MA-201

MID-SEM ASSIGNMENT

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GROUP: GROUP-10

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Que 1:- $\cos(z) = i$, find values of x and y.

Ans:- We know; $z = x + iy$

$\cos(z) = i$

$\Rightarrow \cos(z) = \dfrac{e^{iz} + e^{-iz}}{2} = i$

$e^{iz} + \dfrac{1}{e^{iz}} = 2i \Rightarrow e^{2iz} - 2ie^{iz} + 1 = 0$

Put $e^{iz} = w$

$w^2 - 2iw + 1 = 0$ (completing the whole square)

$w^2 - 2iw - 1 = -2$

$(w-i)^2 = -2$

$w - i = \pm i\sqrt{2}$

$w = i \pm i\sqrt{2}$

$w = i(1 \pm \sqrt{2})$

Case 1:-

$w = i(1-\sqrt{2})$

$w = (-i)(\sqrt{2}-1)$

$w = e^{i(2k\pi - \frac{\pi}{2})} \cdot (\sqrt{2}-1)$ $(k \in z)$

$e^{iz} = e^{i(2k\pi - \frac{\pi}{2})} \cdot (\sqrt{2}-1)$

$iz = \log(e^{i(2k\pi - \frac{\pi}{2})}) + \log(\sqrt{2}-1)$ [taking $\log_e$ both sides)

$iz = i\pi(2k - \frac{1}{2}) + \log(\sqrt{2}-1)$

$z = \pi(2k - \frac{1}{2}) - i\log(\sqrt{2}-1)$

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Comparing with $z = x+iy$

$x = \pi\left(2k - \frac{1}{2}\right)$; $y = -\log_e(\sqrt{2}-1)$ —— ①

Case 2:-

$w = i(\sqrt{2}+1)$

$e^{iz} = e^{i\left(2k\pi + \frac{\pi}{2}\right)} \cdot (1+\sqrt{2})$, $k \in \mathbb{Z}$

$iz = \log\left(e^{i\left(2k\pi+\frac{\pi}{2}\right)}\right) + \log(1+\sqrt{2})$ (taking $\log_e$ both sides)

$iz = i\pi\left(2k + \frac{1}{2}\right) + \log(1+\sqrt{2})$

$z = \pi\left(2k+\frac{1}{2}\right) - i\log(1+\sqrt{2})$

Comparing with $z = x+iy$

$x = \pi\left(2k+\frac{1}{2}\right)$; $y = -\log_e(\sqrt{2}+1)$ ⟶ ②

Combining both ① and ②

$$\boxed{\begin{array}{l} x = \pi\left(2k - \frac{1}{2}\right) ;\ y = -\log_e(\sqrt{2}-1) \\ \text{and. } x = \pi\left(2k+\frac{1}{2}\right) ;\ y = -\log_e(\sqrt{2}+1) \end{array}}$$

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[signature]

Que 2:- $f(z) = \left(\frac{z}{\bar{z}}\right)^2$. find $\lim_{z \to 0} f(z)$.

Ans:- For finding the limit, we will first check whether limit exsists or not.

For finding limit we will traverse along different paths to make sure that limit is same for every path.

a) if we approach from the imaginary axis (i.e y axis)

$z = x + iy$

$x = 0$

$$\Rightarrow \lim_{z \to 0} f(z) = \lim_{y \to 0} f(z) = \left(\frac{iy}{-iy}\right)^2 = 1$$

b) If we approach along the line $y = mx$

$z = x + iy$.

$$\Rightarrow f(z) = \left(\frac{z}{\bar{z}}\right)^2 = \left(\frac{x + imx}{x - imx}\right)^2$$

$$= \left(\frac{1 + im}{1 - im}\right)^2 = \frac{1 - m^2 + 2im}{1 - m^2 - 2im}$$

The limit in this case is dependepend upon m

$\Rightarrow$ The overall limit is path dependent

$\Rightarrow$ $\boxed{\text{Limit does not exsists.}}$

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Que 3:- If $\log(z) = \ln(r) + i\theta\ (r>0,\ \frac{3\pi}{4} < \theta < \frac{11\pi}{4})$. Show $\log(i^2) \neq 2\log(i)$.

Ans:- Given: $r > 0$; $\frac{3\pi}{4} < \theta < \frac{11\pi}{4}$

$$\Rightarrow i = \cos\left(\frac{\pi}{2}\right) + i\sin\left(\frac{\pi}{2}\right) = e^{i\frac{\pi}{2}}$$

$$i^2 = e^{i\pi} \quad \text{(squaring both sides)}$$

$$= \cos\pi + i\sin\pi$$

$$\Rightarrow \log(i^2) = \log e^{i\pi} = i\pi \quad \text{———} \quad ①$$

Now;

$$2\log i = 2\log e^{i\frac{\pi}{2}} = i\pi \longrightarrow ②$$

But under the given conditions $\frac{3\pi}{4} < \theta < \frac{11\pi}{4}$

so $i \neq e^{i\pi/2}$ for the given domain

$$\Rightarrow i = e^{i\frac{3\pi}{2}}$$

$$\Rightarrow 2\log i = 2\log e^{i\frac{3\pi}{2}}$$

$$= i\,3\pi \longrightarrow ③$$

From ①, ② and ③ ; it clearly follows that

$$2\log i \neq i\pi \quad \text{(for the given range)}$$

Hence

$$\boxed{\log(i^2) \neq 2\log(i)} \quad \text{if } \left(\frac{3\pi}{4} < \theta < \frac{11\pi}{4}\right)$$

Hence Proved.

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Que 4: C be an arc of circle $|z|=2$ with positive orientation from $z=2$ to $z=2i$. Show that $\left|\int_C \frac{\bar{z}+1}{z^4+1}dz\right| \leq \frac{\pi}{5}$

Ans: Let ε_k be any point on arc (z_k, z_{k-1})

$$\int_c f(z)\,dz = \lim_{n\to\infty} \Sigma f(\varepsilon_k)(z_k - z_{k-1})$$

$$\left|\int_c f(z)\,dz\right| \leq \lim_{n\to\infty} \sum_{k=1}^{n} |f(\varepsilon_k)||z_k - z_{k-1}|$$

Let $|f(\varepsilon_k)| \leq N$ for all (ε_k)

$$\Rightarrow \left|\int_c f(z)\,dz\right| \leq N \lim_{n\to\infty} \sum_{k=1}^{n} |z_k - z_{k-1}|$$

$$\leq Nl \quad (l = \text{length of the contour})$$

(arc of circle $|z|=2$ with +ve orientation from $z=2$ to $z=2i$)

If our $f(z)$ is continuous and there exists any N such that $|f(z)| \leq N$; the question has a upper limit.

$$|f(z)| = \left|\frac{\bar{z}+1}{z^4+1}\right| = \frac{|\bar{z}+1|}{|z^4+1|} \leq \frac{|\bar{z}|+|1|}{|z^4|-|1|} \quad \left(\text{using } \Delta \text{ inequality}\right)$$

$$\Rightarrow \leq \frac{2+1}{16-1} = \frac{3}{15} = \frac{1}{5}$$

$$|f(z)| \leq \left(\frac{1}{5}(N)\right)$$

Now, arc length $= \frac{2\pi R}{4} = \frac{2\times\pi\times 2}{4} = \pi = l$

$$\Rightarrow \left|\int_c f(z)\,dz\right| \leq \frac{1}{5}\times\pi \Rightarrow \boxed{\left|\int_c f(z)\,dz\right| \leq \frac{\pi}{5}}$$

Hence Proved.

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Que 5:-
$$\int_{c:|z+1+i|=2} \frac{z+i}{z^2+2iz-4}\,dz$$

Ans:-
$$\int_{|z+i+1|=2} \frac{z+i}{z^2+2iz-1-3}\,dz = \int_c \frac{z+i}{(z+i)^2-3}\,dz$$

Let $w = z+i$

$dw = dz + 0$

$c = |z+1+i| = |w+i|$

$$\Rightarrow \int_{|w+1|=2} \frac{w}{w^2-3}\,dw$$

$$\Rightarrow \int_{|w+1|=2} \frac{w}{(w+\sqrt{3})(w-\sqrt{3})}\,dw \quad (\text{rationalising})$$

[the one point of discontinuity in the given domain is $-\sqrt{3}$]

$$= \int_{|w+1|=2} \frac{w(w-\sqrt{3})}{(w+\sqrt{3})}\,dw$$

Using the Cauchy Integral formula.

$$f(z_0) = \frac{1}{2\pi i}\int_c \frac{f(z)}{(z-z_0)}\,dz$$

So: $f(w) = \dfrac{w}{w-\sqrt{3}}$ $\Rightarrow$ $\displaystyle\int \frac{f(w)}{w-w_0} = 2\pi i \times f(w_0)$

$$\Rightarrow \int_{|z+1+i|} \frac{z+i}{z^2+2iz-4} = 2\pi i \times f(-\sqrt{3}) = \cancel{2}\pi i \times \left(\frac{-\cancel{\sqrt{3}}}{-\cancel{2}\cancel{\sqrt{3}}}\right)$$

$$= \pi i$$

So.
$$\boxed{\int_{|z+1+i|=2} \frac{z+1}{z^2+2iz-4}\,dz = \pi i}$$

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Que6: $g(z) = z^{1/2} = \sqrt{r}\, e^{i\theta/2}$; $r > 0$ $\quad -\pi/2 < \theta < \frac{3\pi}{2}$

C = partially oriented boundary of half disk $0 \leq r \leq 1$; $0 < \theta \leq \pi$

$$f(z) = \begin{cases} g(z) & z \neq 0 \\ 0 & z = 0 \end{cases}$$

Calculate $f(z)\,dz$ by parametrization :

Ans:- Calculating $\lim\limits_{z \to 0} f(z)$

$$= \frac{f(z) - f(0)}{z - 0} = \frac{f(z)}{z} = \frac{\sqrt{r}\, e^{i\theta/2}}{r e^{i\theta}}$$

$$= \frac{1}{\sqrt{r}\, e^{i\theta/2}}$$

$$\lim_{r \to 0} \frac{1}{\sqrt{r}\, e^{i\theta/2}} \to +\infty$$

So, as $r \to 0$, the limit tends to positive infinity which implies that the limit does not exsist.

This implies that at 0 the derivative of f ceases to exsist.

$\Rightarrow$ f is analytic at 0

$\Rightarrow$ Cauchy Goursat Theorem cannot be applied to calculate the given integral.

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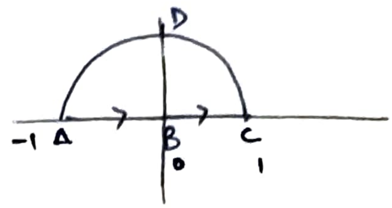
We divide C into 3 parts

$AB = C_1$
$ABC = C_2$
$BC = C_3$



For C_1: $z = t$; $-1 \le t < 0$; $dz = dt$.

$\therefore f(z) = \sqrt{t}, \ -1 \le t < 0$

For C_2: $z = e^{i\theta}$; $0 \le \theta \le \pi$; $dz = ie^{i\theta} d\theta$

$\therefore f(z) = e^{i\theta/2}$; $0 \le \theta \le \pi$

For C_3: $z = t$; $0 < t \le 1$, $dz = dt$

$\therefore f(z) = \sqrt{t}$; $0 < t \le 1$

$$I = \int_C f(z)\,dz$$

$$= \int_{C_1} f(z)dz + \int_{C_2} f(z)dz + \int_{C_3} f(z)dz$$

$$= \int_{-1}^{0} \sqrt{t}\,dt + \int_0^{\pi} e^{i\theta/2} \cdot ie^{i\theta} d\theta + \int_0^1 \sqrt{t}\,dt$$

$$= \frac{2}{3}\left[t^{3/2}\right]_{-1}^{0} + i\int_0^{\pi} e^{i3\theta/2} d\theta + \frac{2}{3}\left[t^{3/2}\right]_0^1$$

$$= \frac{2}{3} + \frac{2}{3}\frac{i}{i}\left[e^{i3\theta/2}\right]_0^{\pi} + \frac{2}{3}i = \frac{2}{3}(1+i) + \frac{2}{3}\left(e^{i\frac{3\pi}{2}} - e^{i\cdot 0}\right)$$

$$= \frac{2}{3}(1+i) - \frac{2}{3}(1+i) = 0$$

$$\boxed{\int_C f(z)\,dz = 0}$$

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Que 7:- Showing $|f(z)|$ attains it minimum value on the boundary of D if $|z| \le 2$ and is analytic and non zero at all points in D.

Ans:- Let us define a function $g(z)$ over C as

$$g(z) = \frac{1}{f(z)}$$

As $f(z)$ is not zero anywhere so $g(z)$ is continuous and by the maximum modulus principle which states that if $g(z)$ is a complex valued function of one or more complex variables and is differentiable then the modulus $|g(z)|$ cannot exhibit a strict local maximum that is properly within the domain of g

$\Rightarrow$ $|g(z)|$ has a maximum in R which is only at the boundary of R.

$\Rightarrow$ $|f(z)|$ has a minimum value in R which occurs at the boundary of R.

$\Rightarrow$ $|f(z)|$ is either zero but as it is given non zero; so $|f(z)|$ has a minimum value on the boundary of R.

Hence Proved.

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Que 8:- Find minimum modulus of $Re(z^2)$ where : $\{z = x+iy : 2 \le x \le 3, 1 \le y \le 3\}$

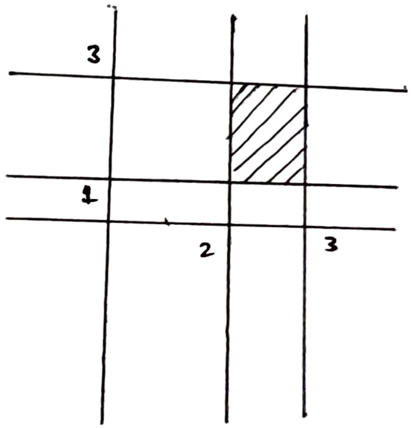
Ans:-

$z = x + iy$ (squaring)

$z^2 = (x+iy)^2 = x^2 - y^2 + 2ixy$

Now, we need to calculate the $Re(z^2)$ which is $x^2 - y^2$

And the maximum value of modulus $|x^2 - y^2|$ is possible in two cases as x and y are independent of each other



Case 1:-

x is maximum

y is minimum

$x_{max} = 3$

$y_{min} = 1$

$\therefore |Re(z^2)| = 8$

Case 2:-

x is minimum

y is maximum

$y_{max} = 3$

$x_{min} = 2$

$|Re(z^2)| = 5$

$\Rightarrow$ So, the maximum modulus of $Re(z^2)$ over given region $= \boxed{8.}$

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