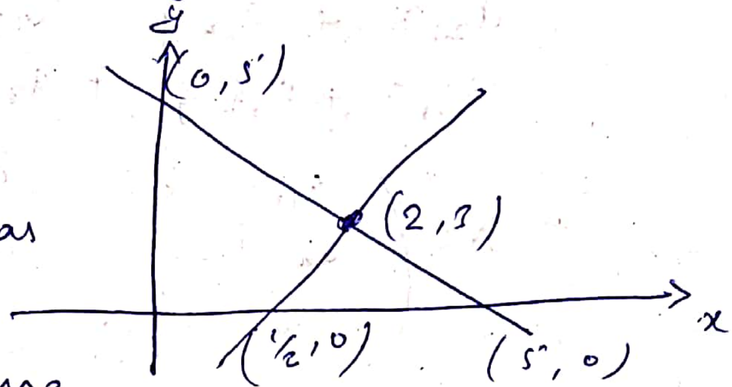
MA102

One of the main aim of linear algebra is to solve a given system of linear system

$$2x - y = 1$$
$$x + y = 5$$



the linear system has a unique solution

If the eqn is one then infinite many solution as x changes with y.

Matrix — arrays of number

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}_{m \times n}$$

$m =$ row
$n =$ column

Addition of Matrix

$A \rightarrow a_{ij}$ $1 \le i \le m$, $1 \le j \le n$

$B \Rightarrow b_{ij}$ $1 \le i \le m$, $1 \le j \le n$

different

$C = A + B$

$\Rightarrow c_{ij} = a_{ij} + b_{ij}$

If order are same

Multiplication of Matrix

$A_{m \times \boxed{n} \times B_{\boxed{n} \times k}} = C_{m \times k}$

must same

But

$B_{n \times \boxed{k} \cdot A_{\boxed{m} \times n}}$ different is not define

$$C = \begin{pmatrix} a_{11}b_{11} + a_{12}b_{21} \ldots a_{1n}b_{n1} & \cdots \\ \vdots & \\ \vdots & \end{pmatrix}$$

Linear System / System of Linear equations

$$a_{11}x + a_{12}y + a_{13}z = b_1$$
$$a_{21}x + a_{22}y + a_{23}z = b_2$$
$$a_{31}x + a_{32}y + a_{33}z = b_3$$

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}_{3\times3} \longrightarrow \text{coffecient matrix}$$

$$\tilde{A} = \left(\begin{array}{ccc:c} a_{11} & a_{12} & a_{13} & b_1 \\ a_{21} & a_{22} & a_{23} & b_2 \\ a_{31} & a_{32} & a_{33} & b_3 \end{array}\right)_{3\times4} \longrightarrow \text{Augmented matrix}$$

$$X = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \longrightarrow \text{unknown vector}$$

Gauss ~~Le~~ Elimination Method / Gauss-Jordan Elimination Method

Three basic operations

- Interchange any two rows (equation), which doesn't affect the solution of system.
- We can multiply throughout any equation by a non-zero number
- We can add an equation (from the system) to a multiple of another equation (of same system).

Consider the equation

$$2x - 5y + 4z = -3$$
$$x - 2y + z = 5$$
$$x - 4y + 6z = 10$$

Augmented Matrix

$$\left(\begin{array}{ccc|c} 2 & -5 & 4 & -3 \\ 1 & -2 & 1 & 5 \\ 1 & -4 & 6 & 10 \end{array}\right)$$

$R_3 \to R_3 - R_2$
$R_2 \to 2R_2 - R_3$

$$\left(\begin{array}{ccc|c} 2 & -5 & 4 & -3 \\ 0 & 1 & -2 & 13 \\ 0 & -2 & 5 & 35 \end{array}\right)$$

$R_3 \to R_3 + 2R_2$

$$\left(\begin{array}{ccc|c} 2 & -5 & 4 & -3 \\ 0 & 1 & -2 & 13 \\ 0 & 0 & 1 & 31 \end{array}\right)$$

75×5
375
127
48

$\boxed{z = 31}$

$y - 2z = 13$

$y - 62 = 13$

$\boxed{y = 75}$

$2x - 5y + 4z = -3$

$2x - 375 + 4 \times 31 = -3$

$2x - 375 + 124 = -3$

$2x = 148$ $\boxed{x = 74}$ $\longrightarrow$ Gauss elimination

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 124 \\ 0 & 1 & 0 & 75 \\ 0 & 0 & 1 & 31 \end{array}\right)$$ Gauss ~~Jota~~ Jordan method

$A\bar{x} = \bar{b}$

~~AA^{-1}~~ $A^{-1}A\,\bar{x} = A^{-1}\bar{b} \Rightarrow I\bar{x} = A^{-1}\bar{b}$

$$\boxed{\bar{x} = A^{-1}\bar{b}}$$

if A is invertiable matrix.

Inverse of Matrix A

$B_{m \times n}$ is said to be inverse of A if

$BA = I$ and $AB = I$

B can be denoted as A^{-1}.

Inverse of a Matrix is always unique.

Proof: B & C are two ~~invertible~~ inverse matrix of A

so $BA = I$ & $AB = I$

$CA = I$ & $AC = I$

$B = BI = B(AC) = (BA)C$

$B = C$

So Inverse of matrix is unique.

Gauss - Jordan elimination (to solve the linear system and finding inverse)

$2x + 8y + 4z = 2$

$2x + 8y + z = 5$

$4x + 10y - z = 1$

$$\underbrace{\begin{pmatrix} 2 & 8 & 4 \\ 2 & 8 & 1 \\ 4 & 10 & -1 \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} x \\ y \\ z \end{pmatrix}}_{\bar{x}} = \underbrace{\begin{pmatrix} 2 \\ 8 \\ 1 \end{pmatrix}}_{b}$$

$A\bar{x} = \bar{b}$

$\Rightarrow A\bar{x} = A^{-1}b$

$(A : I)$

$$\left(\begin{array}{ccc:ccc} 2 & 8 & 4 & 1 & 0 & 0 \\ 2 & 5 & 1 & 0 & 1 & 0 \\ 4 & 10 & -1 & 0 & 0 & 1 \end{array}\right)$$

$R_1 \rightarrow \frac{1}{2} R_1$

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$R_2 \to R_2 - 2R_1$, $R_3 \to R_3 - 4R_1$

$$\left(\begin{array}{ccc|ccc} 1 & 4 & 2 & 1/2 & 0 & 0 \\ 0 & -1 & -1 & -1/3 & -1/3 & 0 \\ 0 & -6 & -9 & -2 & 0 & 1 \end{array}\right)$$

$R_1 \to R_1 - 4R_2$, $R_3 \to R_3 + 6R_2$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 0 & -2 & -5/6 & 4/3 & 0 \\ 0 & 1 & 1 & 1/3 & -1/3 & 0 \\ 0 & 0 & -3 & 0 & -2 & 1 \end{array}\right)$$

$R_3 \to -1/3\, R_3$.

$$\left(\begin{array}{ccc|ccc} 1 & 0 & -2 & -5/6 & 4/3 & 0 \\ 0 & 1 & 1 & 1/3 & -1/3 & 0 \\ 0 & 0 & -1 & 0 & -2/3 & 1/3 \end{array}\right)$$

$R_1 \to R_1 + 2R_3$ & $R_2 \to R_2 + R_3$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -5/6 & 8/3 & -2/3 \\ 0 & 1 & 0 & 1/3 & -1 & 1/3 \\ 0 & 0 & 1 & 0 & -2/3 & -1/3 \end{array}\right)$$

$$A^{-1} = \begin{pmatrix} -5/6 & 8/3 & -2/3 \\ 1/3 & -1 & 1/3 \\ 0 & -2/3 & -1/3 \end{pmatrix}$$

Inverse for the System

$AA^{-1} = I = A^{-1}A$ $\qquad \bar{X} = A^{-1}\bar{b} = \begin{pmatrix} 11 \\ -4 \\ 3 \end{pmatrix}$

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LU-Factorization

$A = LU \rightarrow$ upper triangular Matrix

$\downarrow$

Lower triangular Matrix

$$A_{3\times3}\,\bar{x}_{3\times1} = \bar{b}_{3\times1}$$

$$(L_{3\times3}U_{3\times3})\,\bar{x}_{3\times1} = \bar{b}_{3\times1}$$

$$L\,(U_{3\times3}\,\bar{x}_{3\times1}) = \bar{b}$$

first solve

$$L\bar{c} = \bar{b} \quad \& \text{ then}$$

$$U\bar{x} = c$$

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{pmatrix}$$

$R_2 \rightarrow R_2 - R_1$

change the identity in R_2 as R_2 change and replace first nonzero entry below of R_1 in $R_2 = -1$ by -1

$$\sim \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 2 & 3 \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{E_1} \underbrace{\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{pmatrix}}_{A}$$

$R_3 \rightarrow R_3 - R_1$

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 2 \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}}_{E_2} \underbrace{\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 2 & 3 \end{pmatrix}}_{E_1 A}$$

$$R_3 \longrightarrow R_3 - R_2$$

$$\sim \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}}_{E_3} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 2 \end{pmatrix}$$

$$\Downarrow$$

$$U$$

$$\Rightarrow \quad U = E_3 E_2 E_1 A$$

E_i : are elementary matrix.

$$E_1^{-1} E_2^{-1} E_3^{-1} U = A$$

$$\boxed{L = E_1^{-1} E_2^{-1} E_3^{-1}}$$

$$E = \begin{pmatrix} 1 & 0 & 0 \\ e_{21} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\boxed{\boxed{LU = A}}$$

$$E^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ -e_{21} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

~~Test~~ Tutorials

Linear System of equations

→ consistent — unique sol^n or infinite solution

$$x + y + z = 4$$
$$2x + 5y - 2z = 3$$
$$x + 7y - 7z = 5$$

$$Ax = b$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 5 & -2 \\ 1 & 7 & -7 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \\ 5 \end{bmatrix}$$

$R_2 \to R_2 - 2R_1$, $R_3 \to R_3 - R_1$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 0 & 3 & -4 & -5 \\ 0 & 6 & -8 & 1 \end{array}\right]$$

$R_3 \to R_3 - \frac{1}{3} \times 6R_2$

$\Rightarrow R_3 \to R_3 - 2R_2$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 0 & 3 & -4 & -5 \\ 0 & 0 & 0 & 11 \end{array}\right]$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 3 & -4 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ -5 \\ 11 \end{bmatrix}$$

$x + y + z = 4$

$3y - 4z = -5$

$0 \cdot z = 11$

$0 = 11$ → no happen

So it is inconsistent

for no solution

Q)

$$2\sin x - \cos y + 3\tan z = 3$$
$$4\sin x + 2\cos y - 2\tan z = 10$$
$$6\sin x - 3\cos y + \tan z = 9$$

3×3

$$\left[\begin{array}{ccc|c} 2 & -1 & 3 & 3 \\ 4 & 2 & -2 & 10 \\ 6 & -3 & 1 & 9 \end{array}\right]$$

$R_2 \to R_2 - 2R_1$

$R_3 \to R_3 - 3R_1$

$$\left[\begin{array}{ccc|c} 2 & -1 & 3 & 3 \\ 0 & 4 & -8 & 4 \\ 0 & 0 & -8 & 0 \end{array}\right]$$

$-8z_1 = 0$ $\quad z_1 = 0$

$4y_1 - 8z_1 = 4$

$4y_1 - 0 = 4$

$y_1 = 1$

$2x_1 - y + 3z = 3$

$2x_1 - 1 + 0 = 3$

$2x_1 = 4$

$\boxed{x_1 = 2}$

$\tan z_1 = 0$

$z = 0$

$\sin x = 2$

$\hookrightarrow$ no soln

$\cos y = 1$

System has no soln

-6+8

Q)

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & -2 \\ 3 & +4 & 5 \end{bmatrix} \qquad b = \begin{bmatrix} 1 \\ 4 \\ 7 \end{bmatrix}$$

Using LU method for solution of system

$$|A| = (30-16) + 1(-\cancel{12}) + 2(4-10)$$

$$|A| = 14 - \cancel{12} - 12 = 0$$

so singular

$R_2 \longrightarrow R_2 - 2R_1$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 3 & 4 & 5 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{E} A$$

$R_3 \rightarrow R_3 - 3R_1$

$$\begin{bmatrix} 1 & 0 & \cancel{1} \\ 0 & 2 & 0 \\ 0 & 4 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 3 & 4 & 5 \end{bmatrix} \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix}}_{f}$$

$$\begin{bmatrix} 1 & 0 & \cancel{1} \\ 0 & 2 & 0 \\ 0 & 4 & 2 \end{bmatrix} = FEA$$

$R_3 \rightarrow R_3 - 2R_2$

$$\begin{bmatrix} 1 & 0 & \cancel{1} \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & +2 & \cancel{1} \end{bmatrix}}_{G} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & \cancel{1} \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} = (GFE)A$$

$$U = \begin{bmatrix} 1 & 0 & \cancel{1} \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \qquad L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix}$$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & +2 & 1 \end{bmatrix}$$

$$LU = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 1 \\ 2 & 2 & 2 \\ 3 & 4 & 5 \end{bmatrix}$$

$Ax = b$

$LUx = b$

$Lc = b$ $\qquad c = Ux$

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix} c = \begin{bmatrix} 1 \\ 4 \\ 7 \end{bmatrix}$$

$c_1 = 1$ $\qquad 2c_1 + c_2 = 4$

$c_2 = 2$

$3c_1 + 2c_2 + c_3 = 7$

$3 + 4 + c_3 = 0$

$c_3 = 0$

$$\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$x + 0 + z = 1$

$2y = 2$

$2z = 0$

$$\boxed{\begin{aligned} x &= 1 \\ y &= 1 \\ z &= 0 \end{aligned}}$$

Q. $A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$ $\quad b = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

$\cancel{R_2 \rightarrow R_2 - R_1}$

$\begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & \end{bmatrix}$

$U = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

$L = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$

$A' = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

$x = 1 \quad y = -1$

$z = 1$

$\cancel{x+}$ $\boxed{\begin{aligned} y + z &= 0 \\ x + y &= 0 \\ x + y + z &= 1 \end{aligned}}$

Q) $A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 3 & 6 & 10 \\ 1 & 4 & 10 & 20 \end{bmatrix}$

Q) $A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 4 & 2 \\ 6 & 3 & 5 \end{bmatrix}$

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Q.) $A = \begin{bmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{bmatrix}$

find the inverse of matrix using Gauss-Jordan Method?

$A = L U$

(Lower triangular matrix 1's on the diagonal)

(upper triangular matrix with non-zero entries on the diagonal)

$$\Rightarrow \begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix} \begin{pmatrix} a & x & x \\ 0 & b & x \\ 0 & 0 & c \end{pmatrix}$$

Cholesky factorization

A is ~~skew~~ symmetric if $\boxed{A = A^T}$

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \quad A^T = \begin{pmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{pmatrix}$$

Eg) $A = L$ → lower triangular matrix

$$A = \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ e_{21} & 1 & 0 \\ e_{31} & e_{32} & 1 \end{pmatrix}}_{L} \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{U}$$

$$A = \begin{pmatrix} 1 & 0 & 0 \\ e_{21} & 1 & 0 \\ e_{31} & e_{32} & 1 \end{pmatrix}$$

$R_2 \to R_2 - e_{21} R_1$

~~$R_3 \to R_3 - e_{32}$~~

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ e_{31} & e_{32} & 1 \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ -e_{21} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{E} \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ e_{21} & 1 & 0 \\ e_{31} & e_{32} & 1 \end{pmatrix}}_{A}$$

$R_3 \longrightarrow R_3 - e_{31} R_1$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & e_{32} & 1 \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -e_{31} & 0 & 1 \end{pmatrix}}_{\to E_2} \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ e_{31} & e_{32} & 0 \end{pmatrix}}_{\to E_1 A}$$

$R_3 \longrightarrow R_3 - e_{32} R_2$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -e_{32} & 1 \end{pmatrix}}_{\to E_3} \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -e_{31} & 0 & 1 \end{pmatrix}}_{\to E_2}$$

$$\underbrace{\begin{pmatrix} 1 & 0 & 0 \\ e_{21} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{E_1} \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ e_{21} & 1 & 0 \\ e_{31} & e_{32} & 1 \end{pmatrix}}_{A}$$

$$A = \begin{pmatrix} 0 & 1 \\ 2 & 3 \end{pmatrix} \sim \begin{pmatrix} 2 & 3 \\ 0 & 1 \end{pmatrix} = UI$$

Permutation matrix of A

$$\begin{pmatrix} 0 & 1 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\underbrace{\begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}}_{P} \begin{pmatrix} 0 & 1 \\ 2 & 3 \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ 0 & 1 \end{pmatrix}$$

$$\boxed{PA = LU}$$

$$\boxed{A \neq LU}$$

o can

$$\begin{pmatrix} 0 & 1 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$$

$$P \begin{pmatrix} 0 & 1 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = P \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 3 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} b_2 \\ b_1 \end{pmatrix}$$

$$\boxed{\begin{aligned} 2x + 3y &= b_2 \\ y &= b_2 \end{aligned}}$$

$$\begin{pmatrix} 0 & 1 \\ 2 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix} \begin{pmatrix} b & c \\ 0 & f \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix} \to L \qquad \begin{pmatrix} b & c \\ 0 & f \end{pmatrix} \to U$$

$$\begin{pmatrix} 0 & 1 \\ 2 & 3 \end{pmatrix} = \begin{pmatrix} b & c \\ ab & ac+f \end{pmatrix}$$

$b = 0$ $\qquad$ $c = 1$

$ab = 2$ $\qquad$ $ac + f = 3$

$a = \infty$ $\qquad$ $a + f = 3$

→ not possible

So LU factorize is not possible.

$$\begin{pmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$A : (R_1 \leftrightarrow R_2)$

$$A \to P \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$A \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix} \quad R_1 \leftrightarrow R_2$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} = PA$$

$R_3 \to R_3 - R_1$

$$\to \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix} = E$$

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} = U$$

$EPA = U$

$$\boxed{PA = LU}$$

$A\bar{x} = b$

$PA\hat{x} = Pb$

$$(LU)\bar{x} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$L(U\bar{x}) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Cholesky Decomposition (factorization)

A – symmetric matrix

$$A = A^T$$

A is also positive-definite

$\boxed{x^T A x > 0}$, $\forall\, x \in V$ — Defined in vector space.

$$A = \begin{pmatrix} 1 & -1 & 2 \\ -1 & 5 & -4 \\ 2 & -4 & 6 \end{pmatrix} \quad A^T = \begin{pmatrix} 1 & -1 & 2 \\ -1 & 5 & -4 \\ 2 & -4 & 6 \end{pmatrix}$$

$R_2 \to R_2 + R_1 \longrightarrow E_1$

$R_3 \to R_3 - 2R_1 \longrightarrow E_2$

$R_3 \to R_3 + \frac{1}{2}R_2 \longrightarrow E_3$

$$\begin{pmatrix} 1 & -1 & 2 \\ 0 & 4 & -2 \\ 0 & 0 & 1 \end{pmatrix} = U$$

$$E_1 = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad E_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{pmatrix}$$

$$E_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \frac{1}{2} & 1 \end{pmatrix}$$

$$U = E_3 E_2 E_1 A$$

$$L = E_1^{-1} E_2^{-1} E_3^{-1}$$

$$L = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 2 & -1/2 & 1 \end{pmatrix}$$

$$A = LU$$

$$A = \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 2 & -1/2 & 1 \end{pmatrix}}_{L} \underbrace{\begin{pmatrix} 1 & -1 & 2 \\ 0 & 4 & -2 \\ 0 & 0 & 1 \end{pmatrix}}_{U}$$

$$A = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 2 & -1/2 & 1 \end{pmatrix} I \begin{pmatrix} 1 & -1 & 2 \\ 0 & 4 & -2 \\ 0 & 0 & 1 \end{pmatrix}$$

$$A = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 2 & -1/2 & 1 \end{pmatrix} \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{D} \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/4 & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{D^{-1}} \underbrace{\begin{pmatrix} 1 & -1 & 2 \\ 0 & 4 & -2 \\ 0 & 0 & 1 \end{pmatrix}}_{U}$$

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$$A = \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 2 & -1/2 & 1 \end{pmatrix}}_{L} \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{D} \underbrace{\begin{pmatrix} 1 & -1 & 2 \\ 0 & 1 & -1/2 \\ 0 & 0 & 1 \end{pmatrix}}_{L^T}$$

$$\boxed{A = LDL^T}$$

$$A = \underbrace{L\sqrt{D}}_{C}\,\underbrace{\sqrt{D}\,L^T}_{C^T} \qquad (\sqrt{D} = \sqrt{D}^T)$$

$\hookrightarrow$ ~~diagon~~ diagonal matrix

$$\boxed{A = C\cancel{C}^T} \qquad \sqrt{D} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\boxed{C = L\sqrt{D}}$$

$$C = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 2 & -1/2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$C = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 2 & 0 \\ 2 & -1 & 1 \end{pmatrix}$$

$$\boxed{A = \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ -1 & 2 & 0 \\ 2 & -1 & 1 \end{pmatrix}}_{C} \underbrace{\begin{pmatrix} 1 & -1 & 2 \\ 0 & 2 & -1 \\ 0 & 0 & 1 \end{pmatrix}}_{C^T}}$$

$$\boxed{A = CC^T}$$

$$A = \begin{pmatrix} 25 & 15 & -5 \\ 15 & 18 & 0 \\ -5 & 0 & 11 \end{pmatrix} \quad A = \begin{pmatrix} 9 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 2 & 8 \end{pmatrix}$$

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 2 & 7 \end{pmatrix}$$

Vector Spaces (vector space of any domain on a particular operation ($\star$ and $\star$))

A real (complex) vector space V, over $\mathbb{R}$ or ($\mathbb{C}$), consists of vectors together with a rule of vector addition and multiplication ~~of~~ by scalars ($\mathbb{R}$ or $\mathbb{C}$) such that V is closed under vector addition and scalar multiplication, that is to say,

whenever $v_1, v_2 \in V$ and $\alpha \in \mathbb{R}$ or $\mathbb{C}$ (scalar)

then

$$\underbrace{v_1 + v_2}_{\text{vectors}} \in V$$

and

$\alpha v \in V$, for every $v \in V$. In addition, to this above rule for "vector addition" and "scalar multiplication" are required to satisfy ~~the~~ following properties:-

1) $u + v = v + u$, for all $u, v \in V$

2) $u + (v + w) = (u + v) + w$, for all $u, v, w \in V$

3) there is a "zero vector (0)" $\in V$, such that

$$v + 0 = v, \text{ for all } v \in V$$

4) for every $v \in V$, there exist a unique vector $-v \in V$ such that

$$v + (-v) = 0$$

5) $1v = v$, $v \in V$, $1 \in \mathbb{R}$ or $(\mathbb{C})$

6) $(\alpha\beta) v = \alpha(\beta v)$; $v \in V$, $\alpha, \beta \in \mathbb{R}$ or $(\mathbb{C})$

7) $\alpha(v+\omega) = \alpha v + \beta \omega$, $v, \omega \in V$,
$\alpha \in \mathbb{R}$ or $(\mathbb{C})$

8) $(\alpha+\beta) v = \alpha v + \beta v$, $v \in V, \alpha, \beta \in \mathbb{R}$ or $\mathbb{C}$

Ex 1. $V = \mathbb{R}^2$ $\quad f = \mathbb{R}$

$(v_1, v_2) \in V$ (plane of ~~cor~~ x-y plane)

$\hat{v}_1 = (1, 2)$ $\quad v_2 = (2, 3)$

$$v_1 + v_2 = (1,2) + (2,3) := (1+2, 2+3)$$
$$= (3, 5) \in \mathbb{R}^2$$

$$\alpha \cdot v_1 = (2 \times 1, 2 \times 2) = (2, 4) \in \mathbb{R}^2$$

as it is an real vector space. It follows the rule

vector addition

$$(x_1, x_2) + (y_1, y_2) = (x_1 + y_1, x_2 + y_2)$$

scalar multiplication

$$\alpha(x_1, x_2) = (\alpha x_1, \alpha x_2)$$

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(*) For $u, v \in \mathbb{R}^2$. $(u_1, u_2) \in \mathbb{R}^2$
$u_1 \in \mathbb{R}, u_2 \in \mathbb{R}$

$u + v = (u_1, u_2) + (v_1, v_2)$
$= (u_1 + v_1, u_2 + v_2) \in \mathbb{R}^2$

$\mathbb{R}^2$ is closed under vector addⁿ

$u_1, v_1, v_2, u_2 \in \mathbb{R}$
$u_1 + v_1 \in \mathbb{R}$
$u_2 + v_2 \in \mathbb{R}$

For $\alpha \in \mathbb{R}$ and $u \in V$,
if $u = (u_1, u_2) \Rightarrow$

$\alpha \in \mathbb{R}$
$u_1 \in \mathbb{R}, u_2 \in \mathbb{R}$
$\alpha u_1 \in \mathbb{R}, \alpha u_2 \in \mathbb{R}$

$\alpha u = (\alpha u_1, \alpha u_2)$
$\alpha u \in \mathbb{R}^2$

$\mathbb{R}^2$ is closed under scalar multiplic-ation.

1) $u + v = v + u$ for $u, v \in \mathbb{R}^2$

$u + v = (u_1, u_2) + (v_1, v_2)$
$= (u_1 + v_1, u_2 + v_2)$
$= (v_1 + u_1, v_2 + u_2)$
$= (v_1, v_2) + (u_1, u_2)$

as $v_1, u_1 \in \mathbb{R}$ commutative hold $\boxed{u_1 + v_1 = v_1 + u_1}$

$\boxed{u + v = v + u.}$

2) for $v, u \,\&\, w \in \mathbb{R}^2$

$u + (v + w) = (u_1, u_2) + [(v_1, v_2) + (w_1, w_2)]$
$= (u_1, u_2) + (v_1 + w_1, v_2 + w_2)$

$= (u_1 + (v_1 + w_1), u_2 + (v_2 + w_2))$

for $x, y, z \in \mathbb{R}$

$x + (y+z) = (x+y) + z$

$= ((u_1 + v_1) + w_1, (u_2 + v_2) + w_2)$

$= (u + v) + w$

3) there is a $(0,0) \in \mathbb{R}^2$

$(u_1, u_2) + (0,0) = (u_1, u_2)$

4) for $u = (u_1, u_2) \in \mathbb{R}^2 \exists (-u_1, -u_2) \in \mathbb{R}^2$

$\exists$

$(u_1, u_2) + (-u_1, -u_2)$

$= (u_1 + (-u_1), u_2 + (-u_2))$

$= (0, 0)$

5) $1 \cdot u = 1(u_1, u_2)$

$= (1 \cdot u_1, 1 \cdot u_2)$

$= (u_1, u_2) = u$

6) $(\alpha\beta)u = (\alpha\beta)\cdot(u_1, u_2)$

$= ((\alpha\beta)u_1, (\alpha\beta)u_2)$

$= (\alpha(\beta u_1), \alpha(\beta u_2))$

$$\boxed{(\alpha\beta)u = \alpha(\beta u)}$$

8) $u+v = (u_1+v_1+1, u_2+v_2+1)$

$\alpha u = (\alpha u_1, \alpha u_2)$

Cholesky decomposition

$$A = \begin{bmatrix} 25 & 15 & -5 \\ 15 & 18 & 0 \\ -5 & 0 & 11 \end{bmatrix} = CC^T$$

$A = LU = LIU = LDD^{-1}U$

$= (LD)(D^{-1}U)$

$= (LD)(U')$

$= (L\sqrt{D})(\sqrt{D^T}\,U')$

$= CC^T$

$\boxed{D^{-1}U = U'}$ and $\boxed{D = D^T}$

$D^{-1}U = U'$

$D = D^T$

$R_2 \longrightarrow R_2 - 3/5\, R_1$

$R_3 \to R_3 + 1/5\, R_1$

$\cancel{15} \times 3/5$ (5)

$\cancel{25} \times 3/5$ (5)

$\cancel{15} \times 3/5$ (3)

$\cancel{5} \times 3/5$

$$\begin{bmatrix} 25 & 15 & -5 \\ 0 & 9 & 3 \\ -5 & 0 & 11 \end{bmatrix}$$

$$\begin{bmatrix} 25 & 15 & -5 \\ 0 & 9 & 3 \\ 0 & 3 & 10 \end{bmatrix}$$

$\cancel{A = E_1}$ $E_1 = \begin{bmatrix} 1 & 0 & 0 \\ -3/5 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1/5 & 0 & 1 \end{bmatrix}$

$R_3 \rightarrow R_3 + 1/3 R_2$

$\begin{bmatrix} 25 & 15 & -5 \\ 0 & 9 & -3 \\ 0 & 0 & 9 \end{bmatrix}$ 10-1

$E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1/3 & 1 \end{bmatrix}$

$\cancel{A = U E_1 E_2 E_3}$

$U = A E_1 E_2 E_3$

$A = LU$

$L = E_3^{-1} E_2^{-1} E_1^{-1}$

$L = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1/3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1/5 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 3/5 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$L = \begin{bmatrix} 1 & 0 & 0 \\ 3/5 & 1 & 0 \\ -1/5 & 1/3 & 1 \end{bmatrix}$

$A = LU$

$$D = \begin{bmatrix} 25 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix} \longrightarrow \text{from diagonal of } U$$

$$D^{-1} = \begin{bmatrix} 1/25 & 0 & 0 \\ 0 & 1/9 & 0 \\ 0 & 0 & 1/9 \end{bmatrix}$$

$$\sqrt{D} = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix} = \sqrt{D^T}$$

$$C = L\sqrt{D} = \begin{bmatrix} 1 & 0 & 0 \\ 3/5 & 1 & 0 \\ -1/5 & 1/3 & 0 \end{bmatrix} \begin{bmatrix} 5 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & 0 & 0 \\ 3 & 3 & 0 \\ -1 & 1 & 0 \end{bmatrix}$$

$$\frac{\cancel{15}\,3}{\cancel{25}\,5}$$

$$\cancel{C^T = \begin{bmatrix} 5 & 0 & 6 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 25 \\ 0 \\ 0 \end{bmatrix}}$$

$\sqrt{D^T}\,U =$

$$D^{-1}U = \begin{bmatrix} 1/25 & 0 & 0 \\ 0 & 1/9 & 0 \\ 0 & 0 & 1/9 \end{bmatrix} \begin{bmatrix} 25 & 15 & -5 \\ 0 & 9 & -3 \\ 0 & 0 & 9 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 3/5 & -1/5 \\ 0 & 1 & -1/3 \\ 0 & 0 & 1 \end{bmatrix}$$

$$C^T = \sqrt{D}\, U'$$

$$= \begin{bmatrix} 5 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & 3/5 & -1/5 \\ 0 & 1 & -1/3 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{matrix} 5 & 3 & -1 \\ 0 & 3 & -1 \\ 0 & 0 & 3 \end{matrix}$$

$$C^T = \begin{bmatrix} 5 & 3 & -1 \\ 0 & 3 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Q)

$$A = \begin{bmatrix} 1 & 4 & 2 \\ -2 & -8 & 3 \\ 0 & 1 & 1 \end{bmatrix} \qquad b = \begin{bmatrix} -2 \\ 32 \\ 1 \end{bmatrix}$$

$$Ax = b$$

$$PA = LU$$

$$Ax = b$$

$$PAx = Pb = b_1$$

$$L\underbrace{Ux}_{c} = b_1$$

$$LC = b_1$$

$$x = \begin{bmatrix} 2 \\ -3 \\ 4 \end{bmatrix}$$

$2x + y + z = 1$
$4x + 6y = 1$
$-2x + 7y + 2z = 1$

$$\left[\begin{array}{ccc|c} 2 & 1 & 1 & 1 \\ 4 & -6 & 0 & 1 \\ -2 & 7 & 2 & 1 \end{array}\right]$$

$R_2 \to R_2 - 2R_1$, $R_3 \to R_3 + R_1$

$1 - 1/8$

$$\left[\begin{array}{ccc|c} 2 & 1 & 1 & 1 \\ 0 & -8 & -2 & -1 \\ 0 & 8 & 3 & 2 \end{array}\right]$$

$R_1 \to R_1 + 1/8 R_2$, $R_3 \to R_3 + R_2$

$$\left[\begin{array}{ccc|c} 2 & 0 & 3/4 & 7/8 \\ 0 & -8 & -2 & -1 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

$R_1 \to R_1 - 3/4 R_3$, $R_2 \to R_2 + 2R_3$

$$\left[\begin{array}{ccc|c} 2 & 0 & 0 & 1/8 \\ 0 & -8 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

$R_1 \to R_1/2$
$R_2 \to R_2/-8$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 1/16 \\ 0 & 1 & 0 & -1/8 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

Vector Space

$V(F)$, $+$, $\times$

$x, y \in V$, $x+y \in V$

1) $x+y = y+x$

2) $x+(y+z) = (x+y)+z$

3) $x+0 = 0+x = 0$

4) $x + x^{-1} = 0$

$\longrightarrow$ additive inverse of x

$\alpha \in F$, $x \in V$ $\alpha x \in V$

5) $\alpha(\beta x) = (\alpha\beta)x$

6) $(\alpha+\beta)x = \alpha x + \beta x$

7) $\alpha(x+y) = \alpha x + \alpha y$

8) $1x = x$ $1 \in F$

Q) $\mathbb{R}^n(\mathbb{R})$, Vector space over $\mathbb{R}$

$x = (x_1, x_2 \ldots\ldots x_n)$

$x+y = (x_1, x_2 \ldots\ldots x_n) + (y_1, y_2 \ldots\ldots y_n)$

$= (x_1+y_1, x_2+y_2 \ldots\ldots x_n+y_n)$

if $x_1 \in \mathbb{R}$ $y_1 \in \mathbb{R}$

So $x_1+y_1 \in \mathbb{R}$

So

$x+y \in \mathbb{R}$

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$\alpha x = \alpha(x_1, x_2 \ldots\ldots x_n)$

$= (\alpha x_1, \alpha x_2 \ldots\ldots \alpha x_n)$

if $x_1, x_2, \ldots x_n \in \mathbb{R}$

then

$\alpha x_1, \alpha x_2 \ldots \alpha x_n \in \mathbb{R}$

~~then~~

So

$\alpha x \in \mathbb{R}$

Q. $\mathbb{R}^2(R)$

~~additive identity (e_1, e_2)~~

or $\cancel{(a_1, a_2) + (e_1, e_2) = (a_1, a_2)}$

$(a_1 + e_1$

$\mathbb{R}^2(R)$ ~~ver~~ $\mathbb{R}^2$ vector space on $\mathbb{R}$

if we defined addition like

$(a_1, a_2) + (b_1, b_2) = (a_1 + b_1, 0) \in \mathbb{R}^2$

$\alpha(a_1, a_2) = (\alpha a_1, \alpha a_2)$

(1)

$(a_1, a_2) + (b_1, b_2) = (a_1 + b_1, 0)$

$(b_1, b_2) + (a_1, a_2) = (b_1 + a_1, 0)$

$= (a_1 + b_1, 0)$ $\quad a_1 + b_1 = b_1 + a_1$ for all $a_1, b_1 \in \mathbb{R}$

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2) Let $e \in (e_1, e_2)$ is identity elemen

for identity element

$a + e = a$

$(a_1, a_2) + (e_1, e_2) = (a_1, a_2)$

$(a_1 + e_1, 0) = (a_1, a_2)$

$a_1 + e_1 = a_1 \Rightarrow e_1 = 0$

$a_2 = 0$

~~e_2 may be possible~~

that

$e_2 \in \mathbb{R}$

So for different element we have different ~~element~~ identity

for identity element for any system it should be single

So

It is not hold

So it is not vector space.

$\mathbb{R}^3(\mathbb{R})$

$(a_1, a_2, a_3) + (b_1, b_2, b_3) = (a_1 b_1, a_2 b_2, a_3 b_3) \in \mathbb{R}^3$

Let (e_1, e_2, e_3) identity element

$(a_1, a_2, a_3) + (e_1, e_2, e_3)$

$= (a_1 e_1, a_2 e_2, a_3 e_3) \in \mathbb{R}^3$

$= (a_1, a_2, a_3)$

$e_1 = 1 \quad e_2 = 1, \quad e_3 = 1$

$(1, 1, 1)$ is identity element for all vector

It have one identity

So it is a vector space.

Eg) On $\mathbb{R}^+ = \{x \in \mathbb{R} \mid x \geq 0\}$ is a vector space over $\mathbb{R}$

addition for any $u, v \in \mathbb{R}^+$, $u + v = uv > 0$ (closed under addition)

Scalar multiplication:

For any $u \in \mathbb{R}^+$ and $\alpha \in \mathbb{R}$

$\alpha u = u^{\alpha} > 0$

(Closed under scalar multiplication) $2^{-2} = \frac{1}{2}$

1) For $u, v \in \mathbb{R}^+$,

$$u + v = uv = vu = v + u$$

(commutative)

2) For $u, v, w \in \mathbb{R}^+$

$$u + (v + w) = u(vw)$$
$$= (uv)w$$
$$= (u + v) + w$$

(associative)

3) For $u \in \mathbb{R}^+$, there is

$$u + 1 = u \cdot 1 = u \quad \text{(identity)}$$

4) For $u \in \mathbb{R}$

$\exists$ $\frac{1}{u} \in \mathbb{R}^+$

$\exists$ $u + \frac{1}{u} = 1$

(additive identity)

5) $1 \cdot u = u^1 = u$

$1 \in \mathbb{R}$

1 is multiplicative identity

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6) For $\alpha, \beta \in \mathbb{R}$ and ~~abc~~ $u \in \mathbb{R}^+$

$$(\alpha\beta)u = u^{(\alpha\beta)} = u^{\beta\alpha}$$

$$\alpha(\beta u) = \alpha(u^\beta) = (u^\beta)^\alpha$$

$$= u^{\beta\alpha}$$

$$(\alpha\beta)u = \alpha(\beta u)$$

7) for $\alpha \in \mathbb{R}$, $u, v \in \mathbb{R}^+$

$$\alpha(u+v) = \alpha(uv) = (uv)^\alpha$$

$$= u^\alpha v^\alpha$$

$$= v^\alpha u^\alpha$$

$$= (vu)^\alpha$$

$$= \alpha(vu)$$

$$= \alpha(v+u)$$

not same

8) $(\alpha+\beta)u = u^{\alpha+\beta}$

$$= u^\alpha u^\beta$$

$$= u^\alpha + u^\beta$$

$$= u^\beta + u^\alpha$$

$$= u^\beta u^\alpha$$

$$= u^{\beta+\alpha}$$

$$= (\beta+\alpha)u$$

So $\mathbb{R}^+(\mathbb{R})$ is a vector space

Assignment

```
# include <stdio.h>
int main ( )
{
    int number;
    Printf (" Enter your number");
    Scanf ("%d", number);
    Switch (number)
    {
      case 91 ... 100 ;
       Printf (" Your grade is AA");
       Break;
      case 81 ... 90 ;
       Printf (" Your grade is AB");
       Break;
      case 71 ... 80 ;
       Printf (" Your grade is BB);
       Break;
      case 61 ... 70 ;
       Printf (" Your grade is BC);
       Break;
      case 60 ... 50 ;
       Printf (" Your grade is CC);
       Break;
      case 50 ... 40 ;
       Printf (" Your grade is R);
       Break;
```

```
Default
Printf (" Invalid input");
}

return 0;

}
```

Assignment

```
# include < stdio.h>
int main( )
{   int number;
    Printf(" Enter any number");
    Scanf ("% d", number);
    int i=1, y=1
  while ( i <=number);
  {
        y = y*i ;
      i++;

  }
  Printf (" factarical of is
                    %d", number);

  Return 0;

}
```

$F(\mathbb{R}, \mathbb{R})$ = Set of all functions from $\mathbb{R}$ to $\mathbb{R}$

$= \{ f : \mathbb{R} \to \mathbb{R} \}$

$F(\mathbb{R}, \mathbb{R})$ is a vector space over $\mathbb{R}$

For $f, g \in F(\mathbb{R}, \mathbb{R})$

i.e. $f : \mathbb{R} \to \mathbb{R}$ and

<u>Addition Operation</u> $g : \mathbb{R} \to \mathbb{R}$.

$$f + g := \underbrace{(f+g)}_{\text{new function}}(x) = f(x) + g(x)$$

for all $x \in \mathbb{R}$

$f+g : \mathbb{R} \to \mathbb{R}$

Vector $F(\mathbb{R}, \mathbb{R})$ is closed under addition Operation

<u>Scalar Operation</u>

for any $f \in F(\mathbb{R}, \mathbb{R})$

and α is a scalar $\alpha \in \mathbb{R}$

$$\alpha f := \alpha f(x) \ : \ \forall x \in \mathbb{R}$$

$\Rightarrow \alpha f : \mathbb{R} \to \mathbb{R}$ and hence

$$\alpha f \in F(\mathbb{R}, \mathbb{R})$$

Zero function 0

$$0 : \mathbb{R} \to \mathbb{R} \ ; \ 0(x) = 0 \ , \forall x \in \mathbb{R}$$

- Identity element

: $f(x) = 0 \ , \forall x \in \mathbb{R}$

such than

$\cancel{f(x)}$ $g \in F(\mathbb{R}, \mathbb{R})$

$$g + f := (g+f)(x) = g(x) + f(x)$$

$$= g(x)$$

$f(x) = 0 \ \forall x \in \mathbb{R}$

$f \in F(\mathbb{R}, \mathbb{R})$

$\hookrightarrow$ is identity element.

Inverse :

$f : \mathbb{R} \to \mathbb{R}$ ~~$\& x \in \mathbb{R}$~~

Then $\exists$ $-f : \mathbb{R} \to \mathbb{R}$

$$f + (-f) := (f + (-f))(x) = f(x) + (-f(x)) \ \forall x \in \mathbb{R}$$

$$= 0$$

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Subspace of a vector Space

A ~~only~~ subset ~~of~~ U of V (vector space) is called a subspace of V (vector or space) if

1) $0 \in U$, $0 \in V$
2) whenever two element $u, v \in U$

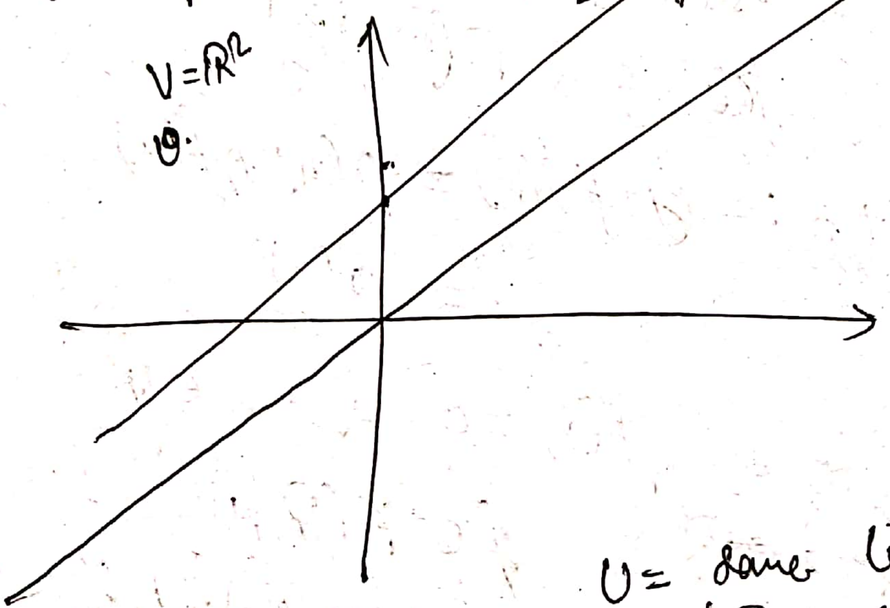
$$u + v \in U \subseteq V$$

3) for $\alpha \in \mathbb{R}$ & $\mathbb{C}$ and $u \in U$

$$\alpha u \in U \subseteq V$$

Z = Same line passing through ~~origin~~ (0,1) and (-1,0) $y = x+1$.

Interpretation



U = Same line passing through origin

(1,1) & and (2,2)

$(1,1) + (2,2) = (3,3) \in U \subseteq V$

$(3,3) \in V$

$\alpha \in \mathbb{R}$

$\alpha(1,1) = (\alpha, \alpha) \in U \subseteq V$ $\alpha(1,1) \in V$

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$$\underset{\in U}{(0,0)} + \underset{\in U}{(1,1)} = (1,1) \in U \subseteq V$$

So it is ~~only~~ subspace of $\mathbb{R}^2$

$$\boxed{Z \text{ is not subspace of } \mathbb{R}^2 \text{ as } (0,0) \notin Z}$$

Examples

1) Take :

$U = \{ (a,b) \in \mathbb{R}^2 \,/\, ax + by = 0 \} \subseteq \mathbb{R}^2$ (vector space)

1) $(0,0) \in U$ $\quad$ ($0 \cdot x + 0 \cdot y = 0$
$0 = 0$)

2) For $u, v \in U$ such that

$u = (a_1, b_1)$

$v = (a_2, b_2)$

$u + v = (a_1, b_1) + (a_2, b_2)$

$= (a_1 + a_2, b_1 + b_2)$

$a_1 x + b_1 y = 0$

$a_2 x + b_2 y = 0$

$(a_1 + a_2) x + (b_1 + b_2) y = 0$

$\Rightarrow (a_1 + a_2, b_1 + b_2) \in U$

$u + v = (a_1, b_1) + (a_2, b_2)$

$= (a_1 + a_2, b_1 + b_2) \in \mathbb{R}^2$

$\Rightarrow (u, v) \in U$

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3) $u \in U$ such that $u = (a, b)$

as $u \in U$

$ax + by = 0$

multiply with α ~~[illegible]~~ if $\alpha \in \mathbb{R}$

on both side

$a(\alpha x) + b(\alpha y) = 0$

$\Rightarrow (\alpha a, \alpha b) \in U$

$\Rightarrow \alpha(a, b) \in U$

$\alpha u \in U$

$\Rightarrow U$ is subspace of $\mathbb{R}^2$

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Take

$U = \{(a, b, c) \in \mathbb{R}^3 / ax + by + cz = 0\}$

$\subseteq (\mathbb{R}^3,$ usual add^n and scalar multiplication over field $\mathbb{R})$

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A polynomial of degree 'n' in the variable x

$$a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots\dots a_n x^n, \; a_n \neq 0$$

$$a_i\text{'s} \in \mathbb{R}, \; i = 0 \text{ to } n$$

P_2 = set of all polynomials of degree ≤ 2

degree 2-degree polynomial (quadratic)

degree 1 - degree polynomial (linear)

degree 0 - polynomial (constant)

$\mathbb{R}(\mathbb{R})$ is a vector space

addition

For

$$p(x) = a_0 + a_1 x + a_2 x^2 \in P_2, \; q(x) = b_0 + b_1 x + b_2 x^2 \in P_2$$

Then

$$p(x) + q(x) = (a_0 + b_0) + (a_1 + b_1)x + (a_2 + b_2)x^2$$

Scalar multiplication

for $\alpha \in \mathbb{R}$, $p(x) \in P_2$

$$\alpha\, p(x) = (\alpha a_0)x + (\alpha a_1)x + (\alpha a_2)x^2$$

$$\in P_2$$

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Zero polynomial

$(0 + 0x + 0x^2) = 0(x)$

$P(x) + 0(x) = P(x)$

zero is identity element.

Inverse

$a_0 + a_1 x + a_2 x^2 \in \mathbb{R}$

There exist

$-a_0 - a_1 x - a_2 x^2$

$= (-a_0) + (-a_1)x + (-a_2)x^2$

$\in \mathbb{R}_2$

if $a_0, a_1, a_2 \in \mathbb{R}$ then $-a_0, -a_1, -a_2 \in \mathbb{R}$

$\Rightarrow$ P_2 = the set of is a vector space over $\mathbb{R}$

Is P_2 is a subspace of $C(\mathbb{R}, \mathbb{R})$

continuous function $\mathbb{R} \to \mathbb{R}$

$\Rightarrow$ P_2 is a subspace of $C(\mathbb{R}, \mathbb{R})$

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P = the set of all polynomials over $\mathbb{R}$

$$P(x) = a_0 + a_1 x + a_2 x^2 + \dots\dots + a_n x^n$$
$$Q(x) = b_0 + b_1 x + b_2 x^2 + \dots\dots + b_m x^m$$

Addition

$$p(x) + Q(x) = (a_0+b_0) + (a_1+b_1)x + (a_2+b_2)x^2 + \dots\dots + \cancel{(a_n+b_n)}x^n + a_n x^n + b_m x^m$$

Ex)

$M_{n\times m}(\mathbb{R})$ = the set of all matrics of $n\times m$

is a vector space over $\mathbb{R}$.

Ex)

$M_{2\times 2}(\mathbb{R})$ is a vector ~~sqa~~ space over $\mathbb{R}$

NI = the set of all non invertiable matrixes (2×2)

If NI a subspace of $M_{2\times 2}(\mathbb{R})$

$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \in NI$ $\quad$ $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \in NI$

$\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \in NI$

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \notin NI$$

which is not a ~~sp~~ subspace.

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Q) which of the following subsets of $M_{3\times3}(\mathbb{R})$ are subspace?

i) $G :=$ the invertible (3×3) matrices

2) $D =$ the diagonal 3×3 matrices

3) $U =$ the upper triangular 3×3 matrices.

reason

1) $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \notin G$

so it is not subspace of $M_{3\times3}(\mathbb{R})$.

Q) which of the following are subspaces of P_2?

a) $A_1 = \{ P(x) \mid P(0) = 2 \}$

b) $A_2 = \{ P(x) \mid P(x) = 0 \}$

c) $A_3 = \{ P(x) \mid P'(1) = P(2) \}$

d) $A_4 = \{ P(x) \mid \int_0^1 P(x)\,dx = 0 \}$

e) $A_5 = \{ P(x) \mid P(-x) = -P(x) \}$

$p(x) = \sin x \qquad q(x) = \cos x$

$p(-x) = -\sin x \in A_5$

Tuitorial

1) i) $\mathbb{R}^2(\mathbb{R})$

$(x_1, x_2) + (y_1, y_2) = (x_1 + x_2, 0)$

Let (e_1, e_2) is the identity

$(x_1, x_2) + (e_1, e_2) = (x_1, x_2)$

$(x_1 + e_1, 0) = (x_1, x_2)$

$\cancel{x_1} + e_1 = \cancel{x_1}$ $\quad$ $x_2 = 0$

$e_1 = 0$ $\quad$ if $x_2 = 2$

$2 = 0$

which is not possible

So $\mathbb{R}^2(\mathbb{R})$ does not have identity element.

ii) $\mathbb{R}^3$ $\quad (x_1, x_2, x_3) + (y_1, y_2, y_3)$

$= (x_1 y_1, x_2 y_2, x_3 y_3)$

Let (e_1, e_2, e_3) is identity

$(x_1, x_2, x_3) + (e_1, e_2, e_3)$

$= (x_1, x_2, x_3)$

$(x_1 e_1, x_2 e_2, x_3 e_3) = (x_1, x_2, x_3)$

$e_1 = 1 \quad e_2 = 1 \quad e_3 = 1$

identity $(1, 1, 1)$

Let inverse of this (e_1', e_2', e_3')

$$(x_1, x_2, x_3) + (e_1', e_2', e_3') = (1, 1, 1)$$

$$(x_1 e_1', x_2 e_2', x_3 e_3') = (1, 1, 1)$$

$$e_1' = \frac{1}{x_1}, \quad e_2' = \frac{1}{x_2}$$

if $x_1 \cancel{or} x_2 \cancel{or} x_3 \neq 0$

then we can get inverse

but $\cancel{(x_1 \neq x_2 \neq 0)}$) have no inverse

So it does not have ~~identity~~ inverse for $(0, 0, 0)$

3) $V = R^2$

$\cancel{a_1 + b}$ $(a_1, a_2) + (b_1, b_2) = (a_1 + b_2, a_2 + b_1)$

$$\alpha(a_1, a_2) = (\alpha a_1, \alpha a_2) \quad \alpha \in R.$$

Identity element

$$(a_1, a_2) + (e_1, e_2) = (a_1, a_2)$$

$$(a_1 + e_2, a_2 + e_1) = (a_1, a_2)$$

$$e_2 = 0 \quad e_1 = 0$$

$(0, 0)$ is identity element

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~~Ider~~ Inverse

$(a_1, a_2) + (e_1, e_2) = (0, 0)$

$(a_1 + e_2, a_2 + e_1) = (0, 0)$

$a_1 = -e_2 \quad e_1 = -a_2$

$e_2 = -a_1 \quad e_1 = -a_2$

every element has its increase

$\cancel{(a_1, a_2) + ((b_1, b_2) + (c_1, c_2)) = (a_1, a_2)}$

$(a_1, a_2) + (b_1, b_2) = (a_1 + b_2, a_2 + b_1)$

$(b_1, b_2) + (a_1, a_2) = (b_1 + a_2, a_1 + b_2)$

$(a_1, a_2) + (b_1, b_2) \neq (b_1, b_2) + (a_1, a_2)$

not a vector space

4) $\mathbb{R}^2(\mathbb{R})$

$(a_1, a_2) + (b_1, b_2) = (a_1 + b_1 + 1, a_2 + b_2 + 1)$

$\alpha(a_1, a_2) = (\alpha a_1, \alpha a_2)$

Identity element

$(a_1, a_2) + (e_1, e_2) = (a_1, a_2)$

$(a_1 + e_1 + 1, a_2 + e_2 + 1) = (a_1, a_2)$

$\therefore (-1, -1)$ is identity element

Inverse

$(a_1\ a_2) + (e'_1, e'_2) = (-1, -1)$

$(a_1 + e'_1 + 1\ ,\ a_2 + e'_2 + 1) = (-1, -1)$

$e'_1 = -1 - a_1$

7) $(\alpha+\beta)(a_1, a_2) = ((\alpha+\beta)a_1\ ,\ (\alpha+\beta)a_2)$

$= (a_1\alpha + a_1\beta\ ,\ a_2\alpha + a_2\beta)$

$\alpha x + \beta x = \alpha(a_1, a_2) + \beta(a_1, a_2)$

$= (a_1\alpha + a_1\beta + 1,\ \alpha a_2 + \beta a_2 + 1)$

~~Vector space~~

8) $\alpha[(a_1, a_2) + (b_1, b_2)] = \alpha[(a_1+b_1+1, a_2+b_2+1)]$

$= [\alpha a_1 + \alpha b_1 + \alpha, \alpha a_2 + \alpha b_2 + \alpha]$

$\alpha(a_1, a_2) + \alpha(b_1, b_2) = (\alpha a_1, \alpha a_2) + (\alpha b_1, \alpha b_2)$

$= (\alpha a_1 + \alpha b_1 + 1\ ,\ \alpha a_2 + \alpha b_2 + 1)$

$\alpha[(a_1, a_2) + (b_1, b_2)] \neq \alpha(a_1, a_2) + \alpha(b_1, b_2)$

So not a vector space

Scanned by CamScanner

4) $V(\mathbb{R})$ $(V, \oplus, \otimes)$

i) $\alpha \otimes 0 = 0 \quad \forall \alpha \in \mathbb{R}, 0 \in V$

$\alpha 0 = \alpha(0 + 0)$ → identity property

$\alpha 0 = \alpha 0 + \alpha 0$ Adding inverse both side

$0 = \alpha 0 + 0$

$\Rightarrow \boxed{\alpha 0 = 0}$

ii) $0u = 0 \quad \forall u \in V$ & $0 \in \mathbb{R}$

$0u = (0+0)u$ $\quad 0 \in \mathbb{R}$, $u \in V$

$0u = 0u + 0u$

adding inverse both side

$(-0u) + (0u) = (0u + (-0u)) + 0u$

$0 = 0 + 0u$

$\boxed{0u = 0}$

iii) $(-1)u = -u, \forall u \in V$

$\cancel{(-1)u = (-1)(u+0)}$

$(-1)u + u = (-1)u + 1 \cdot u$
$= (-1+1)u$
$= 0 \cdot u$
$= 0$

$\Rightarrow \boxed{(-1)u = -u}$

iv)
$$\alpha u = 0$$
$$\Rightarrow \alpha = 0 \text{ OR } u = 0$$
$$\Downarrow \qquad\qquad \Downarrow$$
$$\mathbb{R} \qquad\qquad V$$

5) $V = C[0,1]$ $\quad F = \mathbb{R}$

↳ set of continuous function

$[0,1] \longrightarrow \mathbb{R}$

$$C[0,1] = \{ f : [0,1] \to \mathbb{R},\ f \text{ is continuous} \}$$

$$(f+g)(x) = f(x) + g(x)$$
$$(\alpha f)(x) = \alpha f(x)$$

1) $(f+g)(x) = f(x) + g(x) = g(x) + f(x)$

as $g(x) \in \mathbb{R}$ and $f(x) \in \mathbb{R}$

2) $[f+(g+h)](x) = f(x) + (g+h)(x)$
$$= f(x) + g(x) + h(x)$$
$$= (f+g)(x) + h(x)$$
$$= [(f+g)+h](x)$$

3) $f(x) \in V$ $\quad 0(x) = 0 \longrightarrow$ zero function

$$(f+0)(x) = f(x) + 0(x)$$
$$= f(x) + 0$$
$$= f(x)$$

$\Rightarrow$ $0(x) = 0$ is zero function

4) $f(x) \in V$ $\quad g(x) = -f(x)$

$\cancel{(f+(-f))(x)}$

$$(f+g)(x) = f(x) + g(x)$$
$$= f(x) - f(x)$$
$$(f+g)(x) = 0$$

$\Rightarrow$ $g(x) = -f(x) \in V$
so inverse also exist

5) $\alpha \in \mathbb{R}$ $\quad f(x) \in V$

$(\alpha f)(x) = \alpha f(x) \in V$

6) $\alpha, \beta \in \mathbb{R}$ $\quad f(x) \in V$

$$[(\alpha+\beta) f](x) = [\alpha f + \beta f](x)$$
$$= \alpha f(x) + \beta f(x)$$
$$= (\alpha f)(x) + (\beta f)(x)$$
$$= (\alpha f + \beta f)(x)$$

8)
$$\alpha(f+g)(x) = (\alpha f+\alpha g)(x)$$
$$= \alpha f(x) + \alpha g(x)$$
$$= (\alpha f)(x) + (\alpha g)(x)$$
$$\alpha(f+g)(x) = \alpha f(x) + \alpha g(x)$$
$$\alpha(f+g)(x) = (\alpha f + \alpha g)(x)$$

9) $f(x) \in V \quad 1 \in \mathbb{R}$

$$1 \cdot f(x) = f(x)$$

~~So~~ It is a vector space.

$U \subseteq V$, V is nonempty.

1) $0 \in U$
2) $u+v \in U$, $\forall u, v \in U, V$
3) $\alpha u \in U, V$, $\forall \alpha \in \mathbb{R}$, $u \in U, V$

U is a ~~subsets~~ subspace of V.

Linear Span

For vectors

$v_1, v_2, v_3 \ldots\ldots v_n \in V$ (a vector space).

☆ we say that the vectors $v_1, v_2 \ldots\ldots v_n$ span full vector space V, i.e

$L(\{v_1, v_2 \ldots\ldots v_n\}) = V$, if for all $\underline{v \in V}$

$v = \alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 + \ldots\ldots \alpha_n v_n$

; $\alpha_i \in \mathbb{R}/\mathbb{C}$ $i = 1 \ldots n$

Ex) $V = \mathbb{R}^2$

Span $= S = \{(0,1), (1,0)\}$

Consider

$\alpha_1(0,1) + \alpha_2(1,0)$ for all $\alpha_1, \alpha_2 \in \mathbb{R}$

$= (\alpha_1, \alpha_2) \in \mathbb{R}^2$

$L(S) = \mathbb{R}^2$

S span whole vector space

Proof :) $(x, y) \in \mathbb{R}^2 \Rightarrow (x, 0) + (0, y) \in \mathbb{R}^2$

$x(1,0) + y(0,1) \in \mathbb{R}^2$

$(1,0) = \alpha_1 \quad \alpha_2 = (0,1)$

$x, y \in \mathbb{R}$

$x\alpha_1 + y\alpha_2 \in V$

So

x, y span whole $\mathbb{R}^2$

$S_1 = \{(1,0)\ (0,1)\} \quad V = \mathbb{R}^2$

$L(S_1) = \mathbb{R}^2$

S_1 span V

$S_2 = \{(1,0), (1,1)\} =$

$L(S_2) = \mathbb{R}^2$

S_2 span V

for α_1, α_2

for $(a,b) = \mathbb{R}^2$

$(a,b) = \alpha_1(1,0) + \alpha_2(1,1)$

$(a,b) = (\alpha_1 + \alpha_2, \alpha_2)$

$\alpha_2 = b$ & $a = \alpha_1 + \alpha_2$

$\alpha_1 = a - \alpha_2$

$\alpha_1 = a - b$

$\alpha_1(1,1) + \alpha_2(1,1)$

$= (a-b)(1,0) + b(1,1)$

$= (a-b, 0) + (b, b)$

$= (a,b) \in \mathbb{R}^2$

Ex| $x+y=1$
$x-y=2$

$$x\begin{pmatrix}1\\1\end{pmatrix}+y\begin{pmatrix}1\\-1\end{pmatrix}=\begin{pmatrix}1\\2\end{pmatrix}$$

$$x(1,1)+y(1,-1)=(1,2) \quad \text{-(i)}$$

For $(1,2)\in\mathbb{R}^2$ does there exist

$x, y \in \mathbb{R}$

such that (i) hold

$\alpha_1 = x = 3/2$, $\alpha_2 = y = -1/2$

$\Rightarrow$ $(1,2) \in$ Linear span of $(1,1), (1,-1)$ ~~which~~.

OR

wheather

$(1,2) \in L\{(1,1), (1,-1)\}$
$\downarrow$
Linear span of set

Tutorial
Q-12(ii)

wheather If $(1,1,0) \in L\{(1,2,3), (1,1,-1), (3,5,5)\}$

Then should α_1, α_2 & $\alpha_3 \in \mathbb{R}$

$(1,1,0) = \alpha_1(1,2,3) + \alpha_2(1,1,-1) + \alpha_3(3,5,5)$

$(1,1,0)' = \alpha_1(1,2,3)' + \alpha_2(1,1,-1)' + \alpha_3(3,5,5)'$

$(1,1,0) = (\alpha_1+\alpha_2+3\alpha_3,\ 2\alpha_1+\alpha_2+5\alpha_3,\ 3\alpha_1-\alpha_2+5\alpha_3)$

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$\alpha_1 + \alpha_2 + 3\alpha_3 = 1$

$2\alpha_1 + \alpha_2 + 5\alpha_3 = 1$

$3\alpha_1 - \alpha_2 + 5\alpha_3 = 0$

$$\begin{pmatrix} 1 & 1 & 3 \\ 2 & 1 & 8 \\ 3 & -1 & 5 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

if no sol^n

$\Rightarrow$ no α_1, α_2 & α_3

If infinite sol^n/unique for the ~~the~~ above equation

$\Rightarrow$ infinite many sol^n/unique α_1, α_2 & α_3 for the above equation

So $(1,1,0) \notin L(S)$ vector

So

$(1,1,0) \in L(S)$

no sol^n

So $(1,1,0) \notin L\{(1,1,3), (2,1,-5), (3,-1,5)\}$

if $(0,0,0) \in L\{(1,1,3), (2,1,-5), (3,-1,8)\}$

So $\exists$ infinite many α_1, α_2 and α_3 so $(0,0,0) \in L\{(1,1,3), (2,1,-5), (3,-1,5)\}$

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For any subset $S \subseteq V \neq \{0\}$

$L(S) = \{$ all possible finite linear combinations$\}$

$S = \{v_1 \ldots\ldots\ldots v_n \ldots\ldots\}$

$L(S) = \alpha_1 v_1 + \alpha_2 v_n$ ✓ (Linear span)

$= \alpha_1 u_1 + \alpha_2 u_2 + \alpha_3 u_3 + \ldots \alpha_n v_n$ ✓ (Linear span)

$= \alpha_i$

★ $S \subseteq V$ (vector space)

$\downarrow$

is not finite

$L(S) = \{$ all possible finite linear combinations of the element of $S\}$

$= \left\{ \sum_{i=1}^{k} \alpha_i u_i \;\middle|\; \text{for } u_1 \ldots u_k \in S \;\&\; \alpha_1 \ldots \alpha_k \in \mathbb{R} \text{ or } \mathbb{C} \right\}$

(K is not fixed)

★ If S is finite $\{v_1 \ldots v_n\} = S \subseteq V$

$L(S) = \left\{ \sum_{i=1}^{n} \alpha_i v_i \;\middle|\; \alpha_i \in \mathbb{R} \text{ or } \mathbb{C},\ i = 1 \text{ to } n \right\}$

Remarks

1) For any subset S of a vector space V, its linear span $L(S)$ is always a subspace of V

2) $L(S)$ is the smallest subspace of V which contains the set S.
i.e suppose T is another subspace of V which is also contains in S

Then

$$L(S) \subseteq T$$

Linearly independent

★ The elements $v_1, v_2 \ldots\ldots v_n \in V$ is called Linearly independent if the equations

$$\alpha_1 v_1 + \alpha_2 v_2 + \text{------} \alpha_n v_n = 0 \quad \text{--- (1)}$$

$$\Rightarrow \text{iff } \alpha_1 = \alpha_2 = \alpha_3 = \text{----} \alpha_n = 0$$

i.e eqn (1) has only the zero solution.

if $V = \mathbb{R}^2$, $v_1 = (1,0)$, $v_2 = (0,1)$

$$(1,0) = \alpha(0,1)$$

$$\alpha_1(1,0) + \alpha_2(0,1) = (0,0)$$

$$(\alpha_1, 0) + (0, \alpha_2) = (0,0)$$

$$(\alpha_1, \alpha_2) = (0,0)$$

$$\alpha_1 = 0 , \alpha_2 = 0$$

↳ So v_1, v_2 are linearly independent

$v_1 = (1,1)$ $v_2 = (2,2)$

$$\alpha_1 v_1 + v_2 \alpha_2 = 0$$

$$(\alpha_1, \alpha_1) + (2\alpha_2, 2\alpha_2) = (0,0)$$

$$(\alpha_1 + 2\alpha_2 , \alpha_1 + 2\alpha_2) = (0,0)$$

$$\Rightarrow \alpha_1 = -2\alpha_2$$

$\Rightarrow$ it have $\alpha_1 \neq 0$, $\alpha_2 \neq 0$ also are sam other the $\alpha_1 = \alpha_2 = 0$

$\Rightarrow$ So, v_1, v_2 are linearly dependent.

Basis of a vector space V

The elements $v_1, v_2 \ldots\ldots v_n$ of a vector space V is said to be a basis of V if

1) The linear span is vector space.

$$L\{v_1, v_2 \ldots\ldots v_n\} = V$$

2) $v_1, v_2 \ldots\ldots v_n$ are linearly independent.

★ means that every vector in V can be uniquely written as a linear combination of $v_1, v_2 \ldots v_n$ i.e

$$v = \alpha_1 v_1 + \alpha_2 v_2 + \ldots\ldots \alpha_n v_n$$

where $\alpha_1 \ldots \alpha_n$ are unique.

Suppose if we have

Linear span v has not unique representation

$$v = \alpha_1 v_1 + \alpha_2 v_2 + \ldots \alpha_n v_n$$

$, \alpha_i\text{'s} \in \mathbb{R}$ or $\mathbb{C}$

$$v = \beta_1 v_1 + \beta_2 v_2 + \ldots \beta_n v_n$$

$, \beta_i\text{'s} \in \mathbb{R}$ or $\mathbb{C}$

Subtracting first - second

$$(\alpha_1-\beta_1)v_1 + (\alpha_2-\beta_2)v_2 + \cdots (\alpha_n-\beta_n)v_n = 0$$

If $v_1, v_2 \ldots\ldots v_n$ are independent

so

$\alpha_i - \beta_i = 0$, $\forall$ $i = 1$ to n

$\Rightarrow \alpha_i = \beta_i$ $(\because v_1, v_2 \ldots v_n$ are linear independent)

$\Rightarrow$ Therefore soln of v is unique.

Any two vector in $\mathbb{R}^2$ through origin, which doesn't lie on the same line is a basis for $\mathbb{R}^2$

$(1,0)$ $(0,1)$ $\rightarrow$ are basis of $\mathbb{R}^2$

as

$$(a,b) = a(1,0) + b(0,1) = (a,b) \in \mathbb{R}^2$$

$\rightarrow$ Linear span

$$\alpha_1(1,0) + \alpha_2(0,1) = 0$$

$\alpha_1 = \alpha_2 = 0$

$\rightarrow$ independent

$(1,1)$ & $(0,1)$ $\rightarrow$ are basis of $\mathbb{R}^2$

☆ for ~~...~~ elements $v_1, v_2, \ldots v_n$ of ~~...~~ a vector V can form different basis. But number of element in each ~~base~~ basis remain change. If element increase then ~~they~~ are linear combination of other to give same number of elements for basis

Dimension of vector space

The no of elements in a given basis of a vector space V is known as the dimension of V

$$\boxed{\dim_{R/C}(V) = n = \dim V}$$

$\{(1,0,0), (0,1,0), (0,0,1)\}$ is basis ~~of~~ for R^3

$$\boxed{\dim(V) = 3}$$

Tutorial

7) vi) $S = \{(a, b, c, d) ; a \le b\}$

$(0, 0, 0, 0) \in S$

$\alpha = (a_1\ b_1\ c_1\ d_1)$ $\beta = (a_2\ b_2\ c_2\ d_2)$

$\Rightarrow$ $a_1 \le b_1$ $a_2 \le b_2$

$a_1 + a_2 \le b_1 + b_2$

$\Rightarrow$ $\alpha + \beta \in S$

$\alpha \in \mathbb{R}$. $x = (a_1\ b_1\ c_1\ d_1)$

$\alpha x = \alpha(a_1\ b_1\ c_1\ d_1)$
$= (\alpha a_1, \alpha b_1\ \alpha c_1\ \alpha d_1)$

$\Rightarrow$ $\alpha a_1 \le \alpha b_1$

which is not true fore $\underline{\alpha < 0}$

Hence it is not subspace of Vector.

, i.e $\alpha S \notin S$

8) ii) $\{p \in P : \deg p = 4\} = S$

$P(x) = 0$ $\rightarrow \frac{1}{x^{\infty}}$ degree is ∞

$P(x) \notin S$

so it is not subspace

9) i) $S = \{ A \in \mathbb{R}^{2\times2} \mid A \text{ is diagonal} \}$

$\hookrightarrow$ subspace

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \in \mathbb{R}$$

vii) Ex $A = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ $B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

$$A+B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \notin S$$

not a subspace

10)

W_1 & W_2 are the subspace of vector space

& $W_1 \cup W_2$ is also subspace

Let neither, $W_1 \not\subset W_2$ nor $W_2 \not\subset W_1$

$W_1 \not\subset W_2 \Rightarrow \exists\, x \in V,\ x \in W_1$ & $x \notin W_2$

$W_2 \not\subset W_1 \Rightarrow \exists\, y \in V,\ y \notin W_1$ & $y \in W_2$

$x \in W_1 \Rightarrow x \in W_1 \cup W_2$

&

$y \in W_2 \Rightarrow y \in W_1 \cup W_2$

$\Rightarrow x+y \in W_1 \cup W_2$

either $x+y \in W_1$ or $x+y \in W_2$

if $x+y \in W_1 \rightarrow W_1$ i.e is subspace

$$\therefore y = \overbrace{(x+y)}^{\to W_1} \overbrace{(-x)}^{\to W_2} \in W_1 \rightarrow \text{as } \cancel{\text{sup}} \text{ subspace}$$

$$\therefore y \in W_1$$

Hence it is false

So $W_1 \subseteq W_2$ or $W_2 \subseteq W_1$

11) W_1 and W_2 subspace of V

$W_1 + W_2 = V$

and

$W_1 \cap W_2 = \{0\}$

for each $u \in V$ $\exists$ unique vectors $u_1 \in W_1$ and $u_2 \in W_2$ such that

$u_1 + u_2 = u$

Let ~~this~~ u_1, u_2 is not unique

~~$u_1, v_1 \in W_1$ and $u_2\ \cancel{v_2} \in W_2$~~

~~such that~~ ~~$u_1 + u_2 = u$~~

~~$u_1 + u_2$~~ $= u \in V$

$u = u_1 + v_1$ — (1)

$u_1 + v_1 = u \in V$

Let $\exists$ w_1', w_2' such that $x = w_1' + w_2'$ — (2)

$$\Rightarrow w_1 + w_2 = w_1' + w_2'$$

$$\underbrace{(w_1 - w_1')}_{\to W_1} + \underbrace{(w_2 - w_2')}_{\to W_2} = 0$$

$$w_1 - w_1' \in W_1 \cap W_2 = \{0\}$$

$$\Rightarrow \quad w - w_1' = 0$$

$$w = w'$$

13)

$V = \mathbb{C}^2$

$(1+i, 1-i) \in L[(1+i, 1), (1, 1-i)]$

$\Downarrow$

$x \qquad S = (\overset{\to u}{1+i, 1}), (\overset{\to v}{1, 1-i})$

$\exists \; \alpha, \beta \in \mathbb{C}$

Such that

$$x = \alpha u + \beta v$$

$$(1+i, 1-i) = \alpha(1+i, 1) + \beta(1, 1-i)$$

$$(1+i, 1-i) = (\alpha + \alpha i, \alpha) + (\beta, \beta - \beta i)$$

$$(1+i, 1-i) = (\alpha + \beta + \alpha i, \alpha + \beta - \beta i)$$

$\alpha + \beta + \alpha i = 1 + i$

$\alpha + \beta = 1$

$\beta = 1$

$\alpha + \beta = 1$

$\alpha + \beta - \beta i = 1 - i$

So it is true. $\alpha = 1 + 2i \quad \beta = 1 - i$

Q) Show that $L(S)$ is subspace of Vector $V(F)$ & $S \subseteq L(S)$

$S = \phi \to$ Linear combination is $\underline{\underline{0}}$

$\Rightarrow L(S) = \{0\} \subseteq V$

$S \neq \phi \quad \exists \; v \in S$

$v = 1 \cdot v \in L(S)$

$\downarrow$

Linear ~~comy~~ ~~comb~~ combination of v

$v \in L(S)$

$\Rightarrow S \subseteq L(S)$

Q 14) ii) $M \subset N \Rightarrow$ Proved that $L(M) \subset L(N)$

As $S \subseteq L(S)$

$\Rightarrow N \subseteq L(N) \qquad M \subseteq L(M)$

$M \subset N \subseteq L(N)$

$M \subseteq L(N)$

$L(M) \subseteq L(N)$

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ii) M is a subspace of $V \Rightarrow L(M) = M$

As $M \subseteq L(M)$

Let $x \in L(M)$

$$x = \sum_{i=1}^{n} \alpha_i u_i \quad \text{where } u_i \in M,\ \alpha_i \in F$$

$x \in M$

$L(M) \subseteq M$

$\Rightarrow \underline{L(M) = M}$

iii) $\underline{L(L(M)) = L(M)}$

Eg: $\mathbb{C} = \{x + iy \mid x, y \in \mathbb{R}\}$

is a vector space over $\mathbb{R}$

and a vector space over $\mathbb{C}$

Let $x+iy$ be ~~can~~ any general element of $\mathbb{C}$

$$x + iy = \cancel{(x+0) + iy +}\ \underset{\text{vector of } \mathbb{C}}{x \cdot 1 + i \cdot y} \quad (\in \mathbb{R},\ \in \mathbb{R})$$

(Linear combination on $\mathbb{R}$ of $(1, i)$)

$\{1, i\} \subseteq S \subseteq \mathbb{C}$

$L\{(1, i)\} = \mathbb{C}$

(Linear span $\mathbb{C}$ over $\mathbb{R}$)

$\alpha_1 \cdot 1 + \alpha_2 i = 0$ (L.I of $(1, i)$ over $\mathbb{R}$)

$\Rightarrow \alpha_1 = 0$ and $\alpha_2 = 0$

$(1, i)$ is a basis for $\mathbb{C}$ over $\mathbb{R}$

$$\boxed{\dim_{\mathbb{R}} \mathbb{C} = 2}$$

Let $x + iy \in \mathbb{C}$

$x + iy = (x+iy) \cdot 1$ (span of $\mathbb{C}$ over $\mathbb{C}$)

($x+iy$ → from field ($\mathbb{C}$); 1 → vector from $\mathbb{C}$)

$(x, y) = (x, 0) + (0, y) = x(1,0) + y(0,1)$

$L\{\{1\}\} = \mathbb{C}$ → linear independent

$$\boxed{\text{dimension}_{\mathbb{C}}(\mathbb{C}) = 1}$$

~~$x + iy = x + 0 + iy + 0$~~

$x + iy = x + 0 + iy + 0$
$= x(1+0) + y(0+i)$
$= 1 \cdot x + i \cdot y$

A subset S of V is said to be L.I

if ~~for all possible~~ we consider its all possible finite linear combination of its elements

$L(S) := \left\{ \sum_{i=1}^{k} \alpha_i u_i \;\middle|\; \alpha_i \in \mathbb{R} \text{ or } \mathbb{C} \text{ and } u_i \in S \text{ and } k \in \mathbb{R} \right\}$

and whenever

$\alpha_1 u_1 + \alpha_2 u_2 + \ldots \alpha_k u_k = 0; \forall k$

$\Rightarrow \quad \alpha_1 = \alpha_2 = \cdots\cdots \alpha_k = 0$

Ex) $\mathbb{R}^n = \{ (x_1, x_2 \ldots\ldots x_n) \mid x_i\text{'s} \in \mathbb{R} \}$

is a vector space over $\mathbb{R}$

(under usual addition and

scalar multiplication)

Take $(x_1, x_2 \ldots\ldots x_n) \in \mathbb{R}^n$

$(x_1, x_2, x_3 \ldots\ldots x_n) = (x_1, 0 \ldots\ldots 0) + (0, x_2, 0 \ldots 0)$

$+ (0, 0, x_3 \ldots\ldots 0)$

$+ \ldots\ldots (0, 0, 0 \ldots\ldots x_n)$

$\bar{x} = \underset{\alpha_1}{x_1}(1, 0, 0 \ldots\ldots 0) + \underset{\alpha_2}{x_2}(0, 1, 0 \ldots\ldots 0)$

$+ \underset{\alpha_3}{x_3}(0, 0, 1 \ldots\ldots 0)$

$+ \ldots\ldots \underset{\alpha_n}{x_n}(0, 0 \ldots\ldots 1)$

$\Rightarrow \boxed{\bar{x} = \alpha_1 v_1 + \alpha_2 v_2 + \ldots\ldots \alpha_n v_n}$

$\Rightarrow \quad L(S) = \mathbb{R}^n$

$S = \{ (1, 0 \ldots\ldots 0), (0, 1 \ldots\ldots 0) \ldots\ldots, (0, 0 \ldots\ldots 1) \}$

$\underline{S \subset \mathbb{R}^n}$

$\alpha_1 v_1 + \alpha_2 v_2 + \ldots\ldots \alpha_n v_n = 0$

$\alpha_1 (1, 0, 0 \ldots\ldots 0) + \alpha_2 (0, 1 \ldots\ldots 0)$

$+ \ldots\ldots \alpha_n (0, 0 \ldots\ldots 1) = 0$

$(\alpha_1, \alpha_2 \dots \alpha_n) = (0, 0, 0, \dots 0)$

$\Rightarrow \alpha_1 = \alpha_2 = \alpha_3 = \dots \alpha_n = 0$

$\Rightarrow S$ is a basis for $\mathbb{R}^n$ over $\mathbb{R}$

$$\boxed{\text{dimen}_{\mathbb{R}}(\mathbb{R}^n) = n}$$

1) $\mathbb{C}^n = \{ (z_1, z_2 \dots z_n) / z_i \in \mathbb{R},\ i = 1 \text{ to } n \}$ is a vector space over $\mathbb{C}$

2) $M_2(\mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle/ a, b, c, d \in \mathbb{R} \right\}$ is a vector space over $\mathbb{R}$

3) $P_2 = \{ a + bx + cx^2 / a, b, c \in \mathbb{R} \}$ is a vector space over $\mathbb{R}$

$(z_1, z_2 \dots z_n)$

$$M := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{R}) \;\middle|\; a+d = 0 \right\} \subseteq M_2(\mathbb{R})$$

$\Downarrow$ 2×2 matrices of $\mathbb{R}$

- M is a subspace of $M_2(\mathbb{R})$
  ↳ clear visualize.

- basis of M

i.e

some $S \subseteq M$

$\exists$

i) $L(S) = M$

and ii) S is linear independent.

Basis for $M_2(\mathbb{R})$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{R})$$

$$\begin{pmatrix} a & 0 \\ c & d \end{pmatrix} + \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \in M_2(\mathbb{R})$$

$$\cancel{\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} + \begin{pmatrix} a & 0 \\ c & d \end{pmatrix}}$$

$$\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ c & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & d \end{pmatrix} \in M_2(\mathbb{R})$$

$$a\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + b\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + c\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + d\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \in M_2(\mathbb{R})$$

$a v_1 + b v_2 + c v_3 + d v_4 \in M_2(R)$

$M_2(R) \subseteq L(v_1, v_2, v_3, v_4)$

$M_2(R) \supseteq L(S)$

$$\boxed{L(v_1, v_2, v_3, v_4) = M_2(R)}$$

Consider

$$\alpha_1\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix} + \alpha_2\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix} + \alpha_3\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix} + \alpha_4\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix} = 0$$

$$\begin{pmatrix}\alpha_1 & 0\\ 0 & 0\end{pmatrix} + \begin{pmatrix}0 & \alpha_2\\ 0 & 0\end{pmatrix} + \begin{pmatrix}0 & 0\\ \alpha_3 & 0\end{pmatrix} + \begin{pmatrix}0 & 0\\ 0 & \alpha_4\end{pmatrix} = \begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix}$$

$$\begin{pmatrix}\alpha_1 & \alpha_2\\ \alpha_3 & \alpha_4\end{pmatrix} = \begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix}$$

$$\Rightarrow \alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 0$$

So $M_2(R)$ have basis $\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix} \begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix} \begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix} \begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}$

$\Rightarrow$ dimension of $M_2(R) = 4$

~~for M~~

Take $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M$ therefore $a+d=0$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{R}) = \begin{pmatrix} a & b \\ c & -a \end{pmatrix}$$ condition for $(a+d=0)$

$$\begin{pmatrix} a & b \\ c & -a \end{pmatrix} = a\begin{pmatrix} 1 & b \\ c & -1 \end{pmatrix}$$

$$= a\left[\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ c & 0 \end{pmatrix} + \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix}\right]$$

$$= a\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ c & 0 \end{pmatrix} + \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix}$$

$$= a\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + c\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + b\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$= a v_1 + c v_3 + b v_4$$

$$\alpha_1 \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \alpha_2 \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + \alpha_3 \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = 0$$

$$\begin{pmatrix} \alpha_1 & 0 \\ 0 & -\alpha_1 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ \alpha_2 & 0 \end{pmatrix} + \begin{pmatrix} 0 & \alpha_3 \\ 0 & 0 \end{pmatrix} = 0$$

$$\begin{pmatrix} \alpha_1 & \alpha_3 \\ \alpha_2 & -\alpha_1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$\boxed{\dim(M) = 3}$ $\qquad \alpha_1 = \alpha_2 = \alpha_3 = 0$

General Linear system

$$\left.\begin{array}{l} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \vdots \qquad \vdots \qquad \qquad \vdots \qquad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{array}\right\}$$ System of n variables and m equations

$$x_1\begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{pmatrix} + x_2\begin{pmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{pmatrix} + \cdots x_n\begin{pmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ \vdots \\ b_m \end{pmatrix} \quad \text{—(I)}$$

$$\Rightarrow \boxed{\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & \vdots & \text{-----} & \vdots \\ \vdots & \vdots & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}}$$

eq^n (I) is equivalent to saying for a vector $(b_1, b_2, \ldots, b_m) \in \mathbb{R}^m$, if there exist $x_1, x_2, \ldots, x_n \in \mathbb{R}$ $\exists$

$$(b_1, b_2, \ldots, b_m) = x_1(a_{11}, a_{21}, \ldots, a_{m1}) + x_2(a_{12}, a_{22}, \ldots, a_{m2}) + \cdots + x_n(a_{1n}, a_{2n}, \ldots, a_{mn})$$

$$(b_1, b_2 \cdots b_m) = x_1 (a_{11}, a_{21} \cdots a_{m1})$$

$$+ x_2 (a_{12}, a_{22} + \cdots a_{m2})$$

$$+ x_n (a_{1n}, a_{2n} \cdots a_{mn}) v_m$$

$$(b_1, b_2 \cdots b_n) \in L\{(v_1, v_2 \cdots v_n)\}$$

$\boxed{\text{if } x_1 \cdots x_n \text{ exist then } (v_1, v_2, \cdots v_n) \text{ span the } (b_1, b_2 \cdots b_m) \in \mathbb{R}^m}$

$\Downarrow$

asking a $x_1 \cdots x_n$ means whether

$(v_1, v_2 \cdots v_n)$ is a span of

$(b_1, b_2 \cdots b_m) \in \mathbb{R}^m$

For a given $m \times n$ matrix $A_{m \times n}$

$$C(A) := \{\text{column space of } A_{m \times n}\}$$

$$C(A) := \{\alpha_1 (a_{11}, a_{21} \cdots a_{m1}) + \alpha_2 (a_{12}, a_{22} \cdots a_{m2})$$

$$\cdots \alpha_n (a_{1n}, a_{2n} \cdots a_{mn}) \;/\; \alpha_i\text{'s} \in \mathbb{R},\ i = 1 \text{ to } n\}$$

$C(A') = \{$ Linear Span of column vector of $A\}$

$C(A) = L(\{$ set of column vector$\})$

$C(A) \subseteq A_{m \times n}$

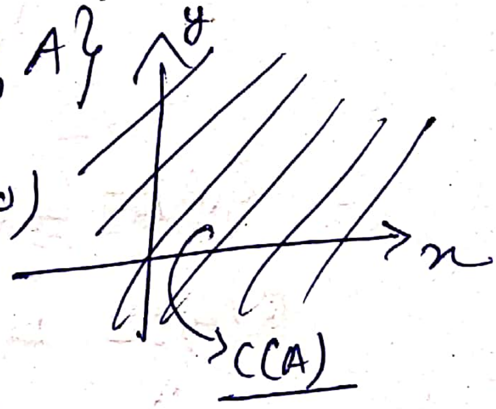
and also

$C(A)$ is a subspace of $A_{m \times n}$ matrix.

Eg)

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}_{3 \times 2}$$

$C(A) := \{$ column space of $A\}$



$C(A) := x(1,0,0) + y(0,1,0)$

$= (x, y, 0) \in \mathbb{R}^3$

$C(A) = \{(x, y, 0) \mid x, y \in \mathbb{R}\}$

$\Rightarrow$ $C(A) \subseteq \mathbb{R}^3$ and $C(A)$ is subspace of $\mathbb{R}^3$

any value of $(x, y, z) \notin C(A)$ is not a soln of this system.

Some Important subspace

For a given $m \times n$ matrix $A_{m \times n}$, $A^T_{n \times m}$ we can define the following subspaces:

$C(A)$ — column space of A

$R(A)$ — row space of A

$N(A^T)$ — null space of the matrix A^T

$N(A)$ — null space of A

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{pmatrix}_{3 \times 2}$$

$C(A) := L(\{(a_{11}, a_{21}, a_{31}), (a_{12}, a_{22}, a_{32})\})$ is a subspace of $\mathbb{R}^3$

$R(A) := L(\{(a_{11}, a_{12}), (a_{21}, a_{22}), (a_{31}, a_{32})\})$

subspace of $\mathbb{R}^2$

$N(A) = \{(x_1, x_2) \in \mathbb{R}^2 \,/\, \underset{3\times 2 \times 1}{A\bar{x}} = \bar{0}_{3 \times 1}\}$

is a subspace of $\mathbb{R}^2$

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Suppose $(x_1, x_2), (y_1, y_2) \in N(A)$

$$(x_1 + x_2) + (y_1, y_2) = (x_1 + y_1, x_2 + y_2) \in N(A)$$

$$A \cdot \begin{pmatrix} x_1 + y_1 \\ x_2 + y_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$ check wheather it is true or not

$c \in \mathbb{R}$

$$A \cdot \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$N(A^T) := \{ (y_1, y_2, y_3) \in \mathbb{R}^3 \mid A^T_{2\times 3} y_{3\times 1} = 0_{2\times 1} \}$$

$\hookrightarrow$ subspace of $\mathbb{R}^3$

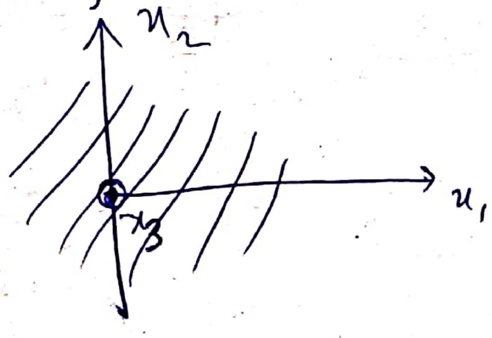
Ex)
$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}_{3\times 2}$$

$$C(A) = L(\{(1,0,0), (0,1,0)\})$$

$$C(A) = \{ x_1(1,0,0) + x_2(0,1,0) \mid x_1, x_2 \in \mathbb{R} \}$$

$$C(A) = \{ (x_1, x_2, 0) \mid x_1, x_2 \in \mathbb{R} \}$$

$= x_1 x_2$-plane



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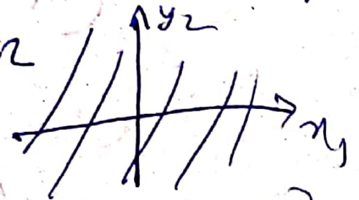
linearly dependent

$R(A) = L(\{(1,0),(0,1),(0,0)\})$

are linearly dependent

$= \{y_1(1,0) + y_2(0,1) + y_3(0,0) \mid y_1, y_2, y_3 \in \mathbb{R}\}$

$= \{(y_1, y_2) \mid y_1, y_2 \in \mathbb{R}\} = \mathbb{R}^2$



$$N(A) = \left\{ (y_1, y_2) \in \mathbb{R}^2 \;\middle|\; \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right\}$$

$= \{(y_1, y_2) \in \mathbb{R}^2 \mid y_1 = 0, y_2 = 0\}$

$= \{(0,0)\}$ is subspace of $\mathbb{R}^2$

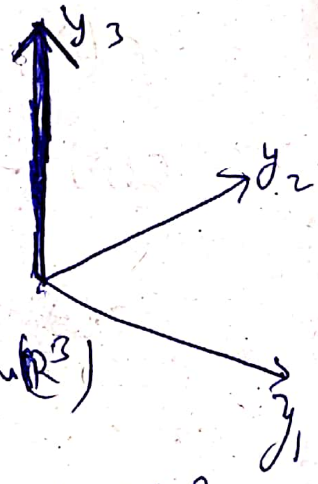
$$N(A^T) = \left\{ (y_1, y_2, y_3) \in \mathbb{R}^3 \;\middle|\; \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\}$$

$= \{(y_1, y_2, y_3) \in \mathbb{R}^3 \mid y_1 = 0, y_2 = 0\}$

$= \{(0, 0, y_3) \mid y_3 \in \mathbb{R}$

$= \{y_3(0,0,1) \mid y_3 \in \mathbb{R}\}$

$= L(\{(0,0,1)\})$



$\left.\begin{array}{l} \dim C(A) = 2 \\ \dim N(A^T) = 1 \end{array}\right\} 3 = \dim(\mathbb{R}^3)$

$\left.\begin{array}{l} \dim \text{ of } R(A) = 2 \\ \dim N(A) = 0 \end{array}\right\} 2 = \dim(\mathbb{R}^2)$

Note :- for $A_{m \times n}$ matrix

$$\dim C(A) + \dim N(A^T) = \dim R^m$$
$$= m$$

$$\dim R(A) + \dim N(A) = \dim R^n$$
$$= n$$

Reduced raw-echelon form (of a matrix)

we say a matrix A is in reduced raw echelon form if it satisfies the following properties:

1) if there is a non-zero entries in ~~one or~~ a raw, then the first nonzero is 1 (known as leading 1 or pivot) of this ~~raw~~ raw.

2) if column contains a leading 1, then rest of the entries in that column are 0.

3) if a raw contains a leading 1, then every raw above it contains a leading 1 to left. (this says that if there are raws with 0's only then they will appear at the bottom of the matrix).

$$A = \begin{pmatrix} 1 & -1 & 0 & 0 & 4 & 2 \\ 0 & 0 & 1 & 0 & -1 & 2 \\ 0 & 0 & 0 & 1 & -1 & 3 \end{pmatrix}_{3\times 8}$$

(ii) (iii) (i)

Yes, it A is RREF?

$A = (0, 1, 2, 3, 4)_{1\times 4}$

$$A = \begin{pmatrix} 1 & 2 & 0 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 \end{pmatrix}_{3\times 4}$$

$\hookrightarrow$ No (iii)

$T(x_1, x_2) = (-x_2, x_1)$

$T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ it is linear map

$$T[(x_1, x_2)] = T[(x_1, 0) + (0, x_2)]$$
$$= T[x_1(1,0) + x_2(0,1)]$$
$$= x_1 T(1,0) + x_2 T(0,1)$$

$T(1,0) = (0,1)$ $\quad T(0,1) = (-1,0)$

$\cancel{T = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}}$

$A(T(1,0), T(0,1))$

$\boxed{A(T(\text{vector of set } b), T(\text{vector of set } b), \ldots T(\text{vector of set } b))}$

$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

$$Ax = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -x_2 \\ x_1 \end{pmatrix}$$

$\boxed{\text{dim of } \mathbb{R}^n \text{ (domain)} = \text{no of column of } A}$

A → corresponding matrix

Linear Independent, Basis, RREF

Ex. Can we have two example of vector space having the same dimension

$\dim_{\mathbb{R}}(\mathbb{R}^2) = 2$ $\quad \dim_{\mathbb{R}}(P_1) = 2$

S is a basis for V.

$L(S) = V$

S is linearly independent

A basis for $\mathbb{R}^n$

Standard basis

$\{(1,0,0 \ldots\ldots 0), (0,1,0 \ldots 0) \ldots (0,0 \ldots 1)\}$

for $\mathbb{R}^n$

$(x_1, x_2 \ldots x_n) = (x_1, 0 \ldots 0) + (0, x_2 \ldots 0)$

$+ \ldots + (0, 0 \ldots x_n)$

$= x_1(1,0 \ldots 0) + x_2(0,1 \ldots 0)$

$+ \ldots x_n(0,0 \ldots 1)$

$P = \{$ set of all polynomial over $\mathbb{R}\}$

$P(x) = a_0 + a_1 x + \ldots\ldots a_n x^n + \ldots$

Basis for P_2 (n is not fixed)

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Let

$v_1, v_2 \dots v_n$ be n-vectors in $\mathbb{R}^m$

Consider

$$A = \begin{pmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{pmatrix}_{m \times n}$$

$$\Downarrow$$

$$RREF \rightarrow \begin{pmatrix} & & \\ & & \\ & & \end{pmatrix}$$

if $Rank(A) = n \Rightarrow \{v_1, v_2 \dots v_n\}$ is linear independent

$rank(A) < n \hookrightarrow$ is linear dependent

$$A = \begin{pmatrix} 1 & 2 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{pmatrix}$$

or $\{(1,1,1), (2,2,2), (1,2,3)\} \in \mathbb{R}^3$ RREF

$$A = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$Rank(A) = 2$

$Rank(A) < 3$ so linear dependent

Coordinate of a vector

$(1,2,3)$ (w.r.t standard basis)

$(1,2,3) = 1(1,0,0) + 2(0,1,0) + 3(0,0,1)$

is $\underline{1,2,3}$

$(1,2,-1)$ w.r.t $\{(2,1,0)\ (2,1,1)\ (2,2,1)\}$

$(1,2,-1) = \alpha_1(2,1,0) + \alpha_2(2,1,1) + \alpha_3(2,2,1)$.

$\alpha_1, \alpha_2, \alpha_3$ are coordinate of $(1,2,-1)$ w.r.t to the given basis

$x_1+x_2+2x_3=0$, $2x_2+x_3=0$

$x_1-x_2+x_3=0$.

$x_1+x_2+2x_3+0\cdot x_4 + 2x_1+2x_2+x_3+0\cdot x_4$

$+ x_1-x_2+x_3+0\cdot x_4$

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The no of solutions of a linear system $A\bar{x} = b$

$$2x_1 + 4x_2 - 2x_3 + 2x_4 + 4x_5 = 2$$
$$x_1 + 2x_2 - x_3 + 2x_4 + 0 \cdot x_5 = 4$$
$$3x_1 + 6x_2 - 2x_3 + x_4 + 9x_5 = 1$$
$$5x_1 + 10x_2 - 4x_3 + 5x_4 + 9x_5 = 9$$

or

$$\left(\begin{array}{ccccc|c} 2 & +4 & -2 & 2 & 4 & 2 \\ 1 & 2 & -1 & 2 & 0 & 4 \\ 3 & 6 & -2 & 1 & 9 & 1 \\ 5 & 10 & -4 & 5 & 9 & 9 \end{array}\right)$$

$R_1 \to \frac{1}{2} R_1$

$\downarrow$

$R_2 \to R_2 - R_1$ and $R_3 \to R_3 - 3R_1$

~~$R_3 \to R$~~ and $R_4 \to R_4 - 5R_1$

$\downarrow$

$R_1 \to R_1 + R_2$, $R_4 \to R_4 - R_2$

$\downarrow$

$R_1 \to R_1 + R_3$, $R_2 \to R_2 + R_3$, $R_4 \to R_4 - 2R_3$

$$\sim \left(\begin{array}{ccccc|c} 1 & 2 & 0 & 0 & 3 & 2 \\ 0 & 0 & 1 & 0 & -1 & 4 \\ 0 & 0 & 0 & 1 & -2 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$\Downarrow$ RREF (Agumented matrix)

free variable

$$x_1 + 2x_2 + \cdots\cdots 3x_5 = 2$$
$$+x_3 \quad -x_5 = 4$$
$$x_4 \quad -2x_4 = 3$$
$$0 = 0$$

(non free variable)
(leading variable)

$x_1 = 2 - 2t_1 - 3t_2$
$x_2 = t_1$
$x_3 = 4 + t_2$
$x_4 = 3 + 2t_2$
$x_5 = t_2$

This system has infinite many solution.

a)
$$\left(\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right) \Rightarrow \text{no sol}^n/\text{inconsistent}$$

$x_1 + 2x_2 = 0$
$x_3 = 0$
$\boxed{0 = 1} \rightarrow$ not possible
$0 = 0$

a)
$$\left(\begin{array}{ccc|c} 1 & 2 & 0 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$x_1 + 2x_2 = 1$
$x_3 = 2$
$0 = 0$
consistent

$\Downarrow$ infinite many sol^n

a)
$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right)$$

consistent

$\Downarrow$

unique sol^n

Rank of a matrix $A_{m \times n}$

It is the number of leading ~~no~~ one's (or pivot) in Raw Reduced Echelon matrix of A.

$$\left(\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$\text{Rank}(A) = 2$ $\quad$ $\text{Rank}(\overset{\circ}{A}|b) = 3$

$$\left(\begin{array}{ccc|c} 1 & 2 & 0 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$\text{Rank}(A) = 2$ $\quad$ $\text{Rank}(\overset{\circ}{A}|b) = 2$

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right)$$

~~Rank~~

$\text{Rank}(A) = 3$ $\quad$ $\text{Rank}(\overset{\circ}{A}|b) = 3$

Consistent system

☆ whenever rank $(A) =$ rank $(A|\bar{b})$

$\Rightarrow$ the system $A\bar{x} = \bar{b}$ has a sol^n.

Inconsistent system

☆ If rank $(A) \neq$ rank $(A|\bar{b})$, then ~~the~~ system has no solution.

Linear Transformation of vector Spaces $(T: \mathbb{R}^n \to \mathbb{R}^m)$

$$A_{m\times n} \vec{x}_{n\times 1} = \vec{b}_{m\times 1}$$ (general Linear system)

Let $A_{2\times 2} \bar{x}_{2\times 1} = \bar{b}_{2\times 1}$

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$$

fixed (i.e. $a_{11}, a_{12}, a_{21}, a_{22}$ are given)

Function over $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ give $\begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$

is like a

i.e $\underset{2\times 2}{A} : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$

for $A_{m\times n}\bar{x}_{n\times 1} = \bar{b}_{m\times 1}$

$A_{m\times n} : \mathbb{R}^n \longrightarrow \mathbb{R}^m$

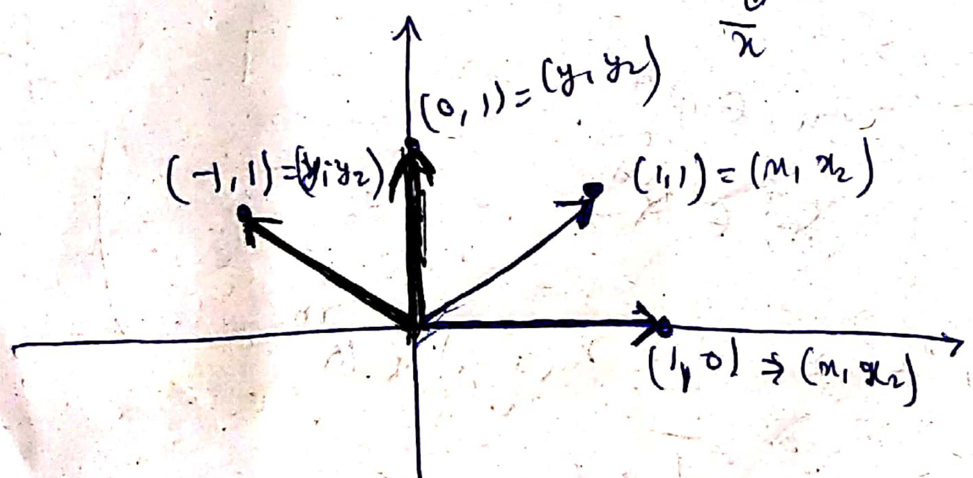
$\uparrow$ $(x_1, x_2 \ldots\ldots x_n)$; $\downarrow$ $(b_1, b_2 \ldots\ldots b_m)$

$\Rightarrow$ Any matrix of order $m\times n$ gives us a map from $\mathbb{R}^n \to \mathbb{R}^m$

Ex) $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}_{2\times 2} \longrightarrow$ Rotation

$A : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$

$\underset{\bar{x}}{(x_1, x_2)} \longrightarrow \underset{\bar{y}}{(y_1 \quad y_2)} = A\bar{x}$



$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

$\hookrightarrow$ which result the rotation of vector by 90°

$\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \rightarrow$ increase the vector length not ~~the~~ change the direction of vector

$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \rightarrow$ donat change anything

$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \rightarrow$ Reflection w.r.t $(y=x)$

Linear transformation of vector spaces

$$A_{m\times n} : \mathbb{R}^n \longrightarrow \mathbb{R}^m$$

$$\underbrace{(0,0,0 \ldots 0)}_{n \text{ terms}} \longrightarrow \underbrace{(0,0, \ldots 0)}_{m \text{ terms}}$$

$(x_1, x_2 \ldots \ldots x_n) \in \mathbb{R}^n$

$(y_1, y_2 \ldots \ldots y_n) \in \mathbb{R}^n$

$\underbrace{(x_1+y_1, x_2+y_2 \ldots \ldots x_n+y_n)}_{\bar{x}+\bar{y}} \in \mathbb{R}^n$

☆ $A(0_{m\times 1}) = 0_{m\times 1}$

☆ $A(\bar{x}+\bar{y}) = A\bar{x} + A\bar{y}$

★ $A(c\bar{x}) = cA(\bar{x})$
If this hold then $A_{m\times n}$ is Linear transformation

Q) $T: \mathbb{R}^n \longrightarrow \mathbb{R}^m$

Ans) Any Linear transformation from $\mathbb{R}^n \longrightarrow \mathbb{R}^m$ corresponds to matrix $A_{m\times n}$

$$\boxed{T(\bar{x}) = A\bar{x}}$$

Ex) $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$

$T: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$

$T(x_1, x_2) = (x_1, x_2)$

1) $T(0,0) = (0,0)$

2) $T((x_1, x_2) + (y_1, y_2)) = T(x_1, x_2) + T(y_1, y_2)$

$T(x_1 + y_1, x_2 + y_2) = (x_1 + y_1, x_2 + y_2)$
$\Downarrow$
$\mathbb{R}^2$
$= (x_1, x_2) + (y_1, y_2)$
$= T(x_1, x_2) + T(y_1, y_2)$

3) $T(c(x_1, x_2)) = T(cx_1, cx_2)$
$= (cx_1, cx_2)$
$= c(x_1, x_2)$
$= cT(x_1, x_2)$

So T is a Linear transformation of $\mathbb{R}^n \cdot \mathbb{R}^m$

Standard basis of $\mathbb{R}^2$

$$\{(1,0)\ (0,1)\}$$

$$\cancel{T(x_1, x_2)}\ (x_1, x_2)$$

$$= (x_1, 0) + (0, x_2)$$

$$= x_1(1,0) + x_2(0,1)$$

$$T(x_1, x_2) = T\big(x_1(1,0) + x_2(0,1)\big)$$

$$= T(x_1(1,0)) + T(x_2(0,1)) \rightarrow \text{Linear Transformation Properties}$$

$$= x_1 \underbrace{T(1,0)}_{(1,0)} + x_2 \underbrace{T(0,1)}_{(0,1)}$$

$$T\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}1\\0\end{pmatrix}_{2\times 1} \qquad \begin{pmatrix}0\\1\end{pmatrix}_{2\times 1} = T\begin{pmatrix}0\\1\end{pmatrix}$$

all different are combin

$$T = \begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}$$

$$T\begin{pmatrix}x_1\\x_2\end{pmatrix} = \begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}\begin{pmatrix}x_1\\x_2\end{pmatrix}$$

$$\boxed{A\begin{pmatrix}x_1\\x_2\end{pmatrix} = \begin{pmatrix}x_1\\x_2\end{pmatrix}}$$