# Lecture. 7

05. 10. 2020

- Some properties of a power series
- Examples of Taylor/Maclaurin series Expansion (How to find coefficients $a_n$?)
- Uniqueness of power series expansion of analytic $f^n$s.
- Laurent's series expansion (in annular domain)
- Examples of Laurent's series expansion
- Uniqueness of series expansion (in annular domain)
- Proof of Laurent's series expansion (Only the case when $z_0 = 0$);

- <u>Some properties of power series</u> (without proof)

Consider the power series $\sum_{n=0}^{\infty} a_n(z-z_0)^n$.

Suppose $C : |z-z_0| = R$ is the circle of convergence.

Then $S(z) := \sum_{n=0}^{\infty} a_n(z-z_0)^n$

is a well-defined function $\forall z$, $|z-z_0|<R$

**Theorem I** The function $S(z)$ is continuous $\forall z$ in $|z-z_0|<R$.

**Theorem II** Let $C_1$ be any contour lying inside of $C$. Then

$$\int_{C_1} S(z)\,dz = \sum_{n=0}^{\infty} a_n \int_{C_1} (z-z_0)^n\,dz$$

**Theorem III** The function $S(z)$ is analytic at each point $z$, $|z-z_0|<R$ and

$$S'(z) = \sum_{n=1}^{\infty} n\cdot a_n (z-z_0)^{n-1} \quad \forall z,\ |z-z_0|<R$$

## Examples :

① $f(z) = e^z$ : is analytic everywhere.

So, $f(z)$ has a Taylor series expansion $\forall z$ around any $z_0$. For $z_0 = 0$,

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} z^n$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} z^n \quad \forall z.$$

$f^{(n)}(z) = e^z$

$\Rightarrow f^{(n)}(0) = 1$

② $f(z) = z^2 e^{3z}$ (analytic everywhere)

Find Maclaurin series expansion.
(i.e. $z_0 = 0$).

Method-I : write $f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} \cdot z^n$

and find $f^{(n)}(0)$

Method. II . Use the expansion of $e^z$ from ①

$$f(z) = z^2 e^{3z} = z^2 \sum_{n=0}^{\infty} \frac{1}{n!} (3z)^n$$

$$= \sum_{n=0}^{\infty} \frac{3^n z^{n+2}}{n!} \quad \forall z$$

Question : How do we know that the expression in Method II is the Taylor series expansion i.e. the coeff. are exactly $\frac{f^{(n)}(z_0)}{n!}$.

(In other words, do both the methods give same answer ?)

Answer : Yes, they are same due to the following theorem..

Theorem IV (Uniqueness of Taylor series) : If

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n \quad \forall z \text{ with } |z-z_0|<R$$

for some $z_0$ and for some $R$. (this expression means $\sum_{n=0}^{\infty} a_n (z-z_0)^n$ converges to $f(z)$ $\forall z$, $|z-z_0|<R$)

Then the series is Taylor series expansion of $f(z)$ in the domain $|z-z_0|<R$.

(means $a_n = \frac{f^{(n)}(z_0)}{n!}$).

This theorem enables us to find Taylor series expansion by Method II.

Examples continued . . .

(3) $f(z) = \sin z$

By method I, $f(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{2n+1}$.

$f^{(n)}(z) = \sin z$ or $\cos z$

$\Rightarrow f^{(n)}(0) = 0$ if $n$ even

$f^{(2n+1)}(0) = (-1)^n$

By method II, write $\sin z = \frac{e^{iz} - e^{-iz}}{2}$ and use the expansion of $e^z$ around $z_0 = 0$.

Ex: Complete this and verify your solution.

(4) $f(z) = \cos z$.

We have $\sin z = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{2n+1}$

By Theorem III, $\cos z = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} (2n+1) \cdot z^{2n}$

$= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} z^{2n} \quad \forall z.$

By Theorem IV above is the Maclaurin series of $\cos z$.

(5) Ex : Write Maclaurin series for

$\sinh z$, $\cosh z$

$\quad\; \|$ $\qquad\qquad \|$

[ $-i\sin iz$ $\qquad$ $\cos(iz)$ : Use previous Egs ]

Ex.

(6) Write Maclaurin series for

| | | | Ans |
|---|---|---|---|
| (a) | $f(z) = \dfrac{1}{1-z}$ | $\lvert z\rvert < 1$ ; $z_0 = 0$ | $\sum z^n$ |
| (b) | $f(z) = \dfrac{1}{1+z}$ | $\lvert z\rvert < 1$, $z_0 = 0$ | $\sum (-1)^n z^n$. |
| (c) | $f(z) = \dfrac{1}{z}$ | $\lvert z-1\rvert < 1$ | $\sum (-1)^n (z-1)^n$ |

around $z_0 = 1$.

[Try to avoid method I as much as possible. for (C), use (a).]

(7) $f(z) = \dfrac{1+2z^2}{z^3+z^5}$

analytic except at $z^3+z^5=0$

i.e. $z^3(1+z^2)=0$

i.e $z=0$, $z=\pm i$

Caution

Cannot write Maclaurin series for $f(z)$.

We can still write some series expansion for $f(z)$ when $z\neq 0$ & $z\neq\pm i$.

$$f(z) = \frac{1+2z^2}{z^3+z^5} = \frac{1+2z^2}{z^3(1+z^2)} = \frac{1+z^2+z^2}{z^3(1+z^2)}$$

$$= \frac{1}{z^3}\left(1+\frac{z^2}{1+z^2}\right)$$

$$= \frac{1}{z^3}\left(1+z^2\cdot\left(\sum_{n=0}^{\infty}(-1)^n z^{2n}\right)\right)$$

$$= \frac{1}{z^3}\left(1+z^2\left(1-z^2+z^4-z^6+\dots\right)\right)$$

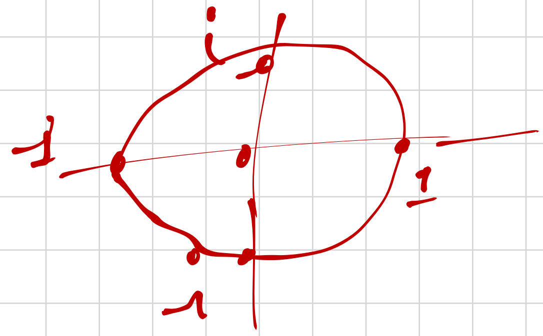
$$= \frac{1}{z^3}+\frac{1}{z}-z+z^3-z^5+\dots$$

$$\frac{1}{1+z} = \sum_{n=0}^{\infty}(-1)^n z^n$$

this equality holds for $0<|z|<1$ $\left[\begin{matrix} z\neq 0 \\ z\neq\pm i\end{matrix}\right]$

Such series expansion are known as Laurent series.
as L

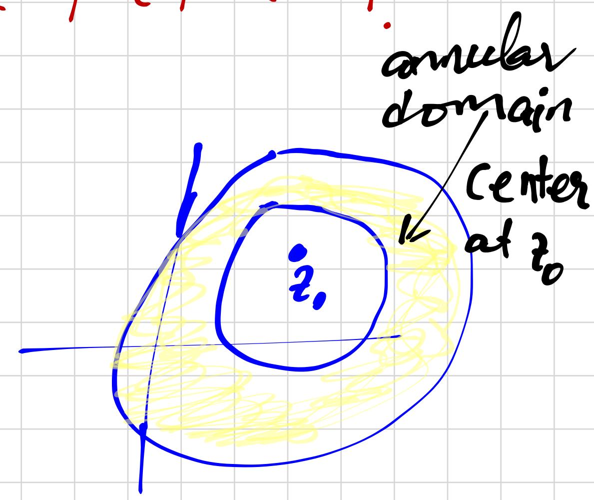
→ $f(z) = \dfrac{1+2z^2}{z^3+z^5}$ : not analytic at $z=0$




but still analytic in $0 < |z| < 1$.

More generally,

→ Suppose some $f(z)$ : analytic in an annular domain




**Laurent's Theorem** : Suppose $f(z)$ is analytic throughout an annular domain centered at $z_0$ $R_1 < |z - z_0| < R_2$. Then at each point $z$ in the annular domain:

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n}$$

$$= a_0 + a_1 (z - z_0) + a_2 (z - z_0)^2 + \dots + \frac{b_1}{(z - z_0)} + \frac{b_2}{(z - z_0)^2} + \dots$$

where $a_n = \dfrac{1}{2\pi i} \displaystyle\int_C \frac{f(z)\,dz}{(z - z_0)^{n+1}}$ & $b_n = \dfrac{1}{2\pi i} \displaystyle\int_C \frac{f(z)\,dz}{(z - z_0)^{-n+1}}$

and $C$ : positively oriented simple closed contour around $z_0$ and lying inside of the annular domain

Examples :

$$f(z) = \frac{1+2z^2}{z^3+z^5} = \frac{1}{z^3} + \frac{1}{z} - z + z^3 - z^5 + \cdots$$

$0 < |z| < 1$

is a (Laurent) series expansion with

$a_0 = 0, \quad a_1 = -1, \quad a_2 = 0, \quad a_3 = 1, \quad \ldots$

$b_1 = 1, \quad b_3 = 1, \quad b_n = 0$ if $n \neq 1, 3$.

But we have not calculated $a_n$ & $b_n$ using the integration formula.

How do we know it is Laurent series expansion.

Theorem : (uniqueness of Laurent series) around $z_0$.

Suppose a series

$$\sum_{n=-\infty}^{\infty} c_n (z-z_0)^n = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n}$$

converges to $f(z)$ for all $z$ in an annular domain $R_1 < |z-z_0| < R_2$. Then it is the Laurent series expansion of $f(z)$ in $R_1 < |z-z_0| < R_2$.

[And so calculating $a_n$ and $b_n$ by other methods give the same values].

## Examples

① $f(z) = \dfrac{\sin z}{z^2}$ : not analytic at $z=0$.
: analytic in $0<|z|<\infty$

Maclaurin series of $\sin z$ is

$$\sin z = \left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots\right) \quad \forall z.$$

Now for $0<|z|<\infty$

$$\frac{\sin z}{z^2} = \frac{1}{z^2}\left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots\right)$$

$$= \frac{1}{z} - \frac{z}{3!} + \frac{z^3}{5!} - \frac{z^5}{7!} \cdots$$

$b_1 = 1 \quad b_n = 0 \ \forall n \geq 2$

$a_{2n} = 0$,

$a_{2n-1} \underset{\text{(Verify)}}{=} \dfrac{(-1)^n}{(2(n+1)-1)!} \qquad \forall n \geq 1.$

So $\dfrac{\sin z}{z^2} = \dfrac{1}{z} + \sum \dfrac{(-1)^n}{(2(n+1)-1)!} \qquad 0<|z|<\infty.$

② $f(z) = e^{\frac{1}{z}}$ : analytic in $0<|z|<\infty$.

__Use__: $e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots \quad \forall z$

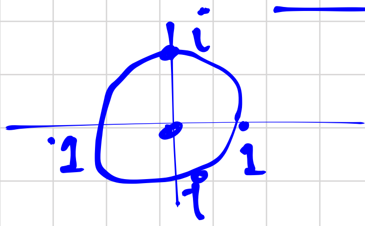
__So__ $e^{\frac{1}{z}} = 1 + \frac{1}{z} + \frac{1}{2!\,z^2} + \frac{1}{3!\,z^3} + \cdots \quad \forall z \neq 0$

is Laurent series expansion in $0<|z|<\infty$.

③ $f(z) = \frac{1}{z^2+1} \qquad z \neq i, -i$

: analytic $\forall z,\ z \neq i, -i$

Ex* can write Taylor series expansion for $\underline{z_0 = 0}$ and $|z| < 1$. [Use Method. II and $\frac{1}{1+z}$]




* can write Laurent series expansion for $\underline{z_0 = i}$ and $0 < |z-i| < 2$

* ~~Si~~ for $\underline{z_0 = -i}$ and $0 < |z-(-i)| < 2$

NOTE: Generally, we do not use the integration formula for finding $a_n$ and $b_n$ in Laurent series expansion.

Infact, we use the values of $a_n$ & $b_n$ to compute integrals.

Let's write Laurent series expansion of

$f(z) = \frac{1}{z^2+1}$ in a deleted neighbourhood (annular domain) around $z_0 = i$.

$$= \frac{1}{(z-i)(z+i)} = \sum_{n=0}^{\infty} a_n (z-i)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z-i)^n}$$

$$= \frac{1}{(z-i)} \left[ \frac{1}{2i + z - i} \right]$$

$$= \frac{1}{(z-i)} \frac{1}{2i\left(1 + \frac{z-i}{2i}\right)}$$

$$= \frac{1}{(z-i)2i} \sum_{n=0}^{\infty} (-1)^n \left(\frac{z-i}{2i}\right)^n$$

in $\left|\frac{z-i}{2i}\right| < 1$

Recall:

$\frac{1}{1+z} = \sum (-1)^n z^n \quad |z| < 1$

so $\frac{1}{1+(z-i)} = \sum (-1)^n (z-i)^n$

$|z-i| < 1$

$$f(z) = \frac{1}{(z-i)\,2i} \sum_{n=0}^{\infty} \left(\frac{-1}{2i}\right)^n (z-i)^n$$

$|z-i| < |2i|$

i.e. $|z-i| < 2$

$$= \frac{1}{2i(z-i)} - \sum_{n=0}^{\infty} \left(\frac{-1}{2i}\right)^{n+1} (z-i)^n .$$

$|z-i| < 2$

is Laurent series expansion around $z_0 = i$ in $0 < |z-i| < 2$

Thus, $a_0 = \frac{(-1)^n}{(2i)^{n+1}}$

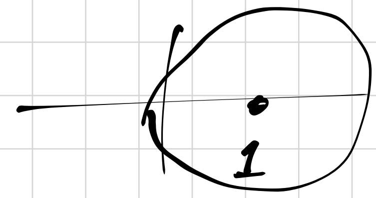
$b_1 = \frac{1}{2i}$, $b_n = 0 \ \forall n \geq 2$.

$= \frac{1}{2\pi i}\int_C f(z)\,dz$ for any simple closed contour $C$ lying inside of the domain $0 < |z-i| < 2$

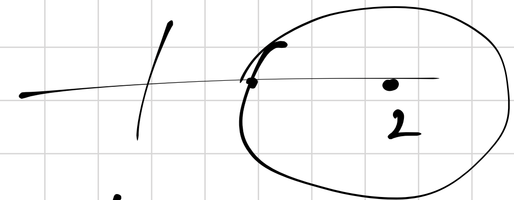
Ex :- Write Laurent series exp. around $z_0 = -i$. (Write the annular domain)

**Tutorial problem**: Write the Laurent series expansion of $f(z) = \dfrac{1}{(z-1)(z-2)}$

(i) in $0 < |z-1| < 1$

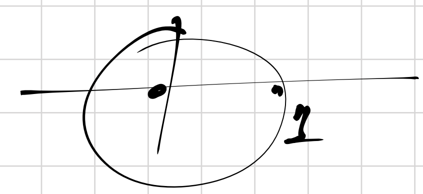



(ii) in $0 < |z-2| < 1$

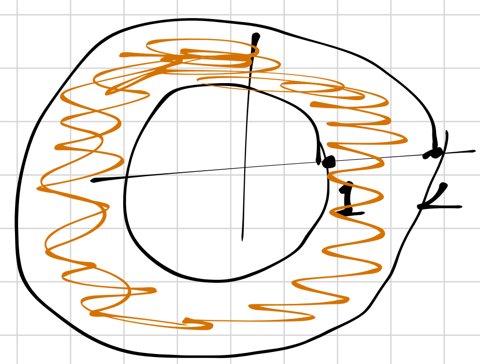



(iii) $f(z)$ is analytic in $|z| < 1$.

so write Taylor series in $|z| < 1$.




(iv) Laurent series in $1 < |z| < 2$.




(v) $2 < |z| < \infty$

**Summary**: ① Taylor series is a special case of Laurent series with $b_n = 0$.

(We can verify that if $f(z)$ is analytic in $|z - z_0| < R$, then

$b_n$

$$b_n = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z-z_0)^{-n+1}} dz$$

$$= \frac{1}{2\pi i} \int_C f(z)(z-z_0)^{n-1} dz$$

$= 0$ by Cauchy's G. Th$^m$.

② If $f(z)$ is analytic in a disk $|z-z_0|<R$ except at the point $z_0$, then we can write Laurent series expansion in deleted neighbourhood.

$0<|z-z_0|<R$.

③ If $f(z)$ is analytic at each point $z$ in the <u>exterior</u> of <u>the circle</u> $|z-z_0|=R_1$, then we can write Laurent series exp in $R_1<|z-z_0|<\infty$

④ If $f(z)$ is analytic everywhere except at $z_0$, then $f(z)$ has Laurent series expⁿ $\forall z$ in $0<|z-z_0|<\infty$

( i.e $\forall z$ except at $z=z_0$ :

⑤ $f(z)$ is analytic at $z_0$ $\overset{\text{we proved}}{\Longrightarrow}$ $f(z)$ has Taylor series expansion in a nbd of $z_0$.
$\Longleftarrow$ (See Th III)

[Your doubts]

**Def$^n$: Holomorphic function :-**

(on wikipedia (as you said)) : is a complex valued function of complex variable $f(z)$ such that $f'(z_0)$ exists in a neighbourhood of $z_0$ for all $z_0$ in the domain of definition of $f(z)$.

which is equivalent to say that $f(z)$ is analytic at each point $z_0$ of its domain of its definition

which is to say that $f(z)$ is analytic function.

So, these two definitions are same for $f(z)$.

We have discussed that an analytic function is
1. infinitely differentiable
2. has Taylor series expansion around each pt.

The two terms analytic function and holomorphic functions are used interchangeably for complex valued functions of complex variables.

But Analytic function in general (for real $f^n$) is a function which can be given by its Taylor series in a nbd of each point in its domain of definition.

[So, a differentiable (real) valued function may not have Taylor series expansion]

Eg: square root: $R^+ \to R^+$

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[In fact, infinitely differentiable f's may not have Taylor series expansion]