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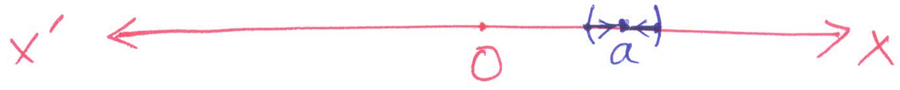
Rough

For real valued functions on $\mathbb{R}$:

$y = f(x)$, $x \in \mathbb{R}$

Let $a \in \mathbb{R}$



$(a-\epsilon, a+\epsilon)$

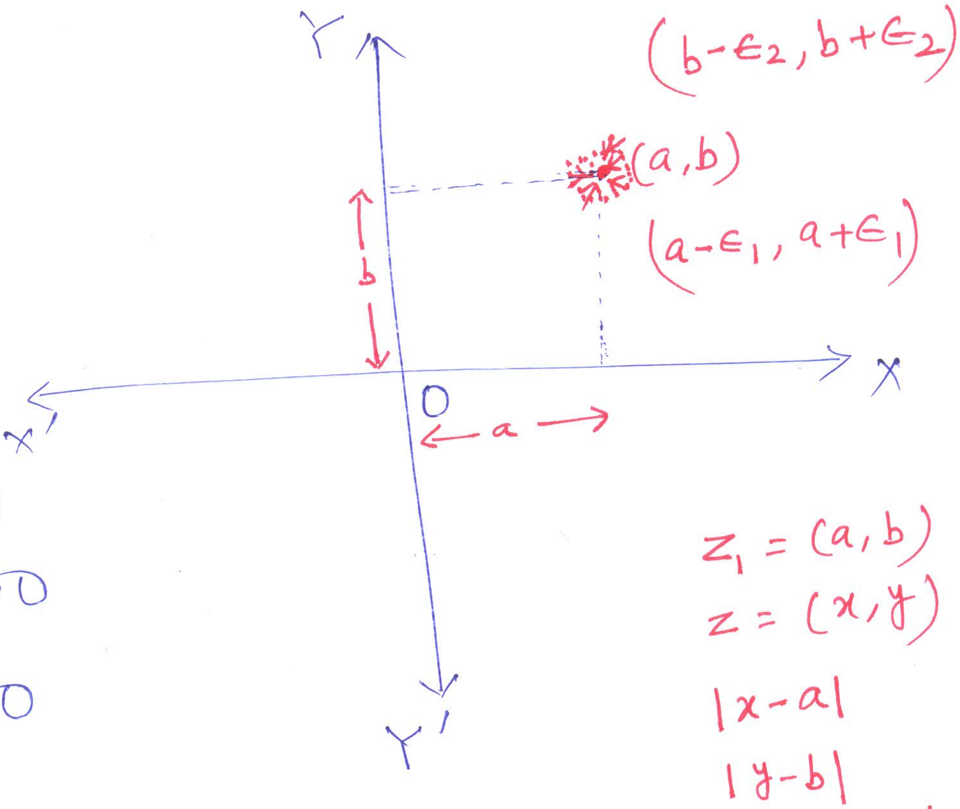
$[a-\epsilon, a+\epsilon]$

L.H. limit

R.H. limit

LHD

RHD



$|x-a| = \delta(a) \to 0$

$|y-b| = \delta(b) \to 0$

or $\delta a, \delta b \to 0$

$\delta z_1 = \delta a + i\delta b$

$\delta z = \delta x + i\delta y$.

$z_1 = (a,b)$

$z = (x,y)$

$|x-a|$

$|y-b|$

$|z-z_1| < \epsilon$ & $|z-z_1| \le \epsilon$

Ultimately represents a circle.

Neighbourhood of z_1 can consider as a circle or a rectangular region or a square.

P-24 Rough

Taylor's Theorem:

Let $f(x)$ be a function of x which can be expanded in powers of x and let the expansion be differentiable term by term any number of times. Then

$$f(a+h) = f(a) + h f'(a) + \frac{h^2}{\lfloor 2} f''(a) + \cdots + \frac{h^n}{\lfloor n} f^n(a) + \cdots\cdots$$

[Taylor's Series]

Theorem: If a function $f(x)$ possesses derivative of all orders in the interval $]a, a+h[$, then for every n, however large, there corresponds a Taylor's expansion with Lagrange's form of remainder, namely

$$f(a+h) = f(a) + h f'(a) + \frac{h^2}{\lfloor 2} f''(a) + \cdots + \frac{h^{n-1}}{\lfloor n-1} f^{n-1}(a) + R_n$$

where $R_n = \frac{h^n}{\lfloor n} f^n(a+\theta h)$, $0<\theta<1$.

Let us suppose $R_n \to 0$ as $n \to \infty$, then

$$f(a+h) = f(a) + h f'(a) + \frac{h^2}{\lfloor 2} f''(a) + \cdots + \frac{h^n}{\lfloor n} f^n(a) + \cdots$$

Taylor's theorem for functions of two variables:

Let $f(x,y)$ be a function of two variables x and y defined on $D \subseteq \mathbb{R}^2$. If $f(x,y)$ has continuous partial derivatives of nth order in some neighbourhood of the point $(a,b) \in D$ and if $(a+h, b+k)$ is any point of this neighbourhood, then there exists some θ in $(0,1)$ such that

$$f(a+h, b+k) = f(a,b) + \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right) f(a,b) + \frac{1}{\lfloor 2}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^2 f(a,b)$$
$$+ \cdots + \frac{1}{\lfloor n-1}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^{n-1} f(a,b) + \frac{1}{\lfloor n}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^n f(a+\theta h, b+\theta k).$$

Functions of a Complex Variable

A curve is a particular kind of geometrical configuration. For complex function theory, it is important to consider a curve as having an addition structure, viz. a specific parametric representation.

$$x = x(t), \quad \text{and} \quad y = y(t) \text{ in a plane.}$$

$$z = z(t) = x(t) + i\,y(t), \quad \text{where } z = x + iy.$$

If for each value of the complex variable $z = x + iy$ in a given region D, we have one or more values of $w = u + iv$, then w is said to be a function of z and we write

$$w = u(x,y) + i\,v(x,y) = f(z)$$

where u, v are real valued functions of x and y.

Partial Derivatives: (Recalled):

$z = f(x,y)$ defined on $D \subseteq \mathbb{R}^2$

We have a point $(a,b) \in D$, then

Limits:

$$\lim_{(x,y) \to (a,b)} f(x,y) = \lim_{\substack{x \to a \\ y \to b}} f(x,y)$$

$$\lim_{x \to a} \lim_{y \to b} f(x,y)$$

$$\lim_{y \to b} \lim_{x \to a} f(x,y)$$

$$\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)$$

(a,b), a neighbouring point $(a+h, b+K)$, then

$$\left.\frac{\partial f(x,y)}{\partial x}\right)_{(a,b)} = \lim_{h\to 0} \frac{f(a+h,b)-f(a,b)}{h}$$

$$\left.\frac{\partial f(x,y)}{\partial y}\right)_{(a,b)} = \lim_{K\to 0} \frac{f(a,b+K)-f(a,b)}{K}.$$

Homogeneous functions:

$$a_0 x^n + a_1 x^{n-1} y + a_2 x^{n-2} y^2 + \cdots + a_n y^n.$$

Euler's Theorem on homogeneous functions:

If $u = f(x,y)$ is a homogeneous function of order n in x and y, then $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu.$

Total Derivative: $u = f(x,y)$, where $x = \phi(t)$ and $y = \psi(t)$.

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial u}{\partial y}\cdot\frac{dy}{dt}$$

Implicit function: $f(x,y) = c$.

$$0 = \frac{df}{dx} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\cdot\frac{dy}{dx}$$

$$\therefore \frac{dy}{dx} = -\frac{\partial f}{\partial x}\Big/\frac{\partial f}{\partial y}.$$

Derivative of f(z) :-

Let $w = f(z)$ be a single-valued function of the variable $z = x+iy$. Then the derivative of $w = f(z)$ is defined to be

$$\frac{dw}{dz} = f'(z) = \lim_{\delta z \to 0} \frac{f(z+\delta z) - f(z)}{\delta z}$$

provided the limit exists and has the same value for all the different ways in which δz approaches zero.

Analytic or Holomorphic or Regular Function :-

A single valued function is said to be analytic at a point if it is differentiable everywhere in some neighbourhood of the point.

A function $f(z)$ which is single-valued and possesses a unique derivative with respect to z at all points of a region D, is called an analytic or a regular function of z in that region.

A point at which an analytic function ceases to possess a derivative is called a singular point of the function.

or,

If $f(z)$ is not analytic at a point z_0, then z_0 is called the singular point of $f(z)$.

The real & imaginary parts of an analytic function are called conjugate functions. The relation between two conjugate functions is given by C-R equations.

Th :- The necessary and sufficient conditions for the derivative of the function $w = u(x,y) + iv(x,y) = f(z)$ to exist for all values of z in a region D, are

(i) $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ are continuous functions of x and y in D

(ii) $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$.

The relation (ii) are known as Cauchy-Riemann equations or briefly C-R equations.

Proof :- (a) Condition is Necessary :-

Let δu and δv be the increments of u and v respectively corresponding to the increment δx and δy of x and y, so that $\delta z = \delta x + i\delta y$.

If $f(z)$ possesses a unique derivative at $P(z)$, then

$$f'(z) = \lim_{\delta z \to 0} \frac{f(z+\delta z) - f(z)}{\delta z}$$

$$= \lim_{\delta z \to 0} \frac{(u+\delta u) + i(v+\delta v) - (u+iv)}{\delta z}$$

$$= \lim_{\delta z \to 0} \left(\frac{\delta u}{\delta z} + i\frac{\delta v}{\delta z} \right)$$

Since δz can approach zero in any manner, we can first assume δz to be wholly real and then wholly imaginary.

When δz is wholly real, then, $\delta y = 0$ and $\delta z = \delta x$

$$\therefore f'(z) = \lim_{\delta x \to 0} \left(\frac{\delta u}{\delta x} + i\frac{\delta v}{\delta x} \right)$$

$$= \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} \longrightarrow (1)$$

When δz is wholly imaginary, then $\delta x = 0$ and $\delta z = i\delta y$

$$\therefore f'(z) = \lim_{\delta y \to 0} \left(\frac{\delta u}{i\delta y} + i\frac{\delta v}{i\delta y} \right) = \frac{1}{i}\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$$

$$= \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y} \longrightarrow (2)$$

Now the existence of $f'(z)$ requires the equality of (1) and (2).

$$\therefore \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}$$

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \longrightarrow (3)$$

Thus the necessary conditions for the existence of the derivative of $f(z)$ is that the C-R equations should be satisfied.

(b) Condition is sufficient :-

Suppose $f(z)$ is a single valued function possessing partial derivatives $\frac{\partial u}{\partial x}$, u_y; v_x, v_y at each point of the region & that the C-R equations are satisfied.

Then by Taylor's theorem for functions of two variables

$$f(z+\delta z) = u(x+\delta x, y+\delta y) + iv(x+\delta x, y+\delta y)$$

$$= u(x,y) + \left(\frac{\partial u}{\partial x}\delta x + \frac{\partial u}{\partial y}\delta y\right) + \cdots$$

$$+ i\left[v(x,y) + \left(\frac{\partial v}{\partial x}\delta x + \frac{\partial v}{\partial y}\delta y\right) + \cdots\right]$$

$\therefore\ f(z+\delta z) = f(z) + \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)\delta x + \left(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right)\delta y$

[Omitting terms beyond the first powers of δx & δy]

or, $f(z+\delta z) - f(z) = \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)\delta x + \left(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right)\delta y$

Now using the C-R equations, replace $\frac{\partial u}{\partial y}$ and $\frac{\partial v}{\partial y}$ by $-\frac{\partial v}{\partial x}$ and $\frac{\partial u}{\partial x}$ respectively.

Then
$$f(z+\delta z) - f(z) = \left[\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right]\delta x + \left[-\frac{\partial v}{\partial x} + i\frac{\partial u}{\partial x}\right]\delta y$$
$$= \left[\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right]\delta x + \left[i\frac{\partial v}{\partial x} + \frac{\partial u}{\partial x}\right]i\delta y$$
$$= \left[\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right](\delta x + i\delta y)$$
$$= \left[\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right]\delta z$$

$$\therefore\ f'(z) = \lim_{\delta z \to 0}\frac{f(z+\delta z) - f(z)}{\delta z} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} \text{ or } \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}$$

which by (1) or (2) proves the sufficiency of conditions.

Note :- 1. Thus, if u and v are real single-valued functions of x and y such that $\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial y}$, $\frac{\partial v}{\partial x}$, $\frac{\partial v}{\partial y}$ are continuous throughout a region R, then the Cauchy-Riemann equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \longrightarrow (1)$$

are both necessary and sufficient conditions for the function $f(z) = u + iv$ to be analytic in R. The derivative of $f(z)$ is then, given by (1) or (2).

2. This theorem also restated as

The necessary Conditions for $f(z)$ to be analytic :-

St :- The necessary conditions that a function $f(z) = u(x,y) + iv(x,y)$ be analytic in a domain D is that in D, u and v satisfy the Cauchy-Riemann equations i.e. $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$; $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$.

Sufficient Conditions for $f(z)$ to be analytic :-

St :- Let $f(z) = u(x,y) + iv(x,y)$ be defined in a domain D. Then $f(z)$ is analytic in D if $u(x,y)$ and $v(x,y)$ have continuous partial derivative and satisfy the Cauchy Riemann equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

at all points of the domain D.

Remarks :- In view of Cauchy-Riemann equations the derived function $f'(z)$ can be expressed more concisely as

$$f'(z) = \frac{\partial}{\partial x}(u+iv) = -i\frac{\partial}{\partial y}(u+iv)$$
$$\Rightarrow f'(z) = \frac{\partial f}{\partial x} = -i\frac{\partial f}{\partial y}.$$

Polar form of Cauchy - Riemann equations

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}, \quad \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}.$$

Derivation: We have

$x = r\cos\theta$, $y = r\sin\theta$, so that $r^2 = x^2+y^2$ and $\theta = \tan^{-1}\frac{y}{x}$.

Now, $\frac{\partial r}{\partial x} = \frac{x}{r} = \cos\theta$

$$\frac{\partial r}{\partial y} = \sin\theta.$$

$$\frac{\partial \theta}{\partial x} = \frac{1}{1+\left(\frac{y}{x}\right)^2}\left(-\frac{y}{x^2}\right) = -\frac{\sin\theta}{r}$$

and $\frac{\partial \theta}{\partial y} = \frac{1}{1+\left(\frac{y}{x}\right)^2}\left(\frac{1}{x}\right) = \frac{\cos\theta}{r}$.

Now, $\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial x} + \frac{\partial u}{\partial \theta}\cdot\frac{\partial \theta}{\partial x}$

$$= \frac{\partial u}{\partial r}\cos\theta - \frac{\partial u}{\partial \theta}\frac{\sin\theta}{r} \longrightarrow (1)$$

Similarly, $\frac{\partial u}{\partial y} = \frac{\partial u}{\partial r}\sin\theta + \frac{\partial u}{\partial \theta}\frac{\cos\theta}{r} \longrightarrow (2)$

$$\frac{\partial v}{\partial x} = \frac{\partial v}{\partial r}\cos\theta - \frac{\partial v}{\partial \theta}\frac{\sin\theta}{r} \longrightarrow (3)$$

$$\frac{\partial v}{\partial y} = \frac{\partial v}{\partial r}\sin\theta + \frac{\partial v}{\partial \theta}\frac{\cos\theta}{r} \longrightarrow (4)$$

Since Cauchy - Riemann equations in cartesian form are

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}; \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \longrightarrow (5)$$

Substituting in these equations from above, we have

$$\frac{\partial u}{\partial r}\cos\theta - \frac{\partial u}{\partial \theta}\frac{\sin\theta}{r} = \frac{\partial v}{\partial r}\sin\theta + \frac{\partial v}{\partial \theta}\frac{\cos\theta}{r} \longrightarrow (6)$$

and $$\frac{\partial u}{\partial r}\sin\theta + \frac{\partial u}{\partial \theta}\frac{\cos\theta}{r} = -\frac{\partial v}{\partial r}\cos\theta + \frac{\partial v}{\partial \theta}\frac{\sin\theta}{r} \longrightarrow (7)$$

Multiplying (6) by $\cos\theta$ and (7) by $\sin\theta$ and adding, we get

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \longrightarrow (8)$$

Again multiplying (6) by $\sin\theta$ and (7) by $\cos\theta$ and subtracting, we get

$$\frac{\partial u}{\partial \theta} = -r\frac{\partial v}{\partial r} \longrightarrow (9)$$

Thus, the C-R equations in polar form are

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \text{ and } \frac{\partial u}{\partial \theta} = -r\frac{\partial v}{\partial r}.$$

Note: Differentiating (8) partially w.r.t. r, we have

$$\frac{\partial^2 u}{\partial r^2} = -\frac{1}{r^2}\frac{\partial v}{\partial \theta} + \frac{1}{r}\frac{\partial^2 v}{\partial \theta\,\partial r} \longrightarrow (10)$$

Differentiating (9) partially w.r.t. θ, we get

$$\frac{\partial^2 u}{\partial \theta^2} = -r\frac{\partial^2 v}{\partial r\,\partial \theta} \longrightarrow (11)$$

Hence using (8), (10) and (11), we have

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = \frac{1}{r^2}\frac{\partial v}{\partial \theta} + \frac{1}{r}\frac{\partial^2 v}{\partial \theta\,\partial r} + \frac{1}{r}\left(\frac{1}{r}\frac{\partial v}{\partial \theta}\right)$$
$$+ \frac{1}{r^2}\left(-r\frac{\partial^2 v}{\partial r\,\partial \theta}\right) = 0 \quad \left(\because \frac{\partial^2 v}{\partial \theta\,\partial r} = \frac{\partial^2 v}{\partial r\,\partial \theta}\right)$$

C-R equations :

Cartesian - $u_x = v_y$ and $u_y = -v_x$.

Polar : $\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}$, $\frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$

Simple derivation: If (r, θ) be the coordinate of a point whose cartesian coordinates are (x, y), then $z = x+iy = re^{i\theta}$

$\therefore$ $u + iv = f(z) = f(re^{i\theta})$, where u and v are now expressed in terms of r and θ.

Differentiating it partially w.r.t. r and θ, we have

$$\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r} = f'(re^{i\theta}) \cdot e^{i\theta}$$

and

$$\frac{\partial u}{\partial \theta} + i\frac{\partial v}{\partial \theta} = f'(re^{i\theta}) \cdot rie^{i\theta}$$

$$= ri\left(f'(re^{i\theta})e^{i\theta}\right) = ri\left(\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}\right)$$

Equating real and imaginary parts, we get

$$\frac{\partial u}{\partial \theta} + i\frac{\partial v}{\partial \theta} = -r\frac{\partial v}{\partial r} + i\,r\frac{\partial u}{\partial r}$$

$$\therefore \quad \frac{\partial u}{\partial \theta} = -r\frac{\partial v}{\partial r} \quad \text{and} \quad \frac{\partial v}{\partial \theta} = r\frac{\partial u}{\partial r}$$

$$\therefore \quad \boxed{\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \quad \text{and} \quad \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}}$$

It is noted that

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0 \quad \left(\because \frac{\partial^2 v}{\partial \theta \partial r} = \frac{\partial^2 v}{\partial r \partial \theta}\right).$$

Ex 1 : Show that the function $f(z) = z^n$, where n is a positive integer is an analytic function.

Soln: We have $f'(z) = \lim_{\delta z \to 0} \frac{f(z+\delta z) - f(z)}{\delta z} = \lim_{\delta z \to 0} \frac{(z+\delta z)^n - z^n}{\delta z}$

Now, $f'(z)$ exists provided the above limit exists and is independent of the manner in which δz approaches zero.

$$\therefore \quad f'(z) = \lim_{\delta z \to 0}\left[nz^{n-1} + \frac{\delta z}{2}n(n-1)z^{n-2} + \cdots + \delta z^{n-1} z\right]$$

$$= nz^{n-1}.$$

Hence $f'(z)$ exists for all finite values of z.

Applying the above formula for $z, z^2, z^3, \ldots$ and we see that a polynomial

$$f(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n$$

is an analytic function of z. Moreover, a rational function

$$f(z) = \frac{a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n}{b_0 + b_1 z + b_2 z^2 + \cdots + b_n z^n}$$

is an analytic function of z throughout any finite domain in the complex plane where the denominator does not vanish.

Remark: 1. A single valued function which is defined and differentiable at each point of a domain D is said to be analytic in that domain.

A function is said to be analytic at a point if its derivative exists not only at that point but in some neighbourhood of that point.

2. The term **Holomorphic function** is also used to denote analyticity in a domain.

3. If a function is analytic at some point in every neighbourhood of a point z_0 except at z_0 itself, then the point z_0 is called an **isolated singularity** of $f(z)$. A function is said to have a removable singularity at a point z_0 of the domain of definition of the function if the function is not analytic at z_0 but can be made analytic by merely assigning a suitable value to the function at the point.

4. Real and imaginary parts of an analytic function satisfy Laplace's equation.

Proof: Let the function $f(z) = u + iv$ be analytic in some domain D. Then by Cauchy - Rieman equations given

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

Now, assume that the second order partial derivatives of u and v exist and are continuous functions of x and y (it is proved that when f is analytic, the partial derivatives of u and v of all orders exist and are continuous functions of x and y). We, then have

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x \partial y}, \quad \frac{\partial^2 u}{\partial y^2} = -\frac{\partial^2 v}{\partial y \partial x} \quad \text{so that} \quad \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

Similarly,

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0.$$

Harmonic function:

Any function of x, y which has continuous partial derivatives of the first and second orders and satisfies Laplace's equation is called a Harmonic function.

If $f(z) = u + iv$ be analytic, then u, v both are harmonic functions since they satisfy Laplace's equation. In such a case, u and v are called conjugate harmonic function or simply conjugate functions.

5. If harmonic functions u and v satisfy Cauchy-Riemann equations, then $u + iv$ is an analytic function.

6. Orthogonal System:

Two family of curves $u(x,y) = c_1$ and $v(x,y) = c_2$ are said to form an orthogonal system if they intersect at right angles at each of their point of intersection.

Diff. $u(x,y) = c_1$, we get

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dx} = 0$$

$$\therefore \frac{dy}{dx} = -\frac{\partial u}{\partial x} \Big/ \frac{\partial u}{\partial y} = m_1 \text{ (say)}.$$

Similarly from $v(x,y) = c_2$, we get

$$\frac{dy}{dx} = \frac{\partial v}{\partial x} \Big/ \frac{\partial v}{\partial y} = m_2, \text{ say}.$$

Now, the two families of curves will intersect orthogonally if $m_1 m_2 = -1$

$$\therefore \frac{\partial u}{\partial x} \cdot \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \cdot \frac{\partial v}{\partial y} = 0.$$

Example: If $w = f(z) = u + iv$ be an analytic function of $z = x + iy$, show that the curves $u = $ const., $v = $ const. represented on the z-plane intersect at right angles.

Sol^n:- $f(z) = u + iv$ is regular function of z, then functions u and v satisfy Cauchy-Riemann equations $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$.
Multiplying these, we get

$$\frac{\partial u}{\partial x} \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \cdot \frac{\partial v}{\partial y} = 0$$

which is the condition that the curve $u = $ const. and $v = $ const. intersect at right angles as shown above.

Note :- Hence if $f(z)$ is regular function of z, then the curves

$$u = R[f(z)] = \text{const.}$$

and $v = I[f(z)] = \text{const.}$

form an orthogonal system i.e. they intersect at right angles.

Ex: Show that a harmonic function satisfies the formal differential equation $\frac{\partial^2 u}{\partial z \partial \bar{z}} = 0$.

Sol^n:- If u is a harmonic function, then

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

Now, $x = \frac{1}{2}(z+\bar{z})$, $y = \frac{1}{2i}(z-\bar{z})$

$$\therefore \frac{\partial u}{\partial \bar{z}} = \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial \bar{z}} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial \bar{z}} = \frac{1}{2}\frac{\partial u}{\partial x} - \frac{1}{2i}\frac{\partial u}{\partial y}$$

and
$$\frac{\partial^2 u}{\partial z \partial \bar{z}} = \frac{1}{2}\frac{\partial^2 u}{\partial x^2}\cdot\frac{\partial x}{\partial z} + \frac{1}{2}\frac{\partial^2 u}{\partial y \partial y}\cdot\frac{\partial y}{\partial z} - \frac{1}{2i}\frac{\partial^2 u}{\partial y^2}\frac{\partial y}{\partial z} - \frac{1}{2i}\frac{\partial^2 u}{\partial x \partial y}\cdot\frac{\partial x}{\partial z}$$

$$= \frac{1}{2}\cdot\frac{\partial^2 u}{\partial x^2}\cdot\frac{1}{2} + \frac{1}{2}\cdot\frac{\partial^2 u}{\partial y \partial x}\cdot\frac{1}{2i} - \frac{1}{2i}\frac{\partial^2 u}{\partial y^2}\cdot\frac{1}{2i} - \frac{1}{2i}\frac{\partial^2 u}{\partial x \partial y}\cdot\frac{1}{2}$$

$$= \frac{1}{4}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) = 0 \quad (\because u \text{ is harmonic}).$$

Proved.

Method of constructing a regular function: (Milne-Thomson's Method):

Since $f(z) = u(x,y) + iv(x,y)$ and $x = \frac{1}{2}(z+\bar{z})$, $y = \frac{1}{2i}(z-\bar{z})$, we may write

$$f(z) = u\left[\tfrac{1}{2}(z+\bar{z}), \tfrac{1}{2i}(z-\bar{z})\right] + iv\left[\tfrac{1}{2}(z+\bar{z}), \tfrac{1}{2i}(z-\bar{z})\right] \quad (1)$$

We may regard (1) as a formal identity in two independent variables $z, \bar{z}$. On putting $\bar{z} = z$, we get $f(z) = u(z,0) + iv(z,0)$.

Now, $f'(z) = \frac{\partial w}{\partial x} = u_x - iv_x = u_x - iu_y$ (by C-R equations).

Let $u_x = \phi_1(x,y)$, $u_y = \phi_2(x,y)$; then

$$f'(z) = \phi_1(x,y) - i\phi_2(x,y) = \phi_1(z,0) - i\phi_2(z,0).$$

Integrating, we get $f(z) = \int \phi_1(z,0)\,dz - i\int \phi_2(z,0)\,dz + C$ where C is arbitrary constant.

Similarly, if $v(x,y)$ be given, we have

$$f(z) = \int[\psi_1(z,0) + i\psi_2(z,0)]\,dz + C'$$

where $v_y = \psi_1(x,y)$ and $v_x = \psi_2(x,y)$.

Application of Cauchy-Riemann equations to find harmonic conjugate:

Theorem: If $f(z) = u + iv$ is an analytic function, where both $u(x,y)$ and $v(x,y)$ are conjugate functions, given one of these, say $u(x,y)$ to find the other $v(x,y)$.

Proof: Since v is a function of two real variables x and y, therefore,

$$dv = \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy$$

$$= -\frac{\partial u}{\partial y}dx + \frac{\partial u}{\partial x}dy \quad \text{(By C-R equations)}$$

The R.H.S. of this equation is of the form $Mdx + Ndy$ where $M = -\frac{\partial u}{\partial y}$ and $N = \frac{\partial u}{\partial x}$.

Therefore $\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}\left(-\frac{\partial u}{\partial y}\right) = -\frac{\partial^2 u}{\partial y^2}$

and $\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial^2 u}{\partial x^2}$

Now, since u is harmonic function, therefore it satisfies Laplace's equation, i.e.

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

$$\Rightarrow \frac{\partial^2 u}{\partial x^2} = -\frac{\partial^2 u}{\partial y^2}$$

$$\Rightarrow \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.$$

Thus $-\frac{\partial u}{\partial y}dx + \frac{\partial u}{\partial x}dy$ satisfies exact differential equation. Therefore

$$dv = -\frac{\partial u}{\partial y}dx + \frac{\partial u}{\partial x}dy$$

can be integrated and we can get $v(x,y)$.

If $f(z) = u(x,y) + iv(x,y)$ is analytic, then v is called a harmonic conjugate of u. Since $if = i(u+iv) = -v+iu$ is analytic whenever f is analytic, we have the anti-symmetric property that v is a harmonic conjugate of u if and only if u is a harmonic conjugate of $-v$.

Ex:- Show that $u(x,y) = e^{-x}[x\sin y - y\cos y]$ is harmonic and find its harmonic conjugate $v(x,y)$ such that $f(z) = u + iv$ is analytic.

Sol^n:- We have
$$u(x,y) = e^{-x}[x\sin y - y\cos y] \longrightarrow (1)$$

$$\frac{\partial u}{\partial x} = e^{-x}\sin y - xe^{-x}\sin y + ye^{-x}\cos y$$

$$\therefore \frac{\partial^2 u}{\partial x^2} = -2e^{-x}\sin y + xe^{-x}\sin y - ye^{-x}\cos y$$

$$\frac{\partial u}{\partial y} = xe^{-x}\cos y - e^{-x}\cos y + ye^{-x}\sin y$$

$$\frac{\partial^2 u}{\partial y^2} = -xe^{-x}\sin y + 2e^{-x}\sin y + ye^{-x}\cos y$$

Thus we get $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$. So that $u(x,y)$ is harmonic.

From Cauchy - Reimann equations

$$\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x} = e^{-x}\sin y - xe^{-x}\sin y + ye^{-x}\cos y \longrightarrow (2)$$

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} = -xe^{-x}\cos y + e^{-x}\cos y - ye^{-x}\sin y \longrightarrow (3)$$

Integrating (2) w.r.t. y, treating x as constant, we get

$$v(x,y) = ye^{-x}\sin y + xe^{-x}\cos y + g(x) \longrightarrow (4)$$

where $g(x)$ is an arbitrary real function of x. From (3) and (4), we obtain

$$-\cancel{ye^{-x}\sin y} + \cancel{e^{-x}\cos y} - \cancel{xe^{-x}\cos y} + g'(x) = \cancel{e^{-x}\cos y} - \cancel{xe^{-x}\cos y} - \cancel{ye^{-x}\sin y}$$

$$\therefore g'(x) = 0$$

So that $g(x) = c$ (an arbitrary real constant)

Hence from (4), we have

$$v(x,y) = e^{-x}(y\sin y + x\cos y) + c.$$

This is the required harmonic conjugate of $u(x,y)$.

Example: ~~If~~ $u(x,y) = e^x(x\cos y - y\sin y)$, find the analytic function $u+iv$.

Sol^n:- We have

$$u(x,y) = e^x(x\cos y - y\sin y) \longrightarrow (1)$$

$$\therefore \frac{\partial u}{\partial x} = e^x(x\cos y - y\sin y) + e^x(\cos y)$$
$$= xe^x\cos y - ye^x\sin y + e^x\cos y$$

$$\therefore \frac{\partial^2 u}{\partial x^2} = e^x\cos y + xe^x\cos y - ye^x\sin y + e^x\cos y$$

$$\frac{\partial u}{\partial y} = e^x(-x\sin y - \sin y - y\cos y)$$
$$= -xe^x\sin y - e^x\sin y - ye^x\cos y$$

$$\therefore \frac{\partial^2 u}{\partial y^2} = -xe^x\cos y - e^x\cos y + ye^x\sin y - e^x\cos y$$

$$\therefore \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \cancel{e^x\cos y} + \cancel{xe^x\cos y} - \cancel{ye^x\sin y} + \cancel{e^x\cos y} - \cancel{xe^x\cos y} - \cancel{e^x\cos y} + \cancel{ye^x\sin y} - \cancel{e^x\cos y} = 0$$

Therefore, $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$. So that $u(x,y)$ is harmonic.

Now, $dv = \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy = -\frac{\partial u}{\partial y}dx + \frac{\partial u}{\partial x}dy$

$$= e^x(x\sin y + \sin y + y\cos y)dx + e^x(x\cos y - y\sin y + \cos y)dy$$

$$\therefore v = \int_{y\ \text{Const.}} e^x(x\sin y + \sin y + y\cos y)dx + \int(\text{those terms which do not contain } x)dy + c$$

$$= (x\sin y + \sin y + y\cos y)e^x - e^x\sin y + c$$
$$= e^x(x\sin y + y\cos y) + c, \text{ where } c \text{ is a constant.}$$

$$\therefore f(z) = u + iv = e^x[x\cos y - y\sin y + ix\sin y - iy\cos y] + ci$$
$$= e^x(x+iy)(\cos y + i\sin y) + ci$$
$$= e^{x+iy}.(x+iy) + ci = ze^z + ci. \quad \underline{\text{Ans}}.$$

<u>Second method</u>: $v = xe^x\sin y + ye^x\cos y - e^x\cancel{\cos y}\sin y + e^x\sin y + C$

$= xe^x\sin y + ye^x\cos y + C$ Ans (by Diff. $\frac{\partial u}{\partial x}$ w.r.t. y)