Poisson Distribution and Joint Probability

Thursday, February 4, 2021 8:54 AM

Poisson Distribution

$k = 0, 1, 2, 3, \dots$

X is a Poisson r.v, then pmf given by

$$P_X(k) = e^{-\lambda} \cdot \frac{\lambda^k}{k!} \qquad k = 0, 1, 2, \dots$$

$$\sum_{k=0}^{\infty} e^{-\lambda} \cdot \frac{\lambda^k}{k!} = e^{-\lambda} \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} = e^{-\lambda} \cdot e^{\lambda} = 1$$

$$E[X] = \sum_{k=0}^{\infty} k \frac{e^{-\lambda} \lambda^k}{k!} = \lambda \sum_{k=1}^{\infty} \frac{e^{-\lambda} \lambda^{k-1}}{(k-1)!} = \lambda e^{-\lambda} \cdot e^{\lambda} = \boxed{\lambda}$$

$$E[X^2] = \text{-----} \quad \text{---} = \lambda(\lambda+1)$$

$$Var(X) = E[X^2] - (E[X])^2 = \lambda(\lambda+1) - \lambda^2 = \boxed{\lambda}$$

Poisson as limiting case of Binomial Distribution

$$\boxed{np = \lambda}$$

$$\lim_{n \to \infty} P(X = k) = \lim_{n \to \infty} \binom{n}{k} p^k (1-p)^{n-k}$$

$$= \lim_{n \to \infty} \frac{n!}{k!\,(n-k)!} \left(\frac{\lambda}{n}\right)^k \left(1 - \frac{\lambda}{n}\right)^{n-k}$$

$$= \lim_{n \to \infty} \underbrace{\frac{n}{n} \cdot \frac{n-1}{n} \cdot \frac{n-2}{n} \cdots \frac{n-k+1}{n}} \quad \underline{\frac{\lambda^k}{k!}} \quad \underline{\left(1 - \frac{\lambda}{n}\right)^n} \cdot \underline{\left(1 - \frac{\lambda}{n}\right)^{-k}}$$

$$= \lim_{n\to\infty} \underbrace{\frac{\cdot}{n} \cdot \frac{\cdot}{n} \cdot \frac{\cdot}{n} \cdots \frac{\cdot}{n}}_{1} \quad \underline{k!} \quad \underbrace{\left(\quad n \right)}_{\downarrow \; e^{-\lambda}} \quad \underbrace{\left(\quad n \right)}_{\downarrow \; 1}$$

$$= \boxed{e^{-\lambda} \cdot \frac{\lambda^k}{k!}} \longrightarrow \text{Poisson with parameter } \lambda.$$

Negative Binomial Distribution

Tossing a coin with $P(H) = p$ independently.

X = denote the no of tosses required to get 'r' Hs.

$$P(X = n) = \binom{n-1}{r-1} p^r (1-p)^{n-r} \qquad n = r, r+1, r+2, \ldots$$

$$\sum_{n=r}^{\infty} P(X = n) = \ldots = 1.$$

$$E[X] = \underline{\underline{\quad ?? \quad}}$$

Joint Probability Distributions

Consider two r.v. X and Y associated with same experiment. The joint pmf of X and Y is defined by,

$$p_{X,Y}(x, y) = P(X = x, Y = y)$$

Example

Tossing a fair coin 4 times, $\Omega = \{H, T\}^4$

Let X = no of Hs in first 3 tosses

Y = no of Hs in last 2 tosses

$$X(\omega = TTHH) = 1 \qquad Y(\omega) = 2$$

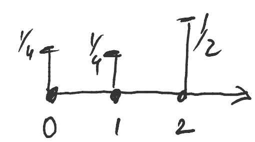
$$P(X=1, Y=1) = 3/16$$

$\{HTTH, THTH, TTHT\}$

$Y \backslash X$	0	1	2	3	
	$2/16$	$6/16$	$6/16$	$2/16$	
2	0	$1/16$	$2/16$	$1/16$	$4/16$
1	$1/16$	$3/16$	$3/16$	$1/16$	$8/16$
0	$1/16$	$2/16$	$1/6$	0	$4/16$

$p_{X,Y}(x,y)$

$$\underline{p_X(x)} = P(X=x)$$



$$P(X=0) = P(X=0, Y=0) + P(X=0, Y=1) + P(X=0, Y=2) = 2/16$$

A = Both X and Y are even

$$P(A) = \sum_{(x,y) \in A} p_{X,Y}(x,y)$$

Given $p_{X,Y}(x,y)$

want to find $p_X(x) = \sum_y p_{X,Y}(x,y)$

$$p_X(x) = P(X=x) = P\left(\bigcup_y \{X=x, Y=y\}\right)$$

Marginal Distribution

$$= \sum_{y} P(X = x, Y = y)$$

$$= \sum_{y} p_{X,Y}(x, y)$$

$$p_Y(y) = \sum_{x} p_{X,Y}(x, y)$$

$$p_{X,Y,Z}(x, y, z) = P(X = x, Y = y, Z = z)$$

$$p_X(x) = \sum_{y} \sum_{z} p_{X,Y,Z}(x, y, z)$$

$$p_Y(y) =$$

$$p_Z(z) =$$