



ICS141: Discrete Mathematics for Computer Science I

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Lecture 8

Chapter 2. Basic Structures

2.1 Sets

2.2 Set Operations



Sets so far...

- Previously...
 - Literal set $\{a, b, c\}$ and set-builder notation $\{x \mid P(x)\}$
 - Basic properties: unordered, distinct elements
 - infinite sets **N**, **Z**, **Z⁺**, **Q**, **R**
- Today
 - \in relational operator, and the empty set \emptyset
 - Venn diagrams
 - Set relations $=$, \subseteq , \subset , \supset , $\not\subset$, etc.
 - Cardinality $|S|$ of a set S
 - Power sets $P(S)$
 - Cartesian product $S \times T$
 - Set operators: \cup , \cap , $-$

Basic Set Relations: Member of

- $x \in S$ (“ x is in S ”) is the proposition that object x is an *Element* or *member* of set S .
 - e.g. $3 \in \mathbf{N}$,
 $a \in \{x \mid x \text{ is a letter of the alphabet}\}$
 - Can define set equality in terms of \in relation:
$$\forall S, T: S = T \leftrightarrow [\forall x (x \in S \leftrightarrow x \in T)]$$

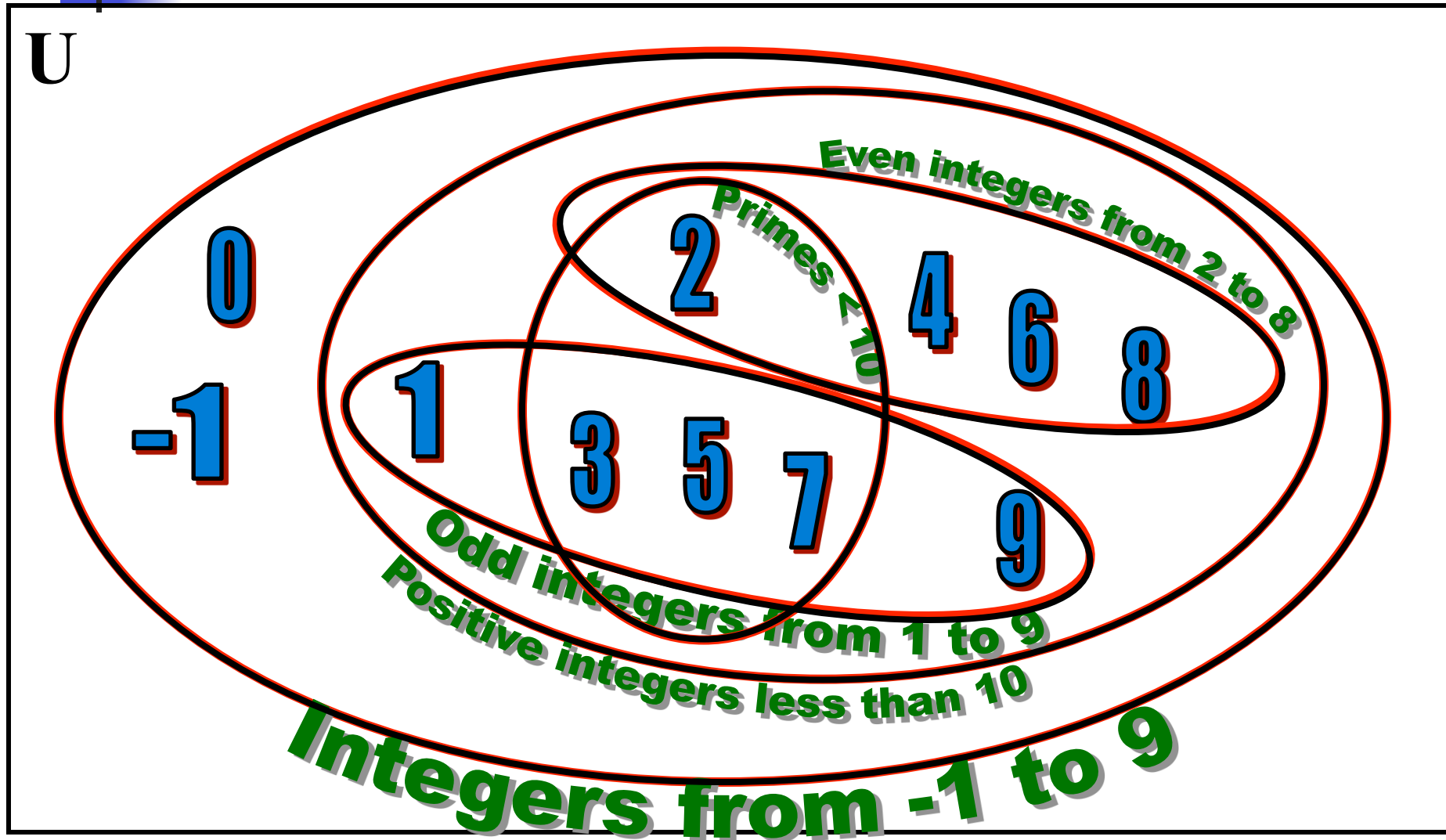
“Two sets are equal iff they have all the same members.”
- $x \notin S \equiv \neg(x \in S)$ “ x is not in S ”



The Empty Set

- \emptyset (“null”, “the empty set”) is the unique set that contains no elements whatsoever.
- $\emptyset = \{ \} = \{x \mid \mathbf{False}\}$
- No matter the domain of discourse, we have the axiom $\neg \exists x: x \in \emptyset$.
- $\{ \} \neq \{ \emptyset \} = \{ \{ \} \}$
 - $\{ \emptyset \}$ it isn't empty because it has \emptyset as a member!

Venn Diagrams

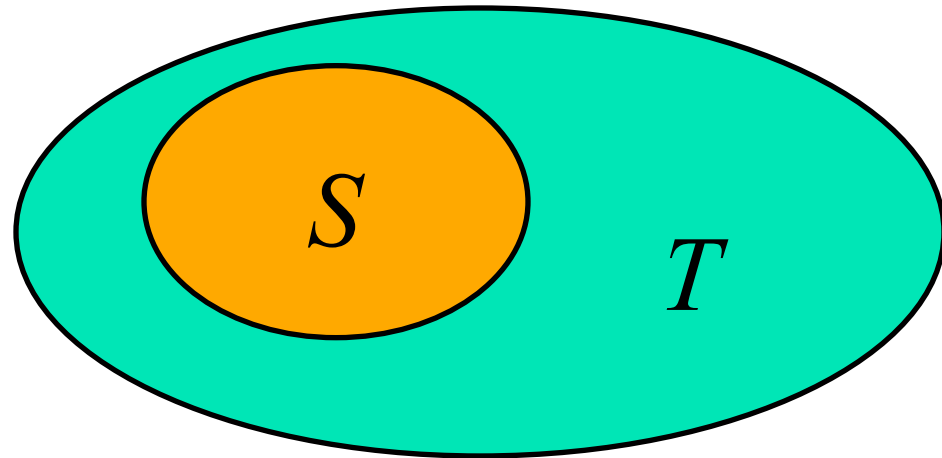


Subset and Superset

- $S \subseteq T$ (“ S is a subset of T ”) means that every element of S is also an element of T .
- $S \subseteq T \equiv \forall x (x \in S \rightarrow x \in T)$
- $\emptyset \subseteq S$, $S \subseteq S$
- $S \subseteq T$ (“ S is a superset of T ”) means $T \subseteq S$
- Note $(S = T) \equiv (S \subseteq T \wedge T \subseteq S)$
 $\equiv \forall x (x \in S \rightarrow x \in T) \wedge \forall x (x \in T \rightarrow x \in S)$
 $\equiv \forall x (x \in S \leftrightarrow x \in T)$
- $S \not\subseteq T$ means $\neg(S \subseteq T)$, i.e. $\exists x (x \in S \wedge x \notin T)$

Proper (Strict) Subsets & Supersets

- $S \subset T$ (“ S is a proper subset of T ”) means that $S \subseteq T$ but $T \not\subseteq S$. Similar for $S \supset T$.
- Example:
 $\{1, 2\} \subset \{1, 2, 3\}$



Venn Diagram of $S \subset T$

Sets Are Objects, Too!

- The objects that are elements of a set may *themselves* be sets.

- Example:

Let $S = \{x \mid x \subseteq \{1, 2, 3\}\}$

then $S = \{ \emptyset,$

$\{1\}, \{2\}, \{3\},$

$\{1, 2\}, \{1, 3\}, \{2, 3\},$

$\{1, 2, 3\} \}$

- Note that $1 \neq \{1\} \neq \{\{1\}\} !!!!$

 **Very
Important!**

Cardinality and Finiteness

- $|S|$ (read “the *cardinality* of S ”) is a measure of how many different elements S has.
- *E.g.*, $|\emptyset| = 0$, $|\{1, 2, 3\}| = 3$, $|\{a, b\}| = 2$,
 $|\{\{1, 2, 3\}, \{4, 5\}\}| = \underline{2}$
- If $|S| \in \mathbf{N}$, then we say S is *finite*.
Otherwise, we say S is *infinite*.
- What are some infinite sets we’ve seen?

N, Z, Q, R

The *Power Set* Operation

- The **power set** $P(S)$ of a set S is the set of all subsets of S . $P(S) = \{x \mid x \subseteq S\}$.
- Examples
 - $P(\{a, b\}) = \{ \emptyset, \{a\}, \{b\}, \{a, b\} \}$
 - $S = \{0, 1, 2\}$
 $P(S) = \{ \emptyset, \{0\}, \{1\}, \{2\}, \{0, 1\}, \{0, 2\}, \{1, 2\}, \{0, 1, 2\} \}$
 - $P(\emptyset) = \{ \emptyset \}$
 - $P(\{\emptyset\}) = \{ \emptyset, \{\emptyset\} \}$
- Note that for finite S , $|P(S)| = 2^{|S|}$.
- It turns out $\forall S (|P(S)| > |S|)$, e.g. $|P(\mathbb{N})| > |\mathbb{N}|$.



Ordered n -tuples

- These are like sets, except that duplicates matter, and the order makes a difference.
- For $n \in \mathbf{N}$, an *ordered n -tuple* or a *sequence* or *list of length n* is written (a_1, a_2, \dots, a_n) . Its *first* element is a_1 , its second element is a_2 , etc.
- Note that $(1, 2) \neq (2, 1) \neq (2, 1, 1)$. ← Contrast with sets' $\{\}$
- Empty sequence, singlets, pairs, triples, quadruples, quintuples, ..., n -tuples.

Cartesian Products of Sets

- For sets A and B , their **Cartesian product** denoted by $A \times B$, is the set of all ordered pairs (a, b) , where $a \in A$ and $b \in B$. Hence,

$$A \times B = \{ (a, b) \mid a \in A \wedge b \in B \}.$$

- E.g. $\{a, b\} \times \{1, 2\}$
 $= \{ (a, 1), (a, 2), (b, 1), (b, 2) \}$

- Note that for finite A, B , $|A \times B| = |A||B|$.
- Note that the Cartesian product is **not** commutative: i.e., $\neg \forall A, B (A \times B = B \times A)$.
- Extends to $A_1 \times A_2 \times \dots \times A_n$
 $= \{ (a_1, a_2, \dots, a_n) \mid a_i \in A_i \text{ for } i = 1, 2, \dots, n \}$



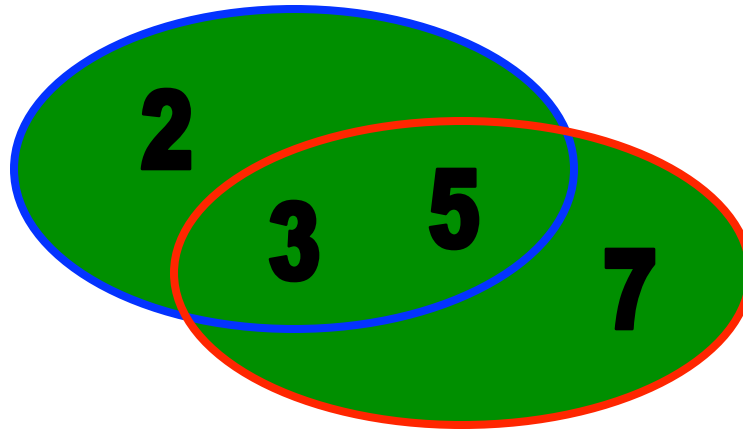
The Union Operator

- For sets A and B , their **union** $A \cup B$ is the set containing all elements that are either in A , **or** (“ \vee ”) in B (or, of course, in both).
- Formally, $\forall A, B: A \cup B = \{x \mid x \in A \vee x \in B\}$.
- Note that $A \cup B$ is a **superset** of both A and B (in fact, it is the smallest such superset):
 $\forall A, B: (A \cup B \subseteq A) \wedge (A \cup B \subseteq B)$

Union Examples

- $\{a, b, c\} \cup \{2, 3\} = \{a, b, c, 2, 3\}$
- $\{2, 3, 5\} \cup \{3, 5, 7\} = \{2, 3, 5, 3, 5, 7\}$
 $= \{2, 3, 5, 7\}$

Required Form



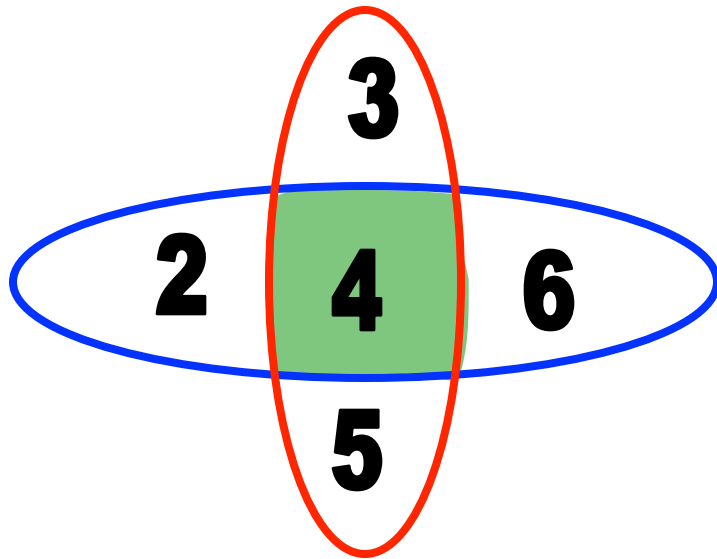


The Intersection Operator

- For sets A and B , their ***intersection*** $A \cap B$ is the set containing all elements that are simultaneously in A **and** (“ \wedge ”) in B .
- Formally, $\forall A, B: A \cap B = \{x \mid x \in A \wedge x \in B\}$.
- Note that $A \cap B$ is a **subset** of both A and B (in fact it is the largest such subset):
$$\forall A, B: (A \cap B \subseteq A) \wedge (A \cap B \subseteq B)$$

Intersection Examples

- $\{a, b, c\} \cap \{2, 3\} = \underline{\emptyset}$
- $\{2, 4, 6\} \cap \{3, 4, 5\} = \underline{\{4\}}$



Think “The intersection of University Ave. and Dole St. is just that part of the road surface that lies on *both* streets.”

Disjointedness

- Two sets A , B are called **disjoint** (i.e., unjoined) iff their intersection is empty. ($A \cap B = \emptyset$)
- Example: the set of even integers is disjoint with the set of odd integers.





Inclusion-Exclusion Principle

- How many elements are in $A \cup B$?

$$|A \cup B| = |A| + |B| - |A \cap B|$$

- Example: How many students in the class major in Computer Science or Mathematics?

- Consider set $E = C \cup M$,

$C = \{s \mid s \text{ is a Computer Science major}\}$

$M = \{s \mid s \text{ is a Mathematics major}\}$

- Some students are joint majors!

$$|E| = |C \cup M| = |C| + |M| - |C \cap M|$$

Inclusion-Exclusion Principle

- How many elements are in $A \cup B$?

$$|A \cup B| = |A| + |B| - |A \cap B|$$

- Example: How many students in the class major in mathematics?

- Consider the set of students in the class who are either majoring in mathematics or are members of the chess team.

$C = \{\text{students who are members of the chess team}\}$

$M = \{\text{students who are majoring in mathematics}\}$

How many students are in $C \cup M$?

$$|C \cup M| = |C| + |M| - |C \cap M|$$



Set Difference

- For sets A and B , the ***difference of A and B*** , written $A - B$, is the set of all elements that are in A but not B .

- Formally:

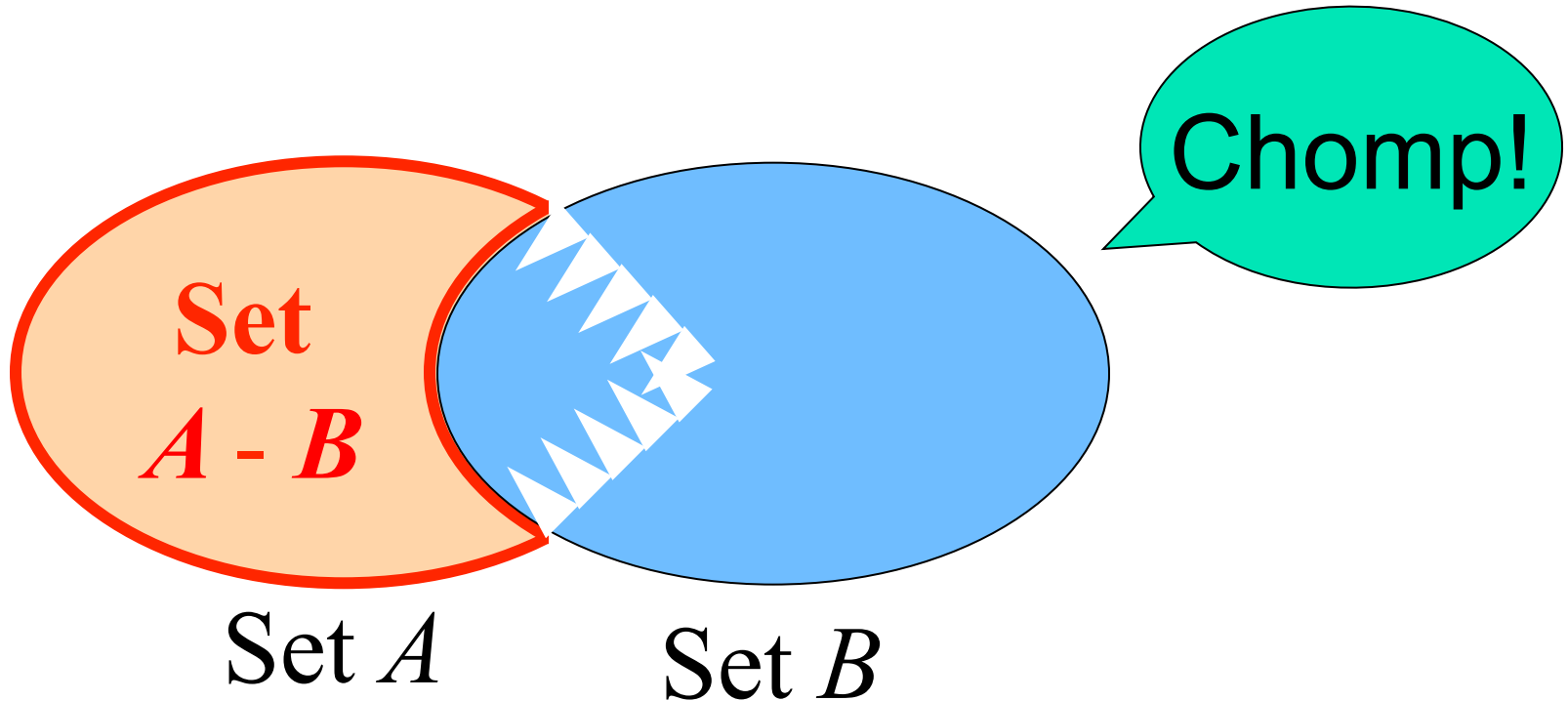
$$\begin{aligned} A - B &= \{x \mid x \in A \wedge x \notin B\} \\ &= \{x \mid \neg(x \in A \rightarrow x \in B)\} \end{aligned}$$

- Also called:

The ***complement of B with respect to A*** .

Set Difference: Venn Diagram

- $A - B$
is what's left after B “takes a bite out of A ”



Set Difference Examples

- $\{ \cancel{1}, \cancel{2}, \cancel{3}, 4, \cancel{5}, 6 \} - \{2, 3, 5, 7, 9, 11\} =$
 $\underline{\{1, 4, 6\}}$

- $\mathbf{Z} - \mathbf{N} = \{ \dots, -1, 0, 1, 2, \dots \} - \{0, 1, \dots \}$
 $= \{x \mid x \text{ is an integer but not a natural } \#\}$
 $= \{ \dots, -3, -2, -1 \}$
 $= \{x \mid x \text{ is a negative integer}\}$



Set Complements

- The *universe of discourse* (or the *domain*) can itself be considered a set, call it U .
- When the context clearly defines U , we say that for any set $A \subseteq U$, the **complement** of A , written as \overline{A} , is the complement of A with respect to U , *i.e.*, it is $U - A$.

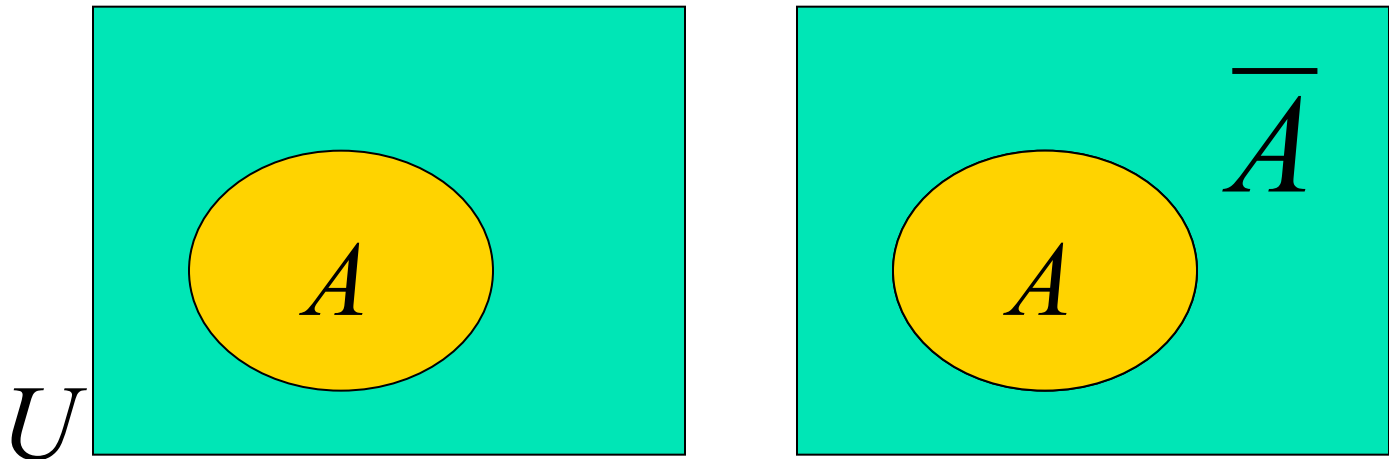
- *E.g.*, If $U = \mathbf{N}$,

$$\overline{\{3, 5\}} = \{0, 1, 2, 4, 6, 7, \dots\}$$

More on Set Complements

- An equivalent definition, when U is obvious:

$$\overline{A} = \{x \mid x \notin A\}$$





Interval Notation

- $a, b \in \mathbf{R}$, and $a < b$ then
 - $(a, b) = \{x \in \mathbf{R} \mid a < x < b\}$
 - $[a, b] = \{x \in \mathbf{R} \mid a \leq x \leq b\}$
 - $(a, b] = \{x \in \mathbf{R} \mid a < x \leq b\}$
 - $(-\infty, b] = \{x \in \mathbf{R} \mid x \leq b\}$
 - $[a, \infty) = \{x \in \mathbf{R} \mid a \leq x\}$
 - $(a, \infty) = \{x \in \mathbf{R} \mid a < x\}$