ICS141: Discrete Mathematics for Computer Science I

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Lecture 19

Chapter 3. The Fundamentals

3.8 Matrices

3.8 Matrices

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- A ***matrix*** is a rectangular array of objects (usually numbers).
- An $m \times n$ ("m by n") matrix has exactly m horizontal rows, and n vertical columns.

$$\begin{bmatrix} 2 & 3 \\ 5 & -1 \\ 7 & 0 \end{bmatrix}$$

A 3×2 matrix

- Plural of matrix = *matrices* (say MAY-trih-sees)
- An $n \times n$ matrix is called a *square* matrix

Applications of Matrices

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- Tons of applications, including:
  - Solving systems of linear equations
  - Computer Graphics, Image Processing
  - Games
  - Models within many areas of Computational Science & Engineering
  - Quantum Mechanics, Quantum Computing
  - Many, many more…

Row and Column Order

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- The rows in a matrix are usually indexed 1 to *m* from top to bottom.
- The columns are usually indexed 1 to *n* from left to right.
- Elements are indexed by row, then by column.

$$\mathbf{A} = [a_{ij}] = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

Matrix Equality

- Two matrices **A** and **B** are considered equal iff they have the same number of rows, the same number of columns, and all their corresponding elements are equal.

$$\begin{bmatrix} 3 & 2 \\ -1 & 6 \end{bmatrix} \neq \begin{bmatrix} 3 & 2 & 0 \\ -1 & 6 & 0 \end{bmatrix}$$

Matrix Sums

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- The *sum* **A** + **B** of two matrices **A**, **B** (which **must** have the same number of rows, and the same number of columns) is the matrix (also with the same shape) given by adding corresponding elements of **A** and **B**.

$$\mathbf{A} + \mathbf{B} = [a_{ij} + b_{ij}]$$

$$\begin{bmatrix} 2 & 6 \\ 0 & -8 \end{bmatrix} + \begin{bmatrix} 9 & 3 \\ -11 & 3 \end{bmatrix} = \begin{bmatrix} 11 & 9 \\ -11 & -5 \end{bmatrix}$$

Matrix Products

- For an $m \times k$ matrix **A** and a $k \times n$ matrix **B**, the *product* **AB** is the $m \times n$ matrix:

$$\mathbf{AB} = \mathbf{C} = [c_{ij}]$$

$$c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{ik}b_{kj} = \sum_{\ell=1}^{k} a_{i\ell}b_{\ell j}$$

- *I.e.*, the element of **AB** indexed (i, j) is given by the vector *dot product* of the i-th row of **A** and the j-th column of **B** (considered as vectors).
- **Note:** Matrix multiplication is not commutative!

Matrix Product Example

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$$\begin{bmatrix} 0 & 1 & -1 \\ 2 & 0 & 3 \end{bmatrix} \cdot \begin{bmatrix} 0 & -1 \\ 2 & 0 \\ 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0\cdot 0 + 1\cdot 2 + (-1)\cdot 1 & 0\cdot(-1) + 1\cdot 0 + (-1)\cdot 0 \\ 2\cdot 0 + 0\cdot 2 + 3\cdot 1 & 2\cdot(-1) + 0\cdot 0 + 3\cdot 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 3 & -2 \end{bmatrix}$$

Matrix Product Example

$$A = \begin{bmatrix} 1 & 0 & 2 \\ 3 & 1 & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$$

- Because **A** is a 2×3 matrix and **B** is a 2×2 matrix, the product **AB** is not defined.

Matrix Multiplication: Non-Commutative

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- Matrix multiplication is not commutative!
- **A**: $m \times n$ matrix and **B**: $r \times s$ matrix
  - **AB** is defined when $n = r$
  - **BA** is defined when $s = m$
  - When both **AB** and **BA** are defined, generally they are not the same size unless $m = n = r = s$
  - If both **AB** and **BA** are defined and are the same size, then **A** and **B** must be square and of the same size
  - Even when **A** and **B** are both $n \times n$ matrices, **AB** and **BA** are not necessarily equal

Matrix Product Example

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$$\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$$

$$\mathbf{AB} = \begin{bmatrix} 3 & 2 \\ 5 & 3 \end{bmatrix} \qquad \mathbf{BA} = \begin{bmatrix} 4 & 3 \\ 3 & 2 \end{bmatrix}$$

Matrix Multiplication Algorithm

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procedure *matmul*(matrices **A**: $m \times k$, **B**: $k \times n$)

for i := 1 **to** m $\Theta(m)\cdot\{$

  for j := 1 **to** n **begin** $\Theta(n)\cdot($

    $c_{ij} := 0$ $\Theta(1)+$

    for q := 1 **to** k $\Theta(k)\cdot$

      $c_{ij} := c_{ij} + a_{iq}b_{qj}$ $\Theta(1))\}$

  end {**C** = [c_{ij}] is the product of **A** and **B**}

What's the Θ of its time complexity?

Answer: $\boldsymbol{\Theta(mnk)}$

Identity Matrices

- The ***identity matrix*** *of order n*, $\mathbf{I}_n$, is the rank-n square matrix with 1's along the upper-left to lower-right diagonal, and 0's everywhere else.

$$\mathbf{I}_n = [\underbrace{\delta_{ij}}_{\text{Kronecker Delta}}] = \left[\begin{cases} 1 \text{ if } i = j \\ 0 \text{ if } i \neq j \end{cases} \right]_{\forall 1 \leq i, j \leq n} = \left.\underbrace{\begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}}_{n}\right\} n$$

Matrix Inverses

- For some (but not all) square matrices $\mathbf{A}$, there exists a unique multiplicative ***inverse*** $\mathbf{A}^{-1}$ of $\mathbf{A}$, a matrix such that $\mathbf{A}^{-1}\mathbf{A} = \mathbf{I}_n$.
- If the inverse exists, it is unique, and $\mathbf{A}^{-1}\mathbf{A} = \mathbf{A}\mathbf{A}^{-1}$.
- We won't go into the algorithms for matrix inversion...

Powers of Matrices

If **A** is an $n \times n$ square matrix and $p \geq 0$, then:

- $\mathbf{A}^p = \underbrace{\mathbf{AAA}\cdots\mathbf{A}}_{p \text{ times}}$ (and $\mathbf{A}^0 = \mathbf{I}_n$)

- Example:
$$\begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix}^3 = \begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix}$$
$$= \begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 3 & 2 \\ -2 & -1 \end{bmatrix}$$
$$= \begin{bmatrix} 4 & 3 \\ -3 & -2 \end{bmatrix}$$

Matrix Transposition

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- If $\mathbf{A} = [a_{ij}]$ is an $m \times n$ matrix, the ***transpose*** of **A** (often written $\mathbf{A}^t$ or $\mathbf{A}^T$) is the $n \times m$ matrix given by $\mathbf{A}^t = \mathbf{B} = [b_{ij}] = [a_{ji}]$ $(1 \le i \le n, 1 \le j \le m)$

$$\begin{bmatrix} 2 & 1 & 3 \\ 0 & -1 & -2 \end{bmatrix}^t = \begin{bmatrix} 2 & 0 \\ 1 & -1 \\ 3 & -2 \end{bmatrix}$$

Symmetric Matrices

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- A square matrix **A** is ***symmetric*** iff **A** = **A**^{t}. *I.e.*, $\forall i, j \le n$: $a_{ij} = a_{ji}$.
- Which of the below matrices is symmetric?

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{bmatrix} \quad \begin{bmatrix} -2 & 1 & 3 \\ 1 & 0 & -1 \\ 3 & -1 & 2 \end{bmatrix} \quad \begin{bmatrix} 3 & 0 & 1 \\ 0 & 2 & -1 \\ 1 & 1 & -2 \end{bmatrix}$$

Zero-One Matrices

- Useful for representing other structures.
  - *E.g.*, relations, directed graphs (later on)
- All elements of a *zero-one* matrix are either 0 or 1.
  - E.g., representing **False** & **True** respectively.
- The ***join*** of **A**, **B** (both $m \times n$ zero-one matrices):
  - $\mathbf{A} \vee \mathbf{B} = [a_{ij} \vee b_{ij}]$
- The ***meet*** of **A**, **B**:
  - $\mathbf{A} \wedge \mathbf{B} = [a_{ij} \wedge b_{ij}] = [a_{ij}\, b_{ij}]$

Join and *Meet* Example

$$\boldsymbol{A} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \text{ and } \boldsymbol{B} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}$$

$$\boldsymbol{A} \vee \boldsymbol{B} = \begin{bmatrix} 1 \vee 0 & 0 \vee 1 & 1 \vee 0 \\ 0 \vee 1 & 1 \vee 1 & 0 \vee 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

$$\boldsymbol{A} \wedge \boldsymbol{B} = \begin{bmatrix} 1 \wedge 0 & 0 \wedge 1 & 1 \wedge 0 \\ 0 \wedge 1 & 1 \wedge 1 & 0 \wedge 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

Boolean Products

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- Let $\mathbf{A} = [a_{ij}]$ be an $m \times k$ zero-one matrix and $\mathbf{B} = [b_{ij}]$ be a $k \times n$ zero-one matrix,
- The ***boolean product*** of **A** and **B** is like normal matrix multiplication, but using ∨ instead of +, and ∧ instead of × in the row-column "vector dot product":

$$\mathbf{A} \odot \mathbf{B} = \mathbf{C} = [c_{ij}] = \left[\bigvee_{\ell=1}^{k} a_{i\ell} \wedge b_{\ell j} \right]$$

Boolean Products Example

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- Find the Boolean product of **A** and **B**, where

$$\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

$$\mathbf{A} \odot \mathbf{B} = \begin{bmatrix} (1 \wedge 1) \vee (0 \wedge 0) & (1 \wedge 1) \vee (0 \wedge 1) & (1 \wedge 0) \vee (0 \wedge 1) \\ (0 \wedge 1) \vee (1 \wedge 0) & (0 \wedge 1) \vee (1 \wedge 1) & (0 \wedge 0) \vee (1 \wedge 1) \\ (1 \wedge 1) \vee (0 \wedge 0) & (1 \wedge 1) \vee (0 \wedge 1) & (1 \wedge 0) \vee (0 \wedge 1) \end{bmatrix}$$

$$= \begin{bmatrix} 1 \vee 0 & 1 \vee 0 & 0 \vee 0 \\ 0 \vee 0 & 0 \vee 1 & 0 \vee 1 \\ 1 \vee 0 & 1 \vee 0 & 0 \vee 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

Boolean Powers

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- For a square zero-one matrix **A**, and any $k \geq 0$, the *k-th Boolean power of* **A** is simply the Boolean product of k copies of **A**.

$$\mathbf{A}^{[k]} = \underbrace{\mathbf{A} \odot \mathbf{A} \odot \cdots \odot \mathbf{A}}_{k \text{ times}}$$

$$\mathbf{A}^{[0]} = \mathbf{I}_n$$

Matrices as Functions

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- An $m \times n$ matrix $\mathbf{A} = [a_{ij}]$ of members of a set S can be encoded as a partial function

$$f_{\mathbf{A}}: \mathbf{N}\times\mathbf{N} \rightarrow S,$$

such that for $i < m$, $j < n$, $f_{\mathbf{A}}(i, j) = a_{ij}$.

- By extending the domain over which $f_{\mathbf{A}}$ is defined, various types of infinite and/or multidimensional matrices can be obtained.