Linear 1^{st} order ODEs

General form

$$a(x)\, y' + b(x)\, y = c(x)$$

Standard form

$$\boxed{y' + p(x)\, y = q(x)}$$

Solution → follow two steps

If $u(x) = e^{\int p(x)\,dx}$ — ①

Then $y(x)\, u(x) = \int q(x)\, u(x)\, dx + c$ — ②

Keywords to learn in this lecture

- Formula to solve
  $y'(x) = p\, y + q(x)$
  $y(t_0) = y_0$; where p is constant
  and its extension for system of equations
- steady-state solution and transient solution
- Input - response and superposition rule of linearity.
- Geometrical meaning of $y' = f(x, y)$ and IVPs.

Heat conduction model

$$\frac{dT}{dt} = k(T_e - T)$$

$$\Rightarrow \frac{dT}{dt} = -kT + kT_e$$

Concentration diffusion model

$$\frac{dc}{dt} = k(c_e - c)$$

Fluid mixing model

$$\frac{dx}{dt} + \frac{r}{V}\, x = r\, c_e$$

Population growth

$$\frac{dx}{dt} = a\, x$$

Bank A/c interest

$$\frac{dx}{dt} - I(t)\, x(t) = q(t)$$

RC- Circuits

$$\boxed{\frac{dq}{dt} = i(t)}$$

$q(t)$: charge on C at time t.

By Kirchhoff's Law

$$Ri + \frac{q}{C} = V(t)$$

$$\Rightarrow \boxed{\frac{dq}{dt} + \frac{1}{RC}\, q = \frac{1}{R}\, V(t)}$$

NOTE that ODE is not written in standard form

Solve:

ODE → $\boxed{x' = px + q(t)}$ where p is a constant and $x' \equiv \frac{dx}{dt}$

I/C → $x(t_0) = x_0$

First solve ODE

$$\frac{dx}{dt} = px + q(t)$$

$$\Rightarrow \quad x' - px = q(t) \quad \text{[standard form]}$$

Find IF $\quad u(t) = e^{-\int p\,dt} = e^{-pt}$ $\quad$ ∵ p is a constant

Then solution

$$\boxed{x(t)\,u(t) = \int u(t)\,q(t)\,dt + C}$$

$$\Rightarrow \quad x(t)\,e^{-pt} = \int e^{-pt} q(t)\,dt + C$$

$$= \int e^{-ps} q(s)\,ds + C$$

$$\Rightarrow \quad x(t) = e^{pt} \int e^{-ps} q(s)\,ds + e^{pt} C$$

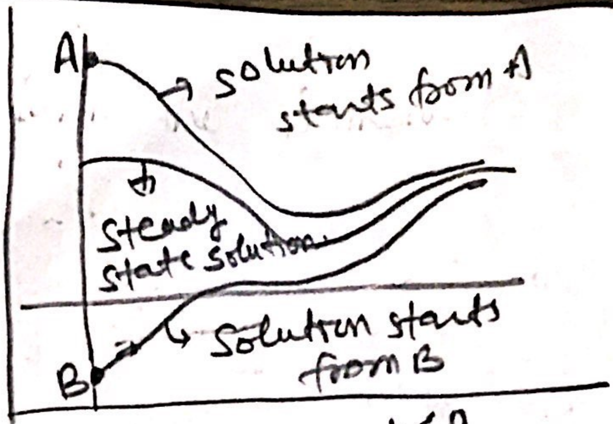
Since, initial time is t_0, we can write solution $x(t)$ in the form of definite integral

$$x(t) = e^{pt} \int_{t_0}^{t} e^{-p(s)} q(s)\,ds + e^{pt} C_0$$

put $x(t_0) = x_0$ and obtain $C_0 = x_0$

$$\boxed{x(t) = e^{pt} x_0 + \int_{t_0}^{t} e^{p(t-s)} q(s)\,ds}$$

Thus, for IVP



$$\frac{dx}{dt} \equiv x' = px + q(t)$$

$$x(t_0) = x_0$$

Solution is

$$x(t) = \underbrace{e^{pt} x_0}_{\to 0 \text{ (If } p<0 \text{ and } t\to\infty)} + \boxed{\int_{t_0}^{t} e^{p(t-s)} q(s)\, ds}$$

[called transient solution]

in case $p<0$ called steady-state Solution

In engineering models: these keywords are very famous as in many ODEs term $p<0$

$\boxed{\text{steady state solution} \\ \text{transient solution}}$

[See Heat conduction, Concentration diffusion, mixing problems, RC-circuits.

Remember! keywords for linear ODE

$$y' = py + q(t)$$

are valid when $p<0$. [p is a negative constant]

But this $(p<0)$ is not the case in all situations: Normally $p>0$ in models in economics and biology

See bank Account model

population model / decay model

One more reason to define formula

$$x(t) = e^{pt}x_0 + \int_{t_0}^{t} e^{p(t-s)} q(s)\, ds$$

for IVP $\begin{cases} x' = px + q(t) \\ x(t_0) = x_0 \end{cases}$

is that it works in system of equation.

$$\dot{X} = AX + f(t)$$
$$X(0) = X_0$$

Its solution is

$$\boxed{X(t) = e^{At}X_0 + \int_{t_0}^{t} e^{A(t-s)} f(s)\, ds}$$

details will come after 2 weeks.

$$\dot{x} = 2x + 3y + f_1(t)$$
$$\dot{y} = 5x + 2y + f_2(t)$$

$$|||$$

$$\dot{X} = AX + f(t)$$

where

$$X = \begin{bmatrix} x \\ y \end{bmatrix}$$

$$A = \begin{bmatrix} 2 & 3 \\ 5 & 2 \end{bmatrix}$$

$$f = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$$

Always use algorithm to solve scalar linear ODE after writing it in standard form.

Another Imp keywords for

$$y' + p(x)y = q(x)$$

are

Input $\rightarrow$ RHS term $q(x)$

response $\rightarrow$ solution $y(x)$.

---

one Imp property of linear equations.

If response for

$q_1 \xrightarrow{\text{is}} y_1$

and $q_2 \xrightarrow{\text{is}} y_2$

Then response for $q_1 + q_2 \xrightarrow{\text{is}} y_1 + y_2$

and $\alpha q_1 \xrightarrow{\text{is}} \alpha y_1$

[This is called **superposition rule of linearity**] Imp keyword.

Easy to prove

↳ check yourself!)

$\boxed{\text{Find response of } y' + ky = kq(t) \text{ when } q(t) = \cos\omega t}$ $k > 0$ is constant.

|||

Solve $\boxed{\dfrac{dy}{dt} + ky = k\cos\omega t}$ $\boxed{k > 0 \text{ is constant}}$

Apply Algorithm

$$\frac{dy}{dt} + ky = k\cos\omega t \qquad \text{(standard form)}$$

$$IF = e^{kt}$$

so $$y e^{kt} = k\int e^{kt}\cos\omega t \, dt + c$$

⋮

OR Use a trick due to superposition rule

$\cos\omega t \rightsquigarrow y_1(t)$

$\sin\omega t \rightsquigarrow y_2(t)$

then $\cos\omega t + i\sin\omega t \rightsquigarrow y_1 + iy_2$

OR $e^{i\omega t} \rightsquigarrow \tilde{y} \, (= y_1 + iy_2)$

so first find $\tilde{y}$ for $e^{i\omega t}$ and then find real part y_1 of $\tilde{y}$.

Scanned with CamScanner

$$y' + ky = k\cos\omega t \quad — (1)$$

complexify the ODE

$$\tilde{y}' + k\tilde{y} = k\,e^{i\omega t} \quad — (2)$$

Now solve (2)

$$\tilde{y}\,e^{kt} = k\int e^{kt}\,e^{i\omega t}\,dt + c$$

$$= k\int e^{(k+i\omega)t}\,dt + c$$

$$= \frac{k}{k+i\omega}\,e^{(k+i\omega)t} + c$$

$$\Rightarrow \quad \tilde{y} = \frac{k}{k+i\omega}\,e^{i\omega t} + c\,e^{-kt}$$

$$= \frac{k\,(k-i\omega)}{k^2+\omega^2}\,(\cos\omega t + i\sin\omega t) + c\,e^{-kt}$$

If $\tilde{y} = y + iu$ then the required response is y and that is

$$y(t) = \frac{k}{k^2+\omega^2}\,\mathrm{Re}\left[(k-i\omega)(\cos\omega t + i\sin\omega t)\right] + c\,e^{-kt}$$

$$= \frac{k}{k^2+\omega^2}\left[k\cos\omega t + \omega\sin\omega t\right] + c\,e^{-kt}.$$

A meaningful assumption

GEOMETRICAL METHOD OR Graphical method

ODE has one and only solution passing through any point (x_0, y_0)

↳ All other ODEs are not of practical importance

- We will read in details under what conditions this assumption holds.

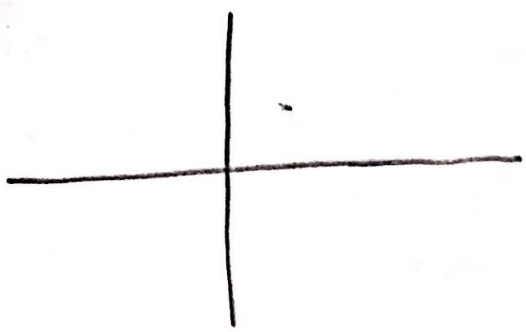
	Analytical meaning	geometrical names
ODE →	$y' = f(x,y)$	Direction field
Solution →	A function $y(x)$ that satisfies ODE	integral/solution curve.

Geometrically problem is – we know $\frac{dy}{dx}$ in the x-y plane, i.e. given by $f(x,y)$; and we have to find y.

$\boxed{\text{Slope is given; function is to be calculated}}$

→ How to draw direction field for $y' = f(x,y)$ with the help of computer (computer method)

1. Pick (x,y) – (equally spaced)
2. Compute $f(x,y)$
3. draw a small line segment at (x,y) having slope $f(x,y)$



By using pen and paper (manual method)

step1 pick slope c

step2 plot $\underbrace{f(x,y) = c}_{\text{called isocline}}$ $\longrightarrow$

MA101
level curve of
f at level
value c.

step3 draw small line segment
on above isoclines having
slope c.

Example 1

$$\frac{dy}{dx} = 1 + x - y$$

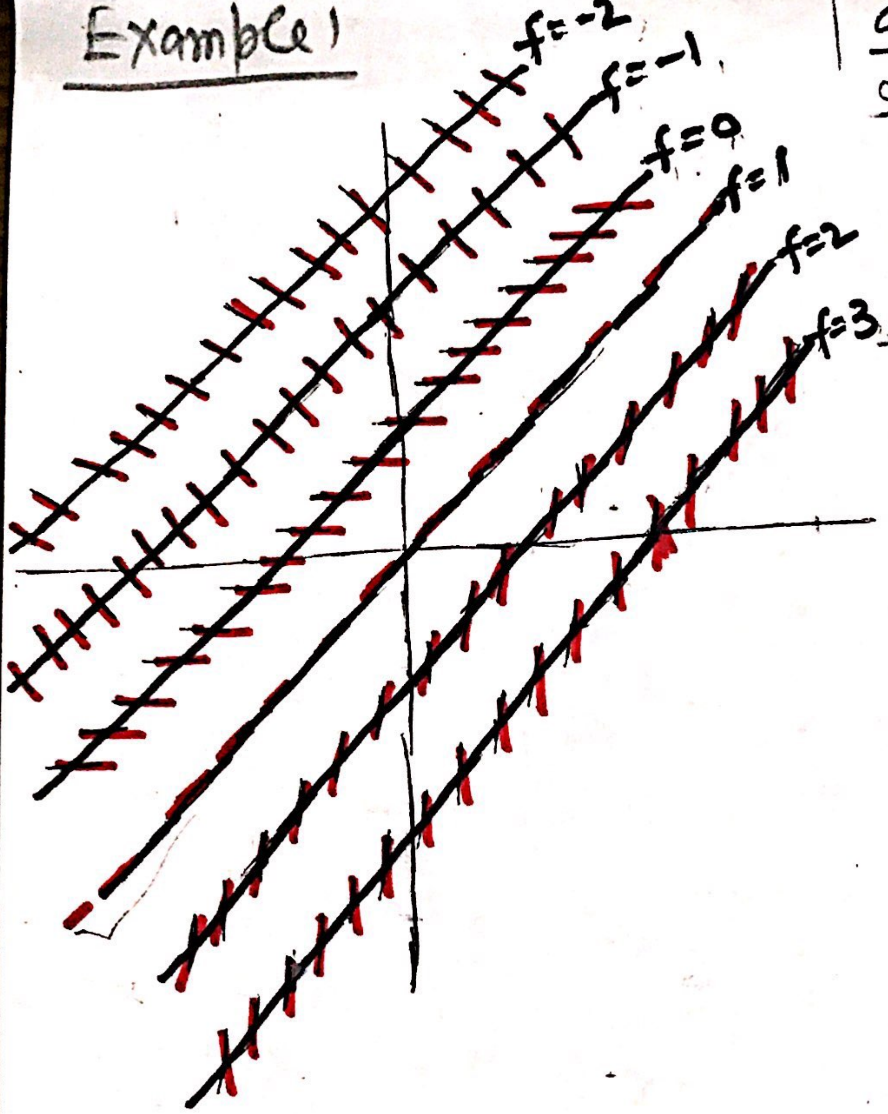


figure indicating first two steps.

Black lines → isoclines

Red line segments → direction field

Step 1

Plot some isoclines

(i) $f = -2$

$1 + x - y = -2$

$\Rightarrow y = x + 3$

(ii) $f = -1$

$1 + x - y = -1$

$\Rightarrow y = x + 2$

(iii) $f = 0$

$1 + x - y = 0$

$\Rightarrow y = x + 1$

(iv) $f = 1 \equiv y = x$

(v) $f = 2 \equiv y = x - 1$

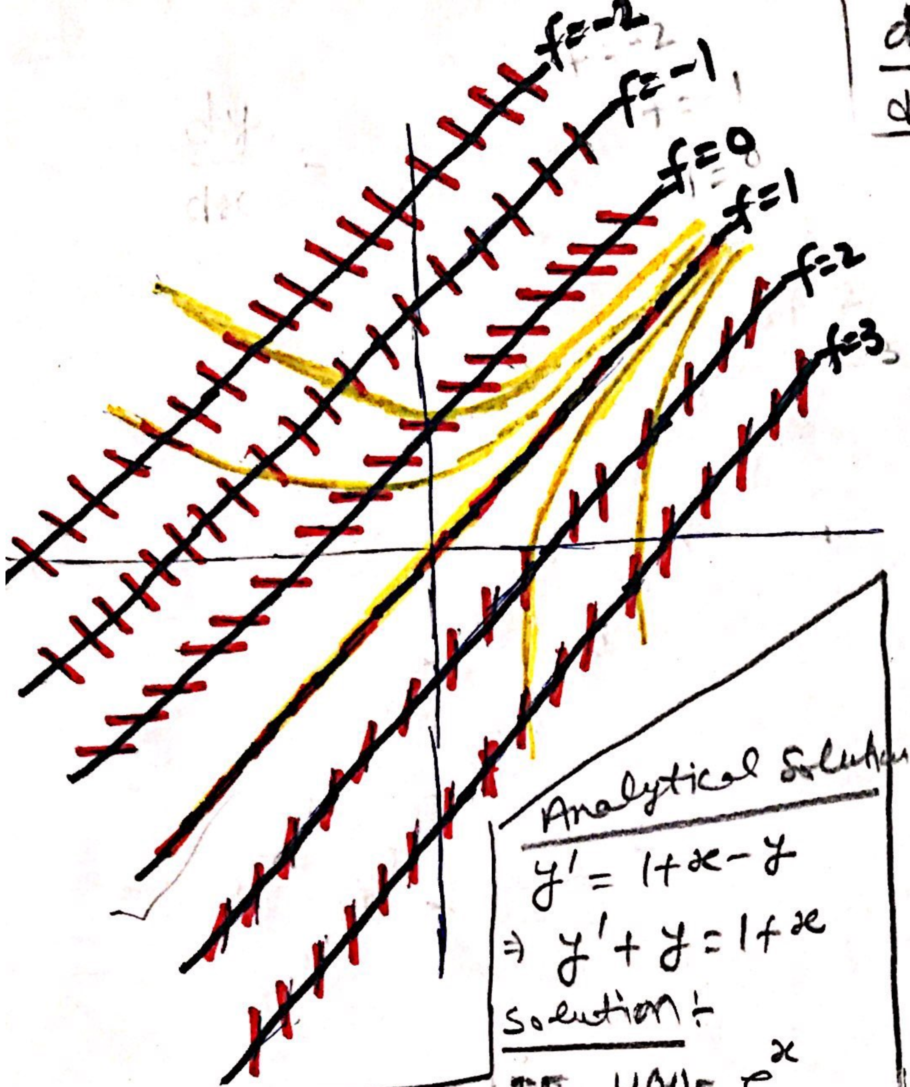
(vi) $f = 3 \equiv y = x - 2$

Step 2

draw direction field on isoclines

step 3

draw a few integral curves.



$$\boxed{\frac{dy}{dx} = 1 + x - y}$$

Step 1

Plot some isoclines

(i) $f = -2$

$1 + x - y = -2$

$\Rightarrow y = x + 3$

(ii) $f = -1$

$1 + x - y = -1$

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(v) $f = 2 \equiv y = x - 1$

(vi) $f = 3 \equiv y = x - 2$

Step 2

draw direction field on isoclines

step 3

draw a few integral curves.

Analytical Solution

$y' = 1 + x - y$

$\Rightarrow y' + y = 1 + x$

Solution :-

IF $\mu(x) = e^{x}$

$y(x)\,e^{x} = \int (1+x)\, e^{x}\, dx + C$

$\Rightarrow y(x) = \underbrace{x + Ce^{-x}}$

$\hookrightarrow x$ as $x \to \infty$

Yellow curves

$\hookrightarrow$ Integral /Solutions curves.

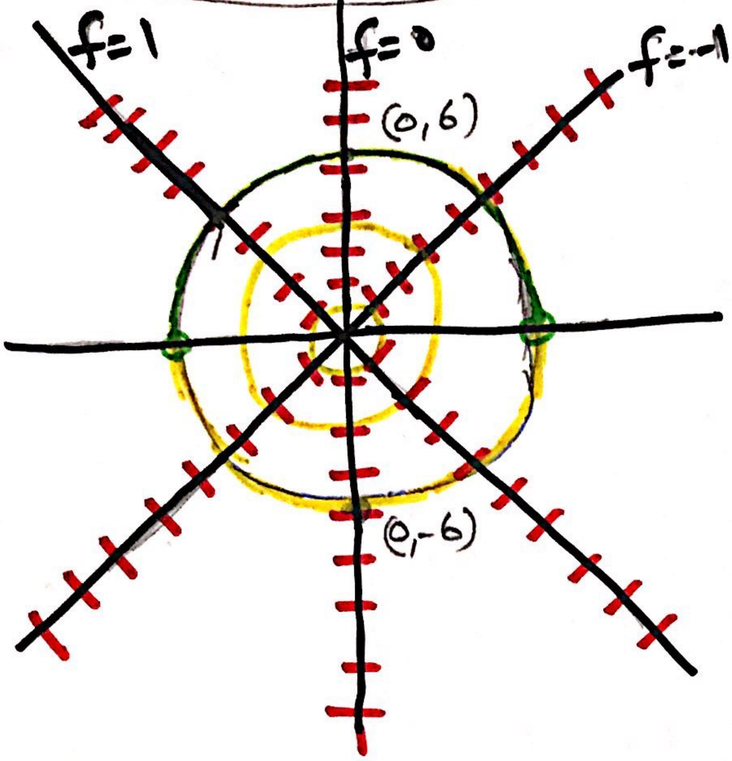
Remember

- Two integral curves cannot cross
- Two Integral curves cannot touch

Geometrical method gives some feeling how analytical solution behaves/Qualitative properties

Example 2

$$\boxed{\frac{dy}{dx} = -\frac{x}{y}}$$



Analytic solution

Separable case

$$x\,dx + y\,dy = 0$$

$$\Rightarrow \boxed{x^2 + y^2 = c}$$

geometric view

Step 1 draw isoclines

(i) $f = -1$

$$\Rightarrow -\frac{x}{y} = -1$$

$$\Rightarrow y = x$$

(ii) $f = 0 \Rightarrow x = 0$

(iii) $f = 1 \Rightarrow y = -x$

Step 2 draw direction field

Step 3 draw some integral curves.

See again

$\frac{dy}{dx} = -\frac{x}{y}$; $y(0) = 6 \rightarrow$ IVP1

$\frac{dy}{dx} = -\frac{x}{y}$; $y(0) = -6 \rightarrow$ IVP2

The particular solution of both IVPs is

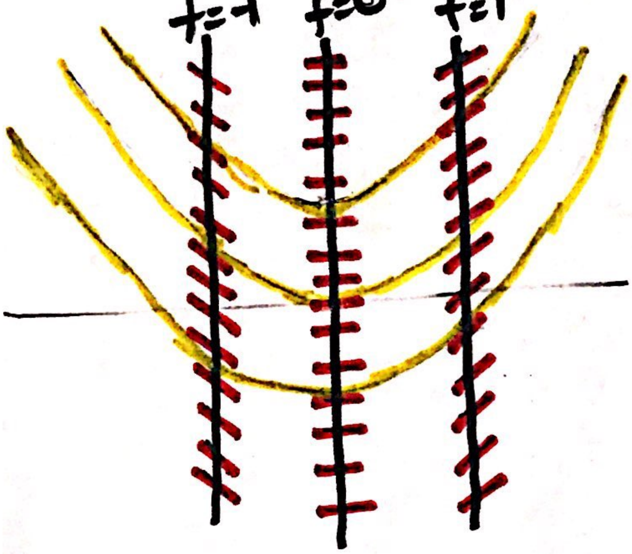
$x^2 + y^2 = 36$.

This gives that upper part of circle $x^2+y^2=36$ is an explicit solution of IVP 1; and lower part of the circle is an explicit solution of IVP 2. Moreover interval of validity for both is $(-6, 6)$.

Example 3 $\frac{dy}{dx} = f(x)$

particular example

$$\frac{dy}{dx} = 2x$$



Analytic solution
(separation case)

$$y = \int f(x)\,dx + c$$

Using fundamental thm, we write

$$\boxed{y = \int_a^x f(t)\,dt + c}$$

↳ provided initial data is given at $x = a$ i.e. $y(a) = c$

Step1 draw isoclines

(i) $f = -1 \Rightarrow x = -\frac{1}{2}$

(ii) $f = 0 \Rightarrow x = 0$

(iii) $f = 1 \Rightarrow x = \frac{1}{2}$

Step2 draw direction field

Step3 draw a few integral curves.

In case $y = f(x)$

1. Isoclines are $\parallel^{al}$ to y-axis.
2. If we know one integral curve then its vertical translation is also an integral curve.

Example 4 $\frac{dy}{dx} = f(y)$

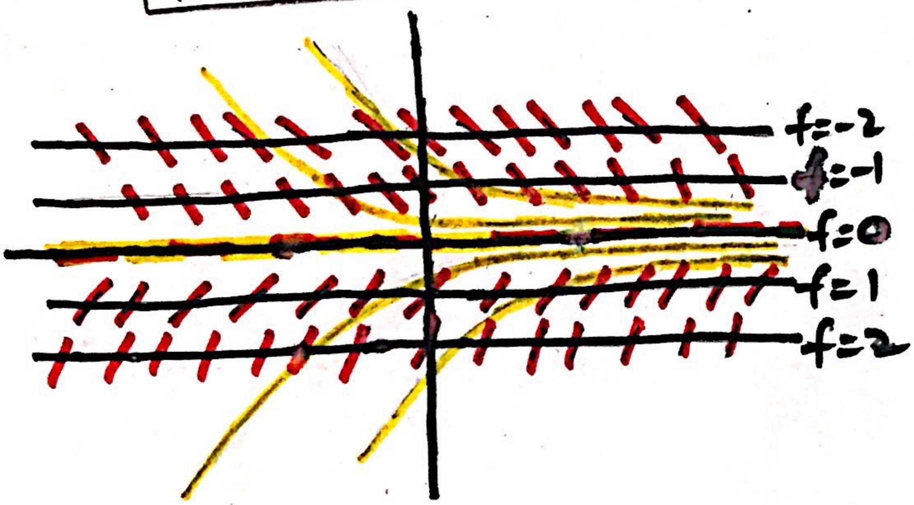
Analytic solution
Use method for separable equations.
Use solver for linear equation if $f(y) = ay$

Such equations are known as Autonomous ODEs.
when $x \rightsquigarrow$ time(t) then other famous name is time-invariant ODEs.

Particular example

$$\frac{dy}{dx} = -3y$$

Solution $\boxed{\text{Analytic}} \to y(x) = ce^{-3x}$
$\boxed{\text{geometrical}}$



Step 1 draw isoclines
(i) $f = -2 \Rightarrow y = \frac{2}{3}$
(ii) $f = -1 \Rightarrow y = \frac{1}{3}$
(III) $f = 0 \Rightarrow y = 0$
(iv) $f = 1 \Rightarrow y = -\frac{1}{3}$
(v) $f = 2 \Rightarrow y = -\frac{2}{3}$

Step 2 direction field

Step 3 integral curves

Some Important facts about $\boxed{y' = f(y)}$

1. (Definition) If $f(y_0) = 0$ at some value y_0 of y, Then y_0 is called a critical point and curve $y = y_0$ is called a steady state solution of ODE.
2. Isoclines are $\parallel^{al}$ to x-axis
3. If we know one integral curve then its horizontal translation is also an integral curve.

④ $\left(\frac{\text{Definition}}{\text{Result}}\right)$ The critical point $y = y_0$ is called stable for ODE $y' = f(y)$

If $\left.\frac{df}{dy}\right|_{y=y_0} < 0$ and unstable

If $\left.\frac{df}{dy}\right|_{y=y_0} > 0$.

Stable means $y(t) \to y_0$ as $t \to \infty$ when $|y(t_0) - y_0| < \varepsilon$ for some t_0

Justification of Result

$$\frac{d}{dt}(y(t) - y_0) = \frac{dy}{dt} - \frac{d}{dt} y_0$$

$$= f(y) - 0$$

From ODE $\frac{dy}{dx} = f(y)$ & y_0 is constant

$$= f(y) - f(y_0)$$

by definition of critical pt $f(y_0) = 0$

$$\simeq \frac{df}{dy}(y - y_0)$$

$$< 0 \quad \text{when } \frac{df}{dy} < 0 \text{ and } |y - y_0| > 0$$

$\Rightarrow$ $y(t) - y_0$ is decreasing as t increases

Hence for this example $y = 0$ is a stable critical point.

OR $\boxed{y = 0 \text{ is a stable steady state solution.}}$??

Ex ⑤

$$\frac{dy}{dt} = ay - by^2 \quad (a, b > 0)$$

$$f(y) = ay - by^2$$
$$= y(a - by)$$

$\because f(y) = 0$

$\Rightarrow y = 0, \frac{a}{b}$

About ODE

Remember basic population model

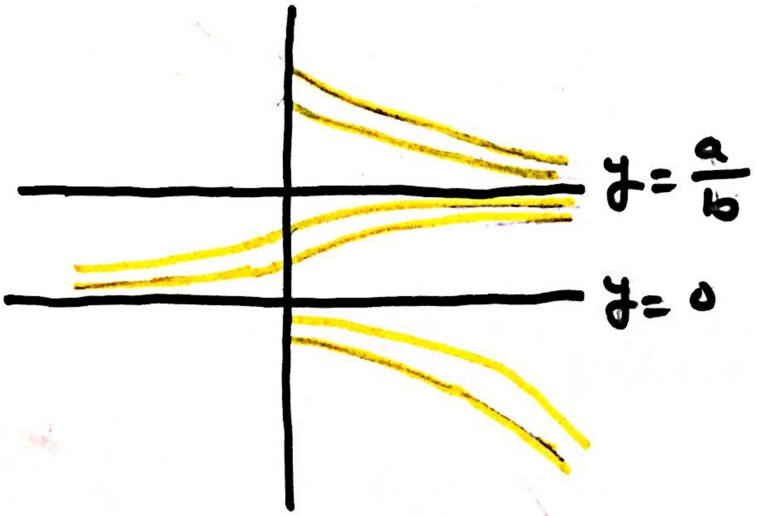
$\frac{dy}{dt} = ky$ $k > 0$ is growth rate (/time)

Actually k declines as y increases because resources are limited. Take $k = a - by$ $(a, b > 0)$

Thus this model is well known as Logistic model for population dynamics.

$$\left.\frac{df}{dy}\right|_{y=0} = a - 2by\Big|_0 = a \Rightarrow y = 0 \text{ is unstable}$$

$$\left.\frac{df}{dy}\right|_{y=\frac{a}{b}} = a - 2by\Big|_{y=\frac{a}{b}} = -2a \Rightarrow y = \frac{a}{b} \text{ is stable}$$



Thus, we can find the nature of solution curves without drawing direction field for $\frac{dy}{dx} = f(y)$

Note that logistic model is non-linear and its analytic sol. is not easy by separation.

Figure 8.4

Read Section 8.1 from SL Ross.

Exercises

Employ the method of isoclines to sketch the approximate integral curves of each of the differential equations in Exercises 1–12.

1. $\dfrac{dy}{dx} = 3x - y.$

2. $\dfrac{dy}{dx} = \dfrac{y}{x}.$

3. $\dfrac{dy}{dx} = \dfrac{y}{x^2}.$

4. $\dfrac{dy}{dx} = x^2 + 2y^2.$

5. $\dfrac{dy}{dx} = \dfrac{3x - y}{x + y}.$

6. $\dfrac{dy}{dx} = \sin x - y.$

7. $\dfrac{dy}{dx} = y^3 - x^2.$

8. $\dfrac{dy}{dx} = \dfrac{3x + 2y + x^2}{x + 2y}.$

9. $\dfrac{dy}{dx} = \dfrac{3x + y + x^3}{5x - y}.$

10. $\dfrac{dy}{dx} = \dfrac{\sin x + y}{x - y}.$

11. $\dfrac{dy}{dx} = \dfrac{(1 - x^2)y - x}{y}.$

12. $\dfrac{dy}{dx} = \dfrac{10(1 - x^2)y - x}{y}.$