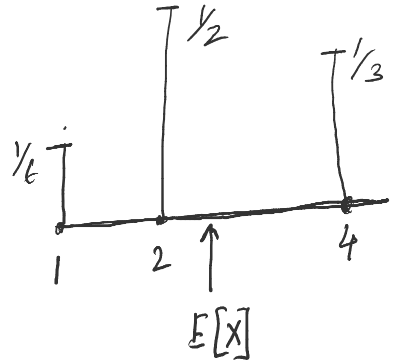
Expectation and Variance

Thursday, January 28, 2021 8:56 AM

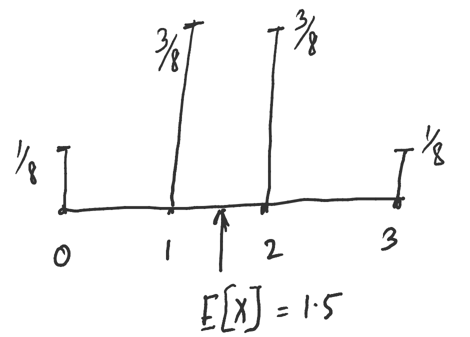
The expected value or Expectation of X with p.m.f $p_X(x)$ is given by

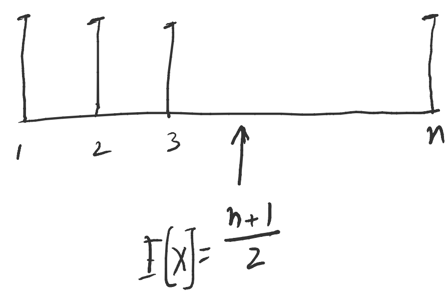
$$\underline{E[X] = \sum_x x\, p_X(x)}$$



$$(1-c)\tfrac{1}{6} + (2-c)\tfrac{1}{2} + (4-c)\tfrac{1}{3} = 0$$

$$c = 1\cdot\tfrac{1}{6} + 2\cdot\tfrac{1}{2} + 4\cdot\tfrac{1}{3} = 2.5$$





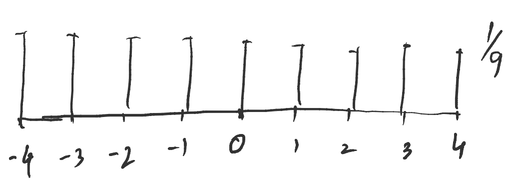
Function of r.v. X, $Y = g(X)$

$$\boxed{E[Y] = \sum_y y\, p_Y(y)}$$

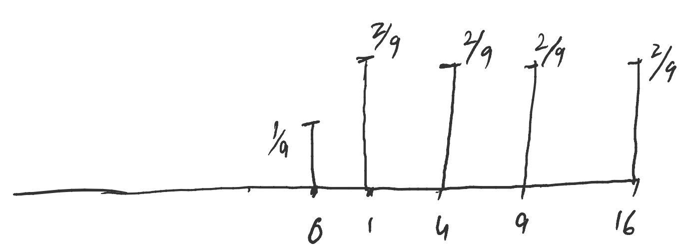
we are given $p_X(x)$

$$\downarrow$$

$p_Y(y)$

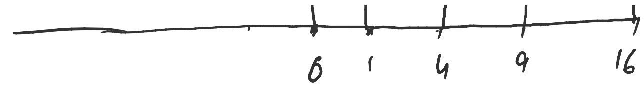


$Y = X^2$



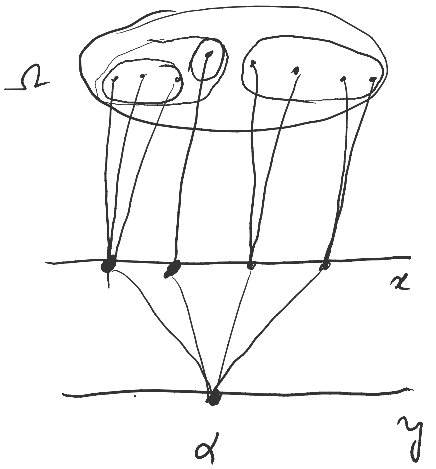
$p_X(x)$

$Y = X^2$



$p_Y(y)$

$$E[Y] = \sum_y y\, p_Y(y) = 0 \cdot \tfrac{1}{9} + 1 \tfrac{2}{9} + 4 \tfrac{2}{9} + 9 \cdot \tfrac{2}{9} + 16 \tfrac{2}{9}$$
$$= \frac{60}{9}$$



$$\boxed{E[Y] = \sum_x \underline{g(x)} \cdot p_X(x)}$$

$Y = X^2$
$$= \sum_{\substack{x \\ -4,-3 \cdots 4}} x^2 p_X(x) = 0 \cdot \frac{1}{9} + 16 \cdot \frac{2}{9} + 9 \frac{2}{9} + \cdots$$
$$= \frac{60}{9}$$

$$E[Y] = \sum_y y\, p_Y(y) = \sum_y y \left(\sum_{\{x \mid g(x) = y\}} p_X(x) \right) = \sum_y \sum_{\{x \mid g(x)=y\}} g(x) \cdot p_X(x)$$

$$= \boxed{E[Y] = \sum_x g(x)\, p_X(x)}$$

$Y = g(X) = \alpha$

$E[\alpha] = \alpha$

$E[\alpha X] = \alpha \cdot E[X]$

$E[X + \beta] = E[X] + \beta$

$$Y = \alpha X + \beta$$

$$E[\alpha X + \beta] = \sum_x (\alpha x + \beta)\, p_X(x) = \sum_x \alpha x\, p_X(x) + \sum_x \beta \cdot p_X(x)$$

$$= \alpha E[X] + \beta$$

(*) $$E[X^2] = \sum_x \underline{x^2}\, p_X(x) \rightarrow \text{2nd moment of } X$$

$$E[X^n] = \sum_x x^n\, p_X(x) \longrightarrow n\text{-th moment of } X$$

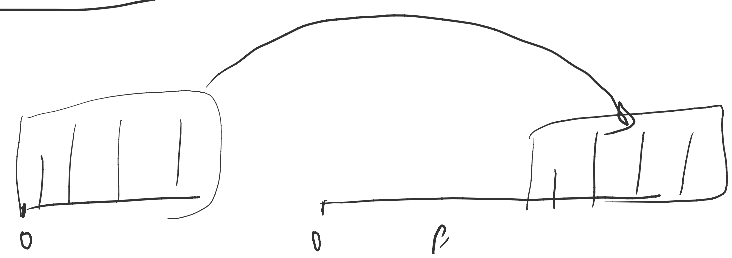
$$E[X] \rightarrow \text{1st moment of } X.$$

$$\boxed{Var(X) = E\left[(X - E[X])^2\right] = \sum_x (x - E[X])^2\, p_X(x)}$$

$$Var(\alpha) = 0$$

$$Var(\alpha X) = \alpha^2 Var(X)$$

$$Var(X + \beta) = Var(X)$$



$$Var(\alpha X + \beta) = \sum_x \left(\alpha x + \cancel{\beta} - \alpha E[X] - \cancel{\beta}\right)^2 p_X(x)$$

$$= \sum_x \alpha^2 (x - E[X])^2\, p_X(x) = \alpha^2 \underline{\sum (\quad)} = \alpha^2 Var(X)$$

$$Var(X) = \sum (x - E[X])^2\, p_X(x)$$

x

$$= \sum_x \left(x^2 - 2x\,E[X] + (E[X])^2\right) p_X(x)$$

$$= \sum_x x^2 p_X(x) - \sum_x \underline{2x\,E[X]\,p_X(x)} + \sum_x \underline{(E[X])^2}\, p_X(x)$$

$$= E[X^2] - 2\,E[X] \cdot E[X] + (E[X])^2$$

$$\mathrm{Var}(X) = E[X^2] - (E[X])^2$$

$$X = \begin{cases} 1 & \text{if } \cancel{B}\ A \\ 0 & \text{o/w } (\Omega - A) \end{cases} \qquad \begin{aligned} p_X(1) &= p \\ p_X(0) &= 1-p \end{aligned}$$

Bernoulli R.V. / 0-1 r.v / Indicator r.v.

$$\boxed{E[X] = 0\cdot(1-p) + 1\cdot p = p}$$

$$\mathrm{Var}(X) = (0-p)^2(1-p) + (1-p)^2 p = (1-p)\,p$$

~~Quicksort~~