Conditional PMF & Expectation

Friday, February 5, 2021 8:58 AM

$$Z = g(X,Y) \qquad p_Z(z) = \sum_{\{x,y \mid g(x,y)=z\}} p_{X,Y}(x,y)$$

$$E[Z] = \sum_z z\, p_Z(z) = \sum_z z \left(\sum_{\{x,y \mid g(x,y)=z\}} p_{X,Y}(x,y) \right)$$

$$= \underbrace{\sum_z \sum_{(x,y) \mid g(x,y)=z}} g(x,y)\, p_{X,Y}(x,y)$$

$$\boxed{E[g(X,Y)] = \sum_{x,y} g(x,y)\, p_{X,Y}(x,y)}$$

$$Z = aX + bY + c$$

$$E[Z] = \sum_{(x,y)} (ax + by + c)\, p_{X,Y}(x,y)$$

$$= \sum_{x\,y} ax\, p_{X,Y}(x,y) + \sum_{x\,y} by\, p_{X,Y}(x,y) + \sum_{x\,y} c \cdot p_{X,Y}(x,y)$$

$$= a \sum_x x \underbrace{\sum_y p_{X,Y}(x,y)} + b \sum_y y \underbrace{\sum_x p_{X,Y}(x,y)} + c \underbrace{\sum_{x\,y} p_{X,Y}(x,y)}$$

$$= a \sum_x x\, p_X(x) + b \sum_y y\, p_Y(y) + c$$

$$= a\,E[X] + b\,E[Y] + c$$

$$X = X_1 + X_2 + \cdots + X_n$$

$$E[X] = \sum_{i=1}^{n} E[X_i] \longrightarrow \text{linearity of expectation.}$$

$$E[X] = \sum_{i=1} E[X_i] \longrightarrow \text{linearity of expectation.}$$

Binomial Random Variable

$$P(X=k) = \binom{n}{k} p^k (1-p)^{n-k} \qquad k = 0, 1, \cdots, n$$

$$E[X] = n \cdot p$$

$$\{ \underset{1st}{\_\_} \; \underset{2nd}{\_\_} \; \_\_ \; \_\_ \; \cdots \; \underset{n\text{-th}}{\_\_} \} \longrightarrow X = \sum_{i=1}^{n} X_i$$

$X_1 \quad X_2 \quad X_3 \qquad\qquad X_n \longrightarrow 0/1$ r.vs

$$X_i = \begin{cases} 1 & \text{if } H \\ 0 & \text{if } T \end{cases}$$

$$E[X] = \sum_{i=1}^{n} E[X_i] = \sum_{i=1}^{n} p = np$$

Conditioning a r.v. on an event A

"A has happed"

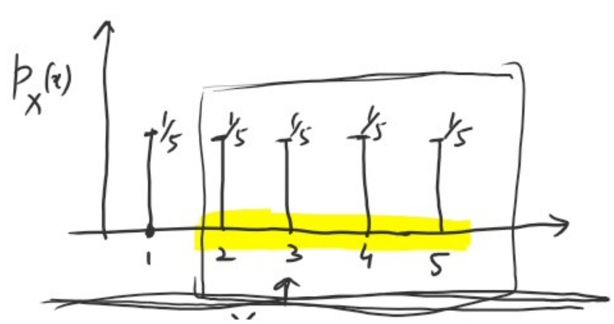
$$p_X(x) = P(X=x)$$

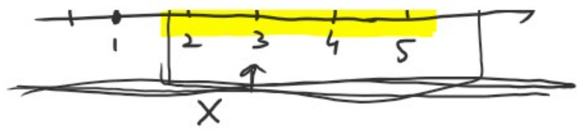
$$p_{X|A}(x) = P(X=x \mid A) = \frac{P(\{X=x\} \cap A)}{P(A)}$$

$\uparrow$

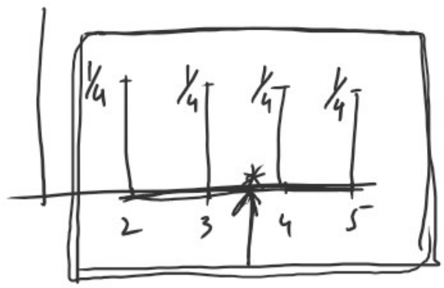
conditional PMF of X

$A = X \geq 2$





$p_{X|A}(x)$ =



$$E[X|A] = \frac{1}{4}\cdot 2 + \frac{1}{4}\, 3 + \frac{1}{4}\cdot 4 + \frac{1}{4}\cdot 5 = 3.5$$

Conditioning a r.v. on other r.v.

y \ x	0	1	2	3
2	0	1/16	2/16	1/16
1	1/16	3/16	3/16	1/16
0	1/16	2/16	1/16	0

$$p_{X|Y}(x,y) = P(X=x \mid Y=y)$$

$Y=2$

$0 \quad \frac{1}{4} \quad \frac{2}{4} \quad \frac{1}{4} \longrightarrow$ adds to 1

$p_X(x|Y=2)$

$$p_{X|Y}(x,y) = P(X=x \mid Y=y) = \frac{P(X=x \cap Y=y)}{P(Y=y)} = \frac{p_{X,Y}(x,y)}{p_Y(y)}$$

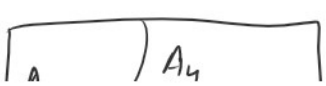
$$\Rightarrow \quad \boxed{p_{X,Y}(x,y) = p_Y(y)\cdot p_{X|Y}(x,y)}$$

$$P(A\cap B) = P(A)\cdot P(B|A)$$

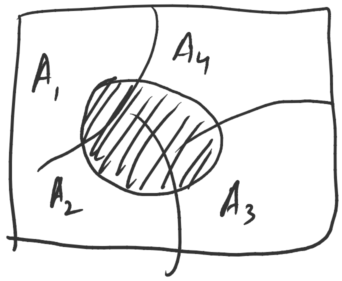
$(Y=y \cap X=x)$ $\quad (Y=y)$ $\quad (X=x|Y=y)$

$$\boxed{p_{X,Y}(x,y) = p_X(x)\cdot p_{Y|X}(y|x)}$$

$Y\in\{y_1, y_2 \cdots y_n\}$ $\quad A_i \equiv \{Y=y_i\}$



$P(B) = P(A)\,P(B|A) + P(A)\,P(B|A)$



$Y \in \{y_1, y_2 \cdots y_n\}$

$$P(B) = P(A_1) \cdot P(B|A_1) + P(A_2) \cdot P(B|A_2) \cdots + P(A_n) \cdot P(B|A_n)$$

$$p_X(x) = p_Y(y_1) \cdot p_{X|Y}(x|y_1) + \cdots + p_Y(y_n) \cdot p_{X|Y}(x|y_n)$$

$$\Rightarrow \boxed{p_X(x) = \sum_y p_Y(y) \cdot p_{X|Y}(x|y)}$$

"Total prob law"

$$E[X|A] = \sum_x x \, p_{X|A}(x)$$