MA225 MIDSEM ASSIGNMENT

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Question 1: A simple undirected graph $G = (V, \mathcal{E})$ with vertex V and edge set $\mathcal{E}$

Solution: Let us assume that:

The graph is connected and the minimum degree of each/any vertex $v \in V$ is d; where $d \geq 1$

According to Question:

X, Y are random sets, with probability of picking any vertex being p.

(a) Find the expected value of $|X| + |Y|$

→ Picking up a random subset of X by including each $v \in V$ independently with p. Let it be X.

Now;

$$E[|X|] = \sum_{i=1}^{|V|} E[X_i] \quad \left(\begin{array}{l} \text{where } X_i = 1 \text{ (if the vertex is selected)} \\ \quad\quad = 0 \text{ (if the vertex is not selected)} \end{array} \right.$$

$$\Rightarrow E[X_i] = p \text{ ; for each } i$$

$$\Rightarrow E[|X|] = \sum_{i=1}^{|V|} p$$

$$E[|X|] = |V| p$$

Let $|V| = n$

$$\Rightarrow \boxed{E[|X|] = np} \text{ ——— (i)}$$

Now, set Y is such set that $Y \cap X = \phi$ and any neighbour of X is not in Y

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For any vertex v;

$$P(v_i \in Y) = (1-p)(1-p)^{degree(v_i)}$$

$\left[(1-p) \to v_i \text{ not included in } x \right.$

$\left.(1-p)^{degree(v_i)} \to \text{no neighbour of } v_i \text{ is included in } x\right]$

$$\Rightarrow P(v_i \in Y) = (1-p)^{degree(v_i)+1}$$

$$\therefore \boxed{E[|Y|] = \sum_{i=1}^{n} (1-p)^{degree(v_i+1}} \quad \text{--- (2)}$$

$E[|X| + |Y|] = E[|X|] + E[|Y|]$. (using property of expectation of random variable)

$$\Rightarrow \boxed{E[|X|+|Y|] = np + \sum_{i=1}^{n} (1-p)^{degree(v_i)+1}} \longrightarrow (3)$$

$(n = |V|)$

(b) For what value of p, the expected value is minimised?

$\Rightarrow$ As; $\min(degree(v_i)) = d. \ \forall v_i \in V$

$$(1-p)^{degree(v_i+1)} \leq (1-p)^{d+1}$$

from eq. 3.

$$E[|X| + |Y|] = np + \sum_{i=1}^{n} (1-p)^{degree(v_i+1)} \leq np + \sum_{i=1}^{n} (1-p)^{d+1}$$

$$\boxed{E[|X| + |Y|] \leq np + n(1-p)^{d+1}} \longrightarrow (4)$$

So, to minimize $E[|X| + |Y|]$;

we need to minimize $np + n(1-p)^{d+1}$

Differentiating w.r.t. p

$$\frac{d}{dp}\left(np + n(1-p)^{d+1}\right) = 0$$

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$$n + n(d+1)(1-p)^d(-1) = 0$$

$$n(1-(d+1)(1-p)^d) = 0$$

$$1 = (d+1)(1-p)^d \quad [\Rightarrow n \neq 0]$$

$$\frac{1}{d+1} = (1-p)^d$$

$$\boxed{p = 1-\left(\frac{1}{d+1}\right)^{1/d}} \text{ — } (5)$$

So, the value of p, for which the expected value is minimised is $\boxed{p = 1-\left(\frac{1}{d+1}\right)^{1/d}}$ where d = min degree of $v_i \in V$

(c) How can you get an upper bound on domination number using expectation argument?

$\Rightarrow$ By mean value theorem; we know for a function f in range (a,b) ~~for~~ some $c \in (a,b)$ then $\frac{f(b)-f(a)}{b-a} = f'(c)$

Now; let

$f(x) = e^{-x}$ for $(a,b) = (0,k)$

$\Rightarrow$ for some c.

$$\frac{e^{-k}-1}{k} = -e^{-c}$$

we already know; for any c between $[0,k]$ $-e^{-c} \geq -1$

$$\Rightarrow \frac{e^{-k}-1}{k} \geq -1 \quad \Rightarrow \boxed{e^{-k} \geq 1-k} \text{ — } (6)$$

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From equation (4).

$$E[|X'|+|Y|] \leq np + n(1-p)^{d+1}$$

Now substituting equation (5).

$$E[|X|+|Y|] \leq np + ne^{-p(d+1)}$$

To get min,
differentiating wrt p.

$$\frac{d}{dp}\left(np + ne^{-p(d+1)}\right) = 0$$

$$n + n(-(d+1))e^{-p(d+1)} = 0$$

$$\Rightarrow p = \frac{\log(1+d)}{(1+d)}$$

Substituting

$$E[|X|+|Y|] \leq n\frac{\log(1+d)}{(1+d)} + ne^{-\log(1+d)}$$

$$\boxed{E[|X|+|Y|] \leq n\left[\frac{1+\log(1+d)}{1+d}\right]}$$

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Que 2:- Let us say:

K_n = Complete graph of n vertices

$R(k,k)$ = smallest integer, such that any 2 coloring of complete graph is either a red K_k or a blue K_k (a monochromatic complete subgraph)

Now, assuming a complete graph of n vertices

In it, let us take any subgraph (g_i) with k vertices

Now,

as g_i is a complete subgraph, so there are $^kC_2$ edges.

$$P(g_i \text{ is red}) = \frac{1}{2}\times\frac{1}{2}\times\frac{1}{2}\times \dots \ (^kC_2 \text{ terms})$$

$$\text{||ly } P(g_i \text{ is blue}) = \frac{1}{2}\times\frac{1}{2}\times\frac{1}{2}\times \dots \ (^kC_2 \text{ terms})$$

$$\Rightarrow P(g_i \text{ is monochromatic}) = \frac{2}{2^{^kC_2}} = 2^{1-{^kC_2}}$$

According to Question:

There are n vertices $\Rightarrow$ ^nC_k subgraphs with k vertices.

Let us assume that

M = Total number of monochromatic K_k

Expected value of M: $E[M]$

$$E[M] = \sum_{i=1}^{^nC_k} P(g_i)$$

$$= {^nC_k}\, 2^{1-{^kC_2}}$$

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Now, (assuming multiple vertex might get removed more than once
let us assume that we remove 1 vertex from each K_K, then from above we can clearly see that no more K_K can be formed from the remaining no of vertices

$\Rightarrow$ remaining vertices $= n - [{}^{n}C_{K}\, 2^{1-{}^{K}C_{2}}]$

$\Rightarrow$ We will need more vertices than $n - {}^{n}C_{K}\, 2^{1-{}^{K}C_{2}}$, so that a monochromatic K_K can be ensured of any 2 coloring of the graph.

Therefore;

$$\boxed{R(K,K) > n - ({}^{n}C_{2})\, 2^{1-{}^{K}C_{2}}}$$

Hence Proved.

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×—×—×— END OF ASSIGNMENT —×—×—×—×
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