CS206 Discrete Maths.

* Set Theory :-

$C \subseteq D, \forall x, x \in C \to x \in D$

$C \subset D, \forall x, \exists x \in D$ s.t. $x \notin C$

$C = D$

$P(C) = \{\{\phi\}, c_1, c_2, \ldots, C\} \Rightarrow$ set of all subsets of C.

Null set $\{\phi\}$

$C \cup D$

$C \cap D$

C / D (Difference) $= C - (C \cap D)$.

$C \Delta D = (C/D) \cup (D/C) = (C \cup D) - (C \cap D)$

$\overline{C}$

$|C \cup D| = |C| + |D| - |C \cap D|$

$|C \cup D \cup E| = |C| + |D| + |E| - |C \cap D| - |D \cap E| - |C \cap E|$
$+ |C \cap D \cap E|$.

Q. $n(F) = 100$; $n(R) = 50$; $n(F \cap R) = 20$.

$n(F \cup R) = 100 + 50 - 20 = 130$

* Fundamentals of Logic :-

Propositions :- Any declarative sentence to which it is meaningful to assign one & only one truth value: True or False.

p: A week has 7 days T
q: A week has more no. of days than a month. F } propositions
r: What is your name? $\longrightarrow$ Not a proposition.
p, q, r are propositional variables.

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★ Connectives :-

Ⓘ Conjunction — 'and' — '$\wedge$'.

Ⓘⓘ Disjunction — 'or' — '$\vee$'.

Ⓘⓘⓘ Negation —— 'not' — '$-$/$\sim$/$\neg$'

Ⓘⓥ Conditional — 'if_ then_' — '$\rightarrow, \Rightarrow$'.

Ⓥ Biconditional — 'iff' — '$\leftrightarrow$, $\Leftrightarrow$'.

{ Equivalence '$\Leftrightarrow$'

1. Conjunction 'p and q' '$p \wedge q$'.

p	q	$p \wedge q$
F	F	F
F	T	F
T	F	F
T	T	T.

2. Disjunction 'p or q' '$p \vee q$'.

p	q	$p \vee q$
F	F	F
F	T	T
T	F	T
T	T	T.

3. Negation $\overline{p}$, $\sim p$, $\neg p$.

p	$\overline{p}$
F	T
T	F

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4. <u>Conditional</u> 'p implies q' $p \to q$

if <u>anticident</u> then <u>consequent</u>.

p	q	$p \to q$
F	F	T
F	T	T
T	F	F
T	T	T

5. <u>Biconditional</u> 'p iff q' $p \leftrightarrow q$. $= (p \to q) \wedge (q \to p)$

p	q	$p \leftrightarrow q$	$p \to q$	$q \to p$
F	F	T	T	T
F	T	F	T	F
T	F	F	F	T
T	T	T	T	T

$f(p,q,r) = (p \wedge q) \vee \bar{r}$.

p	q	r	$p \wedge q$	$\bar{r}$	$(p \wedge q) \vee \bar{r}$
F	F	F	F	T	T
F	F	T	F	F	F
F	T	F	F	T	T
F	T	T	F	F	F
T	F	F	F	T	T
T	F	T	F	F	F
T	T	F	T	T	T
T	T	T	T	F	T

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* Equivalence of proposition p, q

$p \to q \qquad \bar{p} \vee q$

p	q	$f(x)$ $p \to q$	$f(y)$ $\bar{p} \vee q$	$f(x) \to f(y)$	$f(y) \to f(x)$	$f(x) \Leftrightarrow f(y)$
F	F	T	T	T	T	T
F	T	T	T	T	T	T
T	F	F	F	T	T	T
T	T	T	T	T	T	T

$\Rightarrow \quad (p \to q) \equiv (\bar{p} \vee q)$.

$\Rightarrow \quad X \equiv Y$, $X \leftrightarrow Y$ will be always true.

* Tautology :- All true.

* Contradiction / Absurdity :- All false.

* Contigency :- Both true & false.

* LOGICAL INFERENCE *

$p_1 \wedge p_2 \wedge \dots \wedge p_n \longrightarrow q$.

Valid Inference

$\Downarrow$

When it is a tautology.

argument form / inference

$$\left.\begin{matrix} P_1 \\ P_2 \\ \vdots \\ P_n \end{matrix}\right\} \text{premises}$$

$\therefore q$ } conclusion

Rules

On Next Page.

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Rule of inference	Tautological form	Name.
① $\frac{p}{\therefore p \vee q}$	$p \rightarrow (p \vee q)$	Addition.
② $\frac{p \wedge q}{\therefore p}$	$(p \wedge q) \rightarrow p$	Simplification.

p	q	$p \wedge q$	$(p \wedge q) \rightarrow p$
F	F	F	T
F	T	F	T
T	F	F	T
T	T	T	T

③ $p \rightarrow q$
p
$\therefore q$

$(p \rightarrow q) \wedge p \rightarrow q$ Modus ponens.

If u have a password, u can log onto facebook $p \rightarrow q$

You have a password p.

$\therefore$ You can log onto facebook q.

④ $p \rightarrow q$
$\sim q$
$\therefore \sim p$.

$(p \rightarrow q) \wedge \sim q \rightarrow \sim p$ Modus tollens.

If u have a password, u can log onto facebook $p \rightarrow q$

You cannot log onto FB $\sim q$

$\therefore$ You do not have a password $\sim p$.

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5. $\sim p$

$p \vee q$

$\therefore q$

$\sim p \wedge (p \vee q) \rightarrow q$ Disjunction Syllogysm.

The ice-cream is not of vanilla flavour. $\sim p$

The icecream is either vanilla flavour or choc. flav $p \vee q$

$\therefore$ The ice-cream is choc. flavour : q.

6. $p \rightarrow q$

$q \rightarrow r$

$\therefore p \rightarrow r$

$(p \rightarrow q) \wedge (q \rightarrow r) \rightarrow (p \rightarrow r)$ Hypothetical Syllogysm.

If it rains, I shall not go to school. $p \rightarrow q$

If I dont go to school, I wouldn't need to do hw. $q \rightarrow r$

$\therefore$ If it rains, I wouldn't need to do homework $p \rightarrow r$.

7. $(p \rightarrow q) \wedge (r \rightarrow s)$

$p \vee r$

$\therefore q \vee s$.

$(p \rightarrow q) \wedge (r \rightarrow s) \wedge (p \vee r) \rightarrow (q \vee s)$. Constructive dilemma.

If it rains, I will take a leave. $p \rightarrow q$

if it is hot outside, I will go for a shower $r \rightarrow s$

Either it will rain or it is hot outside $p \vee r$

$\therefore$ Either I will take a leave or go for a shower. $(q \vee s)$

8. $(p \rightarrow q) \wedge (r \rightarrow s)$

$\sim q \vee \sim s$

$\therefore \sim p \vee \sim r$

$(p \rightarrow q) \wedge (r \rightarrow s) \wedge (\sim q \wedge \sim s) \rightarrow (\sim p \vee \sim r)$ Destructive dilemma.

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9. $\overline{p \wedge q} \equiv \bar{p} \vee \bar{q}$ Demorgan's Theorem.

$\overline{p \vee q} \equiv \bar{p} \wedge \bar{q}$

10. $p \to q \equiv \bar{q} \to \bar{p}$ Law of contrapositive.

Fallacy.

1. $$\begin{array}{l} p \to q \\ \underline{q} \\ \therefore p \end{array}$$ $(p \to q) \wedge q \to p$ Affirming the consequent.

p	q	p → q	((p → q) ∧ q).
F	F	T	F
F	T	T	T
T	F	F	F
T	T	T	T

If u have the flu, u have a sore throat. $p \to q$

You have a sore throat q.

∴ You have the flu p

2. $$\begin{array}{l} p \to q \\ \underline{\bar{p}} \\ \therefore \bar{q} \end{array}$$ Denying the antecedent.

3. $$\begin{array}{l} \underline{p} \\ \therefore q \end{array}$$ Non-sequitter.

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METHODS OF PROOF

If p then q ; $p \to q$.

$p \to q$

p	q	$p \to q$
F	F	T
F	T	T
T	F	F
T	T	T

1. Trivial proof of $p \to q$:-

Establish 'q' to be true, regardless of truth value of 'p', then $p \to q$ is true.

2. Vacuos proof of $p \to q$:-

If 'p' is false, regardless of 'q' then $p \to q$ is true.

3. Direct proof of $p \to q$:-

Assume 'p' to be true, then based on the assumption show 'q' to be true.

Ex. Show that the square of even number is an even no.

If $\underbrace{n \text{ is even}}_{p}$ then $\underbrace{n^2 \text{ is even}}_{q}$

let, $n = 2k \Rightarrow n^2 = (2k)^2 = 4k^2 = 2(2k^2)$.

$\Rightarrow$ q is true.

4. Indirect proof of $p \to q$:-

Show that $\overline{q} \to \overline{p}$ (i.e. contrapositive is true)

① Assume 'q' to be false

② Show that 'p' is also false from ①.

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e.g. If n^2 is an odd integer then n is an odd integer
$(p \to q)$

If n is an even integer then n^2 is even.
$(\bar{q})$ $(\bar{p})$

5. <u>Proof of $p \to q$ by contradiction</u> :-

$p \to q$ is true if $p \wedge \bar{q}$ is false

$(\because \; p \to q \equiv \bar{p} \vee q)$.

$$\overline{p \to q} \equiv \overline{\bar{p} \vee q} \equiv \bar{\bar{p}} \wedge \bar{q} \equiv p \wedge \bar{q}$$

(a) Assume $p \wedge \bar{q}$ is true.

(b) Discover based on the assumption some conclusion which shows (a) to be false.

(c) Contradiction.

e.g. If n^3+5 is odd, then n is even.
(p) (q)

$\Rightarrow$ (1) n^3+5 is odd and n is odd. $(p \wedge \bar{q})$

(2) $n = (2k+1)$; $n^3+5 = 8k^3+6k^2+6k+6 = 2(4k^3+3k^2+3k+3)$

$\Rightarrow n^3+5$ is even.

(3) $p \wedge \bar{q}$ is false.

6. <u>Proof of $p \to q$ by cases</u>

p is of the form $p_1 \vee p_2 \vee \dots \vee p_n$.

$p_1 \vee p_2 \vee \dots \vee p_n \longrightarrow q$

$p_1 \to q$ & $p_2 \to q$ & ..... & $p_n \to q$.

$p \to q \equiv \bar{p} \vee q$

$(p_1 \vee p_2 \vee \dots \vee p_n) \to q \equiv \overline{(p_1 \vee p_2 \vee \dots \vee p_n)} \vee q$

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$\equiv (\bar{p}_1 \wedge \bar{p}_2 \wedge \ldots \wedge \bar{p}_n) \vee q$

$\equiv (\bar{p}_1 \vee q) \wedge (\bar{p}_2 \vee q) \wedge \ldots \wedge (\bar{p}_n \vee q)$.

$\equiv (p_1 \to q) \wedge (p_2 \to q) \wedge \ldots \wedge (p_n \to q)$.

Q. Let '$\sqcup$' denote minimum operation $a \sqcup b$ gives min of a & b, $a, b \in Z$ Prove that associative operation.

$a \sqcup (b \sqcup c) = (a \sqcup b) \sqcup c \quad \forall a, b, c \; Z$. (Proof by cases).

$\Rightarrow$ If I take 3 integers a, b, c.
The possibilities of $\min(a,b,c)$.

1. $a \le b \le c$
2. $a \le c \le b$
3. $b \le c \le a$
4. $b \le a \le c$
5. $c \le a \le b$
6. $c \le b \le a$

for case 1:

L.H.S. $= a \sqcup (b \sqcup c) = a \sqcup b = a$

R.H.S. $= (a \sqcup b) \sqcup c = a \sqcup c = a$.

$\Rightarrow$ L.H.S. = R.H.S.

Similarly, do for all remaining cases. So, The associativity holds for each of the six cases, it is proved that

$a \sqcup (b \sqcup c) = (a \sqcup b) \sqcup c \quad \forall a, b, c \in Z$.

★ (Exam pe pura likhna hai).

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7. Proof by elimination by cases :-

① p has to be true or q has to be true.
② If we verify 'p' is false.
③ then 'q' is true.

(Disjunctive syllogysm)

$(p \vee q) \wedge \overline{p} \rightarrow q$

8. Conditional Proof :-

$p \rightarrow (q \rightarrow r)$ & $(p \wedge q) \rightarrow r$.

9. Proof of Equivalence :-

$p \leftrightarrow q \equiv (p \rightarrow q) \wedge (q \rightarrow p)$

* Count the no. of integers upto 100 which are not divisible by 2, 3 ~~and~~ or 5.

Ans :- let, U = set of integers x, st $1 \le x \le 100$.

A_1 = set of elements of 'U' divisible by 2.

A_2 = set of elements of 'U' divisible by 3.

A_3 = — " — 5.

So, $\overline{A_1} \cap \overline{A_2} \cap \overline{A_3}$ = set of all elements in 'U' which are not divisible by 2, 3, 5. $= (\overline{A_1 \cup A_2 \cup A_3})$

Again the cardinality, $|\overline{A_1} \cap \overline{A_2} \cap \overline{A_3}| = |\overline{A_1 \cup A_2 \cup A_3}|$

$= |U| - |A_1 \cup A_2 \cup A_3|$

$|U| = 100$; $|A_1 \cup A_2 \cup A_3| = |A_1| + |A_2| + |A_3| - |A_1 \cap A_2|$
$- |A_2 \cap A_3| - |A_1 \cap A_3| + |A_1 \cap A_2 \cap A_3|$

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$|A_1| = \frac{100}{2} = 50$; $|A_2| = \left\lfloor\frac{100}{3}\right\rfloor = 33$; $|A_3| = \frac{100}{5} = 20$.

$|A_1 \cap A_2| = \left\lfloor\frac{100}{6}\right\rfloor = 16$; $|A_1 \cap A_3| = \frac{100}{10} = 10$; $|A_2 \cap A_3| = \left\lfloor\frac{100}{15}\right\rfloor = 6$.

$|A_1 \cap A_2 \cap A_3| = \left\lfloor\frac{100}{30}\right\rfloor = 3$.

$|A_1 \cup A_2 \cup A_3| = 50 + 33 + 20 - 16 - 10 - 6 + 3 = 74$.

$\therefore |\bar{A_1} \cap \bar{A_2} \cap \bar{A_3}| = |U| - |A_1 \cup A_2 \cup A_3| = 100 - 74 = 26$.

Q. Which of the following is not equivalent to $p \leftrightarrow q$.

(a) $(\sim p \vee q) \wedge (p \vee \sim q)$.

(b) $(\sim p \vee q) \wedge (q \rightarrow p)$

(c) $(\sim p \vee q) \vee (p \wedge \sim q)$. ✓

(d) $(\sim p \wedge \sim q) \vee (p \wedge q)$.

$p \Leftrightarrow q = (p \rightarrow q) \wedge (q \rightarrow p)$.

$= (\sim p \vee q) \wedge (\sim q \vee p)$

$= (\bar{p} + q) \cdot (\bar{q} + p)$

$= \bar{p}\bar{q} + p.q$.

(a) $\Rightarrow (\bar{p} + q) \cdot (p + \bar{q}) = \bar{p}p + \bar{p}\bar{q} + qp + q\bar{q}$

$= 0 + \bar{p}\bar{q} + pq + 0$

$= \bar{p}\bar{q} + pq. = p \leftrightarrow q$.

☆ In Exam $\Rightarrow$ use Truth Table.

Q. Find whether $(a \wedge b \wedge c) \rightarrow (c \vee a)$ is tautology or not?

a	b	c	$a \wedge b \wedge c$	$c \vee a$	$(a \wedge b \wedge c) \rightarrow (c \vee a)$
T	T	T	T	T	T
T	T	F	F	T	T
T	F	T	F	T	T
T	F	F	F	T	T
F	T	T	F	T	T
F	T	F	F	F	T
F	F	T	F	T	T
F	F	F	F	F	T.

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$(a \wedge b \wedge c) \rightarrow (c \vee a)$

$\sim (a \wedge b \wedge c) \vee (c \vee a)$.

$= (\overline{a.b.c}) + (c + a)$

$= \bar{a} + \bar{b} + \bar{c} + c + a$

$= 1 + \bar{b} + 1 = 1$

Q. Given the premises

p

$p \rightarrow q$

$s \vee r$

$r \rightarrow \sim q$

arrive at the conclusion

$s \vee t$.

Ans ⇒

Step		Reason
1	p	Premise
2	$p \rightarrow q$	Premise.
3	q	Modus Ponens (using 1 & 2).
4	$r \rightarrow \sim q$	Premise
5	$q \rightarrow \sim r$	Contrapositive
6	$\sim r$	Modus Ponens (using 3 & 5).
7	$s \vee r$	Premise
8	s	disjunctive syllogysm (using 6, 7).
9	$s \vee t$	disjunctive amplification (using 8)

Q. If I get my christmas bonus and my friends are free, I will take a road trip with my friends.

If my friends don't find a job after christmas, then they will be free.

I got my christmas bonus and my friends did not find a job after christmas.

Therefore, I will take a road trip with my friends.

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p = "I get my christmas bonus".
q = "My friends are free"
r = "I will take a road trip with my friends".
s = "My friends find a job after christmas".

- The argument symbollically is as follows :-

Premises.
$(p \wedge q) \to r$
$\sim s \to q$
p
$\sim s$.

Conclusion
$\therefore r$.

Ans.

Step	Reason
1) $\sim s \to q$	premise
2) $\sim s$	premise
3) q	Modus ponens (using 1 & 2)
4) p	premise.
5) $p \wedge q$	conjunction.
6) $(p \wedge q) \to r$	premise.
7) r	Modus ponens (using 5 & 6)

Indirect Proof

Q. If n^3+5 is odd, then n is even.
(p) (q)

$\Rightarrow$ $\bar{q}$ = n is odd. $\Rightarrow n^3+5$ is even $\Rightarrow \bar{p}$ is true.
$n = 2k+1$ $\therefore \bar{q} \to \bar{p}$.
$\Rightarrow n^3+5 = (2k+1)^3+5 = 8k^3+6k^2+6k+6$ Hence, proved.
$= 2(n)$ (p)

Q. If the product of two integers a and b is even then either a is even or b is even.
(q).

$\Rightarrow$ let, p = ab is even.
q = a is even or b is even.

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$(p \to q)$ $\quad \bar{q} \to \bar{p}$ (Taking contrapositive)

$\bar{q}$ = a is odd and b is odd (De Morgan's Law)

$(a = 2m+1 \; ; \; b = 2n+1)$. where, $m, n \in \mathbb{Z}$.

$$ab = (2m+1)(2n+1)$$
$$= 4mn + 2m + 2n + 1$$
$$= 2(2mn + m + n) + 1$$
$$ab = 2(k) + 1 \quad (k \in \mathbb{Z})$$

$\Rightarrow$ ab is odd. $\Rightarrow \bar{p}$ is true. Hence, proved.

☆ Proof by Contradiction :-

Q. The difference of any rational number and any irrational number is irrational.

Ans. $\Rightarrow$ (Suppose not)

suppose $\exists$ a rational number x and an irrational number y such that $(x-y)$ is rational.

By the definition of rational number, $x = \frac{a}{b}$; $(b \neq 0)$ $a, b \in \mathbb{Z}$

$$(x-y) = \frac{c}{d} \; ; \; (d \neq 0) \; ; \; c, d \in \mathbb{Z}.$$

$$\Rightarrow \quad x - y = \frac{c}{d} \Rightarrow \frac{a}{b} - y = \frac{c}{d} \Rightarrow y = \frac{a}{b} - \frac{c}{d}$$

$$\Rightarrow \quad y = \frac{(ad-bc)}{(bd)} = \frac{p}{q} \; ; \; q \neq 0 \; \& \; p, q \in \mathbb{Z}.$$

$\Rightarrow$ y is rational number ; $\Longleftrightarrow$ with given st.

$\therefore$ Hence, proved.

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Proof by equivalence.

Q. Show that $m^2 = n^2$ if and only if $m = n$ or $m = -n$.

⇒ we can write above statement as

$$(m^2 = n^2) \leftrightarrow (m = n) \vee (m = -n).$$

$$(p \leftrightarrow q = (p \to q) \wedge (q \to p))$$

① $(p \to q)$ $(m^2 = n^2) \to (m = n) \vee (m = -n)$

② $(q \to p)$ $(m = n) \vee (m = -n) \to (m^2 = n^2)$

(we have to prove these 2 parts.)

① → Subtract n^2 from both sides of L.H.S.

$$m^2 - n^2 = n^2 - n^2$$

$$\Rightarrow m^2 - n^2 = 0.$$

$$\Rightarrow (m+n)(m-n) = 0.$$

$$\Rightarrow m = -n, n.$$

or $(m = n) \vee (m = -n)$.

② → <u>Proof by cases</u> :-

(a) $m = n$

$\therefore m^2 = n^2$

$\Rightarrow (m = n) \to (m^2 = n^2)$

(b) $m = -n$

$\therefore m^2 = n^2$

$\Rightarrow (m = -n) \to (m^2 = n^2)$

$\therefore (m = n) \vee (m = -n) \to (m^2 = n^2)$.

⇒ Hence, proved.

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PREDICATE LOGIC.

T $3/4$ is a rational no. U = set of real nos.

T $1/2$ is a rational no. $x \in U$

F $\sqrt{2}$ is a rational no, $P(x) \ni x$ is a rational number.

$U \xrightarrow{P(\cdot)} \{T, F\}$. Predicate

$S(x,y) : x+y=5$; $x,y \in U$.

Predicate is a mapping function which connects a universal set to a set of truth values.

$R(x)$: x is a rational no.
$G(y)$: $y > 5$
$S(x,y)$: $x+y=5$.
} $x,y \in U$: set of real numbers.

$E(x)$: x climbed mt. everest.
$C(y)$: y is a lawyer.
} $x,y \in U$: set of human beings

U : universal set / universe of discourse.

★ **Quantifiers :-**

All humans have 2 legs.

Atleast one human has 2 legs.

① UNIVERSAL QUANTIFIER

$\forall$

e.g. $x \in U$, $\forall x, P(x)$

forall x(in U), $P(x)$ is true.

② EXISTENTIAL QUANTIFIER. '$\exists$'

e.g. $x \in U$, $\exists x, P(x)$

There exists x(in U) such that $P(x)$ is true.

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Sr. No.	Proposition		Abbreviation
(1)	$\forall x, F(x)$	$\longrightarrow$	All true.
(2)	$\exists x, F(x)$	$\longrightarrow$	At least one true.
(3)	$\sim[\exists x, F(x)]$	$\longrightarrow$	None true.
(4)	$\forall x, [\sim F(x)]$	$\longrightarrow$	All False.
(5)	$\exists x, [\sim F(x)]$	$\longrightarrow$	At least one false.
(6)	$\sim[\exists x, [-F(x)]]$	$\longrightarrow$	None False.
(7)	$\sim[\forall x, F(x)]$	$\longrightarrow$	Not all true
(8)	$\sim[\forall x, [-F(x)]]$	$\longrightarrow$	Not all false.

1. All true : $\forall x, F(x) \equiv$ None false : $\sim[\exists x\ [-F(x)]]$
2. All false : $\forall x, [-F(x)] \equiv$ None true : $\sim[\exists x, F(x)]$
3. Not all true : $\sim[\forall x, F(x)] \equiv$ At least one false : $\exists x, [-F(x)]$
4. Not all false : $\sim[\forall x, [-F(x)]] \equiv$ At least one true : $\exists x, F(x)$.

$\sim(p \wedge q) \equiv \sim p \vee \sim q$

$\sim(p \vee q) \equiv \sim p \wedge \sim q$

$U = \{a, b, c, d\}$, $x \in U$.

$\forall x, F(x) : F(a) \wedge F(b) \wedge F(c) \wedge F(d)$

$\exists x, F(x) : F(a) \vee F(b) \vee F(c) \vee F(d)$.

1. $\sim[\exists x\ [\sim F(x)]] = \sim[\sim F(a) \vee \sim F(b) \vee \sim F(c) \vee \sim F(d)]$

$= F(a) \wedge F(b) \wedge F(c) \wedge F(d)$

$= \forall x, F(x)$ Hence, Proved.

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2. $\sim[\exists x, F(x)] = \sim[F(a) \vee F(b) \vee F(c) \vee F(d)]$

$= \sim F(a) \wedge \sim F(b) \wedge \sim F(c) \wedge \sim F(d)$

$= \forall x, [-F(x)]$ Hence, Proved.

③. $\exists x, [\sim F(x)] = [\sim F(a) \vee \sim F(b) \vee \sim F(c) \vee \sim F(d)]$

$= \sim[F(a) \wedge F(b) \wedge F(c) \wedge F(d)]$

$= \sim[\forall(x), F(x)]$ Hence, Proved.

④ $\exists x, F(x) = F(a) \vee F(b) \vee F(c) \vee F(d)$.

$= \sim[\sim F(a) \wedge \sim F(b) \wedge \sim F(c) \wedge \sim F(d)]$

$= \sim[\forall x, [-F(x)]]$ Hence, Proved.

★ Proof by example :-

$\exists x, F(x)$ is true, $F(c)$ for some 'c' in U.

Ex. $\exists x$, x is prime : $x \in \mathbb{Z}$.

★ Proof by exhaustion :-

$\forall x, \sim F(x)$: Choose all $c \in U$ & show false.

★ Proof by contradiction / counter. ex. :-

$\forall x, F(x)$ is false, one c is false.

$P(x,y): x+y=8$.

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RULES OF INFERENCE

1. Modus Ponens $\Rightarrow$ $p \wedge (p \to q) \to q$
2. Hypothetical Syllgysm $\Rightarrow$ $(p \to q) \wedge (q \to r) \to p \to r$
3. Demorgan's law $\Rightarrow$ $\overline{p \wedge q} \equiv \sim p \vee \sim q$; $\overline{p \vee q} \equiv \sim p \wedge \sim q$
4. Law of contrapositive $\Rightarrow$ $p \to q \equiv \sim q \to \sim p$.

5. **Universal Specification :-**

$\forall x, P(x) \to$ the quantifier $\forall$ can be used to conclude $P(c)$ to be 'true' for any element 'c' in the universe.

$$\frac{\forall x, P(x)}{\therefore P(c) \text{ for any } c}$$

6. **Universal Generalisation :-**

$$\frac{P(c) \text{ for any arbitrary } c}{\therefore \forall x, P(x)}$$

7. **Existential Specification :-**

$\exists x, P(x)$ then there is an element 'c' in the universe for which $P(c)$ is true.

$$\frac{\exists x, P(x)}{\therefore P(c) \text{ for some } c}$$

8. **Existential Generalisation :-**

$$\frac{P(c) \text{ for same } c}{\therefore \exists x, P(x)}$$

Discrete Maths → Kenneth H. Rosen.

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eg (I) All men are fallible
All kings are men

$\therefore$ All kings are fallible.

let, $M(x)$: x is a man
$K(x)$; x is a king
$F(x)$; x is fallible.

Solution :-
(proof)

$$\begin{array}{l} \forall x, M(x) \to F(x) \\ \forall x, K(x) \to M(x) \\ \hline \therefore \forall x, K(x) \to F(x) \end{array}$$

steps	Formal proof	Reasons
1.	$\forall x, M(x) \to F(x)$	Premise 1
2.	$M(c) \to F(c)$	Rule 5. Step 1.
3.	$\forall x, K(x) \to M(x)$	Premise 2.
4.	$K(c) \to M(c)$	Rule 5, step 3
5.	$\begin{array}{l} K(c) \to M(c) \\ M(c) \to F(c) \\ \hline \therefore K(c) \to F(c) \end{array}$	step 4 & Step 2, Hypothetical Syllogysm.
6.	$\forall x, K(x) \to F(x)$.	Rule 6, step 5.

(II) Lions are dangerous animals
There are lions

$\therefore$ There are dangerous animals

let, $L(x)$: x is a lion
$D(x)$: x is a dangerous animal.

$$\begin{array}{l} \forall x, L(x) \to D(x) \\ \exists x, L(x). \\ \hline \therefore \exists x, D(x). \end{array}$$

steps	formal Proof	Reasons
1.	$\forall x, L(x) \to D(x)$.	Premise 1
2.	$L(c) \to D(c)$	Rule 5, Step 1.
3.	$\exists x, L(x)$.	Premise 2
4.	$L(c)$	Rule 7, step 3
5.	$\begin{array}{l} L(c) \wedge (L(c) \to D(c)) \\ \hline \therefore D(c) \end{array}$	Step 4 & 2 Modus Ponens
6.	$\exists x, D(x)$	Rule 8, step 5.

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3. A student in the class has not read the book
Everyone in this class passed the first exam.
Therefore, someone who passed the first exam has not read the book.

Solution:- (Proof)

let, $S(x)$: x is a student in the class
$R(x)$: x has read the book.
$P(x)$: x has passed the first exam.

$$\exists x,\ S(x) \wedge \sim R(x).$$
$$\forall x,\ S(x) \rightarrow P(x)$$
$$\therefore \exists x,\ P(x) \wedge \sim R(x)$$

Steps.	Formal Proof	Reasons
1.	$\exists x,\ S(x) \wedge \sim R(x)$	Premise 1.
2.	$S(c) \wedge \sim R(c)$	Existential Specification; Step 1.
3.	$\forall x,\ S(x) \rightarrow P(x)$	Premise 2.
4.	$S(c) \longrightarrow P(c)$.	Universal Specification, Step 3.
~~5.~~	~~$(S(c) \Rightarrow P(c)) \wedge (S(c) \wedge \sim R(c))$~~	
5.	$S(c)$	Simplification from 2
6.	$P(c)$	Modus Pones. 5 & 4.
7	$\sim R(c)$	simplification from 2
8.	$P(c) \wedge \sim R(c)$.	Conjunction (6 & 7).
9.	$\exists x,\ P(x) \wedge \sim R(x)$	Existential generalization (8).

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* Proof by example :-

Q. Show that 'some prime numbers are integers' is true.

Ans :- Let, $F(x)$: x is prime.

Rewritting the above statement we get,

$\exists x . F(x)$ where $4 \in \mathbb{Z}$.

Since $F(x)$ holds for $x = 2, 3, 5, 7, \ldots,$ etc.

Hence from proof by example we can say that

$\exists x \, F(x)$ is true.

* Proof by exhaustion :-

Q. Show that difference of two consecutive square numbers betn 25 & 100 is odd.

Ans :- $6^2 = 36$; $7^2 = 49$; $8^2 = 64$; $9^2 = 81$.

$7^2 - 6^2 = 49 - 36 = 13$

$8^2 - 7^2 = 64 - 49 = 15$

$9^2 - 8^2 = 81 - 64 = 17$

So by writing proof by exhaustion, we can say that the difference of two consecutive square nos. betn 25 & 100 is odd.

* Proof by contradiction :-

Q. Show that $x + y = 8$ is not true for all integers.

Ans :- Let, $P(x, y)$: $x + y = 8$.

Assume, $\forall x, y \; P(x, y)$ is true.

when $x = 2, y = 3$

$P(x, y) \neq 8$. $\therefore$ Hence using proof by contradiction it is proved that $x + y = 8$ is not true for all integers.

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Q. Show that the premises (Everyone in this Discrete Maths class has taken a course in computer Science) and (Marla is a student in this class) imply the conclusion that (Marla has taken a course in computer Science).

Ans :- let, D(x) = " x is in this discrete maths class".

C(x) = " x has taken a course in computer science"

Then the premises are

$\forall x\,(D(x) \rightarrow C(x))$

D(Marla).

The conclusion is

C(Marla).

Steps	Reason
1. $\forall x\,(D(x) \rightarrow C(x))$	Premise.
2. $D(\text{Marla}) \rightarrow C(\text{Marla})$	Universal Instantiation from (1)
3. D(Marla)	Premise
4. C(Marla)	Modus ponens from (2) & (3)

Q. Represent the following argument symbollically And decide whether they are valid.

"If x is a lion, then x is carnivorous".

"Moo is not carnivorous". then prove that Moo is not a lion..

⇒ let, L(x) : x is a Lion..

C(x) : x is a Carnivorous.

M(x) : x is a Moo.

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$$\begin{array}{l} \forall x\ L(x) \to C(x). \\ \forall x\ M(x) \to \sim C(x) \\ \hline \forall x\ M(x) \to \sim L(x) \end{array} \equiv \begin{array}{l} \forall x\ L(x) \to C(x). \\ \sim C(Moo). \\ \hline \sim L(Moo). \end{array}$$

Steps	Assertion	Reason
1.	$\forall x(L(x) \to C(x))$	Premise.
2.	$L(Moo) \to C(Moo)$	Universal Instantiation from ①
3.	$\sim C(Moo)$	Premise.
4	$\sim L(Moo)$.	Modus tollens (3 & 4).

Q. "All lions are fierce"
"Some lions do not drink coffee"

"Some fierce creatures do not drink coffee".

Ans:- Let, $L(x)$ = "x is a lion".
$F(x)$ = "x is fierce".
$C(x)$ = "x drinks coffee".

Symbolic representation.

$$\begin{array}{l} \forall x\ (L(x) \to F(x)) \\ \exists x\ (L(x) \land \sim C(x)) \\ \hline \exists x\ (F(x) \land \sim C(x)). \end{array}$$

	Assertion	Reason
1.	$\exists x(L(x) \land \sim C(x))$	Premise
2.	$L(Foo) \land \sim C(Foo)$	Existential Instantiation ①
3.	$L(Foo)$	Simplification from ②
4	$\sim C(Foo)$	Simplification from ②
5.	$\forall x(L(x) \to F(x))$	Premise
6.	$L(Foo) \to F(Foo)$	Universal Instantiation from ⑤.
7.	$F(Foo)$.	Modus Ponens (3 & 6).
8.	$F(Foo) \land \sim C(Foo)$	Conjunction (4 & 7).
9.	$\exists x,\ F(x) \land \sim C(x)$	Existential generalisation ⑧

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MATHEMATICAL INDUCTION

$P(x)$ is a statement which can be either true or false, for all +ve integer 'n'. To prove $P(x)$ to be true for $n \geq n_0$

1. $P(n_0)$ is true.
2. For all $k \geq n_0$, $P(k) \rightarrow P(k+1)$.

Solution : 1. Basis of Induction : $P(n_0)$ is true.

2. Induction Hypothesis. Assume $P(k)$ to be true.
3. Inductive Step : Based on 'step 2' show that $P(k+1)$ is true.

Ex. Show that if n is a +ve integer then

$$1+2+3+\ldots+n = \frac{n(n+1)}{2}.$$

Ans. Let, $P(n)$ be the statement s.t. sum of all positive integers is $\frac{n(n+1)}{2}$.

Step 1. :- $P(1) = \frac{1(1+1)}{2} = \frac{2}{2} = 1.$

Step 2 :- for all $k \geq 1$, $P(k) = \frac{k(k+1)}{2}$.

Step 3 :- $P(k+1) = (1+2+\ldots+k) + k+1$

$$P(k+1) = P(k) + k+1$$

$$= \frac{k(k+1)}{2} + (k+1) = \frac{k(k+1)+2(k+1)}{2}$$

$$P(k+1) = \frac{(k+1)(k+2)}{2} = \frac{(k+1)(k+1+1)}{2}.$$

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Example :- Conjecture a formula for the sum of first n positive odd integers. Then prove using MI :

Ans :- $n = 1, 3, 5, 7, 9$.

$n = 1 \Rightarrow 1 = 1^2$

$n = 2 \Rightarrow 1 + 3 = 4 = 2^2$

$n = 3 \Rightarrow 1 + 3 + 5 = 9 = 3^2$

$n = 4 \Rightarrow 1 + 3 + 5 + 7 = 16 = 4^2$

$n = 5 \Rightarrow 1 + 3 + 5 + 7 + 9 = 25 = 5^2$.

Let, $P(n)$: sum of first 'n' positive odd integers is 'n^2'

$1 + 3 + 5 + \dots + (2n-1) = n^2$.

Step 1 :-

$P(1) = 1 = 1^2$.

Step 2 :- for all $k \geq 1$, $P(k) = k^2$.

Step 3 :- $P(k+1) = \{1 + 3 + 5 + \dots + (2k-1)\} + (2(k+1) - 1)$

$= k^2 + 2k + 2 - 1$

$= k^2 + 2k + 1 = (k+1)^2$.

STRONG MATHEMATICAL INDUCTION

Let $P(n)$ be a statement which is either true or false for each integer 'n'. Then $P(n)$ is true for all +ve integers 'n'. If there is an integer $q \geq 1$ &

1. $P(1), P(2), \dots, P(q)$ are all true.
2. $k \geq q$, the assumption that $P(i)$ is true, where $1 \leq i \leq k$ implies $P(k+1)$ is true.

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Step 1. Basis of Induction $P(1), P(2), \ldots, P(q)$ true.

Step 2. Strong Induction Hypothesis. Assume $P(i)$ to be true. $1 \le i \le k$ for $k \ge q$.

Step 3. Induction step. $P(k+1)$ is true.

Ex. Suppose we have stamps of denominations ₹3 and ₹5. Show that it is possible to make postages of ₹8 or more by using stamps of these denominations only.

$\Rightarrow$

$8 = 3+5$

$9 = 3+3+3$

$10 = 5+5$

$11 = 3+3+5$

$12 = 3+3+3+3$.

step 1. $P(8), P(9), \ldots P(12)$ is true.

Step 2. $P(i)$ is true.

$8 \le i \le k$, $k \ge 12$.

Step 3: $P(k-4)$ is true.

$8 \le k-4 \le k$.

₹ $(k-4)$ + ₹ 5 = ₹ $(k+1)$.

$\Rightarrow$ $P(k+1)$ is true.

Q.1) To Prove: $\left[1^2+2^2+\ldots+n^2 = \frac{n}{6}(n+1)(2n+1)\right]$

Ans:- Basic Step for $P(1)$

L.H.S. $= 1^2 = 1$

R.H.S. $= \frac{1}{6}(1+1)(2\times1+1) = \frac{6}{6} = 1$.

L.H.S. = R.H.S. $\Rightarrow$ $P(1)$ is true.

Inductive Step Assume $P(k)$ is true.

$\Rightarrow$ $1^2+2^2+3^2+ \_\_\_\_ + k^2 = \frac{k}{6}(k+1)(2k+1)$ ..... (1)

we have to prove that $P(k+1)$ is also true when $P(k)$ is also true.

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For $P(k+1)$

L.H.S. $= 1^2+2^2+ \text{---} + k^2 + (k+1)^2 = \frac{k}{6}(k+1)(2k+1) + (k+1)^2$

[using eqn ①]

$= (k+1)\left[\frac{k(2k+1)}{6} + (k+1)\right]$

$= (k+1) \cdot \left[\frac{2k^2+k+6k+6}{6}\right]$

$= \frac{(k+1)}{6}\left[2k^2+7k+6\right] = \frac{(k+1)}{6}(k+2)(2k+3)$.

R.H.S. $= \frac{(k+1)}{6}((k+1)+1)(2(k+1)+1)$

$= \frac{(k+1)}{6}(k+2)(2k+3)$ $\therefore$ L.H.S. = R.H.S.

By M.I. it is proved that $\left[1^2+2^2+ \text{---} + n^2 = \frac{n}{6}(n+1)(2n+1)\right]$.

Q.2) To Prove: n^3-n is divisible by 3 whenever n is a positive integer.

Here: $P(n)$: n^3-n is divisible by 3. $\forall n>0$ (i.e. $n \in N$)

Basic step

For $P(1)$; $1^3-1 = 1-1 = 0$ is divisible by 3.

$\Rightarrow P(1)$ is true.

Inductive step :- Assume $P(k)$ is true

$\Rightarrow k^3-k$ is divisible by 3.

$\Rightarrow k^3-k = 3m$; $(m \in N)$ ——— ①.

Now, we have to prove that $P(k+1)$ is also true where $p(k)$ is true.

For $P(k+1) \Rightarrow (k+1)^3-(k+1) = (k^3+1+3k^2+3k)-(k+1)$.

$= k^3-k+3(k^2+k)$.

$= 3m+3(k^2+k) = 3(m+k^2+k)$.

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$\Rightarrow (k+1)^3 - (k+1)$ is divisible by 3.

$\Rightarrow$ P(k+1) is also true.

Here using M.I. it is proved that n^3-n is divisible by 3.

Q.3) To Prove: $2^n < n!$ for every +ve integer n with $n \geq 4$

Soln:- Basic Step :-

For P(4) ; $2^4 = 16$; $4! = 24$; $2^4 < 4!$.

$\Rightarrow$ P(4) is true.

Inductive Step :-

Assume P(k) is true, where $k \geq 4$

$\Rightarrow \quad 2^k < k!$ —— ①.

Now we have to prove that P(k+1) is true when P(k) is also true.

For P(k+1) $\rightarrow 2^{(k+1)} = 2^k \cdot 2 < k! \cdot 2$ (from ①)

As $k \geq 4$; $\Rightarrow (k+1) \geq 4 \geq 2$.

$\therefore \quad 2^{k+1} < k!(k+1)$

$< (k+1)!$

$\Rightarrow$ P(k+1) is true

Here, using M.I. it is proved that $2^n < n!$ for every +ve integer n with $n \geq 4$.

Q.4) To prove: $[1 + 2 + 2^2 + \text{——} + 2^n = 2^{n+1} - 1]$ for all non-negative integer.

Soln Basic Step For P(0) L.H.S. = 1 ; R.H.S. = $2^{0+1} - 1 = 2^1 - 1$
$= 1$.

L.H.S. = R.H.S.

$\Rightarrow$ P(0) is true.

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Inductive step Assume P(k) is true

$1+2+2^2+ \_\_\_ +2^k = 2^{k+1}-1$. — (i)

and we have to prove that P(k+1) is true when P(k) is true.

For P(k+1) L.H.S. $= 1+2+2^2+ \_\_\_ +2^k+2^{k+1}$

$= 2^{k+1}-1+2^{k+1}$ (using (i)).

$= 2.2^{k+1}-1 = 2^{k+2}-1$.

R.H.S. $= 2^{(k+1)+1}-1 = 2^{k+2}-1$. $\Rightarrow$ $\boxed{L.H.S. = R.H.S}$

$\Rightarrow$ P(k+1) is true.

$\Rightarrow$ Using M.I. it is proved that $[1+2+2^2+ \_\_ +2^n = 2^{n+1}-1]$

Q.5) Let a_n be the sequence defined by $a_1=1$, $a_2=8$ and $a_n = a_{n-1}+2a_{n-2}$ for $n \geq 3$. prove that $a_n = 3.2^{n-1}+2(-1)^n$; $\forall n \in \mathbb{N}$.

Solⁿ:- Basic step

For a_1 ; L.H.S. = 1

R.H.S. $= 3.2^{(1-1)}+2(-1)^1 = 3-2 = 1$.

a_2 ; L.H.S. = 8

R.H.S. $= 3.2^{(2-1)}+2(-1)^2 = 6+2 = 8$.

$\Rightarrow$ a_1, a_2 is true.

Inductive step

$a_{k-1} = 3.2^{(k-1)-1}+2(-1)^{k-1}$ — (i)

Assume a_k and a_{k-1} is true $\Rightarrow$ $a_k = 3.2^{k-1}+2(-1)^k$ — (ii)

Using strong M.I. we have to prove that a_{k+1} is true when a_k is true.

$a_{k+1} = a_k + 2.a_{k-1}$

R.H.S. $= 3.2^{k-1}+2.(-1)^k+2.[3(2^{k-2})+2.(-1)^{k-1}]$

$= 3[2^{k-1}+2^{k-1}]+2[(-1)^k+2(-1)^{k-1}]$

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$= 3 \cdot 2^{k} + 2(-1)^{k+1}$.

L.H.S. $= 3 \cdot 2^{(k+1)-1} + 2(-1)^{k+1} = 3 \cdot 2^{k} + 2(-1)^{k+1}$.

$\therefore$ L.H.S. = R.H.S. $\Rightarrow a_{k+1}$ is true.

$\therefore$ Using Strong M.I. it is proved that - - - -.

Q.6) $3 + 7 + 11 + \dots + (4n-1) = n(2n+1)$.

Ans :- Basic Step :-

for P(1) :- L.H.S. $= 3$; R.H.S $= (1)(2 \times 1 + 1)$
$= 1(3) = 3$.

$\Rightarrow$ L.H.S. = R.H.S. $\Rightarrow$ P(1) is true.

Inductive Step :-

Assume P(n) is true

$\Rightarrow 3 + 7 + 11 + \text{---} + (4n-1) = n(2n+1)$ —— ①

Now using ① we have to prove that P(n+1) is true.

for P(n+1) $\Rightarrow$ L.H.S. $= 3 + 7 + 11 + \text{---} + (4n-1) + (4(n+1)-1)$

$= P(n) + (4(n+1)-1)$

$= n(2n+1) + 4n+3$

$= 2n^2 + n + 4n + 3$

$= 2n^2 + 2n + 3n + 3$

$= 2n(n+1) + 3(n+1)$

$= (n+1)(2n+3)$

$= (2(n+1)+1)(n+1)$

$\Rightarrow$ R.H.S.
$= (n+1)(2(n+1)+1)$

$\therefore$ L.H.S. = R.H.S.

$\therefore$ Hence, proved.

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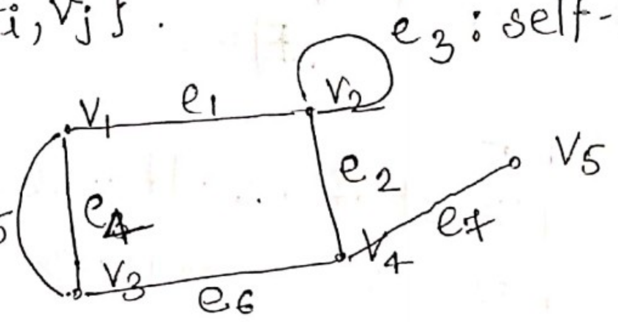
GRAPH THEORY

Defⁿ:- A graph consists of two set of objects $V = \{v_1, v_2, \ldots, v_n\}$ and $E = \{e_1, e_2, \ldots, e_n\}$. $G = (V, E)$

$e_k = \{v_i, v_j\}$.



$G = \{(v_1, v_2, \ldots, v_n) ; (e_1, e_2, \ldots, e_7)\}$.

Simple Graph :- A graph with no parallel edges or self loops.

Incidence : If a 'v_i' is the endpoint of an edge 'e_j' then v_i & e_j are incident on each other.

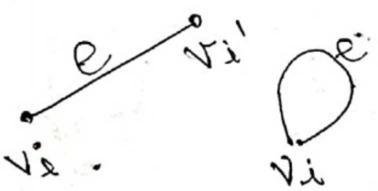
Degree of a vertex : No. of ~~vertex~~ edges incident on a vertex v_i, with self loops counted as 2, is the degree of v_i, $d(v_i)$.

e.g. $d(v_1) = 3$; $d(v_2) = 4$.

Adjacency : Two vertices are adjacent to each other if they have a common edge. e.g. (v_1, v_2)

Two edges are said to be adjacent if they are incident on the same vertex. (e_1, e_5) ; $(e_1 e_4)$.

- If G is a graph with 'e' edges and 'n' vertices $V = \{v_1, v_2, \ldots, v_n\}$.



$$\boxed{\sum_{i=1}^{n} d(v_i) = 2e}$$

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$d(v_1) = 3$
$d(v_2) = 4$
$d(v_3) = 3$
$d(v_4) = 3$
$d(v_5) = 1$

$\sum_{i=1}^{5} d(v_i) = 14 = 2 \times 7$.

Theorem (Handshaking Theorem)

(The no. of vertices of odd degree in a graph is always even).

$G = (V, E)$ where $V = \{v_1, v_2, \ldots, v_n\}$, $|E| = e$.

Proof :-

$V_{odd} \Rightarrow$ set of all odd degree vertices

$V_{even} \Rightarrow$ — " — even degree vertices.

$$\sum_{i=1}^{n} (d(v_i)) = 2e.$$

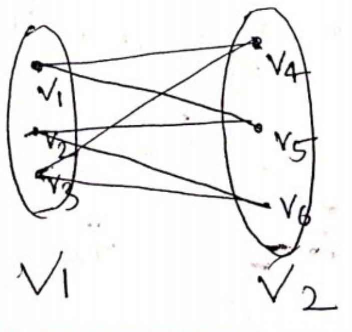
$$\Rightarrow \sum_{v_j \in V_{odd}} d(v_j) + \sum_{v_i \in V_{even}} d(v_i) = 2e.$$

$$\Rightarrow \sum_{v_j \in V_{odd}} d(v_j) = 2e - \sum_{v_i \in V_{even}} d(v_i)$$

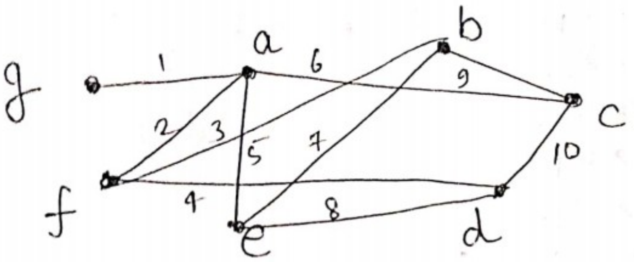
$$\Rightarrow |V_{odd}| = \text{even} \Rightarrow \text{Hence, Proved.}$$

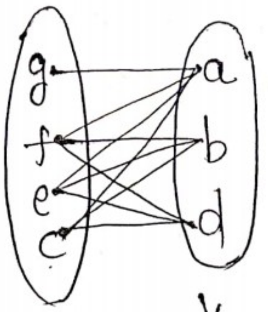
★ **BIPARTITE GRAPH** :-

A simple graph whose vertex set 'V' can be partitioned into 2 disjoint subsets 'V_1' and 'V_2' s.t. every edge in the graph connects a vertex in V_1 & a vertex in V_2 with no edge connects either 2 vertices in V_1 or V_2.



Q.1)

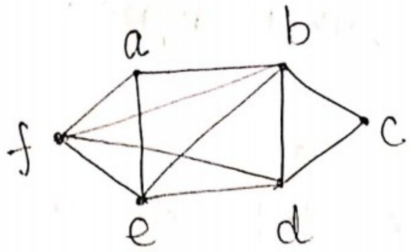




Bipartite

V_1 V_2

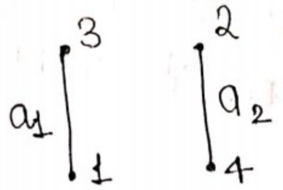
Q.2)



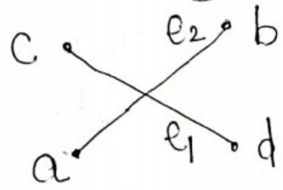
<u>Not Bipartite</u>

★ <u>ISOMORPHISM</u> :-

$G = (V, E)$ V = set of vertices
E = set of edges.



(A)

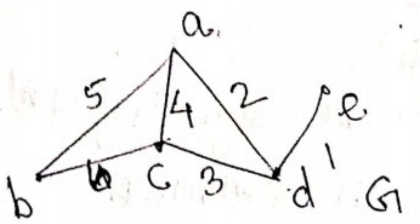


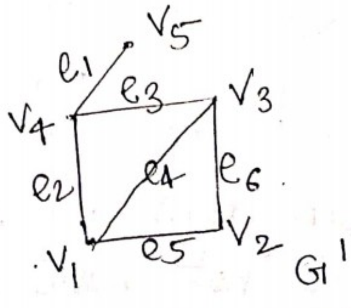
(B)

$1 \to a$ $a_1 \to e_2$
$2 \to c$ $a_2 \to e_1$
$3 \to b$
$4 \to d$

<u>Defn</u> : 2 graph G and G' are said to be isomorphic, if there is a one-to-one correspondence betn the set of vertices & the set of edges s.t, the incidence/adjacen-cy criteria is maintained.

e.g.





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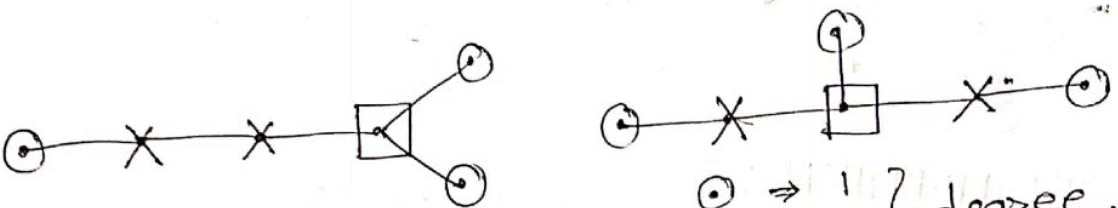
$a \rightarrow v_1$; $1 \rightarrow e_1$
$b \rightarrow v_2$; $2 \rightarrow e_2$
$c \rightarrow v_3$; $3 \rightarrow e_3$
$d \rightarrow v_4$; $4 \rightarrow e_4$
$e \rightarrow v_5$; $5 \rightarrow e_5$
$6 \rightarrow e_6$

1. The no. of edges must be same.
2. The no. of vertices must be same.
3. The no. of vertices with a given degree.



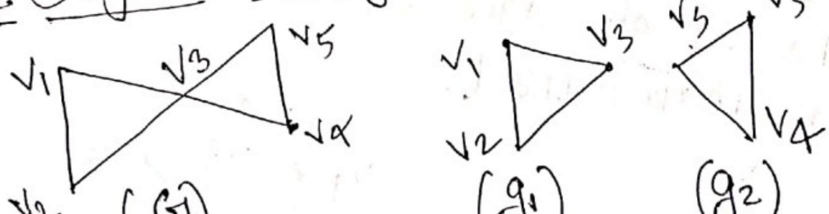
are not isomorphic.

$\odot \Rightarrow 1$, $\times \Rightarrow 2$, $\boxdot \Rightarrow 3$ } degree.

★ SUBGRAPH :- $G=(V,E)$; $V' \subseteq V$; $E' \subseteq E$ with all pairs in E' are in V'; then $G'=(V',E')$ is a sub-graph of G.

1. Every graph is its own subgraph.
2. The subgraph of a subgraph, of a graph G, is a subgraph of G.
3. A single vertex.
4. A single edge with its vertices.

• Edge Disjoint Subgraphs :-



(G1) (g1) (g2)

Two subgraphs are said to be edge-disjoint if they do not have edge in common.

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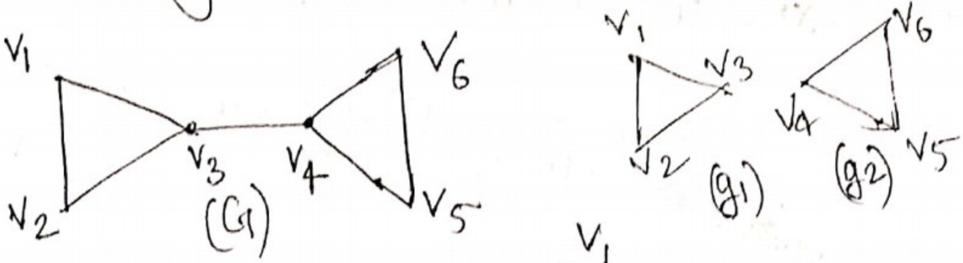
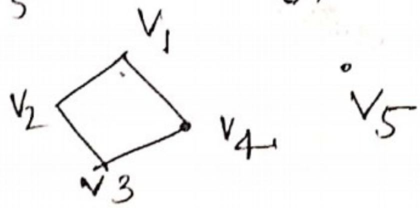
2. Vertex disjoint Subgraphs :-

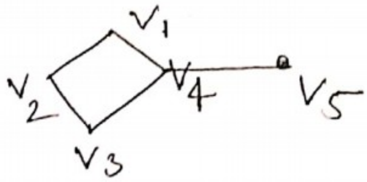
Two edge disjoint subgraphs are said to be vertex disjoint if they do not have a vertex in common.



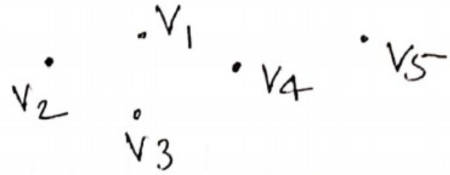
Isolated vertex : vertex with degree '0'



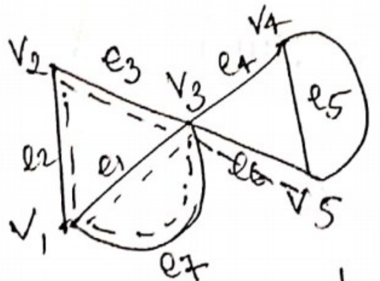
Pendant vertex : degree 1.



Null graph : no edges



Regular graph : all vertices have same degree d. { d-Regular GRAPH.



A walk in a graph is a finite seq of form.

$v_{i0}, e_{j1}, v_{i1}, e_{j2}, \ldots, e_{jk}, v_{jk}$ (vertex and edge both can repeat (walk))

eg. $v_1, e_1, v_3, e_3, v_2, e_2, v_1, e_7, v_3, e_6, v_5$.

Trail : A walk with no repetition of edges.

Closed walk (same Terminal vertices) Open walk (different terminal vertices)

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Path : An open walk with no vertex repeated.

Cycle / ~~Circuit~~ : A closed walk with no repetition of vertex, except the terminals. (A closed path).

Circuit : ~~A closed trail~~

Tutorial Draw a graph such that there is no edge betn the sets if $(A_i \cap A_j \neq 0)$.

Q.1 $A_1 = \{\ldots, -4, -3, -2, -1, 0\}$

$A_2 = \{\ldots, -2, -1, 0, 1, 2\}$

$A_3 = \{\ldots, -6, -4, -2, 0, 2, 4, \ldots\}$

$A_4 = \{-5, -3, -1, 1, 3, 5, \ldots\}$.

$A_5 = \{\ldots, -6, -3, 0, 3, 6, \ldots\}$.

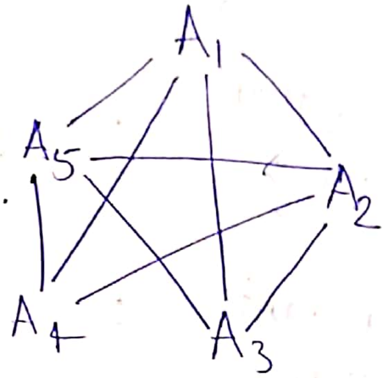


Fig. Solution.

Q.2) What is the total no. of vertices in an undirected connected graph with 27 ~~vertices~~ edges, 6 vertices of deg. 2, 3 vertices of deg. 4 and remaining of deg. 3?

Ans. Total degree = 2 × Number of edges.

$$\Rightarrow \sum_{i=1}^{n} (d(v_i)) = 2e$$

$$\Rightarrow 6 \times 2 + 3 \times 4 + x \times 3 = 2 \times 27$$

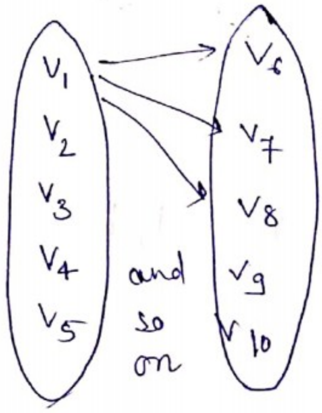
$$\Rightarrow 3x = 54 - (24) = 30$$

$$\Rightarrow \boxed{x = 10}$$

$\therefore$ Total vertices $= 6 + 3 + x = 6 + 3 + 10 = 19$.

Q.3) What is the maximum number of edges present in a bipartite graph with 10 vertices?

⇒



So, max no. of edges

$= 5 \times \cancel{2}5$

$= 25$.

★ **Degree Sequence** :- It is the sequence of degrees of vertices written in non-decreasing order.

A degree sequence is said to be ~~toh~~ valid or graphic if we can construct a graph out of that degree seq.

$S = d_1 \geq d_2 \geq d_3 \geq \dots \geq d_n$.

Necessary Conditions

① $\sum_i d_i = 2e$.

② $\forall i;\ d_i \leq (n-1)$ — n = total no of vertices

ⓐ 3, 3, 3, 3, $\cancel{2}$

ⓑ 5, 4, 3, 2, 1 ✗

ⓒ 4, 4, 3, 2, 1

ⓓ 4, 4, 3, 3, 3 ✗

ⓔ 3, 2, 2, 1, 0

ⓕ 1, 1, 1, 1 ✗

ⓖ 3, 3, 3, 1

$\boxed{\times \Rightarrow \text{Invalid}}$

★ **Havel and Hakimi Theorem** :-

$S = d_1\ d_2 \dots d_n$ such that $d_1 \geq d_2 \geq \dots \geq d_n$ is a valid deg. sequence if $S' = d_2-1, d_3-1, \dots, d_{d_1+1}-1, d_{d_1+2}, \dots, d_n$ is a valid degree sequence. Repeat until $\exists i$ such that d_i is ~~a~~ negative or $\forall i\ d_i = 0$.

if $\exists i\ d_i < 0$ then it is not a valid degree seq.
if $\forall i\ d_i = 0$ then it is a valid degree seq.

(a) 3, 3, 3, 3, 2.

Soln:-
$d_1+1 = 4$ $\quad S_1 = 2,2,2,2$.
$d_1+1 = 3$ $\quad S_2 = 1,1,2 \Rightarrow 2,1,1$
$d_1+1 = 3$ $\quad S_3 = 0,0$.

$\Rightarrow$ Valid degree sequence.

(g) 3, 3, 3, 1.

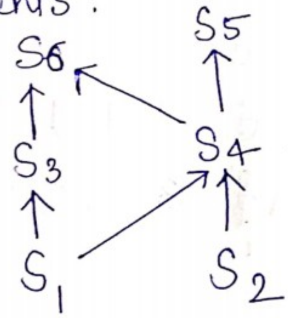
Soln:- $d_1+1 = 4$ $\quad S_1 = 2,2,0$.
$d_1+1 = 3$ $\quad S_2 = 1,-1 < 0$.

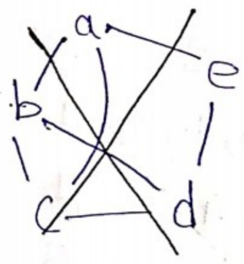
$\Rightarrow$ Not a valid degree sequence.

Q. Draw a precedence graph for given statements.

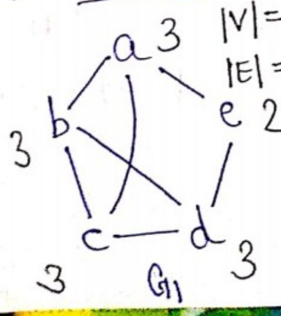
S_1 a=0
S_2 b=1
S_3 c=a+1
S_4 d=b+a.
S_5 e=d+1
S_6: e=c+d.

Ans:-

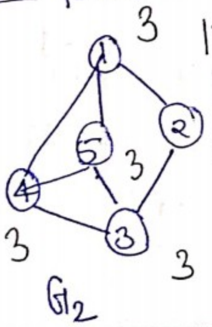




Q. Check Isomorphism.



|V|=5
|E|=7



|V|=5
|E|=7

S_{G_2} = 3 3 3 3 2

S_{G_1} = 3 3 3 3 2

Cycles with length
3 $\Rightarrow$ 2 in both
4 $\Rightarrow$ 2 in both.

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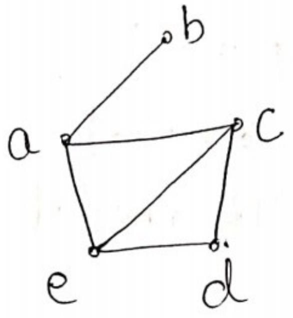
1) One-to-one correspondence

Representing a Graph :-

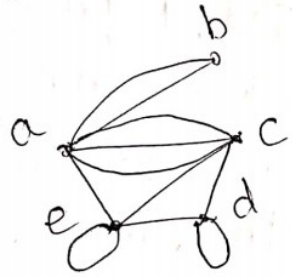


a	b, c, e
b	a
c	a, d, e
d	e, c
e	a, c, d

Adjacency List

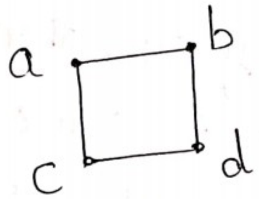
Adjacency Matrix

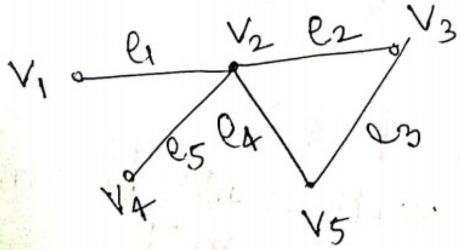
	a	b	c	d	e
a	0	1	1	0	1
b	1	0	0	0	0
c	1	0	0	1	1
d	0	0	1	0	1
e	1	0	1	1	0



	a	b	c	d	e
a	0	2	3	0	1
b	2	0	0	0	0
c	3	0	a	1	1
d	0	0	1	1	1
e	1	0	1	1	1

$$\begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$





Incidence Matrix

	e_1	e_2	e_3	e_4	e_5
v_1	1	0	0	0	0
v_2	1	1	0	1	1
v_3	0	1	1	0	0
v_4	0	0	0	0	1
v_5	0	0	1	1	0

On adding parallel edges & self loops, matrix will still remain binary. (incidence)

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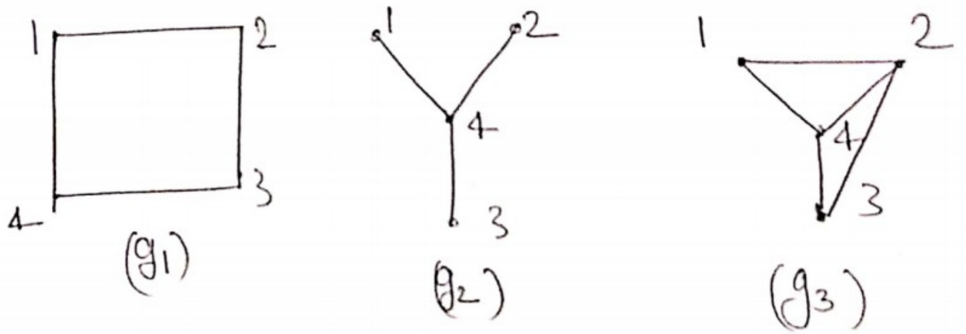
* $G = (V,E)$ $V(G)$; $E(G)$

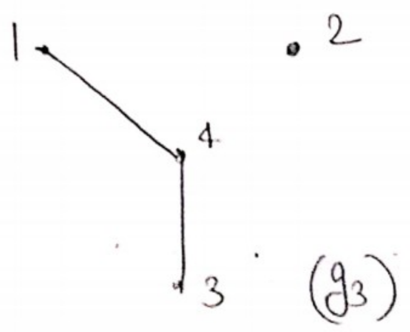
① <u>Union of two graphs</u> g_1 & g_2 is g_3 s.t.

$V(g_3) = V(g_1) \cup V(g_2)$ & $E(g_3) = E(g_1) \cup E(g_2)$.



② <u>Intersection of 2 graphs</u> :-

$V(g_3) = V(g_1) \cap V(g_2)$ & $E(g_3) = E(g_1) \cap E(g_2)$.



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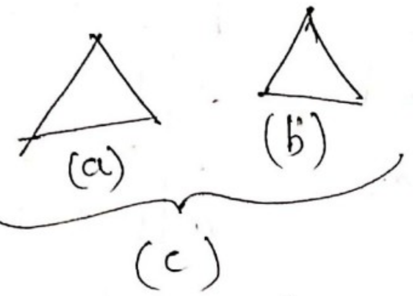
★ Connected Graph :-

A graph is connected if there exists atleast one path betn every pair of vertices.

Disconnected Graph :- Consists of 2 or more connected subgraphs also called as connected components.



a, b ⇒ connected

c ⇒ Disconnected.

Theorem 1 : A graph 'G' is a disconnected graph iff its vertex set 'V' can be partitioned into 2 non-empty disjoint subsets 'V_1' & 'V_2' s.t. there is no edge in 'G' whose one end vertex lies in 'V_1' and other in 'V_2'.

Theorem 2 : If a graph (conn. or disconn.) has exactly 2 vertices of odd degree, then there must be a path joining these 2 vertices.

Proof :- For connected graph, it is obvious.

Let G be a graph with all even degree vertices except 'v_1' and 'v_2'.

'J' a connected component of 'G'.

If $v_1 \in J$, $v_2 \in J$ ($\because$ Total degree of no. of vertices must be even in J)

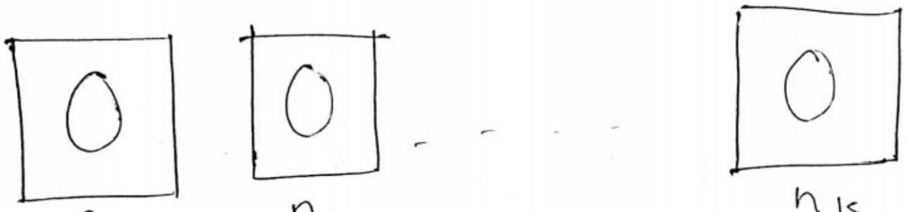
Theorem 3: A simple graph with 'n' vertices and 'k' connected components has at most $\frac{(n-k)(n-k+1)}{2}$ edges.

(Assignment).

Proof :-

$$n_1 + n_2 + \dots + n_k = n \Rightarrow \sum_{i=1}^{k} n_i = n .$$

(where, n_i is number of vertices in 'i'th conn. component



n_1 $\quad$ n_2 $\quad$ n_k

No. of ways of selecting 2 out of $n_1 = {}^{n_1}C_2 = \frac{(n_1)(n_1-1)}{2}$

Basic step :-

If $k=1$; $n_1 = n$. $\Rightarrow$ no. of max edges $= \frac{n(n-1)}{2}$ = L.H.S

$$\text{R.H.S.} = \frac{(n-k)(n-k+1)}{2} = \frac{(n-1)(n-1+1)}{2} = \frac{(n-1)(n)}{2}$$

$\therefore$ L.H.S. = R.H.S. $\Rightarrow$ P(1) is true.

Inductive step :-

Assume P(K) is true and prove that P(k+1) is true. $n_1 + n_2 + \dots + n_k = n$

no. of max possible edges $= \frac{(n-k)(n-k+1)}{2}$

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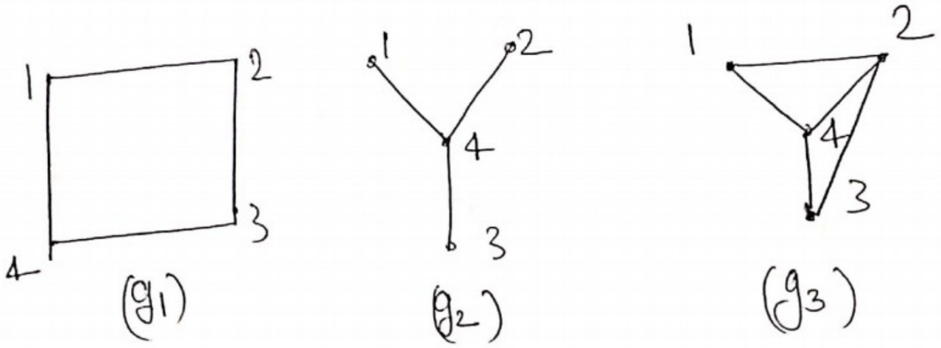
* $G = (V,E)$ $V(G)$; $E(G)$.

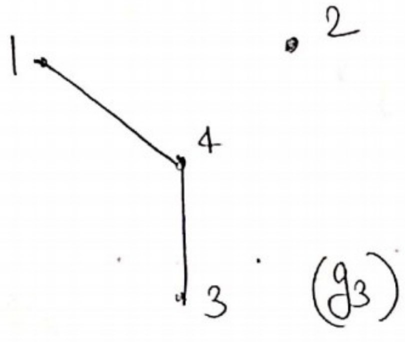
① **Union of two graphs** g_1 & g_2 is g_3 s.t.

$(V_{g_3}) = V(g_1) \cup V(g_2)$ & $E(g_3) = E(g_1) \cup E(g_2)$.



② **Intersection of 2 graphs** :-

$V(g_3) = V(g_1) \cap V(g_2)$ & $E(g_3) = E(g_1) \cap E(g_2)$.



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③ **Ringsum of g_1 & g_2** :- $(g_1 \oplus g_2)$ is a graph g_3 s.t.

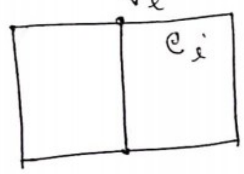
$V(g_1) \oplus V(g_2) = V(g_1) \cup V(g_2)$.

$e \in E(g_1) \oplus E(g_2) \quad [E(g_1) \cup E(g_2)] - [E(g_1) \cap E(g_2)]$

④ **Decomposition** :- Any G can be decomposed into g_1 & g_2 s.t. $g_1 \cup g_2 = G$; $g_1 \cap g_2 = \phi$.

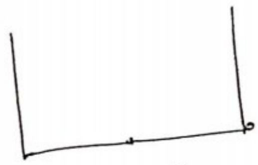
⑤ **Deletion** :-

v_i e_i



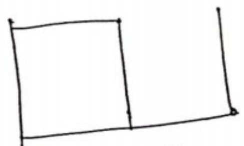
G

(original Graph)



$G - v_i$

(Vertex Deletion).



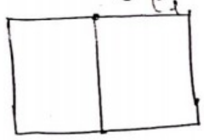
$G - e_i$

(Edge deletion)

⑥ **Fusion** :-

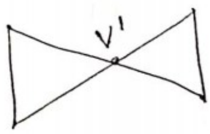
v_i e_i

(G)



v_j

v'



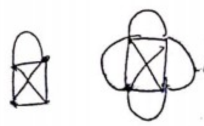
(G_F)

★ **EULER GRAPH** :-

① **Path/Trail** :- A trail which visits every edge of the graph exactly once. [There can be repetition of vertices).

② **Cycle/Circuit** :- An Euler trail whose terminal vertices are the same.

③ **Euler Graph** :- An Euler graph must have an Euler circuit, i.e., a closed trail which visits every edge of the graph exactly once



Non Euler

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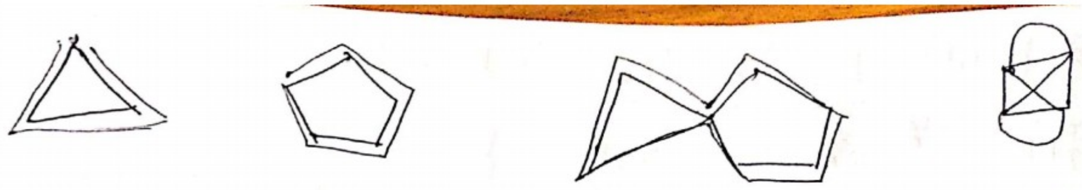
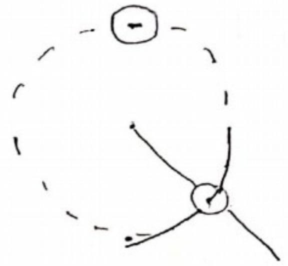
eg.



Theorem :- A given connected graph G is an Euler Graph iff all its vertices are of even degree



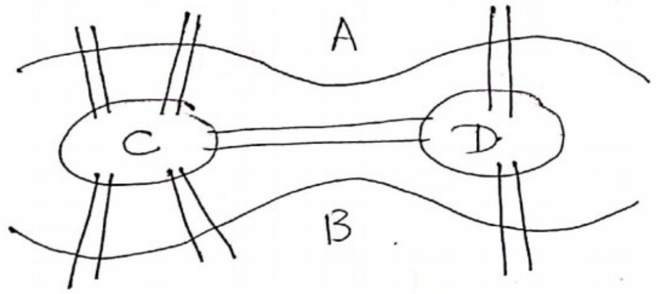
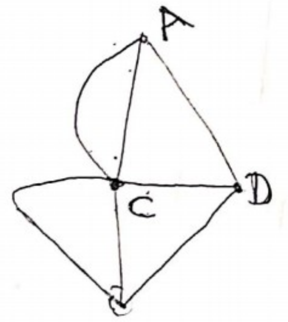


Fig. Konigsberg Bridge Problem



B (Graph) Non Euler

Theorem :- A connected graph is an Euler graph iff it can be decomposed into circuits.

Tutorial

(Q) (a) 4, 4, 4, 3, 2, 1, 0.

$\Rightarrow$ $d_1 + 1 = 5$; 3, 3, 2, 1, 1, 0

$d_1 + 1 = 4$; 2, 1, 0, 1, 0 $\Rightarrow$ 2 1 1 0 0

$d_1 + 1 = 3$; 0, 0, 0, 0 All 0 $\therefore$ ~~Not~~ valid Deg. seq.

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(b) 5, 3, 3, 3, 3, 3.

$\Rightarrow$ $d_1+1 = 6$; 2, 2, 2, 2, 2

$d_1+1 = 3$; 1, 1, 2, 2 $\rightarrow$ 2 2 1 1 (Non-increasing order).

$d_1+1 = 3$; 1, 0, 1, 0 $\Rightarrow$ 1 1 0.

$d_1+1 = 2$; 0, 0.

Valid

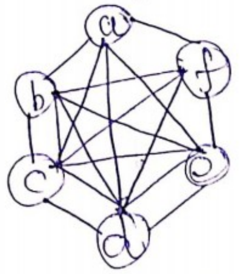
(c) 5, 5, 4, 3, 2, 1

$\Rightarrow$ $d_1+1 = 6$; 4, 3, 2, 1, 0

$d_1+1 = 5$; 2, 1, 0, −1 < 0 Not valid.

Q.) Let G be a complete undirected graph on 6 vertices if vertices of G are labelled then the total number of distinct cycles of length four in G are -

Ans :- a, b, c, d, e, f.



(a, b, c, d, a) $\equiv$ (a, d, c, b, a)

(a, b, d, c, a) $\equiv$ (a, c, d, b, a)

(a, c, b, d, a) $\equiv$ (a, d, b, c, a)

$$^6C_4 = \frac{6\times5}{2} = 15\times3 = \underline{\underline{45}}$$ Total

$$= \frac{6\times5\times4\times3}{4!} = 15$$

★ Completment of Graph. ($\overline{G}$) :-

$G(V, E) \Rightarrow \overline{G}(V, E)$.

- is a graph having same no. of vertices as the original graph and the set of edges are obtained by removing all the existing edges in the graph and drawing an edge betn every pair of vertices which was not there in this graph.

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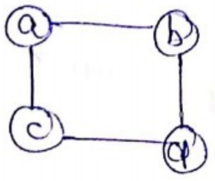
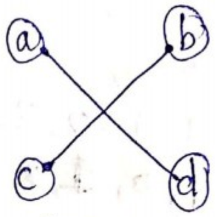


(G)



($\bar{G}$)

★ **Self-Complementary Graph :-**

If it is isomorphic to its complement.

* Find out self-complementary graph of four, five vertices resp.

Ans :-

(four vertices)

(five vertices)

① **Removal of vertex :-**

Remove the vertex as well as associated edges

② **Removal of edge.**

Remove only the edge.

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Cut Vertex / Articulation Point :-

Removal of which disconnects the graph.

Cut edge / Bridge :-

Removal of which disconnects the graph.

Separable Graph :-

If it contains cut vertex.

Non-separable Graph

If it does not contain cut vertex.

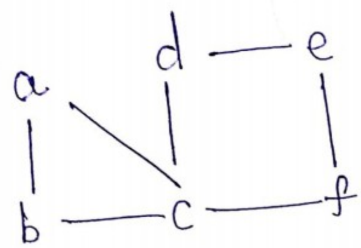
Vertex Connectivity ($K(G)$)

Minimum no. of vertices req. to be removed to disconnect the graph.

Edge Connectivity ($\lambda(G)$)

Minimum no. of edges req. to be removed to disconnect the graph.

Q. (a)



Cut-vertex = c

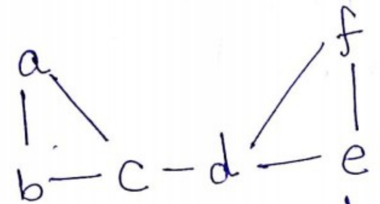
cut-edge = ϕ

Separable = 1

Non-separable = 0.

$K(G) = 1$; $\lambda(G) = 2$.

(b)



Cut-vertex = c, d

Cut-edge = {c, d}

$K(G) = 1$

$\lambda(G) = 0$.

Separable = 1

Non-separable = 0.

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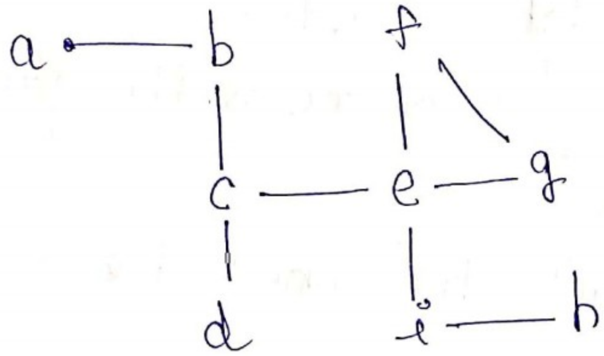
(c)



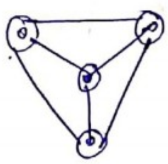
Cut-vertex = b, c, e, i.

Non Cut-edges. = ({e,g}, {e,f}, {f,g}).

$K(G) = 1$ separable = 1

$\lambda(G) = 1$ Non-separable = 0.

* Wheel Graph



(W_3 & W_4).

we are assuming as W_3

Check whether following graphs are separable or not.

C_n ; $n \geqslant 3$; W_n, $n \geqslant 3$; $K_{m,n}$ $m \geqslant 2, n \geqslant 2$.

(Complete graph with n vertices) (Wheel graph with n outward vertices) (complete Bipartite graph with m & n vertices in two disjoint sets resp.)

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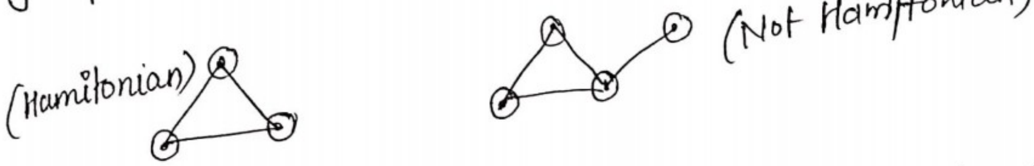
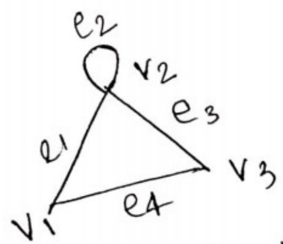
Hamiltonian paths and circuits

Path :- A path which traverses every vertex of a graph exactly once.

circuit :- A Hamiltonian path whose terminal vertices are same.

Graph :- A H. Graph will have one H. circuit i.e., a path which will traverse every vertex of the graph exactly once, except the terminal vertices





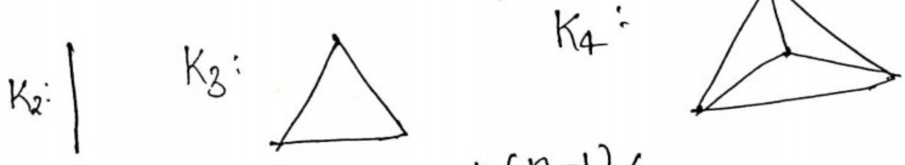
$v_1 e_1 v_2 e_2 v_2 e_3 v_3 e_4 v_1$ Euler Circuit.

$v_1 e_1 v_2 e_3 v_3 e_4 v_1$ Hamiltonian Circuit

(NP complete Problem) (Checking given graph is Hamiltonian or not).

Complete Graph : A simple graph where there exists a unique edge betn every pair of vertices.

K_n.



Total edges $= {}^nC_2 = n(n-1)/2$.

Hint I: ⇑

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Theorem :- In a complete graph with 'n' vertices, there are $\frac{n-1}{2}$ edges disjoint Hamiltonian circuit if n is odd and $n \geq 3$. (Assignment)

Hint I : An 'n' vertex H. circuit will have 'n' edges.

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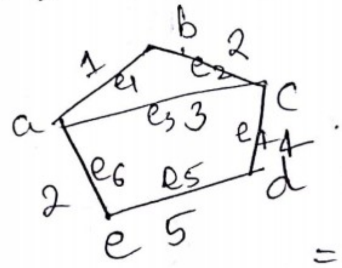
Weighted Graph :- $G = (V, E)$ $w : E \to R^+$

$e_1 \in E$; $w(e_1) \to R^+$



$w(e_1) = w(a,b) = 1$

$w(e_2) = w(b,c) = 2$

$w(e_3) = w(a,c) = 3$

$\vdots$

$G = (\{a, b, \ldots, e\}, \{e_1, e_2, \ldots, e_6\})$

$W : \{e_1, e_2, \ldots, e_6\} \longrightarrow R^+$

Shortest path betn a & c

$a \xrightarrow{1} b,\ b \xrightarrow{2} c$ 3

$a \to c$ 3

$a \xrightarrow{2} e,\ e \xrightarrow{5} d,\ d \xrightarrow{4} c$ 11

★ Dijkstra's Algorithm :-

$s \Rightarrow$ starting node

dist [s] $\leftarrow 0$

for all $v \in V - \{s\}$

  do dist [v] $\leftarrow \infty$

$S \leftarrow \phi$

$Q \leftarrow V$

while $Q \neq \phi$

  do $u \leftarrow$ mindistance (Q, dist)

   $Q \leftarrow Q - \{u\}$ vis [u] = 1

   $S \leftarrow S \cup \{u\}$

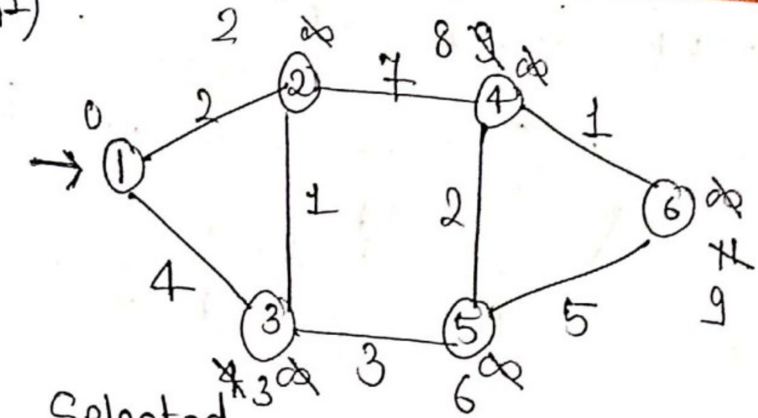
   for all $v \in$ neighbours [u] (non-visited)

    do if dist [v] > dist [u] + w(u,v)

     then dist [v] $\Leftarrow$ dist [u] + w(u,v)

return dist

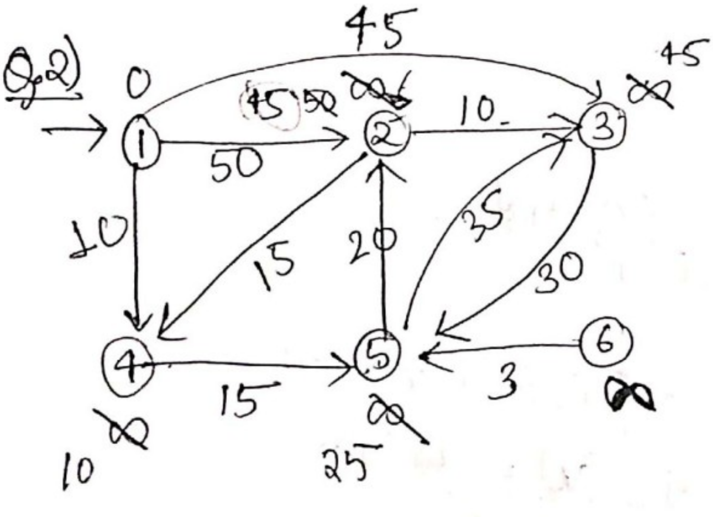
Q.1)



→ indicates (starting node)

Selected nodes ↓	1	2	3	4	5	6	nodes
1	0	∞	∞	∞	∞	∞	1
2		2	4	∞	∞	∞	2
3			3	9	∞	∞	3
5				9	6	∞	4
4				8		11	5
6.						9	6.

Q.2)



Selected vertex.	1	2	3	4	5
1	0	∞	∞	∞	∞
4		50	45	10	∞
5		50	45		25
2.		45	45		
3			45		

v	1	2	3	4	5	6.
dist[v]	0	45	45	10	25	∞

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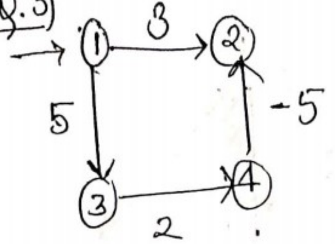
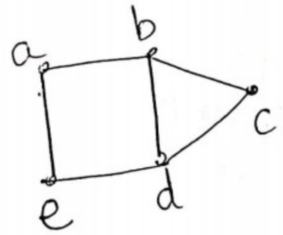
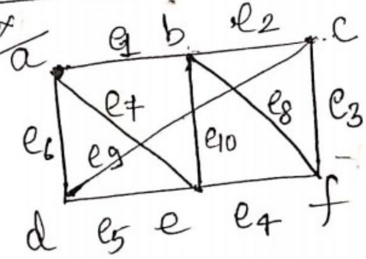
Q.3)





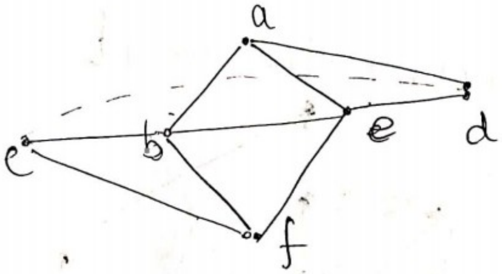
starting vertex



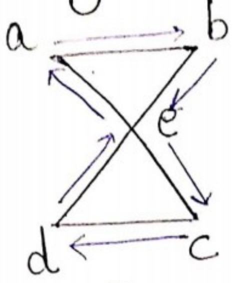
Cycles
- a, b, d, e, a
- b, c, d, b
- a, b, c, d, e, a

$a e_1 b$
$a e_7 e e_{10} b$
$a e_6 d e_9 c e_2 b$
$a e_6 d e_5 e e_{10} b$
$a e_7 e e_4 f e_8 b$
$a e_7 e e_4 f e_3 c e_2 b$
$a e_6 d e_5 e e_4 f e_8 b$.
$a e_6 d e_5 e e_4 f e_3 c e_2 b$.

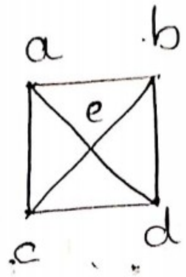
(Assignment) ⟹ Q. Design an algorithm which will take as
input: a simple graph G(V, E) and a starting vertex.
Output: No. of possible
paths
trails
circuits
from the starting vertex.



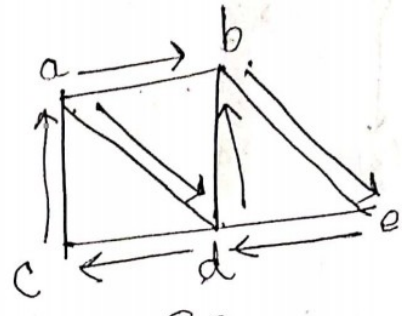
Q. Identify Euler circuits / paths.



G1.
Euler Circuit :
(Euler Graph)

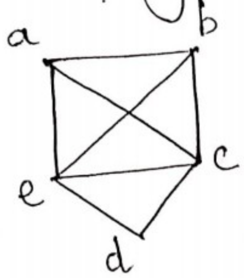


G_2

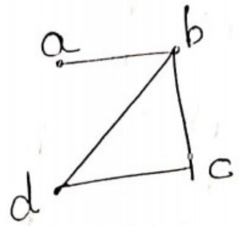


G 3 .
(Euler path / trail)

Q. Identify Hamiltonian circuits / paths.

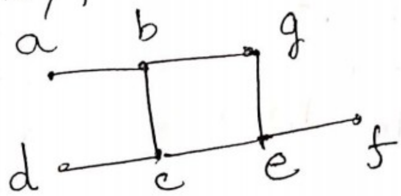


G1
Hamiltonian circuit, path.

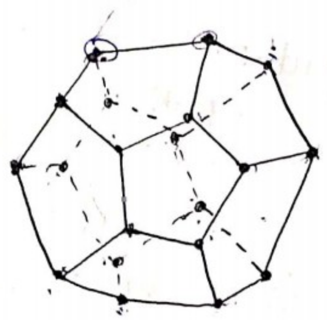


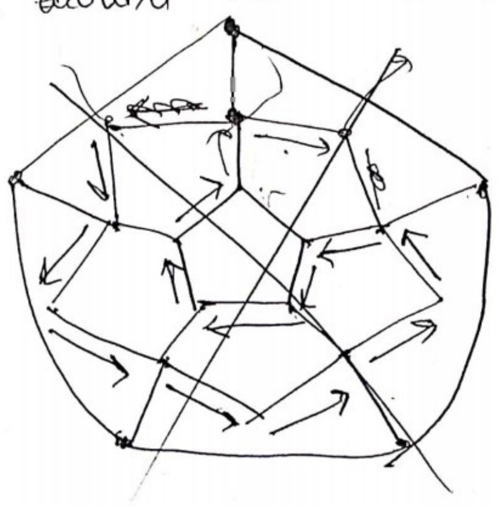
G2

Hamitonian path.

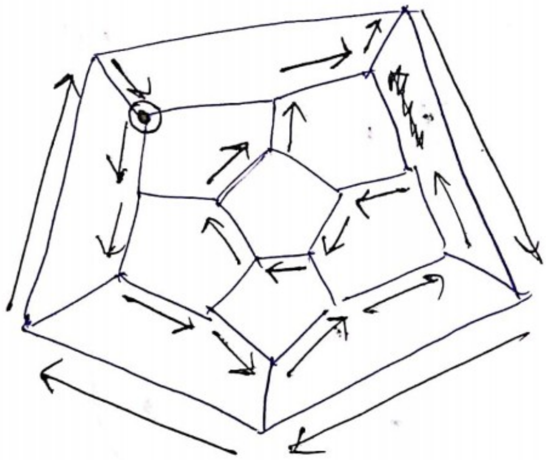


Problem :- A voyage around the world.



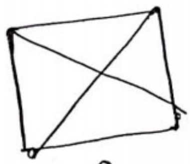


Starting from a vertex visit every vertex exactly once reaching starting vertex again.

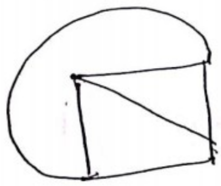


Solution figure

PLANAR GRAPH $G=(V,E)$



G



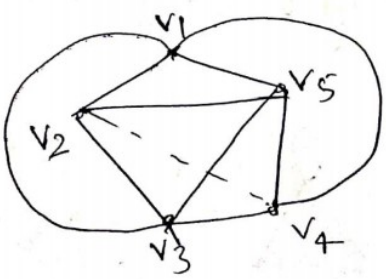
G'

A graph that can be drawn on a plane without any of its edges intersecting. such representation is called the planar representation of a graph.

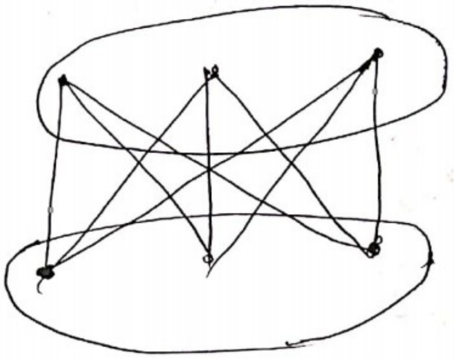
Non-planar graph :- Cannot be drawn w/o a crossover

Kuratowski's Graph

1. Complete graph with 5 vertices K_5.



② Complete bipartite graph K_{33}.

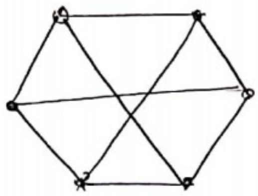


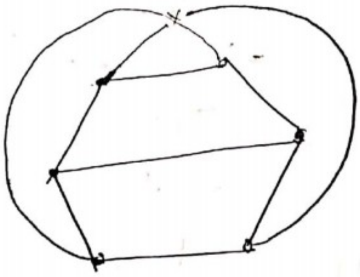
V_1 $\quad V = V_1 \cup V_2$

$V_1 \cap V_2 = \phi$.

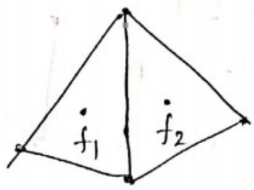
V_2

⇕





① Region/ Face :-



f_3

$f_1; f_2$ finite faces

f_3 Infinite face.

$deg(f_1) = 3$

$deg(f_2) = 3$

$deg(f_3) = 4$.

Degree of a face : $deg(f)$.

The sum of the degree of faces in a graph is twice the number of edges.

★ Euler's Theorem :- A connected planar graph with 'n' vertices and 'e' edges will have

$f = e - n + 2$.

$e=3$
$n=3$

$e-n+2 = 2$
$f = 2$

$e=4$
$n=3$

$e-h+2=3$
$f=3$.

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$e=5$ $\quad e-n+2=4$

$n=3$ $\quad f=4$.

Corollary :- In any simple planar graph with 'f' regions, 'n' vertices, 'e' edges $(e>2)$.

$$\boxed{e \geq \frac{3}{2}f} \; ; \; \boxed{e \leq 3n-6}$$

$f = e-n+2$. $\qquad \deg(f) \geq 3$.

sum of $\deg(f_i) = \underline{2e}$.

$e-n+2 = f$ $\qquad$ sum of $\deg(f) \geq 3f$.

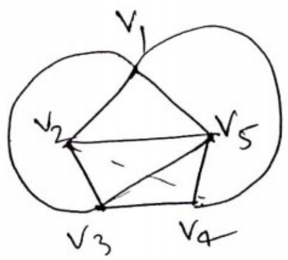
$\Rightarrow 2e \geq 3f$ $\qquad 2e \geq 3f$.

$\Rightarrow 2e \geq 3(e-n+2)$

$\Rightarrow 2e \geq 3e-3n+6$. $\qquad \Rightarrow \boxed{e \geq \frac{3f}{2}}$.

$\Rightarrow \boxed{e \leq 3n-6}$

Hence proved.

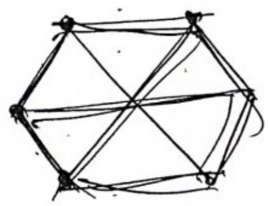


$e=10$; $v=5$ $\qquad e \leq 3n-6$

$10 \leq 15-6$

$10 \leq 9$ Invalid.

non planar.



$e=9$ $\qquad 9 \leq 3(6)-6$.

$n=6$ $\qquad \underline{9 \leq 12}$. valid $\qquad$ can't say anything

$e \quad 9 \geq \frac{3}{2}(5)$

$\underline{9 \geq \frac{15}{2}}$ valid

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~~Mangesh (1800+)~~

★ If a connected planar simple graph has 'e' edges and 'n' vertices ($n \geq 3$) and no circuits of length 3, then $\underline{e \leq 2n-4}$.

Proof :- $\deg(\text{face}) \geq 4$.

$\Rightarrow$ sum of $\deg(\text{faces}) = 2e$.

$\Rightarrow$ sum of $\deg(\text{faces}) \geq 4f$.

$\Rightarrow$ $2e \geq 4f$. $\cancel{e \geq \frac{3f}{2}}$.

$f = e-n+2$.

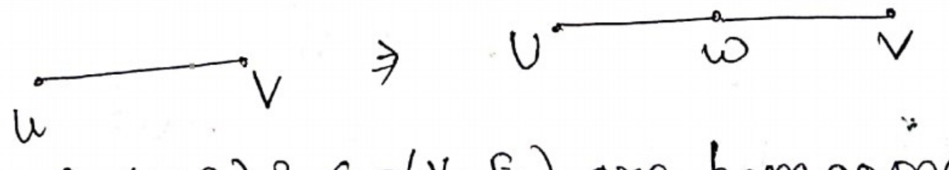
$\Rightarrow$ $2e \geq 4(e-n+2)$

$\Rightarrow$ $2e \geq 4e-4n+8$

$\Rightarrow$ $2e \leq 4n-8$

$\Rightarrow$ $\underline{e \leq 2n-4}$ Hence proved.

★ Elementary subdivision :-



$G=(V,E)$; $G_1=(V_1,E_1)$ & $G_2=(V_2,E_2)$ are homeomorphic graphs ; if G_1 & G_2 can be derived from G using a sequence of elementary subdivision.

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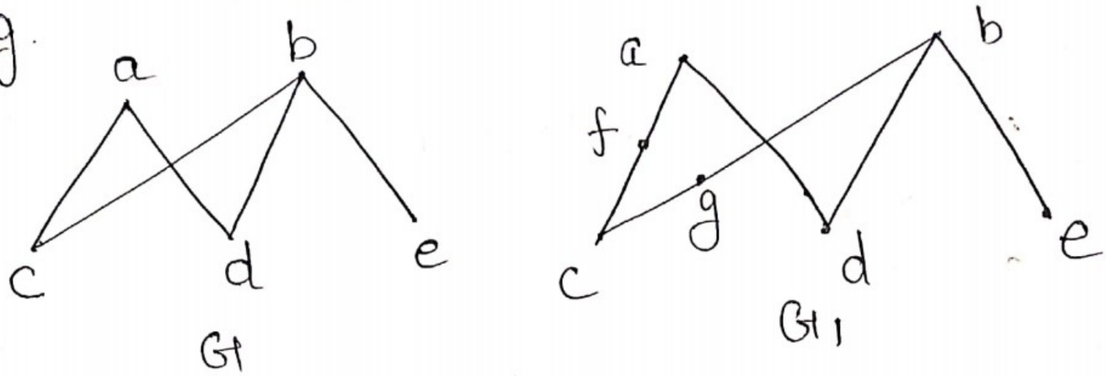
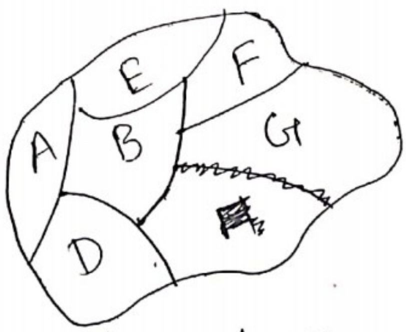
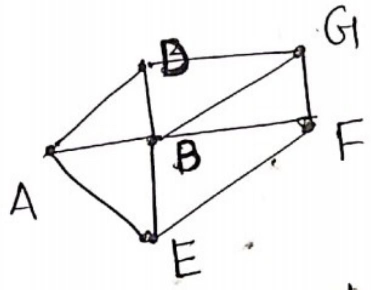
e.g.



are Homeomorphic.

✱ Graph Colouring :-



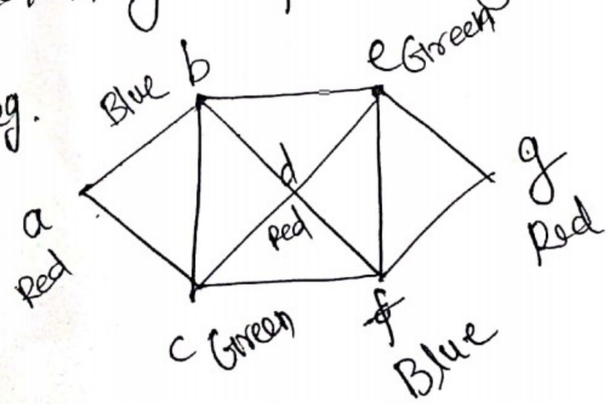


Way of colouring a graph such that no two adjacent vertices have the same colour.

Chromatic Number of a graph - is the least no. of colours used for colouring. Notn : $\chi(G)$.

Four Color Theorem - The least no. of colors for coloring a planar graph is 4 (at max).

eg.



$\chi(G) = 3$.

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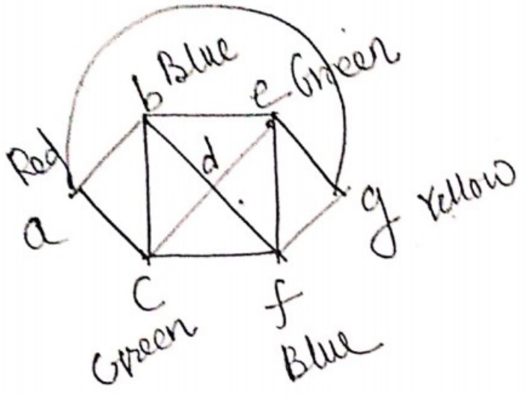
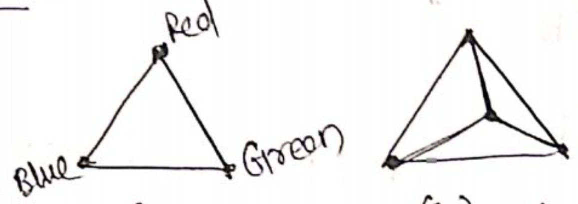


$\chi(G) = 4$.

Q. Chromatic Number of K_n (Complete graph with n vertices).

Ans :-



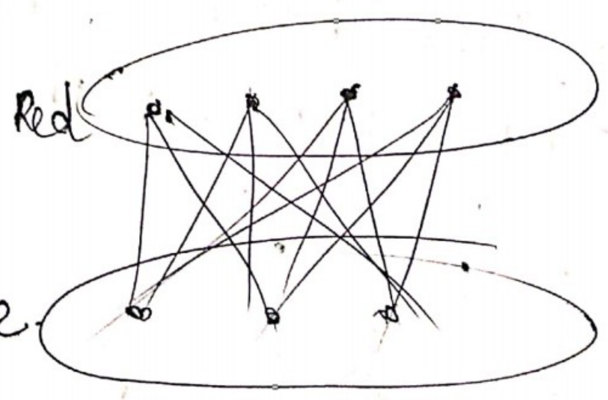
$\chi(G) = 3$ $\chi(G) = 4$ - - - $\chi(G) = n$.

As every vertex is connected to other vertex, so all vertices must be colored differently.

$$\chi(K_n) = n.$$

Q. Chromatic Number of $K_{m,n}$ (Bipartite Graph with m & n elements in disjoint sets of vertices resp.)

Ans :- $\chi(K_{mn}) = 2$.



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Q. Suppose that a connected planar graph has 6 vertices each of degree 4. Into how many regions is the plane divided by planar representation of graph.

$\Rightarrow$ $f = e - n + 2$

$\sum d(v_i) = 2e$

$\Rightarrow 24 = 2e$

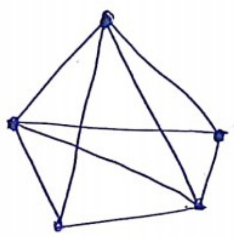
$\Rightarrow \boxed{e = 12}$

$f = 12 - 6 + 2$

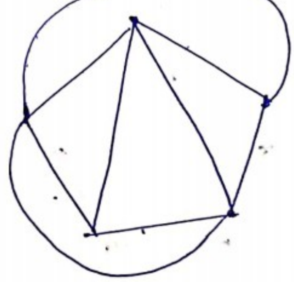
$\Rightarrow \boxed{f = 8}$

Q. Determine planar or not.

(a)



$\Rightarrow$



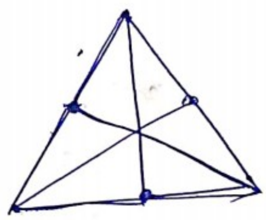
(planar)

$e = 9$; $n = 5$

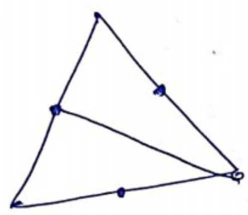
$e \le 3n - 6$

$9 \le 9$

(b)



$\Rightarrow$



(Non planar)

$e = 9$; $n = 5$

(No circuit of length 3).

$e \le 2n - 4$

$9 \le 6$. Invalid.

(circuit board $\Rightarrow$ planar graph)

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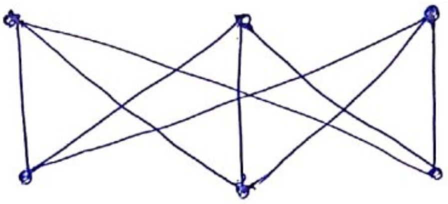
* Kuratowski's Theorem :-

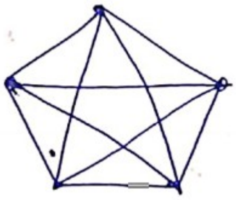
A graph is non-planar if and only if it contains a subgraph homeomorphic to $K_{3,3}$ or K_5.

$K_{3,3}$ ⇒ Complete Bipartite Graph with 3 vertices in each disjoint set.



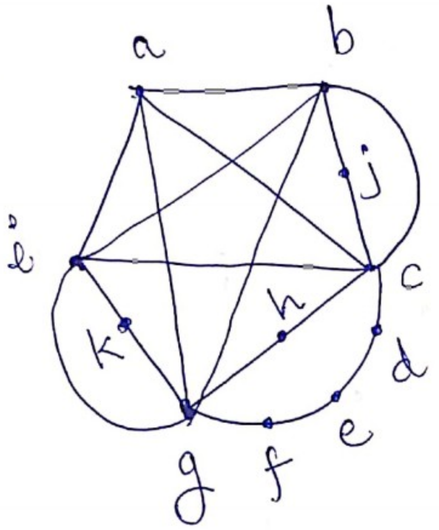
$(K_{3,3})$

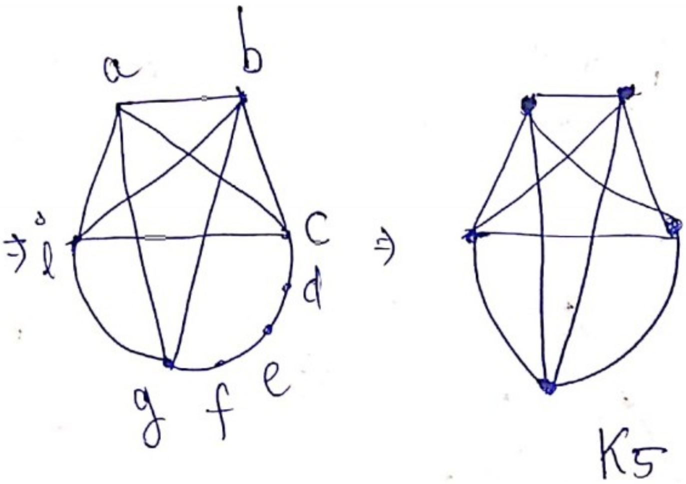
K_5 ⇒ Complete graph of 5 vertices.



(K_5).

Q.





K_5 ⇒ homoegraphic to given graph.

Q. Suppose there are 7 finals to be scheduled. The courses are no. 1 to 7. The following of courses have common students :-

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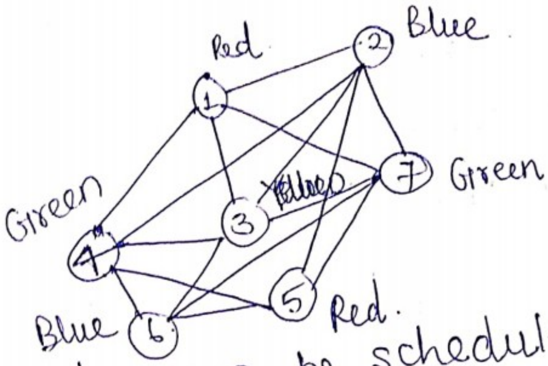
1	2, 3, 4, 7
2	3, 4, 5, 7
3	4, 6, 7
4	5, 6
5	6, 7
6	7

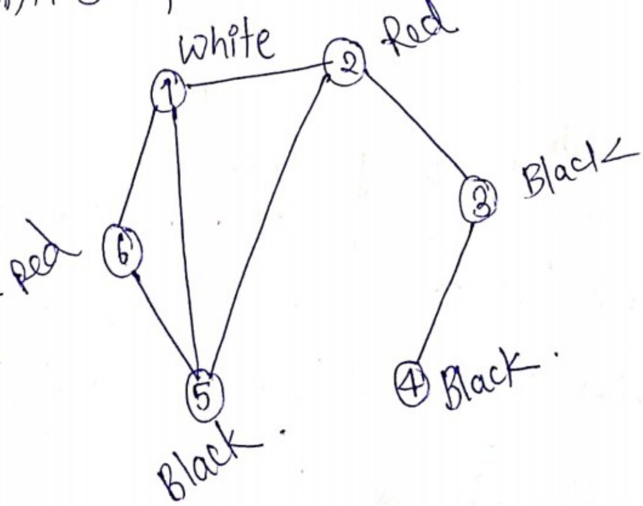


How can the final exam be scheduled s.t. no student has two exams at same day/time?

Time Period	Courses
I (Red)	1, 5
II (Blue)	2, 6
III (Yellow)	3
IV (Green)	4, 7

Q. How many different channels are needed for 6 stations located at the distances shown in the table, two stations can't use the same channel when they are within 150 miles of each other?

	1	2	3	4	5	6
1	–	85	175	200	50	100
2	85	–	125	175	100	160
3	175	125	–	100	200	250
4	200	175	100	–	210	220
5	50	100	200	210	–	100
6	100	160	250	220	100	–



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Channel	Station
I (White)	1
II (Black)	3, 4, 5
III (Red)	2, 6.

Q. A simple graph G has 10 vertices and 21 edges. Find the number of edges in $\overline{G}$.

$\Rightarrow$ Total max poss. edges $= \frac{10(10-1)}{2} = 5 \times 9 = 45$

edges in $\overline{G} = 45 - 21$
$= \underline{\underline{24}}$.

Q. A simple graph G has 30 edges and $\overline{G}$ has 36 edges. Find the number of vertices in G.

$\Rightarrow \frac{n(n-1)}{2} = 66$

$n(n-1) = 132$.

$\Rightarrow n^2 - n - 132 = 0$.

$\Rightarrow n^2 - 12n + 11n - 132 = 0$

$\Rightarrow n(n-12) + 11(n-12) = 0$

$\Rightarrow n = -11, 12$

$\Rightarrow \boxed{n = 12}$.

$\therefore$ Note: $|E(G)| + |E(\overline{G})| = \frac{n(n-1)}{2}$

$\Rightarrow n =$ number of vertices.

* K^{th} power of the Graph G :-

Kth power G^k of an undirected graph G is another graph that has the same set of vertices but in which two vertices are adjacent when their distance in G is at most k.

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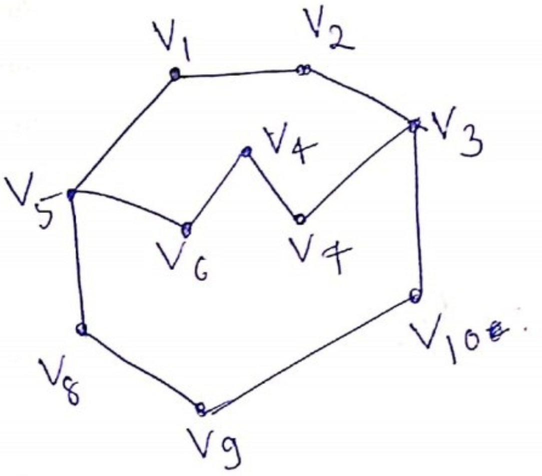
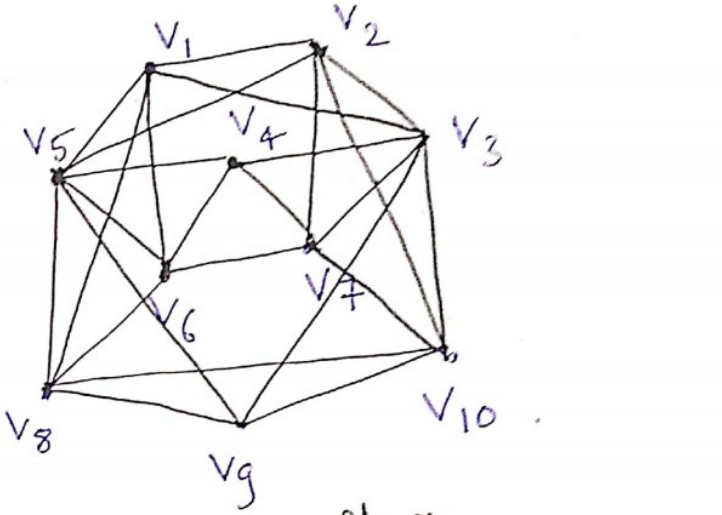
$d(u,v)$ = no. of edges in the shortest path traversed between u and v.

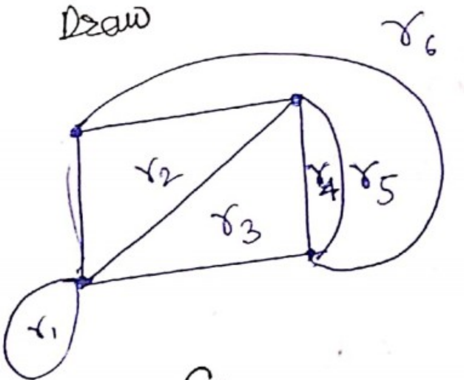


G_1

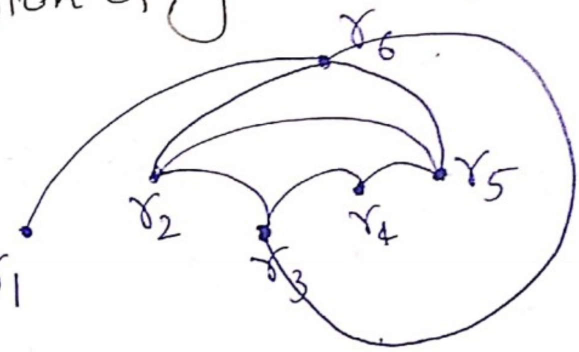


G^2 : Planar.

Q. Find/ Draw Dual Representation of given Graph G_1.



G_1.

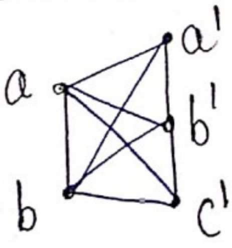


Dual of G_1.

Dual of planar Graph :-

consider each region as vertex. and draw edges between adjacent regions.

Sum of two graphs :-



G_1+G_2 = $V(G_1+G_2) = V(G_1)+V(G_2)$

$E(G_1+G_2) = E(G_1)+E(G_2)$

+ there exists an edge between every pair of vertices (u,v) where $u \in G_1$ & $v \in G_2$.

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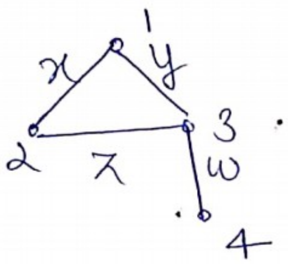
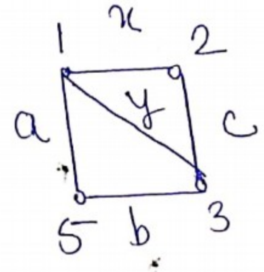
Find Ringsum.



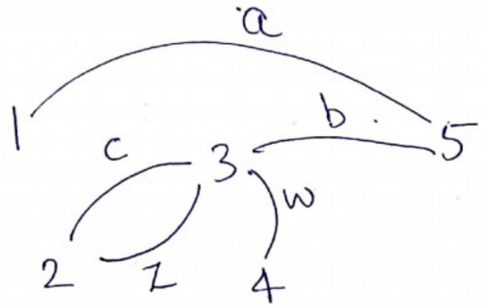
G_1



G_2

$\text{Ringsum}(G_1, G_2) \Rightarrow G_1 \cup G_2 - G_1 \cap G_2$

Ans.

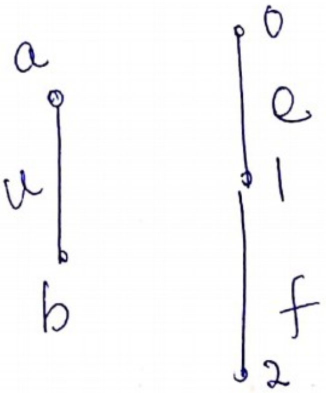


Product of two different graphs.

$G_1 \times G_2$

$$V(G_1 \times G_2) = V(G_1) \times V(G_2)$$

$$E(G_1 \times G_2) = V(G_1) \times E(G_2) \cup V(G_2) \times E(G_1)$$



$V(G_1) = \{a, b\}$; $V(G_2) = \{0, 1, 2\}$

$E(G_1) = \{u\}$; $E(G_2) = \{e, f\}$

$V(G_1 \times G_2) = \{(a,0), (a,1), (a,2), (b,0), (b,1), (b,2)\}$

$V(G_1) \times E(G_2) = \{(a,e), (a,f), (b,e), (b,f)\}$

$V(G_2) \times E(G_1) = \{(0,u), (1,u), (2,u)\}$.

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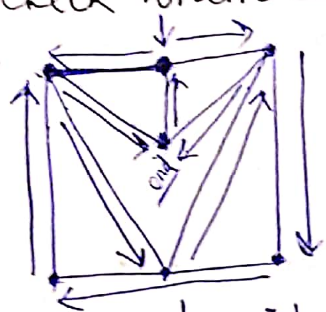
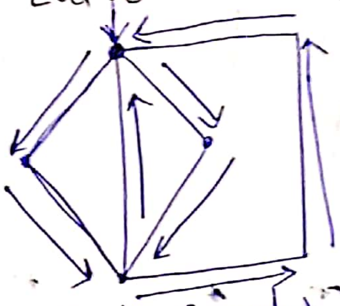
Tut Check whether there are Euler graphs or not. Also check whether Euler circuit, path exists or not.

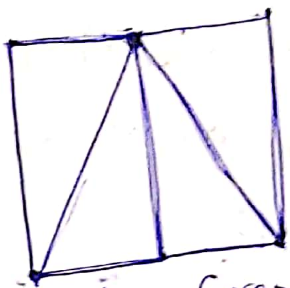
(Q.1)



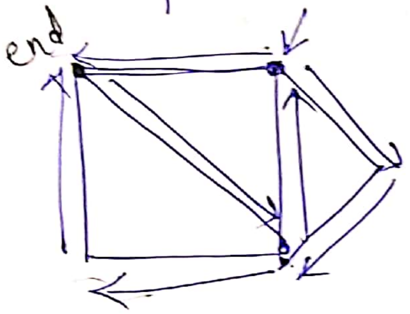
Euler Graph ⇒ No.
Euler circuit ⇒ No.
Euler path ⇒ Yes.



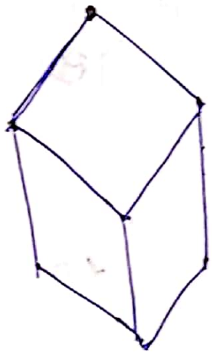
Euler Graph ⇒ Yes.
Euler Circuit ⇒ Yes
Euler Path ⇒ Yes.



Euler Graph ⇒ No
Euler Circuit ⇒ No.
Euler Path ⇒ No.

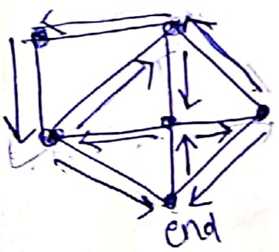


Euler Graph ⇒ No.
Euler Circuit ⇒ No.
Euler Path ⇒ Yes.



Euler Graph ⇒ No
Euler Circuit ⇒ No.
Euler Path ⇒ No

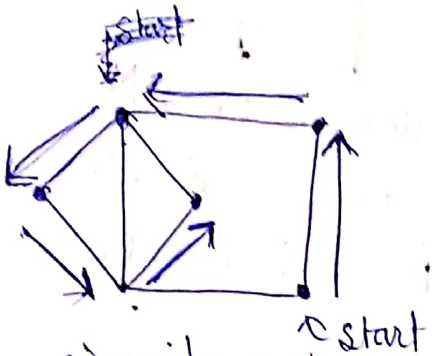
Each and every vertex has even degree ⇒ Euler circuit exists.
Exactly 2 odd degree vertices ⇒ Euler path exists.



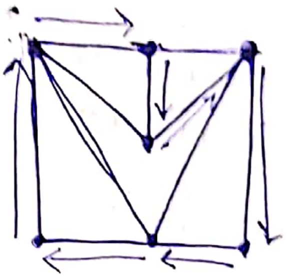
Euler circuit ⇒ No
Euler path ⇒ Yes
Euler graph ⇒ No.

Euler path if exists (for graph having odd degree vertices) ~~starts~~ from one odd degree vertex and ends at other one.

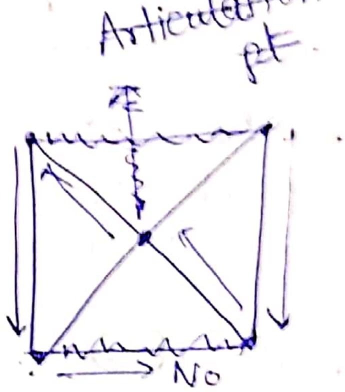
Q. Check whether there are Hamiltonian Graph or not. Also Check whether Hamiltonian Circuit, Hamiltonian Path exists or not.



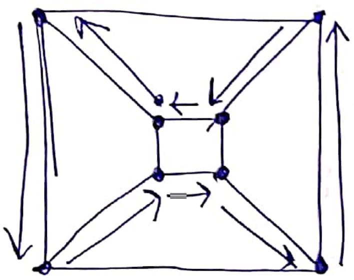
circuit → No
path → Yes
Graph → No



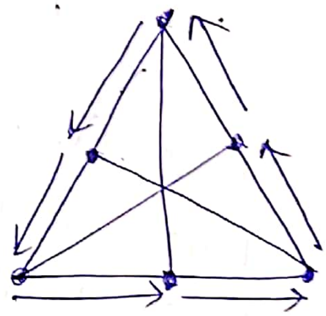
circuit → Yes
path → Yes
Graph → Yes



paths → Yes
→ No.



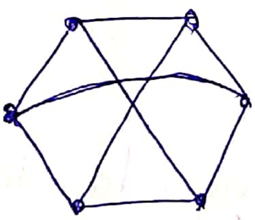
circuit → Yes
path → Yes
Graph → Yes.



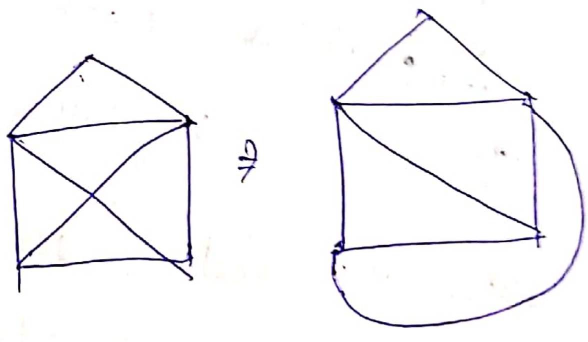
Yes
Yes
Yes.

Tutorial

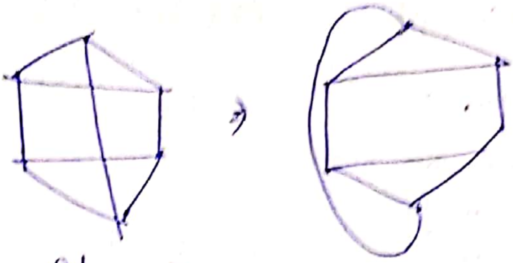
check for Planarity.



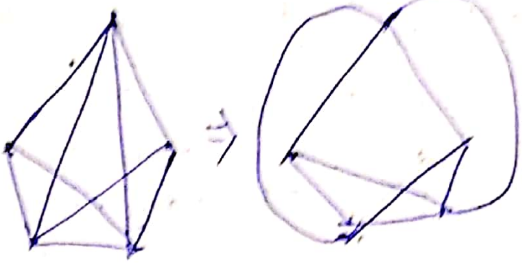
Non-planar



Planar.

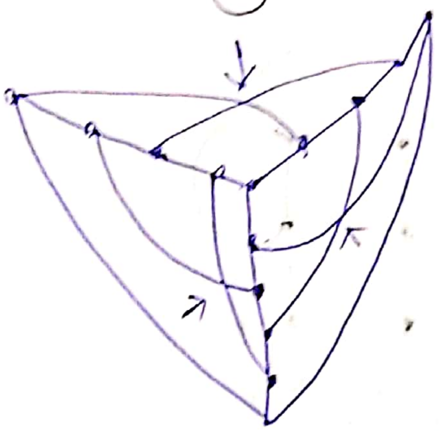


Planar



Non-Planar.

* find Crossing Number $(Cr(G))$.



minimum number of crosses possible

$$\boxed{Cr(G) = 3}$$

Note:-

$$Cr(K_p) \le \frac{1}{4}\left\lfloor\frac{p}{2}\right\rfloor\left\lfloor\frac{p-1}{2}\right\rfloor\left\lfloor\frac{p-2}{2}\right\rfloor\left\lfloor\frac{p-3}{2}\right\rfloor$$

$K_p \Rightarrow$ Complete graph with p vertices

$$Cr(K_{m,n}) \le \left\lfloor\frac{m}{2}\right\rfloor\left\lfloor\frac{m-1}{2}\right\rfloor\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor$$

$K_{m,n} \Rightarrow$ complete Bipartite graph with m & n vertices in two disjoint sets resp.

Q. Find Cr for.

(a) $K_5 \Rightarrow Cr(K_5) \le \frac{1}{4}\left\lfloor\frac{5}{2}\right\rfloor\left\lfloor\frac{4}{2}\right\rfloor\left\lfloor\frac{3}{2}\right\rfloor\left\lfloor\frac{2}{2}\right\rfloor \le 1$.

(b) $K_6 \Rightarrow Cr(K_6) \le \frac{1}{4}\left\lfloor\frac{6}{2}\right\rfloor\left\lfloor\frac{5}{2}\right\rfloor\left\lfloor\frac{4}{2}\right\rfloor\left\lfloor\frac{3}{2}\right\rfloor \le 3$.

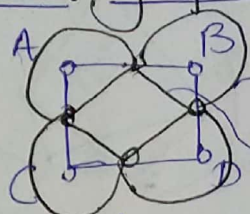
(c) $K_7 \Rightarrow Cr(K_7) \le \frac{1}{4}\left\lfloor\frac{7}{2}\right\rfloor\left\lfloor\frac{6}{2}\right\rfloor\left\lfloor\frac{5}{2}\right\rfloor\left\lfloor\frac{4}{2}\right\rfloor \le 9$.

(d) $K_{3,4} \Rightarrow Cr(K_{3,4}) \le \left\lfloor\frac{3}{2}\right\rfloor\left\lfloor\frac{2}{2}\right\rfloor\left\lfloor\frac{4}{2}\right\rfloor\left\lfloor\frac{3}{2}\right\rfloor \le 2$.

(e) $K_{4,4} \Rightarrow Cr(K_{4,4}) \le \left\lfloor\frac{4}{2}\right\rfloor\left\lfloor\frac{3}{2}\right\rfloor\left\lfloor\frac{4}{2}\right\rfloor\left\lfloor\frac{3}{2}\right\rfloor \le 4$

(f) $K_{5,5} \Rightarrow Cr(K_{5,5}) \le \left\lfloor\frac{5}{2}\right\rfloor\left\lfloor\frac{4}{2}\right\rfloor\left\lfloor\frac{5}{2}\right\rfloor\left\lfloor\frac{4}{2}\right\rfloor \le 16$.

Medial Graph.

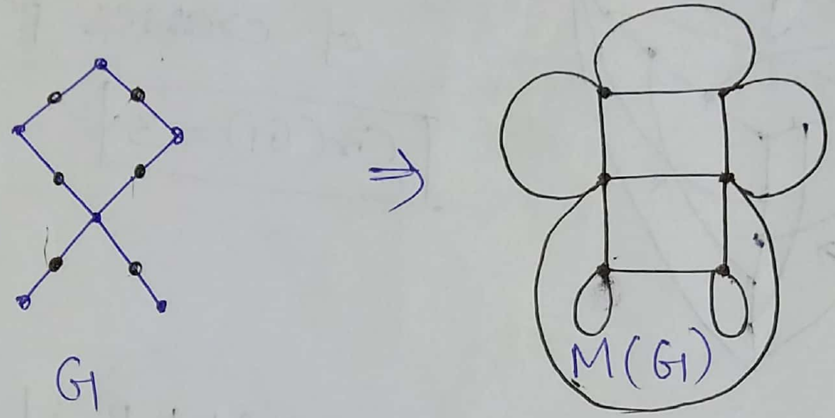


- → Medial Graph (other than square)
- → Original Graph (only square)

↳ due to outer common region

↳ due to inner common face

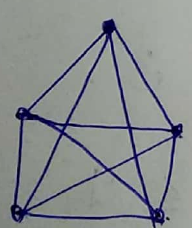
Medial Graph of a planar graph G is a graph denoted by M(G) that represents adjancies betⁿ edges in the faces/regions of G.



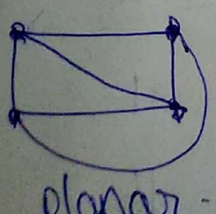
G

Q. Which of these nonplanar graphs, that have the property the removal of any vertex makes it planar.

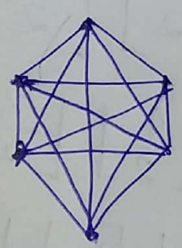
(a) K_5 ✓ (b) K_6 ✗ (c) $K_{3,3}$ ✓ (d) $K_{3,4}$ ✓



⇓

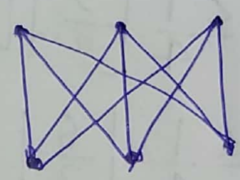


planar

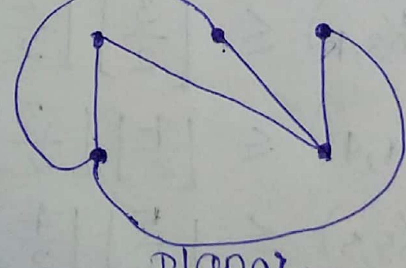


⇓

Not possible

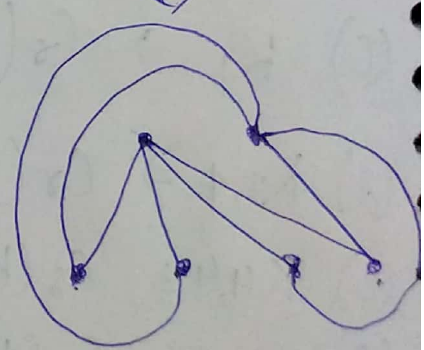


⇓



planar

⇓



planar

Independent Set.

It is a set of vertices S which is a su-bset of V such that no two vertices are adja-cent.

Maximal Independent Set. (MLIS)

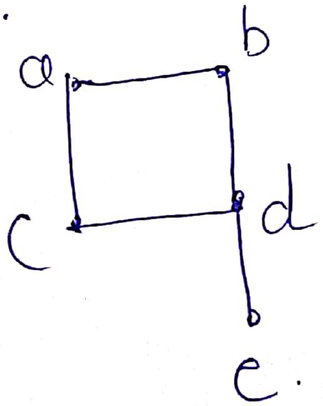
An Independent Set is said to be maximal independent set if no more vertices can be added to it without breaking (violating) Independent set property.

Maximum Independent Set (MMIS)

An Independent Set is said to be maximum independent set if contains maximum possible no. of element for a independent set.

Independence Number :- (IN).

Number of elements present in maximum Indepen-dent set.



MLIS ⇒ {a,d} {c,b,e}

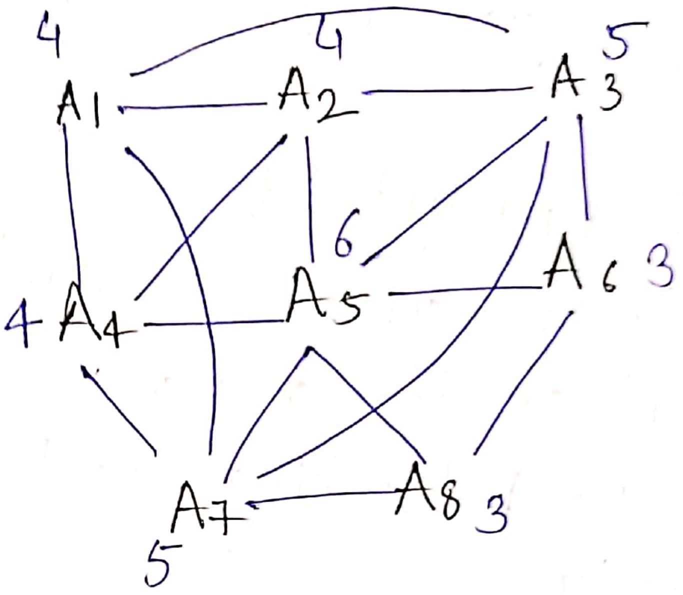
MMIS ⇒ {a,b,e}.

I.N. ⇒ 3.

★ Calculate Chromatic Number. ★ (Welch-Powell Algo.)

Ans (3)

A5 R
A3 G
A7 B
A1 R
A2 B
A4 G
A6 B
A8. G.



★ Vertex Cover.

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RELATIONS

A,B $A \times B$ $R \subseteq A \times B = \{(a,b) \mid a \in A \;\&\; b \in B\}$

ordered pair of sets.

$A = \{1,2,3\}$ $A \times B = \{(1,4),(1,5),(2,4),(2,5),(3,4),(3,5)\}$

$B = \{4,5\}$ $B \times A = \{(4,1),(4,2),(4,3),(5,1),(5,2),(5,3)\}$

$A \times A = \{(1,1),(1,2),(1,3),(2,1),(2,2),(2,3),(3,1),(3,2),(3,3)\}$.

Defn:- For any two sets A and B not necessarily distinct, any subset of $A \times B$ is said to be relation from A to B.

subset of $B \times A$: a relation from B to A.

subset of $A \times A$: a relation on A.

$A = \{0,1,2\}$; $B = \{a,b\}$

$R = \{(0,a),(1,b),(2,a)\}$

$(a,b) \in R$ aRb, $a \in A$ & $b \in B$.

$a \not R b$, $(a,b) \notin R$.

R	a	b
0	✓	×
1	×	✓
2	✓	×

$A = \{1,2,3,4\}$.

$R = \{(a,b) \mid a \text{ divides } b\}$

$R = \{(1,1),(2,2),(3,3),(4,4),(1,2),(1,3),(1,4),(2,4)\}$.

$R_1 = \{(1,1),(1,2),(2,1),(2,2),(3,4),(4,1),(4,4)\}$

$R_2 = \{(1,1),(1,2),(2,1)\}$

$R_3 = \{(1,1),(1,2),(1,4),(2,1),(2,2),(3,3),(4,1),(4,4)\}$

$R_4 = \{(1,1),(2,1),(2,2),(3,1),(4,1),(4,2),(4,4)\}$

$R_5 = \{(3,4)\}$

$R_6 = \{(2,1),(3,1),(3,2),(4,1),(4,2),(4,3)\}$.

Reflexive :- A relation R on a set 'A' is said to be reflexive if $(a,a) \in R$ for all $a \in A$. (R_3)

Irreflexive :- $(a,a) \notin R$ for any $a \in A$. (R_5, R_6).

Symmetric :- $(b,a) \in R$ for $(a,b) \in R$ $\forall a,b \in A$.
(R_2, R_3).

Antisymmetric :- $(a,b) \in R$ & $(b,a) \in R$, $a=b$ $\forall a,b \in A$.
$(a,b) \in R$; $a \neq b$, $(b,a) \notin R$. (R_4, R_5, R_6).

Assymmetric :- $(a,b) \in R$ then $(b,a) \notin R$ $\forall (a,b) \in A$.
(R_5, R_6)

Transitive :- $(a,b) \in R$ & $(b,c) \in R$ $(a,c) \in R$. $\forall a,b,c \in A$.
(R_4, R_5, R_6)

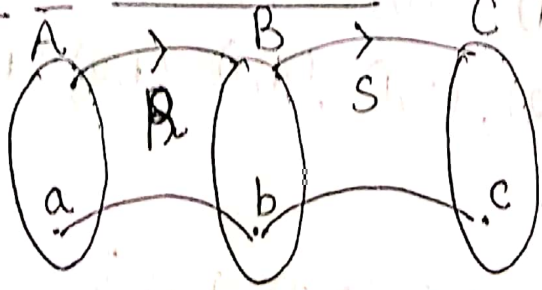
$A = \{1,2,3\}$, $B = \{1,2,3,4\}$.

$R_1 = \{(1,1), (2,2), (3,3)\}$ $\quad R_1 \cup R_2$ $\quad R_1 \cap R_2$

$R_2 = \{(1,1), (1,2), (1,3), (1,4)\}$. $\quad R_1 - R_2 = \{(2,2), (3,3)\}$

$R_2 - R_1$.

Composition of Relations



$R \subseteq A \times B$

$S \subseteq B \times C$.

$a \, R \circ S \, c$ $\exists \, b \in B$ s.t. aRb & bSc.

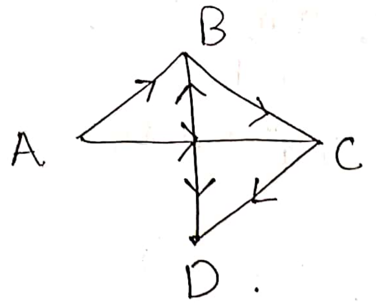
Let $A = \{1,2,3,4\}$ $\quad R \circ S = \{(1,3), (2,4)\}$

$R = \{(1,2), (1,3), (2,4)\}$ $\quad S \circ R = \{(1,4)\}$.

$S = \{(1,2), (2,3), (4,4)\}$

REPRESENTING RELATIONS

$G = \langle V, E \rangle$; $E \subseteq V \times V$.

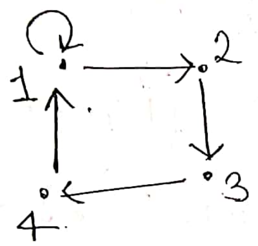


$V = \{A, B, C, D\}$

$E = \{(A,B), (B,C), (C,D), (A,C), (B,D)$
$(D,B)\}$.

$G = \langle V, E \rangle$.

$A = \{1,2,3,4\}$; $R = \{(1,1), (1,2), (2,3), (3,4), (4,1)\}$.



DIGRAPHS

$A = \{a_1, a_2, \text{———}, a_n\}$
$B = \{b_1, b_2, \text{———}, b_m\}$.

R	a_1	a_2	a_3	...	a_n
b_1	m_{11}	m_{12}		...	m_{1n}
b_2	m_{21}			...	m_{2n}
⋮					
b_m	m_{m1}			...	m_{mn}

$m_{ij} = 1 \quad (a_i, a_j) \in R$
$\quad = 0 \quad (a_i, a_j) \notin R$

Relation $\leq$ on the set $\{1, 2, 3, 4\}$.

R	1	2	3	4
1	1	1	1	1
2	0	1	1	1
3	0	0	1	1
4	0	0	0	1

$R_1 = \{(a,b) \mid a \leq b\}$
$R_2 = \{(a,b) \mid a > b\}$
$R_3 = \{(a,b) \mid a = b \text{ or } a = -b\}$.
$R_4 = \{(a,b) \mid a = b\}$.
$R_5 = \{(a,b) \mid a = b+1\}$
$R_6 = \{(a,b) \mid a+b \leq 3\}$

Which of the relations contain each of the pair $(1,1), (1,2), (2,1), (1,-1)$ and $(2,2)$.

$(1,1) \downarrow R_1, R_3, R_4, R_6$
$(1,2) \downarrow R_1, R_6$
$(2,1) \downarrow R_2, R_5, R_6$
$(1,-1) \downarrow R_2, R_3, R_6$
$(2,2) \downarrow R_1, R_3, R_4,$

N-ARY Relation :

$A_1, A_2, \ldots, A_n$ Domain of the relation

$R \subseteq A_1 \times A_2 \times A_3 \ldots \times A_n$

degree = n

Let R be the relation on $N \times N \times N$ consisting of triples (a,b,c) where $a,b,c \in \mathbb{Z}$ & $a<b<c$.

$(1,2,3) \in R$; $(2,4,3) \notin R$.

CLOSURE OF RELATIONS :- $R \subseteq A \times A$.

P: reflexive, symmetric & transitive

Def: The closure of a relation R on a set w.r.t. a given P will be a set R_p s.t. $R \subseteq R_p \subseteq A \times A$.

Reflexive Closure :- $R = \{(1,1), (1,2), (2,1), (3,2)\}$ on $A = \{1,2,3\}$.

$R_p = \{(1,1), (1,2), (2,1), (3,2), (2,2), (3,3)\}$.

$R \subseteq R_p \subseteq A \times A$.

Symmetric Closure :- $R = \{(1,1), (1,2), (2,2), (2,3), (3,1), (3,2)\}$ on $\{1,2,3\}$.

$R_p = \{(1,1), (1,2), \underline{(2,1)}, (2,2), (2,3), (3,1), \underline{(1,3)}, (3,2)\}$

Transitive Closure :- $R = \{(1,3), (1,4), (2,1), (3,2)\}$ on $A = \{1,2,3,4\}$

$R_p' = \{(1,3), (1,4), (2,1), (3,2), (1,2), (2,3), (2,4), (3,1)\}$ ~~(1,1)~~

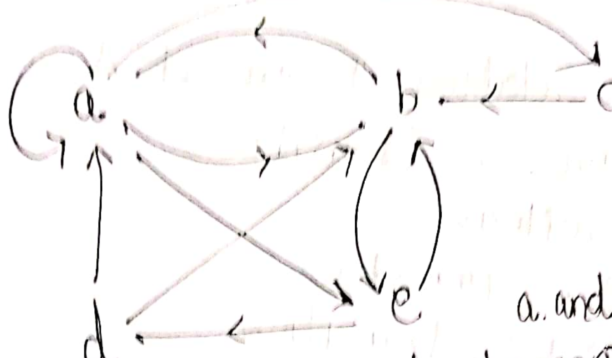
~~(3,3), (3,4)~~

$R_p = \{(1,3), (1,4), (2,1), (3,2), (1,2), (2,3), (2,4), (3,1), \cancel{(3,3)}, (3,4), \cancel{(1,1)}, \cancel{(2,2)},$

R_1 & R_2 M_{R_1} M_{R_2} $M_{R_1 \cup R_2}$ $M_{R_1 \cap R_2}$ ..

Path: A path from a to b is a sequence of edges (x_0, x_1), (x_1, x_2), ..., (x_{n-1}, x_n) where n = non-negative, $x_0 = a$, $x_n = b$.
$x_0, x_1, x_2, \ldots, x_{n-1}, x_n$ length = n.

Cycle: $n \geq 1$, terminal vertices are same.

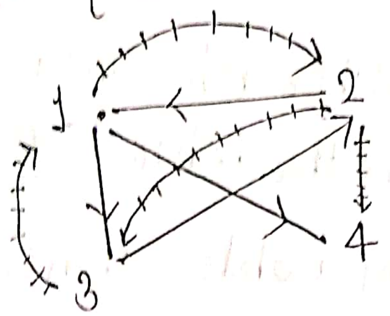


a,b,e,d $(a,b)(b,e)(e,d)$ $L=3$.
a,e,c,d,b $(e,c) \notin E$
b,a,c,b,a,a,b $(b,a)(a,c)(c,b)(b,a)$
$(a,a)(a,b)$ $L=6$.

a and b in a relation R, if there is a sequence of elements $a, x_1, x_2, \ldots, x_{n-1}, \cancel{x_n}\, b$
$(a,x_1) \in R$, $(x_1, x_2) \in R$, ..., $(x_{n-1}, b) \in R$.

for a relation R to be transitive closure the terminal vertices all possible paths of the graph must have a common edge. (cycles are not considered)

$R = \{(1,3), (1,4), (2,1), (3,2)\}$ on $\{1, 2, 3, 4\}$.



$V_1 = (1,3), (3,2) \rightarrow (1,2)$.

$V_2 = (2,1), (1,3) \rightarrow (2,3)$.
$(2,1), (1,4) \rightarrow (2,4)$

$V_3 = (3,2), (2,1) \rightarrow (3,1)$.
$(3,2), (2,1), (1,4), \rightarrow (3,4)$.

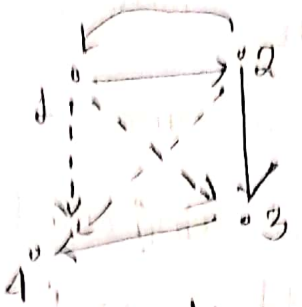
a, b, c all should be different while considering transitive closure.

$R = \{(1,2),(2,3),(3,4),(2,1)\}$ on $\{1,2,3,4\}$.

$v_1 = (1,2)(2,3) \longrightarrow (1,3)$

$v_2 = \cancel{(1,2)}(2,3)(3,4) \longrightarrow (2,4)$

$v_1 = (1,2)^{(1,3)}(2,4) \longrightarrow (1,4)$



Equivalence Relation :- A relation R on set A, if it is symmetric, reflexive & transitive.

$a,b,c \in A$; '$=$' $a=a$, reflexive

$a=b \rightarrow b=a$, symmetric

$a=b$ & $b=c \rightarrow a=c$, transitive.

Partial Orderings :- A relation R on a set A is said to be a partial ordered relation if it is reflexive, antisymmetric, & transitive $(A,R) \rightarrow$ partial ordered set or Poset.

$\geq$ on the set of all +ve integers $\mathbb{Z}$.

$a \geq a$

If $a \geq b$ & $b \geq a$ where $a=b$.

If $a \geq b$ & $b \geq c$ then $a \geq c$.

$(\mathbb{Z}^+, \geq) \rightarrow$ poset

$(A, \preceq)$; $\preceq \in \{\geq, \leq, \subseteq, \ldots\}$

$a, b \in A$.

comparable: if either $a \preceq b$ or $b \preceq a$

INcomparable: if neither $a \preceq b$ nor $b \preceq a$.

If two elements of a poset are comparable, then it will be a total order / linear / simple order.

$(A, \preceq)$ will be a chain.

$(\mathbb{Z}, \leq)$ is a chain ; $\mathbb{Z} = \{1,2,3,4,\ldots\}$.

let R be the relation on the set {0, 1, 2, 3} containing the ordered pairs (0,1), (1,1), (1,2), (2,0), (2,2), (3,0).

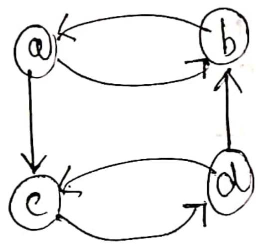
(a) Find the reflexive closure of R.

$\Rightarrow R^{+} = \{(0,1), (1,1), (1,2), (2,0), (2,2), (3,0), (0,0), (3,3)\}$

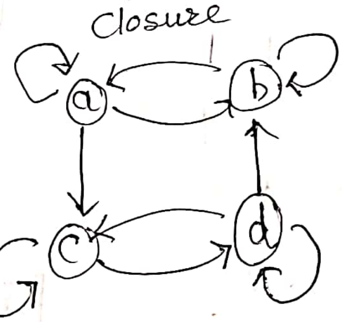
(b) Find the symmetric closure of R.

$\Rightarrow R^{+} = \{(0,1), (1,0), (1,1), (1,2), (2,1), (2,0), (0,2), (2,2), (3,0), (0,3)\}$.

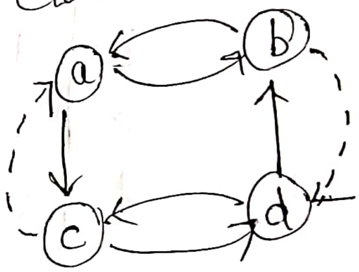
Q2)



Reflexive Closure



Symmetric Closure.



Use Warshall's Algorithm to find the transitive closure of these relations on {1, 2, 3, 4}.

(a) {(1,2), (2,1), (2,3), (3,4), (4,1)}

$$W_0 = \begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & 0 & 1 & 0 & 0 \\ 2 & 1 & 0 & 1 & 0 \\ 3 & 0 & 0 & 0 & 1 \\ 4 & 1 & 0 & 0 & 0 \end{array}$$

p_i q_j
2 2
4
= (2,2), (4,2)

$$W_2 = \begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & 1 & 1 & 1 & 0 \\ 2 & 1 & 1 & 1 & 0 \\ 3 & 0 & 0 & 0 & 1 \\ 4 & 1 & 1 & 1 & 0 \end{array}$$

p_i q_j
1 4
2
4
$\Rightarrow$ (1,4) (2,4) (4,4)

$$W_1 = \begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & 0 & 1 & 0 & 0 \\ 2 & 1 & 1 & 1 & 0 \\ 3 & 0 & 0 & 0 & 1 \\ 4 & 1 & 1 & 0 & 0 \end{array}$$

p_i q_j
1 1
2 2
4 3
$\Rightarrow$ (1,1) (1,2) (1,3) (2,1) (2,2) (2,3) (4,1) (4,2) (4,3)

$$W_3 = \begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & 1 & 1 & 1 & 1 \\ 2 & 1 & 1 & 1 & 1 \\ 3 & 0 & 0 & 0 & 1 \\ 4 & 1 & 1 & 1 & 1 \end{array}$$

p_i q_j
1 1 (1,1)(1,2)(1,3)(1,4)
2 2 $\Rightarrow$ (2,1)(2,2)(2,3)(2,4)
3 3 (3,1)(3,2)(3,3)(3,4)
4 4 (4,1)(4,2)(4,3)(4,4)

$$W_4 = \begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & 1 & 1 & 1 & 1 \\ 2 & 1 & 1 & 1 & 1 \\ 3 & 1 & 1 & 1 & 1 \\ 4 & 1 & 1 & 1 & 1 \end{array}$$

Transitive Closure :- all pairs.

(b) $\{a,b,c,d\}$; $\{(a,d), (b,a), (b,c), (c,a), (c,d), (d,c)\}$.

Ans :-

$$W_0 = \begin{array}{c|cccc} & a & b & c & d \\ \hline a & 0 & 0 & 0 & 1 \\ b & 1 & 0 & 1 & 0 \\ c & 1 & 0 & 0 & 1 \\ d & 0 & 0 & 1 & 0 \end{array}$$

p_i	q_j	
b	d	$\rightarrow$ (b,d) (c,d)
c		

$$W_1 = \begin{array}{c|cccc} & a & b & c & d \\ \hline a & 0 & 0 & 0 & 1 \\ b & 1 & 0 & 1 & 1 \\ c & 1 & 0 & 0 & 1 \\ d & 0 & 0 & 1 & 0 \end{array}$$

p_i	q_j
ϕ	a
	c
	d

$$W_2 = \begin{array}{c|cccc} & a & b & c & d \\ \hline a & 0 & 0 & 0 & 1 \\ b & 1 & 0 & 1 & 1 \\ c & 1 & 0 & 0 & 1 \\ d & 0 & 0 & 1 & 0 \end{array}$$

p_i	q_j	
~~ϕ~~ b	a	$\rightarrow$ (b,a) (b,d)
d	d.	(d,a) (d,d)

$$W_3 = \begin{array}{c|cccc} & a & b & c & d \\ \hline a & 0 & 0 & 0 & 1 \\ b & 1 & 0 & 1 & 1 \\ c & 1 & 0 & 0 & 1 \\ d & 1 & 0 & 1 & 1 \end{array}$$

p_i	q_j	
a	a	$\rightarrow$ (a,a) (a,c) (a,d)
b	c	(b,a) (b,c) (b,d)
c	d	(c,a) (c,c) (c,d)
d		(d,a) (d,c) (d,d)

$$W_4 = \begin{array}{c|cccc} & a & b & c & d \\ \hline a & 1 & 0 & 1 & 1 \\ b & 1 & 0 & 1 & 1 \\ c & 1 & 0 & 1 & 1 \\ d & 1 & 0 & 1 & 1 \end{array}$$

R^+ (Transitive closure) =
$\{$ (a,a) (a,c) (a,d)
(b,a) (b,c) (b,d)
(c,a) (c,c) (c,d)
(d,a) (d,c) (d,d) $\}$

$R_1 = \{(a,b) \mid a-b$ is an integer $\}$ ✓ $(a,b) \in \mathbb{Z}$.

$R_2 = \{(a,b) \mid a-b$ is divisible by $3\}$ ✓

$R_3 = \{(a,b) \mid a-b$ is an odd no. $\}$ Not reflexive. find Equivalence Relation.

$R_4 = \{(a,b) \mid a-b$ is an even no. $\}$ ✓

$A = \{0,1,2,3\}$

$R_1 = \{(0,0), (1,1), (2,2), (3,3)\}$ ✓

$R_2 = \{(0,0), (0,2), (2,0), (2,2), (2,3), (3,2), (3,3)\}$ not reflexive.

$R_3 = \{(0,0), (1,1), (1,2), (2,1), (2,2), (3,3)\}$ ✓

$R_4 = \{(0,0), (1,1), (1,3), (2,2), (2,3), (3,1), (3,2), (3,3)\}$. Not Transitive

$(1,3)(3,2) \rightarrow (1,2)$ is not present.

Find Equivalance Relation.

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

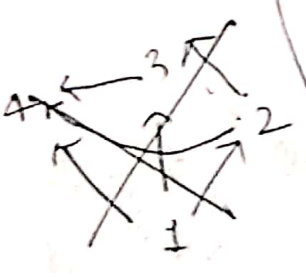
PARTIAL ORDERS

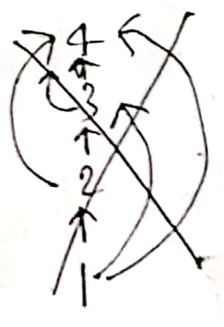
$(A, R) \rightarrow$ poset

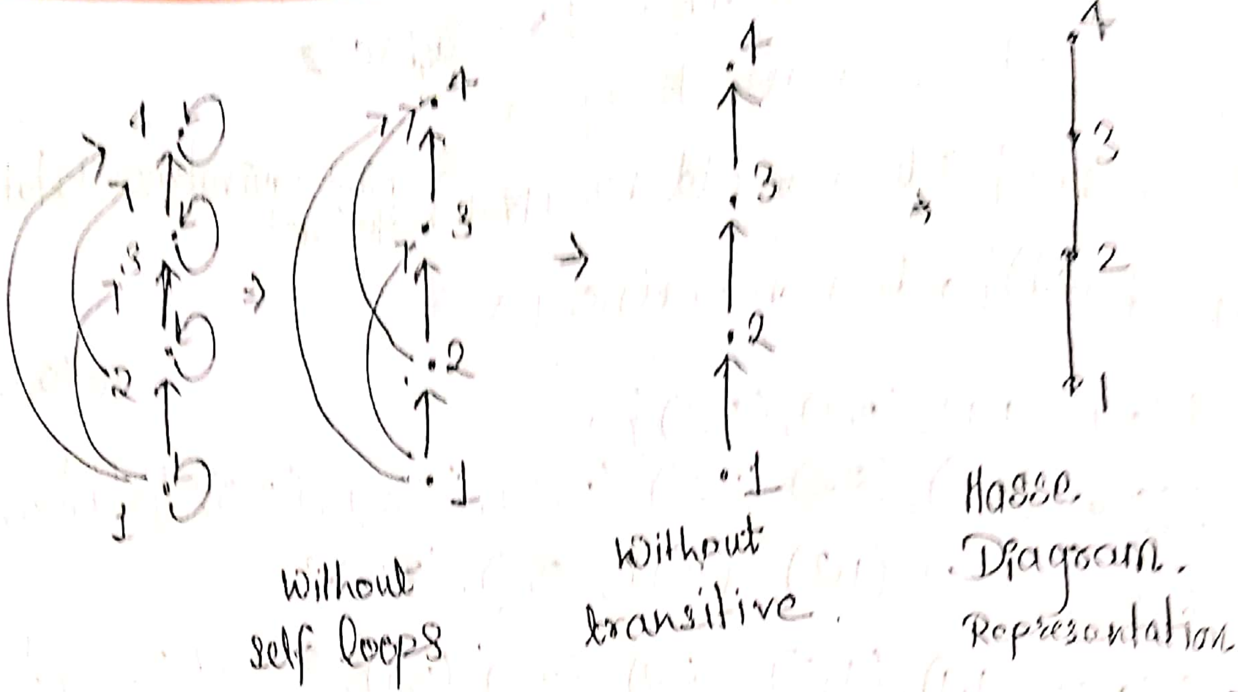
$R \in$ reflexive, antisymmetric & transitive.

Hasse diagrams $\{(1,2,3,4), \leq\}$.

$\leq \{(1,1); (1,2); (1,3); (1,4); (2,2); (2,3); (2,4); (3,4); (3,3); (4,4)\}$.

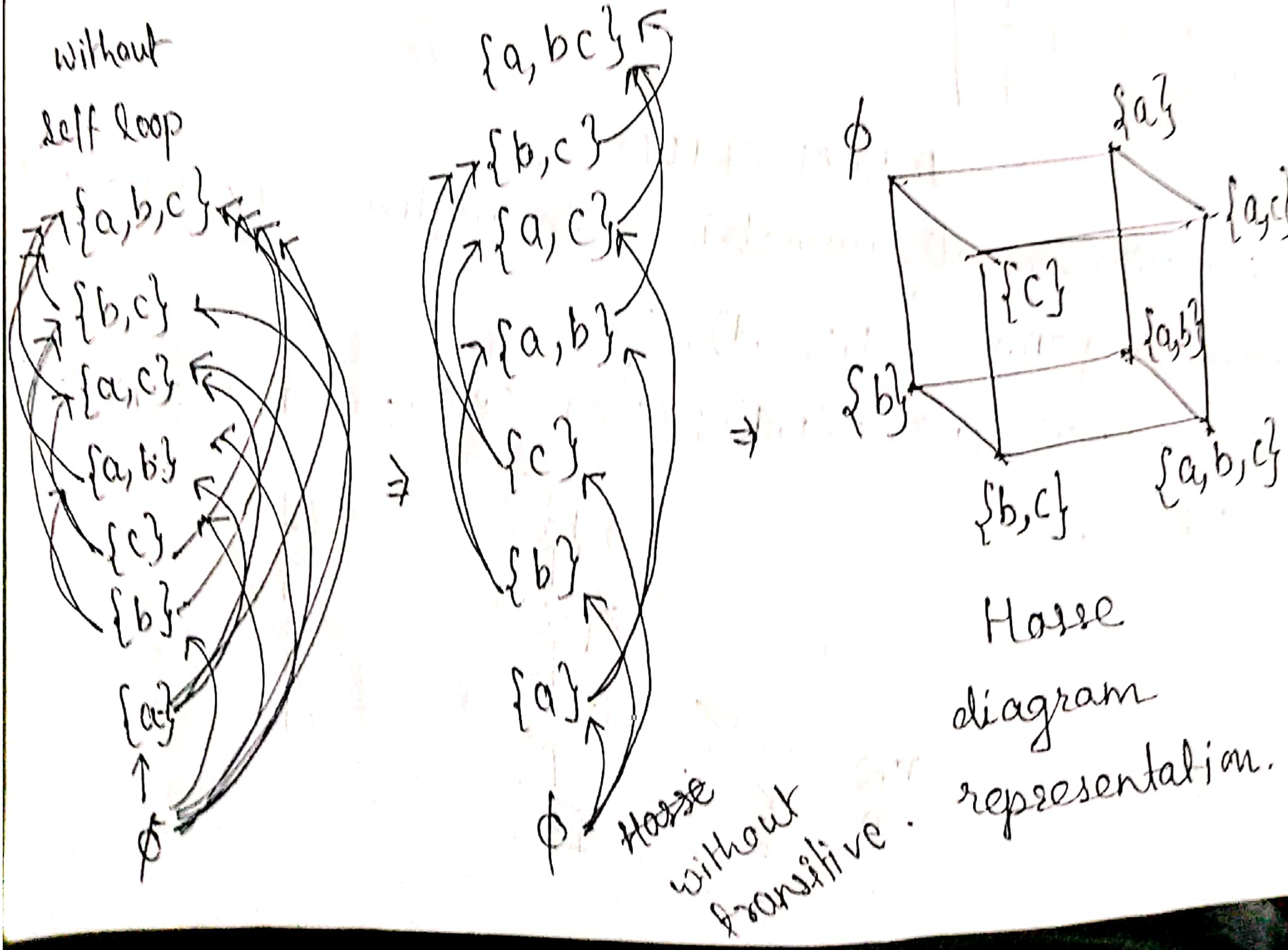






Q Construct the Hasse Diagram of $(P(\{a,b,c\}), \subseteq)$.

$P = \{\phi, \{a\}, \{b\}, \{c\}, \{a,b\}, \{a,c\}, \{b,c\}, \{a,b,c\}\}$



Minimal Element :- $a \in A$ is called minimal, if there does not exist $b \in A$ s.t. bRa.

where (A,R) is a poset ; $R \Rightarrow$ reflexive, antisymmetric, transitive.

e.g. ϕ in $(P(\{a,b,c\}), \subseteq)$.

Maximal element :- $a \in A$, $\nexists\, b \in A$ s.t aRb.

e.g. $\{a,b,c\}$ in $(P(\{a,b,c\}, \subseteq)$

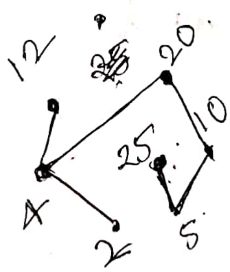
Least element :- $a \in A$ is a least element, if for every $b \in A$. aRb.

e.g. ϕ in $\{P(\{a,b,c\}), \subseteq\}$

Greatest element :- $a \in A$ is greatest element, if for every $b \in A$; bRa e.g. $\{a,b,c\}$ in $\{P(\{a,b,c\}), \subseteq\}$

Q. $(\underset{S}{\{2,4,5,10,12,20,25\}}, |)$.

$R = \{ (2,4); (2,10); (2,12); (2,20); (4,12); (4,20); (5,10)$
$(5,20); (5,25); (10,20);$ and all $(a,a) \}$.



Maximal : aRb not possible for $b \in S$
12, 20, 25

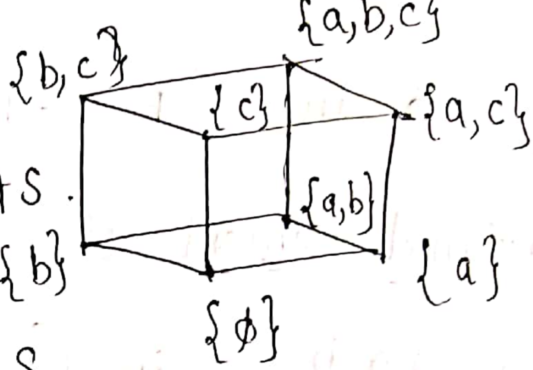
Minimal : bRa not possible for $b \in S$
2, 5

Least : aRb for every $b \in S$: None

Greatest : bRa for every $b \in S$: None.

① Upper Bound

Consider a subset S in (A,R). If there is an element $a \in A$ such that $s R a$ where $s \in S$, then a will be called an upper bound.



② Least Upper Bound {lub} :-

l is an upper bound for subset S.

l will be a lub if $l R z$;

where z is an upper bound for S.

Q. Find the lub of the set $\{3, 9, 12\}$ if they exist, on the poset $(\mathbb{Z}^*, |)$.

Sol^n :- $a \in \mathbb{Z}^*$; $s R a$, where $s \in \{3, 9, 12\}$.

s divides a.

$a \in \{36, 72, \ldots\}$; LCM of 3, 9, 12.

upper bound of the given poset on S.

lub(S) ; $l \in S$ where $l R z$; $z \in \{36, 72, \ldots\}$.

$= 36$.

③ Lower Bound :- Consider a subset S in (A,R). If there is an element $a \in A$ such that $a R s$ where $s \in S$. then a will be called an lower bound.

$\{\phi\}$.

④ Greatest Lower Bound :- (glb).

l is a lower bound for subset S.

l will be a glb if $z R l$ whenever

'z' is the lower bound of 'S'.

Q. Find the glb of set $\overset{S}{\{3, 9, 12\}}$ if they exist on $(\mathbb{Z}^+, |)$.

⇒ $a \in \mathbb{Z}^+$ s.t. a divides $\{3, 9, 12\}$. 1, 3. (GCD/HCF)

glb : 3.

Q. glb and lub of $\{1, 2, 4, 5, 10\}$ on poset $(\mathbb{Z}^+, |)$.

⇒ $a \in \mathbb{Z}^+$ st. a divides $\{1, 2, 4, 5, 10\} \rightarrow \{1\}$.

glb : 1

$a \in \mathbb{Z}^+$ s.t. $\{1, 2, 4, 5, 10\}$ divides a $\rightarrow \{20, 40, 60, \ldots\}$

lub : 20.

LATTICE

A partially ordered set in which every pair of elements has a lub & glb.

(SL) $(L, \preceq)$ where (a, b) lub (a, b) by $a \vee b$ (the join of a & b)

(NSL) glb (a, b) $a \wedge b$ (the meet of a & b)

e.g. poset $(\mathbb{Z}^+, |)$; $a, b \in \mathbb{Z}^+$ $\{a, b\}$ lub ⇒ LCM

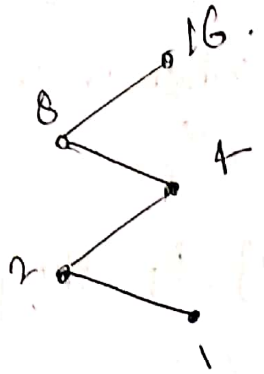
glb ⇒ GCD/HCF.

e.g. Check $(\{1,2,4,8,16\};\,|)$ is lattice or not

Join semi lattice
JSL

meet semi lattice
MSL

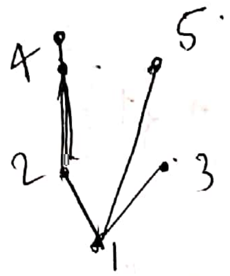
$\Rightarrow$



JSL ✓
MSL ✓
Lattice ✓

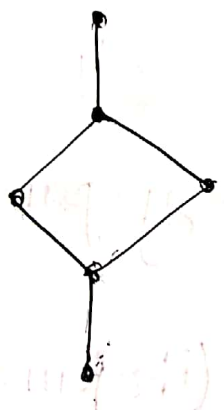
e.g. $(\{1,2,3,4,5\},\,|)$

$\Rightarrow$



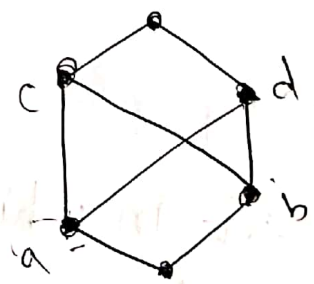
JSL ✗
MSL ✓
Lattice ✗

*



(a)

JSL ✓
MSL ✓
lattice ✓

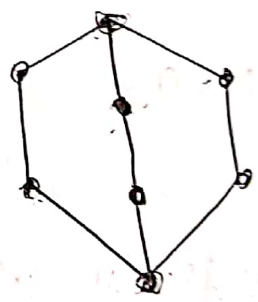


(b)

$a \vee b = \phi$ JSL ✗

$c \wedge d = \phi$ MSL ✗

Lattice ✗



(c)

Lattice ✓

Identify which of the following are <u>posets</u> of $A = \{1,2,3\}$.

$R_1 = \{\ \}$ (Not a poset, Not reflexive)

$R_2 = \{(1,1),(2,2),(3,3)\}$ ✓

$R_3 = \{(1,1),(2,2),(3,3)\},(1,3),(2,3)\}$ ✓

$R_4 = \{(1,1),(2,2),(3,3),(1,2),(2,1)\}$ × Not anti symmetric.

$R_5 = \{(1,1),(1,2),(2,3),(1,3)\}$ × Not reflexive.

$R_6 = A \times A$ ×

$R_1 = \{(a,b) \mid a,b \in \mathbb{Z},\ a<b\}$ × Not reflexive

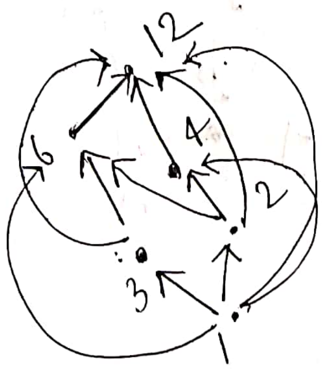
$R_2 = \{(a,b) \mid a,b \in \mathbb{Z},\ a \leq b\}$ ✓

$R_3 = \{(A,B) \mid A,B \in P(X),\ A \subseteq B\}$ ✓

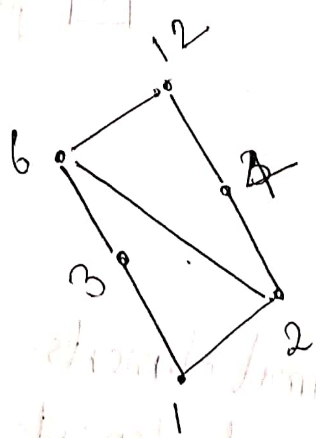
$R_4 = \{(a,b) \mid a,b \in \mathbb{Z},\ b/a \in \mathbb{Z}\}$, ✓

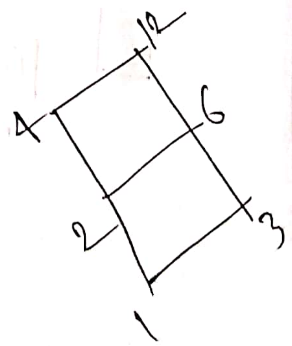
Q. <u>Draw Hasse Diagram</u>.

ⓐ $[\{1,2,3,4,6,12\}, /]$



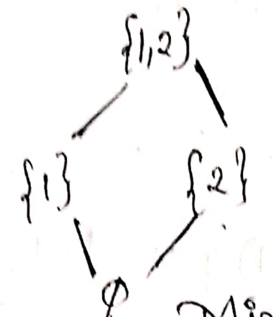
⇒





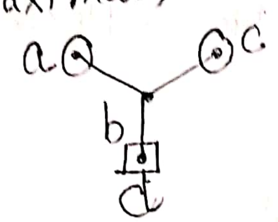
② $[\{\phi, \{1\}, \{2\}, \{1,2\}\}, \subseteq]$

→

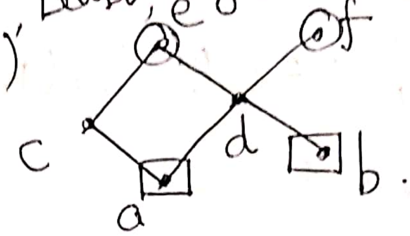


Q. Maximal, Minimal, Least, greatest.

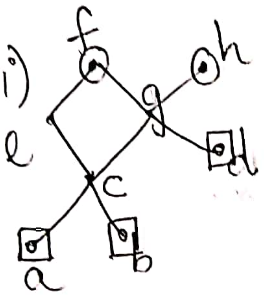
(i)



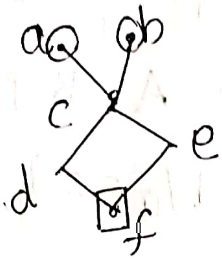
(ii)



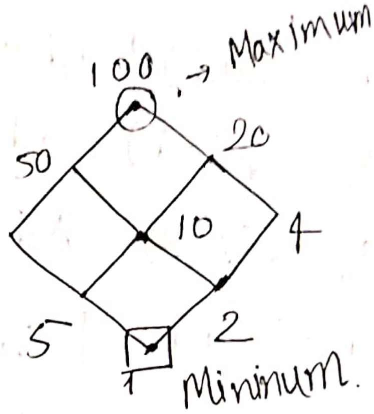
iii)



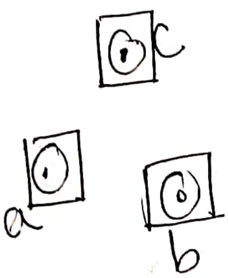
(iv)



(v)



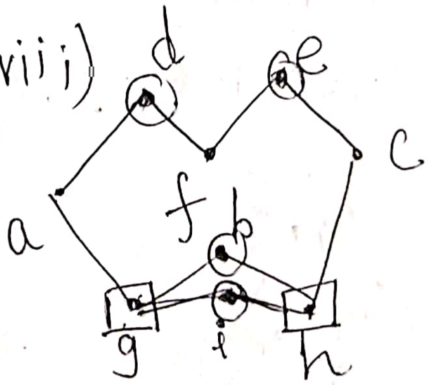
vi)



vii)

Maximum
Minimum

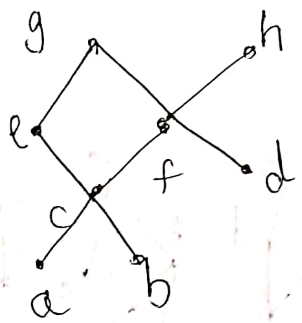
viii)

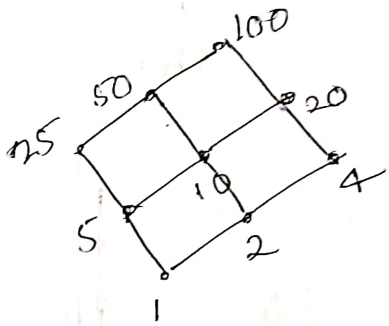


⊙ ⇒ Maximal elements.

⊡ ⇒ Minimal elements.

Q. Lower Bound & Upper Bound

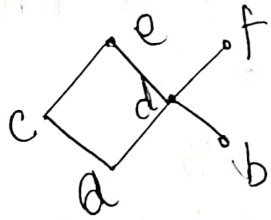


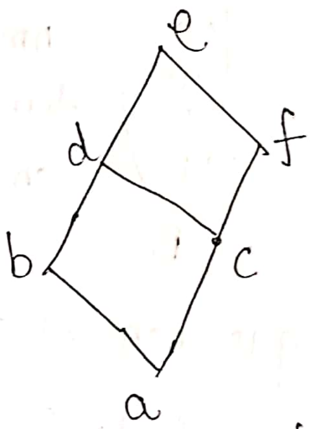


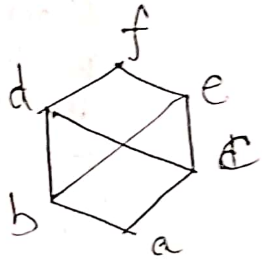
$S_1 = \{e, c\}$.
UB = $\{g, e\}$.
LB = $\{a, b, c\}$.
$S_2 = \{c, f, d\}$.
UB = $\{h, g, f\}$
LB = ϕ

$S_1 = \{50, 10\}$.
UB = $\{50, 100\}$
LB = $\{10, 5, 1, 2\}$.
$S_2 = \{5, 10, 2, 4\}$
UB = $\{20, 100\}$
LB = $\{1\}$.

Q. lub & glb.







$B_1 = \{c, d\}$
UB = e
LB = a
lub = e ; glb = a
$S_2 = \{a, b\}$ UB = $\{d, e, f\}$
LB = $\{\phi\}$
lub = d , glb = ϕ.

$S_1 = \{a, c, f\}$ UB = $\{f, e\}$
LB = $\{a\}$
glb = a
lub = f.

$S_1 = \{d, e\}$
UB = $\{f\}$
LB = $\{a, b, c\}$.
glb = ϕ.
lub = f

$S_1 = \{b, c\}$ UB = $\{d, e, f\}$ LB = $\{a\}$
lub = ϕ , glb = a.

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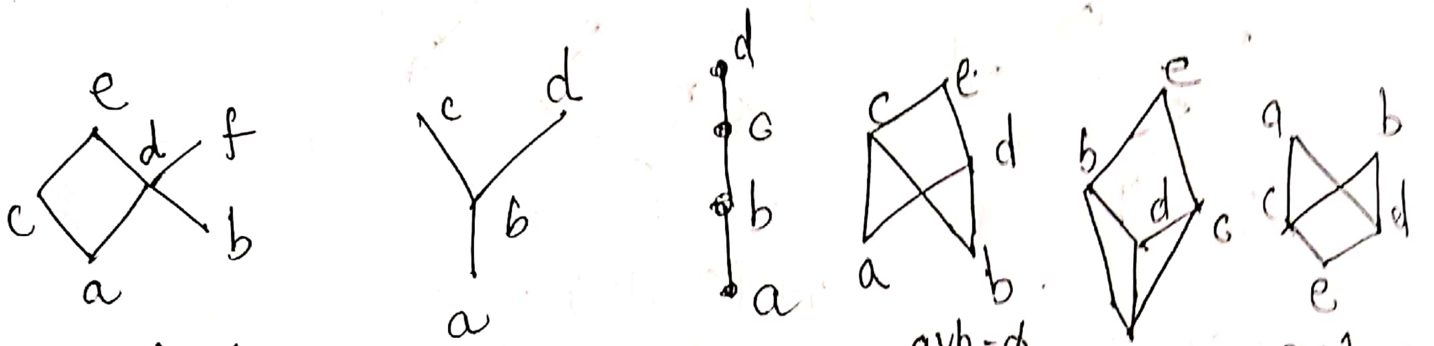
Q. Join Semi-lattice, Meet Semi-lattice.
(∨) (∧)
(lub) (glb).



$e \vee f = \phi$ $a \wedge b = \phi$ — JSL ✗, MSL ✗

$c \vee d = \phi$ — JSL ✗, MSL ✓

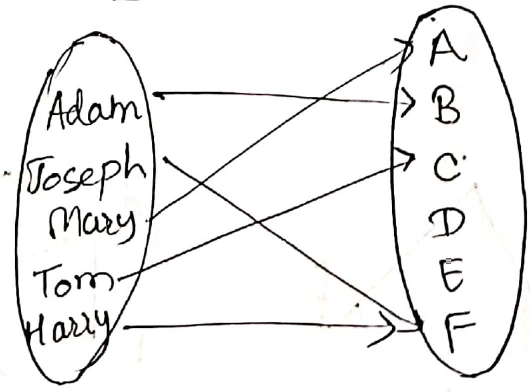
JSL ✓, MSL ✓, Lattice ✓

$a \vee b = \phi$, $a \wedge b = \phi$ — JSL ✗, MSL ✗

JSL ✓, MSL ✓, Lattice ✓

$a \wedge b = \phi$, $a \vee b = \phi$ — MSL ✗, JSL ✗

FUNCTIONS



A   B

A function from A to B is an assignment of exactly one element of B to each element of A.

$f: A \to B$

$f(a) = b$ if b is the unique element of set B which is assigned to $a \in A$.

$f: A \to B$

f(Adam) = B.
f(Joseph) = F.

A ⇒ Domain of 'f'   B ⇒ Co-domain of 'f'

$f(a) = b$; b: image of 'a' ; a: preimage of 'b'.

The range of f will be the set of all images that are assigned to $a \in A$.

{Adam, Joseph, ...} is the domain.

{A, B, C, D, E, F} is the co-domain.

Range : {A, B, C, F}.

Eg. Let $f : \mathbb{Z} \to \mathbb{Z}$ assign the square of an integer to that integer.

$f(x) = x^2$ Domain : $\mathbb{Z}$ Co-domain : $\mathbb{Z}$. Range : $\{0, 1, 4, 9, \ldots\}$.

Two Real valued functions can be added and multiplied. Let f_1 and f_2 are two fns. from $A \to R$.

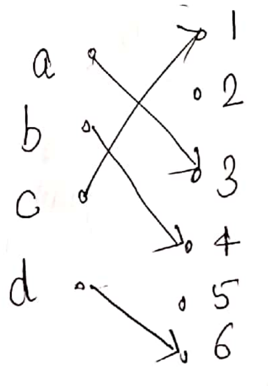
we can define $(f_1 + f_2)x = f_1(x) + f_2(x)$; $f_1f_2(x) = f_1(x).f_2(x)$

Ex. Let f_1 & f_2 be fns. from R to R s.t. $f_1(x) = x^2$ & $f_2(x) = x - x^2$

$(f_1 + f_2)(x) = f_1(x) + f_2(x) = x^2 + (x - x^2) = x$

$f_1f_2(x) = f_1(x). f_2(x) = x^2.(x - x^2) = x^3 - x^4$.

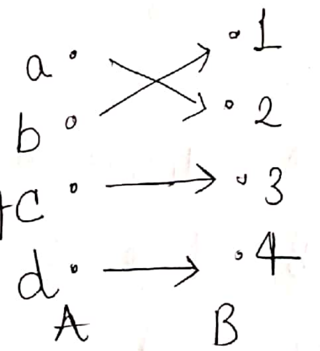
Types of functions



(a) One-to-One / Injective :-

fns. never assign the same value to two different elements

if $f(a) = f(b) \Rightarrow a = b$.



(b) Onto / Surjective :-

for every $b \in B$, there is an element $a \in A$ s.t. $f(a) = b$.

Co-domain = Range.

© One-to-one Correspondence / Bijection :-

If a fn. is both one-to-one and onto then it will be a bijection.

Inverse Functions :-

$f: A \to B$ is an one-to-one correspondence.

$f(a) = b$; $f^{-1} \neq 1/f$.

$f^{-1}: B \to A$ $\boxed{f^{-1}(b) = a}$.

$a \underset{\longleftarrow}{\overset{f}{\longrightarrow}} 1$

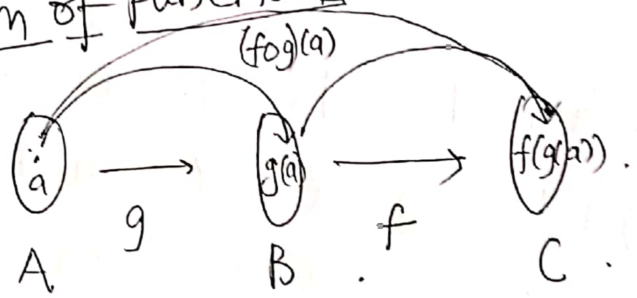
$b \underset{\longleftarrow}{\longrightarrow} 2$

$c \underset{\longleftarrow}{\longrightarrow} 3$

$d \underset{\longleftarrow}{\longrightarrow} 4$.

f^{-1}

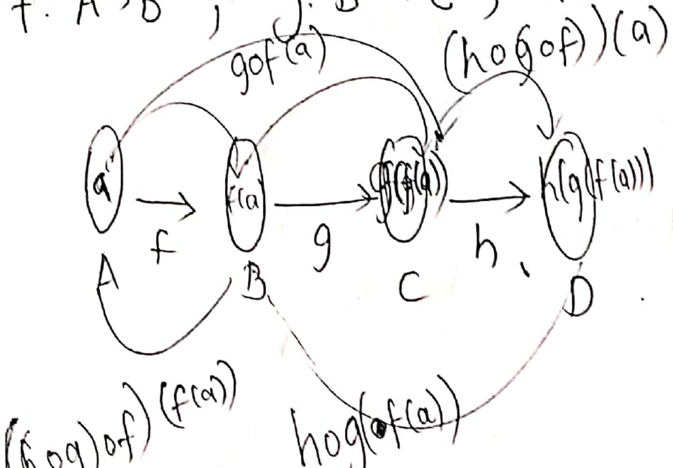
* Composition of Functions :-



Composition of functions is associative.

$f: A \to B$; $g: B \to C$; $h: C \to D$. $(ho(gof))(a)$ & $((hog)of)(a)$ are both fn. from A to D.

$f: A \to D$



Ex. let $A = \{1,2,3\}$, $B = \{r,s\}$, and $C = \{m,n\}$.

$f_1: A \to B$ where $f_1 = \{(1,r), (2,r), (3,s)\}$

$f_2: B \to C$ where $f_2 = \{(r,n) (s,n)\}$.

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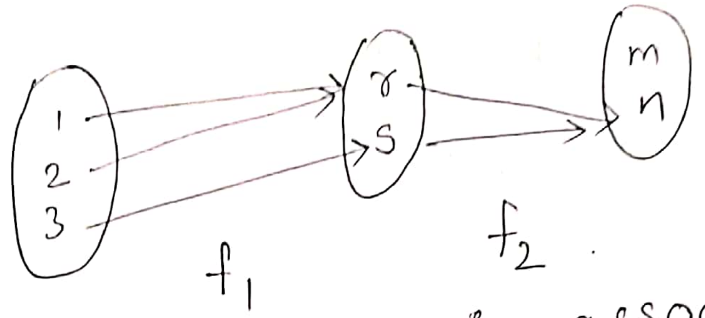
$f_2 \circ f_1 = \{(1,n), (2,n), (3,n)\}$.



Composition of functions is associative.

$f: A \to B$; $g: B \to C$; $h: C \to D$.

$h\circ(g\circ f)$ & $(h\circ g)\circ f$ are both fn. from A to D.

$f(a) = b$, $g(b) = c$, $h(c) = d$.

$(a,b) \in f$ $(b,c) \in g$ $(c,d) \in h$.

$(a,c) \in g\circ f$	$(b,d) \in h\circ g$
$(a,d) \in h\circ(g\circ f)$	$(a,d) \in (h\circ g)\circ f$.

$(h\circ g)\circ f = h\circ(g\circ f)$.

ALGEBRAIC STRUCTURES

Coins deposited	Merchandise
dime, dime	gum
dime, quarter	candy.
quarter, dime.	candy.
quarter, quarter	chocolate.

$A = \{$dime, quarter$\}$.

$B = \{$gum, candy, choclate$\}$

$f: A \times A \to B$. Binary operation on A

$f: A \times A \to A$. Closed binary operation on A.

$\square, \star, *, \ldots$

$\underbrace{(A, \square, *)}$ → set A together with the list of binary operations.

↓ <u>Algebraic Structure</u>

$\star$	dime	quarter
dime	cookies	gum
quarter	gum	candy

$*$	dime	quarter
dime	chips	biscuits.
quarter	biscuits	sandwich.

$(\{\text{dime}, \text{quarter}\}, \star, *)$

<u>Semigroup</u> Let $(A, \square)$ be an algebraic structure if

1. $\square$ is a closed operation w.r.t. A.
2. $\square$ is associative.
   i.e. if $a, b, c \in A$ then $a \square (b \square c) = (a \square b) \square c$.

eg. A = set of all even integers $\{2, 4, 6, \ldots\}$.
$\square$ = + operation.
$(A, +)$ is a semigroup.

<u>Monoid</u> :- 1. $\square$ is a closed operation wrt A.
2. $\square$ is associative.
3. There exists an identity element (e) in A s.t.
$\forall x \in A, \; e \square x = x \square e = x$.

Ex. I: set of all whole nos. (real nos including '0').
$+$: ordinary addition '0' will be 'e'.
$(I,+)$: is a monoid.

<u>Group</u> :- 1. 2. 3.
4. For every element $x \in A$, there exists an inverse element $x^{-1} \in A$ s.t. $x \square x^{-1} = x^{-1} \square x = e$.
e.g. $(I,+)$ where $e=0$; $\forall n \in I, -n \in I$; $n+(-n) = 0 = e$.

<u>Subgroup</u> :- $(A,\square)$ be an algebraic structure & 'B' a subset of A, then $(B,\square)$ will be called a subgroup of A if it is a group by itself.
1. $\square$ is a closed operation w.r.t. B.
2. $\square$ is associative.
3. 'e' must exist in B, as the identity of $(B,\square)$.
4. $\forall\ b \in B,\ b^{-1} \in B$.

e.g I: set of integers ; $+$: addition ; E : set of even integers

<u>Isomorphism</u>

A

$\square$	a	b	c	d
a	a	b	c	d
b	b	a	a	c
c	b	d	d	c
d	a	b	c	d

B.

$*$	α	β	γ	δ
α	α	β	γ	δ
β	β	α	α	γ
γ	β	δ	δ	γ
δ	α	β	γ	δ

a, b, c, d to $\alpha, \beta, \gamma, \delta$ $\quad$ $\square$ to $*$

$(B, *)$ from $(A, \square)$ $\quad$ $f(a) = \alpha$

$f(b) = \beta$

f: one-one & onto function.

$f(a_1 \square a_2) = f(a_1) * f(a_2)$.

~~$f(A,\square)$~~ $\longrightarrow$ ~~$(A,\square)$~~ $\longrightarrow$

(A) $\quad f \quad$ (A) $\quad f$

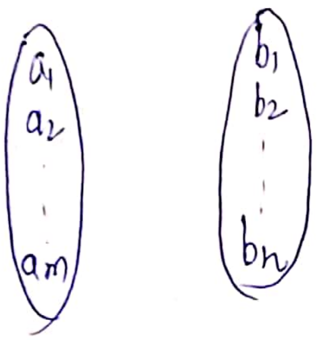
(1-1) $\quad$ (1) $\quad$ (1+1). $\quad$ f: auto morphism. $A \xrightarrow[f]{} B$.

f: isomorphism

Tutorial

1.) let X, Y, Z be the set of sizes x, y, z resp. Let $W = X \times Y$, let E be the set of all subsets of W. find $Z \to E$ no. of functions.

A $\qquad$ B



$f: A \to B$.

$|A| \quad |B|$

no. of fun. from A to B $= |B|^{|A|}$

$\Rightarrow \quad |W| = xy \quad \Rightarrow \quad |E| = 2^{xy}$

no. of functions from $Z \to E$.

$= |E|^{|Z|}$

$= (2^{xy})^z = \underline{\underline{2^{xyz}}}$ Ans.

2.) Total no. of one to one functions from A to B, where $f: A \to B$.

$|A| = m \; ; \; |B| = n$, $\quad \Rightarrow n \quad n-1 \quad n-2 \quad \ldots \quad n-(m-1)$

$\Rightarrow {}^nP_m$.

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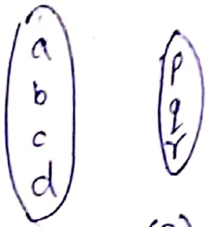
③ Find Total no. of non-onto functions from $A \to B$.

$|A| = 4$; $|B| = 3$.

Ans. let us consider;

S_x = No. of fns. in which 'x' is not is not mapped.



(A) (B)

$n(S_p) = 2^4 = n(S_q) = n(S_r)$;

$n(S_p \cap S_q) = n(S_q \cap S_r) = n(S_p \cap S_r) = 1^4$

$n(S_p \cap S_q \cap S_r) = 0^4 = 0$.

$n(S_p \cup S_q \cup S_r) = n(S_p) + n(S_q) + n(S_r) + n(S_p \cap S_q \cap S_r)$

$- n(S_p \cap S_q) - n(S_q \cap S_r) - n(S_p \cap S_r)$.

$= 2^4 + 2^4 + 2^4 + 0 - (1)^4 - (1)^4 - (1)^4 = 48 - 3 = \underline{45}$.

④. $f: A \to B$.

$|A|$ $|B|$.

One-one function $|A| \leq |B|$.

Onto function $|A| \geq |B|$

Bijection $|A| = |B|$.

⑤ Assuming $g \circ f$ is not empty show that $g \circ f$ is a f from $A \to C$

$= g(f(x))$

$g: B \to C$.

$f: A \to B$

(a, c_1) $g(b) = c_1$ $b \in B$.

(a, c_2) $g(b) = c_2$.

$\times$

$f(a) = b$.

$f(x) = x^2$; $g(x) = 2x + 3$

$fog = f[g(x)] = f[2x+3] = (2x+3)^2$

$gof = g[f(x)] = g[x^2] = 2x^2 + 3$.

	AS	S.G.	Monoid	G	A.G.
N, +	✓	✓	×	×	×
N, −	×	×	×	×	×
N, ×	✓	✓	✓	×	×
N, /	×	×	×	×	×
Z, +	✓	✓	✓	✓	✓
Z, −	✓	×	×	×	×
Z, ×	✓	✓	✓	×	×
Z, /	×	×	×	×	×
R, +	✓	✓	✓	✓	✓
R, −	✓	×	×	×	×
R, ×	✓	✓	✓	×	×
R, /	×	×	×	×	×
e, +	✓	✓	✓	✓	✓
e, ×	✓	✓	×	×	×
o, +	×	×	×	×	×
o, ×	✓	✓	✓	×	×

Q. $(\{1, \omega, \omega^2\}, \times)$; $\omega^3 = 1$

A.S. → ✓

S.G. → ✓

Monoid. → ✓

G → ✓

$\times$	1	ω	ω^2
1	1	ω	ω^2
ω	ω	ω^2	$\omega^3 = 1$
ω^2	ω^2	$\omega^3 = 1$	$\omega^4 = \omega$

Q. $\{(1,-1,i,-i), \times\}$; $i=\sqrt{-1}$

A.S. → ✓

S.G. → ✓

Monoid →

Group →

$\times$	1	-1	i	$-i$
1	1	-1	i	$-i$
-1	-1	1	$-i$	i
i	i	$-i$	-1	1
$-i$	$-i$	i	1	-1

Q. $(N,*)$, $a*b = a^b$.

(a) Semi group.

(b) Not semigroup ✓

(c) Monoid but not G

(d) Group.

$(a*b)*c = (a^b)*c = (a^b)^c = a^{bc}$ —①

$a*(b*c) = a*(b^c) = a^{(b^c)}$ —②

$\underline{1 \neq 2}$.

Q. $(Z,*)$, $a*b = \max(a,b)$

<u>Semigroup</u>

$(a*b)*c = \max(a,b)*c$
$= \max(\max(a,b),c)$

$a*(b*c) = a*\max(b,c)$

$= \max(\cancel{ma}\ a, \max(b,c))$

$a*e = \max(a,e)$

$= a$.

$e = -\infty$

but $-\infty$ does not exist in ~~real~~ Z $\therefore$ not a monoid

★ Addition modulo. $(+_n)$

g. $\{1,2,3,4\}$; $+_5$

$+_5$	1	2	3	4
1	2	3	4	0
2	3	4	0	1
3	4	0	1	2
4	0	1	2	3

(if sum $\geq n$; sum$-n$ $+_n$)

$\{0,1,2,3\}$; $+_4$

$+_4$	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

<u>Group</u>

Identity = 0

Inverse $\frac{0}{0}\ \frac{1}{3}\ \frac{2}{2}\ \frac{3}{1}$

★ Multiplication Modulo ($\times_n$)

$\{0,1,2,3\}, \times_4$

$\times_4$	0	1	2	3
0	0	0	0	0
1	0	1	2	3
2	0	2	0	2
3	0	3	2	1

~~Semi Group~~

Monoid

homomorphism

$$(\mathbb{Z},+) \xrightarrow{f} (\mathbb{Z}_{2n},+)$$

$$f(a * b) = f(a) * f(b)$$

one-one

onto

↓

Isomorphism

★ Homomorphism

$(A,\square)$ and $(B,*)$ are A.S. ; f is a function from A to B $\forall\, a_1, a_2 \in A$; $f(a_1 \square a_2) = f(a_1) * f(a_2)$.

f: Homomorphic from $(A,\square)$ to $(B,*)$.

$(B,*)$ = Homomorphic image of $(A,\square)$

$\square$	α	β	γ	δ	ϵ	ρ
α	α	β	α	α	γ	δ
β	β	α	γ	β	γ	ϵ
γ	α	γ	α	β	γ	ϵ
δ	α	β	β	δ	ϵ	ρ
ϵ	γ	γ	γ	ϵ	ϵ	ρ
ρ	δ	ϵ	ϵ	ρ	ρ	ρ

$*$	1	0	-1
1	1	1	0
0	1	0	-1
-1	0	-1	-1

$f(\alpha)=1$; $f(\beta)=1$; $f(\delta)=0$, $f(\epsilon)=0$, $f(\rho)=-1$, $f(\gamma)=1$.

f is homomorphic from $(\{\alpha,\beta,\gamma,\delta,\epsilon,\rho\},\square)$ to $(\{-1,0,1\},*)$

Congruence Relation :-

$(A, \square)$ is an A.S.

R be an equivalence relation.

R is called congruence relation on A w.r.t. $\square$

if (a_1, a_2) & (b_1, b_2) in R implies $(a_1 \square b_1, a_2 \square b_2)$

is also in R.

$\square$	a	b	c	d
a	a	a	d	c
b	b	a	d	a
c	c	b	a	b
d	c	d	b	a

$\simeq (A, \square)$

	a	b	c	d
a	✓	✓		
b	✓	✓		
c			✓	✓
d			✓	✓

Not a congruence relation.

* Ring, Integral Domain, field

$(A, \square, *)$ is called a ring

① $(A, \square)$ is an Abelian Group.

② $(A, *)$ is semigroup

③ The operation '$*$' is distributive over $\square$

$$a * (b \square c) = (a * b) \square (a * c)$$

$$(b \square c) * a = (b * a) \square (c * a).$$

eg. $(\mathbb{Z}, +, \cdot) \Rightarrow$ Ring $(\mathbb{Z}, +)$ is A.G.

$(\mathbb{Z}, \cdot)$ is S.G.

$a \cdot (b + c) = a.b + a.c$

$(b + c) \cdot a = b.a + c.a$.

$R = (A, \square, *)$

Commutative ring

for ring R ; $\forall x, y \in R$ $x * y = y * x$.

Ring with unity

for ring R $\exists e \in R$ s.t. $x * e = e * x = x$ $\forall x \in R$

Zero divisor of a ring

for ring R, $\exists$ $a, b \in R$

if $a \neq 0, b \neq 0$, then $a * b = 0$ } zero divisor of a ring .

eg $\left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix}, +, \cdot \right\}$; $(a, b, c, d) \in \mathbb{Z}$ ..

Ring without zero divisor $\Rightarrow$ $(\mathbb{Z}, +, \cdot)$

Integral Domain :- A ring R is called Integral Domain if
① It is commutative.
② It has unity.
③ It should be without zero divisor

eg. $(\mathbb{Z}, +, \cdot)$; $(R, +, \cdot)$.

Field :- A ring R is called field if

① It is commutative.

② It has unity.

③ For every non-zero element it has inverse

wrt $*$ in R i.e. for Ring $(A, \square, *)$

$\forall a \in A$, $\exists b \in A$ s.t. $a * b = e = b * a$

e.g. $(R, +, \cdot)$ ✓ field ; $(\mathbb{Z}, +, \cdot)$ × field

★ COUNTING :-

Pigeonhole Principle :- Suppose a flock of 20 pigeons comes home to roost into 19 pigeonholes. So, atleast 1 pigeonhole will have at least 2 pigeons

If $k \in \mathbb{Z}^+$ and $k+1$ or more objects are placed into 'k' boxes, then there is at least 1 box containing 2 or more objects. (Generalization).

Corollary :- A fn. 'f' from a set with 'k+1' or more elements to a set with 'k' elements is not one-to-one (Assignment).

Step 1 :- Suppose none of the 'k' boxes contain more than one object.

Recurrance Relation :-

$\{a_n\}$: seq. of powers of 2 $a_n = 2^n$ where $n = 0, 1, 2, \ldots$

$\begin{cases} a_0 = 1 \\ a_{n+1} = 2a_n \end{cases}$ ⇒ Recurrance Relation.

Recursively defined ~~relati~~ fn. :-

Basic step : Specify the value of f at 0.

Recursive step : Rule for finding f at a particular integer from the value of f at lesser integers ex. $f(0) = 3$; $f(n+1) = 2f(n) + 3$.

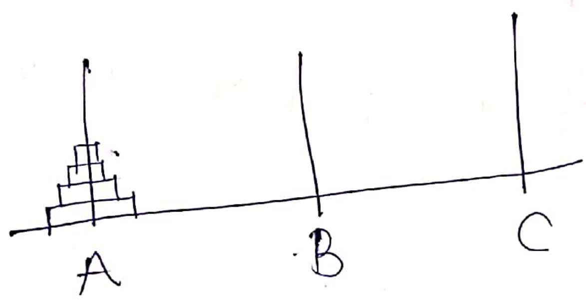
ex. Give the recursive definition of a^x, where 'a' is a non-zero real no. & 'x' is non-negative integer.

→ $f(x) = a.f(x-1)$. $a^0 = 1$

Recursively defined function. RR: $a^{n+1} = a.a^n$.

for $n = 0, 1, 2, 3,$

Any sequence whose terms satisfy the RR is known as the solution to the RR.



Let $\{a_n\}$ be the no. of steps to move 'n' disks from A to B.

$f(n) = 2f(n-1) + 1$. $a_1 = 1$

$\{a_{n-1}\}$ → no. of steps to move $(n-1)$ disks from A to C. $a_n = (a_{n-1}) + 1 + a_{n-1} = 2a_{n-1} + 1$.

Soln. $a_n = 2a_{n-1} + 1$

$= 2(2a_{n-2} + 1) + 1$

$= 2^2 . a_{n-2} + 2 + 1$

$= 2^2(2a_{n-3} + 1) + 2 + 1$

$2^3 a_{n-3} + 2^2 + 2 + 1$

$\vdots$

$= 2^{n-1}.a_1 + 2^{n-2} + 2^{n-3} + \dots + 1$.

Ex. Suppose that the no. of bacteria in a colony doubles every hour. If a colony begins with 5 bacteria, how many will be present in 'n' hours?

Ans :-

$$b(0) = 5.$$

$$b(t) = 2.b(t-1).$$

$$b(n) = 2.b(n-1)$$
$$= 2(2\,b(n-2))$$
$$= 2^2.b(n-2)$$
$$= 2^3.b(n-3).$$

$$b(n) = 2^n.b(0) = 5.2^n \Rightarrow \boxed{b(n) = 5.2^n}$$

Generating Functions

$a_0, a_1, a_2, \dots, a_k, \dots$ an infinite series of real nos.

$$G(x) = a_0 + a_1x + a_2x^2 + \dots + a_k.x^k + \dots$$
$$= \sum_{k=0}^{\infty} a_k.x^k.$$

$\{a_k\}$ where $a_k = 2$; $\quad G(x) = \sum_{k=0}^{\infty} 2.x^k$

$a_k = k+1$; $\quad G(x) = \sum_{k=0}^{\infty} (k+1).x^k.$

Ex. For the sequence, {1, 1, 1, 1, 1 }. Find the generating sequence.

$\Rightarrow$ $G(x) = 1 + x + x^2 + x^3 + x^4$

$$1.\ \frac{(x)^5 - 1}{x-1} = \frac{x^5-1}{x-1}$$

Theorem :- $f(x) = \sum_{k=0}^{\infty} a_k . x^k$ & $g(x) = \sum_{k=0}^{\infty} b_k . x^k$.

$$f(x) + g(x) = \sum_{k=0}^{\infty} (a_k + b_k)\, x^k.$$

$$f(x) . g(x) = \sum_{k=0}^{\infty} \left\{ \sum_{j=0}^{k} a_j b_{k-j} \right\} x^k.$$

ex. $f(x) = \frac{1}{(1-x)^2}$. Find the coefficients $a_0, a_1, \ldots, a_k$ in the expansion $\sum_{k=0}^{\infty} a_k . x^k$.

$\Rightarrow$ $f(x) = (1-x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + 5x^4 + \ldots$

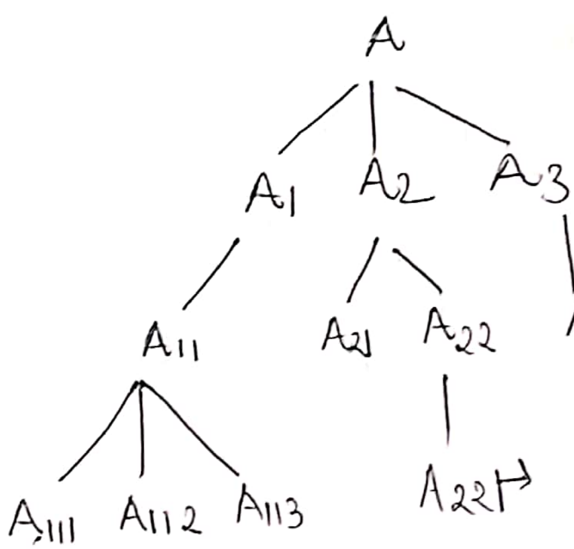
$a_0 = 1$; $a_1 = 2$; $a_2 = 3$; $\ldots$; $a_k = k+1$.

$f_1(x) = \frac{1}{(1-x)}$; $f_2(x) = \frac{1}{(1-x)}$

$$\frac{1}{(1-x)} \cdot \frac{1}{(1-x)} = f_1(x) . f_2(x) = \sum_{k=0}^{\infty} \left\{ \sum_{j=0}^{k} (1) \right\} . x^k$$

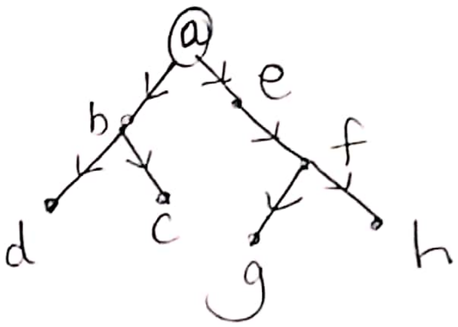
$$= \sum_{k=0}^{\infty} (k+1) . x^k.$$

TREE



Def^n:-
A connected undirected graph with no simple circuits.
⇒ No self loops & no parallel edges
↦ A Tree is a simple graph.

Rooted Tree :- Root is a vertex in a tree and all edges are assumed to move outside from the root.



Rooted Tree

Parent – A_{11} is A_1

child – A_1 is A_{11}

Siblings – A_{22} for A_{21}

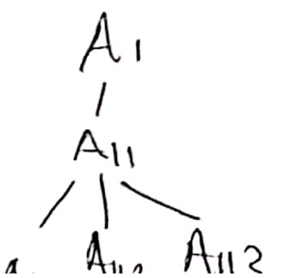
Ancestors – A_{11}, A_1 & A for A_{11}

Descendants - $A_{11}, A_{111}, A_{112}, A_{113}$ for A_1.

Leaf : A vertex with no children A_{111}, A_{112}, A_{13}, A_{21}, A_{221}, A_{31}

Internal vertices : Vertices that have children are called the internal vertices.
$A, A_1, A_2, A_3, A_{11}, A_{22}$

Subtree :- Subtree with root A_1



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Theorem 1: An undirected graph is a tree iff there is a unique simple path between any two of its vertices. (Assignment)

(Computer file system Leaf ⇒ files.
Internal vertices ⇒ subdirectories).

Proof :-

Theorem 2 : A tree with 'n' vertices has n-1 edges

Proof :- **Basis** : n=1 is true.

Induction Hypothesis :

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Generalized Pigeonhole Principle (3 Marks)

$kn+1$ or more → Pigeons.

n → Pigeonhole.

$k+1$ or more pigeons in at least 1 pigeonhole.

Q. Find minimum number of students in a class to be sure that four of them were born in same month.

Ans :- n = no. of months = 12

$k+1 = 4 \Rightarrow k = 4-1 = 3$

$kn+1 = 3(12)+1 = 37$.

minimum number of students are 37.

Note :- (Year has 365 days if not specified in question)

★ Homogeneous Recurrence Relation :-

$a_n = 3 \cdot a_{n-1}$; $a_0 = 5$

$x^n = 3 \cdot x^{n-1}$

$x = 3$

$a_n = A \cdot x^n$

$a_n = A 3^n$

$\Rightarrow a_0 = 5$

$\Rightarrow 5 = A \cdot 3^0$

$\Rightarrow A = 5$

$\Rightarrow \boxed{a_n = 5 \cdot 3^n}$

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Q. $a_n = 2a_{n-1} + 2n$; $a_1 = 6$

(Non-homogeneous recurrence relation of first order) . (5 Marks)

$\Rightarrow$ $$a_n = \underbrace{a_n^{(h)}}_{\text{homogenous solution}} + \underbrace{a_n^{(p)}}_{\text{particular solution}}$$

For Homogenous part we can proceed like this,

$a_n = 2 \cdot a_{n-1}$ $\Rightarrow a_n^{(h)} = A \cdot x^n$

$x^n = 2 \cdot x^{n-1}$ $\Rightarrow \boxed{a_n^{(h)} = A \cdot 2^n}$

$\Rightarrow \boxed{x = 2}$

(On dividing both sides by x^{n-1}).

For Particular part we can proceed like this,

Let $(a_n)^p = cn + d$; be a particular sol^n.

$\Rightarrow (cn+d) = 2[c(n-1)+d] + 2n$

$cn+d = 2cn - 2c + 2d + 2n$.

$\Rightarrow cn+d = 2(cn+d) - 2(c-n)$.

$\Rightarrow 0 = cn - 2c + d + 2n$.

$\Rightarrow 0 = n(c+2) + d - 2c$ —— (1)

$\Rightarrow \cancel{n = \frac{2c-d}{c+2}}$

$c + 2 = 0$; $d - 2c = 0$

$\boxed{c = -2}$; $d = 2c$

$\boxed{d = -4}$

(As n is variable for ① to be valid both $(c+2)$ & $(d-2c)$ must be equal to 0)

$$a_n^{(p)} = cn + d = (-2)n + (-4)$$

$$\Rightarrow \boxed{a_n^{(p)} = -2n - 4}$$

So, the final solution is

$$a_n = a_n^{(h)} + a_n^{(p)}$$

$$a_n = A \cdot 2^n - 2n - 4 \quad \text{—— ②}$$

We know; $a_1 = 6$

Put $n = 1$; in ②

$$a_1 = A \cdot 2^{(1)} - 2(1) - 4$$

$$\Rightarrow 6 = 2A - 6$$

$$\Rightarrow 2A = 12 \Rightarrow \boxed{A = 6}$$

$$\Rightarrow \boxed{a_n = 6 \cdot 2^n - 2n - 4}$$

is the required solution.

Non-homogeneous Recurrence Relation of second order (5 Marks).

$$F_n = AF_{n-1} + BF_{n-2} + f(n).$$

(I) $f(n) = c.x^n$.

$x^2 = Ax + B$ is characteristic eq^n of corresponding homogeneous recurrence relation.

$x_1, x_2 \Rightarrow$ roots of above quad. eq^n.

If $x \neq x_1$ and $x \neq x_2$ then $(a_n)^{(P)} = Ax^n$.

If $x = x_1$ or $x = x_2$ then $a_n^{(P)} = An.x^n$.

If $x = x_1 = x_2$ then $a_n^{(P)} = An^2.x^n$.

Q. $F_n = AF_{n-1} + B.F_{n-2} + f(n)$.

if. $x_1 = 2$; $x_2 = 5$.

$f(n)$	$a_n^{(P)}$
4 $\quad c.x^n \to 4.1^n$	A
5.2^n	$An.2^n$
8.5^n	$An.5^n$
4^n	$A.4^n$
$2n^2+3n+1$	An^2+Bn+C
cn	$cn+d$

eg. $F_n = 3F_{n-1} + 10F_{n-2} + 7.5^n$; $F_0 = 4$
$F_1 = 3$.

For homogenous soln ;

$x^n = 3x^{n-1} + 10.x^{n-2}$

$\Rightarrow x^2 = 3x + 10$

$\Rightarrow \boxed{x_1 = -2; x_2 = 5}$

for Homogeneous part ;

$a_n^{(h)} = A.(x_1)^n + B.(x_2)^n$

$\boxed{a_n^{(h)} = A(-2)^n + B(5)^n}$ ——— (1)

for Particular part ;

$f(n) = 7.5^n = c.x^n \Rightarrow \underline{x=5}$

$x = x_2$ & $x \neq x_1 \Rightarrow a_n^{(P)} = c.n(5)^n$.

$\Rightarrow cn(5)^n = 3[c(n-1).5^{n-1}] + 10[c(n-2)5^{n-2}] + 7.5^n$.

$\Rightarrow 25cn = 3[5c(n-1)] + 10[c(n-2)] + 7(25)$

$\Rightarrow 25\cancel{cn} = 15\cancel{cn} - 15c + 10\cancel{cn} - 20c + 175$.

$\Rightarrow 35c = 175 \Rightarrow c = \frac{175}{35} = 5$.

$\Rightarrow \boxed{c=5}$

$\boxed{a_n^{(P)} = 5n(5)^n}$ ——— (2)

So, the final solution is

$$a_n = a_n^{(h)} + a_n^{(p)}$$

$$a_n = A(-2)^n + B(5)^n + 5n.(5)^n.$$

Using IC. $f_0 = 4$; $f_1 = 3$.

$$4 = A(-2)^0 + B(5)^0 + 5(0)(5)^0$$

$$\Rightarrow \boxed{4 = A + B} \quad \text{—— (I)}$$

$$3 = A(-2)^1 + B(5)^1 + 5(1)(5)^1$$

$$\Rightarrow \quad 3 = -2A + 5B + 25$$

$$\Rightarrow \boxed{2A - 5B = 22} \quad \text{—— (II)}$$

(I) $\times 5$:- $20 = 5A + \cancel{5B}$.

(II) $22 = 2A - \cancel{5B}$

$$42 = 7A \qquad \boxed{A = 6}$$

$$\Rightarrow \boxed{B = -2}$$

$$\boxed{a_n = 6(-2)^n - 2(5)^n + 5n.(5)^n.}$$

$$\sum_{n=0}^{\infty} (n+1).a^n.x^n = \frac{1}{(1-ax)^2}$$

* Solve the following recurrence relating using generating function.

(a) $a_n - 6a_{n-1} + 9a_{n-2} = 0$; $n \geq 2$ $a_0 = 1$ $a_1 = 9$

$\Rightarrow$ Multiplying both sides by x^n. $g(x) = \sum_{n=0}^{\infty} a_n.x^n$.

$$a_n.x^n - 6a_{n-1}.x^n + 9.a_{n-2}.x^n = 0.$$

$$\sum_{n=2}^{\infty} a_n.x^n - 6\sum_{n=2}^{\infty} a_{n-1}.x^n + 9.\sum_{n=2}^{\infty} a_{n-2}.x^n = 0.$$

$$\sum_{n=0}^{\infty} a_n x^n - a_0 - a_1 x - 6x\left[\sum_{n=1}^{\infty} a_{n-1} x^{n-1} - a_0\right]$$

$$+ 9x^2 \sum_{n=2}^{\infty} a_{n-2}.x^{n-2} = 0.$$

$$\Rightarrow g(x) - 1 - 9x - 6x\, g(x) + 6x + 9x^2.g(x) = 0.$$

$$g(x)(9x^2 - 6x + 1) = 9x - 6x + 1 = 1 + 3x.$$

$$\Rightarrow g(x) = \frac{1+3x}{1-6x+9x^2} = \frac{1+3x}{(3x-1)^2}$$

$$= \frac{1}{(3x-1)^2} + \frac{3x}{(3x-1)^2}$$

* Some Useful expansions.

$$\sum_{n=0}^{\infty} (n+1) \cdot a^n \cdot x^n = \frac{1}{(1-ax)^2}.$$

$$\sum_{n=0}^{\infty} (ax)^n = \frac{1}{(1-ax)}$$

$$\sum_{n=0}^{\infty} (-1)^n \cdot x^n = \frac{1}{1+x}$$

$$g(x) = \frac{1}{(3x-1)^2} + 3x \cdot \frac{1}{(3x-1)^2}$$

$$= \sum_{n=0}^{\infty} (n+1) \cdot (3)^n \cdot x^n + (3x) \sum_{n=0}^{\infty} (n+1) \cdot 3^n \cdot x^n$$

$$\left[(n+1) \cdot 3^n\right] + \left[3 \cdot (n) \, 3^{n-1}\right]$$

{ Coeffi. of x^n in above expansion }

$$= (2n+1)3^n.$$

$$g(x) = \sum_{n=0}^{\infty} \left[(2n+1) \cdot 3^n\right] \cdot x^n = \sum_{n=0}^{\infty} a_n \cdot x^n.$$

$$\boxed{a_n = (2n+1) \cdot 3^n}$$

$$a_2 = 6a_1 - 9a_0 = 6(9) - 9(1) = 45.$$

② $a_n - 2a_{n-1} - 3a_{n-2} = 0$; $n \geq 2$; $a_0 = 3$; $a_1 = 1$.

$\Rightarrow$ $a_n \cdot x^n - 2a_{n-1} \cdot x^n - 3a_{n-2} \cdot x^n = 0$.

$$\sum_{n=2}^{\infty} a_n x^n - 2x \cdot \sum_{n=2}^{\infty} a_{n-1} \cdot x^{n-1} - 3x^2 \sum_{n=2}^{\infty} a_{n-2} \cdot x^{n-2} = 0.$$

$$\sum_{n=0}^{\infty} a_n \cdot x^n - a_0 - a_1 x - 2x \left[\sum_{n=1}^{\infty} a_{n-1} \cdot x^{n-1} - a_0 \right]$$

$$-3x^2 \left[\sum_{n=2}^{\infty} a_{n-2} \cdot x^{n-2} \right] = 0.$$

$$g(x) - 3 - x - 2x(g(x) - 3) - 3x^2 \cdot g(x) = 0.$$

$$g(x) - 3 - x - 2x g(x) + 6x - 3x^2 g(x) = 0.$$

$$g(x) - 2x \cdot g(x) - 3x^2 g(x) = 3 + 1x - 6x.$$

$$g(x)\left[1 - 2x - 3x^2\right] = 3 - 5x.$$

$$g(x) = \frac{3-5x}{1-2x-3x^2} = \frac{3-5x}{(1-3x)(1+x)}.$$

$\Rightarrow$ $\dfrac{A}{(1-3x)} + \dfrac{B}{(1+x)}$ $A = 1$; $B = 2$

$$\frac{1}{(1-3x)} + \frac{2}{(1+x)} = \sum_{n=0}^{\infty} 3^n \cdot x^n + 2 \sum_{n=0}^{\infty} (-1)^n \cdot x^n$$

$$= \sum_{n=0}^{\infty} \left[3^n + 2(-1)^n\right] \cdot x^n$$

$$\boxed{a_n = \left[3^n + 2(-1)^n\right]}$$

$a_2 = 2a_1 + 3a_0 = 11$ $a_2 = 3^2 + 2(-1)^2 = 11$

$\star$ (3) $a_n - 3a_{n-1} = n \; ; \; n \geq 1 \; ; \; a_0 = 1$

$$a_n x^n - 3a_{n-1} \cdot x^n = n \cdot x^n$$

$$\sum_{n=1}^{\infty} a_n x^n - 3x \sum_{n=1}^{\infty} a_{n-1} \cdot x^{n-1} = \sum_{n=1}^{\infty} n \cdot x^n$$

$$\sum_{n=0}^{\infty} a_n x^n - a_0 - 3x \sum_{n=1}^{\infty} a_{n-1} \cdot x^{n-1} = \frac{x}{(x-1)^2}$$

$$g(x) - 1 - 3x \cdot g(x) = \frac{x}{(x-1)^2}$$

$$g(x) = \frac{x + (x-1)^2}{(x-1)^2(1-3x)}$$

$$\frac{A}{(x-1)} + \frac{B}{(x-1)^2} + \frac{C}{(1-3x)} = \frac{x^2 - x + 1}{(x-1)^2(1-3x)}$$

$$A = \frac{+1}{4} \; ; \; B = \frac{-1}{2} \; ; \; C = \frac{7}{4}$$

$$g(x) = \frac{+1}{4} \cdot \frac{1}{(x-1)} + \frac{-1}{2} \cdot \frac{1}{(x-1)^2} + \frac{7}{4} \cdot \frac{1}{(1-3x)}$$

$(1-x)^{-1} = 1 + x + x^2 + x^3$ $\quad (1-x)^{-2} = 1 + 2x + 3x^2$

$$= \frac{-1}{4} \sum_{n=0}^{\infty} (1)^n x^n - \frac{1}{2} \sum_{n=0}^{\infty} (n+1)(1)^n \cdot x^n + \frac{7}{4} \cdot \sum_{n=0}^{\infty} (3)^n (x)^n$$

$$g(x) = \sum_{n=0}^{\infty} \left[\frac{-1}{4} - \left(\frac{n+1}{2}\right) + \frac{7}{4} \cdot 3^n \right] \cdot x^n$$

$$\boxed{a_n = \frac{7}{4} \cdot 3^n - \frac{1}{4} - \left(\frac{n+1}{2}\right)}$$

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$$A(x-1)(1-3x) + B(1-3x) + C(x-1)^2$$

$$= A(-3x^2+4x-1) + B(1-3x) + C(x^2-2x+1)$$

$$= x^2(-3A+C) + x(4A-3B-2C) + (-A+B+C).$$

$$= x^2 \cancel{+} - \cancel{2}x + 1$$

$-3A+C = 1$

$4A-3B-2C = -1$

$-A+B+C = \cancel{2}\ 1$

$A = 1/4$

$B = -1/2$

$C = 7/4.$