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Lecture 5.

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Complex sequences: A complex sequence is a function from $\mathbb{N}$: set of natural nos. to the $\mathbb{C}$: set of complex numbers.

$$f : \mathbb{N} \longrightarrow \mathbb{C}, \quad f(n) = z_n.$$

Notation: we write $(z_n)_{n \in \mathbb{N}}$ to denote a sequence.

Eg: $(z_n) = \left(1 + \frac{i}{n}\right)$,

Def^n I We say $\underline{\lim_{n\to\infty} z_n = z}$ or that the sequence $\underline{z_n \text{ converges to } z}$ or $\underline{z_n \to z}$

if there exists a number n_0 s.t.

$$|z_n - z| < \varepsilon \quad \forall \ n \geq n_0.$$

(depends on ε)

In other words, $\forall \ n \geq n_0$ z_n lies in $B_\varepsilon(z_0)$.

II If $\lim_{n\to\infty} z_n$ does not exist, then we say that $\underline{(z_n) \text{ is divergent}}$.

Eg :- $\left(z_n = 1 + \frac{i}{n}\right)_{n\in\mathbb{N}}$ is a convergent seq.

$\left(z_n = 1 + i\cdot n\right)_{n\in\mathbb{N}}$ is divergent.

Facts: (1) limit of sequence (if it exists) is unique

(2) Let $z_n = x_n + i y_n$ & $z = x + iy$

Then $\lim_{n\to\infty} z_n = z$ iff $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} y_n = y$ } real sequen-ces

(3) If $\lim_{n\to\infty} z_n = z$, then $\lim_{n\to\infty} |z_n| = |z|$

(4) (Algebra rules) [using (2) & real sequences]

(i) $\lim_{n\to\infty} (z_n \pm z_n') = \lim_{n\to\infty} z_n \pm \lim_{n\to\infty} z_n'$

(ii) $\lim_{n\to\infty} \overline{z_n} = \overline{\lim_{n\to\infty} z_n}$.

(5) Suppose $\lim_{n\to\infty} z_n = z$.

- Then $\exists\ M > 0$ s.t. $|z_n| \le M \ \forall\ n$.

In other words, $(z_n)_{n\in\mathbb{N}}$ is bounded sequence

Saloni

Proof: ① Suppose $\lim_{n\to\infty} z_n = z_1$ & $\lim_{n\to\infty} z_n = z_2$

Claim 1: $z_1 = z_2$.

if not, take $\varepsilon = |z_1 - z_2| > 0$

Since $z_n \to z_1$,

$\exists\, n_0 \in \mathbb{N}$ s.t. $|z_n - z_1| < \varepsilon/2 \quad \forall n \geq n_0$

Since $z_n \to z_2$,

$\exists\, n_1 \in \mathbb{N}$ s.t. $|z_n - z_2| < \varepsilon/2 \quad \forall n \geq n_1$

Then

$$\varepsilon = |z_1 - z_2| = |z_1 - z_n + z_n - z_2|$$
$$\leq |z_1 - z_n| + |z_n - z_2|$$
$$< \varepsilon/2 + \varepsilon/2$$

$\forall n > \max\{n_1, n_2\}$

$\Rightarrow \varepsilon < \varepsilon$ which is a contradiction

So $\varepsilon = 0$ i.e. $z_1 = z_2$.

② Suppose $\lim_{n\to\infty} z_n = z$

To prove $\lim_{n\to\infty} x_n = x$ & $\lim_{n\to\infty} y_n = y$

Solⁿ:

Let $\varepsilon > 0$ be arbitrary.

[Want $n_0 > 0$ s.t. $|x_n - x| < \varepsilon \ \forall n \geq n_0$]

We have $\exists \, n_0 > 0$ s.t. $|z_n - z| < \varepsilon \ \forall n \geq n_0$

Note $|x_n - x| \leq |z_n - z| < \varepsilon$
$|y_n - y| \leq |z_n - z| < \varepsilon$ } when $n \geq n_0$

So same n_0 works for both the real sequences (x_n) & (y_n).

Conversely given that $x_n \to x$ & $y_n \to x$

To show $z_n \to z = x + iy$

Let $\varepsilon > 0$

[Want $n_0 > 0$ s.t. $|z_n - z| < \varepsilon \ \forall n > n_0$]

We have $\exists \, n_1 > 0$ s.t. $|x_n - x| < \varepsilon/2 \ \forall n \geq n_1$

$\exists \, n_2 > 0$ s.t. $|y_n - y| < \varepsilon/2 \ \forall n \geq n_2$

Choose $n_0 = \max\{n_1, n_2\}$

Then $|z_n - z| = |x_n + iy_n - (x + iy)|$

$= |x_n - x + i(y_n - y)|$

$\leq |x_n - x| + |y_n - y| < \underbrace{\varepsilon/2 + \varepsilon/2}_{= \varepsilon}$

when $n \geq n_0$

So, $|z_n - z| < \varepsilon \quad \forall\, n \geq n_0$

$\Rightarrow \quad z_n \longrightarrow z$.

(3) Given that $\lim_{n\to\infty} z_n = z$.

To show $|z_n| = |z|$

Let $\varepsilon > 0$. Then $\exists\, n_0 > 0$ s.t. $|z_n - z| < \varepsilon$
$\forall\, n \geq n_0$.

Now $\big||z_n| - |z|\big| \leq |z_n - z|$

$< \varepsilon \quad \forall\, n \geq n_0$.

Thus $z_n \longrightarrow z$.

(4) $\lim_{n\to\infty} \underset{x_n+iy_n}{z_n} = \underset{x+iy}{z}$ & $\lim_{n\to\infty} \underset{x_n'+iy_n'}{z_n'} = \underset{x'+iy'}{z'}$

$\Updownarrow$ by (2) $\qquad\qquad$ $\Updownarrow$ by (2)

$\lim_{n\to\infty} x_n = x$ & $\qquad\qquad$ $\lim_{n\to\infty} x_n' = x'$ &

$\lim_{n\to\infty} y_n = y$ $\qquad\qquad$ $\lim_{n\to\infty} y_n' = y'$

Using the theory of real sequences

$x_n \pm x_n' \longrightarrow x \pm x'$ & $y_n \pm y_n' = y \pm y'$

Again using (2),

$(x_n \pm x_n') + i(y_n \pm y_n') \longrightarrow (x \pm x') + i(y \pm y')$

$\underline{\text{L.H.S}}$ is $(z_n \pm z_n')$ & R.H.S. is $(z \pm z')$

⑤ Let $\varepsilon > 0$.

There exists $n_0 > 0$ s.t. $|z_n - z| < \varepsilon \;\forall n \geq n_0$

Now
$$|z_n| = |z_n - z + z|$$
$$\leq |z_n - z| + |z| \quad \forall n \geq n_0$$
$$< \underbrace{\varepsilon + |z|}_{= M_0 \text{ (a constant)}}$$

Take $M = \max\{M_0, |z_1|, \ldots |z_{n_0-1}|\}$

Then $|z_n| \leq M \;\forall n.$

§ Series

For an infinite series $\sum_{n=1}^{\infty} z_n$ — formal notation

define $\underline{\text{partial sums}}$ $S_N = \sum_{n=1}^{N} z_n$
$$= z_1 + \cdots + z_N$$

$\underline{\text{Eg}}$ $S_1 = z_1$

$S_2 = z_1 + z_2$, $S_3 = z_1 + z_2 + z_3$, ...

$\underline{\text{So}}$ we have a sequence of partial sums.

Def^n :- We say that the series $\sum_{n=1}^{\infty} z_n$ converges to a number S if the sequence of partial sums $S_N \to S$.

Then we write $\sum_{n=1}^{\infty} z_n = S$.

Facts ① Suppose $z_n = x_n + i y_n$ $\forall n$ and $S = x + iy$.

Then $\sum_{n=1}^{\infty} z_n = S$ iff $\sum_{n=1}^{\infty} x_n = x$

$\sum_{n=1}^{\infty} y_n = y$

Proof :- (Tutorial-5).

② $\sum_{n=1}^{\infty} z_n$ is a convergent series, then

$\lim_{n\to\infty} z_n = 0$ and thus $(z_n)_{n\in\mathbb{N}}$ is bounded.

Proof :- use ①, $\sum_{n=1}^{\infty} z_n$ is convergent

$\Rightarrow \sum_{n=1}^{\infty} x_n$ and $\sum_{n=1}^{\infty} y_n$ are convergent (real) series

$\Rightarrow x_n \to 0$ & $y_n \to 0$

$$\Rightarrow \quad \underset{\substack{\| \\ z_n}}{x_n + i y_n} \longrightarrow 0.$$

convergence $\Rightarrow$ bounded.

(3) Absolute convergence $\Rightarrow$ convergence.

[Def: Absolute convergence: Say that the series $\sum_{n=1}^{\infty} z_n$ is Absolutely convergent if the series $\sum_{n=1}^{\infty} |z_n|$ is convergent.]

Proof: Suppose $\sum_{n=1}^{\infty} |z_n|$ is convergent

$$\overset{\|}{\sum_{n=1}^{\infty}} \sqrt{x_n^2 + y_n^2}$$

Now $|x_n| \le \sqrt{x_n^2 + y_n^2}$, $\quad |y_n| \le \sqrt{x_n^2+y_n^2}$

By comparison test for real series, we get $\sum_{n=1}^{\infty} |x_n|$ and $\sum_{n=1}^{\infty} |y_n|$ are both convergent.

$\Rightarrow \sum x_n$ and $\sum y_n$ are convergent

(for the last implication, we have used the fact Absolute convergence $\Rightarrow$ convergence for real series.)

Now using Fact 1, we get $\sum_{n=1}^{\infty} z_n$ convergent

Example: 1 $\sum_{n=1}^{\infty} \left(\frac{i}{2}\right)^n$ is convergent. How?

2 $\sum_{n=1}^{n} (2i)^n$ is not convergent. How?

Both the above series are particular cases of $\sum_{n=1}^{\infty} z^n$. For 1, put $z = i/2$

For 2, put $z = 2i$

One can show that (Idea is on next page).

$\sum_{n=1}^{\infty} z^n$ is convergent if $|z| < 1$

is divergent if $|z| > 1$

Consider the partial sums

$$S_N = \sum_{n=1}^{N} z^n = 1 + z + z^2 + z^3 + \dots + z^N$$

Ex: One can show by induction on N, that

$$\boxed{S_N = 1 + z + z^2 + \cdots z^N = \frac{1 - z^{N+1}}{1 - z}}$$

Thus $\lim_{N\to\infty} S_N = \lim_{N\to\infty} \frac{1}{1-z} - \lim_{N\to\infty} \frac{z^{N+1}}{1-z}$

$$= \frac{1}{1-z} - \underbrace{\lim_{N\to\infty} \frac{z^{N+1}}{1-z}}_{??}$$

Ex 2. $\lim_{N\to\infty} \frac{z^{N+1}}{1-z} = \begin{cases} 0 & \text{if } |z| < 1 \\ \text{does not exist} & \text{if } |z| \geq 1. \end{cases}$

So, Finally, $\lim_{N\to\infty} S_N = \begin{cases} \frac{1}{1-z} & \text{if } |z| < 1 \\ \text{does not exist} & \text{if } |z| \geq 1. \end{cases}$

So, we get $\sum_{n=1}^{\infty} z^n = \begin{cases} \frac{1}{1-z} & \text{if } |z| < 1 \\ \text{diverges} & \text{if } |z| \geq 1 \end{cases}$

[End of example.]

Power Series: series of the form

$$\sum_{n=0}^{\infty} a_n (z - z_0)^n = a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + \cdots$$

where $z_0 \in \mathbb{C}$, $a_n \in \mathbb{C}$ $\forall n$.