Method ④ : Solution by substitution method

Case 1 → Bernouilli's Eqn

$$\boxed{\frac{dy}{dx} + p(x)y = q(x)y^a} \quad \begin{pmatrix} a \in \mathbb{R} \\ a \neq 0,1 \end{pmatrix} \rightarrow$$

If $a=0$ ODE is linear
If $a=1$ ODE is separable

$$\Rightarrow \frac{1}{y^a}\frac{dy}{dx} + p(x)\frac{1}{y^{a-1}} = q(x) \quad \text{— (1)}$$

Note that the original equation is non linear

Take $z = \frac{1}{y^{a-1}}$

$$\Rightarrow z = y^{1-a}$$

$$\Rightarrow \frac{dz}{dx} = (1-a)y^{-a}\frac{dy}{dx} = (1-a)\frac{1}{y^a}\frac{dy}{dx}$$

By putting above values in (1), we obtain

$$\frac{1}{(1-a)}\frac{dz}{dx} + p(x)z = q(x)$$

$$\Rightarrow \frac{dz}{dx} + (1-a)p(x)z = (1-a)q(x)$$

solve above linear equation for z and do back substitution for $z = y^{1-a}$.

Ex. $\frac{dy}{dx} = \frac{y}{x} - y^2$

$$\Rightarrow \boxed{\frac{1}{y^2}\frac{dy}{dx} - \frac{1}{x}\frac{1}{y} = -1}$$

Take $z = \frac{1}{y}$ i.e. $\frac{dz}{dx} = -\frac{1}{y^2}\frac{dy}{dx}$

by using above values, the ODE becomes $-\frac{dz}{dx} - \frac{1}{x}z = -1$

$$\Rightarrow \boxed{\frac{dz}{dx} + \frac{1}{x}z = 1}$$

→ This is a linear equation in z and x,

If $u = e^{\int \frac{1}{x}dx} = x$

$$zx = \int x\,dx + c$$

$$\Rightarrow zx = \frac{x^2}{2} + c$$

$$\Rightarrow z = \frac{1}{2}x + \frac{c}{x}$$

Use back substitution!

$$\boxed{\frac{1}{y} = \frac{1}{2}x + \frac{c}{x}}$$ Answer

Case 2

Generalized linear / Bernouilli's Equation

$$\boxed{f'(y)\frac{dy}{dx} + p(x) f(y) = q(x)}$$

Take $z = f(y)$

$$\Rightarrow \frac{dz}{dx} = f'(y)\frac{dy}{dx}$$

Thus, equation reduces to

$$\boxed{\frac{dz}{dx} + p(x) z = q(x)}$$

It is linear. Solve it and use back subtitution to find y.

Example solve $\frac{dy}{dx} = e^{2x} e^{-y} - e^{x}$

$$\Rightarrow e^{y}\frac{dy}{dx} = e^{2x} - e^{x} e^{y}$$

$$\Rightarrow e^{y}\frac{dy}{dx} + e^{x} e^{y} = e^{2x}$$

Take $z = e^{y} \Rightarrow \frac{dz}{dx} = e^{y}\frac{dy}{dx}$

Thus, using the above subtitution, equation reduces to

$$\boxed{\frac{dz}{dx} + e^{x} z = e^{2x}}$$

The above equation is linear.

$\rightarrow$ IF $u(x) = e^{e^{x}}$

$\rightarrow$ $z e^{e^{x}} = \int e^{2x} e^{e^{x}} dx + C$

Take $e^{x} = t$, RHS
$= \int t e^{t} dt + c$
$= t e^{t} - e^{t} + c$
$= e^{x} e^{e^{x}} - e^{e^{x}} + c$

$$\Rightarrow \boxed{z = e^{x} - 1 + c e^{-e^{x}}}$$

Use back subtition to get $\boxed{e^{y} = e^{x} - 1 + c e^{-e^{x}}}$ Ans.

Case 3

Homogeneous ODEs of I^{st} order

$$\boxed{\frac{dy}{dx} = f\left(\frac{y}{x}\right)}$$ ← why homogeneous??

Two types of Substitutions

New var. = Combination of old variables → Direct Substitution

Old variable = Combination of a few old and new variable → Indirect substitution

Take $y = zx$

$$\Rightarrow \frac{dy}{dx} = z + x\frac{dz}{dx}$$

So ODE reduces to

$$z + x\frac{dz}{dx} = f(z)$$

$$\Rightarrow x\frac{dz}{dx} = f(z) - z$$

$$\Rightarrow \boxed{\frac{dz}{f(z)-z} = \frac{dx}{x}}$$ → separable in z and x. solve it and use back substitution.

Example :- Solve :- $xy\frac{dy}{dx} + 4x^2 + y^2 = 0$ — ODE

$$\Rightarrow \frac{dy}{dx} = -\frac{4x^2+y^2}{xy} = -\frac{4+\left(\frac{y}{x}\right)^2}{\left(\frac{y}{x}\right)}$$

Now, by substitution $y = zx$, the ODE reduces to $z + x\frac{dz}{dx} = -\frac{4+z^2}{z}$

$$\Rightarrow x\frac{dz}{dx} = -\frac{4+z^2}{z} - z = -\frac{4+2z^2}{z} = -2\,\frac{2+z^2}{z}$$

$$\Rightarrow \frac{z}{2+z^2}\,dz = -\frac{2}{x}\,dx$$

Its solution is $\frac{1}{2}\ln(2+z^2) + \ln x^2 = c$

$$\Rightarrow \boxed{x^2\sqrt{2+z^2} = c} \Rightarrow z^2 + 2 = \frac{c}{x^4} \xrightarrow{\text{Back Substitution}} \boxed{y^2 = \frac{c}{x^2} - 2x^2}$$ Ans.

Exercise

Find an interval of validity for

$$IVP \equiv \begin{cases} ODE \\ y(2) = -7 \end{cases}$$

$(-\sqrt{114}, \sqrt{114})$

Answer

Case 4

$$\boxed{\frac{dy}{dx} = f(ax+by+c)}$$ a, b, c are constants

Take $z = ax+by+c$

$$\Rightarrow \frac{dz}{dx} = a + b\frac{dy}{dx}$$

Thus ODE reduces to

$$\frac{1}{b}\left(\frac{dz}{dx} - a\right) = f(z)$$

$$\Rightarrow \frac{dz}{dx} = b f(z) + a$$

The above ODE is separable in variables z and x.
Solve it and get y by using back substitution.

Ex. $\frac{dy}{dx} = e^{9y-x}$

Take $z = -x + 9y$

$$\Rightarrow \frac{dz}{dx} = -1 + 9\frac{dy}{dx}$$

Thus ODE reduces to

$$\frac{1}{9}\left(\frac{dz}{dx} + 1\right) = e^{z}$$

$$\Rightarrow \frac{dz}{dx} = 9e^{z} - 1$$

$$\Rightarrow \frac{dz}{9e^{z}-1} = dx = \frac{e^{-z}}{9-e^{-z}}dz = dx$$

Solution: $\ln(9-e^{-z}) = x + c$

$$\Rightarrow -z = \ln(9 - ce^{x})$$

Using back substitution obtain

$$\boxed{y = \frac{1}{9}\left[x - \ln(9 - c\,e^{x})\right]}$$

→ See the ODE is already in separable form

$$e^{-9y}dy = e^{-x}dx$$

Solution is:

$$\boxed{-\frac{1}{9}e^{-9y} = -e^{-x} + c}$$

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Case 5

$$\boxed{(a_1x + b_1y + c_1)\,dx + (a_2x + b_2y + c_2)\,dy = 0}$$

and $\dfrac{a_1}{a_2} = \dfrac{b_1}{b_2} = \dfrac{a}{b}$ (say)

where a_i's, b_i's, c_i's are constants.

Take $z = ax + by$

Then ODE reduces into a separable equation in z and x.

Example. Solve: $(x+2y+3)dx + (2x+4y-1)dy = 0$

Take $z = x+2y \Rightarrow dz = dx + 2\,dy$

Thus equation reduces to

$$(z+3)\,dx + (2z-1)\left(\frac{dz-dx}{2}\right) = 0$$

$$\Rightarrow \frac{7}{2}dx + \frac{1}{2}(2z-1)dz = 0$$

Its solution is $\dfrac{7}{2}x + \dfrac{1}{2}(z^2 - z) = c$

$$\Rightarrow 7x + z^2 - z = c$$

by back substitution

$$7x + (x+2y)^2 - (x+2y) = c$$

$$\Rightarrow \boxed{x^2 + 4y^2 + 4xy + 6x - 2y = c}$$ Ans.

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Case 6

$$\boxed{(a_1x + b_1y + c_1)\,dx + (a_2x + b_2y + c_2)\,dy = 0}$$

$$\frac{a_1}{a_2} \neq \frac{b_1}{b_2}$$

a_i, b_i, c_i are constants

ODE will become homogeneous if you remove constant terms c_1 and c_2 from M and N, respectively.

For, we shift coordinate, i.e.

Take $\quad z_1 = x - h$

$\quad\quad\quad z_2 = y - k$

where h and k are constants such that

$$\left.\begin{array}{l} a_1h + b_1k + c_1 = 0 \\ a_2h + b_2k + c_2 = 0 \end{array}\right) \text{ i.e. } \begin{bmatrix} h \\ k \end{bmatrix} = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}^{-1} \begin{bmatrix} -c_1 \\ -c_2 \end{bmatrix}.$$

By above substitution, ODE reduces to a homogeneous equation in variables z_1 and z_2.

Ex. Solve $\quad (x - 2y + 1)\,dx + (4x - 3y - 6)\,dy = 0$

↳ (Not homogeneous and not exact also).

Take $\quad z_1 = x - h \quad$ where h, k solve the system

$\quad\quad\quad z_2 = x - k$

$$\left.\begin{array}{l} h - 2k + 1 = 0 \\ 4h - 3k - 6 = 0 \end{array}\right) \Rightarrow \begin{bmatrix} h \\ k \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ 4 & -3 \end{bmatrix}^{-1} \begin{bmatrix} -1 \\ 6 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

Thus, by using substitution

$$\left.\begin{array}{l} z_1 = x - 3 \\ z_2 = y - 2 \end{array}\right) \Rightarrow \begin{array}{l} dz_1 = dx \\ dz_2 = dy \end{array}$$

the ODE reduces to

$$(z_1 + 3 - 2z_2 - 4 + 1)\,dz_1 + (4z_1 + 12 - 3z_2 - 6 - 6)\,dz_2 = 0$$

$$\Rightarrow \boxed{(z_1 - 2z_2)\,dz_1 + (4z_1 - 3z_2)\,dz_2 = 0}$$

→ Solve it in z_1 and z_2 and then use back substitution.

Summary

ODE	Substitution	ODE reduces to
$\frac{dy}{dx} + p(x)y = q(x)y^{a}$ (Bernouilli's equations)	$z = \frac{1}{y^{a-1}} = y^{1-a}$	linear ODE
$f'(y)\frac{dy}{dx} + p(x)f(y) = q(x)$ (Generalized linear ODE)	$z = f(y)$	linear ODE
$\frac{dy}{dx} = f\left(\frac{y}{x}\right)$ (Homogeneous ODE)	$y = zx$	separable form in z and x
$\frac{dy}{dx} = f(ax+by+c)$	$z = ax+by+c$	separable form in z and x
$(a_1x+b_1y+c_1)dx + (a_2x+b_2y+c_2)dy$ $\left(\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{a}{b} \text{ say}\right)$	$z = ax+by$	separable form in z and x
Above ODE $\left(\frac{a_1}{a_2} \neq \frac{b_1}{b_2}\right)$	$z_1 = x-h$ $z_2 = y-k$ where $a_1h+b_1k+c_1=0$ $a_2h+b_2k+c_2=0$	homogeneous ODE in z_1 and z_2