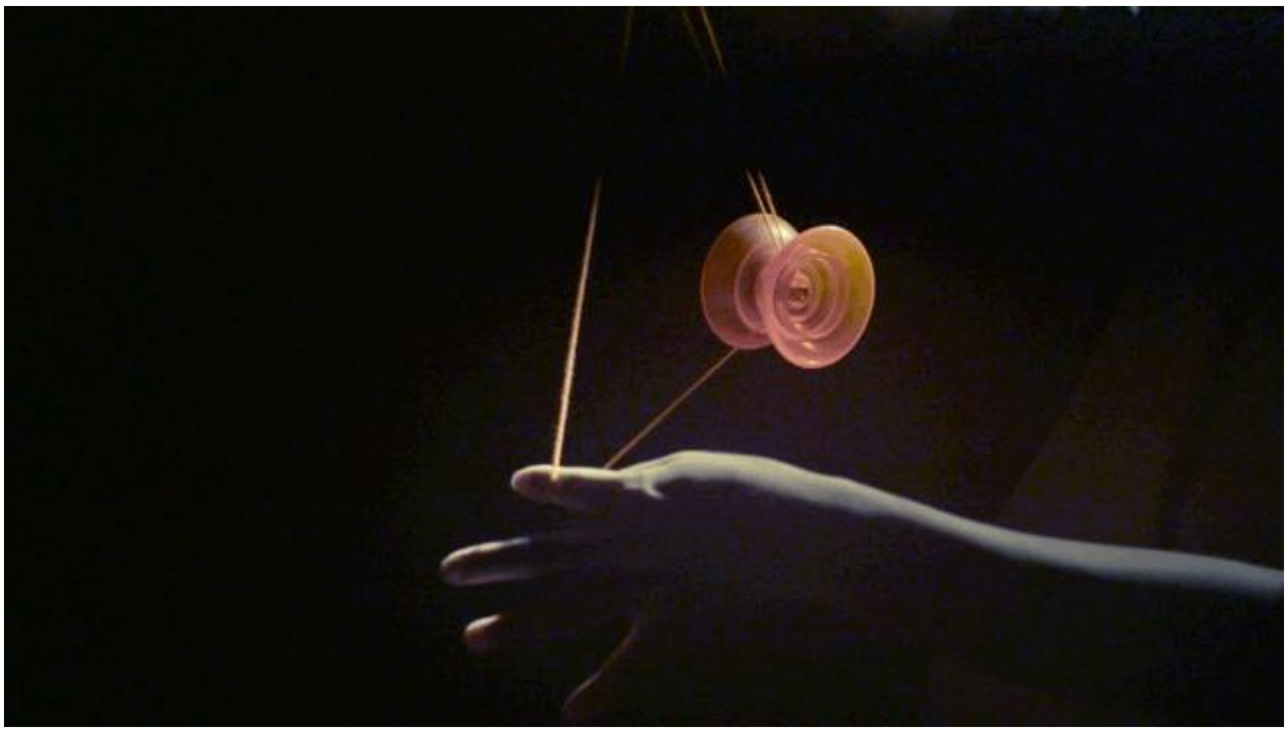
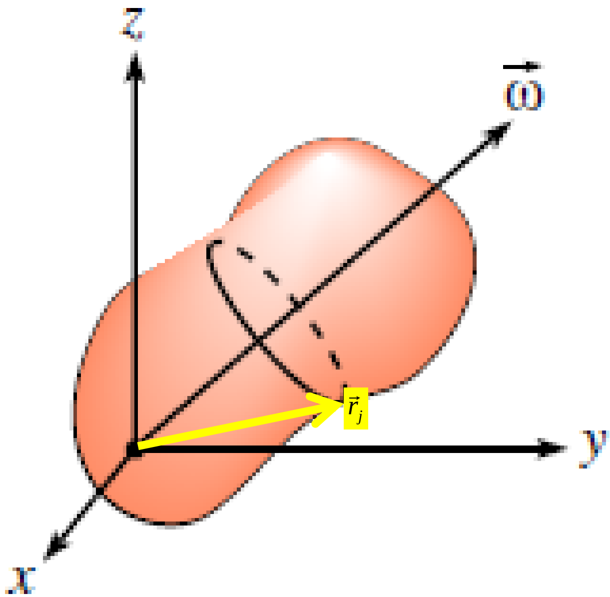
RIGID BODY IN MOTION



Angular momentum of a rigid body



$$[L]=[I][\omega]$$

$$\begin{pmatrix} L_x \\ L_y \\ L_z \end{pmatrix} = \begin{pmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{pmatrix} \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix}$$

$[L]=[I][\omega]$ **Equivalent to** $\vec{p}=m\vec{v}$ **?**

Moment of Inertia Matrix

$$L_x = I_{xx}\omega_x + I_{xy}\omega_y + I_{xz}\omega_z$$

$$L_y = I_{yx}\omega_x + I_{yy}\omega_y + I_{yz}\omega_z$$

$$L_z = I_{zx}\omega_x + I_{zy}\omega_y + I_{zz}\omega_z$$

The fact that *I* is a matrix means that $\vec{L}$ and $\vec{\omega}$ do not necessarily point in the same direction.

Moment of Inertia Matrix

$$[I]=\begin{pmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{pmatrix}$$

$$I_{xx}=\sum m_j\left(y_j^2+z_j^2\right)$$

For a continuous medium,

$$I_{xx}=\int\left(y^2+z^2\right)dm$$

$$[I]=\begin{pmatrix} \int\left(y^2+z^2\right)dm & -\int xydm & -\int zxdm \\ -\int xydm & \int\left(z^2+x^2\right)dm & -\int yzdm \\ -\int zxdm & -\int yzdm & \int\left(x^2+y^2\right)dm \end{pmatrix}$$

Moment of Inertia Matrix

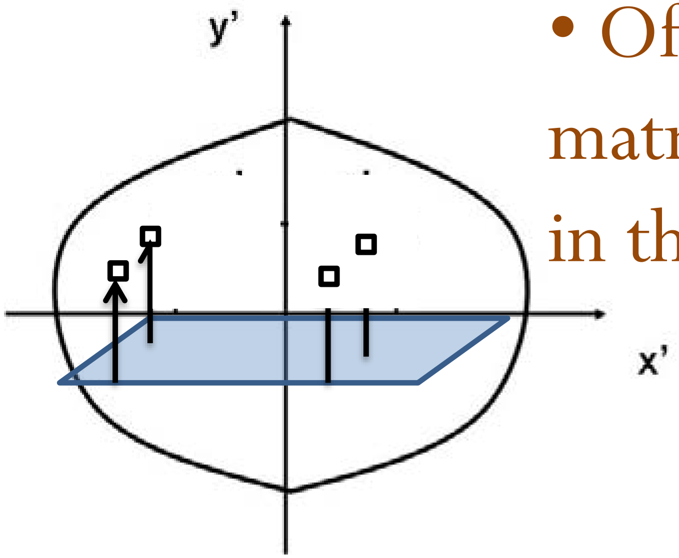
$$[I]=\begin{pmatrix} \int\left(y^2+z^2\right)dm & -\int xydm & -\int zxdm \\ -\int xydm & \int\left(z^2+x^2\right)dm & -\int yzdm \\ -\int zxdm & -\int yzdm & \int\left(x^2+y^2\right)dm \end{pmatrix}$$

- We observe that the quantity in the integrand of diagonal terms is precisely the square of the distance to the x, y and z axis, respectively. They are analogous to the moment of inertia used in two-dimensional case.

- Off diagonal elements in the inertia matrix are a measure of the **imbalance** in the mass distribution.

Moment of Inertia Matrix

$$[I]=\begin{pmatrix} \int\left(y^2+z^2\right)dm & -\int xydm & -\int zxdm \\ -\int xydm & \int\left(z^2+x^2\right)dm & -\int yzdm \\ -\int zxdm & -\int yzdm & \int\left(x^2+y^2\right)dm \end{pmatrix}$$



- Off diagonal elements in the inertia matrix are a measure of the **imbalance** in the mass distribution.

For symmetric object, I_{xy} will be zero

Moment of Inertia Matrix

$$[I]=\begin{pmatrix} \int\left(y^2+z^2\right)dm & -\int xydm & -\int zxdm \\ -\int xydm & \int\left(z^2+x^2\right)dm & -\int yzdm \\ -\int zxdm & -\int yzdm & \int\left(x^2+y^2\right)dm \end{pmatrix}$$

Product of inertia will be non-zero.

Angular momentum and Torque

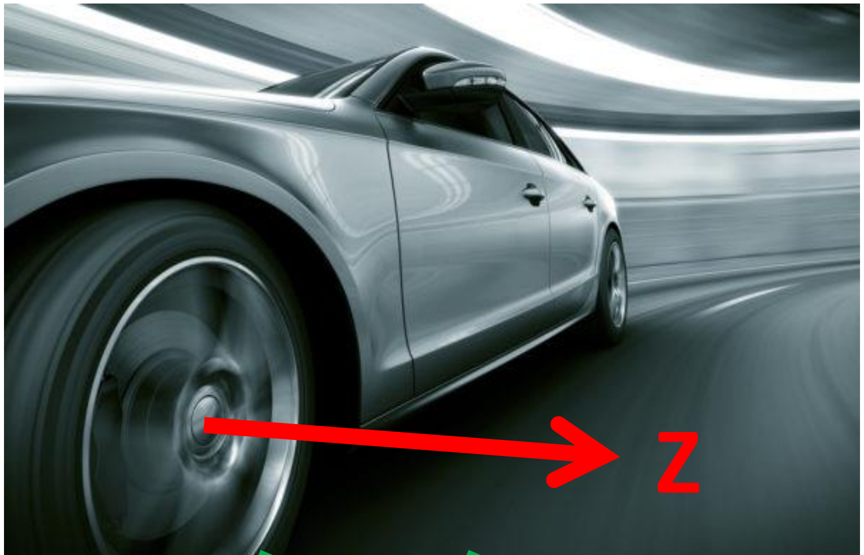
$$\begin{pmatrix} L_x \\ L_y \\ L_z \end{pmatrix} = \begin{pmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{pmatrix} \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix}$$

$$\begin{pmatrix} \tau_x \\ \tau_y \\ \tau_z \end{pmatrix} = \begin{pmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{pmatrix} \begin{pmatrix} \alpha_x \\ \alpha_y \\ \alpha_z \end{pmatrix}$$

$$\tau_z = I_{zx}\alpha_x + I_{zy}\alpha_y + I_{zz}\alpha_z$$

Torque and angular acceleration may not be in the same direction!!!.

Consequence of off-diagonal terms in Moment of Inertia matrix



$$L_z = \cancel{I_{zx}}\omega_x + \cancel{I_{zy}}\omega_y + I_{zz}\omega_z$$

$$\tau_z = \cancel{I_{zx}}\alpha_x + \cancel{I_{zy}}\alpha_y + I_{zz}\alpha_z$$

Balanced Wheel

Consequence of off-diagonal terms in Moment of Inertia Matrix

<u>**Bi-directional coupling**</u>

So, the off-diagonal terms $\Rightarrow$ any attempt to rotate the body by applying a torque about a given axis will not result in a rotation about just that axis, but there will be rotation around the other axes as well.

$$L_z = I_{zx}\omega_x + I_{zy}\omega_y + I_{zz}\omega_z$$

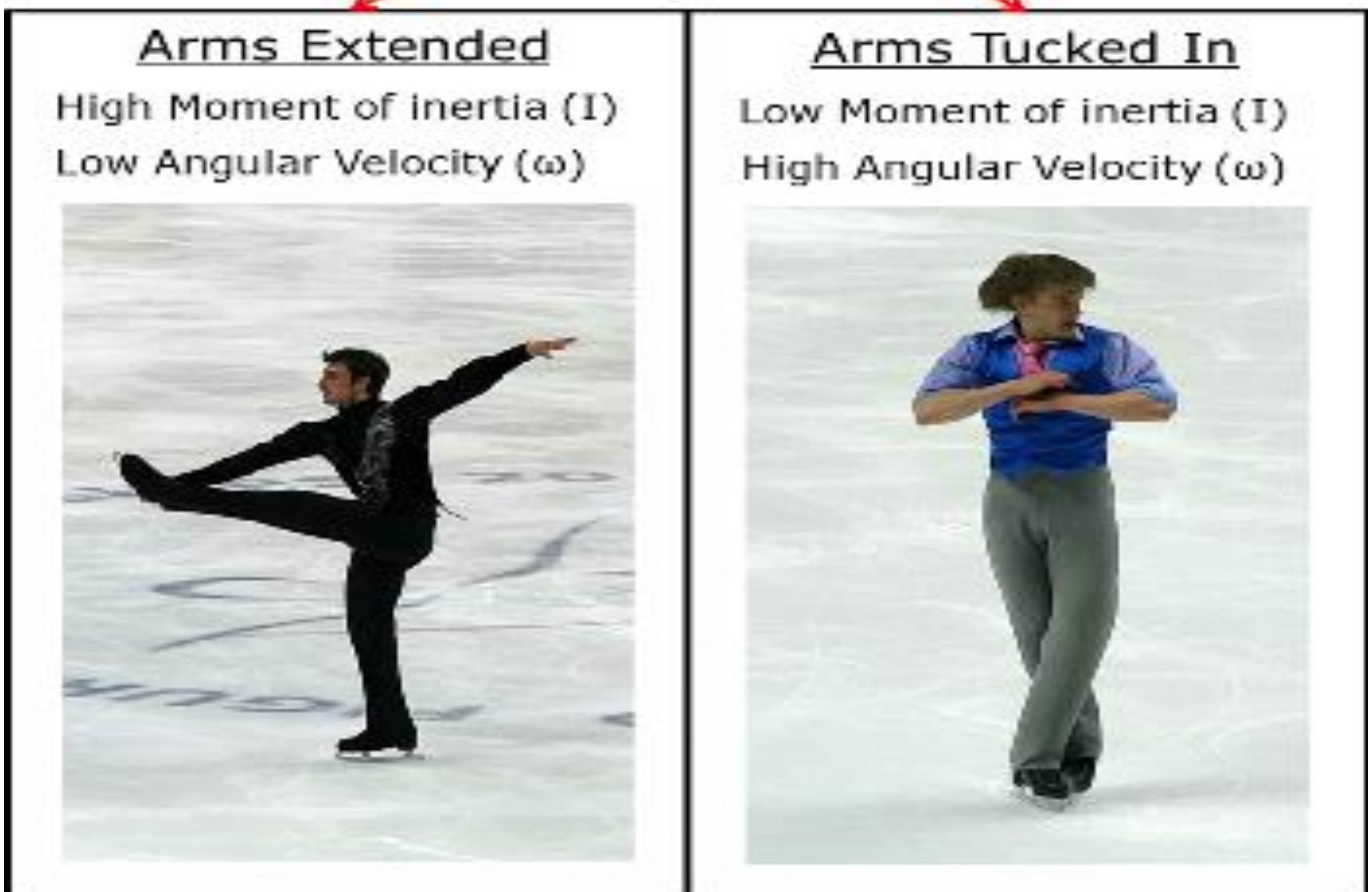
$$\tau_z = I_{zx}\alpha_x + I_{zy}\alpha_y + I_{zz}\alpha_z$$

Off-diagonal elements represent <u>**coupling**</u> of rotation about one axis with other!!

Moment of Inertia Matrix

$$L_i = L_f$$

$$I_i \omega_i = I_f \omega_f$$



Is Moment of Inertia a Scalar? Vector?

Or …. ????

Tensor

Tensor

Tensor represents a physical entity which may be characterized by magnitude and multiple directions simultaneously.

The rank of a tensor is defined by the number of directionality required to describe a component of it.

Tensor

The rank of a tensor is defined by the number of directionality required to describe a component of it.

Temperature, Mass, Potential → '0'

SCALAR

$$Temperature = [T]$$

Force, Velocity, Torque

VECTOR

→ '1'

$$\vec{F} = F_x\hat{e}_x + F_y\hat{e}_y + F_z\hat{e}_z$$

$$\vec{F} = \begin{bmatrix} F_x \\ F_y \\ F_z \end{bmatrix}$$

Tensor

Moment of Inertia

$$[I]=\begin{pmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{pmatrix}$$

'2'

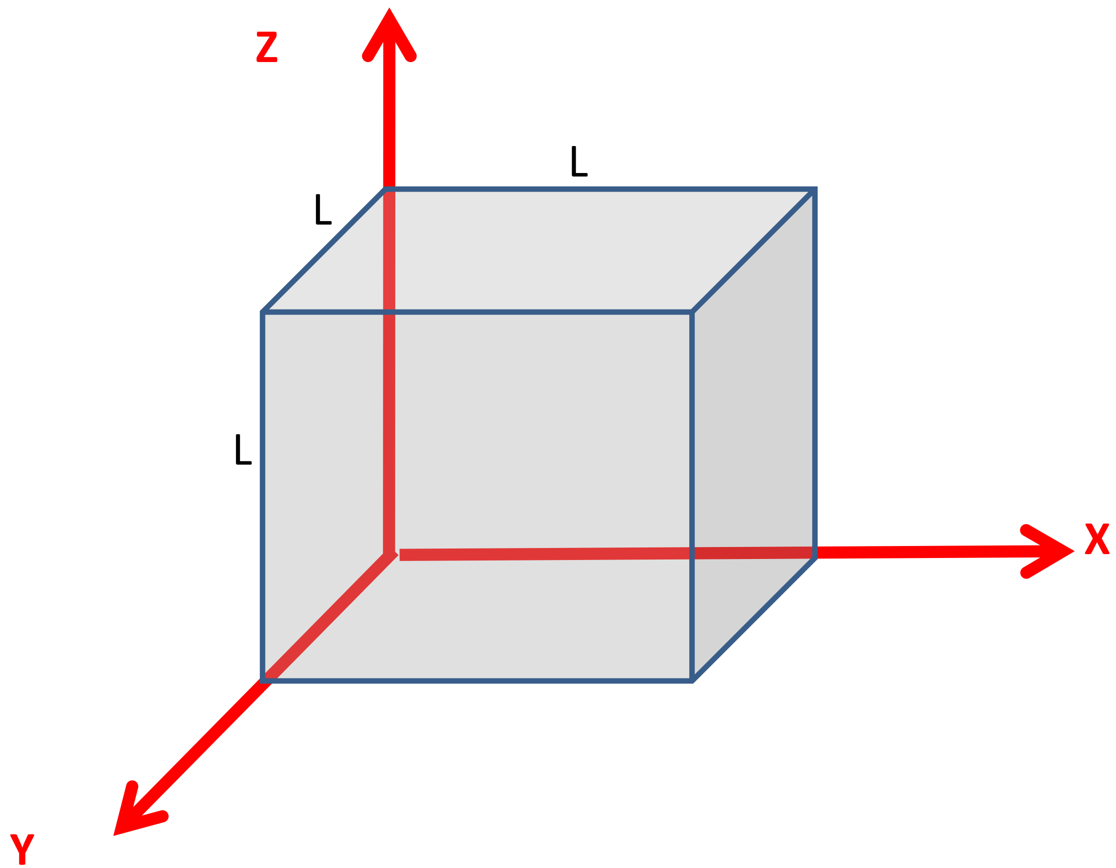
$$\tau_z = I_{zx}\alpha_x + I_{zy}\alpha_y + I_{zz}\alpha_z$$

Torque applied in z – direction need not guarantee angular acceleration only in x-direction, rather it can have y-component, x-component as well. This is due to coupling between two directions (Z-X, Z-Y .....etc).

The rank of a tensor is defined by the number of directionality required to describe a component of it.

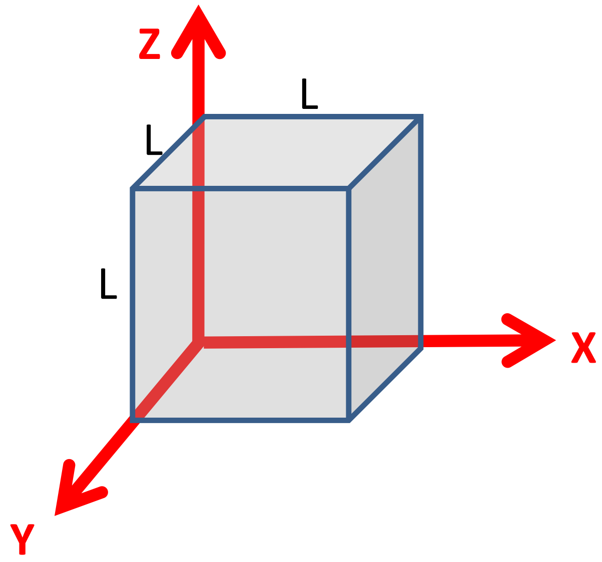
Rank 2 Tensor represents a physical entity which may be characterized by magnitude and bi-directionality.

Find the Moment of Inertia tensor of a cube with Side length L and mass M, with co-ordinate axis parallel to the edges of the cube and the origin at a corner.



Moment of Inertia Tensor

$$[I] = \begin{pmatrix} \int (y^2 + z^2) dm & -\int xydm & -\int zxdm \\ -\int xydm & \int (z^2 + x^2) dm & -\int yzdm \\ -\int zxdm & -\int yzdm & \int (x^2 + y^2) dm \end{pmatrix}$$



Moment of Inertia Tensor

$$\int\left(y^2+z^2\right)dm=\iiint\left(y^2+z^2\right)\rho\,dxdydz$$

$$\iiint\left(y^2+z^2\right)\rho\,dxdydz=\rho L\int_0^L\int_0^L\left(y^2+z^2\right)dydz$$

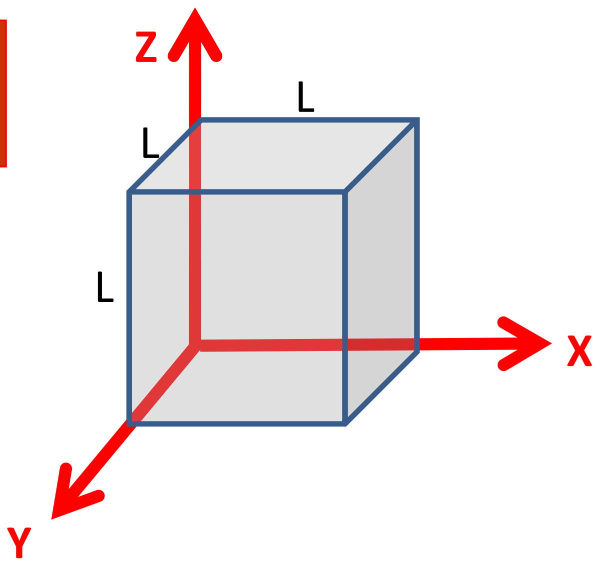
$$\rho L\int_0^L\int_0^L\left(y^2+z^2\right)dydz=\rho L\int_0^L\left(\frac{L^3}{3}+z^2L\right)dz$$

$$\int\left(y^2+z^2\right)dm=\frac{2}{3}L^5\rho=\frac{2}{3}ML^2$$

Moment of Inertia Tensor

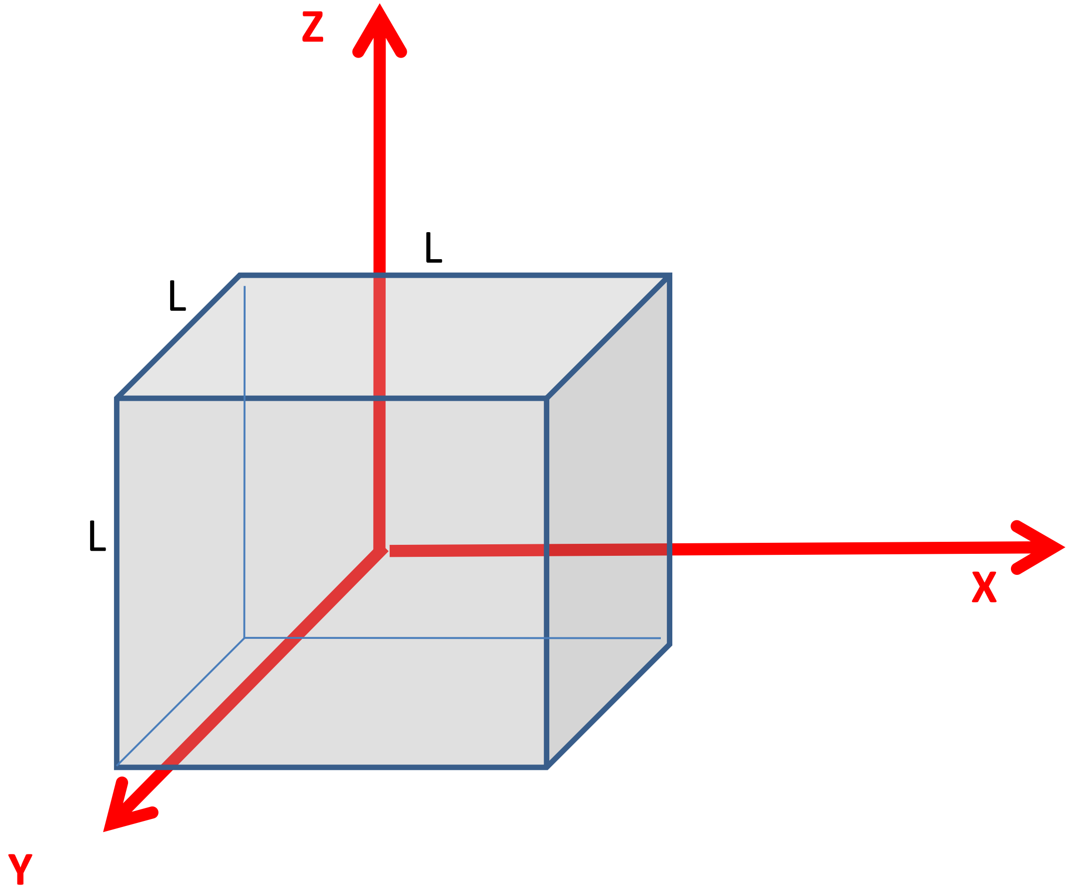
$$-\int (xy)\,dm = -\int xy\rho\,dxdydz$$

$$-\int (xy)\,dm = -\frac{ML^2}{4}$$



$$[I] = ML^2 \begin{pmatrix} 2/3 & -1/4 & -1/4 \\ -1/4 & 2/3 & -1/4 \\ -1/4 & -1/4 & 2/3 \end{pmatrix}$$

Try with origin at the center


Z
L
L
L
X
Y

Moment of Inertia Tensor

$$\int \left(y^2 + z^2 \right) dm = \iiint \left(y^2 + z^2 \right) \rho \, dx\,dy\,dz$$

$$\iiint \left(y^2 + z^2 \right) \rho \, dx\,dy\,dz = \rho L \int_{-L/2}^{L/2} \int_{-L/2}^{L/2} \left(y^2 + z^2 \right) dy\,dz$$

$$\rho L \int_{-L/2}^{L/2} \int_{-L/2}^{L/2} \left(y^2 + z^2 \right) dy\,dz = \rho L \int_{-L/2}^{L/2} \left(\frac{L^3}{12} + z^2 L \right) dz$$

$$\int \left(y^2 + z^2 \right) dm = \frac{1}{6} M L^2$$

Moment of Inertia Tensor

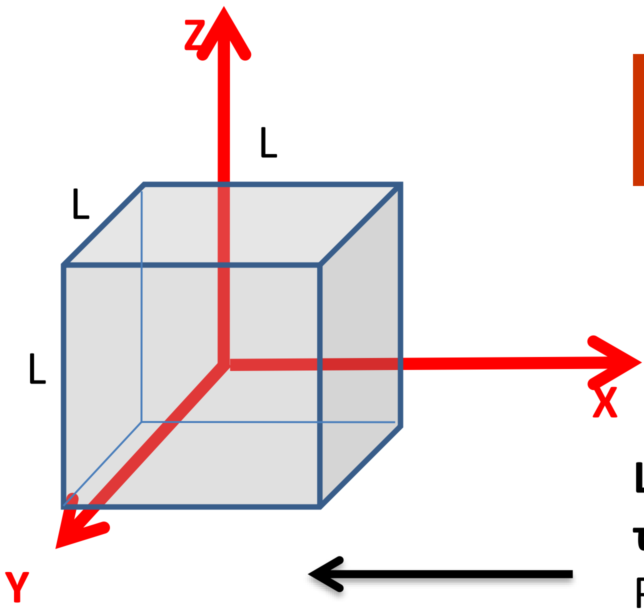
$$-\int (xy)\,dm = -\int xy\rho\,dxdydz$$

$$-\int (xy)\,dm = 0$$

$$\int \left(y^2 + z^2\right) dm = \frac{1}{6} ML^2$$

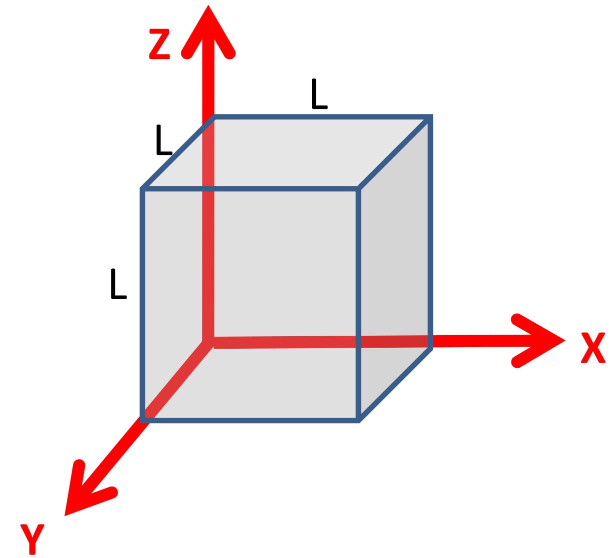
$$[I] = ML^2 \begin{pmatrix} 1/6 & 0 & 0 \\ 0 & 1/6 & 0 \\ 0 & 0 & 1/6 \end{pmatrix}$$

Rotation of a cube along 2 different Axis



$$\vec{L} = [I][\vec{\omega}]$$

$$\vec{\tau} = [I][\vec{\alpha}]$$



L is parallel to **ω**
τ is parallel to **α**
Provided axis of rotation is along one of the axis.

$$[I] = ML^2 \begin{pmatrix} 1/6 & 0 & 0 \\ 0 & 1/6 & 0 \\ 0 & 0 & 1/6 \end{pmatrix}$$

$$[I] = ML^2 \begin{pmatrix} 2/3 & -1/4 & -1/4 \\ -1/4 & 2/3 & -1/4 \\ -1/4 & -1/4 & 2/3 \end{pmatrix}$$

Important Observations

- Choosing a rotation axis at the center of the cube made the Moment of Inertia matrix diagonal
- Torque for rotation is given along a particular axis

L is parallel to $\boldsymbol{\omega}$

$\boldsymbol{\tau}$ is parallel to $\boldsymbol{\alpha}$

NOTE: IN ALL THE DISCUSSIONS TILL NOW TORQUE WAS GIVEN ALONG THE AXIS

What happens if a torque is applied off-axis?

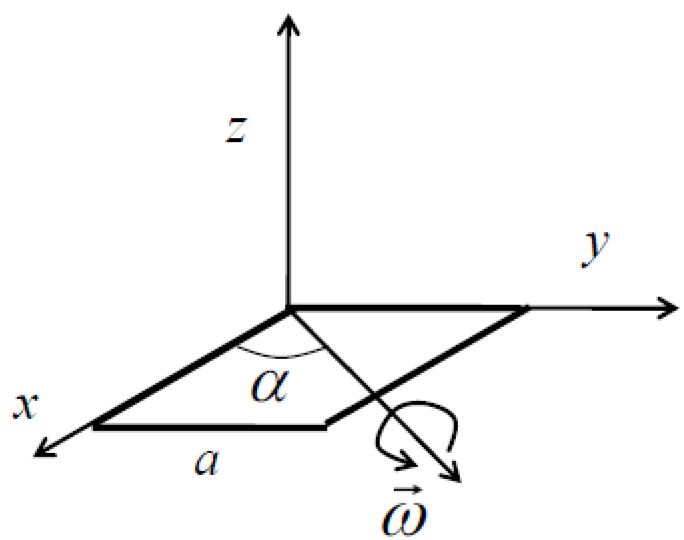
Rotation of a square plate

Consider rotation of a square plate of side a and mass *M* about an axis in the plane of the plate and making an angle α with the x-axis.

(a) What is the angular momentum **L** about the origin?

(b) For what angle **L** and **ω** becomes parallel?

(c) For square plate when the moment of inertia tensor becomes diagonal?



Rotation of a square plate

(a) What is the angular momentum **L** about the origin?

$$\vec{L} = [I][\vec{\omega}]$$

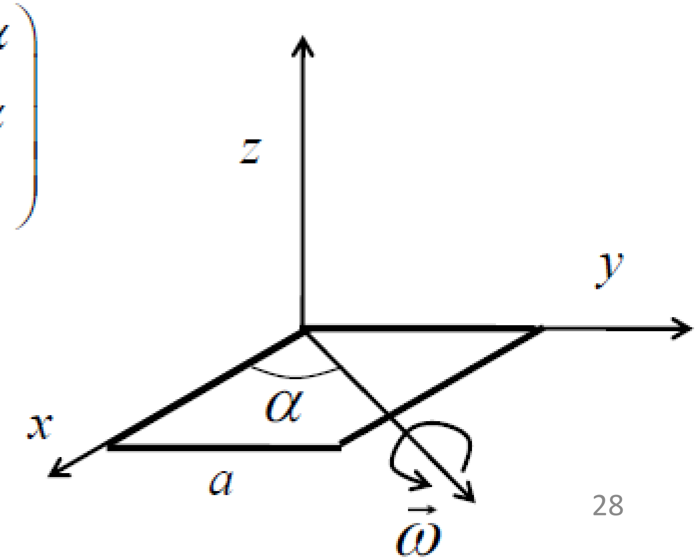
$$[I] = \begin{pmatrix} \int (y^2 + z^2) dm & -\int xydm & -\int zxdm \\ -\int xydm & \int (z^2 + x^2) dm & -\int yzdm \\ -\int zxdm & -\int yzdm & \int (x^2 + y^2) dm \end{pmatrix}$$

$$I_{xx} = \int_{x=0}^{a} \int_{y=0}^{a} \sigma y^2 dxdy = \frac{1}{3}\sigma a^4$$

$$I_{xy} = -\int_{x=0}^{a} \int_{y=0}^{a} \sigma xydxdy = -\frac{1}{4}\sigma a^4$$

$$I_{zz} = \frac{2}{3} Ma^2$$

$$\vec{\omega} = \begin{pmatrix} \omega\cos\alpha \\ \omega\sin\alpha \\ 0 \end{pmatrix}$$



Rotation of a square plate

(a) What is the angular moment L about the origin?

$$I_{xx} = \int_{x=0}^{a}\int_{y=0}^{a} \sigma y^2 dxdy = \frac{1}{3}\sigma a^4$$

$$\vec{\omega} = \begin{pmatrix} \omega\cos\alpha \\ \omega\sin\alpha \\ 0 \end{pmatrix}$$

$$I_{xy} = -\int_{x=0}^{a}\int_{y=0}^{a} \sigma xydxdy = -\frac{1}{4}\sigma a^4$$

$$I_{zz} = \frac{2}{3}Ma^2$$

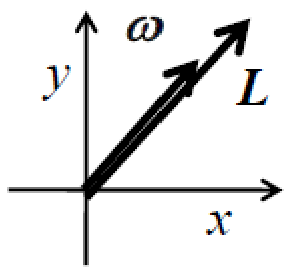
$$\vec{L} = \begin{pmatrix} Ma^2/3 & -Ma^2/4 & 0 \\ -Ma^2/4 & Ma^2/3 & 0 \\ 0 & 0 & 2Ma^2/3 \end{pmatrix}\begin{pmatrix} \omega\cos\alpha \\ \omega\sin\alpha \\ 0 \end{pmatrix} = \begin{pmatrix} Ma^2\omega\left(\frac{1}{3}\cos\alpha - \frac{1}{4}\sin\alpha\right) \\ Ma^2\omega\left(-\frac{1}{4}\cos\alpha + \frac{1}{3}\sin\alpha\right) \\ 0 \end{pmatrix}$$

Rotation of a square plate

(b) For what angle L and ω becomes parallel?

For $\alpha = 45^\circ$,

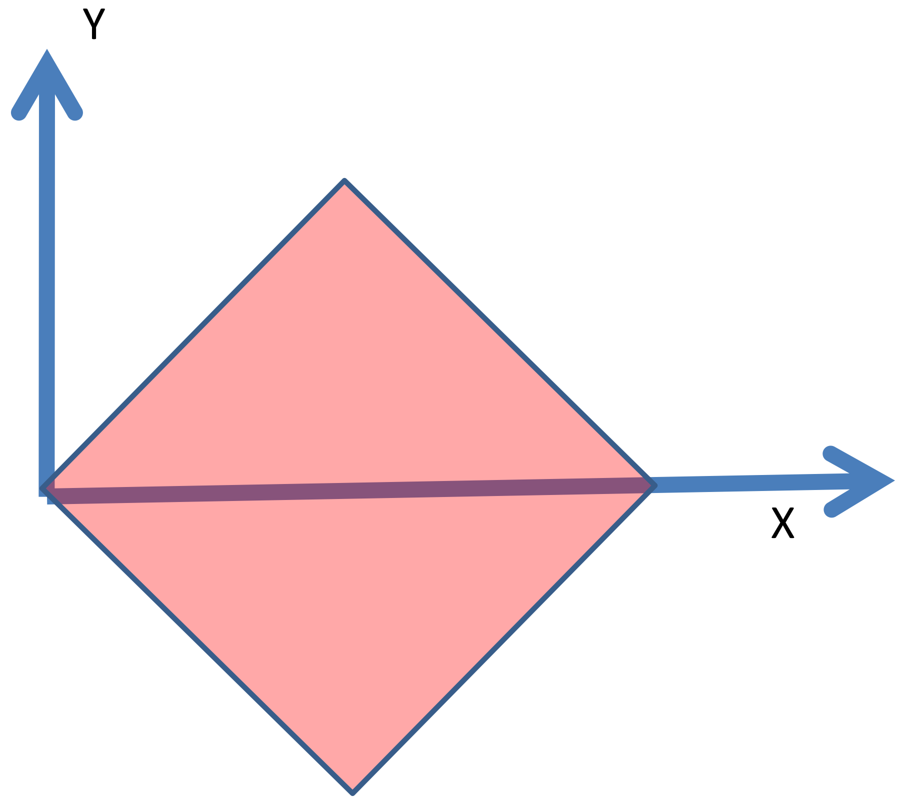
$$\vec{L} = \left(\frac{1}{12\sqrt{2}} Ma^2\omega, \frac{1}{12\sqrt{2}} Ma^2\omega, 0 \right) \quad \text{and} \quad \vec{\omega} = \begin{pmatrix} \omega/\sqrt{2} \\ \omega/\sqrt{2} \\ 0 \end{pmatrix}$$



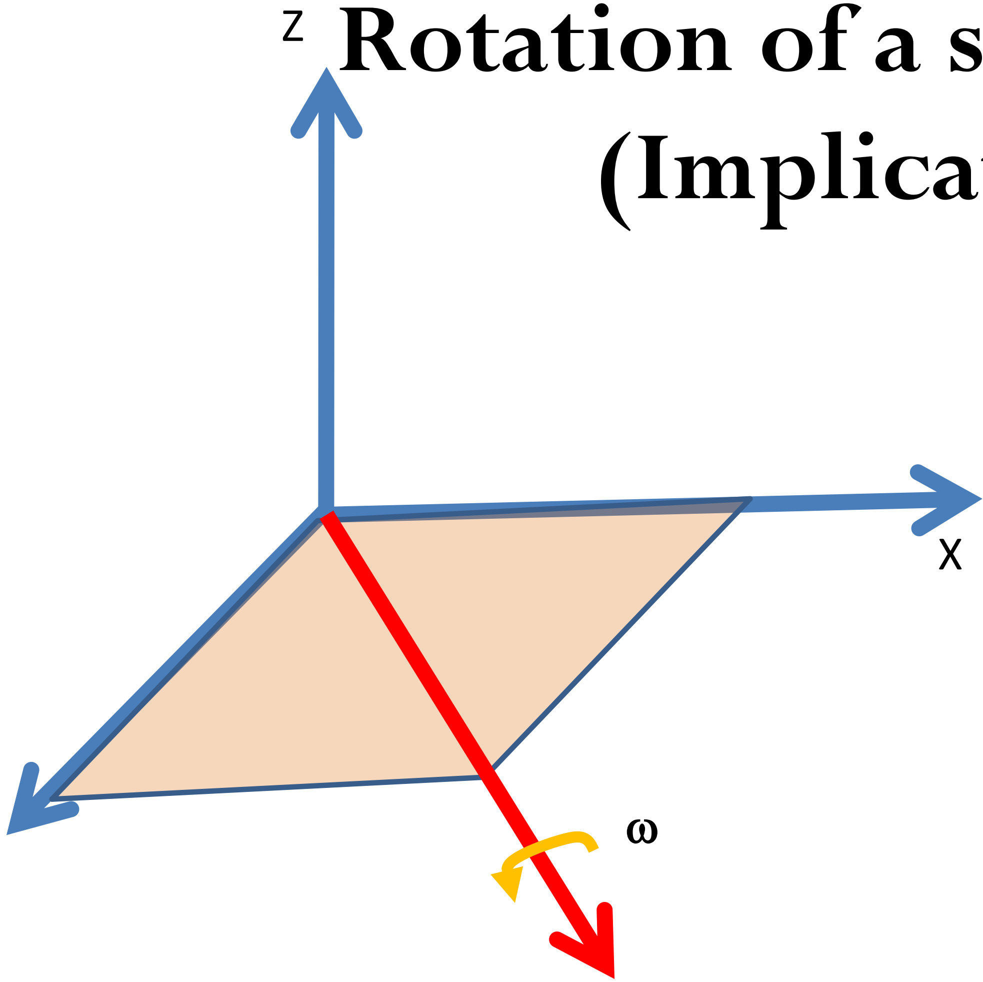
$$\vec{L} = [I][\vec{\omega}] = \lambda\vec{\omega}$$

Rotation of a square plate

<u>(c)</u> for the square plate when the moment of inertia tensor becomes diagonal?



Rotation of a square plate (Implications)

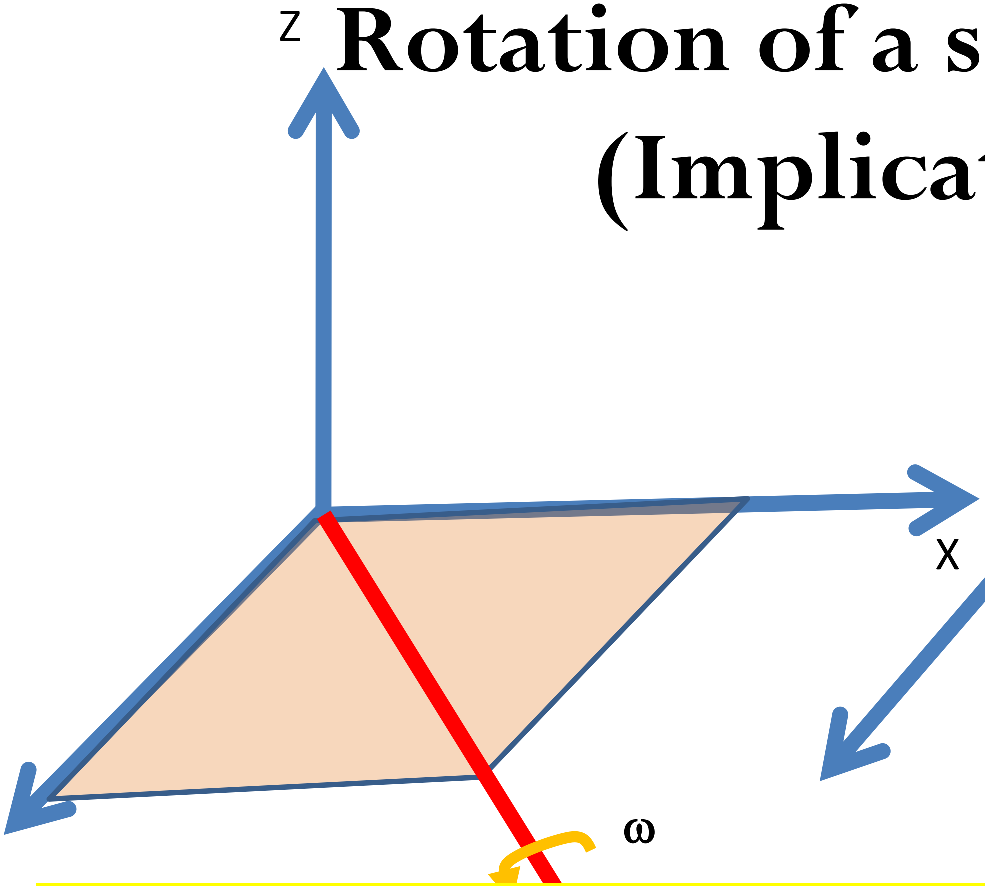


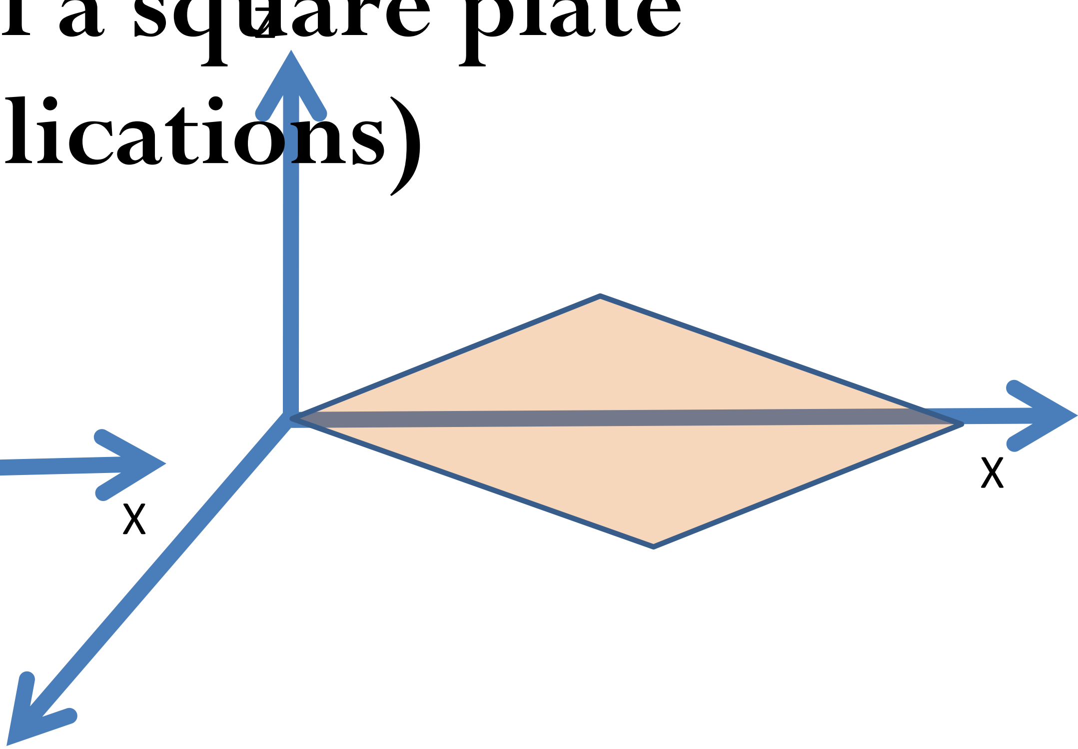
$$[I] = \begin{pmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{pmatrix}$$

$$\vec{L} = [I][\vec{\omega}]$$

L and **ω** are parallel!

Rotation of a square plate (Implications)





Use of symmetry will ensure diagonal Moment of Inertia tensor

$$[I] = \begin{pmatrix} I_{xx} & 0 & 0 \\ 0 & I_{yy} & 0 \\ 0 & 0 & I_{zz} \end{pmatrix}$$

$$L_x = I_{xx}\omega_x$$

$$\tau_x = I_{xx}\alpha_x$$

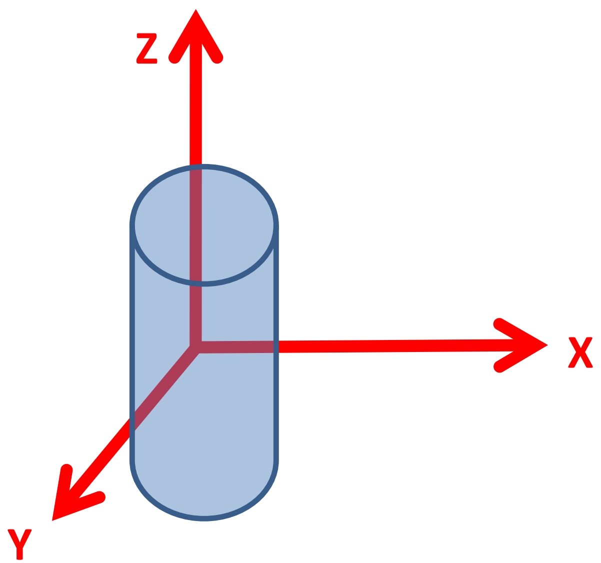
Principal Axis

$$[I] = \begin{pmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{pmatrix}$$

Cumbersome!

Principal axes are the orthogonal axes for Which [I] is diagonal

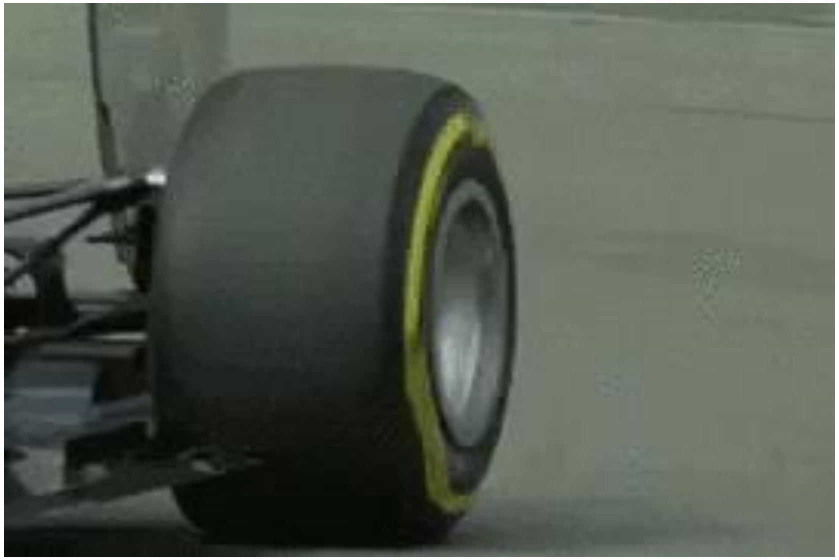
$$[I] = \begin{pmatrix} I_{xx} & 0 & 0 \\ 0 & I_{yy} & 0 \\ 0 & 0 & I_{zz} \end{pmatrix}$$



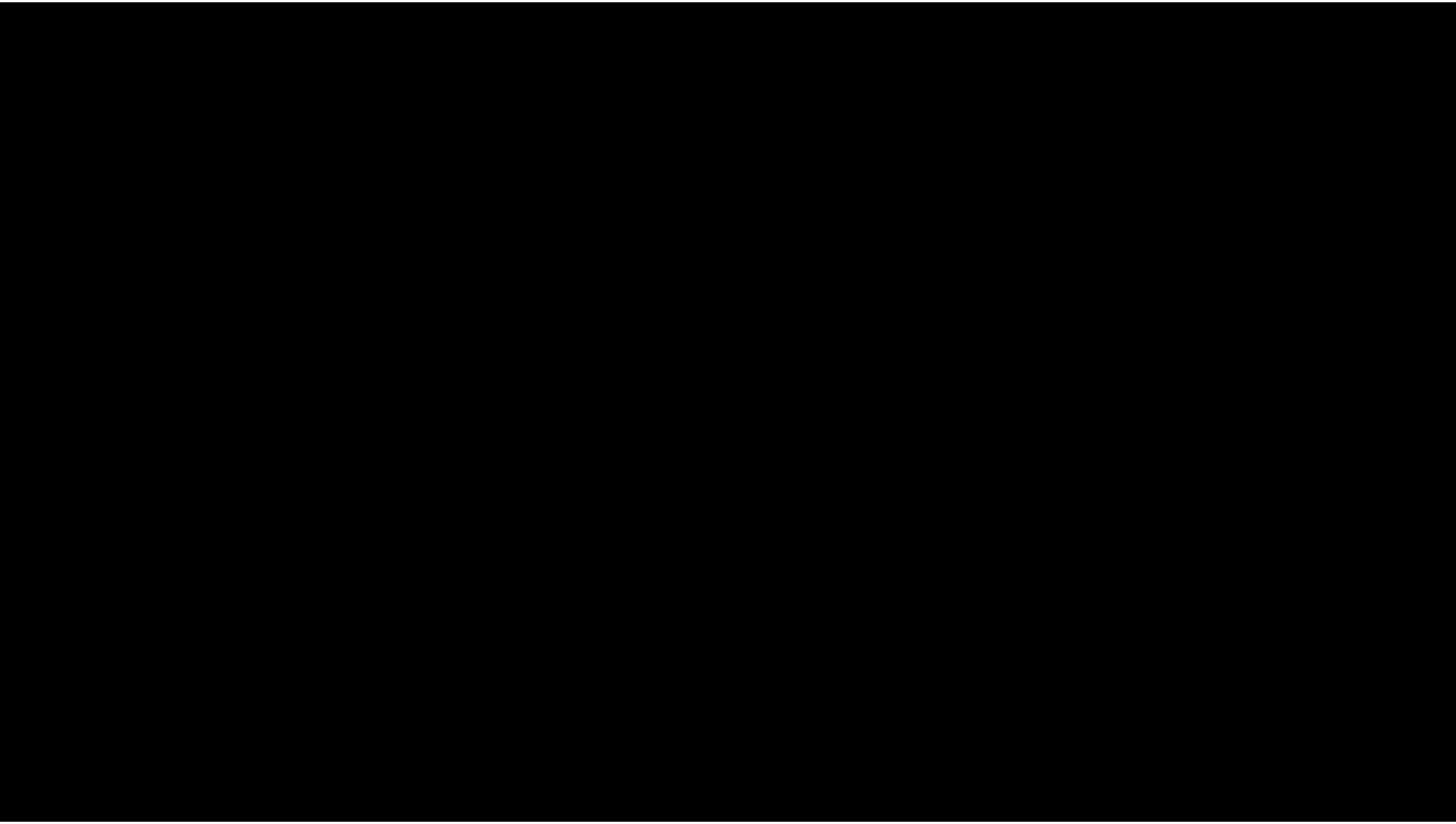
$$L_x = I_{xx}\omega_x \qquad L_y = I_{yy}\omega_y$$

$$L_z = I_{zz}\omega_z$$

Non-zero Product of Inertia

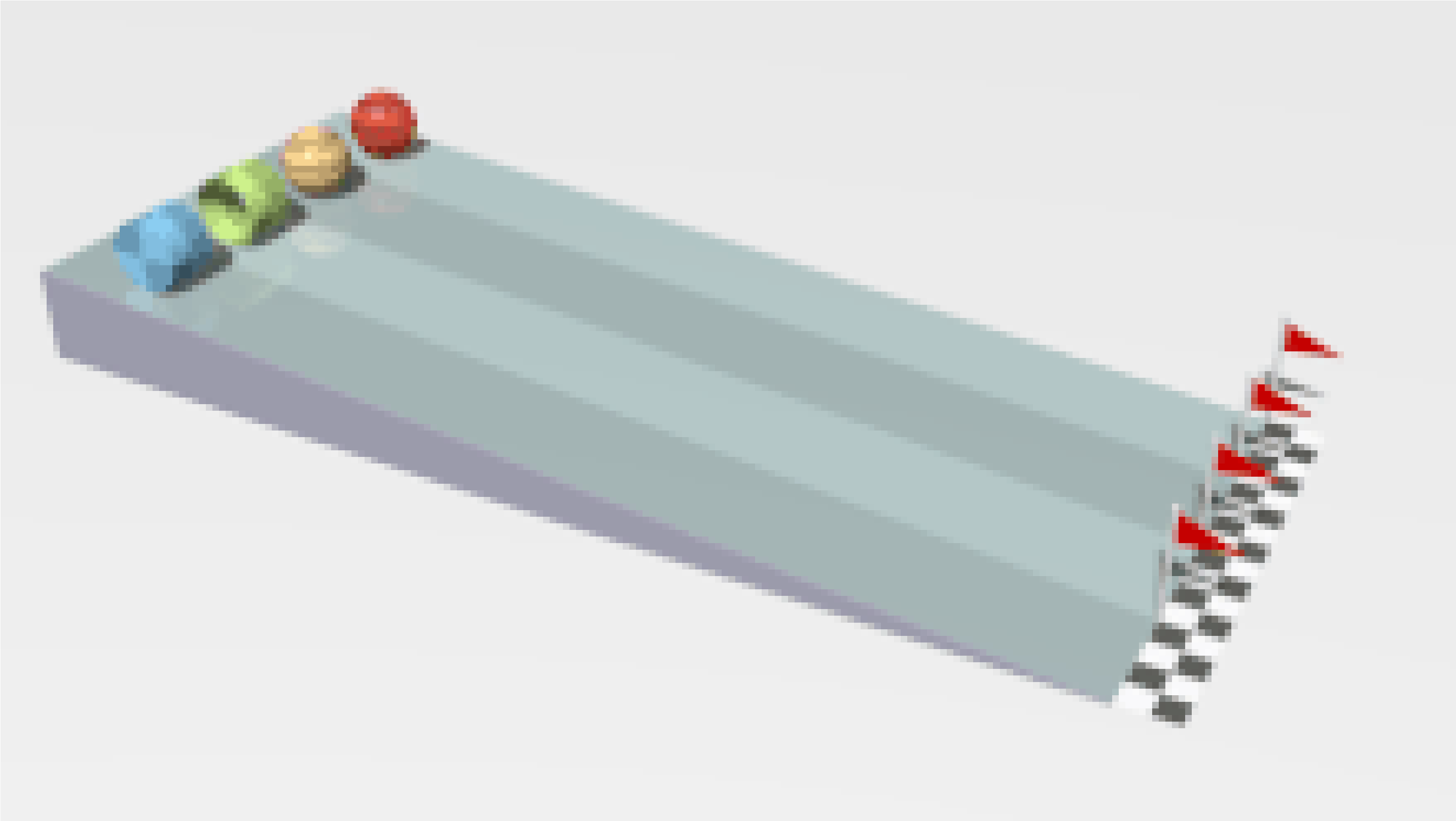


Importance of moment of inertia
Winter Olympics, 2006 Turin, Italy



https://www.youtube.com/watch?v=3FHqYxThI5g

Importance of moment of inertia



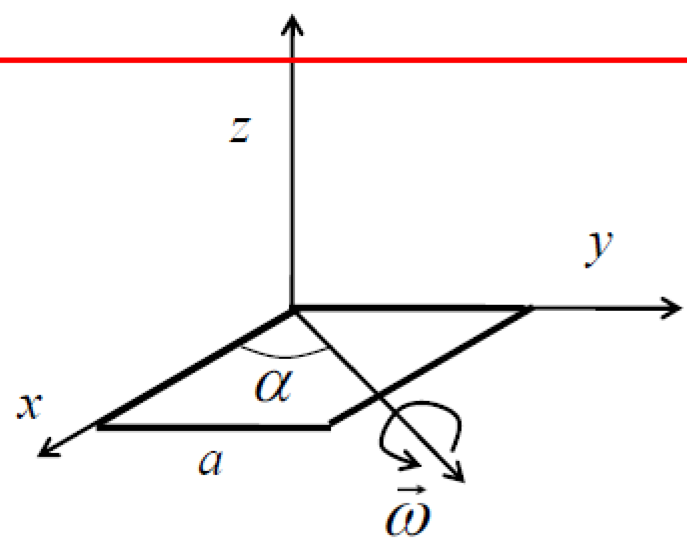
Rotation of a square plate

Consider rotation of a square plate of side a and mass *M* about an axis in the plane of the plate and making an angle α with the x-axis.

(a) What is the angular momentum **L** about the origin?

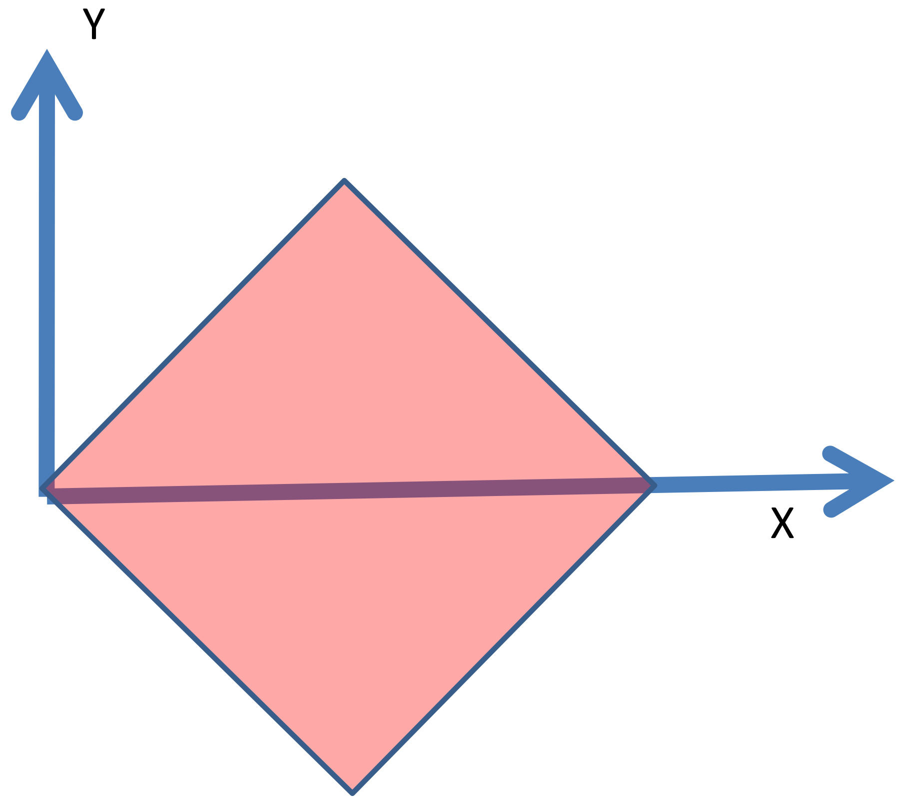
(b) For what angle **L** and **ω** becomes parallel?

(c) For square plate when the moment of inertia tensor becomes diagonal?

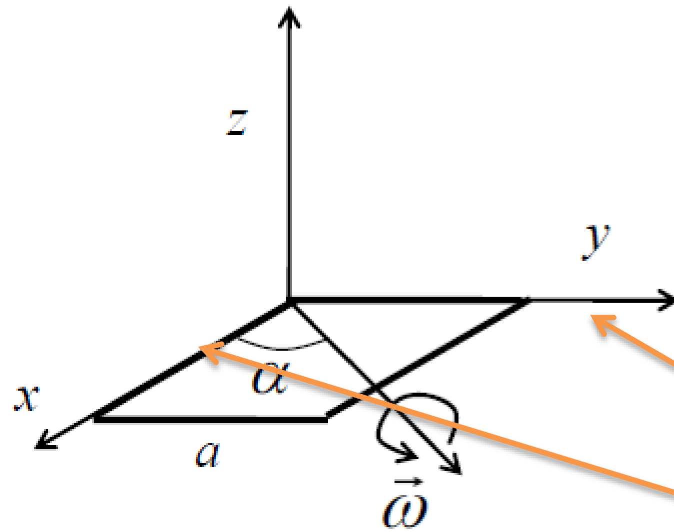


Rotation of a square plate

(c) for the square plate when the moment of inertia tensor becomes diagonal?



Concept behind the problem



Wobble

Torque
Is unequal
Along
X and Y axis

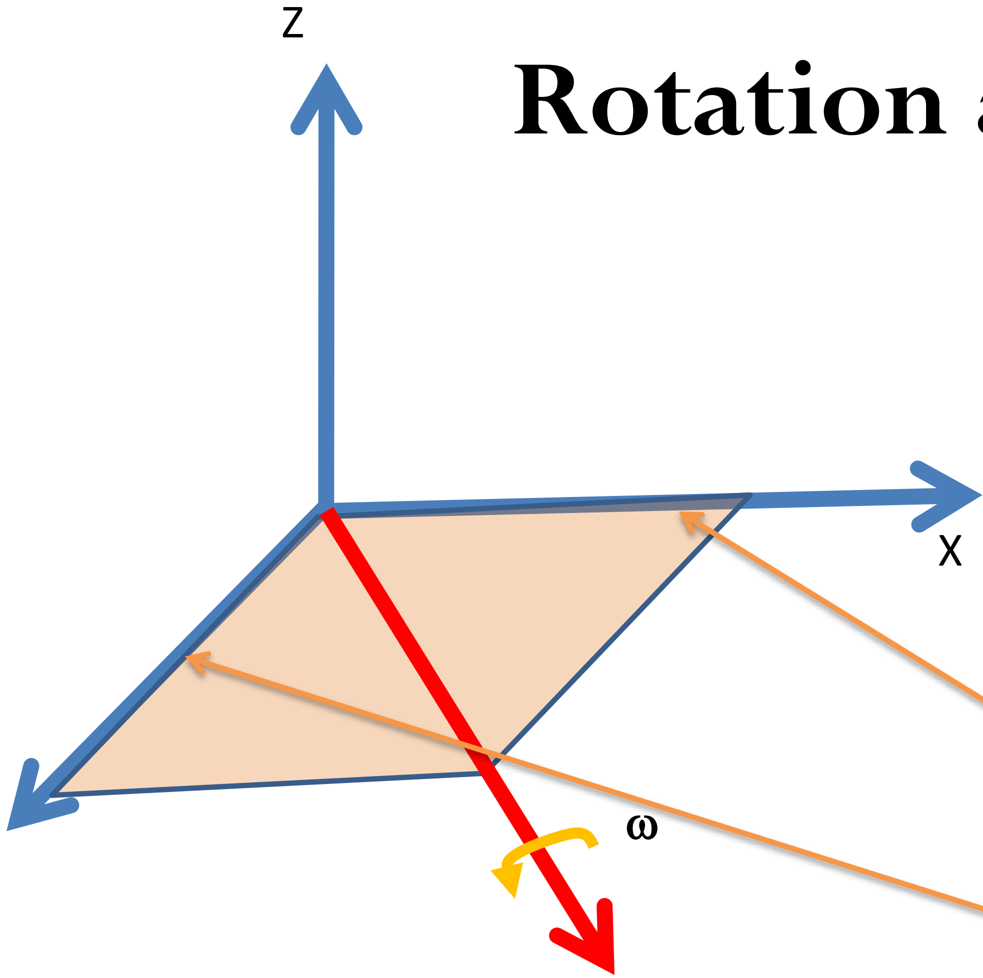
$$\vec{L}=\begin{pmatrix} Ma^2/3 & -Ma^2/4 & 0 \\ -Ma^2/4 & Ma^2/3 & 0 \\ 0 & 0 & 2Ma^2/3 \end{pmatrix}\begin{pmatrix} \omega\cos\alpha \\ \omega\sin\alpha \\ 0 \end{pmatrix}=\begin{pmatrix} Ma^2\omega\left(\frac{1}{3}\cos\alpha-\frac{1}{4}\sin\alpha\right) \\ Ma^2\omega\left(-\frac{1}{4}\cos\alpha+\frac{1}{3}\sin\alpha\right) \\ 0 \end{pmatrix}$$

L is not parallel to **ω**

τ is not parallel to **α**

$$\boldsymbol{\tau}=\begin{bmatrix} Ma^2\left(\frac{1}{3}cos\alpha-\frac{1}{4}sin\alpha\right)\acute{\alpha} \\ Ma^2\left(-\frac{1}{4}cos\alpha+\frac{1}{3}sin\alpha\right)\acute{\alpha} \\ 0 \end{bmatrix}$$

Rotation at 45 degrees

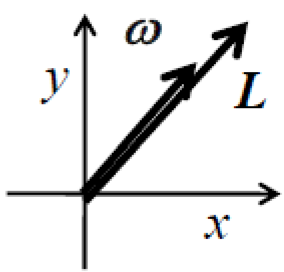


For $\alpha = 45^\circ$,

$$\vec{L} = \left(\frac{1}{12\sqrt{2}} Ma^2 \omega, \frac{1}{12\sqrt{2}} Ma^2 \omega, 0 \right) \quad \text{and} \quad \vec{\omega} = \begin{pmatrix} \omega/\sqrt{2} \\ \omega/\sqrt{2} \\ 0 \end{pmatrix}$$

$$\boxed{\vec{L} = [I][\vec{\omega}] = \lambda\vec{\omega}}$$

L and **ω** are parallel!



$$\boldsymbol{\tau} = \begin{bmatrix} Ma^2 \left(\frac{1}{12\sqrt{2}} \right) \acute{\alpha} \\ Ma^2 \left(\frac{1}{12\sqrt{2}} \right) \acute{\alpha} \\ 0 \end{bmatrix}$$

Torque Is equal Along X and Y axis

Does the Plate Wobble?

Answer is NO

L is parallel to **ω**

τ is parallel to **α**

$$\boldsymbol{\tau} = \begin{bmatrix} Ma^2\left(\frac{1}{12\sqrt{2}}\right)\acute{\alpha} \\ Ma^2\left(\frac{1}{12\sqrt{2}}\right)\acute{\alpha} \\ 0 \end{bmatrix}$$

TORQUE IS BALANCED: Component of torque are equal in X AND Y directions

Note!

- **Whenever ω is parallel to L, choosing the corresponding direction of rotation as co-ordinate axis will be the principal axes.**

How to find principal axis????

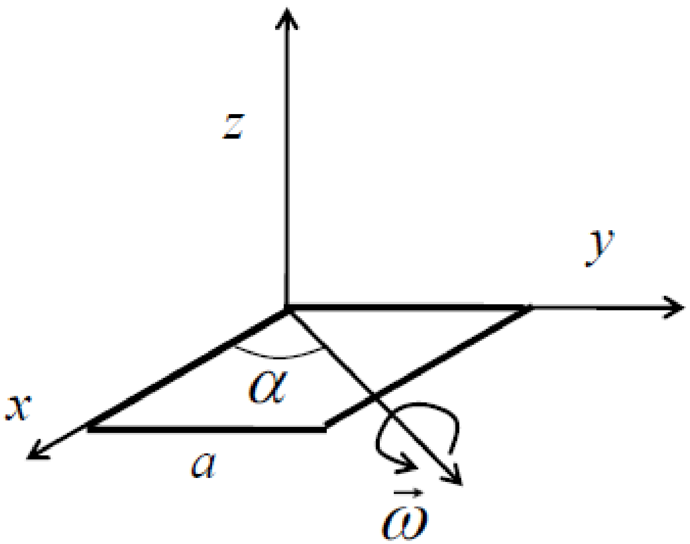
For simple geometries it is easy to find by intuition but not for any general case

How to find diagonalized MI tensor??

Mathematical approach!

How to find principal axis????

$$[I] = \begin{pmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{pmatrix}$$



$$\vec{L} = [I][\vec{\omega}]$$

$$[I][\vec{\omega}] = \lambda\vec{\omega}$$

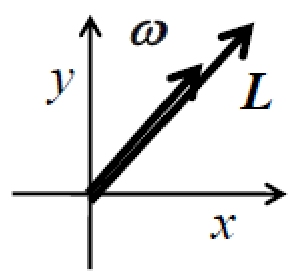
We can write like that when **L** and ω are parallel

Rotation of a square plate

(b) For what angle L and ω becomes parallel?

For $\alpha = 45^\circ$,

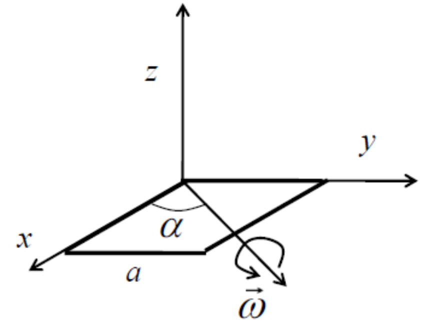
$$\vec{L} = \left(\frac{1}{12\sqrt{2}} Ma^2\omega, \frac{1}{12\sqrt{2}} Ma^2\omega, 0 \right) \quad \text{and} \quad \vec{\omega} = \begin{pmatrix} \omega/\sqrt{2} \\ \omega/\sqrt{2} \\ 0 \end{pmatrix}$$



$$\vec{L} = [I][\vec{\omega}] = \lambda\vec{\omega}$$

How to find principal axis????

$$[I] = \begin{pmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{pmatrix}$$



$$\vec{L} = [I][\vec{\omega}]$$

$$[I][\vec{\omega}] = \lambda\vec{\omega}$$

Unknowns

Step: 1

Find the axis of rotation where **L** and **ω** are parallel.

WE IMPOSE THE CONDITION, TO MAKE L AND ω PARALLEL

$$[I][\vec{\omega}] = \lambda\vec{\omega}$$

Find the unknowns ω and λ

How to find principal axis????

$$[I][\vec{\omega}] = \lambda\vec{\omega}$$

$$\begin{pmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{pmatrix}\begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix} = \lambda\begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix}$$

Insert a unit matrix

$$\begin{pmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{pmatrix}\begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix} = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix}\begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix}$$

$$[MI] - \lambda[1] = 0$$

How to find principal axis????

$$\begin{pmatrix} I_{xx}-\lambda & I_{xy} & I_{xz} \\ I_{yx} & I_{yy}-\lambda & I_{yz} \\ I_{zx} & I_{zy} & I_{zz}-\lambda \end{pmatrix} \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix} = 0$$

Theorem: If **[A][x]=0,** then [A] is non-invertible. This implies A^{-1} does not exist

Hence, $|A|=0$.

MA102 Mathematics - II

MA102	**Mathematics - II**	**3-1-0-8**	**Pre-requisites: nil**

Linear Algebra: Vector spaces (over the field of real and complex numbers). Systems of linear equations and their solutions. Matrices, determinants, rank and inverse. Linear transformations. Range space and rank, null space and nullity. Eigenvalues and eigenvectors. Similarity transformations. Diagonalization of Hermitian matrices. Bilinear and quadratic forms.

How to find principal axis????

$$\begin{pmatrix} I_{xx}-\lambda & I_{xy} & I_{xz} \\ I_{yx} & I_{yy}-\lambda & I_{yz} \\ I_{zx} & I_{zy} & I_{zz}-\lambda \end{pmatrix}\begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix}=0$$

$$\begin{vmatrix} I_{xx}-\lambda & I_{xy} & I_{xz} \\ I_{yx} & I_{yy}-\lambda & I_{yz} \\ I_{zx} & I_{zy} & I_{zz}-\lambda \end{vmatrix}=0$$

Characteristic Equation

How to find principal axis????

$$\begin{vmatrix} I_{xx}-\lambda & I_{xy} & I_{xz} \\ I_{yx} & I_{yy}-\lambda & I_{yz} \\ I_{zx} & I_{zy} & I_{zz}-\lambda \end{vmatrix} = 0$$

Solving characteristic equation result in
$\lambda_1, \lambda_2, \lambda_3.$

$$\begin{pmatrix} I_{xx}-\lambda & I_{xy} & I_{xz} \\ I_{yx} & I_{yy}-\lambda & I_{yz} \\ I_{zx} & I_{zy} & I_{zz}-\lambda \end{pmatrix} \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix} = 0$$

Substitute
$\lambda_1, \lambda_2, \lambda_3$ separately
and solve for ωs

Note!

- $\lambda_1, \lambda_2, \lambda_3$ are called the Eigen values which satisfies the equation

$$[I][\vec{\omega}] = \lambda\vec{\omega}$$

- [ω]'s are called the Eigen vectors.

- For each Eigen values, Eigen vector can be found.

Step2: How to find diagonalized MI tensor??

Diagonalization Theorem,

$$D = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}$$

Diagonalization ensures the rotation axis is along the coordinate axis (Principal axis)

(Will be taught in MA102)

MA102 Mathematics - II

MA102	Mathematics - II	3-1-0-8	Pre-requisites: nil

Linear Algebra: Vector spaces (over the field of real and complex numbers). Systems of linear equations and their solutions. Matrices, determinants, rank and inverse. Linear transformations. Range space and rank, null space and nullity. Eigenvalues and eigenvectors. Similarity transformations. Diagonalization of Hermitian matrices. Bilinear and quadratic forms.

Find principal axis of a rigid body whose moment of inertia tensor is given as

$$I = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$$

$$[I][\vec{\omega}] = \lambda\vec{\omega}$$

$$\begin{vmatrix} 2-\lambda & -1 \\ -1 & 2-\lambda \end{vmatrix} = 0$$

Characteristic Equation

$$\lambda^2 - 4\lambda + 3 = 0$$

$$\lambda_1 = 1, \lambda_2 = 3,$$

Find Principal axis of a rigid body

Use λ_1 in eigenvalue expression:

$$\begin{pmatrix} 2-1 & -1 \\ -1 & 2-1 \end{pmatrix}\begin{pmatrix} \omega_x \\ \omega_y \end{pmatrix} = 0$$

$$\omega_x = \omega_y = a$$

$$[I][\vec{\omega}] = \lambda\vec{\omega}$$

Eigenvalue equation corresponding to λ_{1}:

$$[I][\omega] = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} a \\ a \end{bmatrix} = \begin{bmatrix} a \\ a \end{bmatrix}$$

Eigenvector corresponding to λ_{1}:

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Find Principal axis of a rigid body

Use λ_2 in eigenvalue expression:

$$\begin{pmatrix} 2-3 & -1 \\ -1 & 2-3 \end{pmatrix}\begin{pmatrix} \omega_x \\ \omega_y \end{pmatrix} = 0$$

$$\omega_x = -\omega_y$$

Eigenvector corresponding to λ_2:

$$\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Verify orthogonality of eigenvectors

Finding Principal axis of a rigid body from eigenvectors

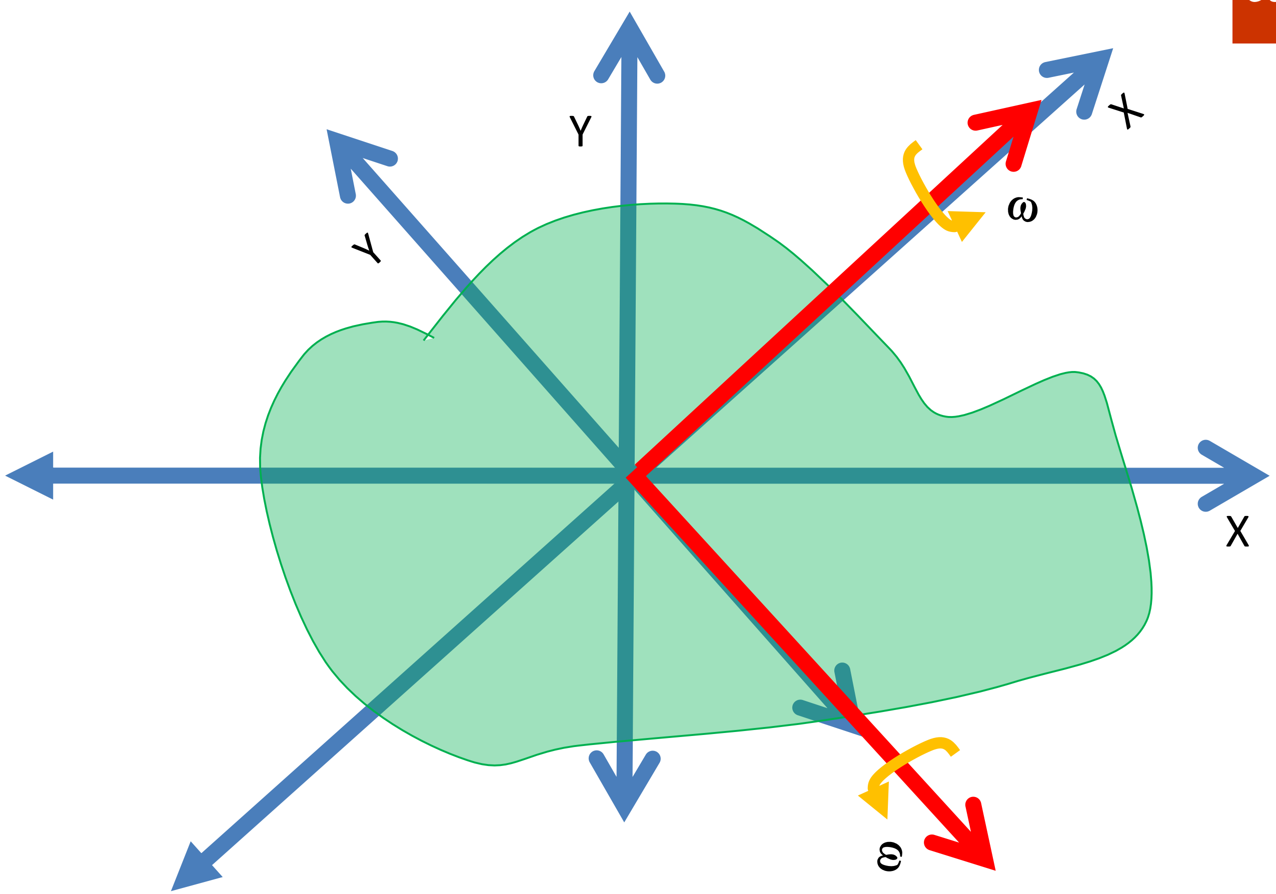
$$[I][\vec{\omega}] = \lambda\vec{\omega}$$

Eigenvector1

$$\omega_x = \omega_y$$

Eigenvector2

$$\omega_x = -\omega_y$$



Moment of Inertia matrix for principal axis

$$\lambda_1 = 1, \lambda_2 = 3,$$

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$$

$$L = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} \omega_x \\ 0 \end{bmatrix} = 1 \begin{bmatrix} \omega_x \\ 0 \end{bmatrix}$$

$$L = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 0 \\ \omega_y \end{bmatrix} = 3 \begin{bmatrix} 0 \\ \omega_y \end{bmatrix}$$

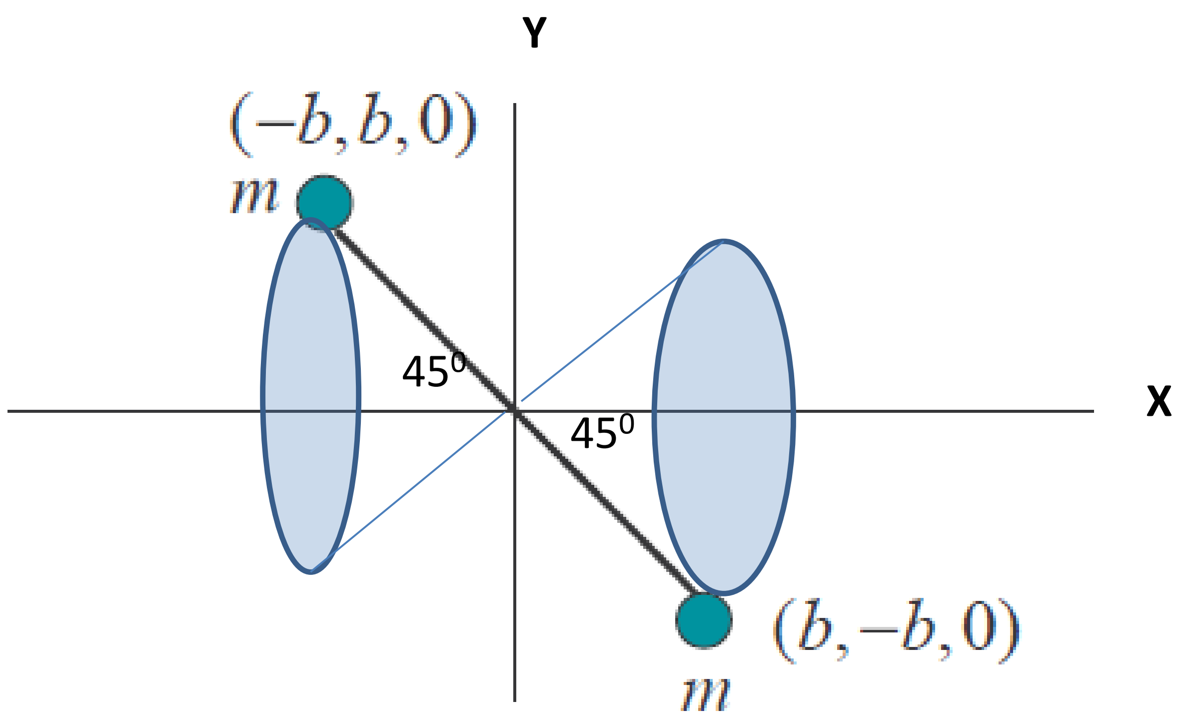
Another example…..

ROTATING DUMBBELL

A dumb-bell is rotated along X and Y axis as shown in figure below

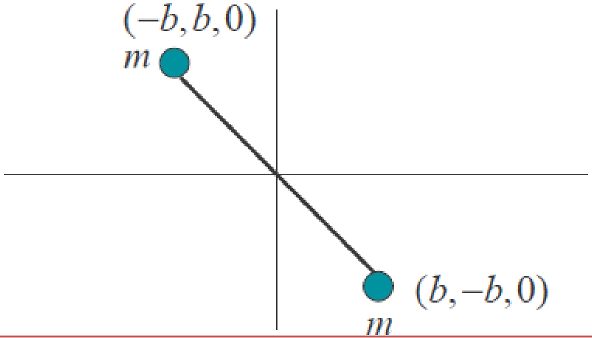
A. Find the Moment of Inertia Tensor

B. Find the principal axis corresponding to it



A. Moment of Inertia Tensor

$$[I]=\begin{pmatrix} \int\left(y^2+z^2\right)dm & -\int xydm & -\int zxdm \\ -\int xydm & \int\left(z^2+x^2\right)dm & -\int yzdm \\ -\int zxdm & -\int yzdm & \int\left(x^2+y^2\right)dm \end{pmatrix}$$



$$\mathbf{I}=m\begin{bmatrix} 2b^2 & 2b^2 & 0 \\ 2b^2 & 2b^2 & 0 \\ 0 & 0 & 4b^2 \end{bmatrix}=2b^2m\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

Finding the Eigenvalues

$$\begin{vmatrix} I_{xx}-\lambda & I_{xy} & I_{xz} \\ I_{yx} & I_{yy}-\lambda & I_{yz} \\ I_{zx} & I_{zy} & I_{zz}-\lambda \end{vmatrix} = 0$$

$$\begin{vmatrix} 2mb^2-\lambda & 2mb^2 & 0 \\ 2mb^2 & 2mb^2-\lambda & 0 \\ 0 & 0 & 4mb^2-\lambda \end{vmatrix} = 0$$

Eigenvalues

$$\begin{vmatrix} 2mb^2-\lambda & 2mb^2 & 0 \\ 2mb^2 & 2mb^2-\lambda & 0 \\ 0 & 0 & 4mb^2-\lambda \end{vmatrix} = 0$$

$$(4mb^2-\lambda)\left[\left(2mb^2-\lambda\right)^2-\left(2mb^2\right)^2\right]=0$$

$$(4mb^2-\lambda)\left[\lambda^2-4mb^2\lambda\right]=0$$

$$(4mb^2-\lambda)\left(\lambda\right)\left(\lambda-4mb^2\right)=0$$

$$\lambda_1=\lambda_2=4mb^2,\lambda_3=0$$

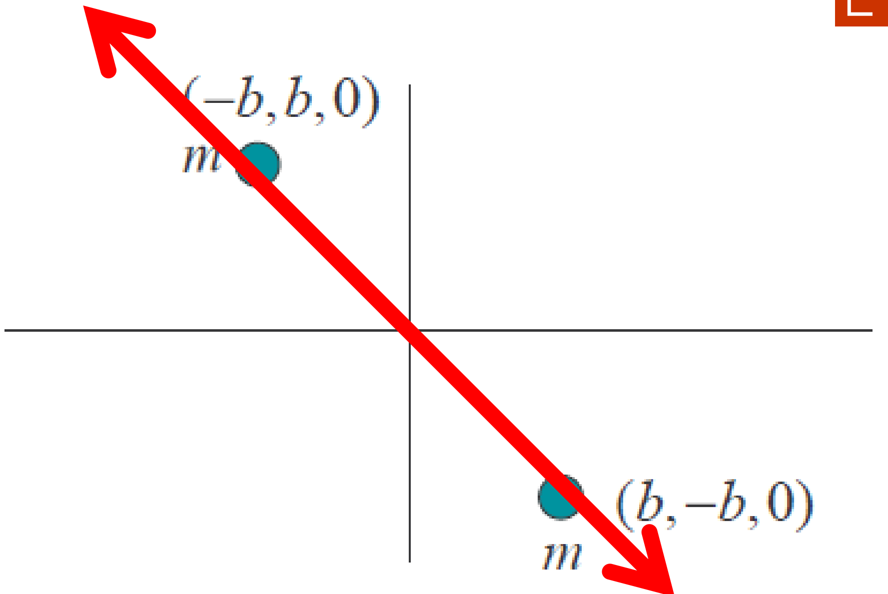
Principal axis and eigenvectors

$$for\ \lambda_3 = 0$$

$$\omega_x = -\omega_y$$

$$\omega_z = 0$$

Eigenvector corresponding to λ_{3}: $\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$



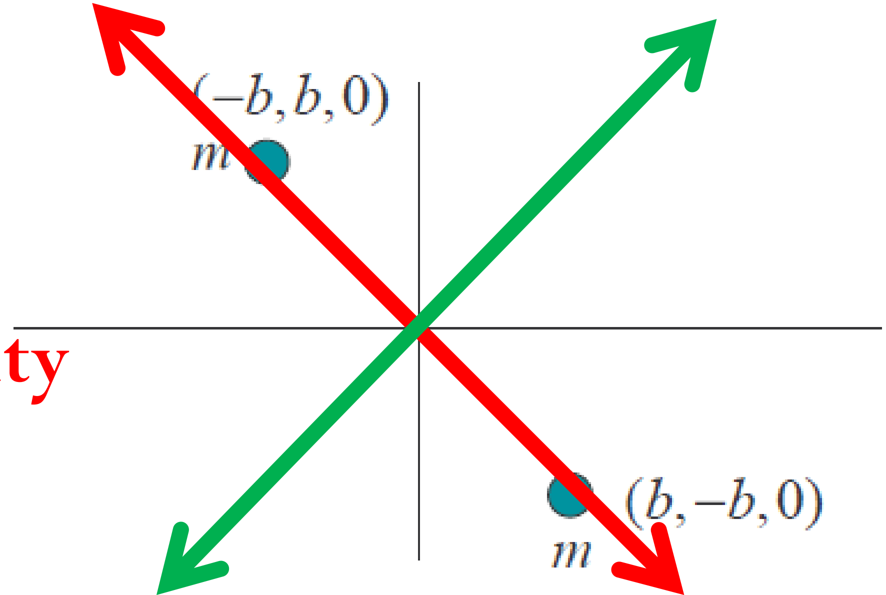
Principal axis and eigenvectors

$$for\ \lambda_1 = \lambda_2 = 4mb^2$$

$$\omega_x = \omega_y$$
$$\omega_z = Anything$$

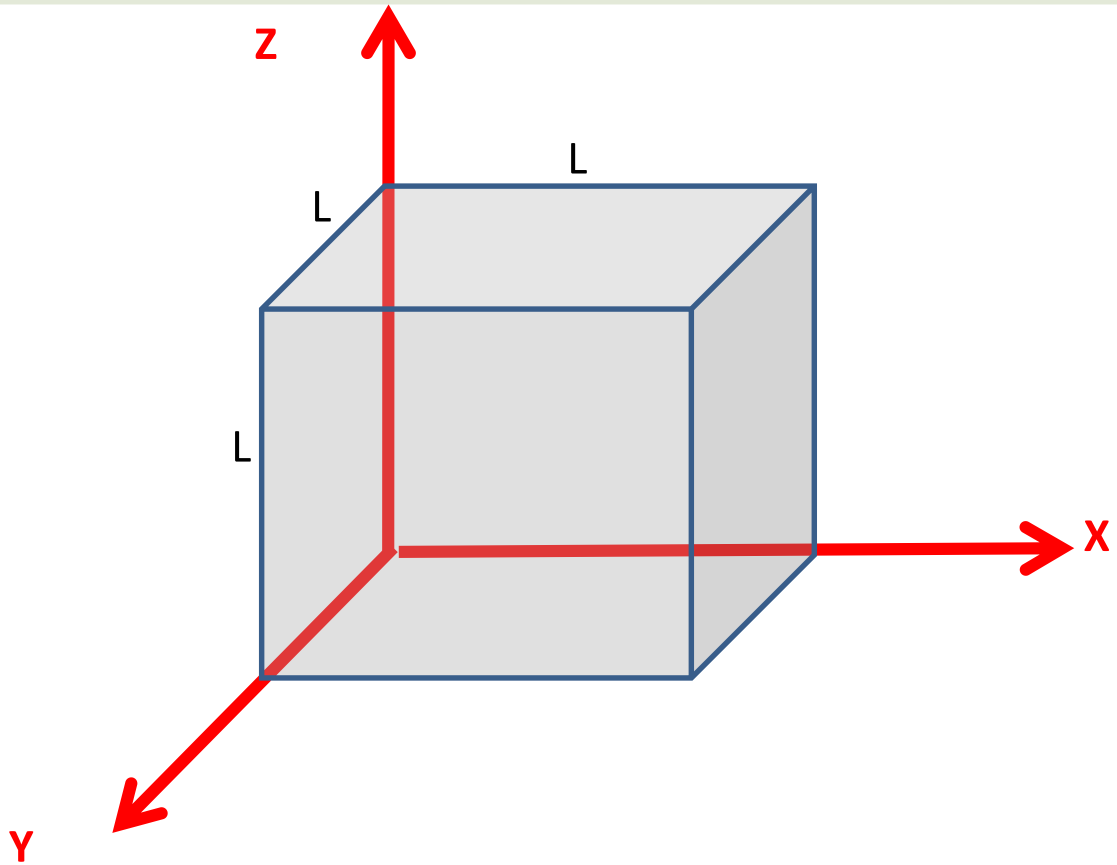
Eigenvector corresponding to λ_1 or λ_2 :

$$\begin{bmatrix} 1 \\ 1 \\ a \end{bmatrix}$$



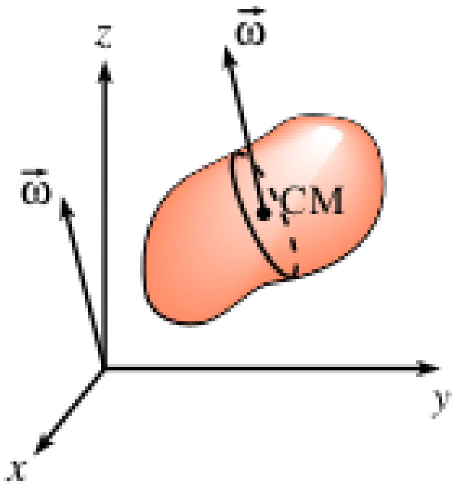
Verify orthogonality of eigenvectors

HW: Find principal axis



$$[I] = ML^2 \begin{pmatrix} 2/3 & -1/4 & -1/4 \\ -1/4 & 2/3 & -1/4 \\ -1/4 & -1/4 & 2/3 \end{pmatrix}$$

Parallel-axis Theorem



CM rotates around the origin with the same angular velocity at which the body rotates around the CM

$$\mathbf{L} = M\mathbf{R} \times (\boldsymbol{\omega} \times \mathbf{R}) + \int \mathbf{r}' \times (\boldsymbol{\omega} \times \mathbf{r}')\, dm$$

Writing the double cross products in matrix form, we get

$$\begin{aligned}
\begin{pmatrix} L_1 \\ L_2 \\ L_3 \end{pmatrix} &= M \begin{pmatrix} Y^2+Z^2 & -XY & -ZX \\ -XY & Z^2+X^2 & -YZ \\ -ZX & -YZ & X^2+Y^2 \end{pmatrix} \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix} \\
&\quad + \begin{pmatrix} \int(y'^2+z'^2) & -\int x'y' & -\int z'x' \\ -\int x'y' & \int(z'^2+x'^2) & -\int y'z' \\ -\int z'x' & -\int y'z' & \int(x'^2+y'^2) \end{pmatrix} \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix} \\
&\equiv (\mathbf{I}_{\mathrm{R}} + \mathbf{I}_{\mathrm{CM}})\boldsymbol{\omega}.
\end{aligned}$$

Kinetic energy: $T = \frac{1}{2}\boldsymbol{\omega} \cdot (\boldsymbol{I_R} + \boldsymbol{I_{CM}})\boldsymbol{\omega} = \frac{\mathbf{1}}{\mathbf{2}}\boldsymbol{\omega} \cdot \boldsymbol{L}$

<u>Few scattered thoughts about rigid body rotation</u>

(1) **<u>Co-ordinate system</u>** (Choice of axis) defines MI tensor. **<u>Axis of rotation</u>** is immaterial in defining MI tensor.

(2) Different co-ordinate axis will result in different MI tensor even if **<u>rotation is physically along same axis!!!</u>**

(3) Different co-ordinate system will only change mathematics! **<u>Rotation dynamics will be same</u>**!

(4) Rotation is concrete (Real), but co-ordinate choice is abstract.

Few scattered thoughts about rigid body rotation

(5) If MI tensor is non-diagonal, **L** and **ω** **need not be** in same direction.

(6) Non-diagonal nature of MI tensor **does not** guarantee non-parallel nature of **L** and **ω. (They can be in different direction)**

(7) Rotation through one axis will induce coupling with other axis if MI tensor is non-diagonal: Bi-directional coupling. MI tensor is dyadic.

Few scattered thoughts about rigid body rotation

(8) If we choose **ω** axis properly**, L** and **ω** can be in same direction even if MI tensor is non-diagonal.

$$L = [I][\vec{\omega}] = \lambda\vec{\omega}$$

$$L_x = I_{xx}\omega_x + I_{xy}\omega_y$$
$$L_y = I_{xy}\omega_x + I_{yy}\omega_y$$

Different

(9) If Co-ordinate axis (Primed axis) is chosen along **ω** axis,

$$L_{x'} = I'_{xx}\omega_{x'}$$

How to find principal axis????

$$L = [I][\vec{\omega}] = \lambda\vec{\omega}$$

$$\begin{pmatrix} I_{xx}-\lambda & I_{xy} & I_{xz} \\ I_{yx} & I_{yy}-\lambda & I_{yz} \\ I_{zx} & I_{zy} & I_{zz}-\lambda \end{pmatrix}\begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix} = 0$$

$$\begin{vmatrix} I_{xx}-\lambda & I_{xy} & I_{xz} \\ I_{yx} & I_{yy}-\lambda & I_{yz} \\ I_{zx} & I_{zy} & I_{zz}-\lambda \end{vmatrix} = 0$$

Characteristic Equation