11/01

	Standard Basis	Dimension
$\mathbb{R}^2$	$\{e_1, e_2\}$ where e_i is the i^{th} col of I_2	2
$\mathbb{R}^n$	$\{e_1, e_2, \ldots, e_n\}$ where e_n is the i^{th} col of I_n Ex In $\mathbb{R}^4$ $S = \left\{ \begin{bmatrix}1\\0\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\0\\1\end{bmatrix} \right\}$ $\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix} = x_1e_1 + x_2e_2 + x_3e_3 + x_4e_4$ It is LI too because none is a multiple of other or a comb. of others.	n

NOTE: $S_1 = \left\{ \begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix} \right\}$ is a basis of $\mathbb{R}^2$.

$S_2 = \left\{ \begin{bmatrix}0\\1\end{bmatrix}, \begin{bmatrix}1\\0\end{bmatrix} \right\}$ is also a basis of $\mathbb{R}^2$

However S_1 & S_2 are considered to be two different basis. In case of

i) S_1 $\begin{bmatrix}x\\y\end{bmatrix} = x\begin{bmatrix}1\\0\end{bmatrix} + y\begin{bmatrix}0\\1\end{bmatrix}$ ii) S_2 : $\begin{bmatrix}x\\y\end{bmatrix} = y\begin{bmatrix}0\\1\end{bmatrix} + x\begin{bmatrix}1\\0\end{bmatrix}$

Coordinates changes.

$\therefore$ BASIS ARE ALWAYS ORDERED.

	Standard Basis	Dimension
$\mathbb{R}^{2\times2}$	$\mathbb{R}^{2\times2} = \left\{ \begin{bmatrix}a & b\\c & d\end{bmatrix} : a, b, c, d \in \mathbb{R} \right\}$ $S = \left\{ \begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}, \begin{bmatrix}0 & 1\\0 & 0\end{bmatrix}, \begin{bmatrix}0 & 0\\1 & 0\end{bmatrix}, \begin{bmatrix}0 & 0\\0 & 1\end{bmatrix} \right\}$	4
$\mathbb{R}^{m\times n}$		$m\times n$

NOTE: There is one-one onto correspondence between $\mathbb{R}^{2\times2}$ and $\mathbb{R}^4$

$\mathbb{R}^{2\times2} \xrightarrow[\text{onto}]{1-1} \mathbb{R}^4$ [linearly isomorphic]

$\begin{bmatrix}a & b\\c & d\end{bmatrix} \xrightarrow[\text{onto}]{1-1} \begin{bmatrix}a\\b\\c\\d\end{bmatrix}$

P_2	$P_2 = \{a + bx + cx^2 : a, b, c \in \mathbb{R}\}$ $S = \{1, x, x^2\}$ $\to$ Can generate any polynomial upto degree 2 S is LI because none is a multiple of other $x = \boxed{x}\,1$ is invalid because x is not a fixed real number. $(\alpha \notin \mathbb{R})$	

NOTE: Let $x, y \in V$ (VS)
x is a multiple of $y \Leftrightarrow x = \alpha y$ where $\alpha \in \mathbb{R}$

$S = \{1, x, x^2\}$ $\quad \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$

For eg: $1 + x^2$

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = 1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 0 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + 1 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \rightarrow (i)$$

So, a set such as
$S = \{1, x, x^2, 1 + x^2\}$ is LD because
$1 + x^2$ can be represented as a combination of $1, x, x^2$ by (i)

P_n	$\{1, x, x^2, \ldots, x^n\}$	$n+1$
$C[a,b]$ / P	$C[a,b]$ is infinite dimensional. P is also infinite dimensional space. PROOF: Let us assume for some finite n. $S = \{1, x, x^2, \ldots, x^n\}$ is the basis Now we know $x^{n+1} + 1 \in P$ But $x^{n+1} + 1$ cannot be generated by S inspite of S being assumed as basis (minimal generator.) $\therefore$ our assumption is wrong. $C[a,b] = \{f : [a,b] \rightarrow \mathbb{R}$ and f is continuos$\}$ It is also infinite dimensional PROOF: Let $S = \{1, x, x^2 \ldots x^n\}$ be a set in $C[a,b]$ Then S is LI. But $f(x) = x^{n+1}$ is a function in $C[a,b]$ s.t $x^{n+1} \notin L[S]$.	∞

TUTORIAL 2:

1) (iv) $\alpha \otimes u = 0 \Rightarrow \alpha = 0$ or $u = 0$

If $\alpha \neq 0 \Rightarrow \alpha^{-1}$ exist

$u = 1 \circledast u = (\alpha^{-1}\alpha) \circledast u = \alpha^{-1} \otimes (\alpha \otimes u)$

$= \alpha^{-1} \otimes 0$

(i.e) $u = 0$

If $\alpha = 0$ then the proof is done in 1.

(6) 9) $(f+g)x = f(x) + g(x)$

$(\alpha f)(x) = \alpha . f(x)$ $\qquad f : [0,1] \to \mathbb{R}$

$S = \{ f \in V : f(1/2) = 0 \}$

Step(i) $0 \in S$ because $0(1/2) = 0$

$S \neq \Phi$ & $0 \in S$

Step(ii) Let $f, g \in S$ f & g are constant

$f(1/2) = 0$ and $g(1/2) = 0$

$f+g$ also continuos

$(f+g) 1/2 = f(1/2) + g(1/2) = 0 + 0$

$\Rightarrow f+g \in S$

Let $\alpha \in \mathbb{R}$ and $f \in S \Rightarrow f(1/2) = 0$

$(\alpha f)(1/2) = \alpha f(1/2) = \alpha . 0 = 0.$

(14) $M \subseteq N$

$\Rightarrow L[M] \subseteq L[N]$

$x \in L[M]$

$\Rightarrow \exists\, x_1, x_2, \ldots x_n \in M$ such that $x = \alpha_1 x_1 + \alpha_2 x_2 + \ldots \alpha_n x_n$

and $\alpha_1, \alpha_2, \alpha_3 \ldots \alpha_n \in F$ *

$\Rightarrow$ as $M \subset N \Rightarrow x_1, x_2, x_3 \ldots x_n \in N$

$\Rightarrow x \in L[N]$ (From *)

ii) M is a subspace of V $\Leftrightarrow$

$M \subseteq L[M]$

$x \in L[M]$

$\exists\, x_i \in M$ and $\alpha_i \in F$ $i = 1, 2, \ldots n$

$$x = \sum_{i=1}^{n} \alpha_i x_i$$

$\Rightarrow x \in M$

$\Rightarrow L[M] \subseteq M \Rightarrow L[x] = M$

M is a subspace.

iii) $L[M] = M \Rightarrow$ M is a subspace

~~$L[M] = M$~~

i) $x_1, x_2 \in M \Rightarrow x_1 + x_2 \in L[M] = M$

$x_1 + x_2 \in M$

ii) and let $\alpha \in F$ $x_i \in M$

$\Rightarrow \alpha x \in L[M] = M$

$\therefore \alpha x \in M$

M is a subspace.

$M \subseteq V$

$N = L[M]$ is a subspace of V

$L[N] = N$

$L[L[M]] = L[M]$

3) $\begin{bmatrix} 1+i \\ 1-i \end{bmatrix} \in L\left[\begin{bmatrix} 1+i \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1-i \end{bmatrix}\right]$

$$\begin{bmatrix} 1+i \\ 1-i \end{bmatrix} = c_1 \begin{bmatrix} 1+i \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 1-i \end{bmatrix}$$

$c_1, c_2 \in \mathbb{C}$

$c_1 = a+ib$

$c_2 = c+id$

$$\begin{bmatrix} 1+i \\ 1-i \end{bmatrix} = (a+ib)\begin{bmatrix} 1+i \\ 1 \end{bmatrix} + (c+id)\begin{bmatrix} 1 \\ 1-i \end{bmatrix}$$

$$\begin{bmatrix} 1+i \\ 1-i \end{bmatrix} = \begin{bmatrix} a-b+c+i(b+a+d) \\ a+c+d+i(b-c+d) \end{bmatrix}$$

$a-b+c=1$

$a+b+d=1$

$a+c+d=1$

$b-c+d=-1$

$$\left[\begin{array}{cccc|c} 1 & -1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & -1 & 1 & -1 \end{array}\right]$$

) $W_1 \cap W_2 = \{0\}$ $\quad W_1 \oplus W_2 = V$

$u \in V$

$u = u_1 + u_2$ $\qquad u = v_1 + v_2$

$u_1 \in W_1,\ u_2 \in W_2$ $\qquad v_1 \in W_1 \quad v_2 \in W_2$

$u_1 + u_2 = v_1 + v_2$

$\underbrace{u_1 - v_1}_{\in W_1} = \underbrace{v_2 - u_2}_{\in W_2}.$

$u_1 - v_1 \in W_1 \cap W_2 = \{0\}$

$u_1 = v_1$

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = I_2$$

$S = \{A : AT = TA\}$ $\qquad (\alpha A)T = T(\alpha A) = ?$

$0 \in S$

$(A+B)T = T(B+A)$?

04/02

Any matrix $A \in \mathbb{R}^{m \times n} / \mathbb{C}^{m \times n}$ can be reduced to the following form by using Type I, Type II and type III elementary operations.

① The ~~points~~ pivots are the first non-zero entries in non-zero rows

② Below each ~~point~~ pivot, all column entries are zero

③ Each pivot lies to the right of the pivot in the row above

④ All zeroes are below non-zero rows

⑤ All pivots are 1.

⑥ Above each ~~point~~ pivot all column entries are zero.

(1–4) – Row echelon form

(1–6) – Row reduced echelon form

Example. (No perfect diagonal)

$$A = \begin{bmatrix} \boxed{1} & 3 & 3 & 2 \\ 2 & 6 & 9 & 7 \\ -1 & -3 & 3 & 4 \end{bmatrix} \sim \begin{bmatrix} \boxed{1} & 3 & 3 & 2 \\ 0 & 0 & \boxed{3} & 3 \\ 0 & 0 & 6 & 6 \end{bmatrix}$$

0 1 ? 3 – If this row existed then we could have swapped with row 2.

Make 3 pivot (2nd row 3rd col)

$$\sim \begin{bmatrix} \boxed{1} & 3 & 3 & 2 \\ 0 & 0 & \boxed{3} & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \text{REF}(A)$$

A form of matrix that satisfies ①–④.

Multiply R_2 by $1/3$

$R_1 = R_1 - 3R_2$

$$\sim \begin{bmatrix} \boxed{1} & 3 & 3 & 2 \\ 0 & 0 & \boxed{1} & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} \boxed{1} & 3 & 0 & -1 \\ 0 & 0 & \boxed{1} & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \text{RREF}(A)$$

The RREF of an invertible matrix is the identity matrix ⇒ which is why we use it in Gauss Jordan

Rank of a matrix : A number r is the rank of $A \in \mathbb{R}^{m\times n} / \mathbb{C}^{m\times n}$ if

① There is atleast one square matrix of order r whose determinant is non-zero.

② If A contains any submatrix of order (r+1) then its det is zero.

For example

REF/doesn't change the determinant
RREF

$$A = \begin{bmatrix} 1 & 3 & 3 & 2 \\ 2 & 6 & 9 & 7 \\ -1 & -3 & 3 & 7 \end{bmatrix} \sim \begin{bmatrix} \boxed{1} & 3 & 3 & 2 \\ 0 & 0 & \boxed{3} & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

→ Det of all 3x3 submatrix is zero
→ Det of all 2x2 submatrix is not zero.

Rank = No. of pivots (non-zero rows in REF/RREF)

Matrix can be seen as a function

$y = f(x)$ $\quad A_{m\times n} : \mathbb{R}^n \mapsto \mathbb{R}^m$

$b = \underset{m\times n}{A}x$

Four fundamental subspaces:

① Column space : = span of columns of A

Column space is a subspace of codomain

↳ L(S) of colums of A in $\mathbb{R}^3$.

Ex: $\begin{bmatrix} \boxed{\begin{matrix}1\\2\\-1\end{matrix}} & \begin{matrix}3\\6\\-3\end{matrix} & \begin{matrix}3\\9\\3\end{matrix} & \begin{matrix}2\\7\\7\end{matrix} \end{bmatrix}_{3\times4}$ $\quad \mathbb{R}^4 \to \mathbb{R}^3$

belongsto

② Row space : Span of rows of A

Row space is a part of Domain

Span of RREF/REF or the original matrix is the same

A basis of Row space [A] = $\left\{ \begin{bmatrix} 1 \\ 3 \\ 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 3 \\ 3 \end{bmatrix} \right\}$ ↗ Non zeros rows in REF/RREF

↳ LI

$= \left\{ \begin{bmatrix} 1 \\ 3 \\ 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 6 \\ 9 \\ 7 \end{bmatrix} \right\}$ → Corresponding pivot rows from original matrix

$= \left\{ \begin{bmatrix} 1 \\ 3 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} \right\}$ → corresponding pivot rows from RREF.

This is the better one because it has many zeroes.

* No. of pivot rows / pivots in REF/RREF = Dimension of rowspace = rank of the matrix.

Basis of colspace:

Columns corresponding to pivots in ~~...~~ REF/RREF ~~..~~ to the original matrix only.

$$\text{Colspace of } A = \left\{ \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 3 \\ 9 \\ 3 \end{bmatrix} \right\}$$

05/02

$$\left[\begin{array}{cccc|ccc} \boxed{1} & 3 & 3 & 2 & 1 & 0 & 0 \\ 2 & 6 & 9 & 7 & 0 & 1 & 0 \\ -1 & -3 & 3 & 4 & 0 & 0 & 1 \end{array}\right] \sim \underset{B \qquad\qquad E_1}{\left[\begin{array}{cccc|ccc} \boxed{1} & 3 & 3 & 2 & 1 & 0 & 0 \\ 0 & 0 & \boxed{3} & 3 & -2 & 1 & 0 \\ 0 & 0 & 6 & 6 & 1 & 0 & 1 \end{array}\right]}$$

$$\sim \underset{REF(A) \qquad\qquad E_2}{\left[\begin{array}{cccc|ccc} \boxed{1} & 3 & 3 & 2 & 1 & 0 & 0 \\ 0 & 0 & \boxed{3} & 3 & -2 & 1 & 0 \\ 6 & 0 & 0 & 0 & 5 & -2 & 1 \end{array}\right]} \sim \underset{S}{\begin{bmatrix} \boxed{1} & 3 & 3 & 2 \\ 0 & 0 & \boxed{1} & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}}$$

$$\begin{bmatrix} \boxed{1} & 3 & 0 & -1 \\ 0 & 0 & \boxed{1} & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Four fundamentals subspace related with $A \in \mathbb{R}^{m \times n}$ (contd.)

Reviewing ① & ②

① $A_{m \times n} : \mathbb{R}^n \to \mathbb{R}^m$

NOTE: Row opr. do not change row However they change col space.

① colspace of A: $col(A) \stackrel{Def}{=}$ span of cols of A

Finding basis: Colums in original A corresponding to pivot cols in REF/RREF.

② $Col(A) = \left\{ \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} \begin{bmatrix} 3 \\ 9 \\ 3 \end{bmatrix} \right\}$

② Row space of A: $row(A) \stackrel{Def}{=}$ span of rows of A

Pivot rows in REF/RREF/ Original matrix

$$row(A) = \left\{ \begin{bmatrix} 1 \\ 3 \\ 0 \\ -1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} \right\}$$

③ Null space: $null(A) \stackrel{def}{=} \{x : Ax = 0\}$

$Ax = 0$

$1x_1 + 3x_2 + 3x_3 + 2x_4 = 0$

$2x_1 + 6x_2 + 9x_3 + 7x_4 = 0$

$-x_1 - 3x_2 + 3x_3 + 4x_4 = 0.$

$Ax = b$
If $b = 0$
Homogeneous system
else
non homogeneous system.

Solution to these three eqns forms the null space.

Solution of three equations is same as solving equs in RREF:

$$\equiv \quad x_1 + 3x_2 - x_4 = 0$$
$$x_3 + x_4 = 0.$$

$$\begin{bmatrix} \boxed{1} & 3 & 0 & -1 \\ 0 & 0 & \boxed{1} & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

PIVOT VARIABLES

FREE VARIABLES

$$\equiv \quad \boxed{\begin{aligned} x_1 &= -3x_2 + x_4 \\ x_3 &= -x_4 \end{aligned}}$$

Pivot variables   Free variables.

* We can choose free variables and find out pivot variables

For eg $x_2 = 1 \quad x_4 = 0$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

$x_2 = 0 \quad x_4 = 1$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}$$

RREF : $\begin{bmatrix} 1 & 3 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Now

$$\begin{bmatrix} \_ \\ 1 \\ \_ \\ 0 \end{bmatrix}$$

since x_3 & x_4 are dependent on x_2 go to col 2 of RREF.

Without changing the order take the negative signs of non-zero row elements and put it in the blank.

$$\begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} \Longleftarrow$$

Similarly if

$$\begin{bmatrix} \_ \\ 0 \\ \_ \\ 1 \end{bmatrix} \rightarrow \text{go to } R_4 \text{ in RREF Repeat steps.} \rightarrow \begin{bmatrix} +1 \\ 0 \\ -1 \\ 1 \end{bmatrix}$$

Now $\begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} +1 \\ 0 \\ -1 \\ 1 \end{bmatrix} \longrightarrow$ BASIS OF NULL SPACE

If $x_2 = \alpha$
$x_4 = \beta$

$$\begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \alpha \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \beta \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \alpha \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + \beta \begin{bmatrix} +1 \\ 0 \\ -1 \\ 1 \end{bmatrix}$$

TO PROVE NULL SPACE IS A SUBSPACE

$\forall \alpha \in \mathbb{R},\ x, y \in S \quad \alpha x + y \in S$

$S = \{x \in \mathbb{R}^n : Ax = 0\}$ (NULL SPACE DEFINITION)

$A(\alpha x + y) = \alpha Ax + Ay$

$= A0 + 0$

$= 0$

$\therefore$ Null space is a ~~sq~~ subspace.

$\therefore$ Null space is a part of domain ($\mathbb{R}^4$) (in this case).

NOTE: Null space is a part of domain

$\begin{pmatrix} \text{Vector from} \\ \text{rowspace} \end{pmatrix} \cdot \begin{pmatrix} \text{Vect from} \\ \text{nullspace} \end{pmatrix} = 0.$

$A_{m \times n} : \mathbb{R}^n \to \mathbb{R}^m$

row(A)

null(A)

rows space $\perp$ nullspace

$$\boxed{\text{If } S_1 \perp S_2 \Rightarrow \{\underbrace{\langle x, y \rangle}_{\text{Dotproduct}} = 0 \ \ \forall x \in S_1 \ \&\ y \in S_2\}}$$

④ Left Null space of A : null(A^T)

$Null(A^T) = \{x : A^T x = 0\}$

$= \{x : x^T A = 0\}$

Null(A^T) = Rows in augmented matrix corresponding to zero rows in A.

Repf

$E_1 A = B$

$E_2 A = REF(A)$

$Null(A^T) = \left\{ \begin{bmatrix} 5 \\ -2 \\ 1 \end{bmatrix} \right\} = [R]_{E_2}.$

$A_{m \times n} : \mathbb{R}^n \to \mathbb{R}^m$

row(A) col(A)

null(A) Null(A^T).

$r = \text{rank}(A) = \text{rank}(A^T) \rightarrow$ PROOF $= \dim \text{col}(A)$ } PROOF

$\parallel$

\# pivots in REF(A) / RREF(A)

$\Updownarrow$

dimension of row space of A $\leftarrow$ which is equal

Definition: dim (Null(A)) is called nullity of A.

Definition: dim (row (A)) / or dim (col(A)) is calld rank of A.

$\Leftrightarrow$ LI cols in A

$\Leftrightarrow$ LI rows in A $\rightarrow$ \# LI rows in A / \# LI col in A is called rank of A

BIG THEORAM:

For any $A \in \mathbb{R}^{m \times n}$ (Rank nullity theoram)

rank + nullity $= n =$ dimension of domain

No zero row in REF $\Rightarrow$ Then left null space does not exist. So dim of null space $= 0$ basis $= \Phi$ $L[\{\Phi\}] = \{0\}$

$A_{m\times n}$ $\boxed{Ax = b}$ linear system of m equations in n unknowns

Solution of 1 exists $\Leftrightarrow b \in \text{col}(A)$

In col space (A) why the remaing rows are LD:

$$\begin{bmatrix} 1 & 3 & 3 & 2 \\ 2 & 6 & 9 & 7 \\ -1 & -3 & 3 & 4 \end{bmatrix} \quad \alpha_1 \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} + \alpha_2 \begin{bmatrix} 3 \\ 9 \\ 3 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \\ -3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 3 \\ 2 & 9 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \\ -3 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} 1 & 3 & 3 \\ 2 & 9 & 6 \\ -1 & 3 & -3 \end{array}\right] \sim \begin{bmatrix} \boxed{1} & 3 & 3 \\ 0 & \boxed{3} & 0 \\ 0 & 0 & 0 \end{bmatrix} \sim \left[\begin{array}{cc|c} \boxed{1} & 3 & 3 \\ 0 & \boxed{3} & 0 \\ 0 & 0 & 0 \end{array}\right]$$

This cannot be nonzero.

1) Solution of ① exists $\Leftrightarrow$ $b \in \text{col} A$ $\Leftrightarrow$ Rank[A] = Rank[A|b]

2) No solution to ① $\Leftrightarrow$ $b \notin \text{col} A$ $\Leftrightarrow$ Rank(A) $\neq$ Rank[A|b]

3) Solution is unique if exist $\Leftrightarrow$ $\text{Null}(A) = \{0\}$ $\Leftrightarrow$ $\text{nullity}(A) = 0$ $\rightarrow$ ⓐ

(Thinking of situation where more than one solution exists.

if $\text{Null}(A) \neq \{0\}$ then there exists $n_1 \in \text{Null}(A)$.

So, let X_1 be a solution then there exist

$$X_2 = X_1 + n_1$$
$$AX_2 = Ax_1 + \cancel{An_1}^{\,0}$$
$$= Ax_1$$
$$= b$$
)

4) From ⓐ we can also conclude that (From big theoram)

rank = n = dimension of domain.

5) Solution of ① exists uniquely $\Leftrightarrow$ Rank(A) = Rank(A|b) = n.

Let $\{x_1, x_2, \ldots x_n\}$ be a given in VS V.

Ques ①:

Check. S is LI/LD

② Check $y \in L(S)$

③ Check S is a basis

Make

$$A = \begin{bmatrix} x_1 & x_2 & x_3 & x \cdots & x_n \\ \downarrow & \downarrow & \downarrow & & \downarrow \end{bmatrix}_{\times n}$$

If Rank(A) = n then

Let $S = \{x_1, x_2, \dots x_n\}$ be a set of vectors in

Ques ① Is S LI/LD ?

If $\text{Rank}(A) = n$ then S is LI

IMPORTANT CONSTRUCTION:

Make a matrix

$$A = \left[\begin{array}{c|c|c|c} x_1 & x_2 & x_3 \cdots & x_n \\ \downarrow & \downarrow & \downarrow & \downarrow \end{array}\right]$$

Ex: Q) Whether S is LI/LD?

$$A = \begin{bmatrix} 1 & 3 & 1 & 1 \\ 1 & 2 & 1 & 2 \\ 1 & 2 & 3 & 6 \\ 0 & 1 & -2 & -5 \end{bmatrix} \cong \begin{bmatrix} \boxed{1} & 3 & 1 & 1 \\ 0 & \boxed{-1} & 0 & 1 \\ 0 & -1 & 2 & 5 \\ 0 & 1 & -2 & -5 \end{bmatrix}$$

$$\cong \begin{bmatrix} \boxed{1} & 3 & 1 & 1 \\ 0 & \boxed{-1} & 0 & 1 \\ 0 & 0 & \boxed{2} & 4 \\ 0 & 0 & -2 & -4 \end{bmatrix} \cong \begin{bmatrix} \boxed{1} & 3 & 1 & 1 \\ 0 & \boxed{-1} & 0 & 1 \\ 0 & 0 & \boxed{2} & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\boxed{\text{Rank} = 3}$

These 4 vectors are linearly dependent as $\text{Rank} \neq n$.

Ex2:

$\{x, x^3 - x, x^4 + x^2, x + x^2 + x^4 + 1/2\}$ in $P4$

$$\begin{bmatrix} 0 & 0 & 0 & 1/2 \\ 1 & -1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

Rank $= \cancel{5}\ \boxed{4}$

Rank $= n$

So, LI

$S = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 2 \\ 0 & 2 \end{bmatrix}$ in $\mathbb{R}^{2\times 2}$

Dimension of $L\{S\} = 2$

$$\begin{bmatrix} 1 & 1 & 2 \\ 1 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 1 & 2 \end{bmatrix}$$

$\text{Rank}\{A_{m\times n}\} \leq \min\{m \times n\}$

$\underbrace{\{e^x, e^{x+1}\}}_{LD}$ $C(\mathbb{R})$ $C: \mathbb{R} \to \mathbb{R}$.

$e^{x+1} = e \cdot e^x$

<u>Wronskian</u>: $W(f_1, f_2) = \begin{vmatrix} f_1(x) & f_2(x) \\ f_1'(x) & f_2'(x) \end{vmatrix}$

$$W(f_1, f_2 \dots f_n) = \begin{vmatrix} f_1 & f_2 & \dots & f_n \\ f_1' & f_2' & \dots & f_n' \\ \vdots & & & \\ f_1^{(n-1)'} & f_2^{(n-1)} & & f_n^{(n-1)'} \end{vmatrix} \neq 0 \text{ at atleast one pt } x_0 \in [a,b]$$

$\Downarrow$

functions are LI

$$\begin{vmatrix} e^x & e^{2x} \\ e^x & 2e^{2x} \end{vmatrix} = \underset{\text{non zero}}{\underset{\downarrow}{e^{3x}}}$$

$\Rightarrow$ LI.

$\{\sin x \quad \sin 2x \quad \dots\dots \quad \sin nx\}$

$c_1 \sin x + c_2 \sin 2x + \dots + c_n \sin nx = 0$

$$\int_{-\pi}^{\pi} \sin mx \sin nx \, dx = 0 \quad m \neq n$$

All $c_1, c_2 \dots c_n = 0$.

NOTE

$\text{Rank}\{A_{m \times n}\} \leq \min\{m, n\}$.

Is $y \in L(S)$

$\equiv$ Find $\alpha_1, \alpha_2, \ldots, \alpha_n$ s.t $\sum_{i=1}^{n} \alpha_i x_i = y$

$\equiv$ Solve $A\alpha = y$.

If $S = \begin{bmatrix} 1 & 1 & 3 \\ 2 & 1 & 5 \\ 3 & -1 & 5 \end{bmatrix}$ Is $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ a part of $L(S)$ ~~of S~~?

$$\left[\begin{array}{ccc|c} 1 & 1 & 3 & 1 \\ 2 & 1 & 5 & 1 \\ 3 & -1 & 5 & 0 \end{array}\right] = \left[\begin{array}{ccc|c} 1 & 1 & 3 & 1 \\ 0 & -1 & -1 & -1 \\ 0 & -4 & -4 & -3 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} \boxed{0} & 1 & 3 & 1 \\ 0 & \boxed{-1} & -1 & -1 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

Not solvable

so $y \notin$ colspace (A)

$y \notin L(S)$

If $\begin{bmatrix} 2 \\ 3 \\ 2 \end{bmatrix}$