Revision

$$y'' + p(x)y' + q(x)y = F(x) \quad \text{—— (\#)} \quad \text{In homogeneous / non-homogeneous}$$

$$y'' + p(x)y' + q(x)y = 0 \quad \text{—— } (*) \quad \text{homogeneous}$$

Th ① Linear combination of n solutions of $(*)$ is also a solution to $(*)$

Th ② If y_1 and y_2 are two solutions to $(*)$ on $[a,b]$.
Then $W(y_1, y_2)$ is either identically zero on $[a,b]$ or never zero on $[a,b]$.

Moreover W is zero on $[a,b]$ $\Leftrightarrow$ y_1 and y_2 are LD on $[a,b]$.

Th ③ Existence and uniqueness theorem for (#) and $(*)$.

Ex.
$$y^{(3)} + 2y'' + 4xy' + x^2 y = 0$$
$$y(2) = 0$$
$$y'(2) = 0$$
$$y''(2) = 0$$

has only trivial solution on $\mathbb{R}$.

Def: Set of 2 LI solutions of $(*)$ is called fundamental set of solutions to $(*)$

Th ④ If y_1 and y_2 are two LI solutions to $(*)$ then general sol. to $(*)$ is; $y = c_1 y_1 + c_2 y_2$

Generalization: Fundamental set of solutions to n^{th} order linear homogeneous equation has n LI solutions. And general solution is their Linear Combination.

Remember Thm. (A) and Thm (B) for $(*)$

Do All exercises of Section 4.1 on page 113
SL Ross (3rd Edition)

↳ many of them, you can solve orally.

- How to find general sol. of 2nd order homogeneous constant coefficient linear ODEs:

(*) — $\boxed{y'' + py' + qy = 0 \;;\; p \text{ and } q \text{ are constants}}$

$\boxed{\text{Guess: } \quad y = e^{mx} \text{ is a solution of (*) for some } m}$

Verify: put $y = e^{mx}$ in (*) and obtain

$$m^2 e^{mx} + pm e^{mx} + q e^{mx} = 0$$

$$\Rightarrow (m^2 + pm + q)e^{mx} = 0$$

Thus e^{mx} is a solution of (*) $\Leftrightarrow$ $\boxed{m^2 + pm + q = 0}$

$$\boxed{m = \frac{-p \pm \sqrt{p^2 - 4q}}{2}}$$

This equation is called auxiliary equation of (*)
(AE)

Thus question is to find roots of this equation.

$$\begin{aligned} & y'' + py' + qy = 0 \\ \equiv \; & D^2y + pDy + qy = 0 \\ \equiv \; & (D^2 + pD + q)y = 0 \\ \equiv \; & Ly = 0 \end{aligned}$$

where $L \equiv D^2 + pD + q$

replace D by m

$m^2 + pm + q$

If m_1 and m_2 are two roots of AE, then there are three possibilities:

Case 1 $m_1 \neq m_2$ and both are real

Case 2 m_1 and m_2 are complex
$m_1 = \alpha + i\beta$ and $m_2 = \alpha - i\beta$

Case 3 $m_1 = m_2$ and both a real

Case 1 $\boxed{(p^2-4q>0)}$

Let $m = m_1, m_2$ be two roots of AE, $m_1 \neq m_2$ and reals.

Then $y_1 = e^{m_1 x}$

$y_2 = e^{m_2 x}$

are two LI solutions to (*)

Case 2 $\boxed{p^2-4q<0}$ Let $m_1, m_2 = a \pm ib$

Then
$$\left.\begin{aligned} y_1 &= e^{(a+ib)x} \\ y_2 &= e^{(a-ib)x} \end{aligned}\right\}$$ are two complex solution.

But we want real solution.

Invoke Thm (A)

If
$$y = e^{(a+ib)x}$$
$$= e^{ax} e^{ibx}$$
$$= e^{ax}(\cos bx + i \sin bx)$$

is a complex solution. Then

$$y_1 = e^{ax} \cos bx$$
$$y_2 = e^{ax} \sin bx$$

are two LI real solutions to (*)

Case 3 $\boxed{p^2-4q=0 \text{ then roots are } m_1=m_2=-\frac{p}{2}}$

If m_1, m_2 are two real and equal roots of AE, i.e.

$$m_1 = m_2 = m \text{ (say)} = -\frac{p}{2}$$

then $y_1 = e^{mx}$ is a solution to ⊛

To find another LI solution, invoke thm (B)

$$\boxed{\begin{array}{l}\text{If } y_1 \text{ is a solution of ⊛} \\ \text{then } y_2 = y_1 \cdot u \text{ is also a solution} \\ \text{where } \quad u = \int \frac{1}{y_1^2} e^{-\int p(x)dx} dx\end{array}}$$

Here $\quad u = \int \frac{1}{e^{2mx}} e^{-px} dx$

$$= \int \frac{e^{-px}}{e^{-px}} dx \quad \left(\because m = -\frac{p}{2}\right)$$

$$= x$$

Thus

$$y_1 = e^{mx}$$

$$y_2 = x e^{mx}$$

are two LI solutions to ⊛

Re-look

$$y'' + py' + qy = 0 \quad (p, q \text{ are constents})$$

Auxilian Equ'n (AE) $\quad m^2 + pm + q = 0$

Find its roots (say m_1, m_2)

Distinct real roots	Conjugate complex roots	Repeated real roots
$m_1 \neq m_2$ (real)	$m_1 = a+ib$ $m_2 = a-ib$	$m_1 = m_2 = m$ real
$y_1 = e^{m_1 x}$ $y_2 = e^{m_2 x}$	$y_1 = e^{ax}\cos bx$ $y_2 = e^{ax}\sin bx$	$y_1 = e^{mx}$ $y_2 = xe^{mx}$
$y = c_1 e^{m_1 x} + c_2 e^{m_2 x}$	$y = e^{ax}(c_1 \cos bx + c_2 \sin bx)$	$y = e^{mx}(c_1 + c_2 x)$

Examples

ODE	AE	roots of AE	General solution
$y'' - 3y' + 2y = 0$	$m^2 - 3m + 2 = 0$ $\Rightarrow (m-1)(m-2) = 0$	$m = 1, 2$	$y = c_1 e^{x} + c_2 e^{2x}$
$y'' - 6y' + 25y = 0$	$m^2 - 6m + 25 = 0$ $\Rightarrow m = \frac{6 \pm \sqrt{36-100}}{2}$	$m = 3 \pm 4i$	$y = e^{3x}[c_1 \cos 4x + c_2 \sin 4x]$
$y'' - 6y' + 9y = 0$	$m^2 - 6m + 9 = 0$ $\Rightarrow (m-3)^2 = 0$	$m = c_1 e^{3x} + c_2 x e^{3x}$ $= e^{3x}[c_1 + c_2 x]$	

Imp Observation $y \equiv 0$ is always a solution of homogeneous equation. And any other solution tends to 0 as $x \to \infty$ when real roots (and real part of complex roots) are negative.

Similarly, we can solve higher order constant coefficient linear homogeneous ODEs.

Case 1 (n distinct real roots): $m_1, m_2, \ldots, m_n$

then $e^{m_1 x}, e^{m_2 x}, \ldots, e^{m_n x}$ are n LI Solutions.

Case 2 If $a \pm bi$ are k-fold roots of the auxiliary equation, then corresponding part of general solution may be written as

$$e^{ax}\left[\left(c_1 + c_2 x + c_3 x^2 + \cdots + c_k x^{k-1}\right)\cos bx + \left(c_{k+1} + c_{k+2} x + c_{k+3} x^2 + \cdots + c_{2k} x^{k-1}\right)\sin bx\right]$$

Case 3 If root m occur k times then corresponding part of general solution may be written as

$$\left(c_1 + c_2 x + \cdots + c_k x^{k-1}\right) e^{mx}$$

The algorithm can be extended to higher order ODEs

Examples.

① $y^{(4)} - 5y'' + 4y = 0$

$(D^4 - 5D^2 + 4)y = 0$

AE $\quad m^4 - 5m^2 + 4 = 0$

$\Rightarrow m = -1, 1, -2, 2$

Thus general sol: $\quad y = c_1 e^{-x} + c_2 e^{x} + c_3 e^{-2x} + c_4 e^{2x}$

② $y^{(4)} - 8y'' + 16y = 0$

AE $\quad m^4 - 8m^2 + 16 = 0$

$\Rightarrow m = -2, -2, 2, 2$

Gen. Sol.: $\quad y = e^{-2x}(c_1 + c_2 x) + e^{2x}(c_3 + c_4 x)$

③ $y^{(4)} - 2y^{(3)} + 2y'' - 2y' + y = 0$

AE $\quad m^4 - 2m^3 + 2m^2 - 2m + 1 = 0$

$\Rightarrow \quad m = 1, 1, \pm i$

Gen. sol: $\quad y = e^{x}(c_1 + c_2 x) + (c_3 \cos x + c_4 \sin x)$

④ If roots of a 4^{th} order homogeneous constant coefficient ODE are

$2+3i,\ 2+3i,\ 2-3i,\ 2-3i$

Then $\quad y = e^{2x}\left[(c_1 + c_2 x)\cos 3x + (c_3 + c_4 x)\sin 3x\right]$

Note: For 2^{nd} order, If roots are $2 \pm 3i$, then $y = e^{2x}\left[c_1 \cos 3x + c_2 \sin 3x\right]$

(4) If roots of 6^{th} order homogeneous constant coefficient ODE are

$$1, 1, \boxed{i, i,} -i, -i$$

Then Gen Sol:- $y = e^{x}(c_1 + c_2 x) + (c_3 + c_4 x)\cos x + (c_5 + c_6 x)\sin x$

(5) If roots of the AE corresponding to a certain 10^{th} order homogeneous linear constant coefficient ODE are

$$4, 4, 4, 4, \underline{2+3i}, 2-3i, \underline{2+3i}, 2-3i, \underline{2+3i}, 2-3i$$

Then

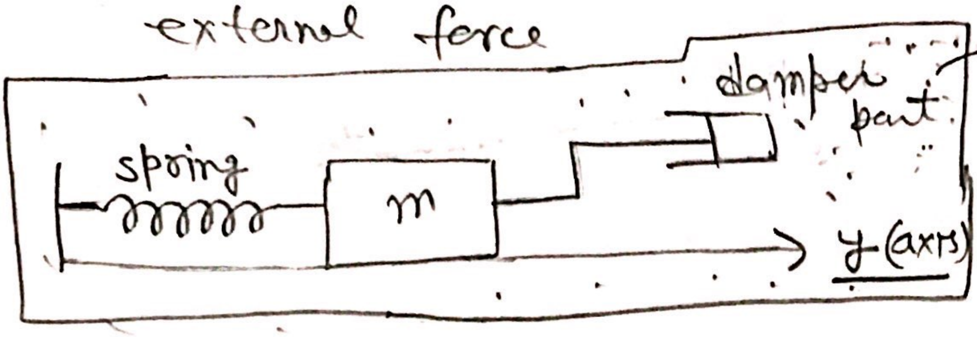
$$y = e^{4x}(c_1 + c_2 x + c_3 x^2 + c_4 x^3) + e^{2x}\left[(c_5 + c_6 x + c_7 x^2)\cos 3x + (c_8 + c_9 x + c_{10} x^2)\sin 3x\right].$$

Practice Problems Section 4.2; page 135

Do Any 10 problems from 1-24.
Any 8 problems from 25-42
All problems from 43-46.

Example

① spring - mass - damper system under some external force



some electric field/force $E(t)$

$$\boxed{m, k, c > 0}$$

$$m\frac{d^2y}{dt^2} = -ky - c\frac{dy}{dt} + E(t)$$

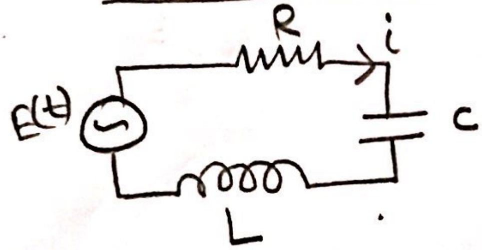
$y(0) = y_0$

$y'(0) = y_1$

ODE: $\boxed{my'' + cy' + ky = E(t)}$

see similarity.

② LRC Circuit



ODE

$$\boxed{Lq'' + Rq' + \frac{1}{C}q = E(t)}$$

OR

differentiate wrt t

$$\boxed{Li'' + Ri' + \frac{1}{C}i = \frac{dE}{dt}}$$

By Kirchhoff's Law

$$L\frac{di}{dt} + Ri + \frac{1}{C}q = E(t)$$

put $i = \frac{dq}{dt}$ and obtain

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{1}{C}q = E$$

$q(0) = q_0$

$\frac{dq}{dt}(0) = i(0) = i_0$

spring-mass-damper ODE

$$my'' + cy' + ky = E(t)$$

Initial data
$y(0) = y_0$ (initial displacement)
$y'(0) = 0$ (No initial velocity)

Case 1 free undamped motion i.e. $\left[\begin{array}{l} E \equiv 0 \\ c = 0 \end{array}\right.$

$$my'' + ky = 0$$

$$\Rightarrow \quad y'' + \frac{k}{m} y = 0 \equiv y'' + \omega_0^2 y = 0$$

Notation $\omega_0^2 = \frac{k}{m}$

Its solution

AE: $m^2 + \omega_0^2 = 0$

$\Rightarrow m = \pm \omega_0 i$

$y = c_1 \cos \omega_0 t + c_2 \sin \omega_0 t$

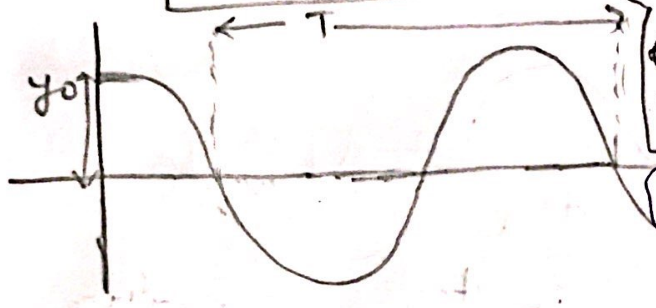
See $y \not\to 0$ as $t \to \infty$

pure oscillation case.

If initial data is
$y(0) = y_0$; $y'(0) = 0$
then $\boxed{y = y_0 \cos \omega_0 t}$

amplitude $= y_0$
period
$T = \frac{2\pi}{\omega_0} = 2\pi\sqrt{\frac{m}{k}}$
frequency
$\frac{1}{T} = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$

bigger k gives motion with smaller period and more frequency



Case 2 free damped motion $\boxed{\text{i.e. } E \equiv 0}$

$$my'' + cy' + ky = 0$$

$$\Rightarrow y'' + 2by' + \omega_0^2 y = 0 \quad \left[\begin{array}{l}\text{Notation} \\ 2b = \frac{c}{m}, \ \omega_0^2 = \frac{k}{m}\end{array}\right]$$

Solution: AE: $m^2 + 2bm + \omega_0^2 = 0$

$$\Rightarrow m = \frac{-2b \pm \sqrt{4b^2 - 4\omega_0^2}}{2} = -b \pm \sqrt{b^2 - \omega_0^2}$$

Subcase 1 $(b > \omega_0)$ (over-damped case)

Then we obtain two distinct real roots and both are $-ive$. Hence $y \to 0$ as $t \to \infty$

Assume :- roots are m_1, m_2 ($m_1, m_2 < 0$ and $m_1 \neq m_2$)

Then $y = c_1 e^{m_1 x} + c_2 e^{m_2 x}$

$$\left.\begin{array}{l} c_1 + c_2 = y_0 \\ m_1 c_1 + m_2 c_2 = 0 \end{array}\right) \Rightarrow \quad c_1 = -\frac{m_2 y_0}{m_1 - m_2}, \quad c_2 = \frac{m_1 y_0}{(m_1 - m_2)}$$

If I/C
$y(0) = y_0$
$y'(0) = 0$

$$y = -\frac{m_2 y_0}{m_1 - m_2} e^{m_1 t} + \frac{m_1 y_0}{m_1 - m_2} e^{m_2 t}$$

(Overdamping)

No Oscillations.

Subcase 2 $(b < \omega_0)$ i.e. $b^2 < \omega_0^2$ ($\because b > 0$) Underdamped Case

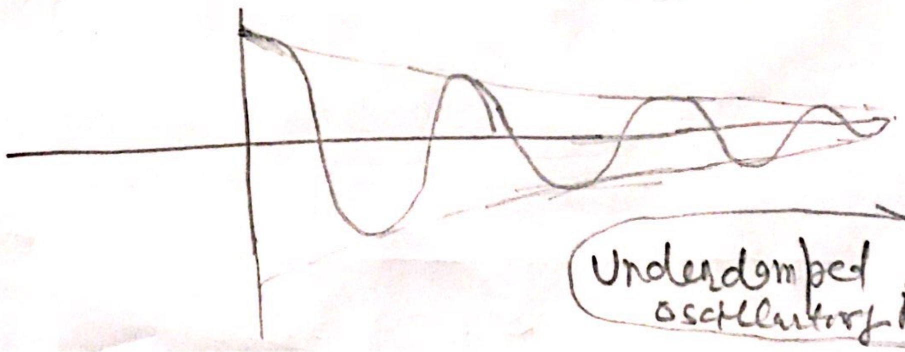
Then we obtain two complex roots (in pair) such that real part is $-b$ (i.e. a negative number)

$$m = -b \pm \alpha i \qquad \alpha = \sqrt{\omega_0^2 - b^2}$$

then $y = e^{-bt}\left[c_1 \cos\alpha t + c_2 \sin\alpha t\right]$

$$\xrightarrow[c_1 = y_0;\ c_2 = 0]{y(0) = y_0,\ y'(0) = 0} \quad y_0 e^{-bt} \cos\alpha t$$

amplitude is $y_0 e^{-bt}$. It decreases as t increases and $y \to 0$ as $t \to \infty$.



(Underdamped oscillatory Motion)

Subcase 3 (Critical damping case) $(b = \omega_0)$

then it is a case of two real and equal roots.

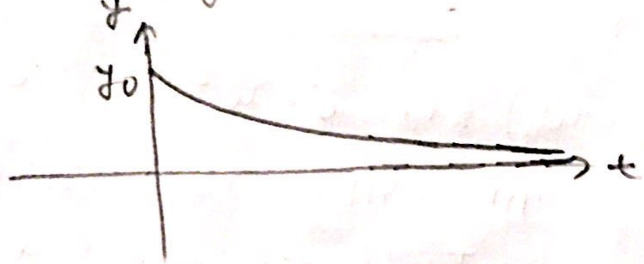
$m = -b, -b$. Then solution is

$$y(t) = e^{-bt}(c_1 + c_2 t) \longrightarrow 0 \text{ as } t \to \infty$$

Further there is not oscillations and damping is just enough to prevent any oscillation.

If $y(0) = y_0$ and $y'(0) = 0$, then we obtain $c_1 = y_0$ and $c_2 = by_0$

and thus $y(t) = y_0 e^{-bt}(1 + bt)$



$\boxed{\text{Read: Sections 5.1 – 5.3.}}$

One trick for finding a solution of variable coefficient 2^{nd} order linear homogeneous ODE

$$y'' + p(x)y' + q(x)y = 0 \quad \text{—— } (*)$$

Here try: $y = x^m$ is a solution to $(*)$

Then

$$m(m-1)x^{m-2} + pmx^{m-1} + qx^m = 0$$

Already done! for constant coefficient case [Algorithm is:

- find AE and its roots
- Write sol. as per roots

Remember! logic was trial solution e^{mx}

$$\Rightarrow \boxed{\left[m(m-1) + mpx + qx^2\right]x^{m-2} = 0}$$

Never true if p & q are constants.

$$\boxed{\begin{aligned} px + qx^2 = 0 &\Leftrightarrow y = x \text{ is a solution to } (*) \\ 2 + 2px' + qx^2 = 0 &\Leftrightarrow y = x^2 \text{ is a solution to } (*) \end{aligned}}$$

The above trick can be combined by Thm (B)

[order-Reduction method]

Example

Solve

$$x^2 y'' - 2x(1+x) y' + 2(1+x)y = 0 \quad — \quad \text{Eq1}$$

$$p(x) = -\frac{2(1+x)}{x} \qquad q(x) = \frac{2(1+x)}{x^2}$$

Since $px + qx^2 = 0$, thus $y = x$ is a solution to (Eq1),

Then, invoke thm (B) for second solution

$$y_2 = y_1 \cdot u$$

when $u = \int \frac{1}{y_1^2} e^{-\int p(x)\,dx} dx$

$$= \int \frac{1}{x^2} e^{+\int(\frac{2}{x}+2)dx} dx$$

$$= \int \frac{1}{x^2} e^{(\ln x^2 + 2x)} dx = \int \frac{1}{x^2} e^{\ln x^2} \cdot e^{2x} dx$$

$$= \frac{1}{2} e^{2x}$$

Thus $y_2 = \frac{1}{2} x \cdot e^{2x}$

and general solution to (Eq1) is

$$y(x) = c_1 x + c_2 \left(\frac{x}{2} \cdot e^{2x}\right)$$

Practice Problems! Do all exercises from 1 to 9 of Section 4.1 on page 124-125 from SL Ross (3rd edition)

Cauchy - Euler equation

$$a_0 x^n y^{(n)} + a_1 x^{n-1} y^{(n-1)} + \cdots + a_{n-1} x y' + a_n y = F(x)$$

($a_0, a_1, a_2, \ldots a_n$ are constants)

By using transformation $x = e^t$ the cauchy - Euler ODE reduces to a constant coefficient ODE — for details read Section 4.5 (pages 164 — 169 of SL Ross).

Otherwise the trick of guessing $y = x^m$ as a solution works nicely for Cauchy - Euler problem. Read a remark given below the Ques. 9 in tutorial sheet for 2nd order

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Example Solve IVP

$$\begin{cases} x^2 y'' - 4x y' + 6y = 0 \\ y(2) = 0 \\ y'(2) = 4 \end{cases}$$

ODE is: $x^2 y'' - 4x y' + 6y = 0$ —— (Eq)

Take $y = x^m$. Then, we obtain

$$x^2 m(m-1) x^{m-2} - 4x \, m \, x^{m-1} + 6 x^m = 0$$

$$\Rightarrow [m(m-1) - 4m + 6] x^m = 0$$

$$\Rightarrow [m^2 - 5m + 6] x^m = 0$$

So x^m is a solution to (Eq) when $m^2 - 5m + 6 = 0$

$\Rightarrow (m-3)(m-2) = 0$

$\Rightarrow m = 2, 3$

Hence $y(t) = c_1 x^2 + c_2 x^3$

By putting initial values we obtain

$$\left.\begin{aligned} 4c_1 + 8c_2 &= 0 \\ 4c_1 + 12c_2 &= 4 \end{aligned}\right\} \Rightarrow \begin{aligned} c_2 &= 1 \\ c_1 &= -2 \end{aligned}$$

Hence solution to IVP is

$$\boxed{y(t) = -2x^2 + x^3}$$

Ans