Oscillations and Waves
Recap

The damped harmonic oscillator

$$F = F_{spring} + F_{friction}$$

$$F = -kx - bv$$

$$m\ddot{x} = -kx - bv$$

$$k, b > 0$$

$$\ddot{x} + 2\gamma\dot{x} + \omega_0^2 x = 0$$

$$2\gamma = \frac{b}{m}, \omega_0{}^2 = \frac{k}{m}$$

Trial solution

$$x(t) = Ae^{\alpha t}$$

The damped harmonic oscillator

$$m\ddot{x} = -kx - bv$$

$$\ddot{x} + 2\gamma\dot{x} + \omega_0^2 x = 0$$

$$2\gamma = \frac{b}{m}, \omega_0{}^2 = \frac{k}{m}$$

Trial solution

$$x(t) = Ae^{\alpha t}$$

$$\alpha^2 + 2\gamma\alpha + \omega_0{}^2 = 0$$

$$\alpha = -\gamma \pm \sqrt{\gamma^2 - \omega_0{}^2}$$

The damped harmonic oscillator

$$\ddot{x} + 2\gamma\dot{x} + \omega_0^2 x = 0$$

$$2\gamma = \frac{b}{m}, \omega_0{}^2 = \frac{k}{m}$$

Trial solution

$$x(t) = Ae^{\alpha t}$$

$$\alpha = -\gamma \pm \sqrt{\gamma^2 - \omega_0{}^2}$$

$$x(t) = e^{-\gamma t}\left[Ae^{\sqrt{\gamma^2-\omega_0{}^2}\,t} + Be^{-\sqrt{\gamma^2-\omega_0{}^2}\,t} \right]$$

$$x(t) = e^{-\gamma t}\left[Ae^{\Omega t} + Be^{-\Omega t} \right]$$

$$\Omega = \sqrt{\gamma^2 - \omega_0{}^2}$$

The damped harmonic oscillator

$$\ddot{x} + 2\gamma\dot{x} + \omega_0^2 x = 0$$

$$2\gamma = \frac{b}{m}, \omega_0{}^2 = \frac{k}{m}$$

$$x(t) = e^{-\gamma t}\left[Ae^{\Omega t} + Be^{-\Omega t}\right]$$

$$\Omega = \sqrt{\gamma^2 - \omega_0{}^2}$$

Case 1: $\Omega^2 < 0$ **(Underdamping)**

Case 2: $\Omega^2 = 0$ **(Critical damping)**

Case 3: $\Omega^2 > 0$ **(Overdamping)**

Underdamping

$$\Omega^2 < 0$$

$$\Omega = \sqrt{\gamma^2 - \omega_0{}^2}$$

$$\gamma = \frac{b}{2m} < \omega_0$$

Light damping

$$\Omega = \sqrt{-(\omega_0{}^2 - \gamma^2)}$$

$$\Omega = i\sqrt{(\omega_0{}^2 - \gamma^2)}$$

$$\Omega = i\widetilde{\omega}$$

$$\alpha = -\gamma + i\widetilde{\omega}, -\gamma - i\widetilde{\omega}$$

$$z(t) = e^{-\gamma t}[Ae^{i\widetilde{\omega}t} + Be^{-i\widetilde{\omega t}}]$$

$$z(t) = e^{-\gamma t}\{(A+B)\cos\widetilde{\omega}t \ + i(A-B)\sin\widetilde{\omega}t\}$$

Solution can be made real by choosing appropriate constants

$$A = C\frac{e^{i\phi}}{2}, B = C\frac{e^{-i\phi}}{2}$$

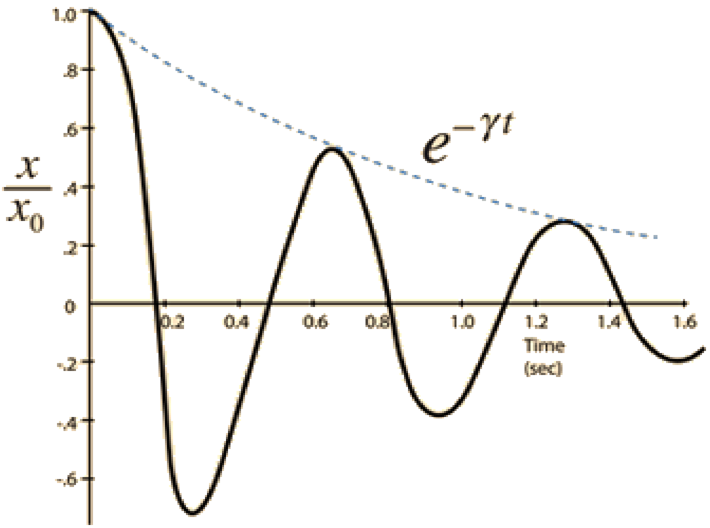
$$\cos\phi = \frac{e^{i\phi}+e^{-i\phi}}{2}, \sin\phi = \frac{e^{i\phi}-e^{-i\phi}}{2i}$$

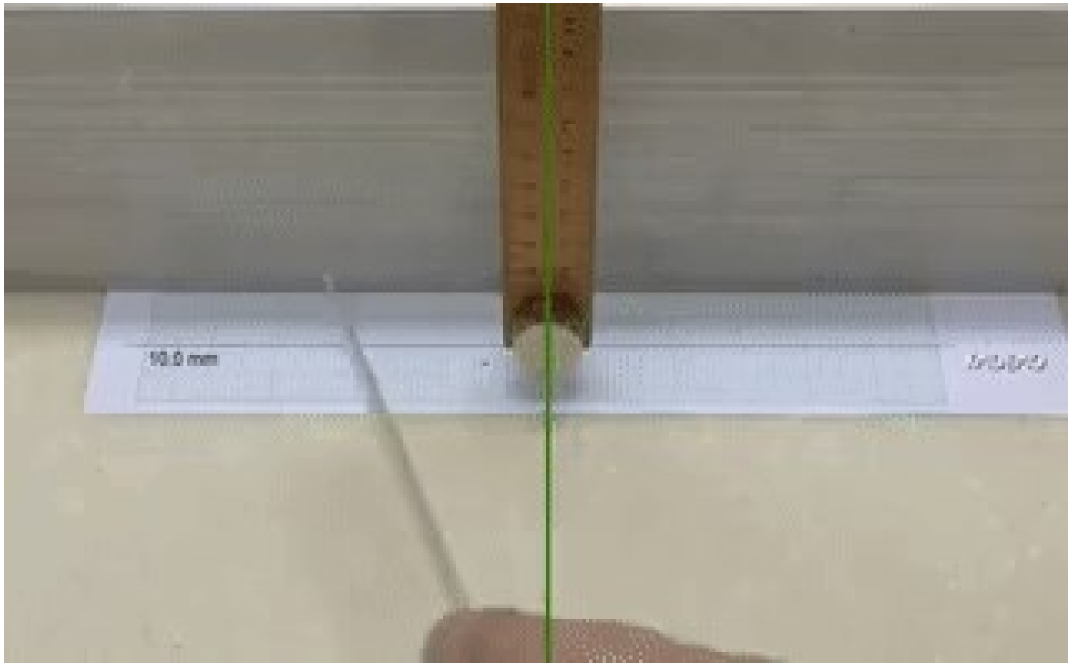
$$\mathrm{x}(t) = Ce^{-\gamma t}\{cos\phi\cos\widetilde{\omega}t \ - sin\phi\sin\widetilde{\omega}t\}$$

$$\mathrm{x}(t) = Ce^{-\gamma t}\cos(\widetilde{\omega}t+\phi)$$

Underdamping: $\Omega^2 < 0$

$$x(t) = Ce^{-\gamma t}\cos(\tilde{\omega} t + \phi)$$





The damped harmonic oscillator

$$\ddot{x} + 2\gamma\dot{x} + \omega_0^2 x = 0$$

$$2\gamma = \frac{b}{m}, \omega_0{}^2 = \frac{k}{m}$$

$$x(t) = e^{-\gamma t}\left[Ae^{\Omega t} + Be^{-\Omega t} \right]$$

$$\Omega = \sqrt{\gamma^2 - \omega_0{}^2}$$

Case 1: $\Omega^2 < 0$ **(Underdamping)**

Case 2: $\Omega^2 = 0$ **(Critical damping)**

Case 3: $\Omega^2 > 0$ **(Overdamping)**

The damped harmonic oscillator

Case 2: $\Omega^2 = 0$ (Critical damping)

$$\alpha = -\gamma \pm \sqrt{\gamma^2 - \omega_0{}^2}$$

$$\Omega = \sqrt{\gamma^2 - \omega_0{}^2}$$

$$x(t) = e^{-\gamma t}\left[Ae^{\Omega t} + Be^{-\Omega t} \right]$$

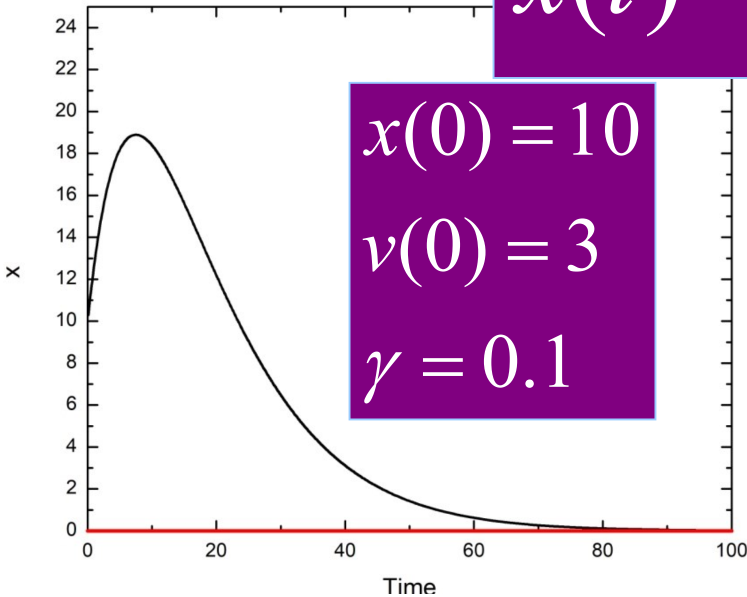
Under critical damping $\alpha = -\gamma$

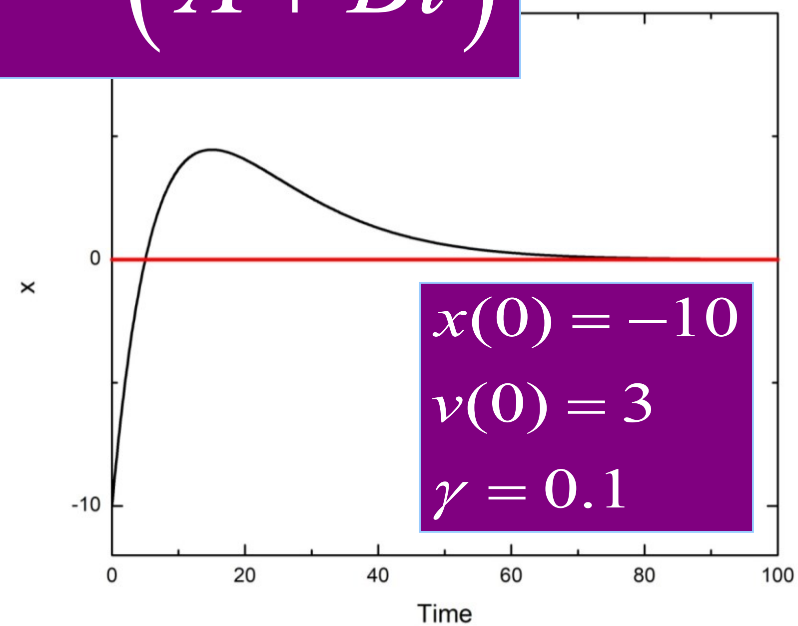
$$x(t) = Ae^{-\gamma t}$$

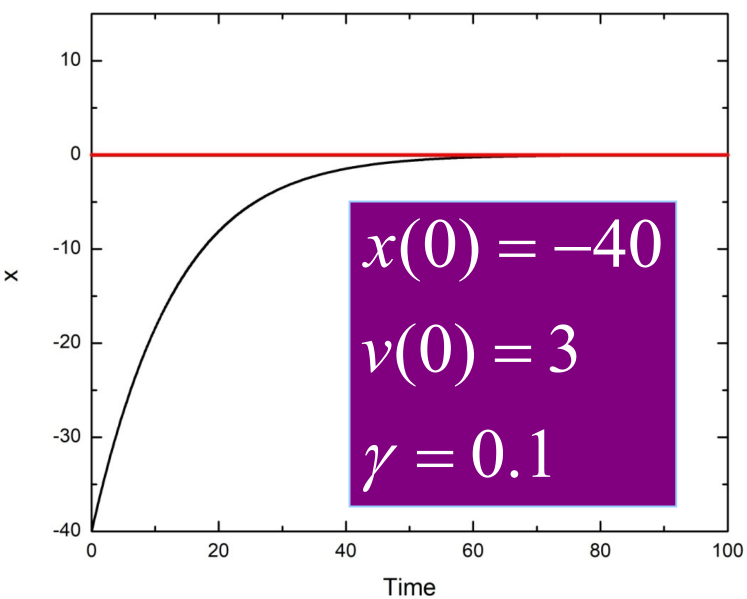
The other solution is $te^{-\gamma t}$

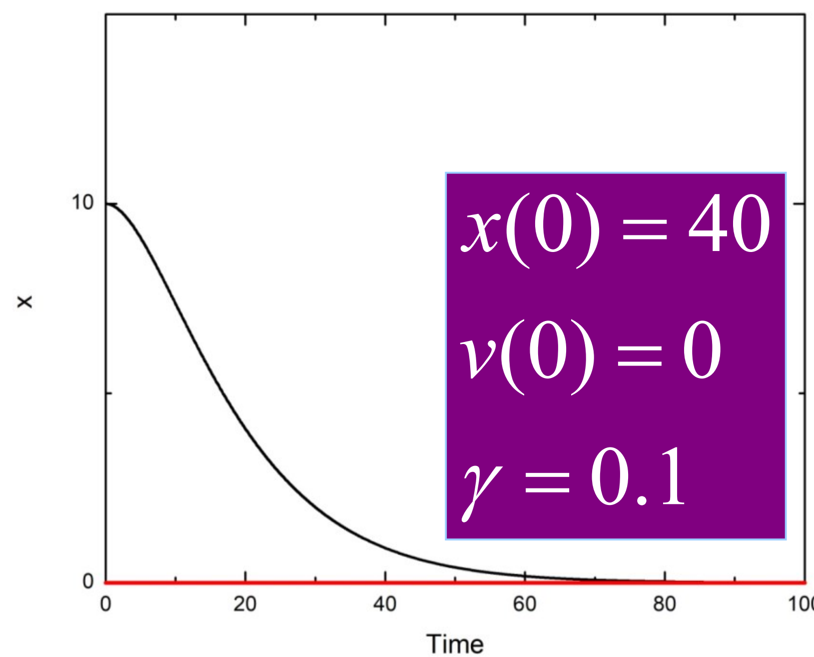
$$x(t) = e^{-\gamma t}\left(A + Bt \right)$$

$$x(t) = e^{-\gamma t}\left(A + Bt\right)$$









The damped harmonic oscillator

$$\ddot{x} + 2\gamma\dot{x} + \omega_0^2 x = 0$$

$$2\gamma = \frac{b}{m}, \omega_0{}^2 = \frac{k}{m}$$

$$x(t) = e^{-\gamma t}\left[Ae^{\Omega t} + Be^{-\Omega t} \right]$$

$$\Omega = \sqrt{\gamma^2 - \omega_0{}^2}$$

Case 1: $\Omega^2 < 0$ **(Underdamping)**

Case 2: $\Omega^2 = 0$ **(Critical damping)**

Case 3: $\Omega^2 > 0$ **(Overdamping)**

Case 3: $\Omega^2 > 0$ (Overdamping)

$$x(t) = Ae^{\alpha t}$$

$$\alpha = -\gamma \pm \sqrt{\gamma^2 - \omega_0{}^2}$$

$$\Omega = \sqrt{\gamma^2 - \omega_0{}^2}$$

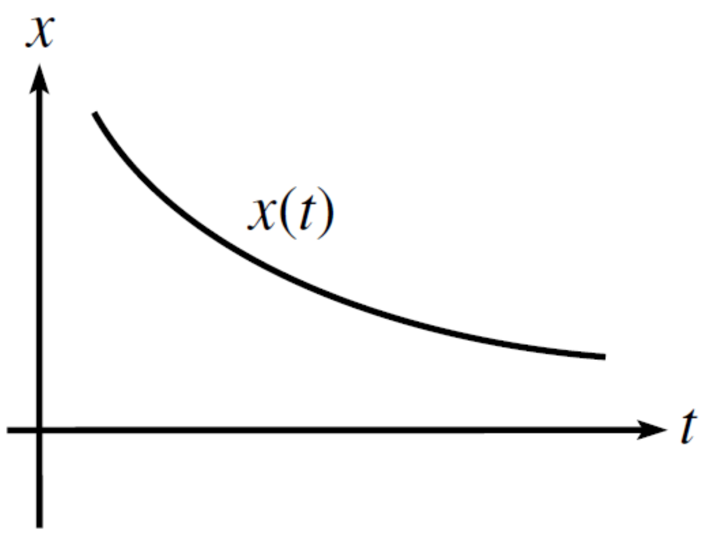
$$\alpha = -\gamma + \Omega, \ \ -\gamma - \Omega$$

$$x(t) = e^{-\gamma t}\left[Ae^{\Omega t} + Be^{-\Omega t} \right]$$

Overdamping

$$x(t) = e^{-\gamma t}\left[Ae^{\Omega t} + Be^{-\Omega t} \right]$$

$$x(t) = \left[\underbrace{Ae^{-(\gamma-\Omega)t}}_{\text{Slow damping}} + \underbrace{Be^{-(\gamma+\Omega)t}}_{\text{Quick damping}} \right]$$

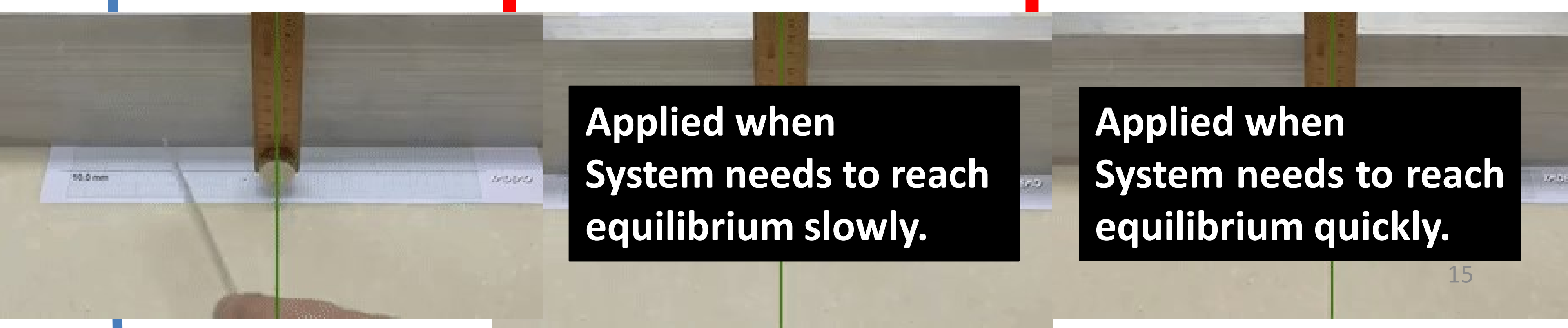


Analysis of damping function f(t)

If we are given a spring with a fixed ω_0

$$\Omega = \sqrt{\gamma^2 - \omega_0{}^2}$$

	Underdamped	Overdamped	Critical Damping
	$\gamma < \omega_0$	$\gamma > \omega_0$	$\gamma = \omega_0$
$f(t)$	$e^{-\gamma t}$	$e^{-(\gamma-\Omega)t}$	$e^{-\omega_0 t}$
		Applied when System needs to reach equilibrium slowly.	Applied when System needs to reach equilibrium quickly.

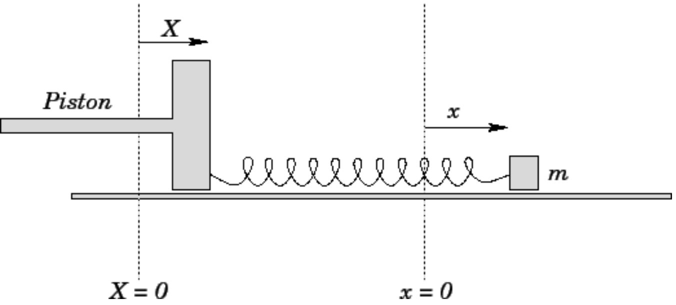


Driven and Damped Oscillations

Forced Harmonic Oscillator

Driven and Damped Oscillations

Forced Harmonic Oscillator



$$F = F_{spring} + F_{friction} + F_{drive}(t)$$

$$F = -kx - bv + F_{dr}(t)$$

$$m\ddot{x} = -kx - bv + F_{dr}(t)$$

$$k, b > 0$$

$$\ddot{x} + 2\gamma\dot{x} + {\omega_0}^2 x = \frac{F_{dr}(t)}{m}$$

$$2\gamma = \frac{b}{m}, {\omega_0}^2 = \frac{k}{m}$$

$$F_{dr}(t) = F_0 e^{i\omega_d t}$$

Driven and Damped Oscillations

Forced Harmonic Oscillator

$$\ddot{x} + 2\gamma\dot{x} + \omega_0{}^2 x = \frac{F_0}{m} e^{i\omega_d t}$$

$$2\gamma = \frac{b}{m}, \omega_0{}^2 = \frac{k}{m}$$

We know the solution will be of oscillatory. Hence the general trial solution is given as

$$x(t) = Ae^{i(\omega_d t - \phi)}$$

$$e^{-i\phi} = \frac{\left(F_0 / mA\right)}{\left(\omega_0{}^2 - \omega_d{}^2\right) + i2\gamma\omega_d}$$

Driven and Damped Oscillations

Forced Harmonic Oscillator

$$e^{-i\phi} = \frac{\left(F_0 / mA\right)}{\left(\omega_0{}^2 - \omega_d{}^2\right) + i2\gamma\omega_d}$$

$$\cos\phi = \frac{\left(F_0 / mA\right)\left(\omega_0{}^2 - \omega_d{}^2\right)}{\left(\omega_0{}^2 - \omega_d{}^2\right)^2 + 4\gamma^2\omega_d{}^2}$$

$$\sin\phi = \frac{\left(F_0 / mA\right)2\gamma\omega_d}{\left(\omega_0{}^2 - \omega_d{}^2\right)^2 + 4\gamma^2\omega_d{}^2}$$

$$\boldsymbol{\phi = \tan^{-1}\left[\frac{2\gamma\omega_d}{(\omega_0^2 - \omega_d^2)}\right]}$$

$$A(\omega_d) = \frac{F_0}{m\sqrt{\left(\omega_0{}^2 - \omega_d{}^2\right)^2 + 4\gamma^2\omega_d{}^2}}$$

Driven and Damped Oscillations

Forced Harmonic Oscillator

$$\ddot{x} + 2\gamma\dot{x} + \omega_0{}^2 x = \frac{F_0}{m} e^{i\omega_d t}$$

$$2\gamma = \frac{b}{m}, \omega_0{}^2 = \frac{k}{m}$$

$$x(t) = Ae^{i(\omega_d t - \phi)}$$

$$x(t) = \frac{F_0}{m\sqrt{\left(\omega_0{}^2 - \omega_d{}^2\right)^2 + 4\gamma^2\omega_d{}^2}} e^{i(\omega_d t - \phi)}$$

Resonance

$$x(t) = \frac{F_0}{m\sqrt{\left(\omega_0{}^2 - \omega_d{}^2\right)^2 + 4\gamma^2\omega_d{}^2}} e^{i(\omega_d t - \phi)}$$

The amplitude of the motion depends on driving frequency

The amplitude will be extremum for $\omega_d = \omega_{res}$

$$\frac{dA}{d\omega_d} = 0$$

$$\frac{dA}{d\omega_d} = -\frac{1}{2} F_0 / m \frac{-4\left(\omega_0{}^2 - \omega_d{}^2\right)\omega_d + 8\gamma^2\omega_d}{\left[\left(\omega_0{}^2 - \omega_d{}^2\right)^2 + 4\gamma^2\omega_d{}^2\right]^{3/2}}$$

Resonance

$$x(t) = \frac{F_0}{m\sqrt{\left(\omega_0{}^2 - \omega_d{}^2\right)^2 + 4\gamma^2\omega_d{}^2}} e^{i(\omega_d t - \phi)}$$

For extremum $\frac{dA}{d\omega_d} = 0$

The condition for extremum : $\omega_d{}^2 = \omega_0{}^2 - 2\gamma^2$

Resonance Frequency

$$\omega_{res} = \sqrt{\omega_0{}^2 - 2\gamma^2}$$

Resonance

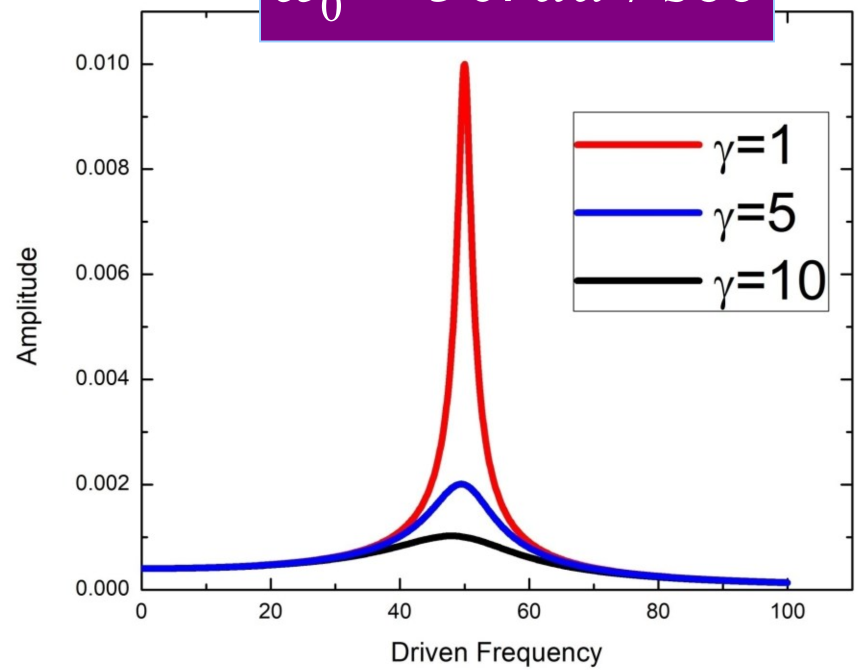
$$x(t) = \frac{F_0}{m\sqrt{\left(\omega_0{}^2 - \omega_d{}^2\right)^2 + 4\gamma^2\omega_d{}^2}} e^{i(\omega_d t - \phi)}$$

$$x(t) = B\left(\omega_d\right) e^{i(\omega_d t - \phi)}$$

$$B(\omega_d) = \frac{F_0}{m\sqrt{\left(\omega_0{}^2 - \omega_d{}^2\right)^2 + 4\gamma^2\omega_d{}^2}}$$

$$\omega_{res} = \sqrt{\omega_0{}^2 - 2\gamma^2}$$

$$\omega_0 = 50 rad / \sec$$



Examples

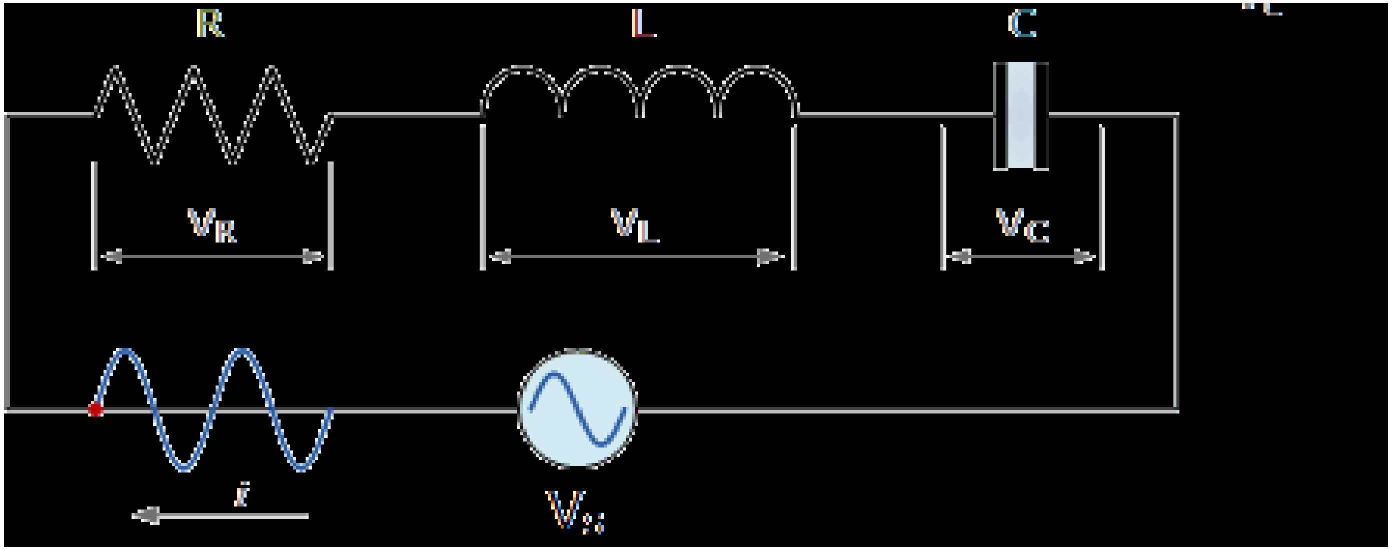
1. Tacoma bridge collapse (Aeroelastic flutter)

Washington, November 7, 1940

Opened for public on July 1, 1940

2. Wine glass Breaking

LCR Circuits



$$L\frac{d^2q}{dt^2}+R\frac{dq}{dt}+\frac{q}{C}=V_0\cos(\omega_d t)$$

$$\frac{d^2q}{dt^2}+\frac{R}{L}\frac{dq}{dt}+\frac{q}{LC}=\frac{V_0}{L}\cos(\omega_d t)$$

Solution of LCR Circuit

Forced Harmonic Oscillator

$$\ddot{x} + 2\gamma\dot{x} + {\omega_0}^2 x = \frac{F_0}{L} e^{i\omega_d t}$$

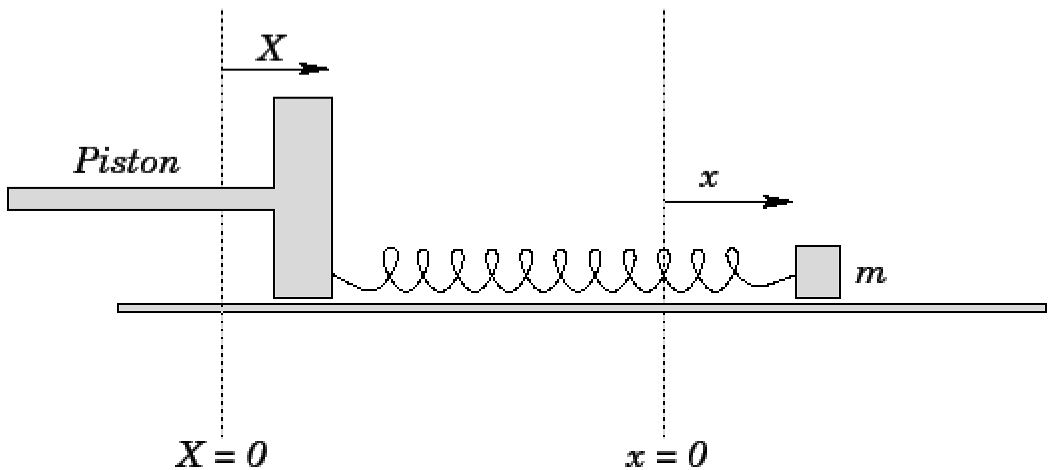
$x \equiv q$ $\quad 2\gamma \equiv \frac{R}{L}$ $\quad \omega_0 \equiv \frac{1}{\sqrt{LC}}$

$$x(t) = \frac{F_0}{L\sqrt{\left({\omega_0}^2 - {\omega_d}^2\right)^2 + 4\gamma^2{\omega_d}^2}} e^{i(\omega_d t - \phi)}$$

LCR circuits can be used as *analog radio tuners*. The circuit only has a strong response when the signal oscillates in the angular frequency range $\omega = 1/\sqrt{LC} \pm R/L$ If L, C and R are properly chosen then the circuit can be made to strongly absorb the signal from a particular radio station, which has a given carrier frequency and bandwidth. In practice, the values of L and R are fixed, while the value of C is varied till the signal from desired radio station is obtained.

Transient Oscillator Response

Driven Damped Harmonic Oscillator



Piston executes simple harmonic oscillation of angular frequency, $\omega>0$, and amplitude $X_0>0$. This system is described by equation

$$\ddot{x} + \gamma \dot{x} + {\omega_0}^2 x = \omega_0^2 X_0 \cos \omega t \quad \text{... (1)}$$

$$\gamma = \frac{b}{m}, \omega_0^2 = \frac{k}{m}$$

Transient Oscillator Response

Driven Damped Harmonic Oscillator equation

$$\ddot{x} + \gamma \dot{x} + {\omega_0}^2 x = \omega_0^2 X_0 \cos \omega t$$ **... (1)**

Solution of above equation is

$$\boldsymbol{x_{ta}(t) = x_0 \cos(\omega t - \varphi)}$$ **... (2)**

$$\boldsymbol{x_0 = \frac{\omega_0^2 X_0}{\left[(\omega_0^2 - \omega^2)^2 + \gamma^2 \omega^2\right]^{1/2}}}$$

$$\boldsymbol{\varphi = \tan^{-1}\left(\frac{\gamma \omega}{\omega_0^2 - \omega^2}\right)}$$

Eq.(1) is second-ordered ordinary differential equation. The general solution of this equation should contain two arbitrary constants. However, Eq. (2) does not contain any arbitrary constants. Therefore, it can not be the general solution.

Transient Oscillator Response

Undriven Damped Harmonic Oscillator equation

$$\ddot{x} + \gamma \dot{x} + {\omega_0}^2 x = 0$$

If we add solution of this equation to Eq.(2), the resultant will still be solution of Eq. (1). The general solution of undriven damped harmonic oscillator equation is

$$x_{tr}(t) = Ae^{-\gamma t/2} \cos \omega_1 t + Be^{-\gamma t/2} \sin \omega_1 t$$

Where A and B are arbitrary constants and $\omega_1 = \left(\omega_0^2 - \frac{\gamma^2}{4}\right)^{1/2}$

The general solution is

$$x(t) = x_{ta}(t) + x_{tr}(t)$$

$$= x_0 \cos(\omega t - \varphi) + Ae^{-\gamma t/2} \cos \omega_1 t + Be^{-\gamma t/2} \sin \omega_1 t$$

... (3)

Transient Oscillator Response

$$x(t) = x_0 \cos(\omega t - \varphi) + Ae^{-\gamma t/2} \cos \omega_1 t + Be^{-\gamma t/2} \sin \omega_1 t$$

Time asymptotic solution

- Oscillates at the driving frequency ω
- Constant amplitude
- Independent of initial conditions
- As time progresses the term becomes dominant

Transient solution

- Oscillates at the frequency ω_1
- Amplitude decays exponentially
- Depends on initial conditions
- As time progresses the term decays away

Transient Oscillator Response

$$x(t) = x_0 \cos(\omega t - \varphi) + A e^{-\gamma t/2} \cos \omega_1 t + B e^{-\gamma t/2} \sin \omega_1 t \quad \dots (3)$$

Lets take the initial conditions be $x(0) = \dot{x}(0) = 0$ to find A and B

$$x(0) = x_0 \cos(\varphi) + A = 0$$

$$A = -x_0 cos\varphi$$

$$\dot{x}(0) = x_0 \omega \sin\varphi - \frac{\gamma}{2} A + \omega_1 B = 0$$

$$B = -x_0 \left[\frac{\omega \sin\varphi + \frac{\gamma}{2}\cos\varphi}{\omega_1} \right]$$

For the driving frequencies close to the resonant frequency $|\omega - \omega_0| \sim \gamma$, we can write

$$x_0 \cong \frac{\omega_0 X_0}{[4(\omega_0 - \omega)^2 + \gamma^2]^{1/2}}$$

$$\sin\varphi \cong \frac{\gamma}{[4(\omega_0 - \omega)^2 + \gamma^2]^{1/2}}$$

$$\cos\varphi \cong \frac{2(\omega_0 - \omega)}{[4(\omega_0 - \omega)^2 + \gamma^2]^{1/2}}$$

Transient Oscillator Response

$$\boldsymbol{x(t) = x_0 \cos(\omega t - \varphi) + A e^{-\gamma t/2} \cos \omega_1 t + B e^{-\gamma t/2} \sin \omega_1 t}$$

Lets take the initial conditions be $x(0) = \dot{x}(0) = 0$ to find A and B **... (3)**

$$x_0 \cong \frac{\omega_0 X_0}{[4(\omega_0 - \omega)^2 + \gamma^2]^{1/2}} \qquad \sin\varphi \cong \frac{\gamma}{[4(\omega_0 - \omega)^2 + \gamma^2]^{1/2}}$$

$$\cos\varphi \cong \frac{2(\omega_0 - \omega)}{[4(\omega_0 - \omega)^2 + \gamma^2]^{1/2}}$$

$$x(t) \cong X_0 \left[\frac{2\omega_0(\omega_0 - \omega)}{[4(\omega_0 - \omega)^2 + \gamma^2]}\right] \left[\cos\omega_1 t - e^{-\gamma t/2} \cos\omega_0 t\right]$$
$$+ X_0 \left[\frac{\omega_0 \gamma}{[4(\omega_0 - \omega)^2 + \gamma^2]}\right] \left[\sin\omega_1 t - e^{-\gamma t/2} \sin\omega_0 t\right] \qquad \text{... (4)}$$

Transient Oscillator Response

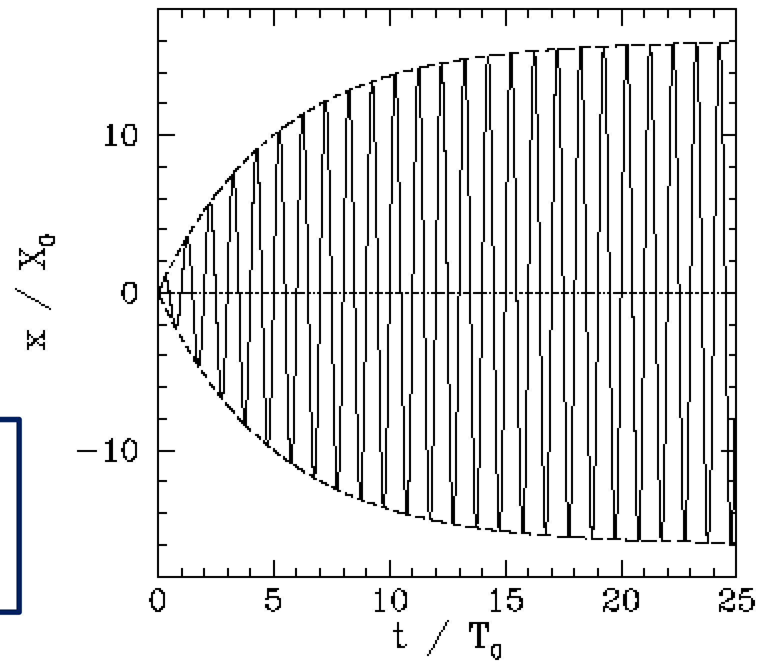
$$x(t) \cong X_0\left[\frac{2\omega_0(\omega_0-\omega)}{[4(\omega_0-\omega)^2+\gamma^2]}\right]\left[\cos\omega_1 t - e^{-\gamma t/2}\cos\omega_0 t\right]$$
$$+X_0\left[\frac{\omega_0\gamma}{[4(\omega_0-\omega)^2+\gamma^2]}\right]\left[\sin\omega_1 t - e^{-\gamma t/2}\sin\omega_0 t\right] \quad \text{... (4)}$$

Case-I: Let the driving frequency equal to resonant frequency $\omega=\omega_0$

$$x(t) = X_0\frac{\omega_0}{\gamma}\left(1-e^{-\gamma t/2}\right)\sin\omega_0 t$$
$$= X_0 Q_f\left(1-e^{-\gamma t/2}\right) sin\,\omega_0 t$$

Where $Q_f = \frac{\omega_0}{\gamma}$

$$T_0 = 2\pi/\omega_0$$
$$Q_f = \omega_0/\nu = 16$$



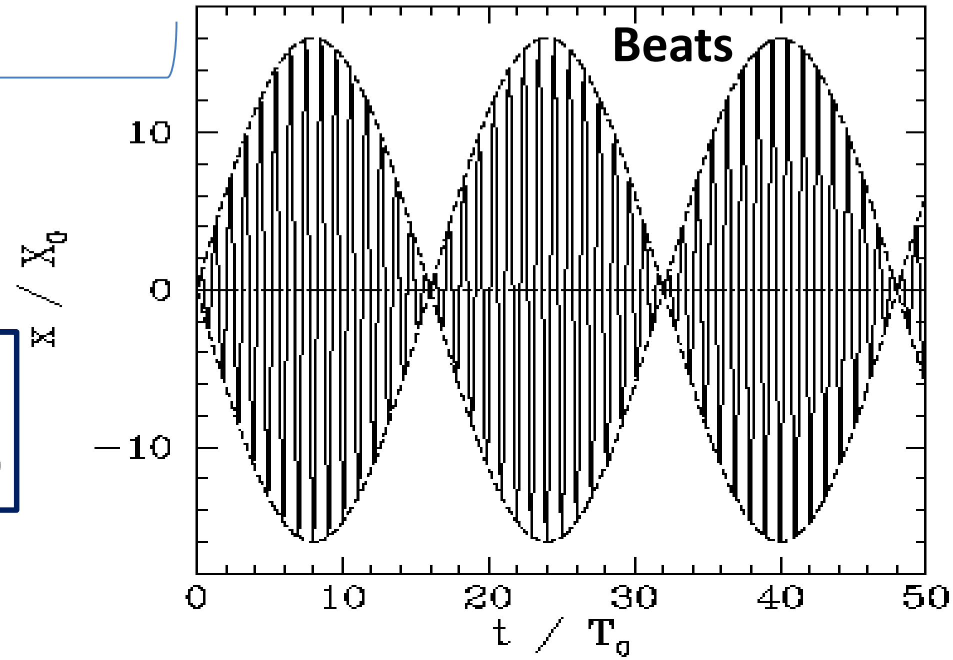
Transient Oscillator Response

$$x(t) \cong X_0 \left[\frac{2\omega_0(\omega_0 - \omega)}{[4(\omega_0 - \omega)^2 + \gamma^2]}\right] \left[\cos \omega_1 t - e^{-\gamma t/2} \cos \omega_0 t\right]$$
$$+X_0 \left[\frac{\omega_0 \gamma}{[4(\omega_0 - \omega)^2 + \gamma^2]}\right] \left[\sin \omega_1 t - e^{-\gamma t/2} \sin \omega_0 t\right] \qquad \text{... (4)}$$

Case-II: No damping γ=0

$$x(t) = \underbrace{X_0 \frac{\omega_0}{(\omega_0 - \omega)} \sin[(\omega_0 - \omega)t/2]}_{A(t)} \sin[(\omega_0 + \omega)t/2]$$

$$T_0 = 2\pi/\omega_0$$
$$\omega_0 - \omega = \omega_0/16$$



Transient Oscillator Response

$$x(t) \cong X_0\left[\frac{2\omega_0(\omega_0-\omega)}{[4(\omega_0-\omega)^2+\gamma^2]}\right]\left[\cos\omega_1 t - e^{-\gamma t/2}\cos\omega_0 t\right]$$
$$+X_0\left[\frac{\omega_0\gamma}{[4(\omega_0-\omega)^2+\gamma^2]}\right]\left[\sin\omega_1 t - e^{-\gamma t/2}\sin\omega_0 t\right] \quad \text{... (4)}$$

Transient solution, needed to produce beats, initially grows (red box), but then damps away leaving behind the constant amplitude time asymptotic solution

$$T_0 = 2\pi/\omega_0$$
$$\omega_0 - \omega = \omega_0/16$$
$$\gamma = \omega_0/16$$