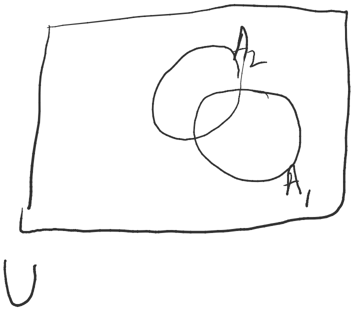
Basic Counting 2

Thursday, January 7, 2021 9:01 AM

Principle of Inclusion & Exclusion

Let $A_1\ A_2 \cdots A_n$ be n subsets of U. We want to calculate no of elements that does not belong to any A_i



$$|U| - \underbrace{|A_1| - |A_2| + |A_1 \cap A_2|}$$

$$\boxed{\left|\bigcap_{i=1}^{n} \bar{A}_i\right| = |U| - \sum_{i} |A_i| + \sum_{i,j} |A_i \cap A_j| - \sum_{ijk} |A_i \cap A_j \cap A_k| + \cdots + (-1)^n |A_1 \cap A_2 \cap \cdots \cap A_n|}$$

PIE

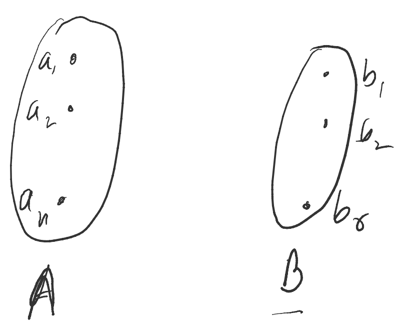
$f: A \longrightarrow B$

$\{a_1\ a_2 \cdots a_n\}$ $\{b_1 b_2 \cdots b_r\}$

$\#$ fns from A to B $= \underbrace{r \cdot r \cdots r}_{n} = r^n$

$\#$ ~~fn~~ one-one fn $A \to B = r(r-1) \cdots (r-n+1) = \frac{r!}{r-n!}$

$\#$ onto fns $=$ Use PIE.



$A_i =$ set of all fns $A \to B$ s.t b_i does not belong to image of f, $1 \le i \le r$

A_i = set of all fns $A \to B$ s.t. b_i does not belong to range of f, ...

U = set of all fns $= r^n$

$|A_i| = (r-1)^n \quad \forall i$

$|A_i \cap A_j| = (r-2)^n \quad \forall i,j$

$\vdots$

$|\bigcap_{i=1}^{n-1} A_i| = 1^n$

$|\bigcap_{i=1}^{n} A_i| = 0$

$\#$ onto fns $A \to B = r^n - \binom{r}{1}(r-1)^n + \binom{r}{2}(r-2)^n - \cdots + (-1)^{r-1}\binom{r}{r-1}(1)^n$

$$= \sum_{k=0}^{r} (-1)^k \binom{r}{k} (r-k)^n$$

$\frac{1}{r!} \sum_{k=0}^{r} (-1)^k \binom{r}{k} (r-k)^n \longrightarrow$ Stirling no of 2nd kind.

Prob <u>Hatcheck Prob</u>

How may ways can a hatcheck girl hand back n hats to n men one to each, with no man getting back his own hat?

An arrangement of first n positive ints in which no number in its correct position is called "<u>Derangement</u>"

$\{1,2,3,4\} \qquad (3\,2\,1\,4) \times \qquad (3\,1\,4\,2) \checkmark$

D ... of $\{1,2,\ldots n\}$

D_n = no of derangements of $\{1, 2, \cdots n\}$

U = set of all permutations $= n!$

A_i = set of all permutations s.t. 'i' is in its correct position.

$|A_i| = (n-1)! \quad |A_i \cap A_j| = (n-2)! \quad \cdots\cdots \quad \left|\bigcap_{i=1}^{n} A_i\right| = (n-n)! = 1$

$$D_n = n!\left[1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots\cdots + (-1)^n \frac{1}{n!}\right]$$

Ex: Try to list down set of all derangements $\{1, 2, 3, 4, 5\}$

Distributing balls into boxes

n balls and r boxes

identical/distinct   identical/distinct

none of the boxes empty

ball \ box	distinct	identical
dist	# onto fns	$\frac{1}{r!}$ * # onto fns $= \left\{ {n \atop r} \right\}$
identical	$\binom{n-1}{r-1}$	$\Pi(n, r)$

box can be empty

ball \ box	dist	identical
dist	r^n	$\sum_{k=1}^{r} \left\{ {n \atop k} \right\}$
identical	$\binom{n+r-1}{n}$	Ex

1 2 3 4 5

0 0 0 0 0

1 2 3	4	5
1st	2nd	3rd
2nd	1st	3rd

permute the boxes

~~[illegible]~~

3A 2B

$$\frac{5!}{3!\,2!}$$