DATE :

(2) $f(z) = e^{\bar{z}} = e^{x-iy}$

$= e^{x}[\cos y - i\sin y]$

$= e^{x}\cos y - ie^{x}\sin y$

$u = e^{x}\cos y$ $\quad$ $v = -e^{x}\sin y$

$u_x = e^{x}\cos y$ $\quad$ $v_x = -e^{x}\sin y$

$u_y = -e^{x}\sin y$ $\quad$ $v_y = -e^{x}\cos y$

$u_x \neq v_y$, $\quad u_y \neq -v_x$ $\quad$ C-R Eq^n Not satisfied.

3. Verify the following

(i) $|e^{2z+i}+e^{iz^2}| \le e^{2x}+e^{-2xy}$

(ii) $|e^{z^2}| \le e^{|z|^2}$

(iii) $|e^{-2z}|<1$ iff $Re\, z>0$.

Soln: (i) Let $z = x+iy$, we have

$$|e^{2(x+iy)+i}+e^{i(x+iy)^2}|$$

$$= |e^{2x+i(2y+1)}+e^{i(x^2-y^2+2ixy)}| \quad (\because |z_1+z_2| \le |z_1|+|z_2|)$$

$$= |e^{2x+i(2y+1)}+e^{-2xy+i(x^2-y^2)}|$$

$$\le |e^{2x+i(2y+1)}|+|e^{-2xy+i(x^2-y^2)}|$$

$$= |e^{2x}|+|e^{-2xy}|$$

$$= e^{2x}+e^{-2xy} \quad (\because e^u>0 \ \forall u\in\mathbb{R})$$

$\therefore |e^{2z+i}+e^{iz^2}| \le e^{2x}+e^{-2xy}$ proved.

(ii) Try it!

(iii) $|e^{-2z}|<1 \Leftrightarrow |e^{-2(x+iy)}|<1 \Leftrightarrow |e^{-2x}|<1$

i.e, $|e^{-2z}|<1 \Leftrightarrow e^{-2x}<1$

$\Leftrightarrow -2x<0$

$\Leftrightarrow x>0$

i.e, $Re(z)>0$

Thus, $|e^{-2z}|<1$ iff $Re(z)>0$.

4. Find all values of z such that

(i) $e^z=2$ (ii) $e^z=1+\sqrt{3}i$ (iii) $e^{2z-1}=1$ (iv) $e^z=-4$ (v) $e^z=\sqrt{3}-i$

Soln: (i) Let $z=x+iy$. Then

$e^z=2 \Rightarrow e^x(\cos y+i\sin y)=2$

$\therefore e^x\cos y=2$ and $e^x\sin y=0$

We have $e^x = 2$ and $\cos y = 1$, $\sin y = 0$

Now,

$e^x = 2 \Rightarrow x = \ln 2$,

$\cos y = 1$ and $\sin y = 0$

$\Rightarrow y = 2k\pi$, where $k \in \mathbb{Z}$.

$\therefore z = \ln 2 + (2k\pi) i$

(ii) $e^z = 1 + \sqrt{3} i$

Let $z = x + iy$. Then

$e^z = 1 + \sqrt{3} i \Rightarrow e^x (\cos y + i \sin y) = 1 + \sqrt{3} i$

$\therefore e^x \cos y = 1$ and $e^x \sin y = \sqrt{3}$.

We have, $e^x = 2$ and $\cos y = \frac{1}{2}$, $\sin y = \frac{\sqrt{3}}{2}$

These determines $x = \ln 2$ and $y = 2k\pi + \frac{\pi}{3}$, $k \in \mathbb{Z}$.

$\therefore z = \ln 2 + (2k\pi + \frac{\pi}{3}) i$.

(iii) $e^{2z-1} = 1$

Soln: Let $z = x + iy$. Then we have

$e^{2(x+iy)-1} = 1$

$\Rightarrow e^{(2x-1) + i(2y)} = 1$

$\Rightarrow e^{2x-1} (\cos 2y + i \sin 2y) = 1$

$\therefore e^{2x-1} \cos 2y = 1$ and $e^{2x-1} \sin 2y = 0$.

Thus, $e^{2x-1} = 1$ and $\cos 2y = 1$, $\sin 2y = 0$

$\Rightarrow 2x - 1 = 0$, $2y = 2k\pi$, $k \in \mathbb{Z}$

$\Rightarrow x = \frac{1}{2}$ and $y = k\pi$, $k \in \mathbb{Z}$.

$\therefore z = \frac{1}{2} + k\pi$, $k \in \mathbb{Z}$

Try rest.

5 (1) $\overline{\exp(iz)} = \overline{\cos z + i\sin z}$

$= \overline{\cos z} + \overline{i\sin z}$

$= \cos\bar{z} - i\sin\bar{z}$

& $\exp(i\bar{z}) = \cos\bar{z} + i\sin\bar{z}$

therefore

$\overline{\exp(iz)} = \exp(i\bar{z})$

$\Leftrightarrow \cos\bar{z} - i\sin\bar{z} = \cos\bar{z} + i\sin\bar{z}$

$\Leftrightarrow \sin\bar{z} = 0$

$\Leftrightarrow z = n\pi, \; n \in \mathbb{Z}$

(2) $e^{z} = e^{x+iy} = e^{x} e^{iy}$

$= e^{x}(\cos y + i\sin y)$

therefore e^{z} is real

$\Leftrightarrow e^{x}\sin y = 0$

$\Leftrightarrow y = n\pi \;, n \in \mathbb{Z}$

& e^{z} is imaginary

$\Leftrightarrow e^{x}\cos y = 0$

$\Leftrightarrow \cos y = 0$

$\Leftrightarrow y = (2n+1)\pi/2 \;, n \in \mathbb{Z}$

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MA201 TA3

6(i) Key Point - $\text{Log } z = \ln|z| + i \text{ Arg } z$

L.H.S $\quad \text{Log}(1+i)^2 = \text{Log}(1+i^2+2i)$

$= \text{Log}(2i)$

$= \ln 2 + i\frac{\pi}{2}$

R.H.S $\quad 2\text{Log}(1+i) = 2(\ln\sqrt{2} + i\frac{\pi}{4})$

$= 2\ln\sqrt{2} + i\frac{\pi}{2} = \text{L.H.S}$

(ii) L.H.S $\quad \text{Log}(-1+i)^2 = \text{Log}(1+i^2-2i)$

$= \text{Log}(-2i)$

$= \ln 2 - i\frac{\pi}{2}$

R.H.S $\quad 2\text{ Log}(-1+i) = 2(\ln\sqrt{2} + i\frac{3\pi}{4})$

$= 2\ln\sqrt{2} + i\frac{3\pi}{2}$

$= \ln 2 + i\frac{3\pi}{2} \neq \text{L.H.S}$

(iii) $\log i^2 = \log(-1) = \ln 1 + i \arg(-1)$

$= \pi i$

$\arg(-1) = \pi \pm 2n\pi$

$n = 0, 1, 2 \ldots$

& $2\log i = 2(\ln 1 + i \arg(i))$

for $n=0$

$= 2i\cdot\frac{\pi}{2}$

$\pi = \arg(-1)$

$\in (\frac{\pi}{4}, \frac{9\pi}{4})$

$= i\pi = \text{R.H.S}$

$$\boxed{\begin{aligned} &\log z = \ln|z| + i \arg z \\ &\arg z = \text{Arg } z \pm 2n\pi \\ &\qquad n = 0, 1, 2 \ldots \end{aligned}}$$

(iv) $\log i^2 = \log(-1) = \ln 1 + i\arg(-1)$
$= i\pi$

$\arg(-1) = \pi \pm 2n\pi$
for $n = 0$
$\arg(-1) = \pi \in \left(\frac{3\pi}{4}, \frac{11\pi}{4}\right)$

& $2\log i = 2(\ln 1 + i\arg i)$
$= 2i\frac{5\pi}{2} = 5\pi i$
$\neq L.H.S.$

$\arg i = \frac{\pi}{2} \pm 2n\pi$
for $n = 1$
$\arg i = \frac{5\pi}{2} \in \left(\frac{3\pi}{4}, \frac{11\pi}{4}\right)$

→ Rest part do by yourself.

7. (i) $(1+i)^i = e^{i\log(1+i)}$
$= e^{i\{\ln\sqrt{2} + i(\frac{\pi}{4} \pm 2n\pi)\}}$
$= e^{i\ln\sqrt{2}} \cdot e^{-(\frac{\pi}{4} \pm 2n\pi)}$
$= e^{i\ln\sqrt{2}} \cdot e^{-\frac{\pi}{4}}$
$= e^{-\frac{\pi}{4} + i\ln\sqrt{2}}$ (Principal value)

(ii) $(-1)^{1/\pi} = \left(e^{i(\pi+2n\pi)}\right)^{1/\pi}$
$= e^{i(1+2n)}$

(iii) $i^i = e^{i\log i} = e^{i\left(i\left(\frac{\pi}{2} \pm 2n\pi\right)\right)}$
$= e^{-\frac{\pi}{2} \mp 2n\pi}$
$= e^{-\frac{\pi}{2}}$ → Principal value

(iv) $\left[\left(\frac{e}{2}\right)(-1-\sqrt{3}i)\right]^{3\pi i}$
$= e^{3\pi i \log\frac{e}{2}(-1-\sqrt{3}i)} = e^{3\pi i(\ln e + i(\frac{4\pi}{3} \pm 2n\pi))}$
$= e^{3\pi i(1 + i(\frac{\pi}{3} \pm 2n\pi))}$
$= e^{3\pi i - \pi^2}$ → Principal value.

8(1) $\sin z = \frac{e^{iz} - e^{-iz}}{2i}$

$= \frac{e^{i(x+iy)} - e^{-i(x+iy)}}{2i}$

$= \frac{e^{-y} e^{ix} - e^{y} e^{-ix}}{2i}$

$= \frac{e^{-y}(\cos x + i\sin x) - e^{y}(\cos x - i\sin x)}{2i}$

$\therefore \overline{\sin z} = \frac{e^{-y}(\cos x - i\sin x) - e^{y}(\cos x + i\sin x)}{-2i}$

$= \frac{e^{-y} e^{-ix} - e^{y} e^{ix}}{-2i}$

$= \frac{e^{-i(x-iy)} - e^{i(x-iy)}}{-2i}$

$= \frac{e^{-i\bar{z}} - e^{i\bar{z}}}{-2i}$

$= \frac{e^{i\bar{z}} - e^{-i\bar{z}}}{2i} = \sin(\bar{z})$

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Similarly, $\overline{(\cos z)} = \cos \bar{z}$

(3) $\cos(iz) = \dfrac{e^{i(iz)} + e^{-i(iz)}}{2}$

$$= \frac{e^{-z} + e^{z}}{2}$$

$$= \frac{e^{-(x+iy)} + e^{(x+iy)}}{2}$$

$$= (e^{-x} e^{-iy} + e^{x} e^{iy})/2$$

$$= \{e^{-x}(\cos y - i\sin y) + e^{x}(\cos y + i\sin y)\}/2$$

$$\Rightarrow \overline{\cos(iz)} = \{e^{-x}(\cos y + i\sin y) + e^{x}(\cos y - i\sin y)\}/2$$

$$= \{e^{-x} e^{iy} + e^{x} e^{-iy}\}/2$$

$$= \{e^{-(x-iy)} + e^{(x-iy)}\}/2$$

$$= \{e^{-\bar{z}} + e^{\bar{z}}\}/2$$

$$= \{e^{i(i\bar{z})} + e^{-i(i\bar{z})}\}/2 = \cos(i\bar{z}) \; \#$$

(4) $\sin(iz) = \dfrac{e^{i(iz)} - e^{-i(iz)}}{2i}$

$$= \frac{e^{-z} - e^{z}}{2i}$$

$$= \{e^{-(x+iy)} - e^{(x+iy)}\}/2i$$

$$=\{e^{-x}e^{-iy}-e^{x}e^{iy}\}/2i$$

$$=\{e^{-x}(\cos y-i\sin y)-e^{x}(\cos y+i\sin y)\}/2i$$

$$\therefore \overline{\sin(iz)}=\{e^{-x}(\cos y+i\sin y)-e^{x}(\cos y-i\sin y)\}/-2i$$

$$=\left(e^{-x}e^{iy}-e^{x}e^{-iy}\right)/-2i$$

$$=\frac{e^{-(x-iy)}-e^{(x-iy)}}{-2i}$$

$$=\frac{e^{-\bar{z}}-e^{\bar{z}}}{-2i}=\frac{e^{\bar{z}}-e^{-\bar{z}}}{2i}$$

$$=\frac{e^{-i(i\bar{z})}-e^{i(i\bar{z})}}{2i}$$

$$=-\frac{\left(e^{i(i\bar{z})}-e^{-i(i\bar{z})}\right)}{2i}$$

$$=-\sin(i\bar{z})$$

Moreover when $z=n\pi i$ then from Question 5 we have

$$\overline{\sin(iz)}=\sin(i\bar{z})$$

(v) $\sin(\bar{z}) = \dfrac{e^{i\bar{z}} - e^{-i\bar{z}}}{2i}$

$= \{e^{i(x-iy)} - e^{-i(x-iy)}\}/2i$

$= (e^{ix}e^{y} - e^{-ix}e^{-y})/2i$

$= \{(\cos x + i\sin x)e^{y} - (\cos x - i\sin x)e^{-y}\}/2i$

$= \cancel{e^{y}\cos x}$

$= \{\cos x(e^{y} - e^{-y}) + i\sin x(e^{y} + e^{-y})\}/2i$

$= \frac{1}{2}\{(-i\cos x)(e^{y} - e^{-y}) + (\sin x)(e^{y} + e^{-y})\}$

$u(x,y) = \frac{1}{2}(\sin x)(e^{y} + e^{-y})$

$v(x,y) = \frac{1}{2}(\cos x)(e^{-y} - e^{y})$

$u_x = \frac{1}{2}(\cos x)(e^{y} + e^{-y})$, $u_y = \frac{1}{2}(\sin x)(e^{+y} - e^{-y})$

$v_x = (-\frac{1}{2})(\sin x)(e^{-y} - e^{y})$, $v_y = \frac{1}{2}(\cos x)\{-e^{-y} - e^{y})$

$u_x \neq v_y$ ⇒ CR-eqn not satisfied

⇒ $\sin(\bar{z})$ is nowhere analytic.

3 (1) $\sin z = \cosh 4$

$$\frac{e^{iz} - e^{-iz}}{2i} = \cosh 4$$

$$\Rightarrow e^{iz} - e^{-iz} = 2i\cosh 4$$

$$\Rightarrow e^{2iz} - 2i(\cosh 4)e^{iz} - 1 = 0$$

i.e. $w^2 - (2i\cosh 4)w - 1 = 0$

$$\Rightarrow w = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{2i\cosh 4 \pm \sqrt{-4(\cosh 4)^2 + 4}}{2}$$

$$= \{i(\cosh 4 \pm \sinh 4)\}$$

$$e^{iz} = ie^4 = e^4(\cos \pi/2 + i\sin \pi/2)$$

$$= e^4 e^{i\pi/2} = e^{(4 + i\pi/2)}$$

$$= e^{(4 + i\pi/2 + 2k\pi i)}$$

$$\Rightarrow iz = 4 + 2k\pi i + \pi i/2$$

$$\Rightarrow z = 2k\pi + \pi/2 - 4i$$

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Remaining Do yourself.