Pure Bending
Lecture 14

Engineering Mechanics - ME102

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Courtesy: TMH

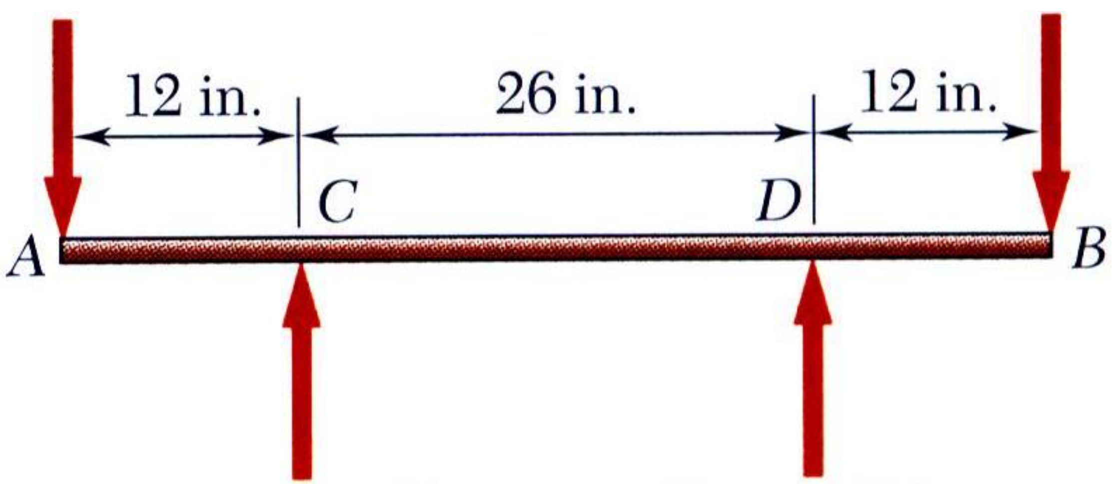
Pure Bending

Can someone bend the measuring scale?

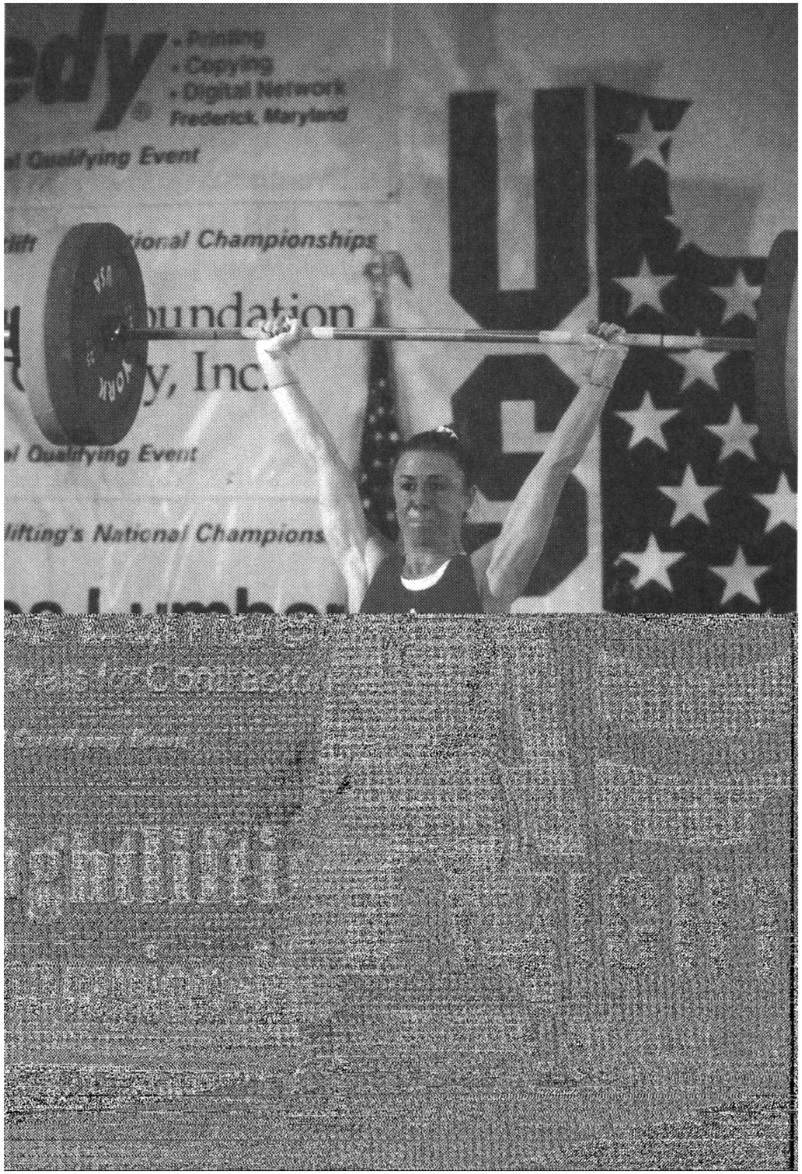
How many forces are applied?

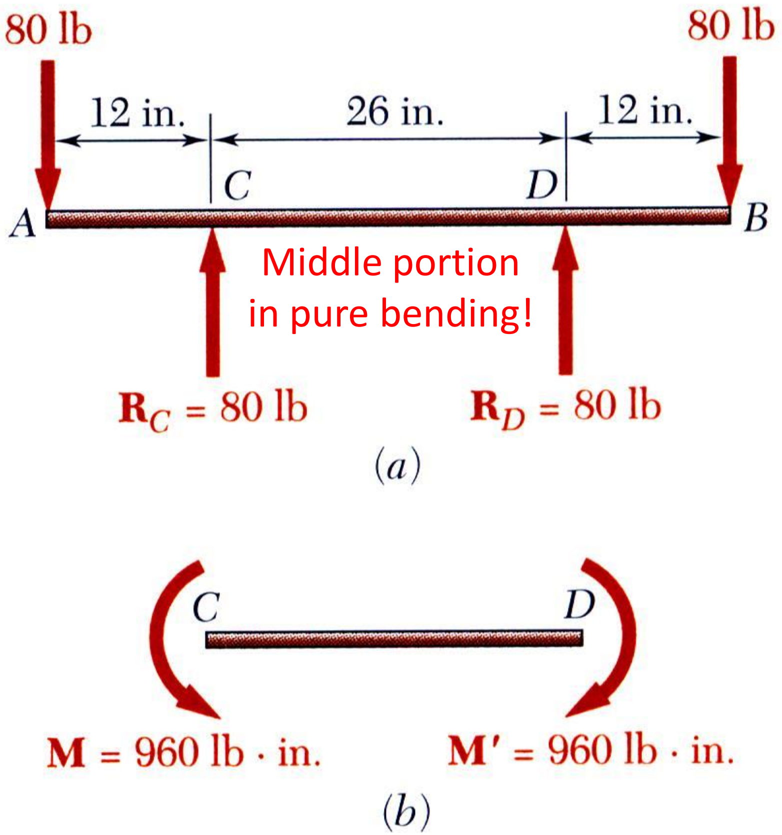
- 1
- 2
- 4

Can you bend the scale with less number of forces?



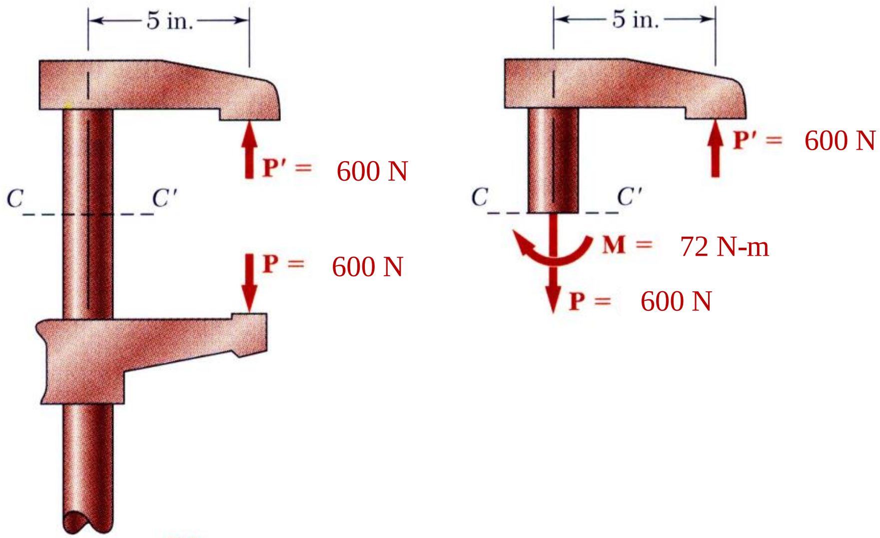
Pure Bending

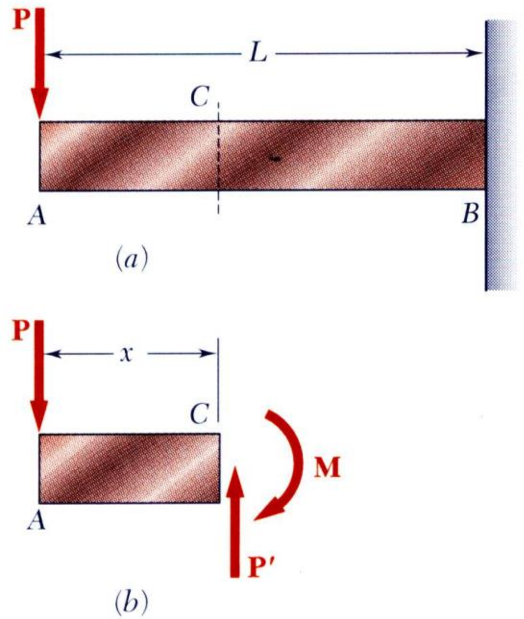




Pure Bending: Prismatic members subjected to ***equal and opposite couples*** acting in the same longitudinal plane

Other Loading Types

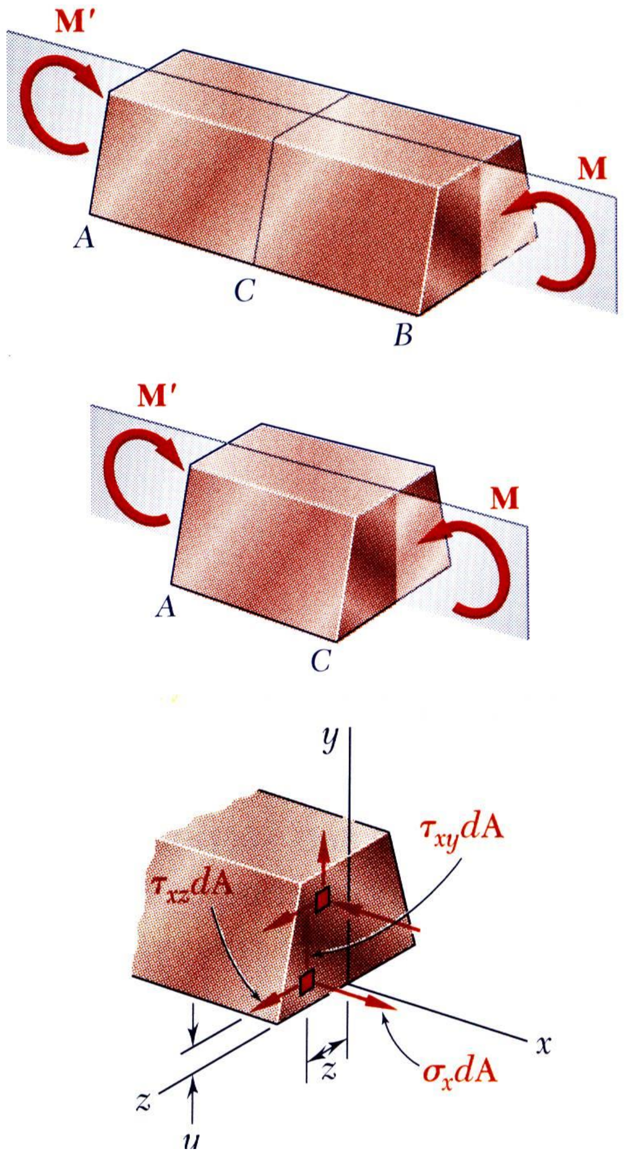




- *Eccentric Loading*: Axial loading which does not pass through section centroid produces internal forces equivalent to an **axial force and a couple**

- *Transverse Loading*: Concentrated or distributed transverse load produces internal forces equivalent to a **shear force and a couple**

Symmetric Member in Pure Bending



- Internal forces in any cross section are equivalent to a couple. The moment of the couple is the section *bending moment*.
- From statics, a couple M consists of two equal and opposite forces.
- The sum of the components of the forces in any direction is zero.
- The moment is the same about any axis perpendicular to the plane of the couple and zero about any axis contained in the plane.
- These requirements may be applied to the sums of the components and moments of the statically indeterminate elementary internal forces.

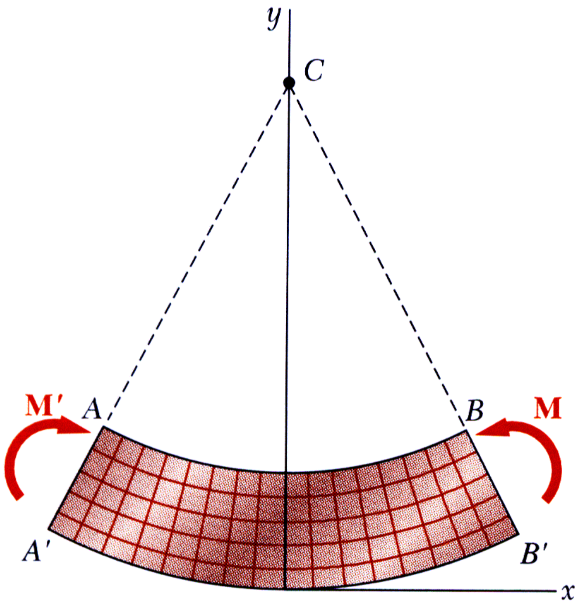
$$F_x = \int \sigma_x \, dA = 0$$

$$M_y = \int z\sigma_x \, dA = 0$$

$$M_z = \int -y\sigma_x \, dA = M$$

M is +ve when bending is concave upwards

Bending Deformations

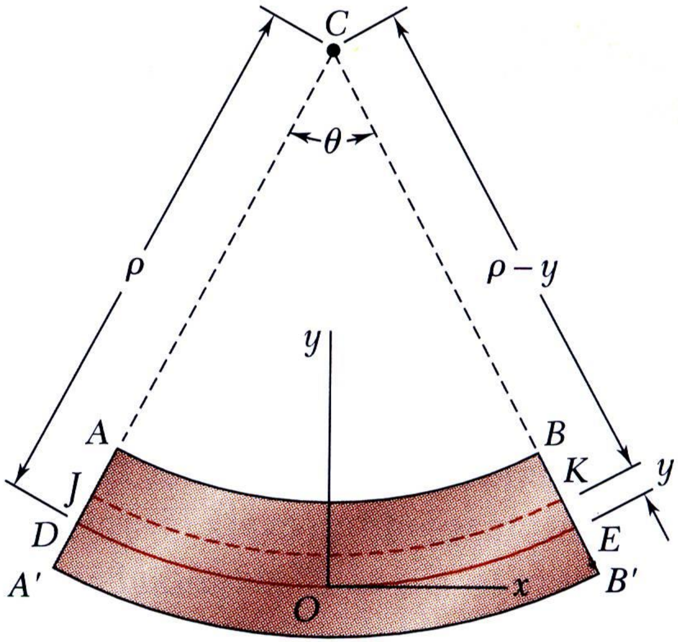


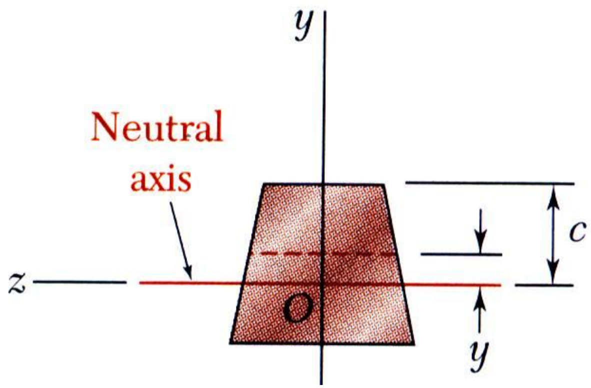
(a) Longitudinal, vertical section (plane of symmetry)

Beam with a plane of symmetry in pure bending:

- member remains symmetric
- bends uniformly to form a circular arc
- cross-sectional plane passes through arc center and remains planar
- length of top decreases and length of bottom increases
- a *neutral surface* must exist that is parallel to the upper and lower surfaces and for which the length does not change
- stresses and strains are negative (compressive) above the neutral plane and positive (tension) below it

Strain Due to Bending





Consider a beam segment of length *L*.

After deformation, the length of the neutral surface remains *L*. At other sections,

$$L' = (\rho - y)\theta$$

$$\delta = L - L' = (\rho - y)\theta - \rho\theta = -y\theta$$

$$\varepsilon_x = \frac{\delta}{L} = -\frac{y\theta}{\rho\theta} = -\frac{y}{\rho} \quad \text{(strain varies linearly)}$$

$$\varepsilon_m = \frac{c}{\rho} \quad \text{or} \quad \rho = \frac{c}{\varepsilon_m}$$

$$\varepsilon_x = -\frac{y}{c}\varepsilon_m$$

Stress Due to Bending

- For a linearly elastic material,

$$\sigma_x = E\varepsilon_x = -\frac{y}{c}E\varepsilon_m$$

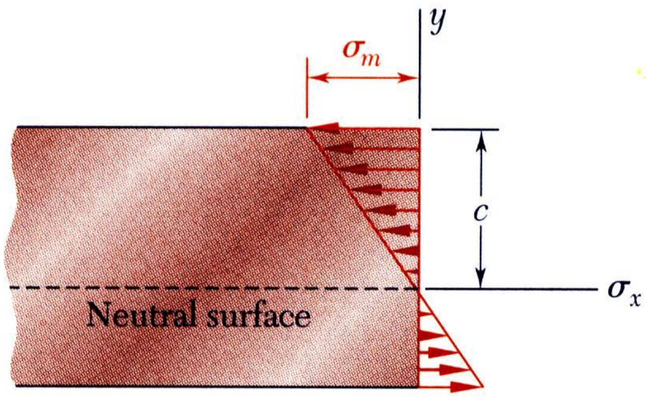
$$= -\frac{y}{c}\sigma_m \quad \text{(stress varies linearly)}$$

- For static equilibrium,

$$F_x = 0 = \int \sigma_x \, dA = \int -\frac{y}{c}\sigma_m \, dA$$

$$0 = -\frac{\sigma_m}{c}\int y \, dA$$

First moment with respect to neutral plane is zero. Therefore, the neutral surface must pass through the section centroid.



- For static equilibrium,

$$M = \int -y\sigma_x \, dA = \int -y\left(-\frac{y}{c}\sigma_m\right) dA$$

$$M = \frac{\sigma_m}{c}\int y^2 \, dA = \frac{\sigma_m I}{c}$$

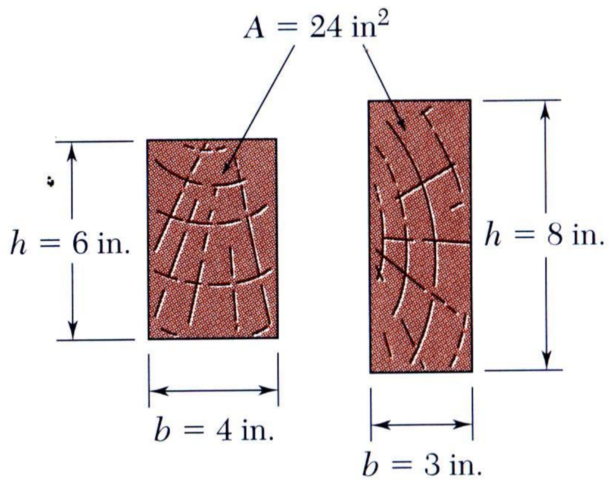
$$\sigma_m = \frac{Mc}{I} = \frac{M}{S}$$

Elastic section modulus

$$\text{Substituting} \quad \sigma_x = -\frac{y}{c}\sigma_m$$

$$\sigma_x = -\frac{My}{I}$$

Beam Section Properties



- The maximum normal stress due to bending,

$$\sigma_m = \frac{Mc}{I} = \frac{M}{S}$$

$$I = \text{section moment of inertia}$$

$$S = \frac{I}{c} = \text{section modulus}$$

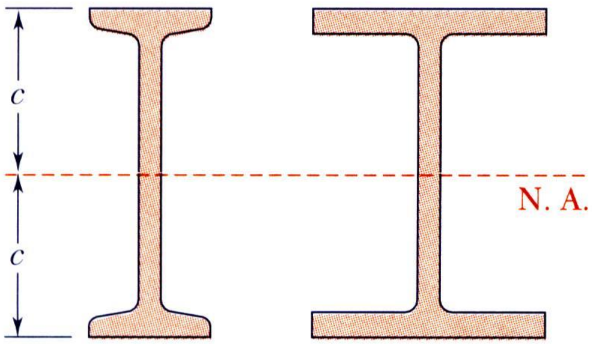
A beam section with a larger section modulus will have a lower maximum stress

- Consider a rectangular beam cross section,

$$S = \frac{I}{c} = \frac{\frac{1}{12}bh^3}{h/2} = \tfrac{1}{6}bh^2 = \tfrac{1}{6}Ah$$

Between two beams with the same cross sectional area, the beam with the greater depth will be more effective in resisting bending.

Beam Section Properties



(a) S-beam (b) W-beam

$$\sigma_m = \frac{Mc}{I} = \frac{M}{S}$$

$I =$ section moment of inertia

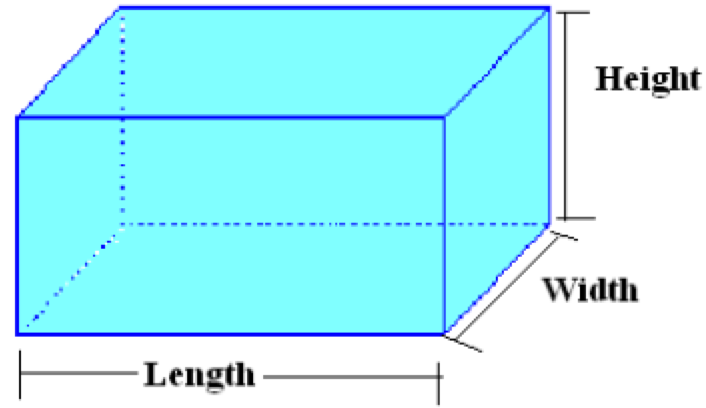
$$S = \frac{I}{c} = \text{section modulus}$$

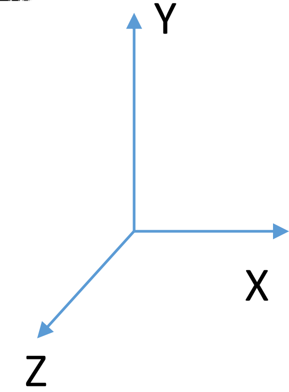
- Structural steel beams are designed to have a large section modulus.
- Preferred since a large portion of their cross section is located far away from the neutral axis.
- Thus, for a given cross sectional area and depth/height, their design provide large value of *I*, and hence *S*.
- Large value of *S* implies that the value of maximum stress in the beam is minimum.

Calculation for a Cuboid

$$\sigma_m = \frac{M_Z H}{I_Z} = \frac{12 M_Z H}{W H^3} = \frac{12 M_Z}{W H^2}$$

$$\sigma_m = \frac{M_X H}{I_X} = \frac{12 M_X H}{L H^3} = \frac{12 M_X}{L H^2}$$





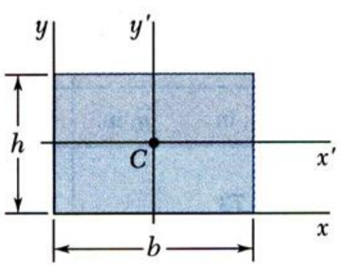
$$If\ W < L,\ M_Z < M_X$$

Which way it will be easy to bend the scale?

$$\sigma_m = \frac{Mc}{I} = \frac{M}{S}$$

$I =$ section moment of inertia

$S = \frac{I}{c} =$ section modulus



$$\bar{I}_{x'} = \frac{1}{12} bh^3$$

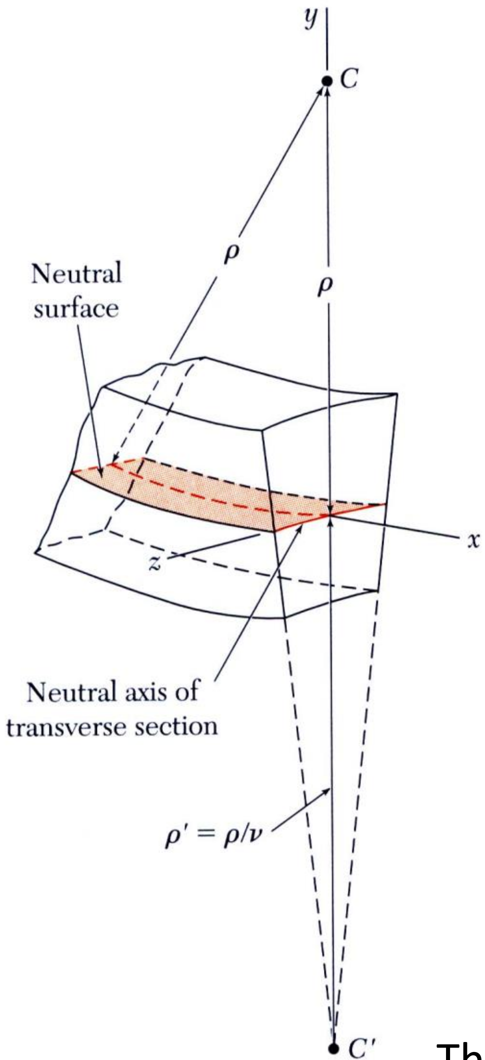
$$\bar{I}_{y'} = \frac{1}{12} b^3 h$$

$$I_x = \frac{1}{3} bh^3$$

$$I_y = \frac{1}{3} b^3 h$$

$$J_C = \frac{1}{12} bh(b^2 + h^2)$$

Deformations in a Transverse Cross Section



- Deformation due to bending moment M is quantified by the curvature of the neutral surface

$$\frac{1}{\rho}=\frac{\varepsilon_m}{c}=\frac{\sigma_m}{Ec}=\frac{1}{Ec}\frac{Mc}{I}$$
$$=\frac{M}{EI}$$

- Although cross sectional planes remain planar when subjected to bending moments, in-plane deformations are nonzero,

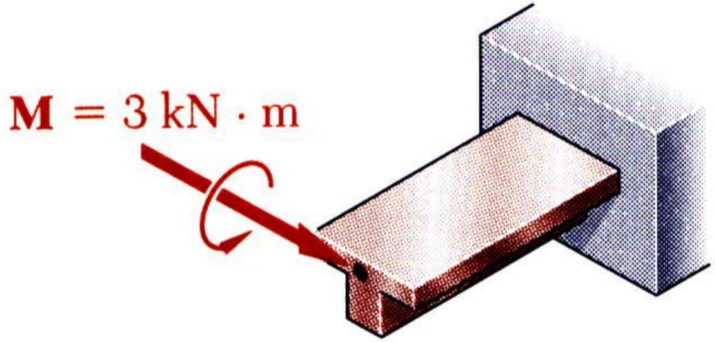
$$\varepsilon_y=-\nu\varepsilon_x=\frac{\nu y}{\rho} \qquad \varepsilon_z=-\nu\varepsilon_x=\frac{\nu y}{\rho}$$

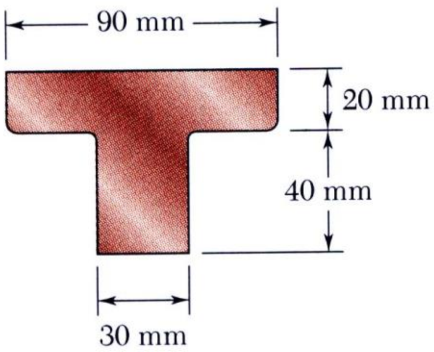
- Expansion above the neutral surface and contraction below it cause an in-plane curvature,

$$\frac{1}{\rho'}=\frac{\nu}{\rho}=\text{anticlastic curvature}$$

This relation shows that that the element located above the neutral plane will expand in both y and z directions and vice-versa for elements located below the neutral axis.

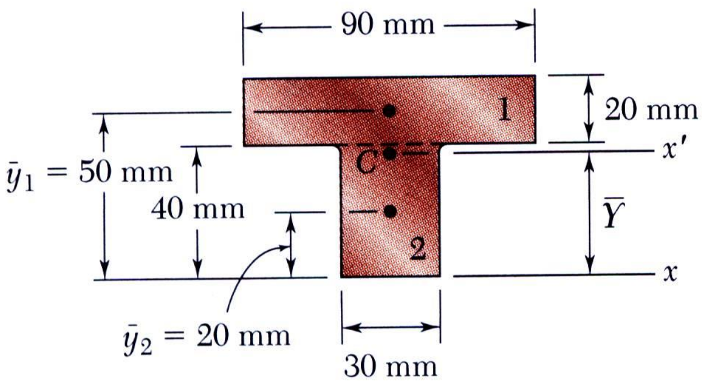
Sample Problem 4.2





A cast-iron machine part is acted upon by a 3 kN-m couple. Knowing E = 165 GPa and neglecting the effects of fillets, determine (a) the maximum tensile and compressive stresses, (b) the radius of curvature.

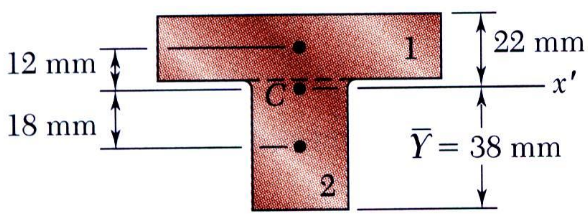
Sample Problem 4.2



SOLUTION:

Based on the cross section geometry, calculate the location of the section centroid and moment of inertia.

	Area, mm^2	$\bar{y}$, mm	$\bar{y}A$, mm^3
1	$20\times 90 = 1800$	50	90×10^3
2	$40\times 30 = 1200$	20	24×10^3
	$\sum A = 3000$		$\sum \bar{y}A = 114\times 10^3$



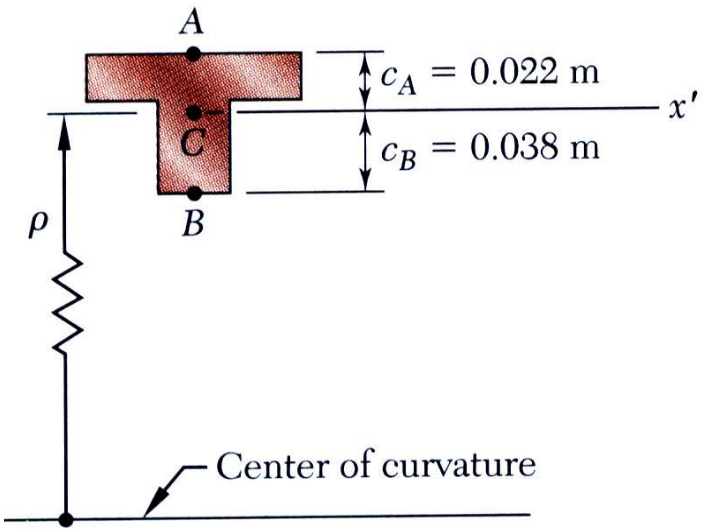
$$\bar{Y} = \frac{\sum \bar{y}A}{\sum A} = \frac{114\times 10^3}{3000} = 38\,\text{mm}$$

$$I_{x'} = \sum\left(\bar{I} + Ad^2\right) = \sum\left(\tfrac{1}{12}bh^3 + Ad^2\right)$$

$$= \left(\tfrac{1}{12}90\times 20^3 + 1800\times 12^2\right) + \left(\tfrac{1}{12}30\times 40^3 + 1200\times 18^2\right)$$

$$I = 868\times 10^3\,\text{mm} = 868\times 10^{-9}\,\text{m}^4$$

Sample Problem 4.2



- Apply the elastic flexural formula to find the maximum tensile and compressive stresses.

$$\sigma_m = \frac{Mc}{I}$$

$$\sigma_A = \frac{M\,c_A}{I} = \frac{3\,\text{kN}\cdot\text{m}\times 0.022\,\text{m}}{868\times 10^{-9}\,\text{mm}^4} \qquad \boxed{\sigma_A = +76.0\,\text{MPa}}$$

$$\sigma_B = -\frac{M\,c_B}{I} = -\frac{3\,\text{kN}\cdot\text{m}\times 0.038\,\text{m}}{868\times 10^{-9}\,\text{mm}^4} \qquad \boxed{\sigma_B = -131.3\,\text{MPa}}$$

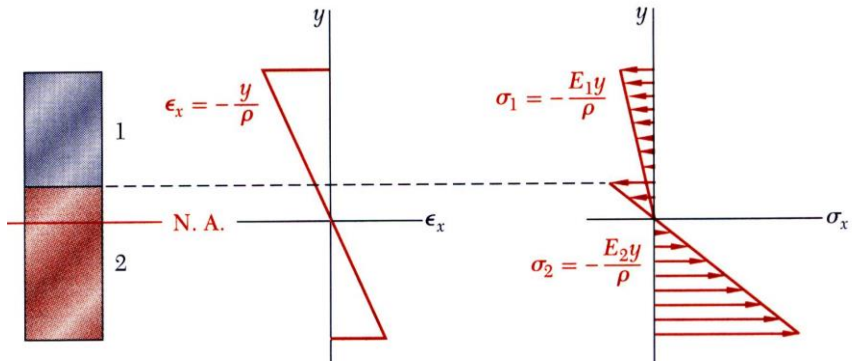
- Calculate the curvature

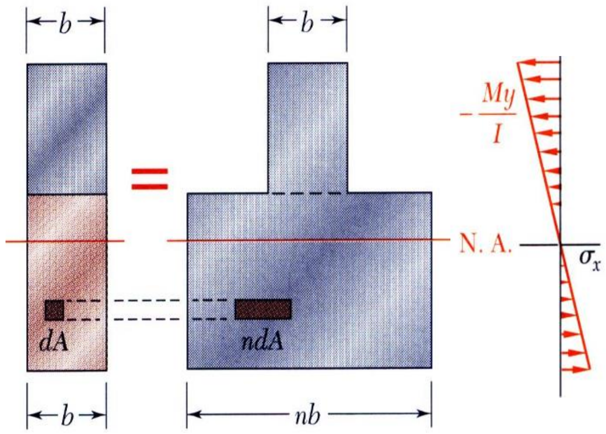
$$\frac{1}{\rho} = \frac{M}{EI}$$

$$= \frac{3\,\text{kN}\cdot\text{m}}{(165\,\text{GPa})\left(868\times 10^{-9}\,\text{m}^4\right)}$$

$$\boxed{\begin{aligned}\frac{1}{\rho} &= 20.95\times 10^{-3}\,\text{m}^{-1}\\ \rho &= 47.7\,\text{m}\end{aligned}}$$

Bending of Members Made of Several Materials





$$\sigma_x = -\frac{My}{I}$$

$$\sigma_1 = \sigma_x \qquad \sigma_2 = n\sigma_x$$

- Consider a composite beam formed from two materials with E_1 and E_2.
- Normal strain varies linearly.

$$\varepsilon_x = -\frac{y}{\rho}$$

- Piecewise linear normal stress variation.

$$\sigma_1 = E_1\varepsilon_x = -\frac{E_1 y}{\rho} \qquad \sigma_2 = E_2\varepsilon_x = -\frac{E_2 y}{\rho}$$

Neutral axis does not pass through section centroid of composite section.

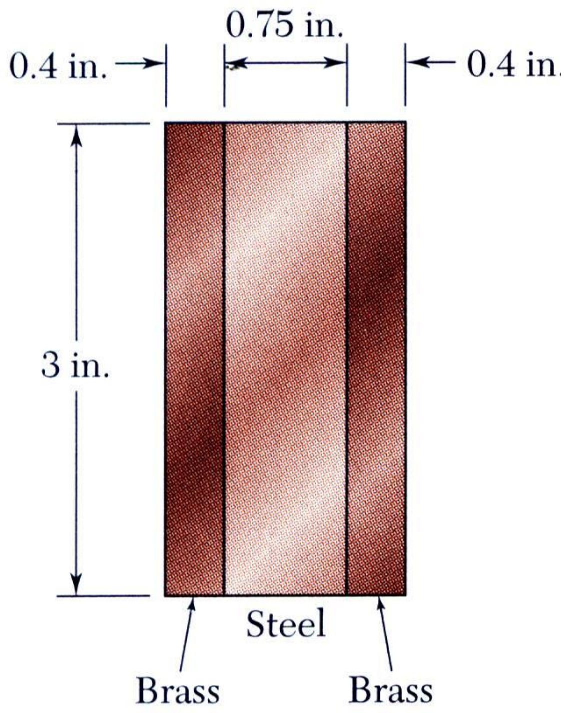
- Elemental forces on the section are

$$dF_1 = \sigma_1 dA = -\frac{E_1 y}{\rho} dA \quad dF_2 = \sigma_2 dA = -\frac{E_2 y}{\rho} dA$$

- Define a transformed section such that

$$dF_2 = -\frac{(nE_1)y}{\rho} dA = -\frac{E_1 y}{\rho}(n\,dA) \qquad n = \frac{E_2}{E_1}$$

Example 4.03

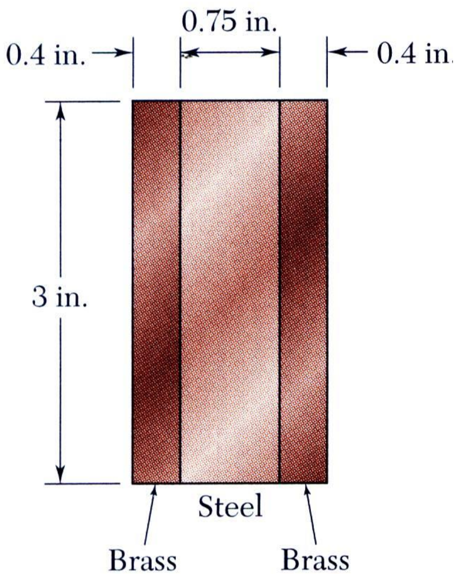


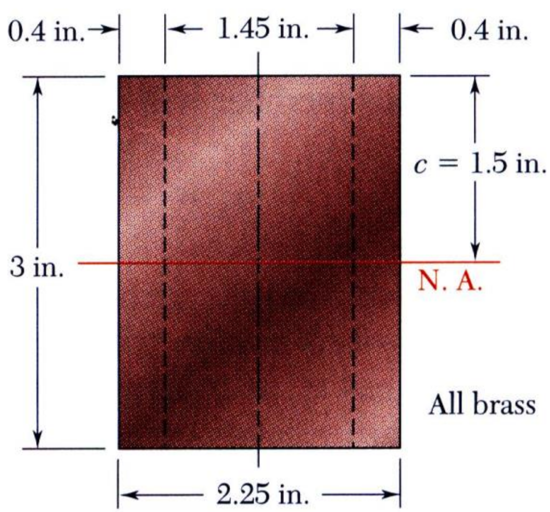
Bar is made from bonded pieces of steel (E_s = 29x10^6 psi) and brass (E_b = 15x10^6 psi). Determine the maximum stress in the steel and brass when a moment of 40 kip*in is applied.

SOLUTION:

- Transform the bar to an equivalent cross section made entirely of brass
- Evaluate the cross sectional properties of the transformed section
- Calculate the maximum stress in the transformed section. This is the correct maximum stress for the brass pieces of the bar.
- Determine the maximum stress in the steel portion of the bar by multiplying the maximum stress for the transformed section by the ratio of the moduli of elasticity.

Example 4.03





SOLUTION:

- Transform the bar to an equivalent cross section made entirely of brass.

$$n = \frac{E_s}{E_b} = \frac{29\times10^6\,\text{psi}}{15\times10^6\,\text{psi}} = 1.933$$

$$b_T = 0.4\,\text{in} + 1.933\times0.75\,\text{in} + 0.4\,\text{in} = 2.25\,\text{in}$$

- Evaluate the transformed cross sectional properties

$$I = \tfrac{1}{12} b_T h^3 = \tfrac{1}{12}(2.25\,\text{in.})(3\,\text{in})^3$$
$$= 5.063\,\text{in}^4$$

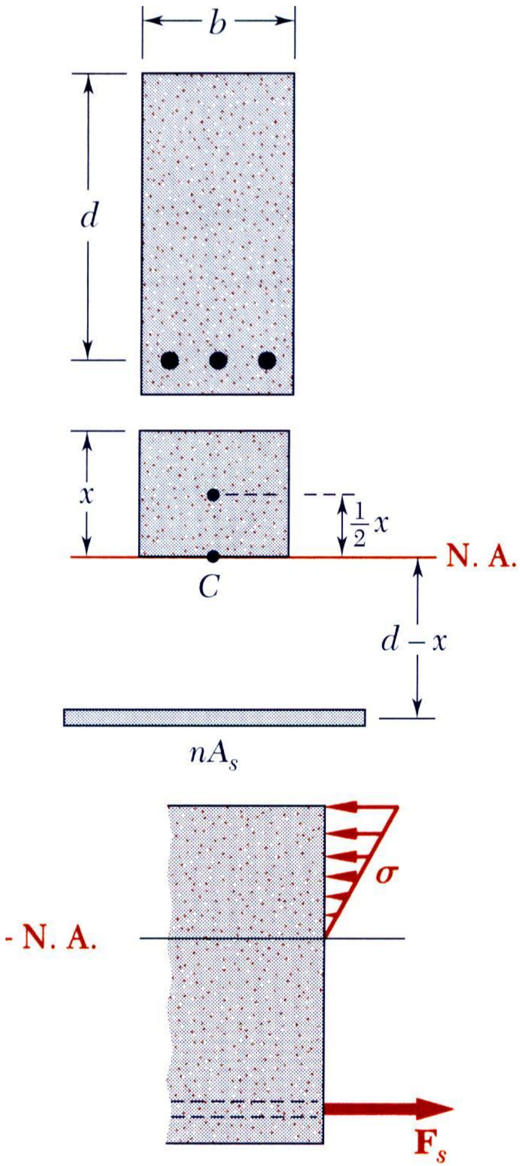
- Calculate the maximum stresses

$$\sigma_m = \frac{Mc}{I} = \frac{(40\,\text{kip}\cdot\text{in})(1.5\,\text{in})}{5.063\,\text{in}^4} = 11.85\,\text{ksi}$$

$$(\sigma_b)_{\max} = \sigma_m$$
$$(\sigma_s)_{\max} = n\sigma_m = 1.933\times11.85\,\text{ksi}$$

$$\boxed{\begin{aligned}(\sigma_b)_{\max} &= 11.85\,\text{ksi}\\ (\sigma_s)_{\max} &= 22.9\,\text{ksi}\end{aligned}}$$

Reinforced Concrete Beams



- Concrete beams subjected to bending moments are reinforced by steel rods.

- **The steel rods carry the entire tensile load below the neutral surface.** The upper part of the concrete beam carries the compressive load.

- In the transformed section, the cross sectional area of the steel, A_s, is replaced by the equivalent area nA_s where $n = E_s/E_c$.

- To determine the location of the neutral axis,

$$(bx)\frac{x}{2} - n A_s(d - x) = 0$$

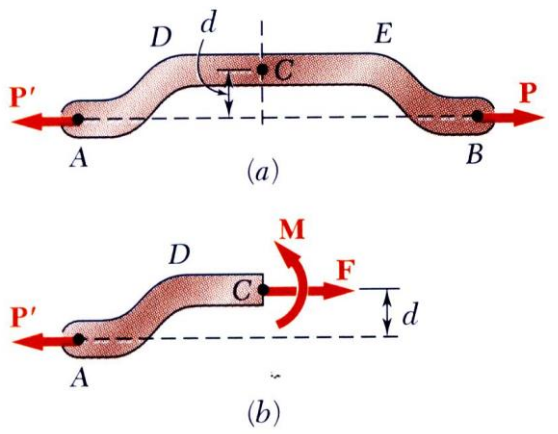
$$\frac{1}{2} b x^2 + n A_s x - n A_s d = 0$$

- The normal stress in the concrete and steel

$$\sigma_x = -\frac{My}{I}$$

$$\sigma_c = \sigma_x \qquad \sigma_s = n\sigma_x$$

Eccentric Axial Loading in a Plane of Symmetry



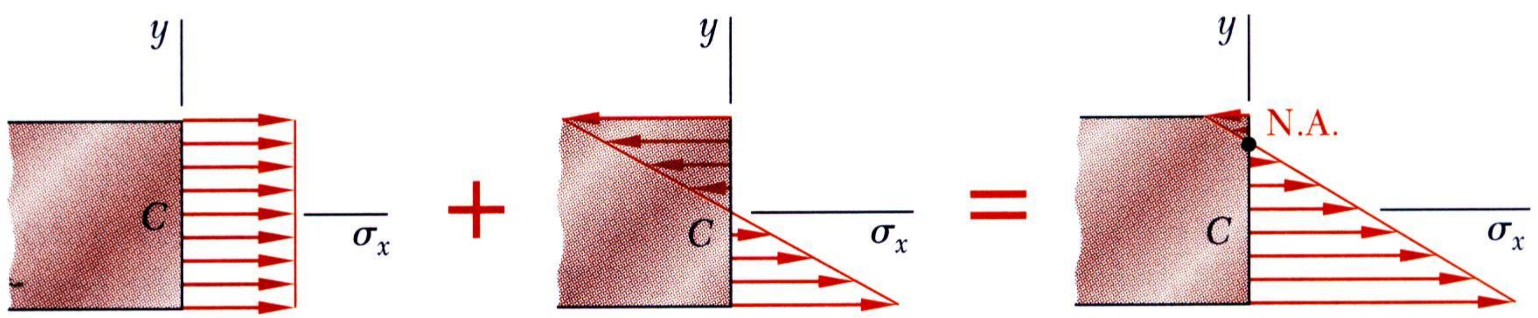
- Eccentric loading

$$F = P$$
$$M = Pd$$

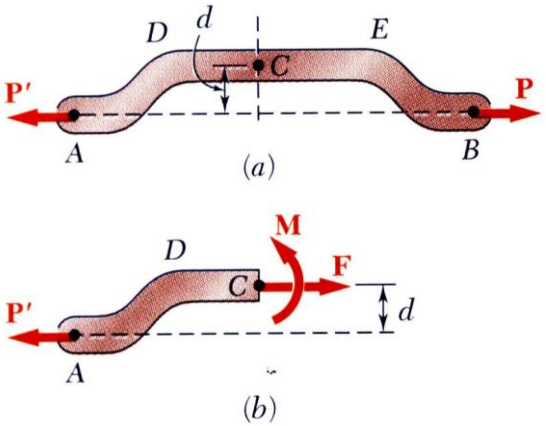
- Stress due to eccentric loading found by superposing the uniform stress due to a centric load and linear stress distribution due a pure bending moment

$$\sigma_x = (\sigma_x)_{\text{centric}} + (\sigma_x)_{\text{bending}}$$
$$= \frac{P}{A} - \frac{My}{I}$$

- Validity requires stresses below proportional limit, deformations have negligible effect on geometry, and stresses not evaluated near points of load application.



Eccentric Axial Loading in a Plane of Symmetry



- Eccentric loading

$$F = P$$
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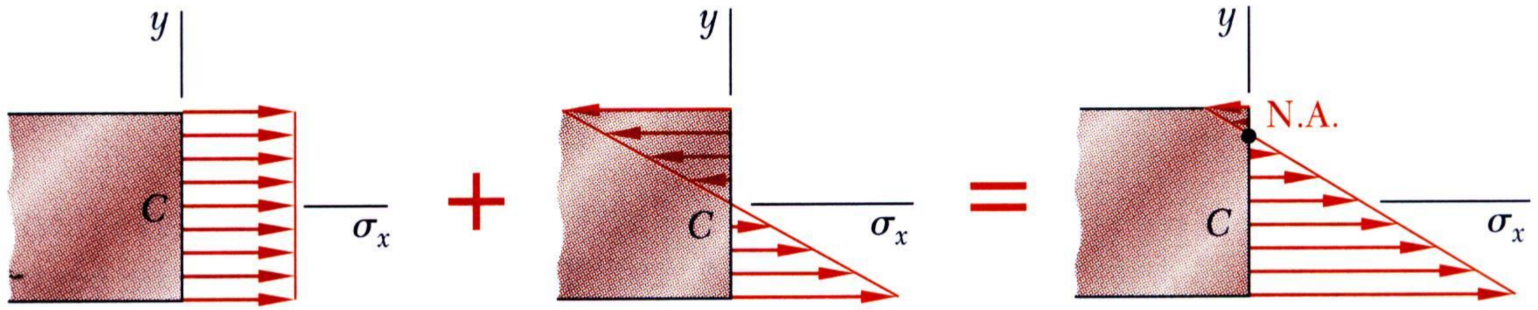
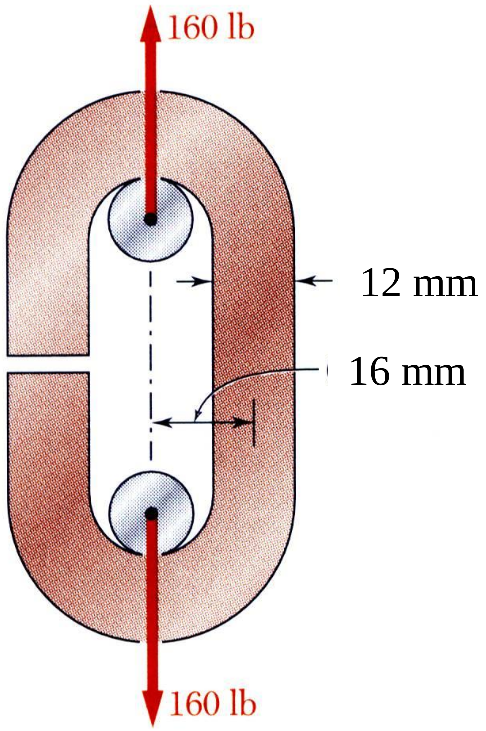
$$\sigma_x = (\sigma_x)_{\text{centric}} + (\sigma_x)_{\text{bending}}$$
$$= \frac{P}{A} - \frac{My}{I}$$

- Validity requires stresses below proportional limit, deformations have negligible effect on geometry, and stresses not evaluated near points of load application.

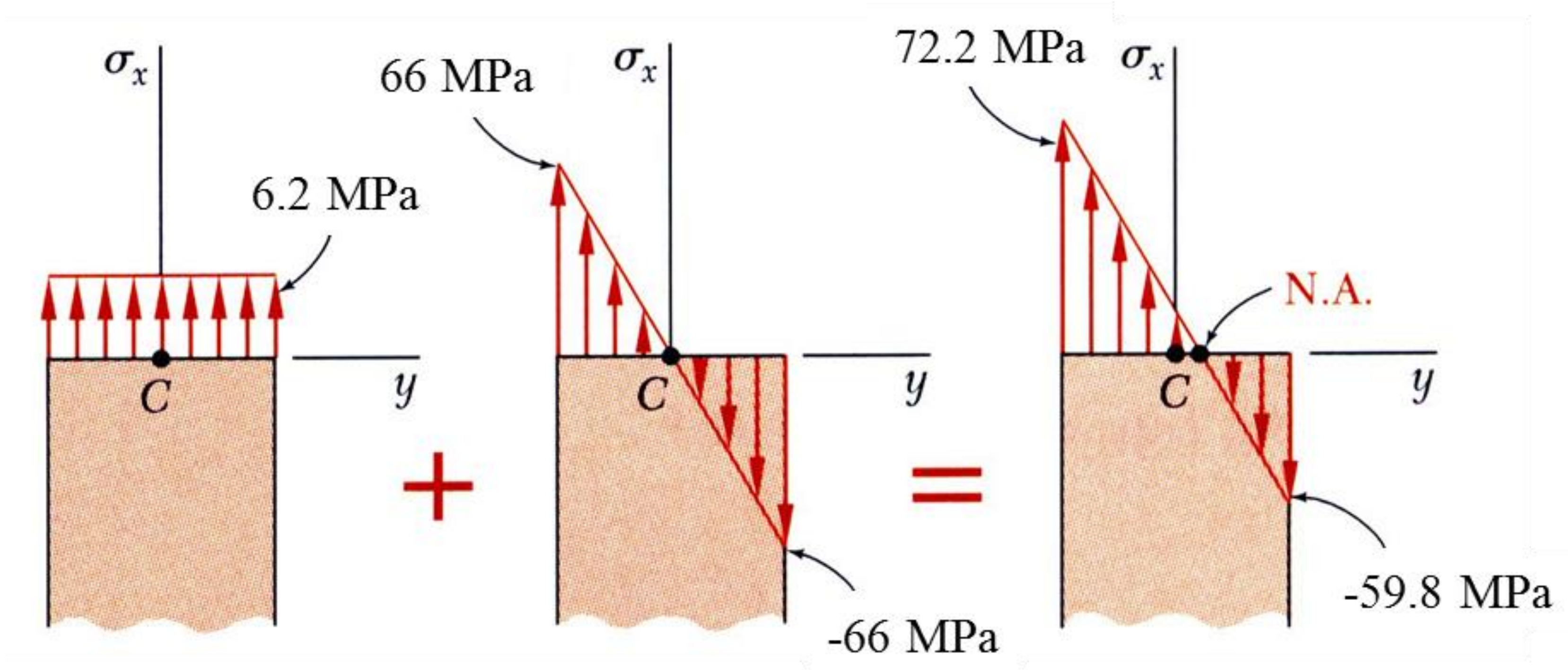


Example 4.07



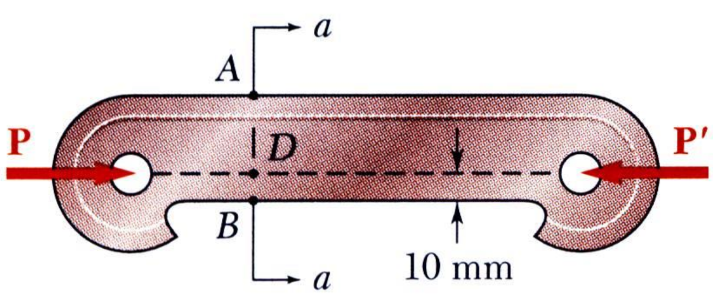
An open-link chain is obtained by bending low-carbon steel rods into the shape shown. For 700 N load, determine (a) maximum tensile and compressive stresses, (b) distance between section centroid and neutral axis

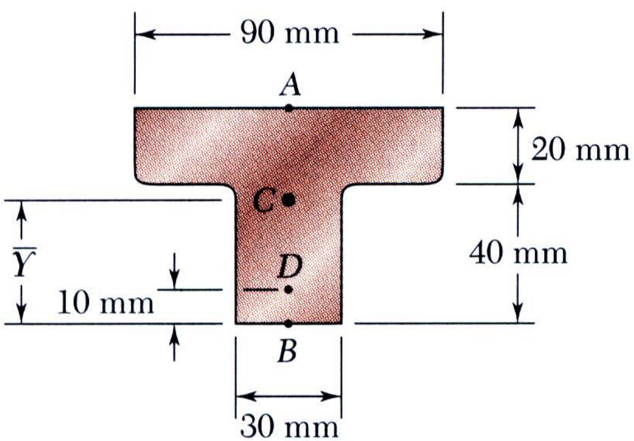
Example 4.07



σ_x
6.2 MPa
C
y
66 MPa
σ_x
C
y
-66 MPa
72.2 MPa
σ_x
N.A.
C
y
-59.8 MPa

Sample Problem 4.8





Section *a–a*

The largest allowable stresses for the cast iron link are 30 MPa in tension and 120 MPa in compression. Determine the largest force *P* which can be applied to the link.

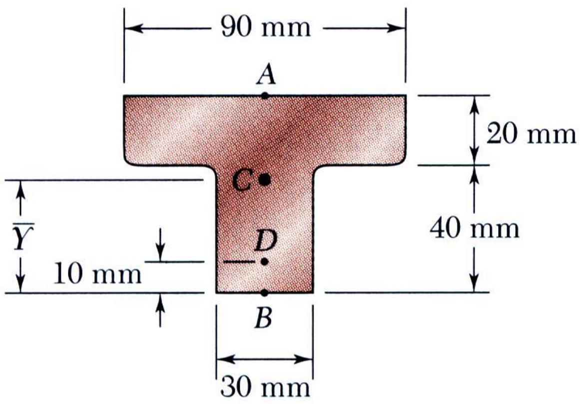
From Sample Problem 4.2

$$A = 3\times10^{-3}\,\mathrm{m}^2$$

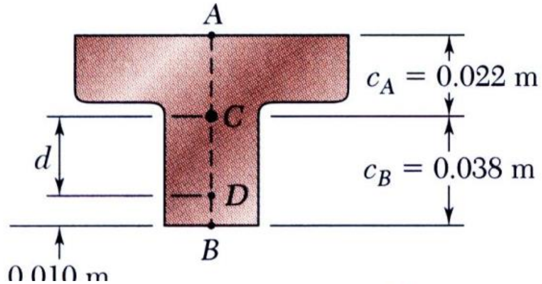
$$\bar{Y} = 0.038\,\mathrm{m}$$

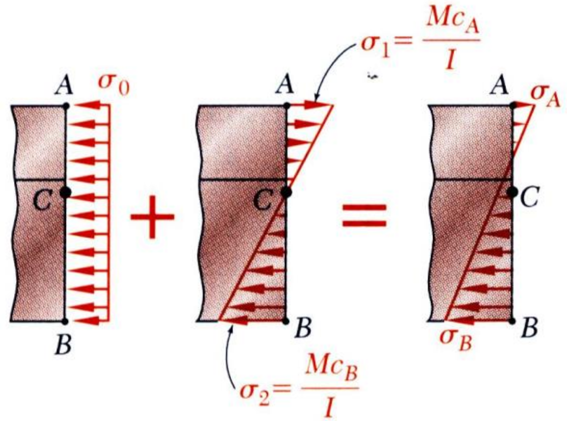
$$I = 868\times10^{-9}\,\mathrm{m}^4$$

Sample Problem 4.8



Section a–a





- Determine an equivalent centric and bending loads.

$$d = 0.038 - 0.010 = 0.028\,\text{m}$$

$$P = \text{centric load}$$

$$M = Pd = 0.028P = \text{bending moment}$$

- Superpose stresses due to centric and bending loads

$$\sigma_A = -\frac{P}{A} + \frac{Mc_A}{I} = -\frac{P}{3\times10^{-3}} + \frac{(0.028P)(0.022)}{868\times10^{-9}} = +377P$$

$$\sigma_B = -\frac{P}{A} - \frac{Mc_A}{I} = -\frac{P}{3\times10^{-3}} - \frac{(0.028P)(0.022)}{868\times10^{-9}} = -1559P$$

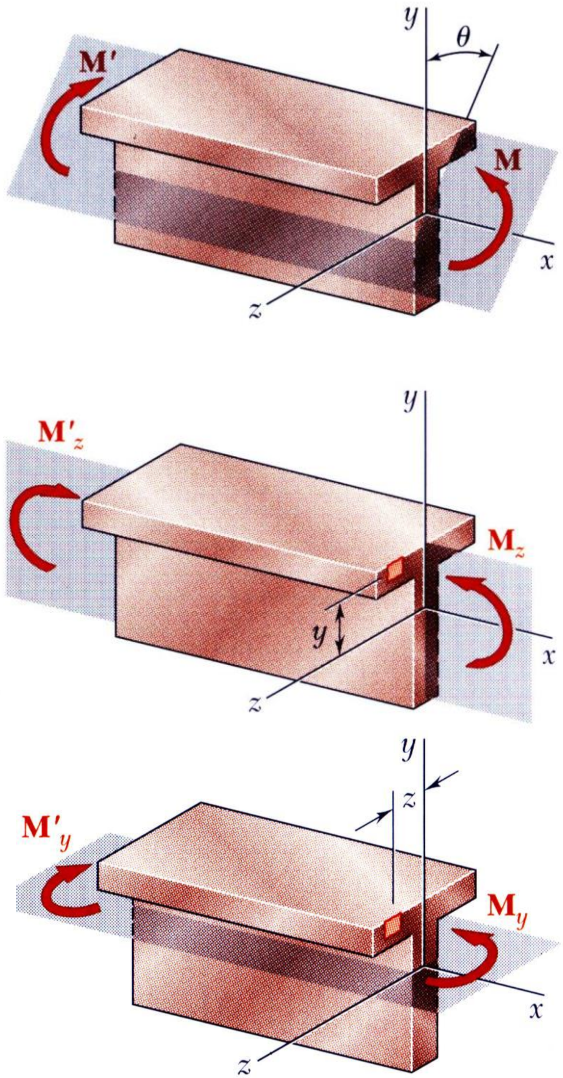
- Evaluate critical loads for allowable stresses.

$$\sigma_A = +377P = 30\,\text{MPa} \qquad P = 79.6\,\text{kN}$$

$$\sigma_B = -1559P = -120\,\text{MPa} \qquad P = 79.6\,\text{kN}$$

- The largest allowable load $\boxed{P = 77.0\ \text{kN}}$

Unsymmetric Bending



Superposition is applied to determine stresses in the most general case of unsymmetric bending.

- Resolve the couple vector into components along the principle centroidal axes.

$$M_z = M\cos\theta \qquad M_y = M\sin\theta$$

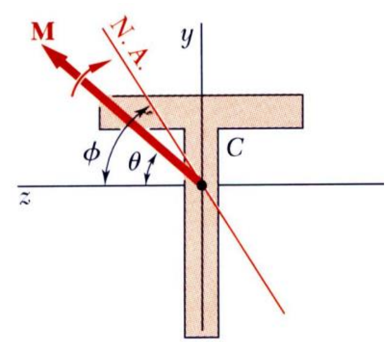
- Superpose the component stress distributions

$$\sigma_x = -\frac{M_z y}{I_z} + \frac{M_y z}{I_y}$$

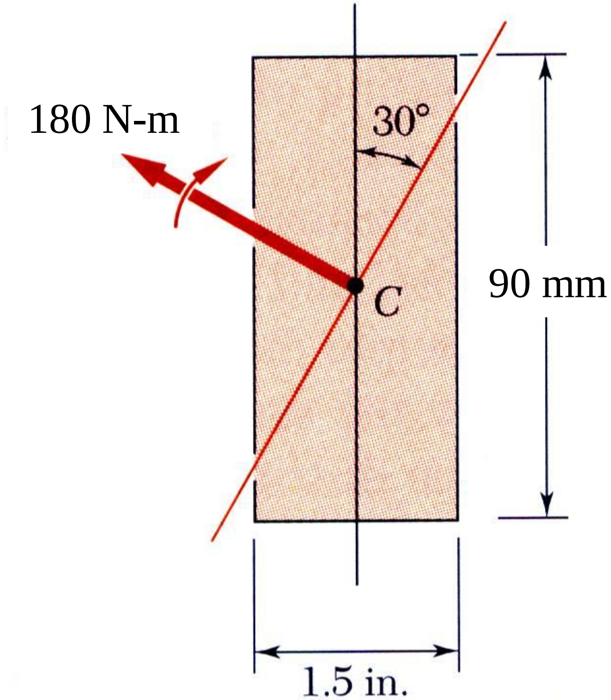
- Along the neutral axis,

$$\sigma_x = 0 = -\frac{M_z y}{I_z} + \frac{M_y z}{I_y} = -\frac{(M\cos\theta)y}{I_z} + \frac{(M\sin\theta)z}{I_y}$$

$$\tan\phi = \frac{y}{z} = \frac{I_z}{I_y}\tan\theta$$



Example 4.08



A 180 N-m couple is applied to a rectangular wooden beam in a plane forming an angle of 30 deg. with the vertical. Determine (a) the maximum stress in the beam, (b) the angle that the neutral axis forms with the horizontal plane.

SOLUTION:

- Resolve the couple vector into components along the principle centroidal axes and calculate the corresponding maximum stresses.

$$M_z = M\cos\theta \qquad M_y = M\sin\theta$$

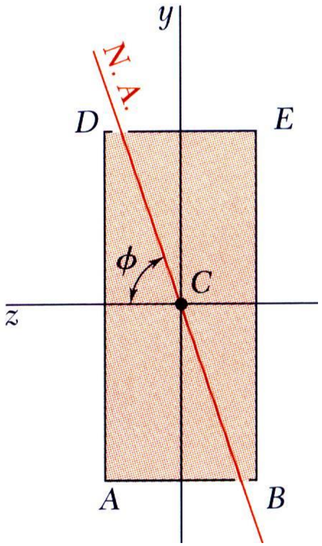
- Combine the stresses from the component stress distributions.

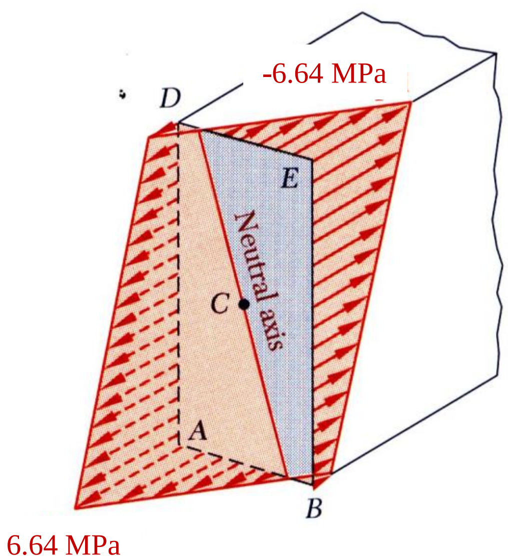
$$\sigma_x = -\frac{M_z y}{I_z} + \frac{M_y y}{I_y}$$

- Determine the angle of the neutral axis.

$$\tan\phi = \frac{y}{z} = \frac{I_z}{I_y}\tan\theta$$

Example 4.08



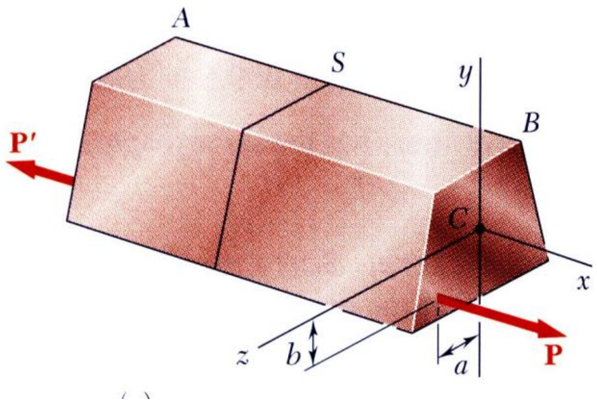


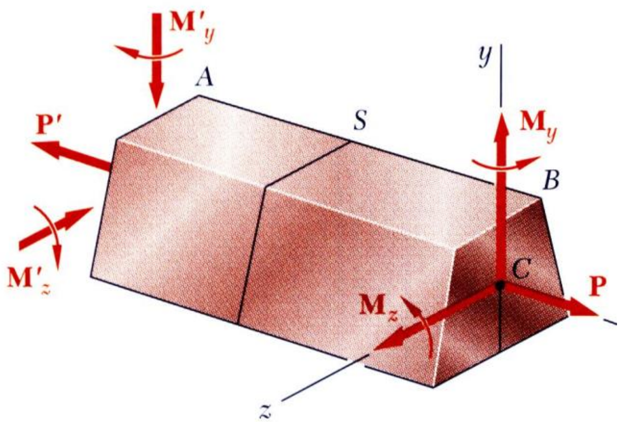
- Determine the angle of the neutral axis.

$$\tan\phi = \frac{I_z}{I_y}\tan\theta = \frac{2.43e-6\,m^4}{0.48e-6\,m^4}\tan 30 = 2.9$$

$$\boxed{\phi = 71^o}$$

General Case of Eccentric Axial Loading





- Consider a straight member subject to equal and opposite eccentric forces.

- The eccentric force is equivalent to the system of a centric force and two couples.

$$P = \text{centric force}$$

$$M_y = Pa \qquad M_z = Pb$$

- By the principle of superposition, the combined stress distribution is

$$\sigma_x = \frac{P}{A} - \frac{M_z y}{I_z} + \frac{M_y z}{I_y}$$

- If the neutral axis lies on the section, it may be found from

$$\frac{M_z}{I_z} y - \frac{M_y}{I_y} z = \frac{P}{A}$$