Few interesting problems

Wednesday, January 13, 2021 4:58 PM

1. Birthday Problem

$p_n \rightarrow$ probability that two person chosen at random have same birthday

p_n high if $p_n > \frac{1}{2}$

Ans: 23

P (no 2 of n persons picked at random have same birthday)

$\frac{1}{365}$

$$= \frac{365 \times 364 \times 363 \times \cdots \times (365-(n-1))}{(365)^n}$$

$$= \left[\frac{365}{365-n}\right]^{365.5-n} \cdot e^{-n}$$

$$\boxed{\begin{array}{c} n! \approx n^n e^{-n} \sqrt{2\pi n} \\ \text{Stirling's Formula} \end{array}}$$

for $n = 23$, $\approx .493$

$p_{23} = \cancel{.507}\ 1 - .493 > .5$

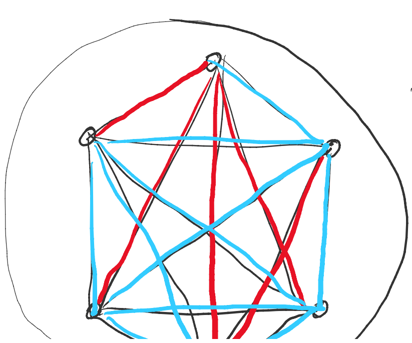
2. Application of Union bound

Graph theory

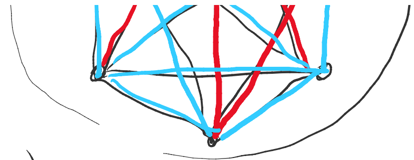
$K_n \rightarrow$ complete graph on n vertices

$K_6 \rightarrow$ coloring with 2 colors

either there is a red K_3 or blue K_3



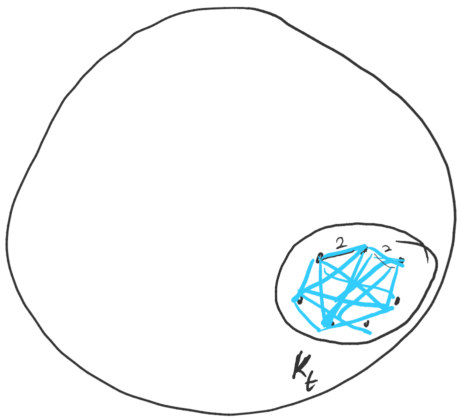
either a red K_4 or a blue K_4



either a red K_t or a blue K_t

n ?

$R(t,t)$ = smallest int n s.t in any two coloring of the edges of K_n either there is a red K_t or blue K_t



Color the edges at random with red and blue

Fix any t vertices

A_i = the subgraph is either blue K_t or red K_t

$$P(A_i) = \frac{2}{2^{\binom{t}{2}}}$$

There are $\binom{n}{t}$ subgraph of t vertices

$A_i \quad 1 \le i \le \binom{n}{t}$

$$\boxed{P\left(\bigcup A_i\right) \le \sum_i P(A_i)} = \boxed{\binom{n}{t} \cdot 2^{1-\binom{t}{2}} < 1}$$

$\Rightarrow$ there must be one coloring s.t. there is no red K_t or blue K_t

$$\binom{n}{t} 2^{1-\binom{t}{2}} < 1$$

Ex

$$\boxed{R(t,t) > \lfloor 2^{t/2} \rfloor}$$

3. Let $x_1, x_2 \cdots x_n$ be n variables s.t. $x_i \in \{1, 2, \cdots m\}$ $m \in \mathbb{N}$

Then $\max\{x_1, x_2, \cdots x_n\} \ge \sum^{n} x_i - \sum \min\{x_i, x_j\}$

Then $\max\{x_1, x_2, \cdots x_n\} \geq \sum_{i=1} x_i - \sum_{i<j} \min\{x_i, x_j\}$

Pf! Bonferroni's inequ.

$\Omega = \{1, 2, 3, \cdots m\}$ and all outcomes are equally likely

$\boxed{x_i \in \Omega}$ $\quad A_i = \underline{\{1, 2, \cdots x_i\}}$ $\quad 1 \leq i \leq n$ for each x_i

we are defining event A_i

$x_1 = 20 \qquad A_1 = \{1, 2, \cdots 20\}$

$$P(A_i) = \frac{|A_i|}{|\Omega|} = \frac{x_i}{m}$$

$x_i = 10 \quad x_j = 20$

$A_i \cap A_j = \{1, 2, \cdots 10\}$

for $i \neq j$ $\quad P(A_i \cap A_j) = \dfrac{\min\{x_i, x_j\}}{m}$

$$P\left(\bigcup_{i=1}^{n} A_i\right) = \frac{\max\{x_1, x_2 \cdots x_n\}}{m}$$

Plug into Bonferroni's ineq

$$P\left(\bigcup_i A_i\right) \geq \sum_i P(A_i) - \sum_{i<j} P(A_i \cap A_j)$$