Ex: Find the value of $\int_{|z|=1} \frac{\sin^6 z}{(z-\pi/6)^3}\,dz$.

Sol^n: Let $f(z)=\sin^6 z$. Obviously $f(z)$ is analytic at all points within and on the circle $|z|=1$.

By the nth derivative formula,

$$f^n(z_0) = \frac{\lfloor n}{2\pi i}\int_C \frac{f(z)\,dz}{(z-z_0)^{n+1}}$$

where C is the circle $|z|=1$:

Taking $f(z)=\sin^6 z$, $z_0=\frac{\pi}{6}$ and $n=2$, we find that

$$\frac{\lfloor 2}{2\pi i}\int_C \frac{\sin^6 z}{\left(z-\frac{\pi}{6}\right)^3} = f''\left(\frac{\pi}{6}\right) = \left[\frac{d^2}{dz^2}(\sin^6 z)\right]_{z=\pi/6}$$

$$= \left[\frac{d}{dz}\left(6\sin^5 z\cos z\right)\right]$$

$$= \left[30\sin^4 z\cos^2 z - 6\sin^6 z\right]_{z=\frac{\pi}{6}}$$

$$= \left[30\times\left(\frac{1}{2}\right)^4\times\frac{3}{4} - 6\times\left(\frac{1}{2}\right)^6\right]$$

$$= \left[\frac{30\times 3}{4\times 16} - \frac{6}{64}\right] = \frac{84}{64} = \frac{21}{16}$$

Thus, $\int_C \frac{\sin^6 z}{\left(z-\frac{\pi}{6}\right)^3}\,dz = \frac{21}{16}\pi i$ Ans.

Q: Find the value of $\frac{1}{2\pi i}\int_C \frac{x^n e^{xz}}{\lfloor n \; z^{n+1}} dz$, where C is any closed contour surrounding the origin.

OR,

Using the integral representation of $f^n(a)$, prove that

$$\left(\frac{x^n}{\lfloor n}\right)^2 = \frac{1}{2\pi i}\int_C \frac{x^n e^{xz}}{\lfloor n \; z^{n+1}} dz$$

where C is any closed contour surrounding the origin.

Hence show that

$$\sum_{n=0}^{\infty}\left(\frac{x^n}{\lfloor n}\right)^2 = \frac{1}{2\pi}\int_0^{2\pi} e^{2x\cos\theta} d\theta .$$

Solⁿ: By Cauchy's integral formula for the n th derivative of f(z) at z = 0, we have

$$f^n(0) = \frac{\lfloor n}{2\pi i}\int_C \frac{f(z)}{z^{n+1}} dz \longrightarrow (1)$$

Taking $f(z) = e^{xz}$, obviously f(z) is analytic at all points within and on closed contour C.

Then $f^n(z) = x^n e^{xz}$

$f^n(0) = x^n$

Substituting in (1), we get

$$\frac{x^n}{\lfloor n} = \frac{1}{2\pi i}\int_C \frac{e^{xz}}{z^{n+1}} dz$$

$$\Rightarrow \left(\frac{x^n}{\lfloor n}\right)^2 = \frac{1}{2\pi i}\int_C \frac{x^n e^{xz}}{\lfloor n \; z^{n+1}} dz \longrightarrow (2)$$

Now, summing both sides of (2) from n = 0 to ∞, we find that

$$\sum_{n=0}^{\infty}\left(\frac{x^n}{\lfloor n}\right)^2 = \sum_{n=0}^{\infty}\frac{1}{2\pi i}\int_C \frac{x^n e^{xz}}{\lfloor n \; z^{n+1}} dz .$$

Since the summation and integration can be interchanged because the series

$$\sum_{n=0}^{\infty} \frac{x^n}{\lfloor n \, z^{n+1}}$$

is uniformly convergent, therefore

$$\sum_{n=0}^{\infty} \left(\frac{x^n}{\lfloor n}\right)^2 = \frac{1}{2\pi i}\int_C e^{xz} \sum_{n=0}^{\infty} \frac{1}{\lfloor n}\left(\frac{x}{z}\right)^n \cdot \frac{dz}{z}$$

$$= \frac{1}{2\pi i}\int_C e^{xz} \, e^{x/z} \, \frac{dz}{z} \longrightarrow (3)$$

Now, if we take C to be the circle $|z|=1$, so that $z = e^{i\theta}$, $dz = ie^{i\theta}d\theta$

We find from (3) that

$$\sum_{n=0}^{\infty} \left(\frac{x^n}{\lfloor n}\right)^2 = \frac{1}{2\pi i}\int_0^{2\pi} e^{x(e^{i\theta}+e^{-i\theta})} \frac{ie^{i\theta}}{e^{i\theta}} d\theta$$

$$= \frac{1}{2\pi}\int_0^{2\pi} e^{2x\cos\theta} d\theta$$

Proved.

9/9/19

Gauss's Mean Value Theorem:

Corollary ①: If $f(z)$ is an analytic function on a domain D and if the circular region $|z-z_0| \leq \rho$ is contained in D, then

$$f(z_0) = \frac{1}{2\pi}\int_0^{2\pi} f(z_0+\rho e^{i\theta})\, d\theta .$$

In other words, the value of $f(z)$ at the point z_0 equals the average of its values on the boundary of the circle $|z-z_0| = \rho$.

Proof: Let C_1 denote the circle $|z-z_0| = \rho$. This equation can be written as

$$z = z_0 + \rho e^{i\theta} \quad (0 \leq \theta \leq 2\pi)$$

so that $dz = \rho i e^{i\theta} d\theta$.

Hence by Cauchy's integral formula, we get

$$f(z_0) = \frac{1}{2\pi i}\int_{C_1} \frac{f(z)}{z-z_0}\, dz$$

$$= \frac{1}{2\pi i}\int_0^{2\pi} \frac{f(z_0+\rho e^{i\theta})\, i\rho e^{i\theta}}{\rho e^{i\theta}}\, d\theta$$

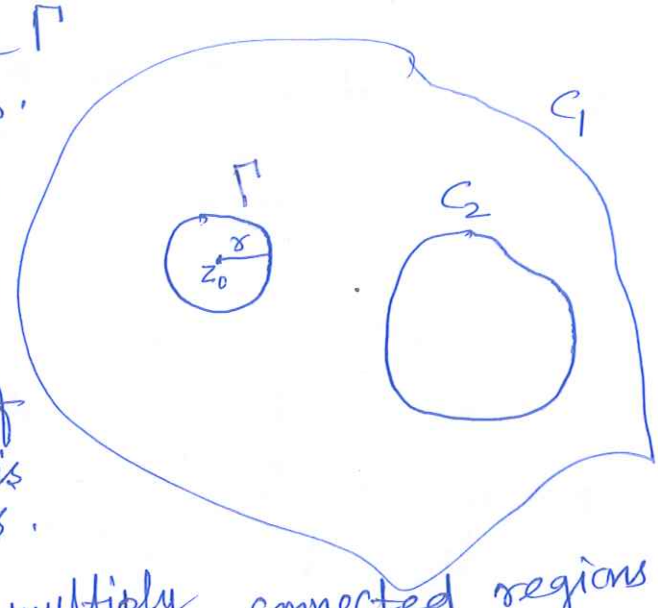
$$= \frac{1}{2\pi}\int_0^{2\pi} f(z_0+\rho e^{i\theta})\, d\theta$$

Corollary: If $f(z)$ is analytic in the region bounded by two closed curves C_1 and C_2 and z_0 is any point in the region, then

$$f(z_0) = \frac{1}{2\pi i}\int_{C_1} \frac{f(z)\,dz}{z-z_0} - \frac{1}{2\pi i}\int_{C_2} \frac{f(z)}{z-z_0}\, dz .$$

(This corollary provides extension of Cauchy's integral formula to multiply connected regions.

Proof:



Proof : Draw a small circle Γ with centre at the point z_0. Consider the function $\frac{f(z)}{z-z_0}$, which is analytic in the region bounded by three curves C_1, C_2 and Γ (because $z-z_0$ is not zero for any value of z in this region) and it is also analytic on these curves.

By Cauchy's theorem for multiply connected regions

$$\frac{1}{2\pi i}\int_{C_1} \frac{f(z)}{z-z_0}\,dz - \frac{1}{2\pi i}\int_{C_2} \frac{f(z)}{z-z_0}\,dz - \int_{\Gamma} \frac{f(z)}{z-z_0}\,dz = 0. \longrightarrow (1)$$

Thus, in view of Cauchy's integral formula, we get

$$f(z_0) = \frac{1}{2\pi i}\int_{\Gamma} \frac{f(z)}{z-z_0}\,dz$$

$$= \frac{1}{2\pi i}\int_{C_1} \frac{f(z)}{z-z_0}\,dz - \frac{1}{2\pi i}\int_{C_2} \frac{f(z)}{z-z_0}\,dz.$$

where integration along each curve is taken in the anti-clockwise direction.

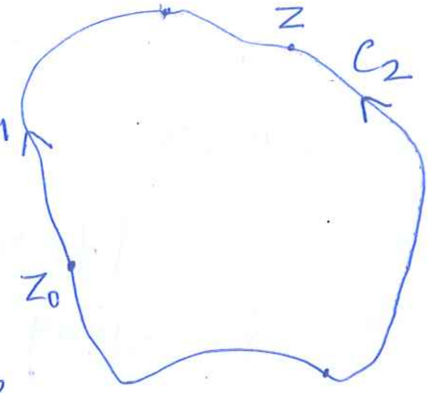
In general if there are more curves $C_3, C_4,$ etc. inside C_1, then we have similarly

$$f(z_0) = \frac{1}{2\pi i}\int_{C_1} \frac{f(z)}{z-z_0}\,dz - \frac{1}{2\pi i}\int_{C_2} \frac{f(z)\,dz}{z-z_0} - \frac{1}{2\pi i}\int_{C_3} \frac{f(z)}{z-z_0}\,dz - \cdots \text{ etc.}$$

9/9/19

Morera's Theorem (Converse of Cauchy's Theorem):

Statement: Let $f(z)$ be continuous function in a simply connected domain G. If $\int_C f(z)\,dz = 0$ along every simple closed contour C in G, then $f(z)$ is analytic in G.



Proof: Let z_0 be a fixed point and z, a variable point in G and let C_1, C_2 be any two continuous rectifiable curves in G joing z_0 to z. Let C denoted the closed contour consisting of C_1 and $-C_2$, so according to given condition

$$\int_C f(z)\,dz = \int_{C_1} f(z)\,dz + \int_{-C_2} f(z)\,dz = 0$$

$$\Rightarrow \int_{C_1} f(z)\,dz = \int_{C_2} f(z)\,dz .$$

This shows that the integral along every curve in G joining z_0 to z is the same. Hence taking ξ as the variable of integration, we have

$$F(z) = \int_{z_0}^{z} f(\xi)\,d\xi \longrightarrow (1)$$

as the integral depends only on z_0 and z.

Let $z+h$ be any point in G near the point z.

Then $$F(z+h) - F(z) = \int_{z_0}^{z+h} f(\xi)\,d\xi - \int_{z_0}^{z} f(\xi)\,d\xi$$

$$= \int_{z}^{z+h} f(\xi)\,d\xi .$$

Now, the above integral is independent of the path joing z and $z+h$. In particular we may choose as path the straight line segment joining z and $z+h$, provided we choose $|h|$ small enough so that this path lies in G.

Thus,

$$\frac{F(z+h)-F(z)}{h} - f(z) = \frac{1}{h}\int_z^{z+h} f(\xi)\,d\xi - \frac{f(z)}{h}\cdot h$$

$$= \frac{1}{h}\int_z^{z+h} [f(\xi)-f(z)]\,d\xi$$

Since $f(\xi)$ is continuous at z, given a positive number ϵ, there exists a $\delta > 0$ such that

$$|f(\xi) - f(z)| < \epsilon \longrightarrow (2)$$

for every ξ satisfying $|\xi - z| < \delta$.

We now choose h s.t. $|h| < \delta$. Then the inequality (2) is satisfied for every point ξ on the line segment joining z and $z+h$. Hence,

$$\left|\frac{F(z+h)-F(z)}{h} - f(z)\right| \le \frac{1}{|h|}\int_z^{z+h} |f(\xi)-f(z)|\,|d\xi|$$

$$< \frac{\epsilon}{|h|}\int_z^{z+h} |d\xi|$$

$$= \frac{\epsilon}{|h|}\cdot|h| = \epsilon \longrightarrow (3)$$

Since ϵ is arbitrary, we get from (3),

$$\lim_{h\to 0} \frac{F(z+h)-F(z)}{h} = f(z).$$

Therefore $F'(z)$ exists and $F'(z) = f(z)$.

Thus, $F(z)$ possesses the derivative $f(z)$ at every point $z \in G$ and consequently $F(z)$ is analytic in G. But the derivative of an analytic function is analytic. $\therefore$ It follows that $f(z)$ is analytic in G.

This completes the proof of the theorem.

Corollary: Let $f(z)$ be continuous in a simply connected domain G and let C be any simple closed contour in G, then a necessary and sufficient condition for $f(z)$ to be analytic in G is that $\int_C f(z)\,dz = 0$.

Proof: The above theorem is simply a combination of two theorems viz. Cauchy's theorem and Morera's theorem.

Cauchy's Inequality:

St: Let $f(z)$ be analytic inside and on the circle $C: |z-z_0| = r$. If $|f(z)| \leq M(r)$ (or $M(r) = \max_{z \in C} |f(z)|$), then

$$|f^n(z_0)| \leq \frac{M(r) \lfloor n}{r^n} \quad (n = 0, 1, 2, 3, \ldots).$$

Since we know that "if $f(z)$ is analytic in a simply connected domain G containing a simple closed contour C, then $f(z)$ has derivatives of all orders at each point z_0 inside C with

$$f^n(z_0) = \frac{\lfloor n}{2\pi i} \int_C \frac{f(z)\,dz}{(z-z_0)^{n+1}}.$$

Hence

$$|f^n(z_0)| \leq \frac{\lfloor n}{2\pi |i|} \int_C \frac{|f(z)|}{|z-z_0|^{n+1}} |dz|$$

$$\leq \frac{\lfloor n}{2\pi r^{n+1}} M(r) \cdot 2\pi r \leq \frac{\lfloor n \, M(r)}{r^n}$$

which completes the proof of Cauchy's inequality.

Poission Integral Formula for a Circle:

St: Let $f(z)$ be analytic in the region $|z| < \rho$ and let $z = re^{i\theta}$ be any point of this region. Then

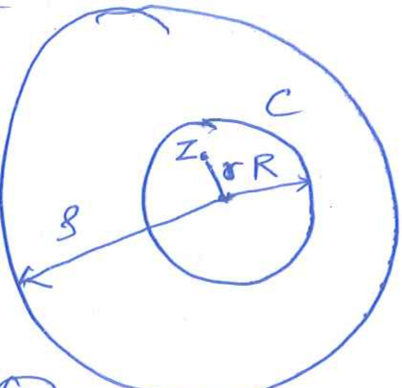
$$f(re^{i\theta}) = \frac{1}{2\pi} \int_0^{2\pi} \frac{(R^2 - r^2) f(Re^{i\phi})}{R^2 - 2Rr\cos(\theta - \phi) + r^2} \, d\phi$$

where R is any number such that $0 < R < \rho$.

Sol^n:- Let C ~~denote~~ be the circle $|z| = R$ such that $r < R < \rho$. As given $z = re^{i\theta}$ is any point of the region $|z| < \rho$ where $r < R < \rho$. Hence by Cauchy's Integral formula, we get

$$f(z) = \frac{1}{2\pi i} \int_C \frac{f(w)}{w-z} \, dw \longrightarrow (1)$$

Now, the inverse of the point z with respect to C is $\frac{R^2}{\bar{z}}$ and lies outside C so that the function $\dfrac{f(w)}{w-\frac{R^2}{\bar{z}}}$ is analytic on and within C. Therefore, by Cauchy-Goursat theorem, we have



$$0 = \int_C \frac{f(w)\,dw}{w-\frac{R^2}{\bar{z}}} \longrightarrow (2)$$

Subtracting (2) from (1), we get

$$f(z) = \frac{1}{2\pi i}\int_C \left[\frac{1}{w-z} - \frac{1}{w-\frac{R^2}{\bar{z}}}\right] f(w)\,dw$$

$$= \frac{1}{2\pi i}\int_C \frac{z - R^2/\bar{z}}{(w-z)(w-R^2/\bar{z})} f(w)\,dw \longrightarrow (3)$$

Now, we write $z = re^{i\theta}$, $w = Re^{i\phi}$. Then $\bar{z} = re^{-i\theta}$, $dw = Rie^{i\phi}\,d\phi$. Substituting these values in (3), we get

$$f(re^{i\theta}) = \frac{1}{2\pi i}\int_0^{2\pi} \frac{\left[re^{i\theta} - \frac{R^2}{r}e^{i\theta}\right] \cdot Rie^{i\phi}\,d\phi}{(Re^{i\phi}-re^{i\theta})\left(Re^{i\phi} - \frac{R^2}{r}e^{i\theta}\right)}$$

$$= \frac{1}{2\pi i}\int_0^{2\pi} \frac{(r^2-R^2)e^{i\theta} f(Re^{i\phi})\, ie^{i\phi}}{(Re^{i\phi}-re^{i\theta})e^{i\theta}e^{i\phi}(re^{-i\theta}-Re^{-i\phi})}\,d\phi$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \frac{(R^2-r^2) f(Re^{i\phi})}{(Re^{i\phi}-re^{i\theta})(Re^{-i\phi}-re^{-i\theta})}\,d\phi$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \frac{(R^2-r^2) f(Re^{i\phi})}{R^2 - 2Rr\cos(\theta-\phi) + r^2}\,d\phi \longrightarrow (4)$$

If we write $f(re^{i\theta}) = u(r,\theta) + iv(r,\theta)$ and $f(Re^{i\phi}) = u(R,\phi) + iv(R,\phi)$ and equate real and imaginary parts in (4), we get

$$u(r,\theta) = \frac{1}{2\pi}\int_0^{2\pi} \frac{(R^2-r^2)\,u(R,\phi)}{R^2 - 2Rr\cos(\theta-\phi) + r^2}\,d\phi \longrightarrow (5)$$

and $$v(r,\theta) = \frac{1}{2\pi}\int_0^{2\pi} \frac{(R^2-r^2)\,v(R,\phi)}{R^2 - 2Rr\cos(\theta-\phi) + r^2}\,d\phi \longrightarrow (6)$$