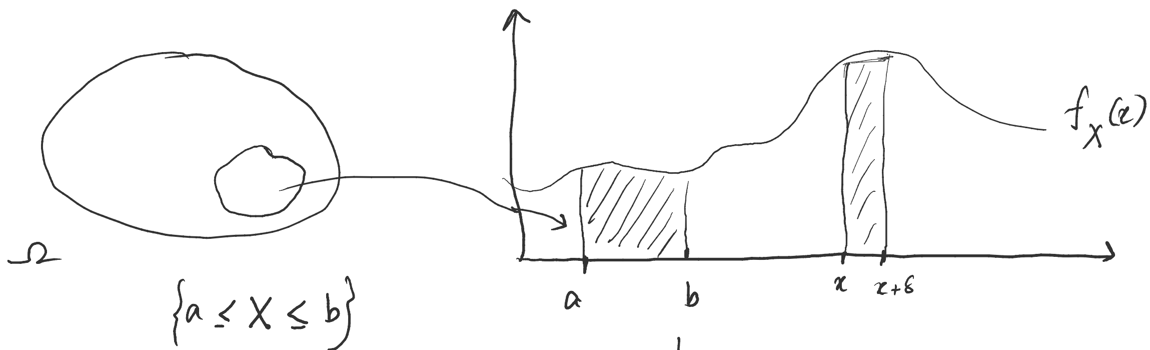
Probability Density and Cumulative Distribution

Thursday, February 11, 2021 8:47 AM

$X \rightarrow$ continuous random variable if X takes any value from $\mathbb{R}$

$X : \Omega \rightarrow \mathbb{R}$



$$P(a \le X \le b) = \int_a^b f_X(x)\,dx \quad \left(\sum_{x \in [a,b]} p_X(x) \right)$$

X can be described in terms of nonnegative fn f_X, $\rightarrow$ (probability density fn)

$$\boxed{P(X = a) = \int_a^a f_X(x)\,dx = 0}$$

$$P(a \le X \le b) = P(a < X < b)$$

(*) ① $f_X(x) \geq 0$ ② $\int_{-\infty}^{\infty} f_X(x)\,dx = 1$

Is it always $f_X(x) \le 1$?

$$P(x \le X \le x+\delta) = \int_x^{x+\delta} f_X(x)\,dx \approx f_X(x)\cdot\delta$$

$$\frac{P(x \le X \le x+\delta)}{\delta} \approx f_X(x)$$

(probability mass per unit interval)

$$P(X \in B) = \int_B f_X(x)\,dx$$ for 'nice' sets B.

(union of intervals)

Expectations

$$E[X] = \int_{-\infty}^{\infty} x \cdot f_X(x)\,dx$$

$Y = g(X)$ $$E[g(X)] = \int_{-\infty}^{\infty} g(x) \cdot f_X(x)\,dx$$

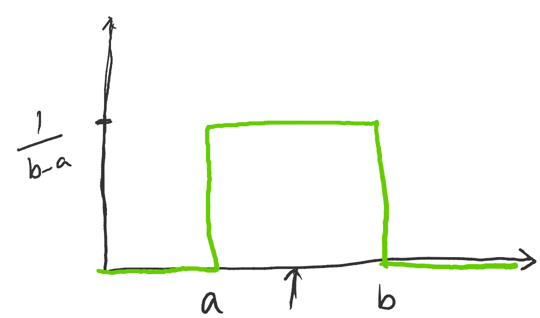
$$E[X^n] = \int_{-\infty}^{\infty} x^n f_X(x)\,dx$$

Variance of X

$$Var(X) = E\left[(X - E[X])^2\right] = \int_{-\infty}^{\infty} (x - E[X])^2 \cdot f_X(x)\,dx$$

$$Var(X) = E[X^2] - (E[X])^2$$

Uniform continuous Ran. vari.



$$f_X(x) = \begin{cases} \frac{1}{b-a} & a \le x \le b \\ 0 & o/w \end{cases}$$

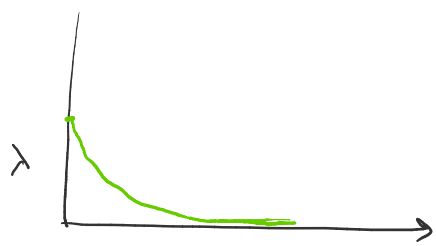
$$E[X] = \int_a^b x f_X(x)\,dx = \frac{b+a}{2}$$

$$Var(X) = \int_a^b \left(x - \frac{a+b}{2}\right)^2 f_X(x)\,dx = \frac{(b-a)^2}{12}$$

$$= E[X^2] - (E[X])^2$$

Exponential Random Variable

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & o/w \end{cases}$$



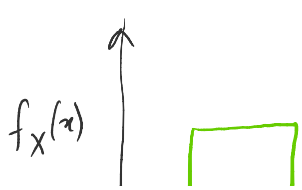
$$\int_{-\infty}^{\infty} f_X(x)\,dx = \int_0^{\infty} \lambda e^{-\lambda x}\,dx = \left[e^{-\lambda x}\right]_0^{\infty} = 1$$

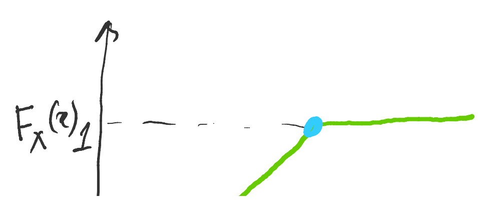
$$P(X \geq a) = \int_a^{\infty} \lambda e^{-\lambda x}\,dx = e^{-\lambda a}$$

$$E[X] = \frac{1}{\lambda}, \quad Var(X) = \frac{1}{\lambda^2}$$

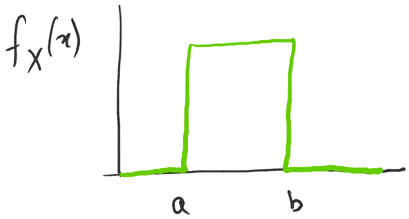
Cumulative Distribution fn. $(F_X(x))$

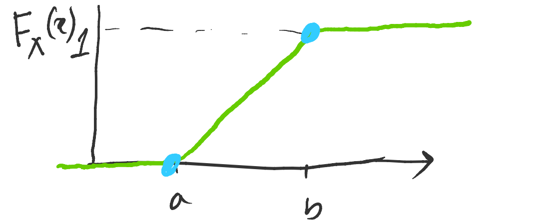
$$F_X(x) = P(X \leq x) = \begin{cases} \sum_{k \leq x} p_X(k) & X \text{ is discrete} \\ \int_{-\infty}^{x} f_X(x)\,dx & X \text{ is continuous.} \end{cases}$$

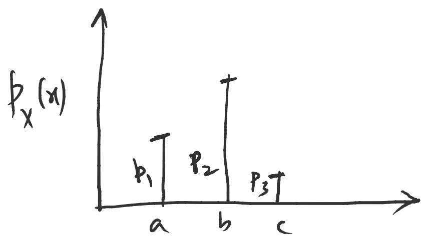


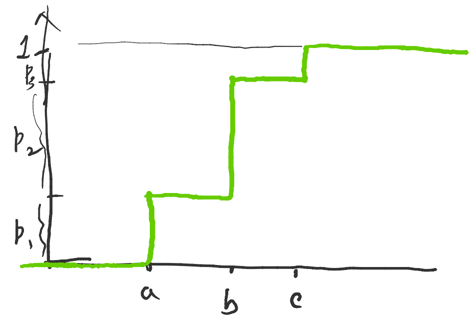


$$\frac{d F_X(x)}{dx} = f_X(x)$$


f_X(x)
a
b


F_X(x)
1
a
b


p_X(x)
p_1
p_2
p_3
a
b
c


1
p_3
p_2
p_1
a
b
c