PARTIAL DERIVATIVE

- $f: R^2 \longrightarrow R$

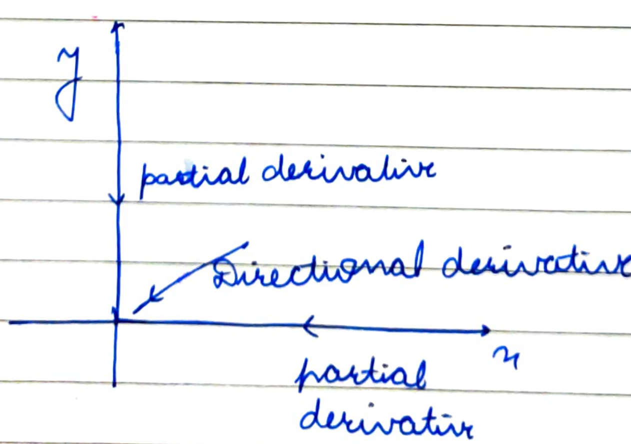
$$\frac{\partial f}{\partial x}(x, y) = \lim_{h \to 0} \frac{f(x_0 + h, y_0) - f(x_0, y_0)}{h}$$

$$\frac{\partial f}{\partial y}(x_0, y_0) = \lim_{k \to 0} \frac{f(x_0, y_0 + k) - f(x_0, y_0)}{k}$$

- $f: R^n \longrightarrow R$

$$e_i = (0, 0, \ldots\ldots, \underset{\substack{\downarrow \\ i^{th} \text{ place}}}{1}, \ldots\ldots, 0)$$

$$\frac{\partial f}{\partial x_i} = \lim_{h \to 0} \frac{f(x_0 + h e_i) - f(x_0)}{h}$$

- 

NOTE: Partial Derivative $\not\Rightarrow$ continuity

$\Downarrow$

Partial Derivatives exist

NOTE: consider $\frac{xy}{x^2+y^2}$. This will have partial derivatives only if its limit for PD exists at (0, 0).

ie to check partial derivability of a f^n, check at all of its discontinuous points. If limit exists, then P.D. exists

DIRECTIONAL DERIVATIVE

- $f: \mathbb{R}^n \longrightarrow \mathbb{R}$, $a \in \mathbb{R}^n$

Let $u \in \mathbb{R}^n$ such that $\|u\| = 1$

the the limit when exists,

- $$D_u f(a) = \lim_{t \to 0} \frac{f(a+tu) - f(a)}{t}$$

$$= \frac{d}{dt} f(a+tu) \Big|_{t=0}$$

= Rate of change of f at a in direction u.

- eg- $f : \mathbb{R}^2 \to \mathbb{R}$

$$f(x,y) = \sqrt{|xy|}$$

then Directional derivative $D_u f(0,0)$ does not exist

for $u_1, u_2 \neq 0$; $u = (u_1, u_2)$ — a vector of norm $= 1$

NOTE: Partial derivative $\not\Rightarrow$ Directional derivative $\not\Rightarrow$ continuity $\not\Rightarrow$ Directional derivative.

→ THEOREM 1 →

- If f is differentiable at $x_0 \in \mathbb{R}^3$ then $D_{\underline{u}} f(x_0)$ exists for all $\underline{u} \in \mathbb{R}^3$

& $$\boxed{D_{\underline{u}} f(x_0) = (f_x(x_0), f_y(x_0), f_z(x_0)) \cdot \underline{u}}$$

dot product

ie $f : \mathbb{R}^3 \to \mathbb{R}$

unit vector

- The above theorem can be generalised for all $\mathbb{R}^n$

→ GRADIENT

- Defined as

$$\nabla f(\underline{x_0}) = (f_x(\underline{x_0}), f_y(\underline{x_0}), f_z(\underline{x_0}))$$

Direction where f increases or decreases most rapidly.

NOTE: $f: \mathbb{R} \to \mathbb{R}$ then $f': \mathbb{R} \to \mathbb{R}$

& $f: \mathbb{R}^3 \to \mathbb{R}$ then $\nabla f: \mathbb{R}^3 \to \mathbb{R}^3$

NOTE: $f: \mathbb{R} \to \mathbb{R}$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

we can write that f is differentiable at x iff $\exists\ \alpha \in \mathbb{R}$ s.t.

scalar — 2 vectors' dot product

$$\left| \frac{f(x+h) - f(x) - \alpha . h}{h} \right| \to 0 \text{ as } h \to 0$$

scalar

where α becomes derivative of f

suppose

$f: \mathbb{R}^3 \to \mathbb{R}$, f is diff. at $\underline{x}$, if $\exists$ $\underline{\alpha} = (\alpha_1, \alpha_2, \alpha_3)$; $\underline{\alpha} \in \mathbb{R}^3$

such that the error f^n.

$$\boxed{\epsilon(H) = \frac{f(\underline{x}+\underline{H}) - f(\underline{x}) - \underline{\alpha}.\underline{H}}{\|H\|} \to 0 \text{ as } \|H\| \to 0}$$

→ THEOREM 2 →

- $f: \mathbb{R}^n \to \mathbb{R}$

If f is differentiable at $\underline{x}$

⇒ f is continuous
& ⇒ f has all its directional derivatives.
& ⇒ f has partial derivatives

→ THEOREM 3 →

- f is differentiable at $\underline{x}$

then
$$\boxed{f'(\underline{x}) = (\alpha_1, \alpha_2, \alpha_3) = \left(\left.\frac{\partial f}{\partial x}\right|_{\underline{x}}, \left.\frac{\partial f}{\partial y}\right|_{\underline{x}}, \left.\frac{\partial f}{\partial z}\right|_{\underline{x}} \right)}$$

→ THEOREM 4 →

- $f: \mathbb{R}^3 \to \mathbb{R}$ such that all partial derivatives exist in a neighbourhood of x_0 and f is continuous at x_0 then f is differentiable at x_0

→ DIFFERENTIABILITY

- consider $f: \mathbb{R} \to \mathbb{R}$

$$\lim_{x \to a} \frac{f(x) - f(a)}{x-a} = f'(a)$$

$$\Rightarrow \lim_{x \to a} \frac{f(x) - f(a) - f'(a)(x-a)}{x-a} = 0$$

Let $h(x) = f'(a)(x-a) + f(a)$.

Then $h(x) \to f(x)$ faster than $x \to a$ (since limit is 0)

- f is differentiable if $\exists$ a linear function $L(x)$ such that

linear f^n $f(x)$

$$\boxed{\lim_{x \to a} \frac{f(x) - f(a) - L(x)}{x-a} = 0}$$

where $\boxed{L(x) = f'(a)(x-a)}$

$f : \mathbb{R}^2 \longrightarrow \mathbb{R}$

$$h(x,y) = f(a,b) + f_x(a,b)\,(x-a) + \cancel{f(x,y)}\, f_y(a,b)\,(\cancel{x}y-b)$$

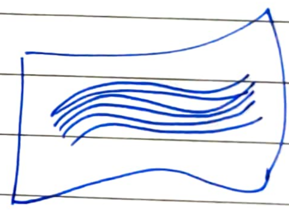
is a good linear approximation of $f(x,y)$ near ~~f(a~~ (a,b)

- Quasilinear 1st order PDE solution

$a(x,y,z)\, z_x + b(x,y,z)\, z_y = c(x,y,z)$ —(i)

- The solution is a surface. ($\mathbb{R}^3$)
- To find solution what we'll do is that we'll find a curve on the surface (which obviously satisfies the PDE) & we take union of all such curves to get the surface.
- Assume

solution surface



— $F(x,y,z) = 0$

and let curve

— $c: (x(s), y(s), z(s))$ ie x, y, z are parametrised.

$\underbrace{\qquad}_{\text{diff. w.r.t. 's'}}$

- now we put the curve parameters in surface

$F(x(s), y(s), z(s)) = 0$

$$\Rightarrow \frac{\partial F}{\partial x}\frac{dx}{ds} + \frac{\partial F}{\partial y}\frac{dy}{ds} + \frac{\partial F}{\partial z}\frac{dz}{ds} = 0$$

now choosing

$$F \equiv u(x,y) - z$$

$$\Rightarrow \underbrace{(u_x, u_y, -1)}_{\text{direction of normal}} \cdot \underbrace{\left(\frac{dx}{ds}, \frac{dy}{ds}, \frac{dz}{ds}\right)}_{\text{direction of tangent of } C} = 0 \quad \text{(ii)}$$

comparing from eqn (i), we can write

$u_x \to z_x \quad u_y \to z_y$

Eqn(i) is $(a, b, c) \cdot (z_x, z_y, -1) = 0 \quad \text{(iii)}$

$\therefore$ from (i) & (ii)

$$\left.\begin{aligned} \frac{dx}{ds} &= a(x(s), y(s), z(s)) \\ \frac{dy}{ds} &= b(x(s), y(s), z(s)) \\ \frac{dz}{ds} &= c(x(s), y(s), z(s)) \end{aligned}\right\}$$

characteristic eqn

System of ODE

or $\boxed{\frac{dx}{a} = \frac{dy}{b} = \frac{dz}{c}} \longrightarrow$ characteristic eqn

Now we can use Picard's Theorem to solve the ~~ODE~~ ODE and get the curve. Further union of curves will give us surface.

characteristic curve

NOTE: Picard's Theorem: (for ODE)

$$y' = f(x, y)$$
$$\Rightarrow y(x_0) = y_0$$

if $f(x, y)$ is Lipschitz, i.e. $|f(x, y_1) - f(x_2, y_2)| \leq M|y_1 - y_2|$

& f is continuous wrt x & y, then $\exists$ unique solution in neighborhood of x_0

NOTE: Further to solve (i),

Theorem

$$a z_x + b z_y = c(x, y, z) \quad \text{(iv)}$$

where $a, b, c \in C'$, $a^2 + b^2 + c^2 \neq 0$

The gen. sol^n of (iv) is given by

$F(u, v) = 0$ where F is arbitrary f^n

and $u(x, y, z) = c_1$, $v(x, y, z) = c_2$ are two LI sol^{n}s of

$$\frac{dx}{a} = \frac{dy}{b} = \frac{dz}{c} = \frac{l_1 dx + l_2 dy + l_3 dz}{l_1 a + l_2 b + l_3 c}$$

eg- $yz\, z_x + xz\, z_y = xy$ ⟶ solution is a surface

$$\therefore \frac{dx}{yz} = \frac{dy}{xz} = \frac{dz}{xy} \qquad (i)$$

$$\Rightarrow \frac{dx}{y} = \frac{dy}{x} \qquad \Rightarrow x\,dx - y\,dy = 0$$

$$\Rightarrow x^2 - y^2 = c_1$$ ⟶ Ц solⁿ of (i)

& also

$$\frac{dy}{z} = \frac{dz}{y} \qquad \Rightarrow y^2 - z^2 = c_2$$ ⟶ Ц solⁿ of (i)

$$\therefore F(x^2-y^2,\ y^2-z^2) = 0$$

is the required surface (where F is arbitrary)

$$\therefore x^2 - y^2 = c_1 (y^2 - z^2)$$ ⟶ Infinite set of solⁿs

is a solution of given PDE.

NOTE: Solution is implicit.

NOTE: If initial conditions are given, we can find c_1.

Eg → $z_t + z z_x = 0$ → Burger's Equation & solⁿ passes thr. $z(x,0) = -x$

$$\frac{dt}{1} = \frac{dx}{z} = \frac{dz}{0}$$

$$\Rightarrow \frac{dt}{1} = \frac{dz}{0} \Rightarrow z = c_1$$

&

$$\frac{dt}{1} = \frac{dx}{z} \Rightarrow \frac{dt}{1} = \frac{dx}{c_1}$$

$$\Rightarrow x - c_1 t = c_2$$

$$\Rightarrow x - zt = c_2$$

General solⁿ

$$F(z, x - zt) = 0$$

$$\Rightarrow z(x,t) = \phi(x - zt)$$

$$\Rightarrow z(x,0) = \phi(x)$$

$$\Rightarrow -\cancel{x} = \phi \times \cancel{x}$$

$$\Rightarrow \phi = -1$$

∴ Solution is

$$z(x,t) = zt - x$$

$$\Rightarrow z = \frac{x}{t-1}$$

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Eg- $u_x = cu + d(x,y)$ ← PDE

continuous & diff.

by Integrating factor (y as parameter)

$$u(x,y) = e^{cx}\left(\int_0^x e^{-c\xi}\, d(\xi,y)\, d\xi + u(0,y)\right)$$

unknown

- $u_x = cu + d(x,y)$ & $\underbrace{u(0,y) = y}_{\text{Initial curve}}$

sol^n is <u>unique</u>

- $u_x = cu$, $u(x,0) = e^{cx}$; $u(x,y) = e^{cx} T(y)$

$e^{cx} = e^{cx}\, T(0)$

$\Rightarrow T(0) = 1$

<u>Infinite sol^n</u>.

- Check for

$u_x = cu$, $u(x,0) = \sin(x)$

Ans <u>no solution</u>

NOTE : In linear ODE, we can't have ∞ sol^n, but we can have ∞ sol^n for PDE

THEOREM: EXISTENCE AND UNIQUENESS THEOREM

- $P(x,y,z)\, z_x + Q(x,y,z)\, z_y = R(x,y,z)$

$P, Q, R \in C'$, $P^2 + Q^2 + R^2 \neq 0$

consider

$x = x_0(s)$, $y = y_0(s)$, $z = z_0(s)$

as initial data curve (continuously diff. on $s \in [a, b]$)

JACOBIAN

If $\boxed{\frac{dy_0}{ds} P(x_0(s), y_0(s), z_0(s)) - \frac{dx}{ds} Q(x_0(s), y_0(s), z_0(s))} \neq 0$

Then $\exists$ a unique solution in a neighborhood of $x = x_0(s)$, $y = y_0(s)$, $z = z_0(s)$ which satisfy $z(x_0(s), y_0(s)) = z_0(s)$

NOTE: An initial data curve does not guarantee a unique solution. If Jacobian is zero, then we may have ∞ sol^n or no sol^n.

- check for last 3 eqs.

* → SINGULAR SOLUTION

- consider

$$\left.\begin{aligned} y' &= 2\sqrt{y} \\ y(0) &= 0 \end{aligned}\right\}$$

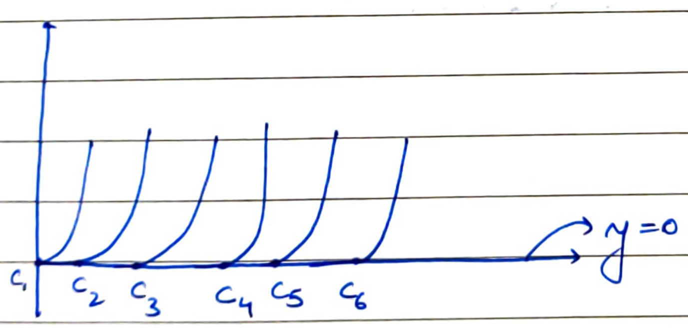
The above ODE has 2 sol^n s

one parameter family of curves

$$y(x) = \begin{cases} (x-c)^2 & x \geq c > 0 \\ 0 & x \leq c \end{cases}$$

and

$$y = 0$$



$y=0$ is tangent to all the solution curves. and is itself a sol^n to the curve.

Such a solution is called singular sol^n to a DE.

NOTE: The envelope of a family of curves which is a solution to DE is called the singular sol^n of DE.

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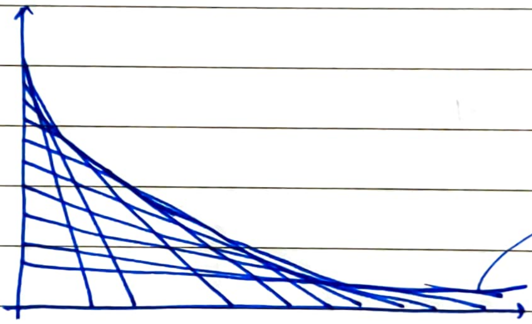
NOTE: Envelope: envelope of a family of curves on a plane is a curve that is tangent to each member of the family at some point.

$$\left.\begin{array}{l} F(x, y, \overset{\text{parameter}}{t}) = 0 \\ \dfrac{\partial F}{\partial t} = 0 \end{array}\right\} \text{Remove 't' by solving these eq}^n\text{s}$$

eg- $\dfrac{x}{t} + \dfrac{y}{11-t} = 1$



→ SINGULAR SOLNS FOR TWO PARAMETER FNS

a) COMPLETE INTEGRAL

A two parameter ~~integral~~ family of sol^n $z = F(x, y, a, b)$ is called complete integral.

b) GENERAL INTEGRAL

Consider $b = \phi(a)$ ← Specific assumptio

$$z = F(x, y, a, \phi(a)) \quad -(i)$$

Then the envelope (if it exists) can be obtained from (i) along with (ii) below

$$\boxed{F_b + F_a \, \phi'(a) = 0} \quad (ii)$$

- Say $a = a(x, y)$

  Then $z = F(x, y, a(x,y), \phi(a(x,y)))$

  is general integral when ϕ is arbitrary.

- If ϕ is specific, then it is called particular integral

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- consider PDE (general)

$$a(x,y,u)\,u_x + b(x,y,u)\,u_y = c(x,y,u)$$

putting $b=1$, $a=u$ and $c=0$, we get Burger's eq^n

$$u_y + u u_x = 0$$

Lets say given initial curve

$$u(x,0) = h(x) \qquad x \in \mathbb{R}$$

putting $x=s$, $y=0$ $\Rightarrow u = h(s)$

We get solution of Burger's equation as

$$\left.\begin{aligned} x(s,t) &= h(s)\,t + s \\ y &= t \\ z = u(s,t) &= h(s) \end{aligned}\right\} \Rightarrow x = h(s)\,y + s \quad \text{(i)}$$

solution $\rightarrow$ $u = h(x - yu)$

Also $\dfrac{dx}{dt} = a$; $\dfrac{dy}{dt} = b$; $\dfrac{du}{dt} = c$

Now we will draw ~~the~~ two solutions on initial curve and take their projection on x-y plane.

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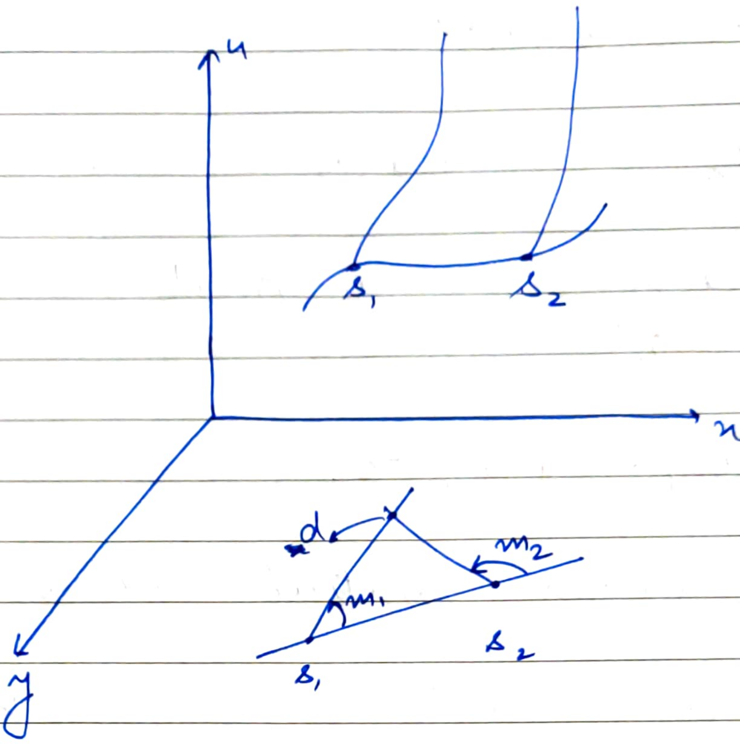


Their projection could be of this form

- If above case happens, then at point d, we have an ambiguity of solution (one solⁿ was along s_1 and one along s_2) $\therefore$ to avoid this, $m_1 \geq m_2$

for $s_1 \leq s_2$

$\therefore$ acc. to above statement

At $s = s_1$, $m_1 = \frac{1}{h(s_1)}$ (from eqⁿ (i) on prev. page)

At $s = s_2$, $m_2 = \frac{1}{h(s_2)}$

$\therefore \frac{1}{h(s_1)} \geq \frac{1}{h(s_2)}$; for $s_1 < s_2$

$\Rightarrow h(s_2) \geq h(s_1)$; $s_1 < s_2$

$h(s_2) \geq h(s_1)$; $s_1 < s_2$

$\therefore$ h should be monotonically increasing or non-decreasing.

- Now consider $y > 0$ and assume h is decreasing f^n ($h' \leq 0$)

$$u = h(x - yu)$$

$$u_x = h'(x-yu)(1 - y u_x)$$

$$\Rightarrow u_x = \frac{h'(x-yu)}{1 + \underbrace{y}_{>0}\,\underbrace{h'(x-yu)}_{<0}}$$

$\therefore$ u_x will become ∞ if $1 + y h'(x - uy) = 0$

This is called gradient catastrophe.

- Examples:

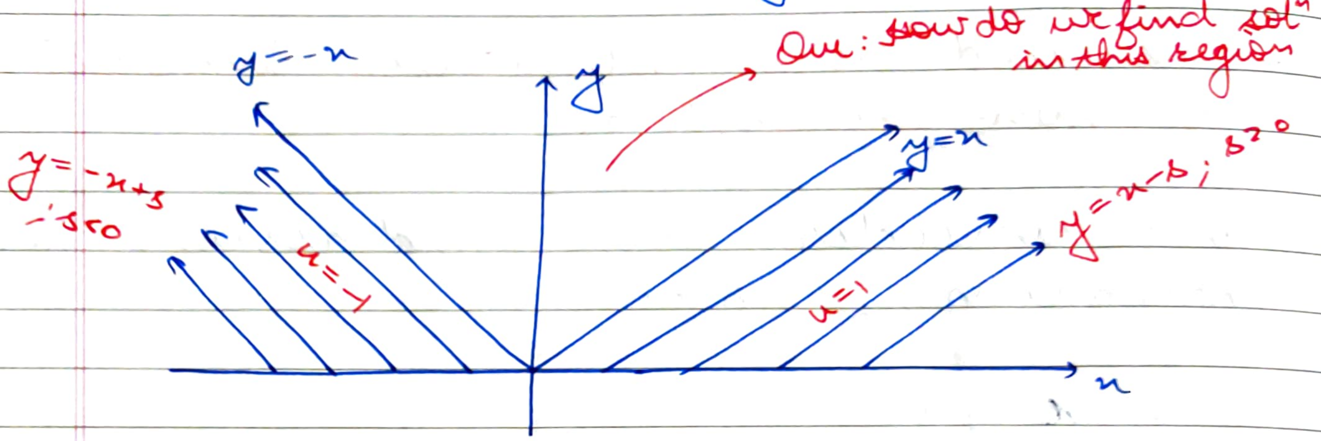
a) $h(x) = \begin{cases} -1 & x < 0 \\ 1 & x \geq 0 \end{cases}$

Base characteristic $y = \frac{x}{h(s)} - \frac{s}{h(s)}$

Suppose $s < 0$, $h(s) = -1$, $\Rightarrow y = -x + s$

& $s > 0$, $h(s) = 1$, $\Rightarrow y = x - s$

In above case, we get projection curves like



Since h is monotonically non-decreasing
∴ we see no intersection

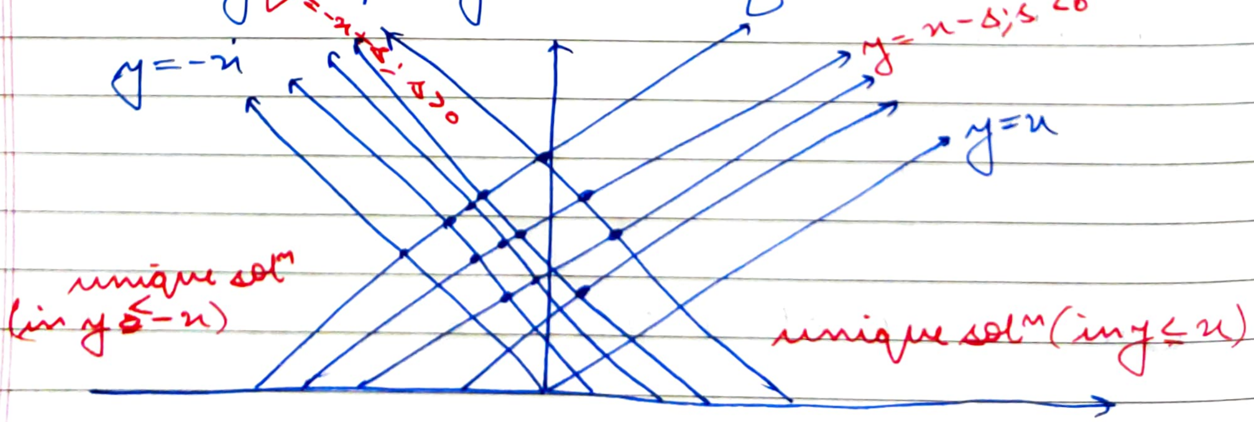
b) $h(x) = \begin{cases} 1 & x<0 \\ -1 & x \geq 0 \end{cases}$

$$y = \frac{x}{h(s)} - \frac{s}{h(s)}$$

$s<0$, $h(s)=1$, $y = x-s$

$s>0$, $h(s)=-1$, $y = -x+s$

∴ we get projection of sol^{n}s like



→ DOMAIN OF DEPENDENCE AND RANGE/DOMAIN OF INFLUENCE

- Consider PDE

$$u_x + u_y = 0$$
$$u(x,0) = \sin x \quad , \quad x \geq 0$$

} soln: $u = \sin(x-y)$ where $x-y \geq 0$

$\frac{dx}{dt} = 1 \Rightarrow x = t + c_1$

$\frac{dy}{dt} = 1 \Rightarrow y = t + c_2$

$\frac{du}{dt} = 0 \Rightarrow u = c_3$

$x = s$

$y = 0$

$u = \sin(s)$

** Second Order Partial Differential Equations

* Semilinear / Linear Second Order PDE

- It is of the form

principle part of PDE

$$\boxed{\underbrace{a(x,y)u_{xx} + 2b(x,y)u_{xy} + c(x,y)u_{yy}} + d(x,y)u_x + e(x,y)u_y + f(x,y)u = g(x,y)}$$

linear 2^{nd} order PDE

- We may take d, e, f as ~~forms~~ f^ns of (x, y, u_x, u_y)
- We can transform the above PDE to other variables to simplify it.
- Consider transformation

$$(x,y) \rightleftarrows (\xi, \eta)$$

For this transformation Jacobian must be non-zero

$$\frac{\partial(\xi,\eta)}{\partial(x,y)} \neq 0, \infty$$

- For this transformation,

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$$\left.\begin{aligned}\frac{\partial u}{\partial x} &= u_\xi \xi_x + u_\eta \eta_x = u_x \\ \frac{\partial u}{\partial y} &= u_\xi \xi_y + u_\eta \eta_y = u_y\end{aligned}\right\} \text{chain rule}$$

now, we will find all terms of partial diff. (u_{xx}, u_{yy}, u_{xy} etc.) in our PDE with help of our eq^{n}s above from chain rule

$$u_{xx} = \frac{\partial (u_\xi \xi_x + u_\eta \eta_x)}{\partial x} = \frac{\partial}{\partial x}(u_x)$$

$$= u_\xi \xi_{xx} + \xi_x \frac{\partial}{\partial x}(u_\xi) + u_\eta \eta_{xx} + \eta_x \frac{\partial}{\partial x}(u_\eta)$$

$$\Rightarrow u_{xx} = u_\xi \xi_{xx} + \xi_x (u_{\xi\xi} \xi_x + u_{\xi\eta} \eta_x) + u_\eta \eta_{xx}$$
$$+ \eta_x (u_{\xi\eta} \xi_x + u_{\eta\eta} \eta_x)$$

Similarly

$$u_{xy} = u_{\xi\xi} \xi_x \xi_y + u_{\xi\eta} (\xi_x \eta_y + \xi_y \eta_x)$$
$$+ u_{\eta\eta} \eta_x \eta_y + u_\xi \xi_{xy} + u_\eta \eta_{xy}$$

On substituting (after finding) all the partial derivatives, we get an eq^n of form

$$\bar{A} u_{\xi\xi} + 2\bar{B} u_{\xi\eta} + \bar{C}_{\eta\eta} + F(\xi, \eta, u, u_\xi, u_\eta) = 0 \quad -(i)$$

$$D_\xi = \frac{\xi_x}{\xi_y} \qquad D_\eta = \frac{\eta_x}{\eta_y}$$

where

$$\bar{A} = a\xi_x^2 + 2b\,\xi_x\xi_y + c\xi_y^2 = [aD_\xi^2 + 2bD_\xi + c]\xi_y^2$$

$$\bar{B} = a\xi_x\eta_x + b[\xi_x\eta_y + \eta_x\xi_y] + c\,\xi_y\eta_y = [aD_\xi D_\eta + b(D_\xi + D_\eta) + c]\,\xi_y\eta_y$$

$$\bar{C} = a\eta_x^2 + 2b\eta_x\eta_y + c\eta_y^2 = [aD_\eta^2 + 2bD_\eta + c]\,\eta_y^2$$

Now, to simplify (i) above, we will make ξ and η such that $\bar{A}$ and $\bar{C}$ are zero (0). (just for simplification)

$$\bar{A} = 0 \Rightarrow [aD_\xi^2 + 2bD_\xi + c]\,\cancel{\xi_y^2} = 0$$

+ve

$$\Rightarrow D_\xi = \frac{-b \pm \sqrt{b^2 - ac}}{a} = \frac{\xi_x}{\xi_y}$$

Similarly for

$$\bar{C} = 0 \Rightarrow D_\eta = \frac{-b \pm \sqrt{b^2 - ac}}{a}$$

Show condn for $\bar{B} \neq 0$

D_ξ and D_η have diff. values. One is generated from +ve sign & one from -ve.

$$\therefore \quad D_{\xi} = \frac{-b-\sqrt{b^2-ac}}{2a} = \frac{\xi_x}{\xi_y}$$

$$D_{\eta} = \frac{-b+\sqrt{b^2-ac}}{a} = \frac{\eta_x}{\eta_y}$$

Now to find $\xi(x,y)$ and $\eta(x,y)$,

consider

$$\xi(x,y) = C_1 \rightsquigarrow \text{constant} \ (\xi \text{ is a family of curves})$$

$$\Rightarrow d\xi = \xi_x dx + \xi_y dy = 0$$

$$\Rightarrow \frac{dy}{dx} = -\frac{\xi_x}{\xi_y} = \frac{b+\sqrt{b^2-ac}}{a}$$

$$\Rightarrow \frac{dy}{dx} = \frac{b+\sqrt{b^2-ac}}{a} \longrightarrow \text{find } \xi(x,y) = C_1$$

We solve integral above & write const. of integ. on RHS and f^n ~~of~~ on LHS. Then LHS $= \xi(x,y)$ & $\xi(x,y) = C_1 \rightsquigarrow$ const of integ.

Similarly find $\eta(x,y)$

$$\eta(x,y) = C_2$$

$$\Rightarrow \frac{dy}{dx} = \frac{b - \sqrt{b^2-ac}}{a}$$

Find $\eta(x,y) = C_2$

- now

If $b^2 - ac = 0$

$\Rightarrow \frac{dy}{dx} = b/a$

$\Rightarrow \xi(x, y) = c_1$

Find η such that $\frac{\partial(\xi, \eta)}{\partial(x, y)} \neq 0$

$\Rightarrow \bar{A} = 0$ (ξ, η are LI)

also $\bar{B} = 0 \rightarrow$ It becomes 0

~~these~~ In this case

ξ and η are called characteristic curves.

- In case of $b^2 - ac < 0$

Then $\left.\begin{array}{l} \xi = c_1 \\ \eta = c_2 \end{array}\right\}$ complex transformations

In this case, we can form

$\left.\begin{array}{l} \alpha = \xi + \eta \\ \beta = i(\xi - \eta) \end{array}\right\}$ Real transformations.

- Therefore, we can conclude

a) $b^2 - ac > 0$

$u_{\xi\eta}$ + lower order terms $= 0$

b) $b^2 - ac = 0$

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→ CAUCHY PROBLEM

- $$a u_{xx} + 2b u_{xy} + c u_{yy} + F(x, y, u, u_x, u_y) = 0 \quad \text{(i)}$$

where a, b, c are fns of x and y

Initial data

$x = \phi(s)$, $y = \theta(s)$ $\quad s \in [s_0, s_1]$

assume on Γ : u, u_x, u_y are given for $s \in [s_0, s_1]$

$$f'(s) = \frac{\partial u}{\partial s} = \frac{\partial u}{\partial x}\frac{dx}{ds} + \frac{\partial u}{\partial y}\frac{dy}{ds} = u_x \phi'(s) + u_y \theta'(s)$$

Let $p = u_x$, $q = u_y$ are given on the curve

$$\frac{dp}{ds} = \frac{d}{ds}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial^2 u}{\partial x^2}\frac{dx}{ds} + \frac{\partial^2 u}{\partial x \partial y}\frac{dy}{ds} \quad \text{(ii)}$$

$$\frac{dq}{ds} = \frac{d}{ds}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial^2 u}{\partial x \partial y}\frac{dx}{ds} + \frac{\partial^2 u}{\partial y^2}\frac{dy}{ds} \quad \text{(iii)}$$

We will solve (i), (ii) and (iii) to find u_{xx}, u_{yy}, u_{xy}

Solving it

$$\begin{bmatrix} a & 2b & c \\ \frac{dx}{ds} & \frac{dy}{ds} & 0 \\ 0 & \frac{dx}{ds} & \frac{dy}{ds} \end{bmatrix} \begin{bmatrix} u_{xx} \\ u_{xy} \\ u_{yy} \end{bmatrix} = \begin{bmatrix} -F \\ \frac{dp}{ds} \\ \frac{dq}{ds} \end{bmatrix}$$

If $\det(M) \neq 0$ then we can find u_{xx}, u_{xy}, & u_{yy} uniquely on Γ

$\therefore$ now,

$u, u_x, u_y, u_{xx}, u_{xy}$, & u_{yy} are known uniquely on the curve. Here, we can find the solⁿ uniquely on the neighborhood of the curve by Taylor's expansion.

$$f(x,y) = f(0,0) + x f_x(0,0) + y f_y(0,0) +$$

$$\frac{x^2}{2!} f_{xx}(0,0) + xy f_{xy}(0,0) + \frac{y^2}{2!} f_{yy}(0,0) \cdots\cdots$$

NOTE: Suppose $\det(M) = 0$, $\therefore$ in that case, expanding $\det(M)$

$$a\left(\frac{dy}{ds}\right)^2 - 2b\frac{dx}{ds}\frac{dy}{ds} + c\left(\frac{dx}{ds}\right)^2 = 0$$

Divide by $\left(\frac{dx}{ds}\right)^2$ and note that $\dfrac{\frac{dy}{ds}}{\frac{dx}{ds}} = \dfrac{dy}{dx}$

$$\therefore a\left(\frac{dy}{dx}\right)^2 - 2b\frac{dy}{dx} + c = 0$$

$$\boxed{\frac{dy}{dx} = \frac{b \pm \sqrt{b^2 - ac}}{a}} \longleftarrow \text{characteristic curve}$$

Charac. eqⁿ of ~~[illegible]~~ PDE

So, if we specify the data on the characteri

However, we can't to expect a unique solⁿ

NOTE:

$-\Delta u = 0$ ⟿ Laplace eqⁿ

$-\Delta u = f$ ⟿ Poisson's eqⁿ.

$$\left.\begin{aligned} -\Delta u &= f \quad x \in \Omega \text{ — Domain} \\ u\big|_{\partial\Omega} &= g \end{aligned}\right] \text{ Dirichlet Problem}$$

$u \in C^2(\Omega) \cap C(\bar{\Omega})$

⟶ u is double derivable & double derivative is continuous

→ D ALEMBERT SOLUTION (WAVE EQN)

- ∞ string

$$\left.\begin{array}{l} y_{tt} - c^2 y_{xx} = 0 \\ \text{Initial curve} \begin{cases} y(x,0) = f(x) \\ y_t(x,0) = g(x) \end{cases} \end{array}\right\} (i) \qquad \begin{array}{c} -\infty < x < \infty \\ t \geq 0 \end{array}$$

y_t represents velocity at posn x at $t=0$

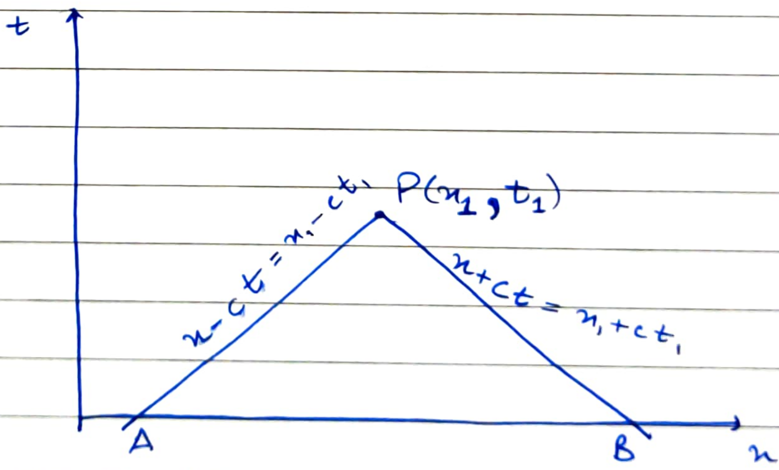
characteristics → $\xi = x - ct$

$\eta = x + ct$

canonical form $u_{\xi\eta} = 0$

- $$\boxed{y(x,t) = \frac{1}{2}\left[f(x+ct) + f(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\,ds}$$

Assume $f \in C^2$ & $g \in C^1$

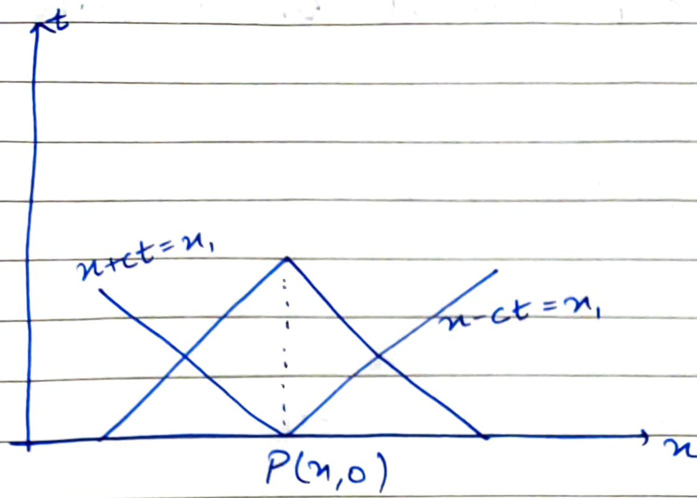


y_1 and y_2 are solution of (i) with $y_i(x,0) = f_i$

$y_{i_t}(x,0) = g_i$

where $f_1 = $ $g_1 = g_2$ in AB

Domain of dependence for the soln at $P(x_1, t_1)$ is AB interval.



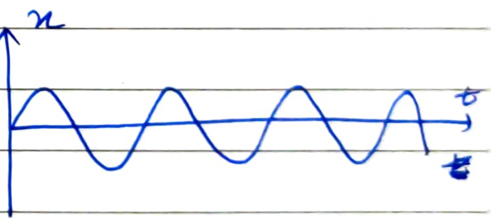
Region of Influence of $P(x,0)$ is inside the line segments

$$x+ct=x_1$$
$$x-ct=x_1$$

- Semi ∞ string

- $y_{tt} - \underbrace{c^2}_{\text{speed of wave}} y_{xx} = 0 \quad , \quad x \in (0,\infty)$

$$t>0$$



Initial cond^n $\begin{cases} y(x,0) = f(x) \\ y_t(x,0) = g(x) \end{cases}$

velocity

Boundary cond^n $\Big[\ y(0,t) = 0 \implies \big(y_t(0,t) = 0\big)$

disp. at $x=0$ is 0

vel at $x=0$ is zero

Extend f and g by odd expansion.

$$F(x) = \begin{cases} f(x), & x > 0 \\ -f(x), & x \leq 0 \end{cases}$$

$$G(x) = \begin{cases} g(x), & x \geq 0 \\ -g(x), & x \leq 0 \end{cases}$$

So now domain is $(-\infty, \infty)$

$$y(x,t) = \frac{1}{2}\left[F(x+ct) + F(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} G(s)\,ds$$

(De Alembert soln)

$$y(x,0) = \frac{1}{2}\left[F(x) + F(x)\right] + 0$$

$$= \frac{1}{2}\left[f(x) + f(x)\right] = f(x)$$

$$y(0,t) = \frac{1}{2}\left[F(ct) + F(-ct)\right] + \frac{1}{2c}\int_{-ct}^{ct} G(s)\,ds$$

$$= 0$$

Similarly show that

$$y_t(x,0) = g(x)$$

If $g=0$, then

$$y(x,t) = \begin{cases} \frac{1}{2}\left[u(x-ct) + u(x+ct)\right], & x \geq ct \\ \frac{1}{2}\left[u(x+ct) - u(ct-x)\right], & x < ct \end{cases}$$

- non-homogenous wave equation
- In such cases, we have a eqⁿ like

$$\left.\begin{aligned} & y_{tt} - c^2 y_{xx} = \underbrace{h(x,t)}_{\text{acceleration}} \\ & \text{IC} \begin{cases} y(x,0) = 0 \\ y_t(x,0) = 0 \end{cases} \end{aligned}\right] \quad (i)$$

$-\infty < x < \infty$
$t > 0$

At $t = s - \Delta s$ $h(x,s)$ is applied on string
at $t = s$, stop $h(x,s)$

Due to Δs time, the string will acquire a velocity $= \Delta s\, h(x,s)$

string's position will be changed to

$\frac{1}{2} h(x,s)\, \Delta s^2$ (can be neglected as Δs is small)

We construct a new PDE

$$w_{tt} - c^2 w_{xx} = 0 \qquad t \geq s \qquad (ii)$$

$$\text{IC}\begin{cases} w(x,s,s) = 0 & \leftarrow \text{displacement} \\ w_t(x,s,s) = h(x,s) & \leftarrow \text{velocity} \end{cases}$$

IC given at $t = s$

We can't directly apply D Alembert's solution to the above PDE. Before it, we need to transform it.

Change of variable

$$w(x,t,s) = \tilde{w}(x, t-s, s)$$

$\tilde{w}$ solves $\longrightarrow \tilde{w}_{tt} = c^2 \tilde{w}_{xx}$, $t \geq 0$

$$\tilde{w}(x,0,s) = 0, \qquad \tilde{w}_t(x,0,s) = h(x,s)$$

— (iii)

By D'Alembert's solⁿ to (iii)

$$\tilde{w}(x,t,s) = \frac{1}{2c} \int_{x-ct}^{x+ct} h(\pi, s)\, d\pi$$

solⁿ of (ii)

$$w(x,t,s) = \tilde{w}(x, t-s, s)$$

$$= \frac{1}{2c} \int_{x-c(t-s)}^{x+c(t-s)} h(\pi, s)\, d\pi$$

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Sol^n of (i)

$$y(x,t) = \int_0^t w(x,t,s)\, ds$$

$$= \int_0^t \tilde{w}(x, t-s, s)\, ds$$

$$y(x,t) = \frac{1}{2c} \int_0^t \int_{x-c(t-s)}^{x+c(t-s)} h(\pi, s)\, ds\, ds \qquad \text{(iv)}$$

Check by differentiability of (iv) that it solves

$$y_{tt} - c^2 y_{xx} = h(x,t)$$

(use Leibnitz Rule)

$$\frac{d}{dx}\left[\int_{a(x)}^{b(x)} f(x,t)\, dt \right] = f(x, b(x))\, b'(x)$$

$$- f(x, a(x))\, a'(x)$$

$$+ \int_{a(x)}^{b(x)} \frac{\partial}{\partial x} f(x,s)\, ds$$

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→ MAXIMUM PRINCIPLE

- $f: \mathbb{R} \to \mathbb{R}$ has a local maximum at $x = x_0$ if $\exists$ $\delta > 0$ and nbd of x_0, say N_δ, such that

$\boxed{f(x_0) \geq f(x)}$ $\forall$ $x \in N_\delta$

- suppose f is diff^n, $f: \mathbb{R} \mapsto \mathbb{R}$

necessary cond^n for max^m

$\boxed{f'(x_0) = 0}$

but this condition is not sufficient

- Sufficient condition

Let $x_0 \in (a, b)$

$f: (a, b) \longrightarrow \mathbb{R}$

$f'(x_0) = 0$

$f''(x_0) < 0$

$\Rightarrow$ maximum is at x_0

- if

$$\left.\begin{aligned} f'(x_0) &= 0 \\ f''(x_0) &> 0 \end{aligned}\right\} \text{minimum at } x_0$$

eg- $f(x) = -x^4 \quad f:(-1,1) \longrightarrow \mathbb{R}$

$$f'(0) = 0 \ , \ f''(0) = 0$$

- If f has max^m at x_0, then

$$f''(x_0) \leq 0 \ \& \ f'(x_0) = 0$$

~~fo~~ but if $f'(x_0)=0$ & $f''(x_0)=0$, then it does not imply a maximum. We need to check the further order derivative.

- maxima for f^ns of 2 variably
- $f : \mathbb{R}^2 \to \mathbb{R}$
- Existence : Domain is closed & bounded
- $f: D \longrightarrow \mathbb{R}$ is continuous

$\Rightarrow f$ has min^m / max^m

- <u>necessary cond^n</u>

Let (x_0, y_0) is an interior point in D

$f: D \rightarrow \mathbb{R}$

f_x, f_y exist in D

Then $\boxed{f_x(x_0, y_0) = 0 = f_y(x_0, y_0)}$

necessary cond^n but not sufficient

- eg- $f: R^2 \to R$, $f(x,y) = xy$

$f_x(0,0) = 0$, $f_y(0,0) = 0$ but $(0,0)$ is neither maxima or minima

- <u>Second derivative test</u>

(x_0, y_0) is an interior point of D

and $f_x(x_0,y_0) = 0 = f_y(x_0,y_0)$

- For second derivative test, we check the Hessian Matrix

$$H = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{vmatrix}$$

a) $H = \left(f_{xx} f_{yy} - f_{xy}^2\right)(x_0, y_0) > 0$

then

i) $f_{xx}(x_0,y_0) > 0 \Rightarrow f$ ~~has~~ min at (x_0,y_0)

ii) $f_{xx}(x_0,y_0) < 0 \rightarrow f$ has max at (x_0,y_0)

b) $H = \left(f_{xx} f_{yy} - f_{xy}^2\right)(x_0,y_0) < 0$

then (x_0,y_0) is <u>saddle point</u>

(ie in nbd of (x_0,y_0) there are 2 pts (x_1,y_1) (x_2,y_2) such that $f(x_0,y_0) > f(x_1,y_1)$ but $f(x_2,y_2) > f(x_0,y_0)$)

c) $H = (f_{xx} f_{yy} - f_{xy}^2)(x_0, y_0) = 0$

no conclusion

- maximum principle : Let $\Omega \subseteq R^2$ be a bounded domain and $u : \Omega \to R$ is continuous. Let u can be extended to boundary of Ω, ie $\partial\Omega$ by continuity. Such a function is called $C(\bar{\Omega})$

Suppose $u \in C(\bar{\Omega}) \Rightarrow$ u has a maxima and minima in Ω

- Weak maximum principle :

Let $\Omega \subseteq R^2$ be bounded domain. Let $u \in C^2(\Omega) \cap C(\bar{\Omega})$

Proof?

$$\Delta u = u_{xx} + u_{yy} = 0$$

Then max value of u in $\bar{\Omega}$ is achieved at boundary $\partial\Omega$.

However, max can be achieved at interior of the domain also.

- Weak minimum principle

similar to weak maximum principle.

Ω bounded and $\underbrace{\Delta u}_{\text{Laplace of } u} = 0$

$$\min_{\bar{\Omega}} u = \min_{\partial\Omega} u$$

→ UNIQUENESS AND STABILITY OF SOLUTION

- We know that we have a unique solⁿ for

$$\left.\begin{aligned} \Delta u &= f \\ u|_{\partial u} &= g \end{aligned}\right\} \text{unique sol}^n$$

Laplace of u

- eg -

$$\left.\begin{aligned} \Delta u &= 0 \\ u|_{y=0} &= 0 \end{aligned}\right\} \quad \Omega = \{(x,y), y > 0\}$$

unbounded

$u=0$ & $u=xy$ both are solⁿs

- Stability

$$\left.\begin{aligned} \Delta u_i &= f \\ u|_{\partial\Omega} &= g \end{aligned}\right\} \quad u_i \in C(\bar{\Omega}) \cap C^2(\Omega)$$

Then, $\boxed{\max_{\bar{\Omega}} |u_1(x) - u_2(x)| \leq \max_{\partial\Omega} |g_1 - g_2|}$

Proof: Let

$w = u_1 - u_2$

$\Rightarrow \Delta w = 0$

Laplace of w

$w|_{\partial u} = g_1 - g_2$

using max^m - min^m principle,

$$\boxed{\min_{\partial\Omega} |g_1 - g_2| \leq |u_1 - u_2| \leq \max_{\partial\Omega} |g_1 - g_2|}$$

→ STRONG MAX^M PRINCIPLE

$\Omega \subseteq \mathbb{R}^2$ (not necessarily bounded)

$\Delta u = 0$, If u attains its max^m in Ω then u is constant

NOTE: It does not talk about where max is attained

eg- $\Delta u = 0$

$u = \log(x^2 + y^2)$, $(x, y) \neq (0,0)$

Strong max^m principle holds
similarly strong min principle also holds

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→ WEAK MAX PRINCIPLE

consider ~~$Lu = \sum_{i,j}^{n}$~~

$$Lu = \sum_{i,j=1}^{n} a_{ij}(x) \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i(x) \frac{\partial u}{\partial x_i} + cu$$

and $\Omega \subseteq \mathbb{R}^n$
↓
bounded

Let $u \in C^2(\Omega) \cap C(\bar{\Omega})$ and let (a_{ij}) symmetric ~~[illegible]~~ and strictly +ve definite.

$\Rightarrow$ L is elliptic