Question No. 01

Fact: let $f(z) = u(x,y) + iv(x,y)$, $z_0 = x_0 + iy_0$ and $w_0 = p + iq$ Then

$$\lim_{z \to z_0} f(z) = w \iff \lim_{(x,y)\to(x_0,y_0)} u(x,y) = p \text{ and } \lim_{(x,y)\to(x_0,y_0)} v(x,y) = q$$

(i) $u(x,y) = \dfrac{xy}{x^2+y^2}$ $\quad v(x,y) = 2xy$

$$\lim_{(x_0,y_0)\to(0,0)} u(x,y) = \lim_{(x_0,y_0)\to(0,0)} \frac{xy}{x^2+y^2}$$

taking path along $y = mx$

$$\Rightarrow \lim_{(x_0,y_0)\to(0,0)} \frac{xy}{(x^2+y^2)} = \lim_{x_0\to 0} \frac{x(mx)}{x^2+m^2x^2} = \frac{m}{1+m^2}$$

which depends upon m so, $\lim_{(x,y)\to(0,0)} u(x,y)$ D.N.E

Hence, $\lim_{z\to 0} f(z)$ D.N.E

(ii) $u(x,y) = \dfrac{xy^3}{x^3+y^3}$ $\quad v(x,y) = \dfrac{x^8}{y^2+1}$

$$\lim_{(x,y)\to(0,0)} \frac{xy^3}{x^3+y^3} = \lim_{x\to 0} \frac{x\,m^3x^3}{x^3+m^3(x^3)} = \lim_{x\to 0} \frac{m^3x}{1+m^3} = 0$$

and idependent ab path so, limit exist.

Similliarly

$$\lim_{(x,y)\to(0,0)} \frac{x^8}{y^2+1} = 0$$

so, $\lim_{z\to 0} f(z) = 0 + i0 = 0$

(iii)
$$\lim_{z\to i} \frac{3z^4-2z^3+8z^2-2z+5}{z-i}$$

$$\Rightarrow \lim_{z\to i} \frac{\cancel{(z-i)}\left((3z^3+(-2+3i)z^2+(5-2i)z+5i\right)}{\cancel{(z-i)}}$$

$$\Rightarrow -3i+2-3i+5i+2-5 = \underline{-1-i} \text{ Ans}$$

(iv)
$$\lim_{z\to 2e^{\pi i/3}} \frac{z^3+8}{z^4+4z^2+16} \left(\frac{0}{0}\right) \quad z_0 = 2e^{i\pi/3} = 2\left(\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)$$

$$\Rightarrow \boxed{1+i\sqrt{3}}$$

$$\lim_{z\to 2e^{i\pi/3}} \frac{(z+2)\cancel{(z^2-2z+4)}}{\cancel{(z^2-2z+4)}(z^2+2z+4)} \quad z_0^3 = 8e^{i\pi} \Rightarrow 8(\cos(\pi)+i\sin\pi) = -8$$

$$\Rightarrow \frac{2e^{i\pi/3}+2}{4e^{i2\pi/3}+2\cdot2e^{i\pi/3}+4} \quad e^{i2\pi/3} = \cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3} = -\frac{1}{2}+i\frac{\sqrt{3}}{2}$$

$$\Rightarrow \frac{1+\sqrt{3}i+2}{4\left(-\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)+2+2\sqrt{3}i+4}$$

$$\Rightarrow \frac{3+\sqrt{3}i}{-2+i2\sqrt{3}+2+2\sqrt{3}i+4} = \frac{3+\sqrt{3}i}{4+4\sqrt{3}i} \text{ Ans}$$

(v)
$$\lim_{z\to 1+i} \frac{z-\bar{z}}{z+\bar{z}} = \frac{(1+i)-(1-i)}{(1+i)+(1-i)} \Rightarrow \frac{2i}{2} = i \text{ Ans}$$

(vi)
$$\lim_{z\to (1-i)} \left(|z|^2 - i|\bar{z}|\right) = 2-i(1+i) = 2-i+1$$
$$= 3-i \text{ Ans}$$

given $\varepsilon>0$, $\exists\ \delta>0$

(2) $\lim_{z\to z_0} f(z) = l$ such that $|f(z)-l| < \varepsilon \ \forall\ |z-z_0|<\delta$

~~given~~ $\varepsilon>0$, $\boxed{\cancel{||f(z)|-|l||<\varepsilon}}$

$\Rightarrow \quad |z-z_0|<\delta$

$\Rightarrow \quad |f(z)-l| < \varepsilon$

But $\quad \big||z_1|-|z_2|\big| \le |z_1-z_2|$

$\Rightarrow ||f(z)| - |l|| \le |f(z)-l| < \varepsilon$

$\Rightarrow \big||f(z)|-|l|\big| < \varepsilon$

$\Rightarrow \forall\ \varepsilon>0,\ \exists\ \delta>0$ s.t

$|z-z_0|<\delta \Rightarrow \big||f(z)|-|l|\big|<\varepsilon$

$\Rightarrow \lim_{z\to z_0} |f(z)| = |f|$

(3) $f(z) = \dfrac{Re(z)}{|z|} = \begin{cases} \dfrac{x}{\sqrt{x^2+y^2}}, & z\neq 0 \text{ i.e } (x,y)\neq(0,0) \\ 0 & (x,y)=0 \end{cases}$

$\lim_{(x,y)\to(0,0)} f(z) = \lim_{(x,y)\to(0,0)} \dfrac{x}{\sqrt{x^2+y^2}}$ taking along $y=mx$

$\Rightarrow \lim_{x\to 0} \dfrac{\cancel{x}}{\cancel{x}\sqrt{1+m^2}} = \dfrac{1}{\sqrt{1+m^2}}$ Limit D.N.E

$\Rightarrow f(z)$ not continuous at origin.

(8) $\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial u}{\partial \theta}\cdot\frac{\partial \theta}{\partial x}$ (chain rule)

$r = \sqrt{x^2+y^2} \qquad \theta = \tan^{-1}(y/x)$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial r^2}\left(\frac{\partial r}{\partial x}\right)^2 + \frac{\partial u}{\partial r}\left(\frac{\partial^2 r}{\partial x^2}\right) + \frac{\partial u}{\partial \theta}\frac{\partial^2 \theta}{\partial x^2} + \frac{\partial^2 u}{\partial \theta^2}\left(\frac{\partial \theta}{\partial x}\right)^2$$

$$+ \frac{\partial^2 u}{\partial r \partial \theta}\left(\frac{-2xy}{r^3}\right)$$

$$= \frac{\partial^2 u}{\partial r^2}\left(\frac{x^2}{r^2}\right) + \frac{\partial u}{\partial r}\left(y^2/r^3\right) - \left(\frac{2xy}{r^3}\right)\left(\frac{\partial^2 u}{\partial r \partial \theta}\right) + \left(\frac{2xy}{r^4}\frac{\partial u}{\partial \theta}\right)$$

$$+ \frac{\partial^2 u}{\partial \theta^2}\left(\frac{x^2}{r^4}\right)$$

Similiarly,

$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 u}{\partial r^2}\, y^2/r^2 + \frac{\partial u}{\partial r}\frac{x^2}{r^3} + \frac{\partial^2 u}{\partial r \partial \theta}\frac{2xy}{r^3} - \frac{\partial u}{\partial \theta}\frac{2xy}{r^4} + \frac{\partial^2 u}{\partial \theta^2}$$

$$\Rightarrow \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0$$

Using, $\frac{\partial^2 r}{\partial x^2} = y^2/r^3 \qquad \frac{\partial^2 r}{\partial y^2} = x^2/r^3$

$\frac{\partial \theta}{\partial x} = -\frac{y}{r^2} \qquad \frac{\partial \theta}{\partial y} = \frac{x}{r^2}$

$\frac{\partial^2 \theta}{\partial x^2} = \frac{2xy}{r^4} \qquad \frac{\partial^2 \theta}{\partial y^2} = \frac{-2xy}{r^4}$

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(4) $f(z) = \sqrt{z} = \sqrt{|z|}\, e^{i(\operatorname{Arg}(z))/2}$

$= (x^2+y^2)^{1/4} e^{\frac{i}{2}\tan^{-1}(y/x)}$

Let $z_0 = -X$, $X > 0$ then

$$\lim_{z\to z_0} f(z) = \lim_{\substack{x\to -X \\ y\to 0}} f(z)$$

$$= \begin{cases} X^{1/2} e^{i\pi/2} & y>0 \\ X^{1/2} e^{-i\pi/2} & y<0 \end{cases}$$

$$= \begin{cases} iX^{1/2} & y>0 \\ -iX^{1/2} & y<0 \end{cases}$$

This shows that limit does not exists.

(5) (i) $f(z) = z^2$ cont everywhere.

(ii) $f(z) = \cot z = \frac{\cos z}{\sin z}$

Clearly it is continuous everywhere except $z = k\pi$, $k \in \mathbb{Z}$.

(iii) $f(z) = \frac{\tanh(z)}{z^2+1} = \frac{-i\tan(iz)}{(z^2+1)}$

$$f(z) = \frac{-i \sin(iz)}{\{\cos(iz)\}(z+i)(z-i)} = \frac{g(z)}{h(z)}$$

Therefore its continuity will fail at only those points where the denominator is zero.

i.e. $\{\cos(iz)\}(z+i)(z-i) = 0$

$\Rightarrow z = \pm i, \frac{-i(2k+1)\pi}{2}$ #

6 (1) $f(z) = \frac{Re(z)}{z+iz} - 2z^2, \quad z_0 = e^{i\pi/4}$

$= \frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}$

Putting $z = x+iy$ we have

$$f(z) = \frac{x}{x+iy+i(x+iy)} - 2(x+iy)^2$$

$$= \frac{x}{(x+iy)(1+i)} - 2(x+iy)^2$$

$$\therefore \lim_{z\to z_0} f(z) = \lim_{\substack{x\to 1/\sqrt{2} \\ y\to 1/\sqrt{2}}} \frac{x}{(x+iy)(1+i)} - 2(x+iy)^2$$

$$= \frac{1/\sqrt{2}}{(\frac{1}{\sqrt{2}}+i\frac{1}{\sqrt{2}})(1+i)} - 2\left(\frac{1}{\sqrt{2}}+i\frac{1}{\sqrt{2}}\right)^2$$

$$= \frac{1/\sqrt{2}}{\frac{1}{\sqrt{2}}(1+i)^2} - (1+i)^2$$

$$= \frac{1}{(1+i)^2} - (1+i)^2 \quad \#$$

(2) $f(z) = \begin{cases} \dfrac{z^3-1}{z^2+z+1}, & |z| \neq 1 \\ \dfrac{-1+i\sqrt{3}}{2}, & |z| = 1 \end{cases}$ $\quad z_0 = \dfrac{1+i\sqrt{3}}{2}$

$$\lim_{z \to z_0} f(z) = \lim_{z \to z_0} \frac{z^3-1}{z^2+z+1} = \lim_{z \to z_0} (z-1)$$

$$= z_0 - 1 = \frac{1+i\sqrt{3}}{2} - 1 = \frac{-1+i\sqrt{3}}{2} = f(z_0)$$

#

(9.) $f(x,y) = \begin{cases} \dfrac{x^3(1+i) - y^3(1-i)}{x^2+y^2}, & z \neq 0 \\ = 0 & , z = 0 \end{cases}$

$f(x,y) = u(x,y) + i v(x,y)$ we get

$u = u(x,y) = \dfrac{x^3 - y^3}{x^2+y^2}$, $v = v(x,y) = \dfrac{x^3+y^3}{x^2+y^2}$

for $z \neq 0$, since u and v are rational fns with non-zero denominators. Hence u and v are cont. when $z \neq 0$

Tacking $x = r\cos\theta$, $y = r\sin\theta$

$u = r(\cos^3\theta - \sin^3\theta)$, $v = r(\cos^3\theta + \sin^3\theta)$

$z \to 0$ implies $r \to 0$

limiting value of u and v is equal to 0. Hence $f(x,y)$ is cont. for all values of (x,y)

Now,

$$u_x = \lim_{x\to 0} \frac{u(x,0) - u(0,0)}{x} = \lim_{x\to 0} \frac{x-0}{x} = 1$$

$$u_y = \lim_{y\to 0} \frac{u(0,y) - u(0,0)}{y} = \lim_{y\to 0} \frac{-y-0}{y} = -1$$

$$v_x = \lim_{x\to 0} \frac{v(x,0) - v(0,0)}{x} = \lim_{x\to 0} \frac{x-0}{x} = 1$$

$$v_y = \lim_{y\to 0} \frac{v(0,y) - v(0,0)}{y} = \lim_{y\to 0} \frac{y-0}{y} = 1$$

Hence $u_x = v_y$, $u_y = -v_x$ and the C-R equations are satisfied.

But, we have

$$f'(0) = \lim_{\Delta z\to 0} \frac{f(0+\Delta z) - f(0)}{\Delta z}$$

$$= \lim_{\Delta z\to 0} \frac{(\Delta x)^3 - (\Delta y)^3 + i\left((\Delta x)^3 + (\Delta y)^3\right)}{(\Delta x)^2 + (\Delta y)^2} \cdot \frac{1}{(\Delta x + i\Delta y)}$$

If $\Delta z \to 0$ along x axis $\Delta y \to 0$ then $f'(0) = 1+i$

if $\Delta z \to 0$ along the curve $y = x$ then $f'(0) = (1+i)/2$

Since the limits are different, $f'(0)$ does not exist.

(10) $$f(z) = \frac{|z^2 - \bar{z}^2|^{1/2}}{2} = \frac{|(x+iy)^2 - (x-iy)^2|^{1/2}}{2}$$

$$= \frac{|(x^2-y^2+2ixy) - (x^2-2ixy-y^2)|^{1/2}}{2}$$

$$= \frac{|4ixy|^{1/2}}{2}$$

$$= \frac{2\sqrt{xy}}{2} = \sqrt{xy}$$

$f(z) = u(x,y) + iv(x,y)$

$u(x,y) = \sqrt{xy}$, $v(x,y) = 0$

$$u_x(0,0) = \lim_{x \to 0} \frac{u(x,0) - u(0,0)}{x} = 0$$

$$u_y(0,0) = \lim_{y \to 0} \frac{u(0,y) - u(0,0)}{y} = 0$$

$v_x(0,0) = 0$

$v_y(0,0) = 0$

$u_x = v_y$, $u_y = -v_x$ C-R equations satisfied

$$f'(0) = \lim_{\Delta z \to 0} \frac{f(\Delta z) - f(0)}{\Delta z} = \lim_{\Delta z \to 0} \frac{\sqrt{\Delta x \Delta y}}{\Delta x + i\Delta y}$$

along x axis $f'(0) = 0$

along $y = x$ curve $f'(0) = \frac{1}{1+i}$

$f'(0)$ does not exist.

(11) $f(z) = \begin{cases} \dfrac{\bar{z}^2}{z}, & z \neq 0 \\ 0, & z = 0 \end{cases}$

$z \neq 0 \quad f(z) = \dfrac{\bar{z}^3}{z\bar{z}} = \dfrac{(x-iy)^3}{x^2+y^2} = \dfrac{x^3-3xy^2}{x^2+y^2} + i\,\dfrac{y^3-3x^2y}{x^2+y^2}$

$u_x(0,0) = \lim\limits_{x \to 0} \dfrac{u(x,0)-u(0,0)}{x} = \lim\limits_{x \to 0} \dfrac{x^3}{x^3} = 1$

$u_y(0,0) = \lim\limits_{y \to 0} \dfrac{u(0,y)-u(0,0)}{y} = 0$

$v_x(0,0) = \lim\limits_{x \to 0} \dfrac{v(x,0)-v(0,0)}{x} = 0$

$v_y(0,0) = \lim\limits_{y \to 0} \dfrac{v(0,y)-v(0,0)}{y} = 1$

$u_x = v_y, \quad v_x = -u_y$, C-R equations are satisfied at origin

$f'(0) = \lim\limits_{\Delta z \to 0} \dfrac{f(\Delta z)-f(0)}{\Delta z} = \lim\limits_{\Delta z \to 0} \dfrac{(\overline{\Delta z})^2}{(\Delta z)^2}$

$\Delta z \to 0$ along x-axis $\Delta y \to 0$ then $f'(0) = 1$

$\Delta z \to 0$ along curve $y = x$ then $f'(0) = \dfrac{(1-i)^2}{(1+i)^2}$

$f'(0)$ does not exist.

(12) $f(z) = xy + yi$

Since,

$$f(z) = u(x,y) + iv(x,y)$$

here

$$u(x,y) = xy \quad \& \quad v(x,y) = y$$

$u(x,y)$ & $v(x,y)$ are polynomial functions. So, these are continuous everywhere and hence $f(z)$ is continuous everywhere.

C-R eq^{ns} -

$$u_x = v_y \quad \& \quad u_y = -v_x$$

We note that,

$$u_x = y, \quad u_y = x$$
$$v_x = 0, \quad v_y = 1$$

we see that

$$u_x \neq v_y \quad , \text{except for } y = 1$$
$$\& \text{ also } \quad u_y \neq -v_x \quad \text{except for } x = 0$$

Hence, CR eq^{ns} are not satisfied anywhere in the z-plane except at the point $x = 0$, $y = 1$, i.e, $z = i$

Therefore, $f(z)$ is not analytic.

(13) $f(z) = x^3 + i(1-y)^3$

$$u(x,y) = x^3, \quad v(x,y) = (1-y)^3 = 1 - y^3 - 3y(1-y)$$
$$= 1 - y^3 - 3y + 3y^2$$

$$u_x = 3x^2 \qquad v_x = 0$$
$$u_y = 0 \qquad v_y = -3y^2 - 3 + 6y$$

We note that

$$u_x \neq v_y \quad \& \quad u_y = -v_x$$

$u_x = v_y$ only when $y = 1$

so, for $z = i$ (sufficient condition for diff:
(i) CR eq^ns hold
(ii) u_x, u_y, v_x, v_y exits and cont.)

$f(z) = u_x + i v_x$
$= 3x^2$

(14) $f(z) = z \cdot \bar{z} = |z|^2$

$z = x + iy$

$\therefore f(z) = x^2 + y^2$

Here, $u(x,y) = x^2 + y^2$, $v(x,y) = 0$

$u(x,y)$ is a polynomial function & $v(x,y) = 0$
i.e., $u(x,y)$ is continuous everywhere and hence $f(z)$ is continuous everywhere.

Again,

$$u_x = 2x, \quad v_x = 0$$
$$u_y = 2y, \quad v_y = 0$$

Note that,

$$u_x \neq v_y \ \& \ u_y \neq -v_x$$

except for origin, ie, $z = 0$
$(x,y) = (0,0)$.

Hence, CR eq^ns are not satisfied anywhere except at origin. Therefore, $f(z)$ is not analytic.

(15) (i) $f(z) = |z| = \sqrt{x^2+y^2}$

Here,

$u(x,y) = \sqrt{x^2+y^2}$

$v(x,y) = 0$

&

$u_x \neq v_y$ and $u_y \neq -v_x$

Hence, CR eq^{ns} are not satisfied anywhere.

So, nowhere differentiable.

(ii) $f(z) = Re(z)$

$u(x,y) = x$, $v(x,y) = 0$

Here,

$u_x = 1$, $v_y = v_x = 0$

$u_x \neq v_y$

Therefore, CR eq^{ns} are not satisfied anywhere.

Hence, Nowhere diff.

(iii) $f(z) = Im(z)$

Here, $u(x,y) = 0$, $v(x,y) = y$

$u_y = u_x = 0$, $v_y = 1$

$v_y \neq u_x$

CR eq^{ns} are not satisfied.

so nowhere differentiable.

(iv) $f(z) = \bar{z} = x - iy$

$u(x,y) = x$, $v(x,y) = -y$

$u_x = 1$, $v_y = -1$

$u_y = 0$, $v_x = 0$

$u_x \neq v_y$ & $u_y = v_x$

→ CR eq^{ns} are not satisfied
→ Nowhere diff.

(v) $f(z) = z - \bar{z} = x + iy - x + iy$
$= 2iy$

$u(x,y) = 0, \quad v(x,y) = 2y$

Here, $u_x = u_y = 0, \quad v_y = 2, \quad v_x = 0$

$u_x \neq v_y$

→ nowhere diff.

* other parts can be done similarly.

(16) 2. $U = 4xy - x^3 + 3xy^2$

$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} = 0 \quad \text{(Laplace eq}^n\text{)} \quad —①$$

$U_x = 4y - 3x^2 + 3y^2$

$U_{xx} = -6x$

$U_y = 4x + 6xy$

$U_{yy} = 6x$

Since, $U_{xx} + U_{yy} = 0$ & $U(x,y)$ has continuous partial derivatives of the first & second order and hence $U(x,y)$ is a harmonic function.

Now, we have to find the harmonic conjugate of $U(x,y)$, i.e., $V(x,y)$.

CR eq^{ns} are

$$u_x = v_y \quad \& \quad u_y = -v_x$$

$$v_x = -u_y = -4x - 6xy$$

Partially integrating with respect to x gives,

$$v(x,y) = -2x^2 - 3x^2y + g(y)$$

Now, partially differentiate this expression and using CR eq^{ns} we have,

$$v_y = -3x^2 + g'(y) = u_x = 4y - 3x^2 + 3y^2$$

Hence,

$$g'(y) = 4y + 3y^2$$

$$\Rightarrow \quad g(y) = 4\frac{y^2}{2} + y^3 + C$$

$$= 2y^2 + y^3 + C$$

where C is a constt.

Therefore,

$$v(x,y) = -2x^2 - 3x^2y + 2y^2 + y^3 + C$$

(17) <u>Proof</u> : Let $f(z) = u+iv$ be an analytic function with constant modulus.

Since $f(z)$ is analytic

$\Rightarrow$ CR Condition is Satisfied

i.e $u_x = v_y$, $u_y = -v_x$ — (1)

Now, $|f(z)| = C$

$u^2+v^2 = C^2$

diff w.r to x,

$$2u\frac{\partial u}{\partial x} + 2v\frac{\partial v}{\partial x} = 0$$

$$\Rightarrow u\frac{\partial u}{\partial x} + v\frac{\partial v}{\partial x} = 0 \longrightarrow (2)$$

Again diff w.r to y.

$$\Rightarrow u\frac{\partial u}{\partial y} + v\frac{\partial v}{\partial y} = 0 \quad — (3)$$

From (3)

$$u\cdot\left(-\frac{\partial v}{\partial x}\right) + v\frac{\partial v}{\partial y} = 0 \longrightarrow (4)$$

$u \times (2) + v \times (4)$ gives.

$$u^2\frac{\partial u}{\partial x} + \cancel{uv\frac{\partial v}{\partial x}} - \cancel{uv\cdot\frac{\partial v}{\partial x}} + v^2\frac{\partial v}{\partial y} = 0$$

$$\Rightarrow u^2 v_y + v^2 v_y = 0 \quad (\text{as } u_x = v_y)$$

$$\Rightarrow (u^2+v^2)(v_y) = 0$$

as $u^2+v^2 = c^2 \neq 0$, $\Rightarrow v_y = 0 \Rightarrow v = k$

Similarly, $v \times (2) + u \times (4)$ gives, $u = k$

$u(x,y)$ and $v(x,y)$ are harmonic.

$\Rightarrow \quad u_{xx} + u_{yy} = 0 \quad \& \quad v_{xx} + v_{yy} = 0$

To show $f(z) = (u_y - v_x) + i(u_x + v_y)$ is analytic in D

Let $g = u_y - v_x$ & $h = u_x + v_y$

To show that $f(z)$ is analytic we show g, h satisfy C-R Eqn

$g_x = u_{yx} - v_{xx} \quad h_y = u_{xy} + v_{yy}$

& $g_y = u_{yy} - v_{xy} \quad h_x = u_{xx} + v_{yx}$

$g_x = u_{yx} - v_{xx} = u_{yx} + v_{yy} = h_y$

$g_y = u_{yy} - v_{xy} = -(u_{xx} + v_{yx}) = -h_x$

Hence, $g_x = h_y, \; g_y = -h_x$ proved

(19) $f(z) = u(x,y) + iv(x,y)$ is analytic at z.

$\Rightarrow$ C-R Eqn Satisfied

$u_x = v_y, \; u_y = -v_x$

Now, To prove
$f(z) = v(x,y) - iu(x,y)$ is analytic at z
$f(z) = h + ig$
where $h = v(x,y)$, $g = -u(x,y)$

$h_x = v_x \qquad g_x = -u_x$
$h_y = v_y \qquad g_y = -u_y$

$h_x = v_x = -u_y = g_y \Rightarrow h_x = g_y$

$h_y = v_y = u_x = -g_x \Rightarrow h_y = -g_x$

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