$f(z) = (1-z)^n - 1$

$= \left(1^n - \binom{n}{1} 1^{n-1} z + \binom{n}{2} 1^{n-2} z^2 - \cdots + (-1)^n z^n\right) - 1$

$= -nz + \frac{n(n-1)}{2} z^2 - \cdots + (-1)^n z^n$ — (2)

Since $z_0, z_1, \ldots, z_{n-1}$ are the roots of $f(z)$, we have

$f(z) = a(z-z_0)(z-z_1) \cdots (z-z_{n-1})$

where a is the leading coefficient of $f(z)$.

Here $a = (-1)^n$ and $z_0 = 0$

$f(z) = (-1)^n z (z-z_1) \cdots (z-z_{n-1})$

$= (-1)^n z \left(z^{n-1} - (z_1+z_2+\cdots+z_{n-1}) z^{n-2} + \cdots + (-1)^{n-1} (z_1 z_2 \cdots z_{n-1})\right)$

$= - (z_1 z_2 \cdots z_{n-1}) z + \cdots + (-1)^n z^n$ — (3)

from (2) and (3)

$z_1 z_2 \cdots z_{n-1} = n$ — (4)

(1) and (4) $\Rightarrow$

$\left(\sin\frac{\pi}{n}\right)\left(\sin\left(\frac{2\pi}{n}\right)\right) \cdots \sin\left(\frac{(n-1)\pi}{n}\right) = \frac{n}{2^{n-1}}$

(proved)

(10) $z^{2n}-1 = \prod_{k=0}^{2n-1}\left(z-\left(\cos\frac{2k\pi}{2n}+i\sin\frac{2k\pi}{2n}\right)\right)$

$= (z-1)\prod_{k=1}^{n-1}\left(z-\left(\cos\frac{k\pi}{n}+i\sin\frac{k\pi}{n}\right)\right)$
$(z+1)\prod_{k=n+1}^{2n-1}\left(z-\left(\cos\frac{k\pi}{n}+i\sin\frac{k\pi}{n}\right)\right)$

$= (z^2-1)\prod_{k=1}^{n-1}\left(z-\left(\cos\frac{k\pi}{n}+i\sin\frac{k\pi}{n}\right)\right)\left(z-\left(\cos\frac{k\pi}{n}-i\sin\frac{k\pi}{n}\right)\right)$

$= (z^2-1)\prod_{k=1}^{n-1}\left[z^2-2z\cos\frac{k\pi}{n}+1\right]$

[as. $\cos\frac{k\pi}{n}+i\sin\frac{k\pi}{n}$ $\quad n+1 \le k \le 2n-1$]

let $k = 2n-j \quad 1 \le j \le n-1$

$\cos\frac{(2n-j)\pi}{n}+i\sin\frac{(2n-j)\pi}{n} = \left(\cos\frac{j\pi}{n}-i\sin\frac{j\pi}{n}\right)$

$\prod_{k=1}^{n-1}\left[z^2-2z\cos\frac{k\pi}{n}+1\right] = 1+z^2+(z^2)^2+\dots+(z^2)^n$

$z=1 \quad \prod_{k=1}^{n-1}\left(2-2\cos\frac{k\pi}{n}\right) = n$

$\Rightarrow 2^{n-1}\prod_{k=1}^{n-1}\left(1-\cos\frac{k\pi}{n}\right) = n$

$\Rightarrow 2^{n-1}\cdot 2^{n-1}\prod_{k=1}^{n-1}\sin^2\frac{k\pi}{2n} = n$

$\Rightarrow \prod_{k=1}^{n-1}\sin\frac{k\pi}{2n} = \frac{\sqrt{n}}{2^{n-1}}$

$\Rightarrow (\sin\pi/2n)\cdot\sin\left(\frac{2\pi}{2n}\right)\dots\sin\left(\frac{\pi(n-1)}{2n}\right) = \frac{\sqrt{n}}{2^{n-1}}$ (proved)