15/01 Problem on LU Decomposition: CHOLESKY DECOMPOSITION:

$$\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \sim \begin{bmatrix} 2 & -1 & 0 \\ 0 & 3/2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \qquad L = \begin{bmatrix} 1 & 0 & 0 \\ -1/2 & 1 & 0 \\ 0 & -2/3 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 2 & -1 & 0 \\ 0 & 3/2 & -1 \\ 0 & 0 & 4/3 \end{bmatrix}$$

Using elementary type II operations $\qquad U = DU'$

$$= \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3/2 & 0 \\ 0 & 0 & 4/3 \end{bmatrix} \begin{bmatrix} 1 & -1/2 & 0 \\ 0 & 1 & -2/3 \\ 0 & 0 & 1 \end{bmatrix} \qquad \boxed{A = U'^T D U'}$$

$= LDU$

where $\boxed{L = U^T}$

$L_{new} = L\sqrt{D}$

$U_{new} = \sqrt{D}\,U$

$= L\sqrt{D}\sqrt{D}\,U$

$= L_{new} U_{new}$

Possible only when

1) Matrix is symmetric

Keywords to be discussed:

- Binary operation: Two inputs and 1 output.
- Group: Let V be a non-empty set. Let + be a binary operation in V

Then $(V, \oplus)$ is called a group of

① $x \oplus y \in V \;\; \forall\, x, y \in V$ (Closure).

② $(x \oplus y) \oplus z = x \oplus (y \oplus z)$ (Associative)

③ $\exists!\, e \in V$ st $x + e = x \;\; \forall\, x \in V$ (existence of identity element)

unique element

④ $\forall\, x \in V\; \exists!\, y \in V$ st $x \oplus y = e$. ~~Additive~~ (exist of inverse)

⑤ $x \oplus y = y \oplus x \;\; \forall\, x, y \in V$

$\mathbb{R} \rightarrow$ Set of all real numbers

$\mathbb{C} \rightarrow$ Set of all complex numbers

$$\mathbb{R}^n = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} : x_i \in \mathbb{R} \quad \forall i = \underbrace{1:1:n}_{1 \text{ to } n} \right. \qquad \begin{bmatrix} 1 \\ 2 \end{bmatrix} \in \mathbb{R}^2$$

$$\mathbb{C}^n = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} : x_i \in \mathbb{C} \quad \forall i = 1:1:n \right\} \qquad \begin{bmatrix} i \\ 2+3i \end{bmatrix} \in \mathbb{C}^2$$

* Internal Binary operation

$$\underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}}_{\text{Binary operation in } R_2} = \begin{bmatrix} x_1 + y_1 \\ x_2 + y_1 \end{bmatrix}$$

↳ Binary operation in R

* External Binary operation

$$\underset{\in R^1}{3} \cdot \underset{R^2}{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}} = \underset{R^2}{\begin{bmatrix} 3x_1 \\ 3x_2 \end{bmatrix}}$$

* Field

(Set of R is a field w.r.t +, x)

Let V be a nonempty set. Let ⊕ and ⊙ be two binary operation. Then (V, ⊕, ⊙) is called field of

① (V, ⊕) is a commutative group. (Let e be identity w.r.t ⊕.)

② (V, ⊙) is a commutative group (e does not have inverse w.r.t ⊙)

③ $(x \oplus y) \odot z = (x \odot z) \oplus (y \odot z)$

④ $x \cdot (y \oplus z) = (x \odot y) \oplus (y \odot z)$

} Distributive property from right & left.

Eg: $(\mathbb{R}, +, \cdot)$, $(\mathbb{C}, +, \cdot)$

Let $(F, +, \cdot)$ be a field. Let V be an non-empty set. Define two binary operations one is internal in V ($\oplus$) and one is external in V (b/w V and F) denoted by $\odot$. Then $(V, \oplus, \odot)$ is a vector field/linear space over field $(F, +, \cdot)$ if

i) $(V, \oplus)$ is a commutative grp.

ii) $\alpha \odot x \in V \ \forall \alpha \in F$ and $x \in V$

iii) $1 \odot x = x \in V$ and 1 is identity of $(F, \cdot)$

iv) $(\alpha_1 \cdot \alpha_2) \odot x = \alpha_1 \odot \alpha_2 \odot x) \ \forall \alpha_1, \alpha_2 \in F$ & $x, y \in V$

v) $(\alpha_1 + \alpha_2) \cdot x = (\alpha \odot x) \oplus (\alpha_2 \odot x) \ \forall \alpha_1, \alpha_2 \in F$ and $x \in V$

Elements of V are called vector. Elements of F are called scalar.

$\oplus$ (Internal binary op in V) - vector addition.

$\odot$ (External binary op in V) - scalar multiplication

If $F \equiv R$ then vector space is called real vector space

$F \equiv \mathbb{C}$ complex vector space

Examples:

i) Vector space: $R^n = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ $x_i \in R$ $R^2 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

Elements of vector space are called vectors

Case 1: $\underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \oplus \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}}_{\text{vector addition}} = \underbrace{\begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 \end{bmatrix}}_{\text{usual addition}}$; $\alpha \odot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \alpha x_1 \\ \alpha x_2 \end{bmatrix}$

Case 2: $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \oplus \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} x_1 + y_2 \\ x_2 + y_1 \end{bmatrix}$; "

Case: $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \oplus \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} x_1 y_1 \\ x_2 y_2 \end{bmatrix}$; "

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* Identity element of case 1 : $\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

* Case 2 is not commutative

* Additive inverse in case 3 is $\begin{bmatrix} 1/x_1 \\ 1/x_2 \end{bmatrix}$

* $\mathbb{R}$ over $\mathbb{R}$ is also a vector space.

- Any field over itself is a vector space
- $\mathbb{R}$ over $\mathbb{C}$ is not a vector space becaus $\alpha \cdot x \in \mathbb{R}$

$P_2 \rightarrow$ set of all polynomials <u>upto</u> order 2 over $\mathbb{R}$

$$\begin{array}{r} a_1x^2 + b_1x + c_1 \\ a_2x^2 + b_2x + c_2 \\ \hline (a_1+a_2)x^2 + (b_1+b_2)x + (c_1+c_2) \end{array}$$

Internal op.

$$\alpha \odot (a_1x^2 + b_1x + c_1) = (\alpha a_1)x^2 + \alpha b_1 x + \alpha c_1$$

External op.

- If P_2 = set of all polynomials of order 2 then P_2 over $\mathbb{R}$ is not a vector space

  Eg: $(x^2+2), (-x^2+2) \in P_2$

  but $x^2+2-x^2+2 = 4 \in \mathbb{R}$

- $C[a,b] = \{ f$ continuos : $[a,b] \rightarrow \mathbb{R}$

  set of all continuos function from $[a,b] \rightarrow R$

  It is also a vector space.

- $(f \oplus g)(x) = f(x) + g(x) \quad \forall x \in [a,b]$
- $(\alpha \odot f)(x) = \alpha f(x) \quad ; \alpha \in \mathbb{R}$

Zero vector (additive identity) is a zero function

Additive inverse is $-f(x)$.

- $C^{(k)}[a,b]$ set of all k times continuosly differentiable function
- $\{0\}$ - trivial vector space

Subspace:

Let $(V, \oplus, \odot)$ be a vector space over $F(\mathbb{R} | \mathbb{C})$

Subspace is a non-empty subset of V, whose elements (i.e. elements of the subset) satisfy all the properties of vector space. (all 10 properties)

Let $S \subseteq V$. Then S is a subspace of if $(S, \oplus, \odot)$ is a vector space in itself w.r.t same field F.

Result:

S is a subspace $\Leftrightarrow$ if and only if

1) $0 \in S$
2) $x \oplus y \in S \; \forall \; x, y \in S$
3) $\alpha \odot x \in S \; \forall \alpha \in F$ and $x \in S$

(or)

S is a subspace $\Leftrightarrow$

1) $x \oplus y \in S \; \forall \; x, y \in S$
2) $\alpha \odot x \in S \; \forall \; \alpha \in F$ and $x \in S$

or

$\Leftrightarrow$

$(\alpha \odot x) \oplus y \in S \; \forall \alpha \in F, \; x, y \in S.$

* Zero vector in S & V are same

* Additive inverse of S & V are same

Ex1 Let $V \subseteq F$ be a vectorspace over the field. suppose $S = \{0\}$ (Zero subspace of V)

Ex2: Let $V = \{[a_{ij}]_{m \times n} \,|\, a_{ij} \in \mathbb{R}\}$ be a vector space over $\mathbb{R}$

suppose
$S = \{A \in V \,|\, A^t = A\}$ $\longrightarrow$ ②

Theoram: A non empty subset W of V is a subspace iff for each $\alpha, \beta \in W$, and $C \in F$, the vector $C\alpha + \beta \in W$

(converse is also true)

PROOF:

It is given W is a non empty set $(W \neq \phi)$, and

$\alpha, \beta \subset W$, $C \in F \Rightarrow C\alpha + \beta \in W$

$(\underbrace{C \odot \alpha}_{\text{scalar multiplication}} \xrightarrow{} \oplus \beta)$ vector addition.

Proove that W is closed in $\oplus$ & $\odot$ by taking $C = 1$ and $\beta = \{0\}$ respectively.

(i) Associative Law

Let $\alpha, \beta, \gamma \in W$

Since α, β, γ are also members of

$(\alpha+\beta)+\gamma \in V$

$(\alpha+\beta)+\gamma = \alpha + (\beta+\gamma) \in W$

(ii) Existence of the zero vector

$\alpha, \beta \in W, C \in F$, $C\alpha + \beta \in W$

$C = -1$, $\beta = \alpha$

$-\alpha + \alpha = 0 \in W$

(iii) Existence of additive inverse.

$\alpha, \beta \in W$ $C \in F$ For each $\alpha \in W$ $C = -1$, $\beta \in 0$

$C = 1$ $\beta = 0$ $\Rightarrow -\alpha \in W$

(iv) Commutative Law

Let $\alpha, \beta \in W$

If $C = 1$ $\alpha + \beta \in W \subseteq V$

In V commutative law holds.

So $\beta + \alpha \in W \subseteq V$

$c.(\alpha+\beta) = c\alpha + c\beta$

For each $\alpha+\beta \in W$ $c\alpha \in W$

$c\beta \in W$

Converse : Suppose W is subspace of V then $c\alpha+\beta \in W$

$\forall$ $c \in F$, $\alpha, \beta \in W$

Since W is a subspace so it is ~~observed~~ closed under vector addition as well as as scalar multiplication

For $c \in F$, $\alpha \in W \Rightarrow c\alpha \in W$

again let $\beta \in W \Rightarrow c\alpha+\beta \in W$ (as W is itself a vector space)

Proof of ② :

Ex: Let $V = \{[a_{ij}]_{n \times n} \mid a_{ij} \in \mathbb{R}\}$ be a vector space over the field $\mathbb{R}$. Prove that $S = \{A \in V \mid A^t = A\}$ is a subspace of V.

PROOF :

$O_{n \times n} \in S$ i.e $S \neq \Phi \Rightarrow$ (S is non empty which allows us to choose $\alpha, \beta \in S$)

Let A and $B \in S$ i.e $A^t = A$ and $B^t = B$

We have to show $(CA+B) \in S$. To show that :

$$(CA+B)^t = (CA)^t + B^t = CA^t + B^t = CA+B.$$

$$\Rightarrow \boxed{CA+B \in S}$$

Generally

Step 1: $S \neq \Phi$ Step 2: $\alpha, \beta \in S$ $c\alpha+\beta \in S$ $\forall c \in F$

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Ex 3: $V = \mathbb{R}^n(\mathbb{R})$

$S = \{x_1, x_2, \dots x_n\} \in \mathbb{R}^n \mid x_1 = 0\}$

PROVE THAT: S is a subspace of V.

Ex 4: $V = \{F : \mathbb{R} \to \mathbb{R}\}$, $V(\mathbb{R})$

$S = \{F \in V \mid F(-x) = F(x) \ \forall x \in \mathbb{R}\}$

Ex 5: $S_1 = \{F \in V \mid f(-x) = -f(x) \ \forall x \in \mathbb{R}\}$

Ex 6: Let V be a vector space over the field F. Suppose W_1 and W_2 are two subspace of V. Prove that $W_1 \cap W_2$ is a subspace of V.

Since W_1 & W_2 are ~~non~~. two subspaces of V so

$0 \in W_1$ and $0 \in W_2 \Rightarrow 0 \in W_1 \cap W_2 \Rightarrow W_1 \cap W_2$ is non-empty

$\boxed{W_1 \cap W_2 \neq \phi}$

So, we can choose $\alpha, \beta \in W_1 \cap W_2$

~~$c\alpha + \beta \in W_1$ $\quad c\alpha + \beta \in W_2$~~

$\Rightarrow \alpha \in W_1$ and $\alpha \in W_2$; $\beta \in W_1$ and $\beta \in W_2$

$(x \in W_1 \cap W_2 \quad x \in W_1 \ \& \ x \in W_2)$

Since

$\Rightarrow \quad c\alpha + \beta \in W_1 \quad ; \quad c\alpha + \beta \in W_2$

$\Rightarrow \quad c\alpha + \beta \in W_1 \cap W_2$

Since $\alpha, \beta \in W_1$ and W_1 is subspace

$c\alpha + \beta \in W_1 \ \forall c \in F$

Similarily $c\alpha + \beta \in W_2 \ \forall c \in F$

$\Rightarrow c\alpha + \beta \in W_1 \cap W_2$

$\Rightarrow W_1 \cap W_2$ is a subspace

Ex 7: $V = \mathbb{R}^2$

$W_1 = \{(x,0) \mid x \in \mathbb{R}\}$, $W_2 = \{(0,x) \mid x \in \mathbb{R}\}$

$W_1 \cup W_2 = \{(x,0), (0,y) \mid x, y \in \mathbb{R}\}$

Is it a subspace?

$(1,0) \in W_1 \cup W_2$

$(0,1) \in W_1 \cup W_2$

$(1,0) \oplus (0,1) = (1,1) \notin (W_1 \cup W_2)$

$\therefore$ Not a subspace.

THEORAM:

Ex 8: Let W_1 and W_2 are two subspaces of $V(F)$. Then $W_1 \cup W_2$ is a subspace of V iff either $W_1 \subseteq W_2$ or $W_2 \subseteq W_1$

~~If $\cup$~~

~~$W_1 \cup W_2 =$~~

If $W_1 \subseteq W_2$ $\quad W_1 \cup W_2 = W_2 \Rightarrow$ subspace

so $W_1 \cup W_2$ is a subspace.

Similarily

Now to prove that: given $W_1 \cup W_2$ is a subspace of V. We have to proove either $W_1 \subseteq W_2$ or $W_2 \subseteq W_1$

Let $\alpha \in W_1$ and $\beta \in W_2$

$\Rightarrow \alpha \in W_1 \cup W_2$ and $\beta \in W_1 \cup W_2$

$\alpha + \beta \in W_1 \cup W_2$

$\therefore$ either $\alpha + \beta \in W_1$ or $\alpha + \beta \in W_2$

i) If $\alpha + \beta \in W_1$ then

$\Rightarrow -\alpha + (\alpha + \beta) \in W_1$ ($\because W_1$ is subspace & $\alpha \in W_1$ $\Rightarrow -\alpha \in W_1$)

$\Rightarrow (-\alpha + \alpha) + \beta \in W_1$

$\Rightarrow 0 + \beta \in W_1$

$\beta \in W_1$

$\therefore W_2 \subseteq W_1$

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Convert to Cholesky decomposition.

$$A = \begin{bmatrix} 1 & -1 & 2 \\ -1 & 5 & -4 \\ 2 & -4 & 6 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 0 & 4 & -2 \\ 0 & -2 & 2 \end{bmatrix} \begin{matrix} R_2+R_1 \\ R_3+(-2)R_1 \end{matrix} \underset{R_2}{\overset{E}{\sim}} \begin{bmatrix} 1 & -1 & 2 \\ 0 & 4 & -2 \\ 0 & 0 & 1 \end{bmatrix}$$

(2)

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} \qquad E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1/2 & 1 \end{bmatrix}$$

$$E_2 E_1 A = U$$

$$A = LU$$

$$L = E_1^{-1} E_2^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 2 & -1/2 & 1 \end{bmatrix}$$

$$U = DL_1$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & -1/2 \\ 0 & 0 & 1 \end{bmatrix}$$

(3)

$$A = LDL^T$$

$$= (L\sqrt{D})(\sqrt{D}L^T) \quad \sqrt{D} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$L_1 = L\sqrt{D} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 2 & 0 \\ 2 & -1 & 1 \end{bmatrix} \quad L_1^T = \begin{bmatrix} 1 & -1 & 2 \\ 0 & 2 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

TUTORIAL 2:

1) $\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} a_1+b_1 \\ 0 \end{bmatrix}$

$$\alpha \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} \alpha a_1 \\ \alpha a_2 \end{bmatrix} \qquad \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} + \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} a_1+e_1 \\ 0 \end{bmatrix} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \qquad \therefore a_1+e_1=a_1$$

$$e_1 = 0$$

$$0 = a_2$$

$\begin{bmatrix} e_1 \\ \alpha \end{bmatrix} \rightarrow$ arbitary $\rightarrow$ not unique $\rightarrow$ Identity element does not exist.

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} a_1 b_1 \\ a_2 b_2 \\ a_3 b_3 \end{bmatrix}.$$

Additive identity : $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

Additive inverse : $\begin{bmatrix} 1/a_1 \\ 1/a_2 \\ 1/a_3 \end{bmatrix}$ X However, this element does not exist for all ~~element~~ a_1, a_2, a_3

If $a_1 / a_2 / a_3 = 0$ then $1/a_1 \,/\, 1/a_2 \,/\, 1/a_3$ does not exist.

$V = \mathbb{R}^2$

$$\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} a_1 + b_2 \\ a_2 + b_1 \end{bmatrix}$$

$$\alpha \cdot \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} \alpha a_1 \\ \alpha a_2 \end{bmatrix}$$

(i) Non-empty ✓

(ii) Associativity $(a \oplus b) \oplus c = a \oplus (b \oplus c)$

$$(a \oplus b) \oplus c = \begin{bmatrix} a_1 + b_2 \\ a_2 + b_1 \end{bmatrix} \oplus \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} a_1 + b_2 + c_2 \\ a_2 + b_1 + c_1 \end{bmatrix}$$

$$a \oplus (b \oplus c) = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \oplus \begin{bmatrix} b_1 + c_2 \\ b_2 + c_1 \end{bmatrix} = \begin{bmatrix} a_1 + b_2 + c_1 \\ a_2 + b_1 + c_1 \end{bmatrix}$$

It will not form a real vector space.

$V = \mathbb{R}^+$

$u \oplus v = uv$

$\alpha \otimes u = u^{\alpha}$

① ✓

② ✓

③ ✓ $e = 1$

④ $y = 1/u$

⑤ ✓

⑥ ✓

⑦ $\alpha * (\beta * u) = \alpha * u^{\beta} = (u^{\beta})^{\alpha} = u^{\alpha\beta} = (\alpha\beta) * (u$

⑧

⑨ $\alpha \odot (u \oplus v) = \alpha \odot (uv)$

$= (uv)^{\alpha}$

$\alpha \cdot (u) + \alpha \cdot (v) = u^{\alpha} \cdot v^{\alpha} = (uv)^{\alpha}$

(10) $(\alpha+\beta) \odot u = u^{\alpha+\beta} = u^{\alpha} \cdot u^{\beta}$

$= (\alpha \circledast u) \oplus (\beta * u)$

$(\mathbb{R}^+, \oplus, \odot)$

$\therefore$ forms a real vector space.

① $\alpha \circledast 0 = 0 \quad \forall \alpha \in F$

② $0 \otimes u = 0 \quad \forall u \in V$

③ $-1 \otimes u = -u \quad \forall u \in V$

④ $\alpha \otimes u = 0 \Rightarrow \alpha = 0 / u = 0$.

defined in

will give result in V

① $\alpha \otimes 0 = \alpha \otimes (0 \oplus 0)$

From V

$= \alpha \otimes 0 \oplus \alpha \otimes 0$

Additive inverse of $\alpha \otimes 0$ is $-(\alpha \otimes 0)$

$\alpha \otimes 0 + (-\alpha \otimes 0) = \alpha \otimes 0 \oplus \underbrace{\alpha \otimes 0 + (-\alpha \otimes 0)}_{0}$

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$\cancel{\text{If}}$ ... subspaces of V

$U_1 \cap U_2$ is also a subspace of V (vector space)

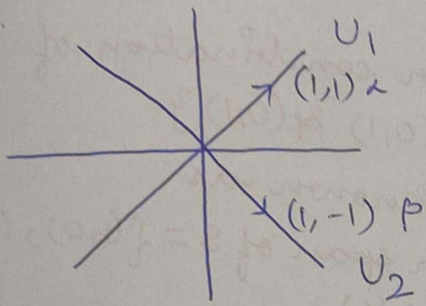
$(U_1 \cap U_2 \cap \ldots \cap U_n \cap \ldots)$ is also a subspace of V

~~Where~~

Is $U_1 \cup U_2$ a subspace of V?

Let us take an example:

Ex:



$\alpha+\beta \notin U_1 \mid \alpha+\beta \notin U_2$

But $\alpha+\beta = (2,0)$ Does not belong to either.

*Defining U_1+U_2

$U_1+U_2 := \{U_1+U_2 \mid u_1 \in U_1 \ \& \ u_2 \in U_2\} = \mathbb{R}^2$

In previous ex

$u_1 = \mathbb{R}$ Line (Subspace) $\quad u_2 = \mathbb{R}$ Line (Subspace) $\quad U_1+U_2 = \mathbb{R}^2$ (Subspace)

So, U_1+U_2 is a subspace of V if U_1 & U_2 are subspaces of V.

PROOF:

1) $0 \in U_1+U_2$ $(\because 0 \in U_1, 0 \in U_2)$

2) For $x, y \in U_1+U_2$, $x+y \in U_1+U_2$

$x = u_1+u_2$ for some $u_1 \in U_1$ & $u_2 \in U_2$

$y = u_1'+u_2'$ for some $u_1' \in V_1$ & $u_2' \in U_2$ ⊛

⊛ imply $x+y = (u_1+u_2)+(u_1'+u_2')$ (As they are all elements of vector space V we can rearrange them)

$= \underbrace{(u_1+u_1')}_{U_1} + \underbrace{(u_2+u_2')}_{U_2}$

$\Rightarrow (x+y) \in (U_1+U_2)$

3) For $\alpha \in \mathbb{R}$ or $\mathbb{C}$ (field), & $\alpha \in U_1+U_2$

$\alpha x = \alpha(u_1+u_2) \Rightarrow \alpha u_1 + \alpha u_2 \in U_1+U_2$

$\uparrow U_1 \quad \uparrow U_2 \quad (\because \alpha$

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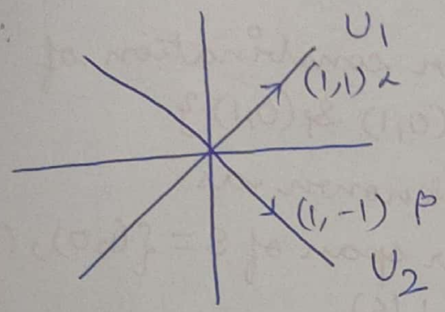
If U_1, U_2 are two ~~sq~~ subspaces of V (vector space
$U_1 \cap U_2$ is also a subspace of V
$(U_1 \cap U_2 \cap \dots \cap U_n \cap \dots)$ is also a subspace of V

~~When is~~
Is $U_1 \cup U_2$ a subspace of V?

Let us take an example:

Ex:



$\alpha+\beta \in U_1 \mid \alpha+\beta \in U_2$
But $\alpha+\beta = (2,0)$
$\downarrow$
Does not belong to either.

2}

*Defining U_1+U_2

$U_1+U_2 := \{ U_1+U_2 \mid u_1 \in U_1 \ \& \ u_2 \in U_2 \} = \mathbb{R}^2$

In previous ex $\quad \Downarrow \mathbb{R} \quad \Downarrow \mathbb{R}$

$U_1 = \mathbb{R}$ Line (Subspace) $\quad U_2 = \mathbb{R}$ Line (Subspace) $\quad U_1+U_2 = \mathbb{R}^2$ (Subspace)

So, U_1+U_2 is a subspace of V if U_1 & U_2 are subspaces of V.

PROOF:

1) $0 \in U_1+U_2$ $\quad (\because 0 \in U_1, 0 \in U_2)$

2) For $x, y \in U_1+U_2$, $x+y \in U_1+U_2$

$x = u_1+u_2$ for some $u_1 \in U_1$ & $u_2 \in U_2$
$y = u_1'+u_2'$ for some $u_1' \in V_1$ & $u_2' \in U_2$ $\Big\}$ (*)

(*) imply $x+y = (u_1+u_2)+(u_1'+u_2')$
$= \underbrace{(u_1+u_1')}_{U_1} + \underbrace{(u_2+u_2')}_{U_2}$ (As they are all elements of vector space V we can rearrange them)

$\Rightarrow (x+y) \in (U_1+U_2)$

3) For $\alpha \in \mathbb{R}$ or $\mathbb{C}$ (field), & $x \in U_1+U_2$

$\alpha x = \alpha(u_1+u_2) \Rightarrow \underset{\uparrow U_1}{\alpha u_1} + \underset{\uparrow U_2}{\alpha u_2} \in U_1+U_2$ $\quad (\because \alpha$

For any $u_1, u_2 \in V$ and $\alpha \in \mathbb{R}$: $\alpha u_1 + u_2 \in V$

Linear span of a set $\mathbb{R}$ (need not be a subspace)

Take $S \subseteq \mathbb{R}^2$ (S is a subset of $\mathbb{R}$)

$\{(1,0),(0,1)\}$

Consider for all $x, y \in \mathbb{R}$

$\boxed{x(1,0) + y(0,1)}$ → linear combination of $\{(0,1) \& (0,1)\}$ is known as linear span of $S = \{(1,0),(0,1)\}$ ($L(S)$)

$= (x,0) + (0,y)$

$= (x,y) \in \mathbb{R}^2$

So,

$$\boxed{L(S) = \{\alpha V_1 + \beta V_2 \mid \alpha, \beta \in \mathbb{R},\ V_1 = (1,0),\ V_2 = (0,1)\}}$$

$L(S)$ is always a subspace of $V = \mathbb{R}^2$

$L(S) = \mathbb{R}^2$

If $S = \{(1,0)\}$

$L(S) = x$ axis. $L(1,0) = x$ axis.

$S = \{1,1\}$ $S = \{(2,2)\}$

$L(S) = x = y$ line $L(S) = x = y$ line

Remarks:

- If $S_1 \neq S_2$ then it may be the case. $L(S_1) = L(S_2)$
- For a subset S of V, $L(S)$ is the smallest subspace of V which contains S. → (Proof in Tut)
- $S \subseteq L(S)$
- Suppose if U is some subspace of V which contains S $\Rightarrow L(S) \subseteq U$.

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~~Tutorial Question~~

Definition of linear span:

$L(S) := \{$all possible finite linear combination of the elements of $S\}$

$$L(S) = \left\{ \sum_{i=1}^{K} \alpha_i V_i \,\middle|\, \alpha_i \in \mathbb{R} \text{ or } \mathbb{C} \right\}$$

Tutorial Question

12) i) $(0,0,0)$

& whether $(0,0,0) \in L(\underbrace{\{(1,2,3),(1,1,-1),(3,5,5)\}}_{S})$

$$(0,0,0) \in \{ x(1,2,3) + y(1,1,-1) + z(3,5,5) \mid x,y,z \in \mathbb{R} \}$$

Does there exist $x, y, z \in \mathbb{R}$

$$x\begin{pmatrix}1\\2\\3\end{pmatrix} + y\begin{pmatrix}1\\1\\-1\end{pmatrix} + z\begin{pmatrix}3\\5\\5\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$$

$$\begin{pmatrix}1 & 1 & 3\\2 & 1 & 5\\3 & -1 & 5\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$$

$$\begin{pmatrix}1 & 1 & 3\\0 & -1 & -1\\0 & 0 & 0\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$$

$y = -z$; $x + y + 3z = 0$

$$(-2z, -z, z) = z(-2, -1, 1)$$

* There are infinitely many (x, y, z) s.t $(0,0,0)$ belongs to $L(S)$.

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Tutorial:

ii) $0 \otimes u = (0+0) \otimes u$

$\Rightarrow \quad 0 \otimes u = 0 \otimes u + 0 \otimes u$

$\Rightarrow \quad 0 \otimes u \oplus \cancel{a} \cdot 0 \otimes u = +0 \otimes u + 0 \otimes u + 0 \otimes u$

$0 = 0 \oplus 0 \otimes u$

$0 = 0 \otimes u.$

(iii) $1 + (-1) = 0$

$(1+(-1)) \otimes u = 0 \otimes u$

$(1+(-1)) \otimes u = 0 \otimes u$

$1 \otimes u \oplus -1 \otimes u = 0$

or $(-u) \oplus u \oplus (-1) \otimes u = -u \oplus 0$

$0 \oplus (-1) \otimes u = -u.$

$(-1) \otimes u = -u$

6) $S =$

$C[0,1] = \{ f : [0,1] \to \mathbb{R} \text{ f is continuos} \}$

$S = \{ f \in V : f(3/4) = 1 \}$

$f, g \in V \quad (f+g)x = f(x) + g(x)$

~~(f)(x) = a f~~.

$(cf)(x)$

$0(x) = 0 \quad \forall x \in [0,1]$

$0(3/4) = 0$

However ~~S~~ $S : f(3/4) = 1.$

$0 \notin S$

$\therefore$ It is not a subspace.

Conditions

Let V be a vector space:

$S \neq \Phi \quad S \subseteq V$

$0 \in S$

$\begin{cases} \text{(i) } \forall\, x_1, x_2 \in S, & x_1 + x_2 \in S \rightarrow ⓐ \\ \text{(ii) } \forall\, \alpha \in F, x \in S, & \alpha x \in S \rightarrow ⓑ \end{cases}$

(iv) $S = \{ f \in V : f(x)$ is zero at finite no. of points $)$

$0(x) = 0 \quad 0 \leq x \leq 1$

0 is zero at infinite no. of points

$0 \notin S$

S is not a vector subspace.

7) iii) $S = \left\{ \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} : c = d \right\} \subseteq \mathbb{R}^4$

$S \neq \Phi \quad 0 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ Here $c = d$

So, $0 \in S$.

$\therefore$ S is non empty.

Now ⓐ

$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \quad y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix}$

Now $x_3 = x_4 \rightarrow$ (i), $y_3 = y_4 \rightarrow$ (ii)

$x + y = \begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 \\ x_3 + y_3 \\ x_4 + y_4 \end{bmatrix}$ Here

$x_3 + y_3 = x_4 + y_4$

We can get the result by (i) & (ii)

So, $x + y \in S$.

Now ⓑ,

$\alpha \in \mathbb{R} \quad x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \in S \Rightarrow x_3 = x_4$

$cx = \begin{bmatrix} cx_1 \\ cx_2 \\ cx_3 \end{bmatrix}$ $\quad cx_3 = cx_4$

$\therefore cx \in S$ $\quad \therefore$ S is a vector subspace

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v) $S = \{(a,b,c,d) : a = 1\}$

$$0 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$ Not a subspace $0 \notin S$

vi) Property (ii) not satisfied

vii) Yes it is a subspace

viii) Property (ii) will not always satisfy

ix)

$$S = \left\{ \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} : a^2 - b^2 = 0 \right\}$$

$a \in S \quad S \leq \mathbb{R}^2$

(i) $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \quad x_1^2 - x_2^2 \quad \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = y \quad y_1^2 - y_2^2$

$$\begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 \\ x_3 + y_3 \\ x_4 + y_4 \end{bmatrix}$$

$$(x_1+y_1)^2 - (x_2+y_2)^2 = x_1^2 - x_2^2 + y_1^2 - y_2^2 + 2(x_1x_2 - y_1y_2)$$
$$= 2(x_1y_1 - x_2y_2)$$

Need not be always zero.

$x \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \; y = \begin{bmatrix} -1 \\ -1 \\ 0 \\ 0 \end{bmatrix}$

(8) $F(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n \quad a_n \neq 0$

$F(x) = 0$

In (ii) & (iii) $0 \notin S$

(9) (vi) $\rightarrow 0 \notin S$

(iv) $\rightarrow 0 \notin S$

(iii) $\rightarrow$ Subspace

(ii) $\rightarrow$ Subspace

(i) $\rightarrow$ Subspace

28/01 Keywords to learn : LI/LD / basis/ dimension/ coordinates / Subspace / span.

NOTE :

$\mathbb{R}^2$ — What are the possible subspaces?

$\underbrace{\mathbb{R}^2, \left\{\begin{matrix}0\\0\end{matrix}\right\}}_{\text{Trivial subspan}}$, Any line that passes through origin

(entire vector space)

linear combination:

$S = \{x_1, x_2, \dots x_n\}$

$\alpha_1 x_1 + \alpha_2 x_2 \Rightarrow$ infinitely many possibilites

$\alpha_1 x_1 + \alpha_2 x_2 + \dots + \alpha_n x_n \Rightarrow$ LINEAR COMBINATION OF VECTORS

Eg: In vector space $\mathbb{R}^2$

$\alpha_1 \begin{bmatrix}1\\2\end{bmatrix} + \alpha_2 \begin{bmatrix}3\\4\end{bmatrix} = \begin{bmatrix}x_1\\x_2\end{bmatrix}$ → This vector is called

The ~~set~~ collection of all possible linear combinations is called linear span.

Coordinates of the linear combination are (α_1, α_2) Eg: $1\begin{bmatrix}1\\2\end{bmatrix} + 2\begin{bmatrix}3\\4\end{bmatrix} = \begin{bmatrix}7\\10\end{bmatrix}$ Coordinates: (1,2)

$\begin{bmatrix}x_1\\x_2\end{bmatrix} = x_1\begin{bmatrix}1\\0\end{bmatrix} + x_2\begin{bmatrix}0\\1\end{bmatrix}$ → For any vector (x_1, x_2) the coordinates are said to be (x_1, x_2) because they can be represented like this.

Linear span of

$S = \{x_1, x_2, \dots x_n\}$

$L[S] = \{\alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 + \dots + \alpha_n x_n : \alpha_1 \dots \alpha_n \in F\}$

Let us take a set $S\left\{\begin{bmatrix}1\\1\end{bmatrix}\right\}$

$L[S]=$ Line $y=x$ which is a subspace of $\mathbb{R}^2$

If $S=\{(0,1),(1,0)\}$

$L[S]=\mathbb{R}^2$

If $S=\{(0,1),(1,1)\}$

$L[S]=\mathbb{R}^2$

Linear span vs linear combination

$\Downarrow$ $\qquad\qquad$ $\Downarrow$

$\sum_{i=1}^{n}\alpha_i x_i$ $\qquad\qquad$ $\sum_{i=1}^{n}\alpha_i x_i$

n can be ∞ $\qquad$ n is finite (only)

Linear span of $S=\{(0,1),(1,0)\}$ $\Rightarrow$ S has finite element

Linear span of $S=$ First quadrant $\Rightarrow$ S has infinite element.

$S=\{x_1,x_2,x_3 \dots\}$

$$L[S]=\left\{\sum_{i=1}^{n}\alpha_i x_i : \alpha_i\in\mathbb{R}(F),\ x_i\in S\right\}$$

$L[S] \downarrow$ Linear span.

$\sum_{i=1}^{n}\alpha_i x_i \downarrow$ Linear combination

LINEARLY DEPENDENT SET (252 point)

Let V be a VS.

Let $S\subset V$ be a subset

Then S is said to be LD set, if for arbitrarily chosen n elements say $x_1,x_2,\dots x_n$ we have n scalars $\alpha_1,\alpha_2,\dots\alpha_n$ (not all zeroes) st

$$\sum_{i=1}^{n}\alpha_i x_i=0$$

For example in $\mathbb{R}^2$

$\alpha_1 x_1 + \alpha_2 x_2 = 0$ Eg: $1\begin{bmatrix}1\\1\end{bmatrix} + 1\begin{bmatrix}-1\\-1\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$

$$\Rightarrow \boxed{x_2 = -\frac{\alpha_1}{\alpha_2} x_1}$$

x_1 is linearly dependent on x_2

Otherwise if $\sum \alpha_i x_i = 0 \Rightarrow \alpha_i = 0$ then S is called Linearly independent.

$\mathbb{R}^2$ is a set. It is a linearly dependent set.

Now, let $S = \{(0,1), (1,0), (1,1)\}$

$1\begin{bmatrix}0\\1\end{bmatrix} + 1\begin{bmatrix}1\\0\end{bmatrix} + (-1)\begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix} \Rightarrow$ So linearly dependent

For LD there must be atleast 1 vector satisfying the condition: Produce zero vector with non-zero combination of α_is.

Now let $S = \{(0,1), (1,0)\}$

$\alpha_1\begin{bmatrix}0\\1\end{bmatrix} + \alpha_2\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$

(α_1, α_2) must be equal to $(0,0)$. (NO OTHER NON-ZERO (α_1, α_2) POSSIBLE)

$\therefore$ S is Linearly independent.

BASIS OF VECTOR SPACE:

A set that is linearly independent and its linear span ~~it~~ is the vectorspace itself is called BASIS OF THE VECTOR SPACE.

For eg: In $\mathbb{R}^2$:

$S = \{(0,1), (1,0)\}$

$\Rightarrow$ Linearly independent

$L[S] = \mathbb{R}^2$

① l.c (linear combination) means finite sum. We say vector x is a linear c. of vectors $\{y_1, y_2, \cdots y_n\}$ if $\exists$ scalars $\alpha_1, \alpha_2, \alpha_3 \cdots \alpha_n$ s.t. $\boxed{x = \sum_{i=1}^{n} \alpha_i x_i}$

Moreover, scalars $(\alpha_1, \alpha_2 \ldots \alpha_n)$ are called coordinates of vector x w.r.t vectors $(y_1, y_2, \cdots y_n)$

② Let S be a set (subset of vectorspace). Then linear span of S

$L[S] = \{$collection of all possible lcs of elements of $S\}$

③ L[S] is the minimal subspace that contains S
(PROOF: TUTORIAL CLASS)

④ vectors $x_1, x_2, \cdots x_n$ are LD (linearly dependent) if $\exists$ scalars $\alpha_1, \alpha_2 \cdots \alpha_n$ (not all zero) s.t.

$$\sum_{i=1}^{n} \alpha_i x_i = \alpha_1 x_1 + \alpha_2 x_2 + \ldots + \alpha_n x_n = 0$$

$$\Rightarrow \sum_{i=1}^{n} \alpha_i x_i = 0 \nRightarrow \alpha_i \neq 0 \; \forall i$$

⑤ Vectors $x_1, x_2, x_3 \ldots x_n$ are LI (linearly independent) if

$$\sum_{i=1}^{n} \alpha_i x_i = 0 \Rightarrow \alpha_i = 0 \; \forall i$$

⑥ Set S is LD (or LI) if arbitarily chosen any finite number of vectors in S are LD (or LI) [atleast one combination must satisfy LD]

⑦ $0 \in S \Rightarrow S$ is LD

$$\alpha_1 \overset{0}{x_1} + \alpha_2 \overset{0}{x_2} + \alpha_3 \overset{0}{x_3} = 0$$

($\alpha_1, \alpha_2, \alpha_3$ can take non-zero arbitary values)

⑧ Any singleton set $\{v\}$; $v \neq 0$ is always LI

⑨ A set S is LD if $\exists$ atleast one vector that can be written as a lc. of others

$$\alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 = 0$$

$$x_3 = -\frac{\alpha_1}{\alpha_3} x_1 + \frac{(-\alpha_2)}{\alpha_3} x_2$$

(10) If $S_1 \subset S_2$ Then
a) S_1 is LD $\Rightarrow$ S_2 is LD
b) S_2 is LI $\Rightarrow$ S_1 is LI
} However converse need not be true

PROOF:
a) Let $S_1 \subset S_2$. Let S_1 be LD
Let $S_2 = S_1 \cup \{y_1, y_2 \dots\}$
Since S_1 is LD, $\exists$ vectors $x_1, x_2, \dots, x_n$ s.t.
$\sum_{i=1}^{n} x_i \alpha_i = 0 \Rightarrow \alpha_i \neq 0. \longrightarrow$ (i)
$\Rightarrow \sum_{i=1}^{n} \alpha_i x_i + \sum_{j=1}^{n} B_j y_j = 0$ for same choice of α_i's as in (i) and $B_j = 0 \; \forall j$.
$\therefore$ S_2 is LD.

b) Let $S_1 \subset S_2$. Let S_2 is LI
Assume S_1 is LD.
Then by part (a) S_2 is LD. Therefore our assumption that S_1 is LD is wrong
Hence S_1 is LI.
(In a vector space any subset is either LD/LI)

(11) Let V be a vector space
Def Let $S \subset V$ (subset)
Then S is called a basis of V if
(i) S is LI
(ii) $L[S] = V$ [S spans/generates full space. V]

Th
(12) The number of elements in any two basis of V are always same. (A set can have infinite basis)

(13) The number of elements in a basis of V is called its dimension.

(14) A basis is
- maximal LI set
- minimal generator

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BASIS:

① $\mathbb{R}^2$

$S = \left\{ \begin{bmatrix}1\\0\end{bmatrix}; \begin{bmatrix}0\\1\end{bmatrix} \right\}$ $\begin{bmatrix}x_1\\x_2\end{bmatrix} = x_1\begin{bmatrix}1\\0\end{bmatrix} + x_2\begin{bmatrix}0\\1\end{bmatrix}$

$S_1 = \left\{ \begin{bmatrix}1\\1\end{bmatrix} \begin{bmatrix}1\\2\end{bmatrix} \right\}$

$$\alpha \begin{bmatrix}1\\1\end{bmatrix} + \beta \begin{bmatrix}1\\2\end{bmatrix} = \begin{bmatrix}x_1\\x_2\end{bmatrix}$$

$$\begin{bmatrix}1 & 1\\1 & 2\end{bmatrix}\begin{bmatrix}\alpha\\\beta\end{bmatrix} = \begin{bmatrix}x_1\\x_2\end{bmatrix}$$

↳ Invertible.

If

$$\begin{bmatrix}1 & 1\\1 & 2\end{bmatrix}\begin{bmatrix}\alpha\\\beta\end{bmatrix} = \begin{bmatrix}2\\3\end{bmatrix} \quad ; \alpha = 1, \beta = 1.$$

So,

$$\begin{bmatrix}\alpha\\\beta\end{bmatrix} = \begin{bmatrix}1 & 1\\1 & 2\end{bmatrix}^{-1}\begin{bmatrix}x_1\\x_2\end{bmatrix}$$

$$= \begin{bmatrix}2 & -1\\-1 & 1\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix} = \begin{bmatrix}2x_1 - x_2\\x_1 + x_2\end{bmatrix}$$

Linear span : $L\left(\begin{bmatrix}\alpha\\\beta\end{bmatrix}\right) = \mathbb{R}^2$ (full $\mathbb{R}^2$)

It is called ~~extended~~ standard easiest basis of $\mathbb{R}^2$

$\mathbb{R}^{m \times n}$

P_n $C[a,b]$

Now,

In $\mathbb{R}^2$. $S = \left\{ \begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix} \right\}$

$\downarrow e_1$ $\downarrow e_2$

Second column of IM.

First column of IM

So,

$\mathbb{R}^n = \{e_1, e_2, \dots e_n\}$ $\boxed{\text{standard basis}}$

Maximal LI? It means that even if one more element is added to the basis it becomes LD as the elements of basis can be expressed ~~ea~~ as a linear comb ~~of~~ which equals to the negative of the added element.

Minimal generator? Even if one element from the basis is removed its linear span cannot generate the entire vector space.