Chapter - 1
Introduction to complex numbers

- $i = \sqrt{-1}$

- complex numbers are represented as $z = x + iy$
  $x, y \in R$

- $C = \{ z \mid z = a + ib \ ; a, b \in R \}$
  $\Downarrow$
  Field

- Addition, subtraction and multiplication are all binary operations, ie their results are included within C itself.

- A complex number $z = x + iy$ can be represented as (x, y), where x and y are known as real part and imaginary part, respectively of z.

$$x = Re(z)$$
$$y = \cancel{Re} \ Im(z)$$

- x and y can be represented on a plane called complex plane or Gauss plane or Argand plane

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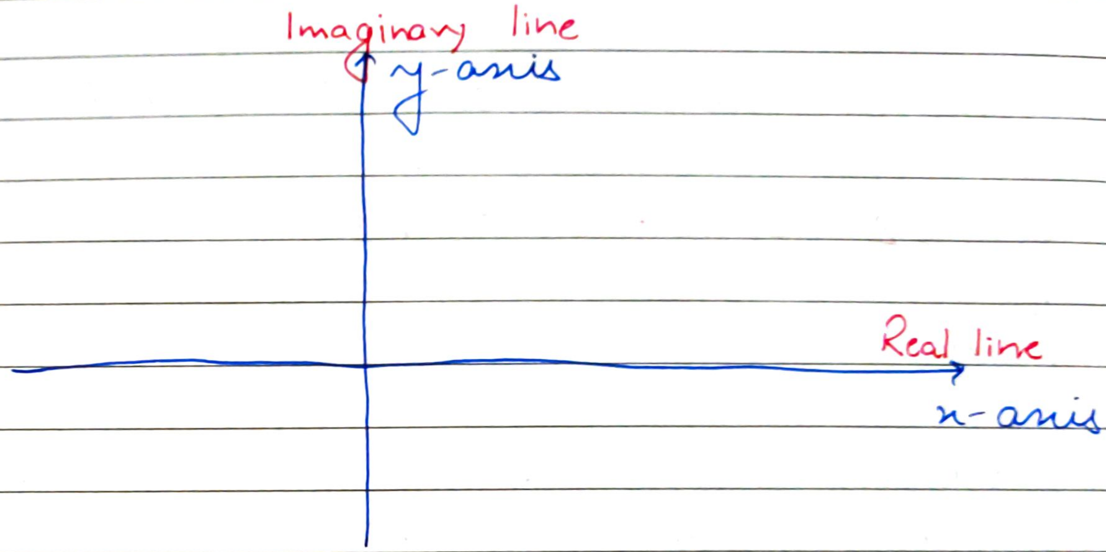
NOTE: The standard x and y axis are taken perpendicular but that is only done to make calculations easy.



- $z_1 z_2 = (x_1x_2 - y_1y_2 \,,\; x_1y_2 + x_2y_1)$

- $\boxed{(\sqrt{-a})(\sqrt{-b}) = -\sqrt{ab}}$

~~$\Rightarrow \sqrt{-a \times -b} = \sqrt{ab}$~~ → ~~wrong~~

$(\sqrt{-a})(\sqrt{-b}) = \sqrt{-a \times -b} = \sqrt{ab}$ ✗ → wrong

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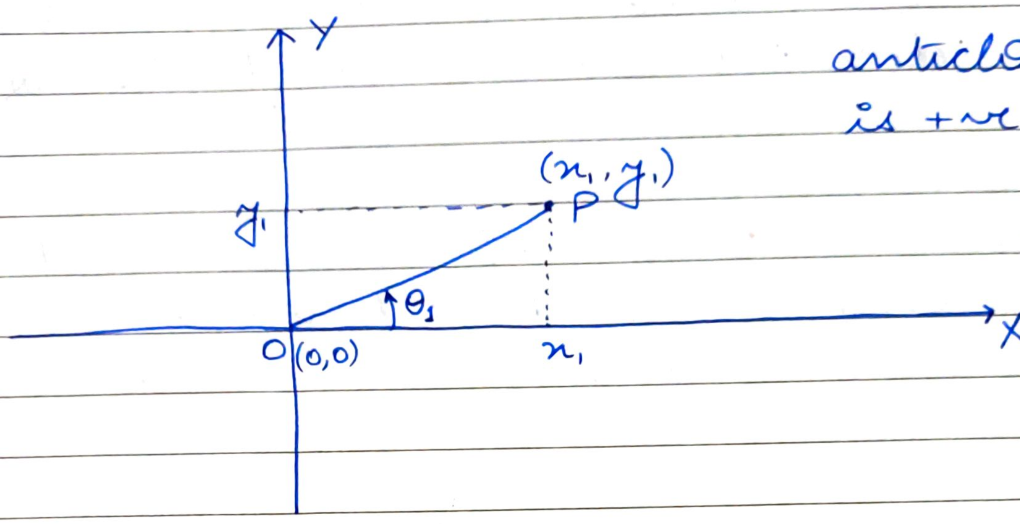
* GEOMETRICALLY (POLAR FORM - EULER FORM)



anticlockwise is +ve directio

Let $z_1 = x_1 + iy_1 = (x_1, y_1)$

$z_2 = x_2 + iy_2 = (x_2, y_2)$

$\boxed{\vec{OP} = z_1}$ $\boxed{OP = |\vec{OP}|}$

imp NOTE: Any complex number z is a vector in argand plane.

now if we write z_1 in polar form. (r, θ) where $r = |\vec{OP}|$

$r = \sqrt{x^2 + y^2}$
↓
modulus/magnitude of complex no.

$amp(z_1) = \theta_1$

- **Modulus** – modulus or absolute value of complex number $z = x + iy$ is denoted by $|z|$ and defined as
$$|z| = \sqrt{x^2 + y^2}$$

- **Amplitude** of z is denoted by $amp(z)$ and defined as
$$amp(z) = 2n\pi + \theta$$
$$= 2n\pi + \tan^{-1}(y/x)$$

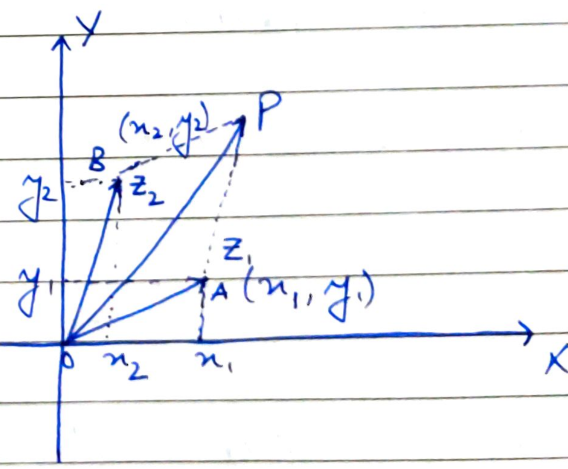
But we restrict a principle value of θ as
$$-\pi \leq \theta \leq \pi \quad \text{(principal argument)}$$

- GEOMETRICAL ADDITION

$z_1 = \vec{OA}$

$z_2 = \vec{OB}$

$z_1 + z_2 = \vec{OP}$



It is similar to vector addition.

- GEOMETRICAL SUBTRACTION

$z_1 - z_2$ can be calculated in a similar fashion. We reverse the direction of z_2 and then perform a vector addition o them.

- GEOMETRICAL MULTIPLICATION

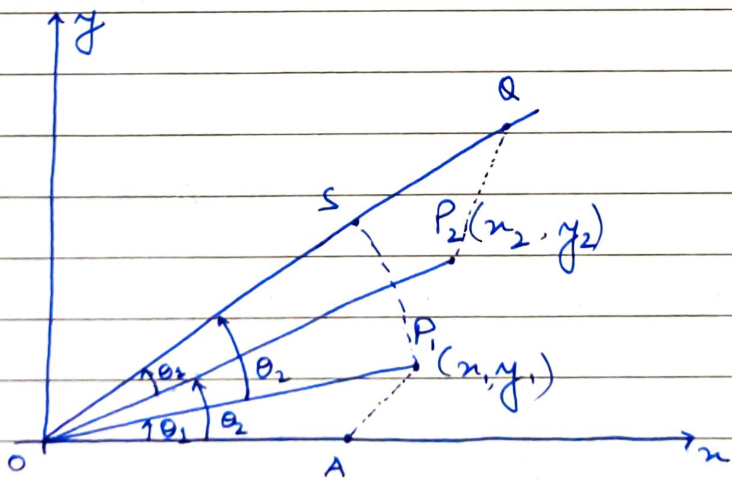
For viewing multiplication geometrically, we will use euler form for simplicity

$$z = r e^{i\theta}$$
$$z = r(\cos\theta + i\sin\theta)$$

Let us consider

$$z_1 = r_1 e^{i\theta_1} = x_1 + i y_1$$
$$z_2 = r_2 e^{i\theta_2} = x_2 + i y_2$$

$$\therefore z_1 z_2 = r_1 r_2 \, e^{i(\theta_1+\theta_2)}$$



$|OA| = 1$

$\angle OAP_1 = \angle OP_2Q$

$\therefore \triangle OAP_1$ is similar to $\triangle OP_2Q$

$\cancel{\Rightarrow \frac{\vec{OP_1}}{\vec{OA}} = \frac{\vec{OP_2}}{\vec{OQ}}}$ $\cancel{\Rightarrow \angle \vec{OP_1} = \frac{\vec{OP_2}}{\vec{OQ}}}$

$\cancel{\Rightarrow \vec{OP_2} =}$ PTO

$$\frac{|\vec{OP_1}|}{|\vec{OA}|} = \frac{|\vec{OQ}|}{|\vec{OP_2}|}$$

$$\Rightarrow |\vec{OP_1}| \cdot |\vec{OP_2}| = |\vec{OQ}|$$

$$\Rightarrow |\vec{OQ}| = |z_1| \cdot |z_2|$$

& we know

$|z_1 z_2| = |z_1| \cdot |z_2|$

$\therefore$ $|\vec{OQ}| = |z_1 z_2|$

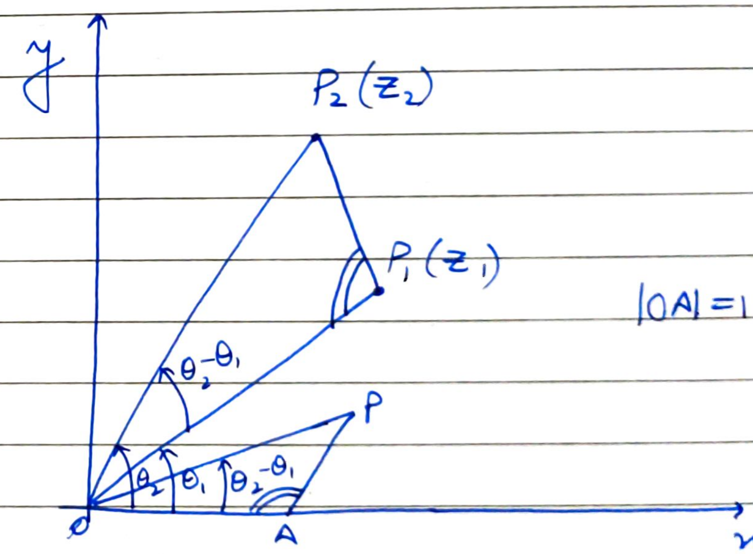
$\Rightarrow$ also amplitude of $\vec{OQ}$

$$amp(z_1 z_2) = \theta_1 + \theta_2$$

- GEOMETRICAL DIVISION

$z_1 = x_1 + iy_1$

$z_2 = x_2 + iy_2$



$|OA| = 1$

$\angle OAP = \angle OP_1P_2$

$\triangle OAP$ is similar to $\triangle OP_1P_2$

$$\frac{|\vec{OP}|}{|\vec{OA}|} = \frac{|\vec{OP_2}|}{|\vec{OP_1}|}$$

$$\Rightarrow |\vec{OP}| = \frac{|z_2|}{|z_1|}$$

$$amp(\vec{OP}) = \theta_2 - \theta_1$$

$$\therefore amp\left(\frac{z_2}{z_1}\right) = \theta_2 - \theta_1$$

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- POWER OF A COMPLEX NUMBER

$z_1 = x_1 + iy_1$; $z_2 = x_2 + iy_2$; $z = x + iy$

$z^2 = (x+iy)^2 = \alpha + i\beta$ (say)

$\Rightarrow x^2 - y^2 + 2ixy = \alpha + i\beta$

$\alpha = x^2 - y^2$

$\beta = 2xy$

similarly we can calculate as per our requirement.

- SQUARE ROOT OF COMPLEX NUMBER

$\sqrt{x+iy} = \alpha + i\beta$ (say)

$\Rightarrow x + iy = (\alpha + i\beta)^2$

$\Rightarrow x = \alpha^2 - \beta^2 \quad -(i)$

$\& \ y = 2\alpha\beta \quad -(ii)$

$(\alpha^2+\beta^2)^2 = (\alpha^2-\beta^2)^2 + 4\alpha^2\beta^2$

$\Rightarrow \alpha^2 + \beta^2 = \sqrt{x^2+y^2}$

$\therefore \ \alpha = \pm\sqrt{\tfrac{1}{2}\left(x + \sqrt{x^2+y^2}\right)}$

$\beta = \pm\sqrt{\tfrac{1}{2}\left(-x + \sqrt{x^2+y^2}\right)}$

only those pairs of (α, β) will be valid which have same sign as $y = 2\alpha\beta$

★ DE-MOIVRE'S THEOREM

- $z^{1/q}$ may have multiple roots depending upon q (ie q roots)

- We know
  $$z = re^{i\theta}$$
  $\therefore$ we can conclude that

$$\boxed{(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta)}$$

- If n = integer, then the above eqn stands

- If n = rational fraction ie $n = p/q$ $q \neq 0$, then we can write

$$(\cos\theta + i\sin\theta)^{p/q} = (\cos(p\theta) + i\sin(p\theta))^{1/q}$$

$$= (\cos(2m\pi + p\theta) + i\sin(2m\pi + p\theta))^{1/q}$$

$$\Rightarrow (\cos\theta + i\sin\theta)^{p/q} = \cos\left(\frac{2m\pi + p\theta}{q}\right) + i\sin\left(\frac{2m\pi + p\theta}{q}\right)$$

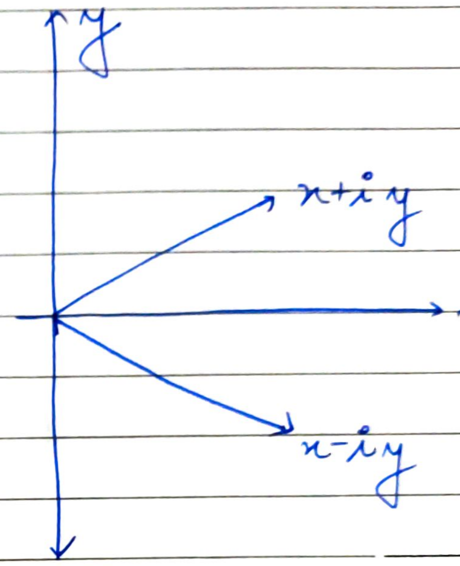
where

$$0 \le m \le q-1$$

→ CONJUGATE OF A COMPLEX NUMBER

$z = x + iy$

Conjugate $\bar{z} = x - iy$



- $|z| = \sqrt{x^2+y^2} = |\bar{z}|$
- $z\bar{z} = |z|^2$
- $\overline{z_1+z_2} = \bar{z_1} + \bar{z_2}$

→ INEQUALITIES

- $|z_1 z_2| = |z_1||z_2|$
- $|z_1 + z_2| \leq |z_1| + |z_2|$
- $|z_1 + z_2|^2 = (z_1+z_2)(\overline{z_1+z_2})$

$$= |z_1|^2 + z_1\bar{z_2} + z_2\bar{z_1} + |z_2|^2$$

$$= |z_1|^2 + |z_2|^2 + 2\,Re(z_1\bar{z_2}) \qquad \left(\because \overline{z_1\bar{z_2}} = \bar{z_1}z_2\right)$$

$$\Rightarrow |z_1+z_2|^2 \leq |z_1|^2 + |z_2|^2 + 2|z_1||z_2|$$

- $z + \bar{z} = 2Re(z)$

→ CAUCHY'S INEQUALITY

$$|x_1y_1 + x_2y_2 + \dots + x_ny_n|^2 \leq \left(\sum_{i=1}^{n}|x_i|^2\right)\left(\sum_{i=1}^{n}|y_i|^2\right)$$

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*→ COMPLEX FUNCTIONS

- consider function

$$z(t) = x(t) + iy(t) \quad ; \quad a \le t \le b$$

$$x, y \in D \subseteq \mathbb{R} \subseteq \mathbb{C}$$

- We will be dealing with

i) Real functions ~~=~~ with real values
ii) Real functions with complex values
iii) complex f^ns with real values
iv) complex f^ns with complex values.

→ DIFFERENTIABILITY OF COMPLEX F_NS →

- Let $w = f(z)$ be a single complex valued function defined on $D(\subseteq \mathbb{C}^*)$ we say $f(z)$ is differentiable at z if

$$\lim_{\delta z \to 0} \frac{f(z+\delta z) - f(z)}{\delta z}$$

exists and all values are same for different pats of approaching δz to 0

$$\lim_{z \to a} \frac{f(z) - f(a)}{z - a}$$

extended domain (∞ included)

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→ ANALYTIC FUNCTION

- A single valued function $w = f(z)$ defined on domain D ($\subseteq \mathbb{C}^*$) is said to be analytic at $z = a \in D$ if $f(z)$ is differentiable at all points of some neighbourhood of a.

St: The necessary and sufficient condition for ~~derivative~~ a single valued function $w = f(z) = u + iv$ ~~to~~ defined on ~~D~~ D to be analytic on D is

i) $\frac{\partial u}{\partial x}, \frac{\partial v}{\partial y}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}$ exist and are continuous, and,

ii) $\boxed{\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}}$ and $\boxed{\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}}$ → C-R Equations

PROOF: Let $w = f(z)$ & $z = x + iy$

$$\therefore w = f(z) = u(x, y) + i v(x, y)$$

$$v, u, y, x \in R$$

Let $w = f(z)$ is analytic (differentiable) on D.

$$\lim_{\delta z \to 0} \frac{f(z + \delta z) - f(z)}{(\delta z)}$$ exists and has unique value, ~~where~~ whatever the path of approaching ~~δ~~ δz to 0.

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Let $w + \delta w = f(z + \delta z)$

Let δw be the change coming due to changes δu and ~~δu~~ δv in u and v at the neighbouring point $(x+\delta x) + i(y+\delta y)$ $= z + \delta z$

$$z + \delta z = (x + iy) + (\delta x + i\delta y)$$

$$= (x + \delta x) + i(y + \delta y)$$

$$w + \delta w = f(z + \delta z) = u(x,y) + \delta u(x,y) + v(x,y) + \delta v(x,y)$$

now

$$\frac{df(z)}{dz} = \lim_{\delta z \to 0} \frac{f(z+\delta z) - f(z)}{\delta z}$$

$$= \lim_{\delta z \to 0} \frac{\cancel{u(x,y)} + \delta u(x,y) + \cancel{iv(x,y)} + i\delta v(x,y) - \cancel{u(x,y)} - \cancel{iv(x,y)}}{\delta z}$$

$$\Rightarrow \frac{df(z)}{dz} = \lim_{\delta z \to 0} \frac{\delta u(x,y) + i\delta v(x,y)}{\delta z} \quad (i)$$

When δz approaching to 0 along real axis then $\delta y = 0$ & $\delta z = \delta x$

$$\therefore \frac{df(z)}{dz} = \lim_{\delta x \to 0} \frac{\delta u(x,y)}{\delta x} + \frac{i\delta v(x,y)}{\delta x}$$

$$\Rightarrow \frac{df(z)}{dz} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} \quad (ii)$$

When δz approaches 0 along imaginary axis then $\delta x = 0$

$\therefore \delta z = i\delta y$

$\therefore$ From (i)

$$\frac{df(z)}{dz} = \lim_{\delta y \to 0} \frac{\delta u(x,y)}{i\delta y} + i\frac{\delta v(x,y)}{i\delta y}$$

$$= -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$$

$$\Rightarrow \frac{df(z)}{dz} = -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} \quad \text{(iii)}$$

NOTE: ~~Here~~ we proved that given condition is necessary. We also need to prove that the conditions are sufficient.
continued on next page

From (ii) & (iii)

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \& \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

These equations are called Cauchy-Riemann Equations

→ TAYLOR THEOREM AND SERIES

- consider

$y = f(x)$

According to Taylor Theorem, we can express

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!}f''(a) + \dots + \frac{h^n}{n!}f^n(a+\theta h)$$

$(0<\theta<1)$

not differentiable further

- An infinitely differentiable function forms a Taylor series

- Taylor Series for f^ns of 2 variables

$z = f(x, y)$ on Δ $(a, b) \in \Delta$

$$f(a+h, b+k) = f(a,b) + \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right) f(a,b) + \frac{1}{2!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^2 f(a,b) + \frac{1}{3!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^3 + \dots + R_n + \dots + R_\infty$$

NOTE: We can ignore higher order terms if 'h' & 'k' are very small

PROOF: continued for sufficiency

Let $w = f(z) = u(x, y) + i v(x, y)$ defined on a domain Δ such that $u(x, y)$ and $v(x, y)$ have continuous partial derivatives of first order and $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$, ie $\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial y}$, $\frac{\partial v}{\partial x}$, $\frac{\partial v}{\partial y}$ exist and continuous.

Then by Taylor's Theorem for two variables function, we have

$$w + \delta w = f(z + \delta z) = u(x + \delta x, y + \delta y) + i v(x + \delta x, y + \delta y)$$

$$= u(x,y) + \left(\delta x \frac{\partial u}{\partial x} + \delta y \frac{\partial u}{\partial y}\right) + \dots +$$

$$i\left[v(x,y) + \left(\delta x \frac{\partial v}{\partial x} + \delta y \frac{\partial v}{\partial y}\right) + \dots\right]$$

now, with assumption that rest of the terms (higher order) are insignificant due to δx & δy being very small

$f(a,b)$

$$\Rightarrow f(z+\delta z) = u(x,y) + iv(x,y) + \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)\delta x$$

$$+ \left(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right)\delta y$$

$$\Rightarrow f(z+\delta z) = f(z) + \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)\delta x + \left(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right)\delta y$$

Using C-R Eqn's

$$\Rightarrow f(z+\delta z) - f(z) = \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)\delta x + \left(i^2\frac{\partial v}{\partial x} + i\frac{\partial u}{\partial x}\right)\delta y$$

$$= \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)\delta x + \left(i\frac{\partial v}{\partial x} + \frac{\partial u}{\partial x}\right)i\,\delta y$$

$$= \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)\left(\delta x + i\,\delta y\right)$$

$$\Rightarrow \lim_{\delta z \to 0} \frac{f(z+\delta x) - f(z)}{\delta z} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial}{\partial x}(u+iv)$$

$$= \frac{\partial f(z)}{\partial x}$$

$$\Rightarrow \frac{d f(z)}{dz} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

$$= \frac{\partial v}{\partial y} + i\left(-\frac{\partial u}{\partial y}\right)$$

$$= -i\frac{\partial u}{\partial y} - \frac{\partial v}{\partial y} i^2$$

$$= -i \frac{\partial}{\partial y}(u+iv)$$

$$\Rightarrow \frac{d f(z)}{dz} = -i\frac{\partial f(z)}{\partial y}$$

$\therefore$ $f(z)$ is analytical in this case. $\therefore$ sufficiency proved.

NOTE: For any analytic f^n $w = f(z) = u(x,y) + iv(x,y)$, the functions $u(x,y)$ & $v(x,y)$ are said to be conjugate to each other.

→ ORTHOGONAL CURVES

- Suppose $u(x,y) = c_1$ and $v(x,y) = c_2$ are parts of an analytic function, $f(z) = u(x,y) + iv(x,y)$

- $u(x,y) = c_1$

$$\Rightarrow \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\frac{dy}{dx} = 0$$

$$\Rightarrow \frac{dy}{dx} = -\frac{\partial u/\partial x}{\partial u/\partial y} \quad (i)$$

& $v(x,y) = c_2$

$$\Rightarrow \frac{\partial v}{\partial x} + \frac{\partial v}{\partial y} \cdot \frac{dy}{dx} = 0$$

$$\Rightarrow \frac{dy}{dx} = -\frac{\partial v}{\partial x} \Big/ \frac{\partial v}{\partial y}$$

- For orthogonality,

$m_1 m_2 = -1$

$$\Rightarrow \boxed{\frac{\partial u}{\partial x} \cdot \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \cdot \frac{\partial v}{\partial y} = 0} \rightarrow \text{condition for orthogonality}$$

→ C-R EQUATIONS IN POLAR FORM

- every analytic f^n satisfies ~~CR equations~~ CR equations.

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \& \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

~~$\frac{\partial u}{\partial x}\left(-\frac{\partial v}{\partial y}\right) + \left(\frac{\partial u}{\partial y}\right)\frac{\partial v}{\partial x} = 0$~~

- $z = x + iy$, $x = r\cos\theta$, $y = r\sin\theta$

$$\Rightarrow z = e^{i\theta} r$$

$$\& \ r^2 = x^2 + y^2 \quad \& \quad \theta = \tan^{-1}(y/x)$$

$$w = f(re^{i\theta})$$

$$\frac{\partial f(z)}{\partial r} = f'(re^{i\theta})\, e^{i\theta} + 0$$

$$\frac{\partial f(z)}{\partial \theta} = f'(re^{i\theta})\, re^{i\theta} \cdot i$$

$$= ir \frac{\partial f}{\partial r}$$

$$\frac{\partial u}{\partial \theta} + \frac{\partial v}{\partial \theta} = ir\left(\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}\right)$$

comparing real & imaginary

$$\therefore \boxed{\frac{\partial v}{\partial \theta} = r\frac{\partial u}{\partial r}} \quad (i)$$

$$\boxed{\frac{\partial u}{\partial \theta} = -r\frac{\partial v}{\partial r}} \quad (ii)$$

NOTE: $|z_1||z_2| = |z_1 z_2|$

TUTORIAL
Assignment 1

1. i) $|z_1 + z_2| \leq |z_1| + |z_2|$

LHS $\quad |z_1 + z_2|^2 = (z_1 + z_2)\,\overline{(z_1 + z_2)}$ $\quad \left(\because z\bar{z} = |z|^2\right)$

$= (z_1 + z_2)(\bar{z}_1 + \bar{z}_2)$

$= z_1\bar{z}_1 + z_2\bar{z}_1 + z_1\bar{z}_2 + z_2\bar{z}_2$

$= |z_1|^2 + z_2\bar{z}_1 + \overline{\bar{z}_1 z_2} + |z_2|^2$

$= |z_1|^2 + 2Re(\bar{z}_1 z_2) + |z_2|^2$

$\leq |z_1|^2 + 2|z_1||z_2| + |z_2|^2$ $\quad \because Re(\bar{z}_1 z_2) \leq |z_1||z_2|$

$\leq \left(|z_1| + |z_2|\right)^2$

iv) $\sqrt{2}\,|z| \geq |Re(z)| + |Im(z)|$

$z = x + iy$

$(|x| + |y|)^2 = |x|^2 + |y|^2 + 2|x||y|$

$\leq |x|^2 + |y|^2 + |x|^2 + |y|^2$

$= 2(|x|^2 + |y|^2) = 2(x^2 + y^2)$

$= 2|z|^2$

$|x| + |y| \leq \sqrt{2}\,(|z|)$

$Re(z) + Im(z) \leq \sqrt{2}|z|$

3 Qu:

Ans: $Re(z_1\bar{z}_2) \leq |z_1\bar{z}_2|$

$z_1 = x_1 + iy_1$; $z_2 = x_2 + iy_2$

$Re(z_1\bar{z}_2) = Re\left((x_1+iy_1)(x_2+iy_2)\right)$

$= x_1x_2 + y_1y_2$

$= \sqrt{(x_1x_2+y_1y_2)^2}$

$\leq \sqrt{(x_1x_2+y_1y_2)^2 + (y_1x_2 - x_1y_2)^2}$

$= |z_1\bar{z}_2|$

They will be equal if

$y_1x_2 = x_1y_2$

$\Rightarrow y_1/x_1 = y_2/x_2$

$\therefore arg(z_1) = arg(z_2)$

Rest of it prove by $z_1 = r_1e^{i\theta_1}$

4 Que:

Ans: $p(z) = \sum_{i=0}^{n} a_i z^i$

As z_1 is root $\Rightarrow p(z_1) = 0$

$p(\bar{z}_1) = \sum_{i=0}^{n} a_i \bar{z}_1^{\,i}$

$= \sum_{i=0}^{n} a_i \overline{z_1^{\,i}} = \overline{\sum_{i=0}^{n} a_i z_i} = \overline{p(z_1)} = \bar{0}$
$= 0$

Hence proved.

5 Que:

iii) $|z-4i| + |z+4i| = 10$

let

$z = x + iy$

$\Rightarrow \sqrt{x^2 + (y-4)^2} + \sqrt{x^2 + (y+4)^2} = 10$

6 Que:

Ans: $|\sin z| = \left| \frac{e^{iz} - e^{-iz}}{2i} \right|$

$z = \pi + i \ln(2+\sqrt{5})$

Solve

;

9 Que:

Ans: i) $\left|\frac{1}{z^4-4z^2+3}\right|$ & $|z|=2$

$|z^4-4z^2+3| = |(z^2-3)(z^2-1)|$

$= |z^2-3||z^2-1|$

$\geq ||z|^2-|3||\,||z|^2-|1||$ from (1)(ii)

$\geq (4-3)|4-1|$

≥ 3

$\Rightarrow \quad 1/3 \geq \frac{1}{|z^4-4z^2+3|}$

10 Que:

Ans: $1+z+z^2+\dots+z^n = \frac{1-z^{n+1}}{1-z}, \; z\neq 1$ (i)

~~For an~~ we prove using induction

For $n=1$

$1+z = \frac{1-z^2}{1-z}$

$= 1+z$

$\therefore$ true

Assume true for k

$\therefore \; 1+z+z^2+\dots+z^k = \frac{1-z^{k+1}}{1-z}$

For $k+1 = n$

$$\underbrace{1 + \cdots + z^{k}}_{\text{substituting from previous step}} + z^{k+1} = \frac{1-z^{k+1}}{1-z} + z^{k+1}$$

$$= \frac{1 - \cancel{z^{k+1}} + \cancel{z^{k+1}} - z^{k+2}}{1-z}$$

$$= \frac{1-z^{k+2}}{1-z}$$

Hence, proved.

i) $1 + \cos\theta + \cdots + \cos n\theta = \frac{1}{2} + \frac{\sin((n+1/2)\theta)}{2\sin(\theta/2)}$

For this
we use $z = e^{i\theta}$ in (i)

$$1 + \cos\theta + \cos(2\theta) + \cdots + \cos(n\theta) + i\{\sin\theta + \sin(2\theta) + \cdots + \sin(n\theta)$$

$$= \frac{1 - \cos((n+1)\theta) - i\sin((n+1)\theta)}{1 - \cos\theta - i\sin\theta}$$

$$= \frac{1 - \cos((n+1)\theta) - i\sin((n+1)\theta)}{2\sin^2\theta/2 - i2\sin\theta/2\cos\theta/2}$$

$$= \frac{(\quad)}{2\sin(\theta/2)(\sin\theta/2 - i\cos\theta/2)} \times \frac{\sin(\theta/2) + i\cos(\theta/2)}{\sin(\theta/2) + i\cos(\theta/2)}$$

$$= \frac{(\quad) \times \sin(\theta/2) + i\cos(\theta/2)}{2\sin(\theta/2)}$$

~~$1+\cos\theta+\ldots+\cos n\theta = (1-\cos(n+1)\theta)\sin\theta/2 + \sin(n+1)$~~

~~8 Que.~~

12 Que.

Ans:

$$z^4+4$$

find roots

$$z^4+4+4z^2-4z^2$$

$$=(z^2+2)^2-4(z^2$$

$$=(z^2+2z+2)(z^2-2z+2)$$

PTO

15 Que.

Ans: $\cancel{z}$ $z^n - 1$ $\cancel{eq}$

$\cancel{= (z-1)(z^{n-1}}$

$= (z-z_1)(z-z_2)(\;)\cdots\cdots(z-z_{n-1})(z-1)$

&

$z^{n-1} - 1 = (z-1)(z^{n-1} + z^{n-2} + \cdots + 1)$

From above 2 eq^{n}s

$\therefore (z-z_1)\cdots\cdots(z-z_{n-1}) = (z^{n-1} + z^{n-1} + \cdots + 1)$

put $z=1$

$\Rightarrow (1-z_1)(1-z_2)\cdots\cdots(1-z_{n-1}) = n$

Hence, proved.

HARMONIC FUNCTION

- A function $z = f(x, y)$ of two variables x and y is said to be a harmonic function if partial derivatives of first and second order of $f(x, y)$ exist and satisfies the Laplace equation.

ie $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial^2 f}{\partial x^2}, \frac{\partial^2 f}{\partial y^2}$ exist

and this eq^n is followed

$$\boxed{\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0}$$

- In case of 3-variable function,

$$\boxed{\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = 0}$$

-

consider analytic f^n $f(z) = u(x, y) + iv(x$

Let $f(z) = u(x, y) + iv(x, y)$ be an analytic defined on D. Then C-R equations exist ie $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ (i) and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ (ii)

Differentiating (i) wrt x partially,

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x \partial y} \qquad \text{(iii)}$$

Differentating (ii) wrt y partially, we get

$$\frac{\partial^2 u}{\partial y^2} = -\frac{\partial^2 v}{\partial y \partial x}$$

Adding (iii) & (iv)

$$\boxed{\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial x^2} = 0}$$

This shows that $u(x, y)$ satisfies Laplace eq^n.

$\therefore$ $u(x, y)$ is harmonic.

Same holds true for v

$\therefore$ For any analytic function, $u(x, y)$ and $v(x, y)$ are harmonic f^n's. They are called harmonic conjugate of each other.

NOTE: $f(z) = u(x, y) + i v(x, y)$

$$z = x + iy \qquad \bar{z} = x - iy$$

$$x = \frac{z + \bar{z}}{2} \qquad y = \frac{z - \bar{z}}{2}$$

$\therefore$ $f(z)$ is a f^n of z and $\bar{z}$.

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Que: If $f(z)=z^2$ Find $\frac{df(z)}{dz}$

Ans: We have

$$f(z) = z^2 = (x+iy)^2$$

$$= x^2 - y^2 + 2ixy$$

$$\frac{df(z)}{dz} = \frac{\partial f(z)}{\partial x} = 2x + 2iy$$

$$= 2(x+iy) = 2z$$

Que: If $f(z) = \mathrm{Im}(z) - iy$. Determine analyticity of $f(z)$?

Ans: We have

$$f(z) = iy$$

$$\therefore \frac{\partial u}{\partial x} = 0, \frac{\partial u}{\partial y} = 0, \frac{\partial v}{\partial x} = 0, \frac{\partial v}{\partial y} = 1$$

$\therefore$ CR eqn's won't hold. $\therefore$ not analytic at any point.

Que: Prove $f(z) = e^z$ is analytic everywhere

Ans: Here, $f(z) = e^z = e^{x+iy}$

$$= e^x e^{iy}$$

$$= e^x(\cos y + i \sin y)$$

$$= \underbrace{e^x \cos y}_{u(x,y)} + i \underbrace{e^x \sin y}_{v(x,y)}$$

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$$\frac{df(z)}{dz} = \frac{\partial f(z)}{\partial x} = e^x \cos y + i e^x \sin y$$

$$= e^z$$

for any(z)

$\therefore f(z)$ is analytic everywhere.

→ THEORIM

- If $f'(z) = 0$ in a domain D, then $f(z)$ is constant in D

$$w = u(r,\theta) + i v(r,\theta)$$

$$r^2 = x^2 + y^2 \quad \tan^{-1}(y/x) = \theta$$

$$w = f(z)$$

$$\frac{dw}{dz} = \frac{\partial w}{\partial x} = \frac{\partial w}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial w}{\partial \theta}\frac{\partial \theta}{\partial x}$$

$$= \frac{-i}{r} e^{-i\theta} \frac{\partial w}{\partial \theta}$$

$$= e^{-i\theta} \frac{\partial w}{\partial r} \left(\frac{\partial u}{\partial \theta} + \frac{i \partial v}{\partial \theta}\right)$$

Que: If $w = \log z$, find $\frac{dw}{dz}$ and determine where w is non-analytic

Ans: $w = u + iv = \log z = \log(x + iv)$

$$= \frac{1}{2}\log(x^2+y^2) + i\tan^{-1}(y/x)$$

$$u = \tfrac{1}{2}\log(x^2+y^2) \quad v = \tan^{-1}(y/x)$$

Prove using CR eqⁿs that it is not analytic at $z=0$

Que: Show that a harmonic function $u(x,y)$ satisfies the formal differential equation

$$\frac{\partial^2 u}{\partial z \partial \bar{z}} = 0$$

Ans:

$$x = \frac{1}{2}(z+\bar{z}) \qquad y = \frac{1}{i2}(z-\bar{z})$$

$$\frac{\partial u}{\partial \bar{z}} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial \bar{z}} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial \bar{z}}$$

$$= \frac{\partial u}{\partial x}\left(\frac{1}{2}\right) + \frac{\partial u}{\partial y}\left(\frac{-1}{2i}\right)$$

$$\frac{\partial^2 u}{\partial z \partial \bar{z}} = \frac{1}{2}\frac{\partial^2 u}{\partial x^2}\cdot\frac{1}{2} + \frac{1}{2}\frac{\partial^2 u}{\partial y \partial x}\left(\frac{1}{2i}\right) - \frac{1}{2i}\frac{\partial^2 u}{\partial y^2}\frac{1}{2i}$$

$$-\frac{1}{2i}\frac{\partial^2 u}{\partial x \partial y}\cdot\frac{1}{2}$$

$$= \frac{1}{4}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

& since it is a harmonic fⁿ.

$\therefore$

$\underline{\underline{=0}}$

Hence, proved.

→ APPLICATION OF C-R EQUATIONS TO FIND THE HARMONIC CONJUGATE →

Let $w = f(z) = u(x,y) + i v(x,y)$ be the analytic function for which $u(x,y)$ is given and we have to obtain $v(x,y)$

Since v is a function of x and y (x and y both are real variables)

$$dv = \frac{\partial v}{\partial x} dx + \frac{\partial v}{\partial y} dy$$

$$\Rightarrow dv = -\frac{\partial u}{\partial y} dx + \frac{\partial u}{\partial x} dy \quad \text{(By CR eq}^{n}\text{s)} \quad -(i)$$

In RHS, it is of form

$M dx + N dy$ where $M = -\frac{\partial u}{\partial y}$

$N = \frac{\partial u}{\partial x}$

We check exactness,

$$\frac{\partial M}{\partial y} = -\frac{\partial^2 u}{\partial y^2} \quad \& \quad \frac{\partial N}{\partial x} = \frac{\partial^2 u}{\partial x^2}$$

NOTE Since u is a harmonic f^n, $\therefore \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial x^2} = 0$

$$\therefore \Rightarrow \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$\therefore$ diff eq^n is exact.

$\therefore$ we can find $v(x,y) + C$

NOTE: If $f(z) = u(x,y) + iv(x,y)$ is analytic then,

$if(z)$ must be analytic

ie $if(z) = -v + iu$ is analytic

$\therefore$ if v is a harmonic conjugate of u, then u is a harmonic conjugate of $-v$ and vice-versa

→ MILNE-THOMSON METHOD (for constructing harmonic conjugate)

Since $f(z) = u(x,y) + iv(x,y)$, where $x = \frac{1}{2}(z+\bar{z})$ and $y = \frac{1}{2i}(z-\bar{z})$

Now we may write $f(z) = u\left[\frac{1}{2}(z+\bar{z}), \frac{1}{2i}(z-\bar{z})\right] + iv\left[\frac{1}{2}(z+\bar{z}), \frac{1}{2i}(z-\bar{z})\right]$

On putting $z = \bar{z}$, we get

$$f(z) = u(z,0) + iv(z,0)$$

Therefore, $f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = u_x + iv_x$

$= u_x - iu_y$

(CR eq^ns)

$$\frac{df(z)}{dz} = \frac{\partial f(z)}{\partial x}$$

$$\therefore f'(z) = \varphi_1(x,y) - i\varphi_2(x,y)$$

$u_x = \varphi_1(x,y)$, $u_y = \varphi_2(x,y)$ then

$$f'(z) = \varphi_2(x,y) - i\varphi_2(x,y) = \varphi_1(z,0) - i\varphi_2(z,0) \quad -(i)$$

Integrating (i), we get

$$f(z) = \int \varphi_1(z,0)\,dz - i\int \varphi_2(z,0)\,dz + \underbrace{C}_{\text{const. of integration}}$$

Similarly if $v(x,y)$ be given, we have

$$f(z) = \int \psi_1(z,0)\,dz + i\int \psi_2(z,0)\,dz + c'$$

where $\psi_1(x,y) = \frac{\partial v}{\partial y}$ and

$\psi_2(x,y) = \frac{\partial v}{\partial x}$

Que: Prove that f^n $f(z)$ defined by

$$f(z) = \begin{cases} \dfrac{x^3(1+i) - y^3(1-i)}{x^2+y^2}, & (z \neq 0) \\ = 0, & (z = 0) \end{cases}$$

is continuous & the C-R eq^{n}s are satisfied at origin yet $f'(0)$ does not exist.

Ans: we have

$$f(z) = \begin{cases} \dfrac{x^3(1+i) - y^3(1-i)}{x^2+y^2} & (z \neq 0) \\ = 0, & z = 0 \end{cases}$$

$$\lim_{z \to 0} f(z) = \lim_{(x,y)\to(0,0)} f(z)$$

$$= \lim_{x \to 0} \lim_{y \to 0} f(z)$$

$$= \lim_{x \to 0} \lim_{y \to 0} \frac{x^3(1+i) - y^3(1-i)}{x^2+y^2}$$

$$= \lim_{x \to 0} \frac{x^{\cancel{3}}(1+i)}{\cancel{x^2}}$$

$$= 0$$

We can also use $x = r\cos\theta$ & $y = r\sin\theta$ & then $z \to 0$ ie $r \to 0$ for calculating limit.

$\therefore$ The function is continuous at $(0,0)$

now checking for differentiability.

$$\lim_{z\to 0} \frac{f(z)-f(0)}{z} =$$

$$\lim_{z\to 0} \frac{(x^3-y^3) + i(x^3+y^3)}{(x^2+y^2)(x+iy)}$$

We check along any line $y = mx$

$$\lim_{z\to 0} \frac{f(z)-f(0)}{z} = \lim_{z\to 0} \frac{\cancel{x^3}(1-m^3) + i\,x^3(1+m^3)}{\cancel{x^3}(1+m^2)(1+im)}$$

For diff. values of m, we get different limit values. $\therefore$ $f'(0)$ does not exist.

Que: ~~Prove that~~ If $f(z) = \dfrac{x^3y(y-ix)}{x^6+y^2}$, $(z \neq 0)$,

$f(0) = 0$. Prove that $\dfrac{f(z)-f(0)}{z} \to 0$ as $z \to 0$

along any radius vector but not as $z \to 0$ in any manner.

Ans: $$f(z) = \frac{x^3y(y-ix)}{x^6+y^2}$$

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We convert into polar form

$$f(z) = \frac{\cancel{r^6}^{\,r^3} \cos^3\theta \sin\theta\,(\sin\theta - i\cos\theta)}{\cancel{r^2}\,(r^4\cos^6\theta + \sin^2\theta)}$$

$$\frac{f(z)-f(0)}{z} = \frac{r^2\cos^3\theta\sin\theta\,(\sin\theta - i\cos\theta)}{(r^4\cos^6\theta+\sin^2\theta)\,(\cos\theta + i\sin\theta)}$$

& $z \to 0 \Rightarrow r \to 0$

$$\therefore \lim_{z\to 0} \frac{f(z)-f(0)}{z} = 0$$

$\therefore$ lim exists along any radial vector

~~...~~ checking ~~...~~ along $y = mx$ ie radial manner

$$\lim_{z\to 0}\frac{f(z)-f(0)}{z} = \lim_{z\to 0}\frac{m x^{\cancel{4}\,\cancel{3}\,2}\, x\,(m-i)}{\cancel{x^2}\,(x^4+m^2)\,\cancel{x}\,(1+im)}$$

$$= \lim_{z\to 0}\frac{m x^2 (m-i)}{(x^4+m^2)(1+im)} = 0$$ always in this manner

Now let $z \to 0$ along $y = x^3$

Then

$$\lim_{z\to 0}\frac{f(z)-f(0)}{z} = \lim_{z\to 0}\frac{\cancel{x^6}\,(x^3 - ix)}{2\cancel{x^6}\,(x+ix^3)}$$

$$= \lim_{z\to 0} \frac{x^3 - ix}{2(x+ix^3)}$$

$$= -\frac{1}{2} i$$

$\therefore \lim_{z\to 0} \frac{f(z)-f(0)}{z}$ does not exist as its diff. for $y = mx$ & $y = x^3$

Que: Show that $u(x,y) = e^{-x}[x\sin y - y\cos y]$ is a harmonic f^n and find its harmonic conjugate $v(x,y)$ such that $f(z) = u(x,y) + i\,v(x,y)$ is analytic.

Ans: We have $u(x,y) = e^{-x}[x\sin y - y\cos y]$

which is continuous & differentiable

$$\frac{\partial u}{\partial y} = e^{-x}[x\cos y + y\sin y - \cos y]$$

$$\frac{\partial u}{\partial x} = -e^{-x}[x\sin y - y\cos y] + e^{-x}\sin y$$

Show $\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial x^2} = 0$ $\longrightarrow$ Laplace's eq^n to prove u harmonic.

Since u is harmonic, $\therefore$ $f(z)$ is analytic.

Using C.R eq^{n}s

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} = -(e^{-x}x\cos y + ye^{-x}\sin y - e^{-x}\cos y)$$

$$\Rightarrow v(x,y) = -\int (e^{-x}x\cos y + ye^{-x}\sin y - e^{-x}\cos y)\,dx + g(y)$$

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Que: Examine the nature of the function $f(z) = \frac{x^2 y^5 (x+iy)}{x^4 + y^{10}}$, $z \neq 0$ and $f(0) = 0$

in a region including the origin.

Ans: we have,

$$f(z) = \frac{x^2 y^5 (x+iy)}{x^4 + y^{10}}$$

$\lim_{z \to 0} f(z) = 0$ for $y = mx$

Computing along $y^5 = x^2$,

$$\lim_{z \to 0} f(z) = \frac{\cancel{x^4}(x + i x^{2/5})}{2\cancel{x^4}} = 0$$

$\therefore$ its continuous but checking differen. ~~along~~ of

$$\lim_{z \to 0} \frac{f(z) - f(0)}{z} = \frac{x^2 m^5 x^5 (x + i mx)}{x^4 + m^{10} x^{10}}$$

$= 0$ $\to$ along $y = mx$

Along $y^5 = x^2$

$$\lim_{z \to 0} \frac{f(z) - f(0)}{z} = \frac{\cancel{x^4}\cancel{(x + i x^{2/5})}}{2\cancel{x^4}\cancel{(x + i x^{2/5})}} = 1/2$$

$\therefore$ $f'(0)$ does not exist.

Que: If $u(x,y) = e^x(x\cos y - y\sin y)$, find the analytic f^n $u + iv$.

Sol^n: $u(x,y) = e^x(x\cos y - y\sin y)$

$$\frac{\partial u}{\partial x} = e^x x\cos y - e^x y\sin y + e^x \cos y$$

$$\frac{\partial u}{\partial y} = -e^x x\sin y - \sin y - y\cos y$$

If $\frac{\partial^2 u}{\partial x^2} = -\frac{\partial^2 u}{\partial^2 y}$ then u is harmonic.

Prove this.

now

$$dv = \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy$$

using C-R Eq^{n}s,

$$dv = -\frac{\partial u}{\partial y}dx + \frac{\partial u}{\partial x}dy \rightarrow \text{exact eq}^n$$

$$\therefore dv(x,y) = -\int \frac{\partial u}{\partial y}dx + \int \frac{\partial u}{\partial x}dy$$

$\Rightarrow dv = +e^x(x\sin y + \sin y + y\cos y)dx + e^x(x\cos y - y\sin y + \cos y)dy$

Integrating it

$\Rightarrow v(x,y) = +\int e^x(x\sin y + \sin y + y\cos y)dx + \int (\,)\,dy + C$

unusual step

$y = \text{const}$

no term taken

indep. since no term

$\Rightarrow v(x,y) = (x\sin y + \sin y + y\cos y)e^x - e^x \sin y$
$+ C$

$= e^x(x\sin y + y\cos y) + C$

$\therefore f(z) = u + iv$

$= e^x(x\cos y - y\sin y) + ie^x(x\sin y + y\cos y) + iC$

$= e^x(\cos y\,(x + iy) + i\sin y\,(-y + ix)) + iC$

$= e^x(\cos y\,(z) - \sin y\,(z)) + iC$

$= ze^x(\cos y - \sin y) + iC$

you can also use Milne Thomson technique.

$u(x,y) = e^x(x\cos y - y\sin y)$

$\frac{\partial v}{\partial y} = e^x(x\cos y - y\sin y + \cos y)$

Integrating w.r.t y

$v(x,y) = \int e^x(x\cos y - y\sin y + \cos y)\,dy + g(x)$

$\frac{\partial v}{\partial x} = \cdots$

$\vdots$ Aage khud kar 😑

Ques: If $f'(z)=0$ in a domain D, then $f(z)$ is constant in D

PROOF: $f'(z)=0$

$$\boxed{\frac{d f(z)}{dz} = \frac{\partial f(z)}{\partial x}} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

$$\Rightarrow \frac{\partial u}{\partial x}=0, \quad \frac{\partial v}{\partial x}=0, \quad \frac{\partial u}{\partial y}=0=\frac{\partial v}{\partial y}$$

$$\therefore \frac{df(z)}{dz}=0$$

we can also prove this as

$$\boxed{\frac{df(z)}{dz} = -i\frac{\partial f(z)}{\partial y}}$$

imp:

• THEOREM

Let $|f(z)|$ be constant in a region where $f(z)$ is analytic. Then $f(z)$ is constant.

PROOF: Let $f(z)=u+iv$ be an analytic f^n such that

$$|f(z)|=c \text{ (constant) (say)}$$

$$u^2+v^2=c^2$$

Differentiating w.r.t x, we get

$$2u\frac{\partial u}{\partial x}+2v\frac{\partial v}{\partial x}=0$$

$$\Rightarrow u\frac{\partial u}{\partial x}+v\frac{\partial v}{\partial x}=0$$

Diff. w.r.t. y,

$$u\frac{\partial u}{\partial y} + v\frac{\partial v}{\partial y} = 0$$

imp

Que: If $w = f(z) = u + iv$ and $u - v = e^x(\cos y - \sin y)$ find w in terms of z.

Ans:

$$u - v = e^x(\cos y - \sin y)$$

$$\frac{\partial u}{\partial x} - \frac{\partial v}{\partial x} = e^x(\cos y - \sin y)$$

$$\frac{\partial u}{\partial y} - \frac{\partial v}{\partial y} = -e^x(\sin y + \cos y)$$

Use C-R eq^{n}s to find $\frac{\partial u}{\partial y}$ or $\frac{\partial u}{\partial x}$ or $\frac{\partial v}{\partial x}$ or $\frac{\partial v}{\partial y}$

After this, find v & u & then proceed.

Chapter - 2
Complex Integration

NOTE: · A point on domain of f^n where the f^n ceases to be analytic, is called a singular point.

· Isolated singular point: a singular point ~~xxx~~ which can't be excluded from the domain.

→ RECTIFIABLE CURVE

· Let L be a continuous curve with equation $z = x(t) + i\,y(t)$, $\alpha \le t \le \beta$

Suppose we divide the interval $[\alpha, \beta]$ into n subintervals $[t_{k-1}, t_k]$, $k = 1, 2, 3, \ldots, n$ by introducing $n-1$ intermediate points $t_1, t_2, \ldots, t_{n-1}$ satisfying the inequalities

$$\alpha = t_0 < t_1 < t_2 < \ldots \ldots < t_{n-1} < t_n = \beta$$

· ~~Let~~ The set $P = \{t_0, t_1, \ldots, t_n\}$ is called a partition and the largest length of the sub-intervals ie largest length among $t_1 - t_0$, $t_2 - t_1$, ..., $t_n - t_{n-1}$ is known as norm of the partition P, denoted by $\|P\|$

· Let $z_0, z_1, z_2, \ldots, z_n$ be the points on the curve corresponding to the values $t_0, t_1, \ldots, t_n$ ie $z(t_k) = z_k$. Clearly the length of the polygon

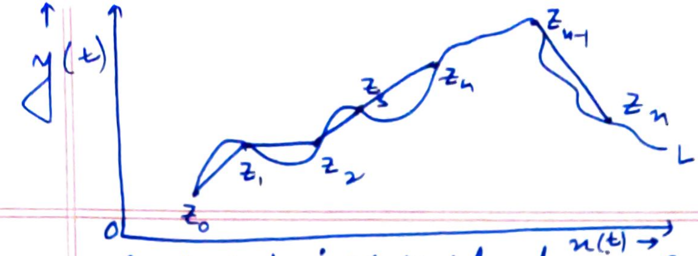


fig (i)

curve L inscribed in graph obtained by joining successively $z_0, z_1, \ldots, z_n$ etc. by straight line segments given in figure (i) is

$$\boxed{\sum_{k=1}^{n} |z_k - z_{k-1}|}$$

- The curve L is said to be rectifiable if $\sup \sum |z_k - z_{k-1}| = l < \infty$, where the least upper bound or supremum is taken over all possible partitions of $[\alpha, \beta]$

supremum

→ CONTOUR

- Let $z = x(t) + i\,y(t)$ —(i)
  where t runs through the interval $\alpha \le t \le \beta$
  and $x(t)$ and $y(t)$ are continuous functions of t representing a continuous arc L in the complex plane.

If equation (i) is satisfied by more than one values of t in the given range, then the point z or (x, y) is a multiple point of the arc.
A continuous arc without multiple point is called jordan arc.

- If for a point z on a Jordan arc, z as expressed in eq^n (i) is single valued and $x(t)$ and $y(t)$ are continuous and if $x'(t)$ & $y'(t)$ are continuous in the range $\alpha \leq t \leq \beta$, then arc is known as regular arc of the Jordan curve.

- Jordan curve consisting of continuous chain of a finite number of regular arcs is called

→ INTEGRATION

- If $f(z)$ is a continuous f^n of the complex variable $z = x + iy$ defined at all points on curve C having end points A and B. Divide C into n parts at the points

$$A = A_0(z_0), A_1(z_1), A_2(z_2), \ldots\ldots, A_n(z_n) = B$$

Let $\delta z_i = z_i - z_{i-1}$ and ξ_i be any point on the arc $A_{i-1}A_i$

then the limit of the sum

$$\sum_{i=1}^{n} f(\xi_i)\,\delta z_i$$

as $n \to \infty$ in such a way that the length of the chord δz approaches to zero, is called the line integral of $f(z)$ taken along curve C.

- $w = f(z) = u(x,y) + iv(x,y)$

where $z = x + iy$

$\Rightarrow dz = dx + i\,dy$

$$\therefore \int_C f(z)\,dz = \int_C \big(u(x,y) + iv(x,y)\big)\big(dx + i\,dy\big)$$

$$= \int_C \big(u(x,y)\,dx - v(x,y)\,dy\big) + i\big(v(x,y)\,dx + u(x,y)\,dy\big)$$

$$\Rightarrow \boxed{\int_C f(z)\,dz = \int_C \big(u(x,y)\,dx - v(x,y)\,dy\big) + i\int_C v(x,y)\,dx + u(x,y)\,dy}$$

line integral of complex f^ns

Que: Using the def. of integral as the limit of sum, evaluate the following:

i) $\int_C dz$ ii) $\int_C |dz|$ iii) $\int_C z\,dz$

iv) $\int_C |z|\,dz$

where C is any rectifiable ~~curve~~ arc joining the points A & B where A & B are complex nos.

Ans:

i) By using def^n of integration as limit of sum,

$$\int_c dz = \lim_{n \to \infty} \sum_{r=1}^{n} 1 \cdot (z_r - z_{r-1})$$

$$= \lim_{n \to \infty} (z_n - z_0)$$

$$= \lim_{n \to \infty} (b - a)$$

$$\boxed{\int_c dz = b - a}$$

ii) $$\int_c |dz| = \lim_{n \to \infty} \sum_{r=1}^{n} 1 \cdot |z_r - z_{r-1}|$$

$$= \lim_{n \to \infty} \left(\underbrace{|z_1 - z_0|}_{\text{length of infinitesimal segment}} + |z_2 - z_1| + \cdots + |z_n - z_{n-1}| \right)$$

$$= \lim_{n \to \infty} \left(\text{arc } z_0 z_1 + \text{arc } z_1 z_2 + \cdots + \text{arc } z_n z_{n-1} \right)$$

applying limit

~~$= \left(\lim_{n \to \infty} \text{arc length of} \right.$~~

$$\int_c |dz| = \text{arc length of } c$$

$$\boxed{\int_c |dz| = \text{arc length of } c}$$

iii) $$\int_C z\,dz = \lim_{n\to\infty} \sum_{r=1}^{n} \xi_r (z_r - z_{r-1})$$

Since ξ_r is arbitrary point in r^{th} arc $z_{r-1} z_r$

Taking $\xi_r = z_{r-1}$ & then $\xi_r = z_r$

then

$$\int_C z\,dz = \lim_{n\to\infty} \sum_{r=1}^{n} z_{r-1} (z_r - z_{r-1}) \qquad (i)$$

Also

$$\int_C z\,dz = \lim_{n\to\infty} \sum_{r=1}^{n} z_r (z_r - z_{r-1}) \qquad (ii)$$

adding (i) and (ii) we get

$$2\int_C z\,dz = \lim_{n\to\infty} \sum_{r=1}^{n} (z_r - z_{r-1})(z_r + z_{r-1})$$

$$= \lim_{n\to\infty} \sum_{r=1}^{n} z_r^2 - z_{r-1}^2$$

$$= \lim_{n\to\infty} (z_n^2 - z_0^2)$$

$$\Rightarrow 2\int_C z\,dz = \lim_{n\to\infty} (b^2 - a^2)$$

$$\Rightarrow \boxed{\int_C z\,dz = \frac{b^2 - a^2}{2}}$$

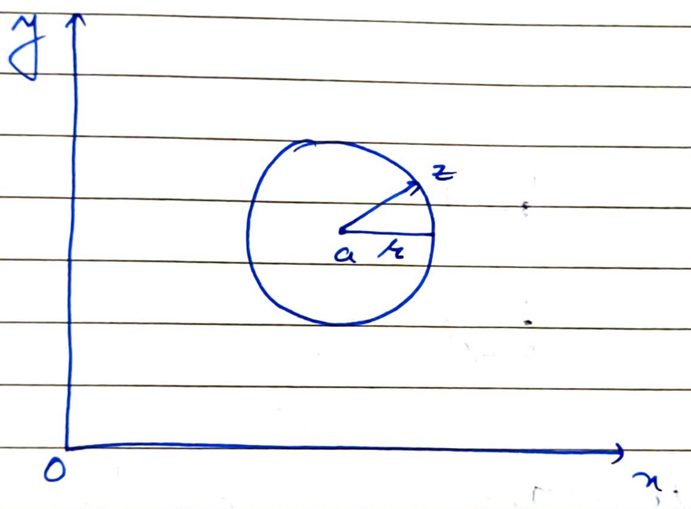
Que: Prove:

i) $\int_C \frac{dz}{z-a} = 2\pi i$

ii) $\int_C (z-a)^n \, dz = 0$ [n, any integer $\neq 1$]

where C is the circle $|z-a| = r$

Ans:



i) $\int_C \frac{dz}{z-a}$

Converting to polar form

$z = re^{i\theta} + a$

$dz = re^{i\theta} i \, d\theta$

$\therefore \int_C \frac{dz}{z-a} = \int_0^{2\pi} \frac{1}{\cancel{re^{i\theta}}} \times \cancel{re^{i\theta}} \, i \, d\theta$

$= i \int_0^{2\pi} d\theta$

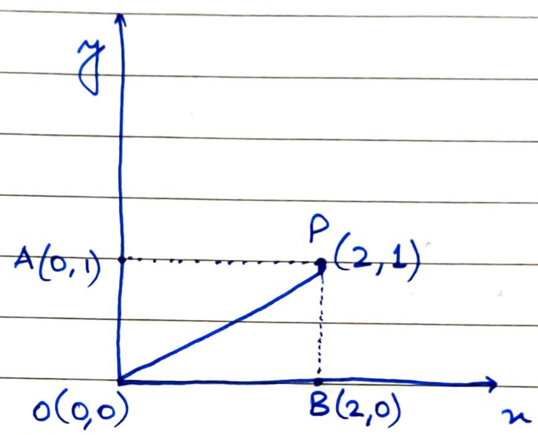
$= 2\pi i$ Hence, proved.

Qu: Evaluate : $\int_0^{2+i} (\bar{z})^2 \, dz$ along

i) the line $y = x/2$

ii) the real axis to 2 and then vertically to $2+i$

Ans:



i) We have to eval. $I = \int_0^{2+i} (\bar{z})^2 dz$ along $y=$

$\Rightarrow x = 2y$

$z = x + iy$

$\Rightarrow z = (2+i)\, y$

$\Rightarrow \bar{z} = (2-i)\, y$

$\Rightarrow I = \int_0^{2+i} (2-i)^2 y^2 \, d[(2+i)y]$

$= \int_0^1 (2-i)^2 (2+i)\, y^2 \, dy$

$$= (2-i)^2(2+i)\ \frac{y^3}{3}\Big|_0^1$$

$$= \frac{(2-i)5}{3}$$

ii) $$I = \int_0^{2+i} (\bar{z})^2 dz = \int_{OB} (\bar{z})^2 dz + \int_{BP} (\bar{z})^2 dz$$

Now at OB, $y=0$

$$\therefore z = x$$

$$\int_{OB} (\bar{z})^2 dz = \int_0^2 x^2 dx = \frac{x^3}{3}\Big|_0^2 = \frac{8}{3}$$

Also

$$\int_{OP} (\bar{z})^2 dz = \int_0^1 (2-iy)^2\, i\,dy = i\int_0^1 (4-4iy-y^2)d$$

$$z = 2+iy$$

$$\therefore \bar{z} = 2-iy$$

$$dz = i\,dy$$

CAUCHY GOURSAT THEOREM

- If $f(z)$ is analytic with a continuous derivative in a simply connected domain G, and C is a closed contour lying in G, then

$$\int_C f(z)\,dz = 0$$

- GREEN'S THEOREM

Let C be a positively oriented piecewise smooth simple closed curve in a plane and let D be the region bounded by C. If L and M are functions of x and y defined on an open region containing D and have continuous partial derivative there, then

$$\oint (L\,dx + M\,dy) = \iint_D \left(\frac{\partial M}{\partial x} - \frac{\partial L}{\partial y}\right) dx\,dy$$

PROOF: Let $f(z) = u(x, y) + i\,v(x, y)$ — (i)

be an analytic function with a continuous derivative in a simply connected domain G, and C is a closed contour lying in G,

then by C-R eq^ns for analytic function

$$f'(z) = u_x + i v_x = v_y - u_y \quad \text{— (ii)}$$

$\forall$ points in the domian

$\therefore$ $f'(z)$ is continuous & the 4 partial derivatives u_x, v_x, u_y, v_y must also be continuous in domain G.

Also $z = x + iy$ $\therefore dz = dx + idy$

$$\therefore \int f(z)\,dz = \int_C (u+iv)(dx+i\,dy)$$

$$= \int_C u\,dx - v\,dy + i\int_C v\,dx + u\,dy \quad \text{(iii)}$$

Therefore by Green's theorem, we have

$$\int_C f(z)\,dz = \iint_G -\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) dx\,dy + i\iint_G \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) dx\,dy$$

Using C-R equations.

→ THEOREM

- If $f(z)$ is continuous on a closed contour C of length l and $|f(z)| \leq M$ for every z on C, then

$$\boxed{\left|\int_C f(z)\,dz\right| \leq \int_C |f(z)|\,|dz| \leq Ml}$$

PROOF.
$$\int_C f(z)\,dz = \lim_{n\to\infty} \sum_{k=1}^{\infty} f(\xi_k)\,(z_k - z_{k-1})$$

where ξ_k is any point on the arc $z_{k-1}\, z_k$

now,

$$\left|\lim_{n\to\infty} \sum_{k=1}^{n} f(\xi_k)\,(z_k - z_{k-1})\right| \leq \lim_{n\to\infty} \sum_{k=1}^{n} |f(\xi_k)|\,|z_k - z_{k-1}|$$

$$\leq \lim_{n\to\infty} M \sum_{k=1}^{n} |z_k - z_{k-1}|$$

$$\Rightarrow \left|\int_C f(z)\,dz\right| \leq \int_C |f(z)|\,|dz| \leq Ml$$

Hence, proved.

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CIRCLE IN COMPLEX PLANE :

- $x^2 + y^2 + 2gx + 2hy + c = 0$

  centre $\equiv (-g, -f)$

  & radius $= \sqrt{g^2 + f^2 - c}$

- In complex plane, the eqⁿ for a circle is

$$|z - z_0|^2 = r^2$$

centre $\equiv z_0$

- The general eqⁿ of a circle in argand plan[e] represented by

$$az\bar{z} + \alpha z + \bar{\alpha}\bar{z} + c = 0$$

where a, c are real constants, α is a complex constant and z is a complex variable

Let $\alpha = a_1 + ia_2$ and $z = x + iy$

$\bar{\alpha} = a_1 - ia_2$ and $\bar{z} = x - iy$

$\therefore$ eqⁿ of circle is

$a(x^2 + y^2) + (a_1 + ia_2)(x + iy) + (a_1 - ia_2)(x - iy) + c = 0$

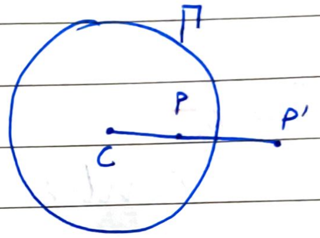
$\Rightarrow a(x^2 + y^2) + 2a_1 x - 2a_2 y + c = 0$

$\therefore$ from this we can deduce

$$\boxed{\text{centre} \equiv \left(-\frac{a_1}{a}, \frac{a_2}{a}\right) = -\frac{\bar{\alpha}}{a}}$$

$$\boxed{\text{radius} = \sqrt{\frac{a_1^2}{a^2} + \frac{a_2^2}{a^2} - c} = \sqrt{\frac{a_1^2 + a_2^2 - ac}{a^2}} = \sqrt{\frac{\alpha\bar{\alpha} - ac}{a^2}}}$$

NOTE: given circle Γ



If $\underline{(CP)(CP') = r^2}$ then P and P' are said to be inverse point of each other.

- If we go to find out the relation w.r.t the circle

$$az\bar{z} + \alpha z + \bar{\alpha}\bar{z} + c = 0 \qquad (i)$$

Let $\underbrace{P}_{z}$ & $\underbrace{P'}_{z'}$ be inverse points wrt (i)

$$\left|z + \frac{\bar{\alpha}}{a}\right|\left|z' + \frac{\bar{\alpha}}{a}\right| = r^2$$

since C, P & P' are co linear,

$$\therefore \quad \arg\left(z + \frac{\bar{\alpha}}{a}\right) = \arg\left(z' + \frac{\bar{\alpha}}{a}\right)$$

$$\Rightarrow -\arg\left(\bar{z} + \frac{\alpha}{a}\right) = \arg\left(z' + \frac{\bar{\alpha}}{a}\right)$$

shows that

$\left(z' + \frac{\bar{\alpha}}{a}\right)\left(\bar{z} + \frac{\alpha}{a}\right)$ is a positive real number

$$\therefore \quad \left|z' + \frac{\bar{\alpha}}{a}\right| \left|\bar{z} + \frac{\alpha}{a}\right| = r^2$$

for center (0,0)

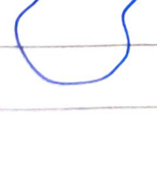
$$|z||z'| = r^2$$

→ CONNECTED REGION

- A region is said to be **connected** if any 2 pts. of region G can be connected by a curve which lies entirely within the curve.

- A connected region is **simply connected** if every closed curve in the region can be shrunk to a point without passing out of the region otherwise the region is said to be **multiply connected** region

eg-



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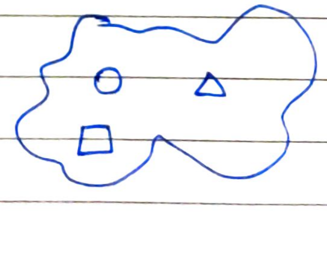
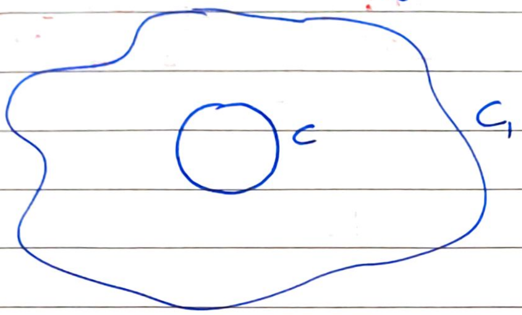


- $$f(z_0) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z - z_0}\, dz$$

- In case of multiply connected regions, where f^n is analytic,



$$\boxed{\int_{C_1} f(z)\,dz - \int_C f(z)\,dz = 0}$$

→ THE DERIVATIVES OF AN ANALYTIC FUNCTION

- THEOREM 1

If a function is analytic ($f(z)$) within and on a simple closed contour c, then its derivative at any point z_0 inside c is given by

$$f'(z_0) = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z - z_0)^2}\, dz$$

- THEOREM 2

The derivative of an analytic function is itself an analytic function.

- THEOREM 3

If a f^n $f(z)$ is analytic on simple closed contour c, then $f(z)$ has derivatives of all orders at each point z_0 inside c with

$$f^n(z_0) = \frac{n!}{2\pi i} \int_c \frac{f(z)}{(z-z_0)^{n+1}} \, dz$$

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PROOF:

$$f(z_0) = \frac{1}{2\pi i} \int_c \frac{f(z)}{z-z_0} \, dz$$

$$z_0 + h \xrightarrow{\text{neighbour}} z_0$$

$$f(z_0+h) = \frac{1}{2\pi i} \int_c \frac{f(z)}{z-z_0-h} \, dz$$

$$\lim_{h\to 0} \frac{f(z_0+h) - f(z_0)}{h} = \lim_{h\to 0} \frac{1}{2\pi i h} \int_c \left(\frac{f(z)}{z-z_0-h} - \frac{f(z)}{z-z_0} \right) dz$$

$$= \lim_{h\to 0} \frac{1}{2\pi i h} \int_c \frac{h\, f(z)}{(z-z_0)(z-z_0-h)} \, dz$$

$$= \frac{1}{2\pi i} \int_c \frac{f(z)}{(z-z_0)^2} \, dz$$

Integral Function (Entire Function)

- A function which is analytic in every finite region of the z plane is called an integral function or entire function

Liouville's Theorem

- If a f^n $f(z)$ is analytic for all finite values of z and is bounded then it is a constant f^n.

PROOF: Let z_1, z_2 be 2 points in the z plane.
Let c be a circle with centre at z_1 and radius R such that the point z_2 is interior to C.
Then by Cauchy's integral formula,

$$f(z_1) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z-z_1} dz$$

$$f(z_2) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z-z_2} dz$$

$$f(z_2) - f(z_1) = \frac{1}{2\pi i} (z_2 - z_1) \int_C \frac{f(z)\,dz}{(z-z_1)(z-z_2)} \quad (i)$$

Now we choose R so large that $|z_2 - z_1| < R/2$

Then since $|z - z_1| = R$ $\therefore$ we have

$$|z-z_2| = |z-z_1+z_1-z_2| = |(z-z_1)-(z_2-z_1)|$$

$$\geq |z-z_1| - |z_2-z_1| \geq R - R/2$$

$$= R/2$$

Also $f(z)$ is bounded, say $|f(z)| \leq M$

Hence, from (i), we get

$$|f(z_2)-f(z_1)| = \left| \frac{(z_2-z_1)}{2\pi i} \int_C \frac{f(z)\,dz}{(z-z_1)(z-z_2)} \right|$$

$$\leq \frac{|z_2-z_1|}{2\pi} \int_C \frac{|f(z)|\,|dz|}{|z-z_1|\,|z-z_2|}$$

$$\leq \frac{|z_2-z_1|}{2\pi} \int_C \frac{M\,|dz|}{R \cdot R/2}$$

$$\leq \frac{M\,|z_2-z_1|}{\pi R^2} \times 2\pi R$$

$$|f(z_2)-f(z_1)| \leq \frac{2\,|z_2-z_1|\,M}{R} \quad \text{-(ii)}$$

Letting $R \to \infty$, we see that the RHS of (ii) $\to 0$ and consequently

$$|f(z_2)-f(z_1)| = 0 \quad \Rightarrow f(z_2)-f(z_1) = 0$$

$$\Rightarrow f(z_2) = f(z_1)$$

Que: Find the value of $\frac{1}{2\pi i} \int_C \frac{x^n e^{xz}}{n!\, z^{n+1}}\, dz$

where C is any closed contour surrounding the origin.

Using the integral representation of $f^n(a)$ prove that

$$\left(\frac{x^n}{n!}\right)^2 = \frac{1}{2\pi i} \int_C \frac{x^n e^{xz}}{n!\, z^{n+1}}\, dz$$

where C is any closed contour surrounding the origin and hence show that

$$\sum_{n=0}^{\infty} \left(\frac{x^n}{n!}\right)^2 = \frac{1}{2\pi} \int_0^{2\pi} e^{2x\cos\theta}\, d\theta$$

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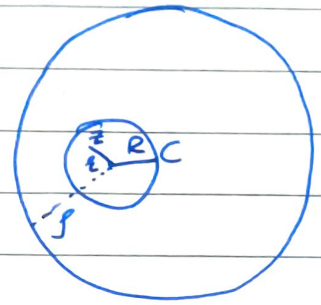
→ POISSON INTEGRAL FORMULA FOR A CIRCLE

$|z| < \rho$ and $0 < r < R < \rho$

$$\boxed{f(re^{i\theta}) = \frac{1}{2\pi}\int_0^{2\pi} \frac{(R^2 - r^2)\, f(Re^{i\varphi})}{R^2 - 2Rr\cos(\theta - \varphi) + r^2}\, d\varphi}$$



PROOF: Let C be a circle $|z| = R$ such that $r < R < \rho$. As given $z = re^{i\theta}$ is any point of the region $|z| < \rho$. Hence by Cauchy Integral formula

$$f(z) = \frac{1}{2\pi i}\int_C \frac{f(w)}{w - z}\, dw \qquad -(i)$$

Now inverse of the point z w.r.t. C is $\frac{R^2}{\bar{z}}$ and lies outside of the circle c

$z\bar{z} = |z|^2$

&

$|z'||z| = R^2$

so that the function $\frac{f(w)}{w - R^2/\bar{z}}$ is analytic within c and $\therefore$ by using Cauchy Goursat Theorem, we have

$$\int_C \frac{f(w)\,dw}{w - R^2/\bar{z}}\, dw = 0 \qquad (ii)$$

Subtracting (i) - (ii)

$$f(z) = \frac{1}{2\pi i}\left(\int_C \frac{f(w)\,dw}{w-z} - \int_C \frac{f(w)\,dw}{w-R^2/\bar{z}}\right)$$

$$= \frac{1}{2\pi i}\left(\int_C \frac{\left(\cancel{f(w)\,w} - f(w)\frac{R^2}{\bar{z}} - \cancel{f(w)\,w} + f(w)\,z\right)}{(w-z)\left(w-\frac{R^2}{\bar{z}}\right)}\right)$$

$$= \frac{1}{2\pi i}\int_C \left(\frac{1}{w-z} - \frac{1}{w-R^2/\bar{z}}\right) f(w)\,dw \quad \text{— (iii)}$$

now we write

$$z = re^{i\theta} \qquad w = Re^{i\phi}$$

$$dw = Ri\,e^{i\phi}\,d\phi$$

From (iii)

$$f(re^{i\theta}) = \frac{1}{2\pi i}\int_0^{2\pi} \frac{\left(re^{i\theta} - R^2/r\, e^{i\theta}\right) Ri\,e^{i\phi}\,d\phi}{\left(Re^{i\phi} - re^{i\theta}\right)\left(Re^{i\phi} - \frac{R^2}{r}e^{i\theta}\right)}$$

steps

$$= \frac{1}{2\pi i}\int_0^{2\pi} \frac{(R^2-r^2)\, f(Re^{i\phi})}{R^2 - 2Rr\cos(\theta-\phi) + r^2}\,d\phi$$

Hence proved

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NOTE: $f(re^{i\theta}) = u(r,\theta) + iv(r,\theta)$

$f(Re^{i\varphi}) = u(R,\varphi) + iv(R,\varphi)$

Que: Find $\displaystyle\int_{|z|=1} \frac{\sin^6 z \, dz}{(z-\pi/6)^3}$

Ans Let $f(z) = \sin^6 z$. Clearly $f(z)$ is analytic in $|z|=1$. Therefore by n^{th} derivative formula

$$f^n(z_0) = \frac{n!}{2\pi i}\int_C \frac{f(z)}{(z-z_0)^{n+1}}\,dz \quad \text{where } |z|<1$$

Taking $f(z) = \sin^6 z$, $z_0 = \pi/6$ and $n=2$.

we find that $\displaystyle\frac{2!}{2\pi i}\int_C \frac{\sin^6 z \, dz}{(z-\pi/6)^3} = f''(\pi/6)$

$$= \frac{d^2}{dz^2}(\sin^6 z)\Big|_{z=\pi/6}$$

$$\Rightarrow I = \left[30\sin^4 z\cos^2 z - 6\sin^6 z\right]_{\pi/6 = z}$$

$$\Rightarrow I = \frac{21}{16}\pi i$$

→ MAXIMUM MODULUS PRINCIPLE

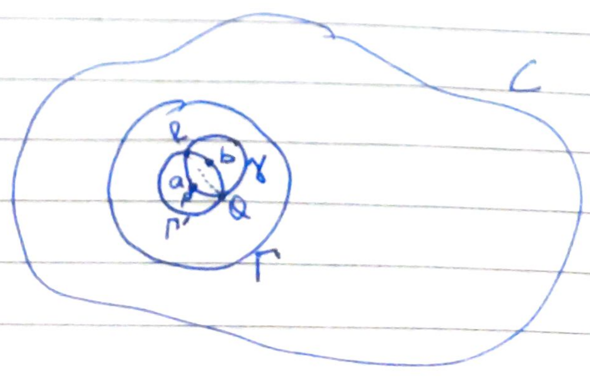
- Let $f(z)$ be analytic within and on a simple closed ~~curve~~ contour C. Then $|f(z)|$ reaches its maximum value on C (and not inside C), unless $f(z)$ is constant

OR,

If M is the maximum ~~minimum~~ value of $|f(z)|$ on and within C, then unless $f(z)$ is constant, $|f(z)| < M$ for every point z within C.

PROOF: Since $f(z)$ is analytic, hence continuous within and on the closed contour C.

It follows that $|f(z)|$ must reach its max value M at some points on or within C. We consider $f(z)$ is not constant in C. Then we wish to prove that $|f(z)|$ takes the value M at some point on C. (attained at a point within C)



Since $|f(z)| = M$ is the max^m value of $|f(z)|$ and $f(z)$ is contt., there exists a point, say b, inside Γ such that

$|f(b)| < M$. ie $f(b) = M - \epsilon$. Since, $|f(z)|$ is continuous at b, for one choice of $\delta > 0$ & $\epsilon > 0$

$$|f(z) - f(b)| < \epsilon/2, \quad \text{whenever } |z-b| < \delta$$

$$|f(z)| - |f(b)| \le \epsilon/2$$

$$|f(z)| < |f(b)| + \epsilon/2$$

$$\Rightarrow |f(z)| < M - \epsilon + \epsilon/2$$

$$\Rightarrow |f(z)| < M - \epsilon/2$$

For all $|z-b| < \delta$ ie for all points z inside a circle γ with centre at b and radius of δ

Now draw a circle Γ' with centre at a such that it passes through b. The arc QR of the circle Γ' lies within γ, so that on this arc, we have

$$|f(z)| < M - \epsilon/2$$

On the remaining portion (arc) of Γ'

$$|f(z)| \le M$$

The radius of Γ' $|b-a| = r$, say

Then by cauchy integral formula, we have

$$f(a) = \frac{1}{2\pi i} \int_{\Gamma'} \frac{f(z)}{z-a}\, dz$$

now on Γ', we have $z - a = re^{i\theta}$

$$\Rightarrow dz = rie^{i\theta} d\theta$$

$$\Rightarrow f(a) = \frac{1}{2\pi i} \int_{\Gamma'} \frac{f(re^{i\theta}+a)\, \cancel{r}i\cancel{e^{i\theta}}\, d\theta}{\cancel{re^{i\theta}}}$$

$$\Rightarrow f(a) = \frac{1}{2\pi} \int_{\Gamma'} f(re^{i\theta} + a)\, d\theta$$

If we measure θ from PQ in the anticlockwise direction, and if $\angle QPR = \alpha$, then

$$f(a) = \frac{1}{2\pi} \int_0^{\alpha} f(a + re^{i\theta})\, d\theta + \frac{1}{2\pi} \int_{\alpha}^{2\pi} f(a + re^{i\theta})\, d\theta$$

$$\Rightarrow |f(a)| \leq \frac{1}{2\pi} \int_0^{\alpha} |f(a+re^{i\theta})|\, d\theta + \frac{1}{2\pi} \int_{\alpha}^{2\pi} |f(a+re^{i\theta})|\, d\theta$$

$$\leq \frac{\alpha}{2\pi}\left(M - \epsilon/2\right) + \frac{M}{2\pi}(2\pi - \alpha) = M - \frac{\alpha\epsilon}{4\pi}$$

This is an absurd result since

M can't be less than $M - \frac{\epsilon\delta}{4\pi}$ $\therefore$ our initial ~~to~~ assumption was wrong.
$\therefore$ maximum value lies of the curve C.

→ MINIMUM MODULUS PRINCIPLE

- If $f(z)$ is analytic on and inside a closed contour C, $|f(z)|$ reaches its minimum value at the curve C itself provided $f(z) \neq 0$

PROOF: similar to above

→ POWER SERIES

- $\{x_n\}$ is a sequence.

Then for given $\epsilon > 0$ $\exists$ a positive integer m s.t

$$\underbrace{|x_n - x|}_{\text{limit of sequence}} < \epsilon , \quad \text{for } n \geq m$$

i.e. $\lim_{n \to \infty} x_n = x$

- x_n is convergent if

$$\sum_{n=1}^{\infty} x_n = y$$

or if $\{s_n\}$ is convergent where $s_n = \sum^{n} x_k$

$$\therefore \lim_{n\to\infty} s_n = s$$

- Cauchy sequence

  - A sequence in which $\{z_n\}$

$$|z_m - z_n| < \epsilon \quad , \quad m, n \geq M_0$$

- Weierstrass M test

  - The series $\sum f_n(x)$ of functions each defined on the same set A converges uniformly on A if

i) $|f_n(x)| \leq M_n$ for every +ve integer n and every $x \in A$, where M_n is a positive constant independent of x

and

ii) the series $\sum M_n$ is ~~constant~~ convergent.

- D' Alembert's ratio Test

$$\sum u_n \quad , \text{ let } L = \left| \frac{u_{n+1}}{u_n} \right|$$

i) If $L < 1$, the series is absolutely convergent
ii) If $L > 1$, the series is absolutely divergent
iii) If $L = 1$, then we can't say anything (test fails)

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Assignment 3

13 Ques i) Show:

$$\sin(z_1+z_2) = \sin z_1 \cos z_2 + \cos z_1 \sin z_2$$

Ans: $$\sin(z_1+z_2) = \frac{\exp(i(z_1+z_2)) - \exp(-i(z_1+z_2))}{2i}$$

$$= \frac{\exp(iz_1)\exp(iz_2) \cancel{\exp(iz_2)} - \exp(-iz_1)\exp(-iz_2)}{2i}$$

$$= \frac{t_1t_2 - \frac{1}{t_1t_2}}{2i} \quad ; \quad t_1 = \exp(iz_1), \ t_2 = \exp(iz_2)$$

$$= \frac{t_1^2t_2^2 - 1}{2i(t_1t_2)}$$

$$= \frac{(t_1^2-1)(t_2^2+1) + (t_1^2+1)(t_2^2-1)}{2i(t_1t_2)}$$

$$= \frac{\left(t_1 - \frac{1}{t_1}\right)\left(t_2 + \frac{1}{t_2}\right)}{2i \times 2} + \frac{\left(t_1 + \frac{1}{t_1}\right)\left(t_2 - \frac{1}{t_2}\right)}{2 \times 2i}$$

ii) Prove: $|\sin z| \geq |\sin x|$ where x is $Re(z)$
& $|\cos z| \geq |\cos x|$

Ans: $$\sin z = \frac{1}{2i}\left(e^{iz} - e^{-iz}\right)$$

$$= \frac{e^{-y}+e^{y}}{2} \sin x + i \cos x \left(\frac{e^{y}-\ldots}{2}\right.$$

$$\Rightarrow |\sin z| \geq |\sin x| \quad \left(\because \frac{e^{-y}+e^{y}}{2} = \frac{\left(e^{-y/2} - e^{\ldots}\right.}{2} \right.$$

v) Prove that $|\sinh y| \leq |\sin z| \leq \cosh y$

Ans: $$\sin z = \sin(x+iy) = \sin x \cosh y + i \cos x \sinh y$$

$$|\sin z|^2 = \sin^2 x \cosh^2 y + \cos^2 x \sinh^2 y$$

$$= \sin^2 x \cosh^2 y + (1-\sin^2 x)\sinh^2 y$$

$$= \sin^2 x + \sinh^2 y \leq 1 + \sinh^2 y = \cosh^2 y$$

6 Que.

iii) $$\log(i^2) = 2\log(i) \qquad \pi/4 < \theta < 9\pi/4$$

$$\log(i^2) = \log(-1) = \log|-1| + i\arg(i^2)$$

$$= 0 + i\pi$$

$$= 2(i\,\pi/2) = 2(i\arg(i))$$

$$\mathrm{Log}\, z = \log|z| + i\arg(z).$$

$$= 2\log(i)$$

(ii) $\cancel{\log(z_1 z_2) = \log(z_1) + \log(z_2)}$

$\cancel{\neq \log(r_1 e^{i\theta_1} r_2 e^{i\theta_2})}$

$\cancel{\neq \log(e^{i(\theta_1+\theta_2)}) + \log r_1 + \log r_2}$

$\cancel{\neq i(\arg(z_1) + \arg(z_2)) + \log r_1 + \log r_2}$

$\cancel{\neq \log z_1 + \log z_2}$

→ THEOREM

- The power series $\sum a_n z^n$ either (i) conver[ges] for all values of z, (ii) converges only for $z=0$, (iii) converges for z in some region in the complex plane.

PROOF: It is sufficient to construct some eg- to show these.

i) $\sum_{n=0}^{\infty} \frac{z^n}{n!}$

ii) $\sum_{n=1}^{\infty} n!\, z^n$

iii) $\sum_{n=0}^{\infty} z^n$

i) $u_n = \frac{z^n}{n!}$; $u_{n+1} = \frac{z^{n+1}}{(n+1)!}$

$$\lim_{n \to \infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \frac{z}{n+1} = 0 < 1$$

$\therefore$ it converges for every value of z

ii) $\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} (n+1)\, z$

$\therefore$ series converges only for $z=0$

iii) converges for region $|z| < 1$

$\frac{u^{n+1}}{u^n} = \frac{z^{n+1}}{z^n} = z$

$\lim_{n\to\infty} z < 1$ for $|z| < 1$

→ ABEL'S THEOREM

- If the power series $\sum_{n=0}^{\infty} a_n z^n$ converges for a particular value of z_0 of z, then it converges absolutely for all values of z for which

$$|z| < |z_0|$$

- RADIUS OF CONVERGENCE

The circle $|z| = R$ which includes interior $|z| < R$ all values of z for which $\Sigma a_n z_n$ power series converges & is called circle of convergence. The radius of this circle is known as radius of convergence.

→ CAUCHY - HADAMARD'S THEOREM

- The radius of convergence R of a power series is given by

$$\frac{1}{R} = \lim_{n \to \infty} |a_n|^{1/n}$$

In practice there is a simpler formula for finding R given by

$$\frac{1}{R} = \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right|$$

→ THEOREM

- Let $\sum_{n=0}^{\infty} a_n z^n$ be a power series and let $\sum_{n=1}^{\infty} n a_n z^{n-1}$ be power series obtained by differentiating the given series terms by terms. Then ~~these~~ derived series has the same radius of convergence as original series.

prove yourself

Qu: Find radius of convergence of:

i) $\sum \frac{z^n}{n!}$ (ii) $\cancel{\sum}(1+1/n)^{n^2} z^n$

iii) $\sum \frac{(n+1)}{(n+2)(n+3)} z^n$ iv) $\sum \frac{(n!)^2}{(2n)!} z^n$

v) $\sum (\log n)^n z^n$ vi) $\sum \frac{z^n}{2^n+1}$

Ans:

i) $\frac{1}{R} = \lim_{n\to\infty} \frac{z^{n+1}}{z^n} \frac{n!}{(n+1)!} = \lim_{n\to\infty} \frac{z}{n+1} = 0$

ii) $a_n = (1+1/n)^{n^2}$

$$\frac{1}{R} = \lim_{n\to\infty} |a_n|^{1/n}$$

$$= \lim_{n\to\infty} (1+1/n)^{n^2/n}$$

$$= e$$

Que: Find the radii of convergence of the following power series.

i) $\sum (3+4i)^n z^n$ ii) $\sum \frac{n\sqrt{2}+i}{1+2in} z^n$

iii) $\sum \frac{(-1)^n}{n} (z-2i)^n$

Ans:

ii) $a_n = \frac{n\sqrt{2}+i}{1+2in}$

$|a_n| = |\text{-- - — -}|$

$|a_n| = \frac{\sqrt{8n^4+6n^2+1}}{1+4n^2}$

$|a_{n+1}| = \text{- - - -}$

$\frac{1}{R} = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$

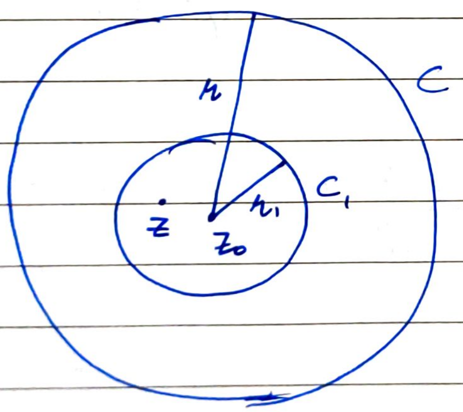
→ TAYLOR THEOREM

- If a function $f(z)$ is analytic at all points within a circle C with centre z_0 and radius r, then at each point z within C_1,

$$f(z) = f(z_0) + (z-z_0) f'(z_0) + \frac{(z-z_0)^2}{2!} f''(z_0) + \ldots\ldots + \frac{(z-z_0)^n}{n!} f^n(z_0) + \ldots\ldots \infty$$

ie $$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n \quad ; \quad a_n = \frac{f^n(z_0)}{n!}$$

PROOF



$$f(z) = \frac{1}{2\pi i} \int_{C_1} \frac{f(\xi)}{(\xi - z)} d\xi \qquad (i)$$

Now

$$\frac{1}{\xi - z} = \frac{1}{(\xi - z_0) - (z - z_0)} = \frac{1}{\xi - z_0}\left(1 - \frac{z-z_0}{\xi - z_0}\right)^{-1}$$

sum of an ∞ GP

$$= \frac{1}{\xi - z_0}\left(1 + \frac{z-z_0}{\xi - z_0} + \left(\frac{z-z_0}{\xi - z_0}\right)^2 + \ldots\ldots + \infty\right)$$

$$= \frac{1}{\xi_1 - z_0}\left(1 + \alpha + \alpha^2 + \alpha^3 + \cdots\cdots + \alpha^n + \alpha^{n+1} + \cdots\cdots \infty\right)$$

where $\alpha = \dfrac{z - z_0}{\xi_1 - z_0}$

$$= \frac{1}{\xi_1 - z_0}\left(1 + \alpha + \alpha^2 + \cdots\cdots + \alpha^n \underbrace{\left(1 + \alpha + \alpha^2 + \cdots\cdots \infty\right)}_{\text{Initial sum}}\right)$$

$$= \frac{1}{\xi_1 - z_0}\left(1 + \alpha + \alpha^2 + \cdots\cdots + \alpha^n \left(\frac{\xi_1 - z_0}{\xi_1 - z}\right)\right)$$

$$= \frac{1}{\xi_1 - z_0} + \frac{z - z_0}{(\xi_1 - z_0)^2} + \frac{(z - z_0)^2}{(\xi_1 - z_0)^3} + \cdots\cdots + \frac{(z - z_0)^{n-1}}{(\xi_1 - z_0)^n}$$

$$+ \frac{(z - z_0)^n}{(\xi_1 - z_0)^n}$$

Multiplying RHS & LHS with $\dfrac{1}{2\pi i} f(\xi_1)$

$$\frac{f(\xi_1)}{2\pi i(\xi_1 - z)} = \frac{f(\xi_1)}{2\pi i(\xi_1 - z_0)} + \frac{(z - z_0) f(\xi_1)}{2\pi i(\xi_1 - z_0)^2} + \cdots\cdots + \frac{(z - z_0)^{n-1} f(\xi_1)}{2\pi i(\xi_1 - z_0)^n}$$

$$+ \frac{(z - z_0)^n}{2\pi i(\xi_1 - z_0)^n} f(\xi_1)$$

Multiplying with $d\xi$ and integrating

$$\frac{1}{2\pi i}\int_{C_1} \frac{f(\xi)\,d\xi}{(\xi - z)} = \frac{1}{2\pi i}\int_{C_1} \frac{f(\xi)\,d\xi}{(\xi - z_0)} + \frac{1}{2\pi i}\int_{C_1} \frac{f(\xi)\,(z-z_0)}{(\xi - z_0)^2}\,d\xi + \cdots$$

$$\cdots + \frac{1}{2\pi i}\int_{C_1} \frac{f(\xi)\,(z-z_0)^{n-1}\,d\xi}{(\xi - z_0)^n} + R_n$$

$$R_n = \frac{1}{2\pi i}\int_{C_1} \frac{f(\xi)\,(z-z_0)^n\,d\xi}{(\xi - z_0)^n(\xi - z)}$$

$$\Rightarrow f(z) = f(z_0) + f'(z_0)(z-z_0) + \frac{f''(z_0)}{2!}(z-z_0)^2 + \cdots$$

$$+ \frac{f^{n-1}(z_0)(z-z_0)^{n-1}}{(n-1)!} + R_n$$

now we need to show that when $n \to \infty$, $R_n \to 0$

since $|z - z_0| = \rho$ $\quad |\xi - z_0| = r_1$

$$|\xi - z| = |(\xi - z_0) - (z - z_0)| \geq |\xi - z_0| - |z - z_0|$$

$$= r_1 - \rho$$

$$|R_n| \leq \frac{1}{2\pi}\int_{C_1} \left|\frac{z - z_0}{\xi - z_0}\right|^n \left|\frac{f(\xi)}{\xi - z}\right| |d\xi|$$

$\because f(\xi) < M$

$$\therefore |R_n| < \frac{M}{2\pi}\left(\frac{\rho}{r_1}\right)^n \left(\frac{1}{r_1-\rho}\right) \cdot (2\pi r_1)$$

$$< M\left(\rho/r_1\right)^n \frac{1}{(1-\rho/r_1)}$$

as $n \to \infty$

$|R_n| \to 0$

Que: Expand $\log(1+z)$ in a Taylor's series about $z=0$ and determine the region of convergence for the result.

Ans. $f(z) = \log(1+z)$ $\quad z \neq -1$

$f'(z) = \frac{1}{1+z}$ $\quad f''(z) = \frac{-1}{(1+z)^2}$ $\quad f'''(z) = \frac{2}{(1+z)^3}$

$$f^n(z) = \frac{(-1)^{n-1}(n-1)!}{(1+z)^n}$$

$$f(z) = \log(1+z) = f(0) + z f'(0) + \frac{z^2}{2!} f''(0) + \cdots$$

$$= 0 + z - \frac{z^2}{2!} + \frac{z^3 (2!)}{3!} - \frac{z^4 (3!)}{4!} \cdots$$

$$= z - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4} + \cdots$$

$$u_n = \frac{(-1)^{n-1} z^n}{n} \quad , \quad u_{n+1} = \frac{(-1)^{n} z^{n+1}}{n+1}$$

$$\lim_{n\to\infty} \left| \frac{u_n}{u_{n+1}} \right| = \lim_{n\to\infty} \left| \frac{n+1}{nz} \right| = \frac{1}{|z|}$$

LAURENT'S THEOREM

- Let $f(z)$ be analytic in the annulus (ring shaped region) between 2 circles C_1 and C_2 with centre z_0 and radii R_1 and R_2 $(R_1 > R_2)$ respectively then at any point z of the annulus,

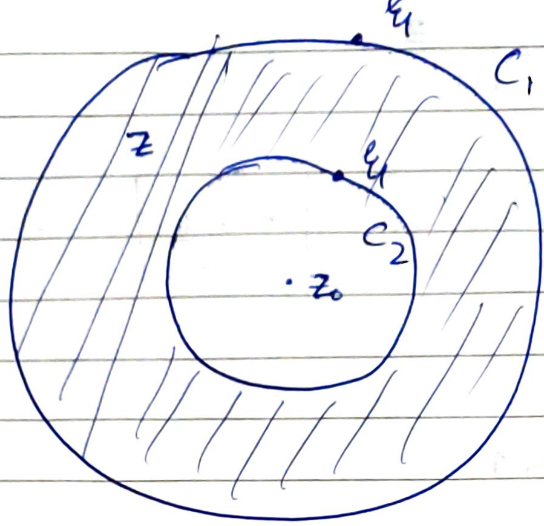
$$\boxed{f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \sum_{n=1}^{\infty} b_n (z-z_0)^{-n}}$$

where $a_n = \dfrac{1}{2\pi i} \displaystyle\int_{C_1} \frac{f(\xi)}{(\xi - z_0)^{n+1}} \, d\xi \quad , \; n = 0, 1, 2, \ldots\ldots$

$$b_n = \frac{1}{2\pi i} \int_{C_2} \frac{f(\xi)}{(\xi - z_0)^{-n+1}} \, d\xi \quad , \; n = 1, 2, 3, \ldots\ldots$$

PTO

PROOF:



For c_1

$$\frac{1}{\xi_1 - z} = \frac{1}{\xi_1 - z_0 - (z - z_0)} = \frac{1}{\xi_1 - z_0}\left[1 - \frac{z - z_0}{\xi_1 - z_0}\right]^{-1}$$

like Taylor's proof

For C_2

$$-\frac{1}{\xi_1 - z} = -\frac{1}{\xi_1 - z_0 - (z - z_0)} = \frac{1}{z - z_0}\left[1 - \frac{\xi_1 - z_0}{z - z_0}\right]^{-1}$$