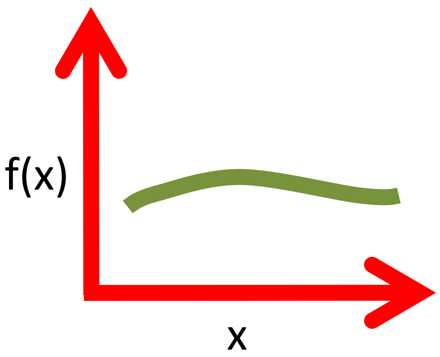
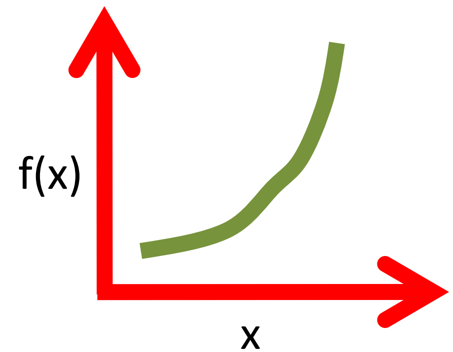
Introduction to Vector Operators

Gradient, Divergence and Curl

Ordinary Derivatives

Ordinary Derivatives





?

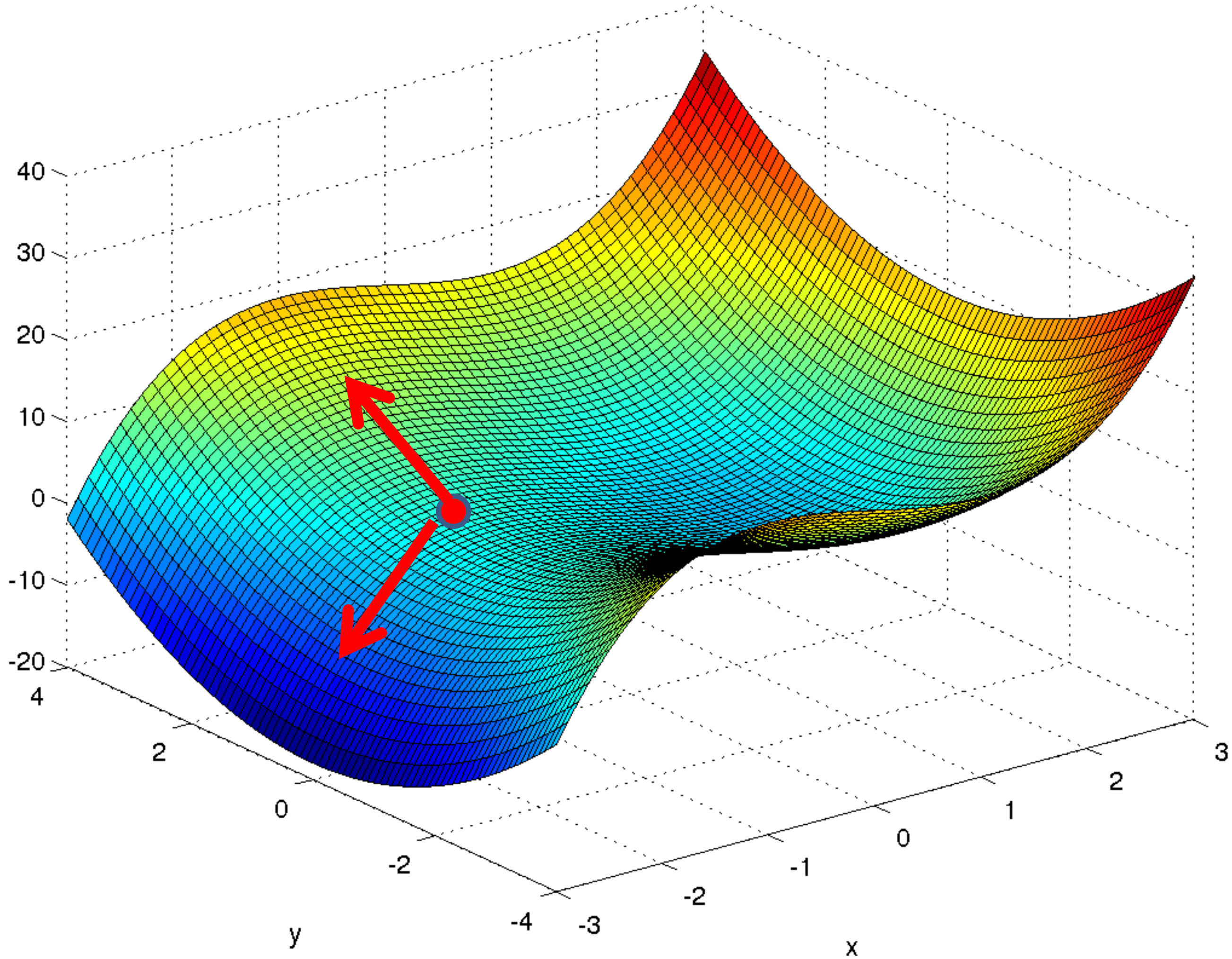
$$d\vec{r} = dr\hat{e}_r + r\,d\hat{e}_r$$

$$df = \left(\frac{df(x)}{dx}\right)dx$$

RECOLLECT (By Taylor Series) for Plane polar

$$d\hat{e}_r(\theta) = \frac{d\hat{e}_r(\theta)}{d\theta}\,d\theta$$

Derivative of a function in 3D



Derivative is directional!

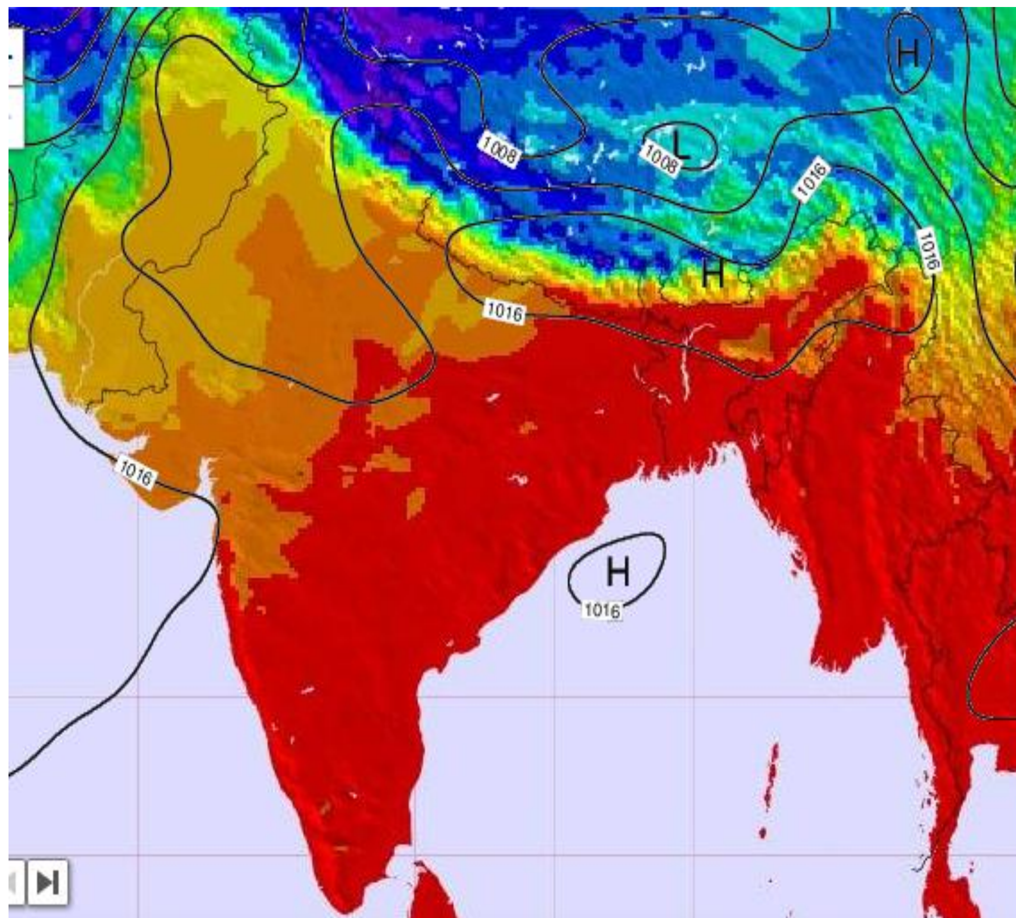
Scalar Field Function

A function of space whose value at each point is a scalar quantity.

Its value at any point is independent of where observer's frame of reference is located and how it is oriented.

Example: Temperature

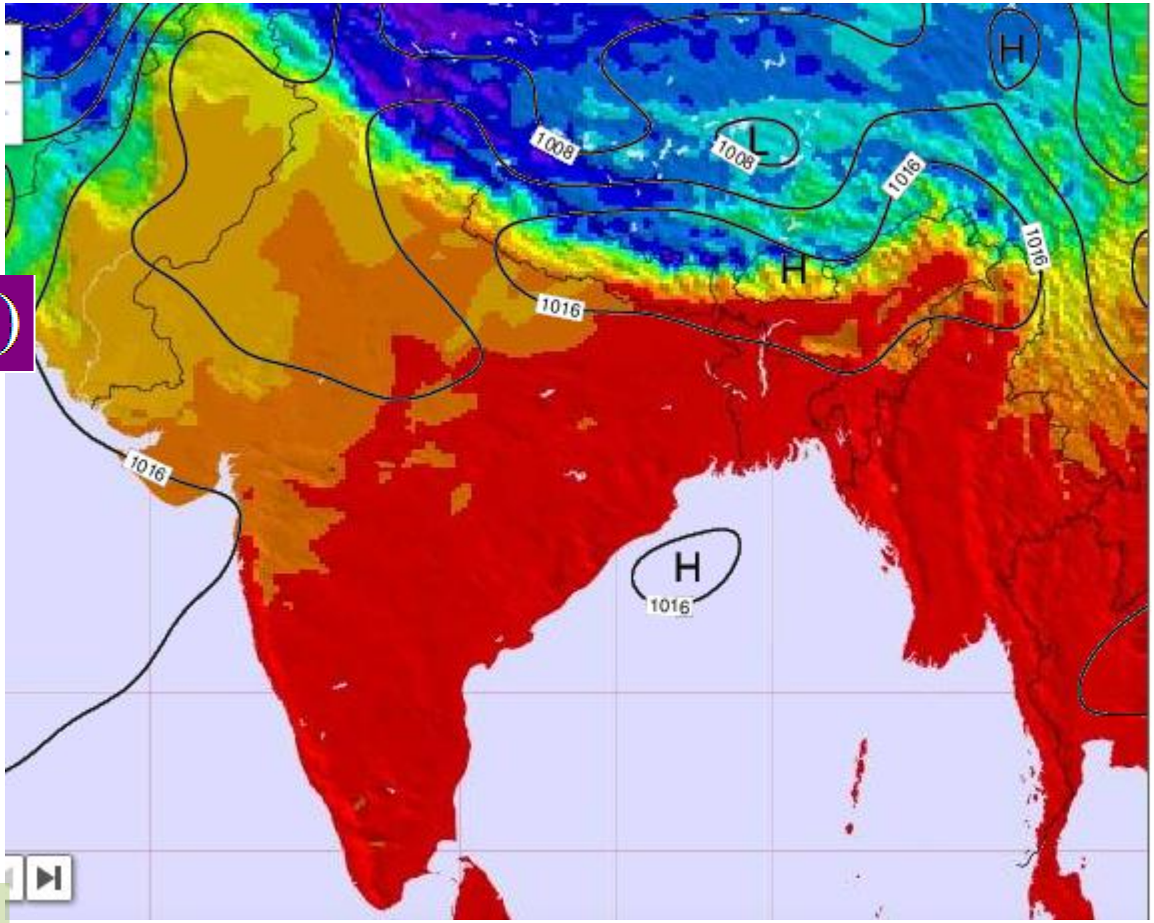
$$T(x, y, z)$$



Gradient of a scalar function

$$T(x, y, z)$$

$$T(x+dx, y+dy, z+dz)$$



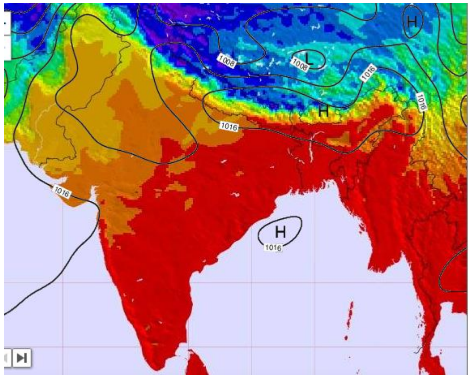
$$\frac{\partial T(x, y, z)}{\partial x}$$

Partial derivative

Gradient of a function

$$df = \left(\frac{df(x)}{dx} \right) dx$$

$$dT = \frac{\partial T}{\partial x} dx + \frac{\partial T}{\partial y} dy + \frac{\partial T}{\partial z} dz$$



$$T(x, y, z)$$

$$dT = \left(\frac{\partial T}{\partial x} \hat{e}_x + \frac{\partial T}{\partial y} \hat{e}_y + \frac{\partial T}{\partial z} \hat{e}_z \right) \bullet \left(dx\hat{e}_x + dy\hat{e}_y + dz\hat{e}_z \right)$$

$$dT = \vec{\nabla} T \bullet d\vec{l}$$

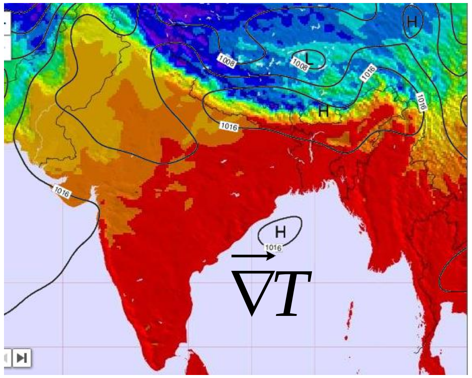
$$\vec{\nabla} T = \left(\frac{\partial T}{\partial x} \hat{e}_x + \frac{\partial T}{\partial y} \hat{e}_y + \frac{\partial T}{\partial z} \hat{e}_z \right)$$

Gradient

Gradient: Geometrical interpretation

$$dT_{\hat{u}} = \vec{\nabla} T \bullet \hat{u}$$

$$dT_{\hat{u}} = \left|\vec{\nabla} T\right| \cos\theta$$



$$T(x, y, z)$$

The gradient $\vec{\nabla} T$ points in the direction of maximum increase of the T.

Let's do an example

Find the gradient of $r = \sqrt{x^2 + y^2 + z^2}$

$$\vec{\nabla} f = \left(\frac{\partial f}{\partial x} \hat{e}_x + \frac{\partial f}{\partial y} \hat{e}_y + \frac{\partial f}{\partial z} \hat{e}_z \right)$$

$$\boldsymbol{\vec{\nabla} r = \frac{\vec{r}}{r}}$$

What would it mean for the gradient $\vec{\nabla} f$ **to vanish?**

Stationary point of *f(x,y,z)*

$\vec{\nabla}$: Vector operator (The del operator)

$$\vec{\nabla} = \left(\hat{e}_x \frac{\partial}{\partial x} + \hat{e}_y \frac{\partial}{\partial y} + \hat{e}_z \frac{\partial}{\partial z} \right)$$

Three ways the operator $\vec{\nabla}$: can act:

$\vec{\nabla} f$ (The gradient)

$\vec{\nabla} \cdot \vec{V}$ (The divergence)

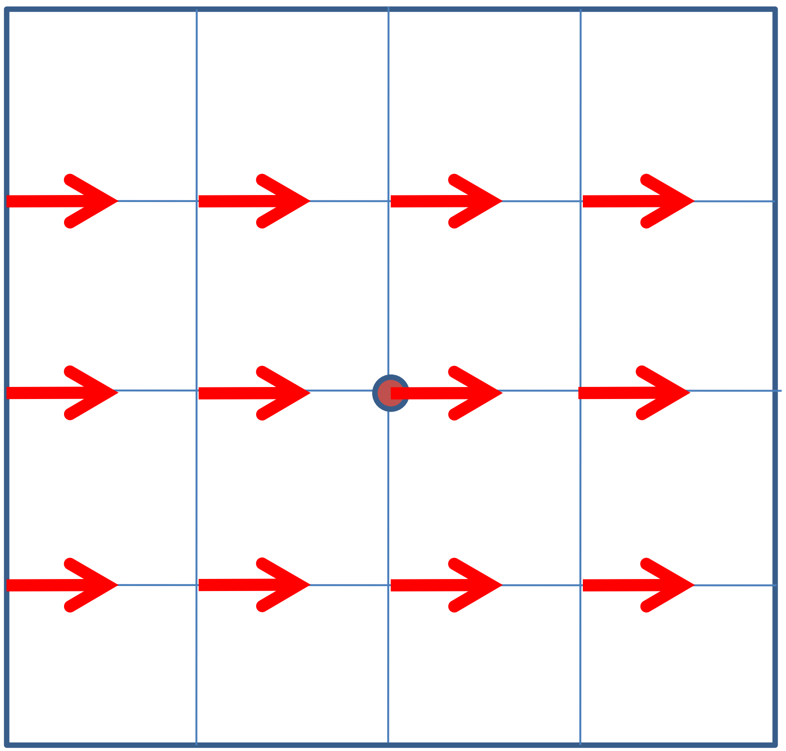
$\vec{\nabla} \times \vec{V}$ (The curl)

In the divergence and curl $\vec{\nabla}$ operates on a VECTOR FIELD .

Examples of vector fields are velocity of fluid flow , electric field, magnetic field etc

Vector Field

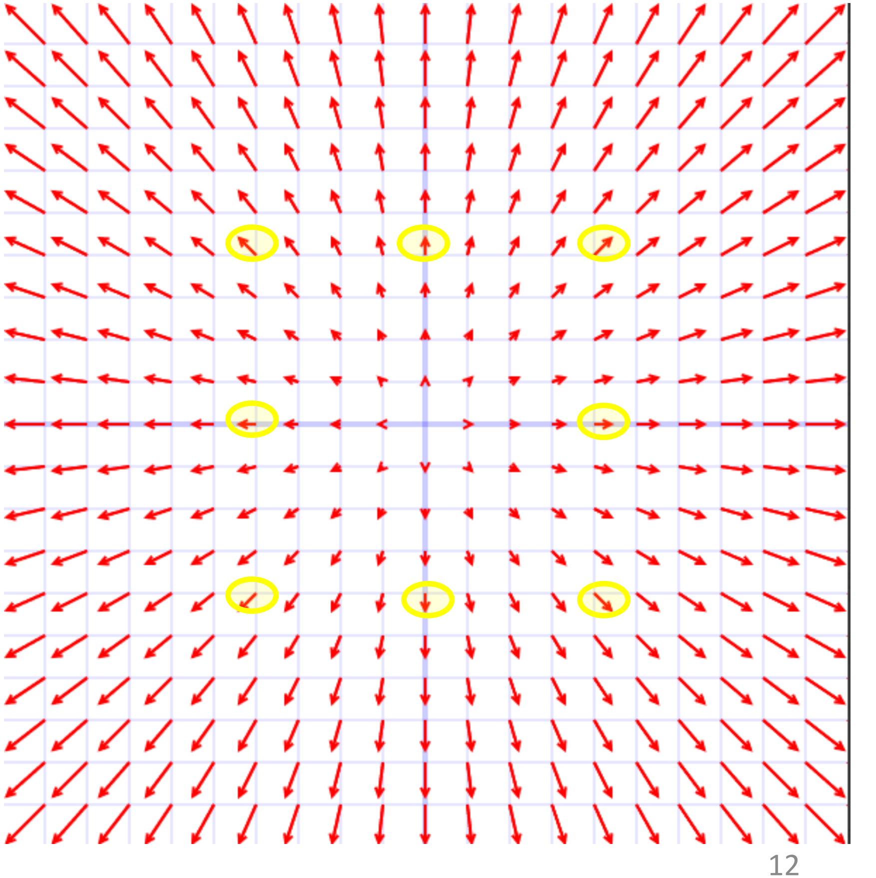
$$\vec{A} = 0.5\hat{e}_x$$



Vector Field

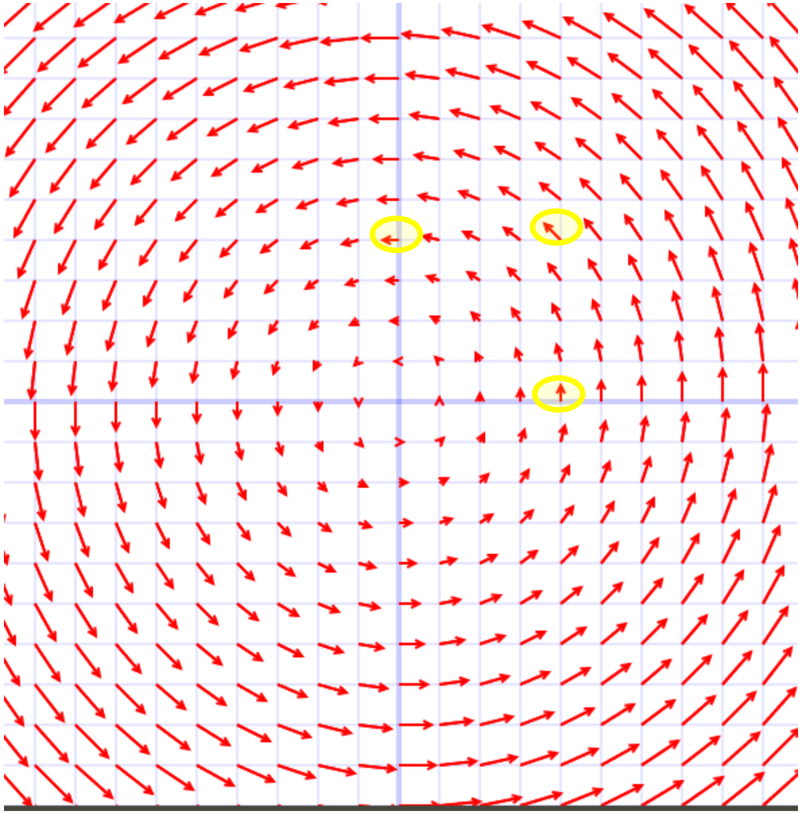
$$\vec{A} = 1\hat{e}_x + 1\hat{e}_y :$$

$$\vec{A} = x\hat{e}_x + y\hat{e}_y :$$



Vector Field

$$\overrightarrow{A} = -y\hat{e}_x + x\hat{e}_y :$$



(4,0)

(4,4)

(0,4)

$\vec{\nabla} \cdot \vec{V}$(The divergence)

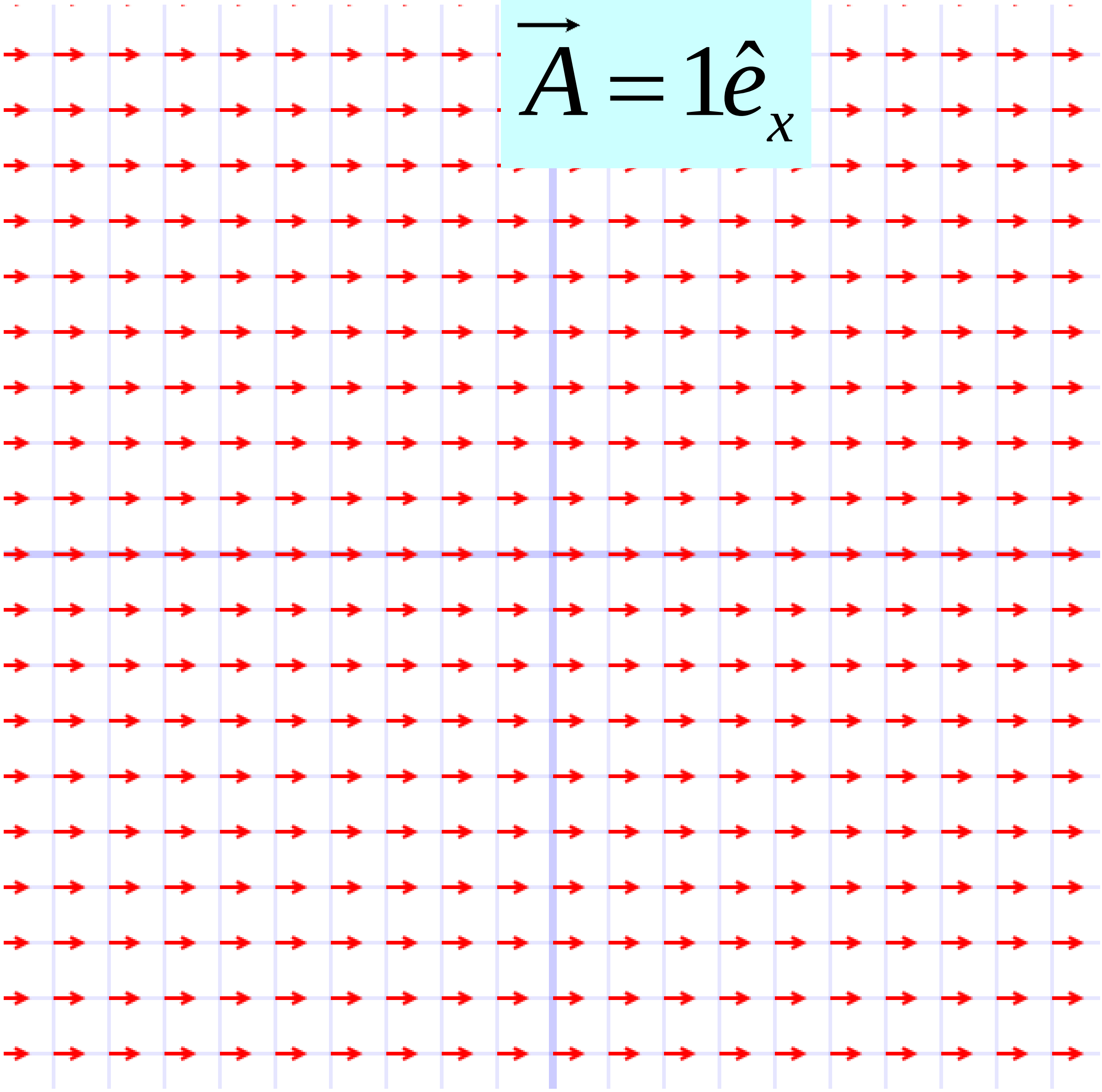
$$\vec{\nabla} \cdot \vec{V} = \left(\hat{e}_x \frac{\partial}{\partial x} + \hat{e}_y \frac{\partial}{\partial y} + \hat{e}_z \frac{\partial}{\partial z} \right) \cdot (\hat{e}_x V_x + \hat{e}_y V_y + \hat{e}_z V_z)$$

$$\vec{\nabla} \cdot \vec{V} = \left(\frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z} \right)$$

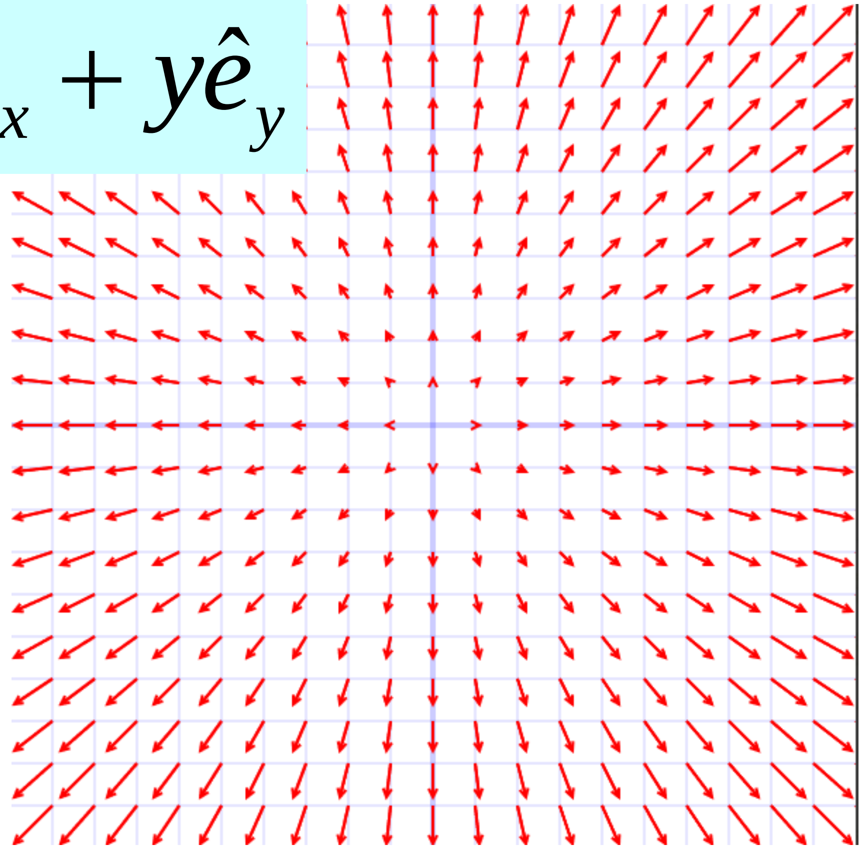
The divergence: Geometrical interpretation

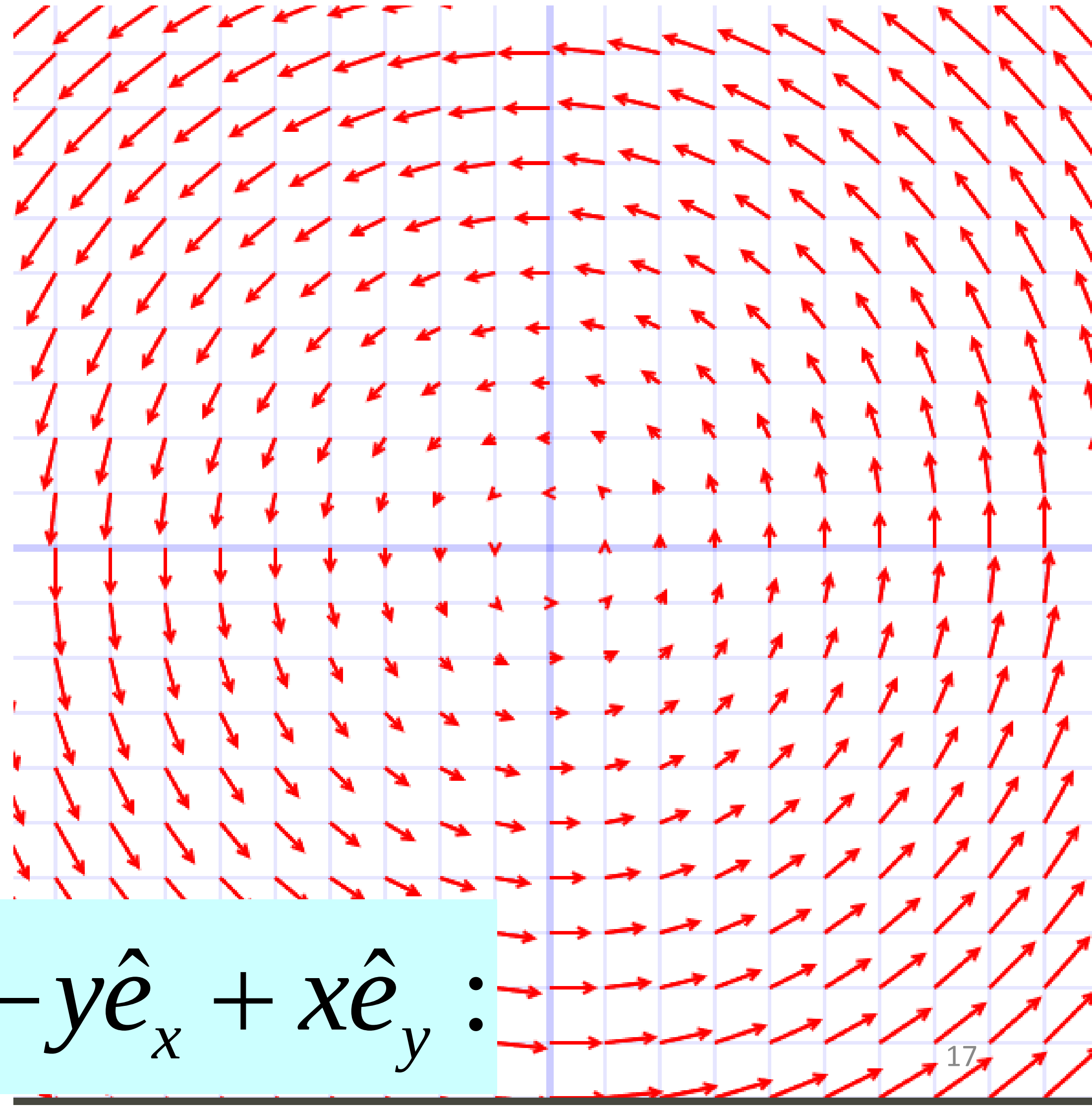
Net flow out through a closed surface

$$\vec{A} = 1\hat{e}_x$$



$$\vec{A} = x\hat{e}_x + y\hat{e}_y$$





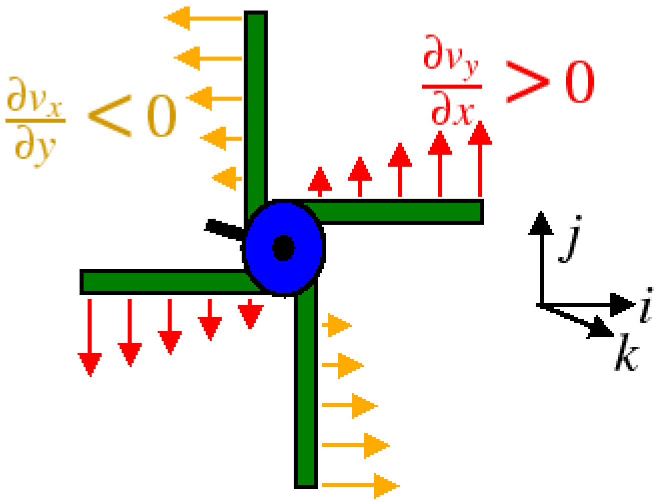
$$\vec{A} = -y\hat{e}_x + x\hat{e}_y :$$

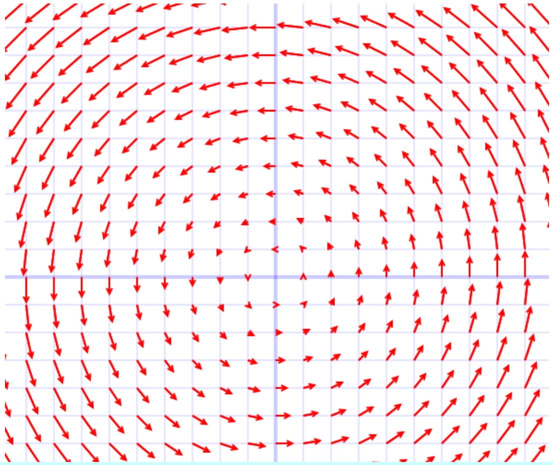
$$\vec{\nabla} \times \vec{V} \text{ (The curl)}$$

$$\vec{\nabla} \times \vec{V} = \hat{e}_x \left(\frac{\partial V_z}{\partial y} - \frac{\partial V_y}{\partial z} \right) + \hat{e}_y \left(\frac{\partial V_x}{\partial z} - \frac{\partial V_z}{\partial x} \right) + \hat{e}_z \left(\frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y} \right)$$

The curl: Geometrical interpretation

Measure of counter clockwise rotation





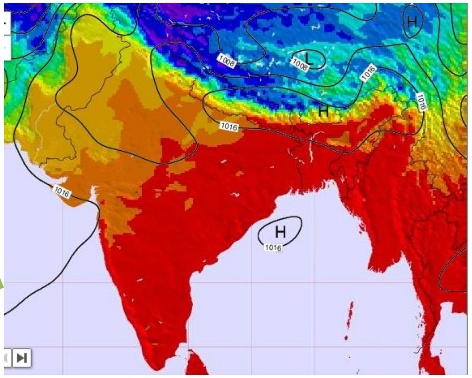
$$\vec{V} = -y\hat{e}_x + x\hat{e}_y :$$

Gradient of a scalar function in different coordinate system

$$dT = \frac{\partial T}{\partial x}dx + \frac{\partial T}{\partial y}dy + \frac{\partial T}{\partial z}dz$$

$$dT = \vec{\nabla}T \bullet d\vec{l}$$

This definition is independent of the coordinate system used



$$T(x, y, z)$$

$$T(r, \theta, \phi)$$

$$T(\rho, \phi, z)$$

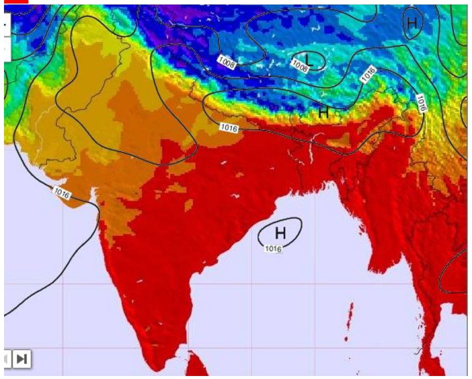
$$dT = \frac{\partial T}{\partial r}dr + \frac{\partial T}{\partial \theta}d\theta + \frac{\partial T}{\partial \phi}d\phi$$

$$dT = \frac{\partial T}{\partial \rho}d\rho + \frac{\partial T}{\partial \phi}d\phi + \frac{\partial T}{\partial z}dz$$

Gradient in cylindrical polar coordinate system

$$dT = \frac{\partial T}{\partial \rho} d\rho + \frac{\partial T}{\partial \phi} d\phi + \frac{\partial T}{\partial z} dz$$

$$dT = \vec{\nabla} T \bullet d\vec{l}$$



$$T(\rho, \phi, z)$$

$$d\vec{l} = d\rho\hat{e}_\rho + \rho d\phi\hat{e}_\phi + dz\hat{e}_z$$

$$dT = \left(\vec{\nabla} T\right) \cdot \left(d\rho\hat{e}_\rho + \rho d\phi\hat{e}_\phi + dz\hat{e}_z\right)$$

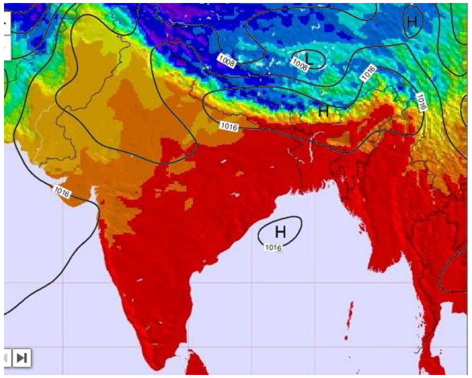
$$\vec{\nabla} T = \left(\frac{\partial T}{\partial \rho}\hat{e}_\rho + \frac{1}{\rho}\frac{\partial T}{\partial \phi}\hat{e}_\phi + \frac{\partial T}{\partial z}\hat{e}_z\right)$$

Gradient

Gradient in cylindrical polar coordinate system

$$dT = \frac{\partial T}{\partial \rho} d\rho + \frac{\partial T}{\partial \phi} d\phi + \frac{\partial T}{\partial z} dz$$

$$dT = \vec{\nabla} T \bullet d\vec{l}$$



$$T(\rho, \phi, z)$$

$$\vec{\nabla} T = \left(\frac{\partial T}{\partial \rho} \hat{e}_\rho + \frac{1}{\rho} \frac{\partial T}{\partial \phi} \hat{e}_\phi + \frac{\partial T}{\partial z} \hat{e}_z \right)$$

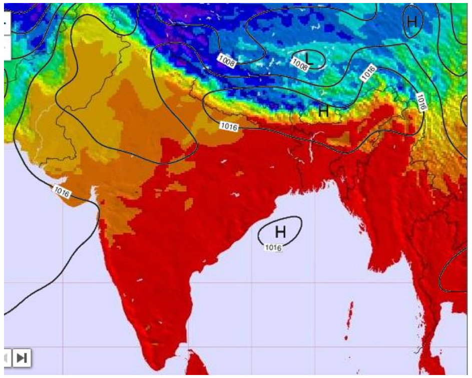
Gradient

$$dT = \left(\frac{\partial T}{\partial \rho} \hat{e}_\rho + \frac{1}{\rho} \frac{\partial T}{\partial \phi} \hat{e}_\phi + \frac{\partial T}{\partial z} \hat{e}_z \right) \cdot \left(d\rho \hat{e}_\rho + \rho d\phi \hat{e}_\phi + dz \hat{e}_z \right)$$

$$\vec{\nabla} = \left(\hat{e}_\rho \frac{\partial}{\partial \rho} + \hat{e}_\phi \frac{1}{\rho} \frac{\partial}{\partial \phi} + \hat{e}_z \frac{\partial}{\partial z} \right)$$

Gradient in spherical polar coordinate system

$$dT = \frac{\partial T}{\partial r}dr + \frac{\partial T}{\partial \theta}d\theta + \frac{\partial T}{\partial \phi}d\phi$$



$$dT = \vec{\nabla}T \bullet d\vec{l}$$

$T(r,\theta,\phi)$

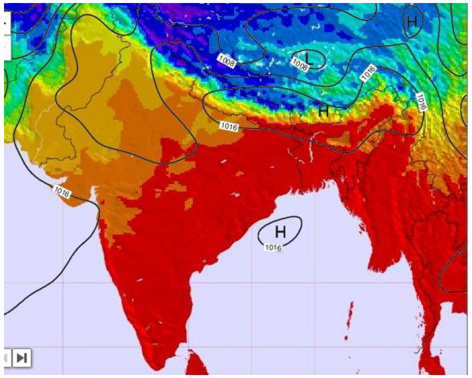
$$d\vec{l} = dr\hat{e}_r + rd\theta\hat{e}_\theta + r\sin\theta d\phi\hat{e}_\phi$$

$$dT = \left(\vec{\nabla}T\right)\cdot\left(dr\hat{e}_r + rd\theta\hat{e}_\theta + r\sin\theta d\phi\hat{e}_\phi\right)$$

$$\vec{\nabla}T = \left(\frac{\partial T}{\partial r}\hat{e}_r + \frac{1}{r}\frac{\partial T}{\partial \theta}\hat{e}_\theta + \frac{1}{r\sin\theta}\frac{\partial T}{\partial \phi}\hat{e}_\phi\right)$$

Gradient

Gradient in spherical polar coordinate system



$$T(r,\theta,\phi)$$

$$dT = \frac{\partial T}{\partial r}dr + \frac{\partial T}{\partial \theta}d\theta + \frac{\partial T}{\partial \phi}d\phi$$

$$dT = \vec{\nabla}T \bullet d\vec{l}$$

$$dT = \left(\frac{\partial T}{\partial r}\hat{e}_r + \frac{1}{r}\frac{\partial T}{\partial \theta}\hat{e}_\theta + \frac{1}{r\sin\theta}\frac{\partial T}{\partial \phi}\hat{e}_\phi \right) \cdot \left(dr\hat{e}_r + r\,d\theta\hat{e}_\theta + r\sin\theta\, d\phi\hat{e}_\phi \right)$$

$$\vec{\nabla} = \left(\hat{e}_r \frac{\partial}{\partial r} + \hat{e}_\theta \frac{1}{r}\frac{\partial}{\partial \theta} + \hat{e}_\phi \frac{1}{r\sin\theta}\frac{\partial}{\partial \phi} \right)$$

$\vec{\nabla}$ Operator in Different coordinate System

Cartesian Coordinate system

$$\vec{\nabla} = \left(\hat{e}_x \frac{\partial}{\partial x} + \hat{e}_y \frac{\partial}{\partial y} + \hat{e}_z \frac{\partial}{\partial z} \right)$$

Cylindrical polar Coordinate system

$$\vec{\nabla} = \left(\hat{e}_\rho \frac{\partial}{\partial \rho} + \hat{e}_\phi \frac{1}{\rho} \frac{\partial}{\partial \phi} + \hat{e}_z \frac{\partial}{\partial z} \right)$$

Spherical polar Coordinate system

$$\vec{\nabla} = \left(\hat{e}_r \frac{\partial}{\partial r} + \hat{e}_\theta \frac{1}{r} \frac{\partial}{\partial \theta} + \hat{e}_\phi \frac{1}{r \sin\theta} \frac{\partial}{\partial \phi} \right)$$

Divergence in Spherical Polar Coordinate System

Spherical polar Coordinate system

$$\vec{\nabla} = \left(\hat{e}_r \frac{\partial}{\partial r} + \hat{e}_\theta \frac{1}{r} \frac{\partial}{\partial \theta} + \hat{e}_\phi \frac{1}{r \sin\theta} \frac{\partial}{\partial \phi} \right)$$

$$\vec{\nabla} \cdot \vec{A} = \left(\hat{e}_r \frac{\partial}{\partial r} + \hat{e}_\theta \frac{1}{r} \frac{\partial}{\partial \theta} + \hat{e}_\varphi \frac{1}{r \sin\theta} \frac{\partial}{\partial \varphi} \right) \cdot \left(\hat{e}_r A_r + \hat{e}_\theta A_\theta + \hat{e}_\varphi A_\varphi \right)$$

$$\hat{e}_r \cdot \frac{\partial}{\partial r} \left(\hat{e}_r A_r + \hat{e}_\theta A_\theta + \hat{e}_\varphi A_\varphi \right) +$$
$$\hat{e}_\theta \cdot \frac{1}{r} \frac{\partial}{\partial \theta} \left(\hat{e}_r A_r + \hat{e}_\theta A_\theta + \hat{e}_\varphi A_\varphi \right) +$$
$$\hat{e}_\varphi \cdot \frac{1}{r \sin\theta} \frac{\partial}{\partial \varphi} \left(\hat{e}_r A_r + \hat{e}_\theta A_\theta + \hat{e}_\varphi A_\varphi \right)$$

Divergence in Spherical Polar Coordinate System

$$\hat{e}_r \cdot \frac{\partial}{\partial r}\left(\hat{e}_r A_r + \hat{e}_\theta A_\theta + \hat{e}_\varphi A_\varphi\right) +$$

$$\hat{e}_\theta \cdot \frac{1}{r}\frac{\partial}{\partial \theta}\left(\hat{e}_r A_r + \hat{e}_\theta A_\theta + \hat{e}_\varphi A_\varphi\right) +$$

$$\hat{e}_\varphi \cdot \frac{1}{r\sin\theta}\frac{\partial}{\partial \varphi}\left(\hat{e}_r A_r + \hat{e}_\theta A_\theta + \hat{e}_\varphi A_\varphi\right)$$

$$\hat{e}_r \cdot \left(\hat{e}_r \frac{\partial A_r}{\partial r} + A_r \frac{\partial \hat{e}_r}{\partial r}\right) +$$

$$\hat{e}_r \cdot \left(\hat{e}_\theta \frac{\partial A_\theta}{\partial r} + A_\theta \frac{\partial \hat{e}_\theta}{\partial r}\right) +$$

$$\hat{e}_r \cdot \left(\hat{e}_\varphi \frac{\partial A_\varphi}{\partial r} + A_\varphi \frac{\partial \hat{e}_\varphi}{\partial r}\right) +$$

Derivatives of Unit Vectors

$$\begin{bmatrix} \hat{e}_r \\ \hat{e}_\theta \\ \hat{e}_\phi \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\phi & \cos\phi & 0 \end{bmatrix} \begin{bmatrix} \hat{e}_x \\ \hat{e}_y \\ \hat{e}_z \end{bmatrix}$$

$$\frac{d\hat{e}_r}{dr} = 0 \qquad \frac{d\hat{e}_\theta}{dr} = 0 \qquad \frac{d\hat{e}_\phi}{dr} = 0$$

$$\frac{d\hat{e}_r}{d\theta} = \hat{e}_\theta \qquad \frac{d\hat{e}_\theta}{d\theta} = -\hat{e}_r \qquad \frac{d\hat{e}_\phi}{d\theta} = 0$$

$$\frac{d\hat{e}_r}{d\phi} = \sin\theta\hat{e}_\phi \qquad \frac{d\hat{e}_\theta}{d\phi} = \cos\theta\hat{e}_\phi \qquad \frac{d\hat{e}_\phi}{d\phi} = -\cos\theta\hat{e}_\theta - \sin\theta\hat{e}_r$$

Divergence and curl in Spherical polar Coordinate system

$$\boxed{\hat{e}_r \cdot \frac{\partial}{\partial r}\left(\hat{e}_r A_r + \hat{e}_\theta A_\theta + \hat{e}_\varphi A_\varphi\right) +}$$

$$\hat{e}_\theta \cdot \frac{1}{r}\frac{\partial}{\partial \theta}\left(\hat{e}_r A_r + \hat{e}_\theta A_\theta + \hat{e}_\varphi A_\varphi\right) +$$

$$\hat{e}_\varphi \cdot \frac{1}{r\sin\theta}\frac{\partial}{\partial \varphi}\left(\hat{e}_r A_r + \hat{e}_\theta A_\theta + \hat{e}_\varphi A_\varphi\right)$$

$$\hat{e}_r \cdot \left(\hat{e}_r \frac{\partial A_r}{\partial r} + \cancel{A_r \frac{\partial \hat{e}_r}{\partial r}}\right) +$$

$$\hat{e}_r \cdot \left(\cancel{\hat{e}_\theta \frac{\partial A_\theta}{\partial r}} + \cancel{A_\theta \frac{\partial \hat{e}_\theta}{\partial r}}\right) +$$

$$\hat{e}_r \cdot \left(\cancel{\hat{e}_\varphi \frac{\partial A_\varphi}{\partial r}} + \cancel{A_\varphi \frac{\partial \hat{e}_\varphi}{\partial r}}\right) +$$

and proceeding further…..

Divergence in Spherical polar Coordinate system

$$\vec{\nabla} \cdot \vec{A} = \left(\frac{1}{r^2} \frac{\partial}{\partial r}\left(r^2 A_r\right) + \frac{1}{r \sin\theta} \frac{\partial}{\partial \theta}\left(A_\theta \sin\theta\right) + \frac{1}{r \sin\theta} \frac{\partial A_\phi}{\partial \phi} \right)$$

Curl in Spherical Polar Coordinate System

Spherical polar Coordinate system

$$\vec{\nabla} = \left(\hat{e}_r \frac{\partial}{\partial r} + \hat{e}_\theta \frac{1}{r} \frac{\partial}{\partial \theta} + \hat{e}_\phi \frac{1}{r \sin\theta} \frac{\partial}{\partial \phi} \right)$$

$$\vec{\nabla} \times \vec{A} = \left(\hat{e}_r \frac{\partial}{\partial r} + \hat{e}_\theta \frac{1}{r} \frac{\partial}{\partial \theta} + \hat{e}_\varphi \frac{1}{r \sin\theta} \frac{\partial}{\partial \varphi} \right) \times \left(\hat{e}_r A_r + \hat{e}_\theta A_\theta + \hat{e}_\varphi A_\varphi \right)$$

Curl in Spherical Polar Coordinate System

$$\vec{\nabla}\times\vec{A}=\frac{\widehat{e}_r}{r\sin\theta}\left(\frac{\partial}{\partial\theta}\left(A_\phi\sin\theta\right)-\frac{\partial A_\theta}{\partial\phi}\right)+$$

$$\frac{\widehat{e}_\theta}{r\sin\theta}\left(\frac{\partial A_r}{\partial\phi}-\sin\theta\frac{\partial\left(rA_\phi\right)}{\partial r}\right)+$$

$$\frac{\widehat{e}_\phi}{r}\left(\frac{\partial\left(rA_\theta\right)}{\partial r}-\frac{\partial A_r}{\partial\theta}\right)$$

Laplacian – a scalar operator

$$\vec{\nabla}^2 f = \vec{\nabla} \cdot (\vec{\nabla} f)$$

Laplacian – In Cartesian coordinate system

$$\vec{\nabla}^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

Laplacian – Spherical Polar Coordinate System

$$\vec{\nabla}^2 f = \vec{\nabla} \cdot (\vec{\nabla} f)$$

$$\vec{\nabla}^2 f = \left(\hat{e}_r \frac{\partial}{\partial r} + \hat{e}_\theta \frac{1}{r}\frac{\partial}{\partial \theta} + \hat{e}_\phi \frac{1}{r\sin\theta}\frac{\partial}{\partial \phi} \right) \cdot \left(\hat{e}_r \frac{\partial f}{\partial r} + \hat{e}_\theta \frac{1}{r}\frac{\partial f}{\partial \theta} + \hat{e}_\phi \frac{1}{r\sin\theta}\frac{\partial f}{\partial \phi} \right)$$

$$\vec{\nabla}^2 f = \left(\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial f}{\partial r} \right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial f}{\partial \theta} \right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 f}{\partial \phi^2} \right)$$

Gauss Divergence' Theorem

Let:

E be a simple solid region and let S be the boundary surface of E, given with positive (outward) orientation. Let $\vec{\mathbf{F}}$ be a vector field whose component functions have continuous partial derivatives on an open region that contains E.

Then,

$$\boxed{\iint_S \vec{F} \cdot d\vec{S} = \iiint_E \vec{\nabla} \cdot \vec{F}\, dV}$$

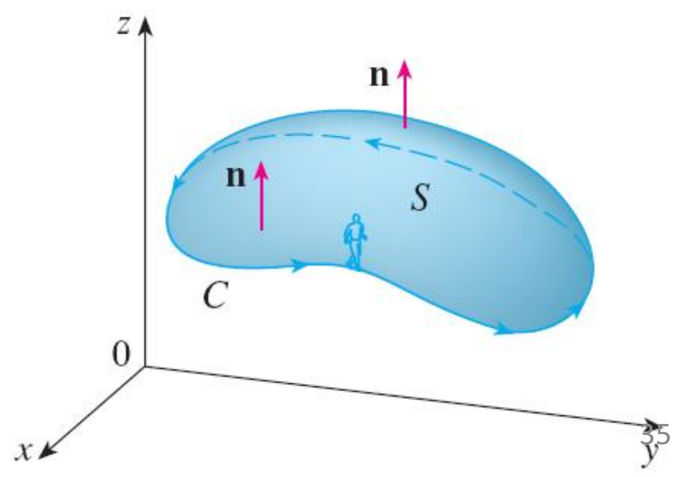
It relates the flux of $\vec{\mathbf{F}}$ across the boundary surface (S) of *E* to the triple integral of the divergence of $\vec{\mathbf{F}}$ over *E*.

Stoke's Theorem

Stokes' Theorem relates a surface integral over a surface S to a line integral around the boundary curve of S (a space curve).

Let:

S be an oriented piecewise-smooth surface bounded by a simple, closed, piecewise-smooth boundary curve C with positive orientation. **F** be a vector field whose components have continuous partial derivatives on an open region in R^3 that contains S.



Stoke’s Theorem

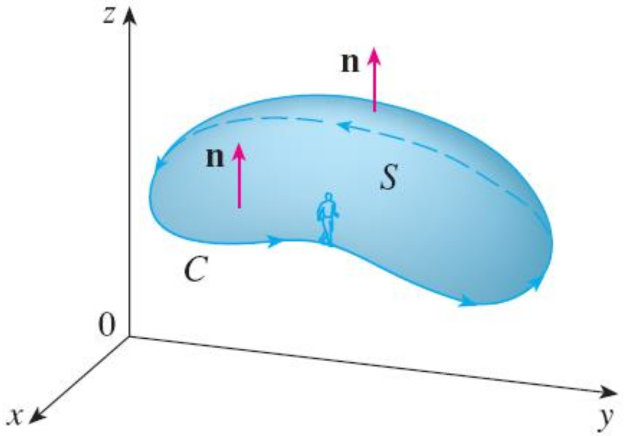
Stokes’ Theorem relates a surface integral over a surface *S* to a line integral around the boundary curve of *S* (a space curve).

Let:

S be an oriented piecewise-smooth surface bounded by a simple, closed, piecewise-smooth boundary curve *C* with positive orientation. **F** be a vector field whose components have continuous partial derivatives on an open region in R^3 that contains *S*.

Then,

$$\int_C \vec{F} \cdot d\vec{r} = \iint_S (\vec{\nabla} \times \vec{F}) \cdot d\vec{S}$$



The line integral around the boundary curve of *S* of the tangential component of **F** is equal to the surface integral of the normal component of the curl of **F**