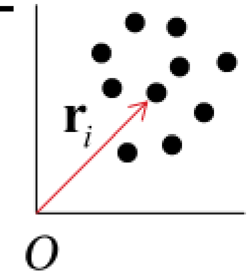
Lagrange’s Equation of Motion

03 Feb 2020

Constraints and Generalized Co-ordinates



In solving mechanical problems, we start with the 2nd law

$$\sum_j \mathbf{F}_{ji} + \mathbf{F}_i^{(e)} = m_i \ddot{\mathbf{r}}_i \qquad (*)$$

In principle, one can solve for $\boldsymbol{r}_i(t)$ (trajectory) for the i^{th} particle by specifying all the external and internal forces acting on it .

However, if constraints are present, these external forces in general are NOT known.

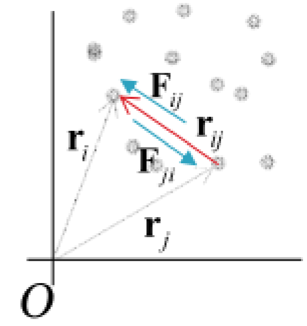
Therefore, we need to understand the various constraints and know how to handle them.

Holonomic constraints can be expressed as a function in terms of the coordinates and time,

$$f\left(\mathbf{r}_1, \mathbf{r}_2, \cdots; t\right) = 0$$

e.g. (a rigid body) → $\left(\mathbf{r}_i - \mathbf{r}_j\right)^2 - c_{ij}^2 = 0$

Difficulties involving Constraints and Solutions



1. Through $f(\mathbf{r}_1, \mathbf{r}_2, \cdots; t) = 0$, the individual coordinates $\boldsymbol{r}_i$ are *no* longer *independent*

   → eqs of motion (*) for individual particles are now *coupled* (not independent)

2. Forces of constraints are not known *a priori* and must be solved as additional unknowns

With **holonomic** constraints:

Prob #1 can be solved by introducing a set of "proper" (independent) Generalized Coordinates

Prob #2 can be treated with: D'Alembert's Principle & Lagrange's Equations

Generalized Co-ordinates

- Without constraints, a system of N particles has $3N$ dof
- With K constraint equations, the # dof reduces to $3N$-K
- With holonomic constraints, one can introduce ($3N$-K) **independent** (proper) generalized coordinates $\left(q_1, q_2, \cdots, q_{3N-K}\right)$ such that:

$$\left.\begin{aligned} \mathbf{r}_1 &= \mathbf{r}_1\left(q_1, q_2, \cdots, q_{3N-K}, t\right) \\ &\vdots \\ \mathbf{r}_N &= \mathbf{r}_N\left(q_1, q_2, \cdots, q_{3N-K}, t\right) \end{aligned}\right\} \text{a point transformation}$$

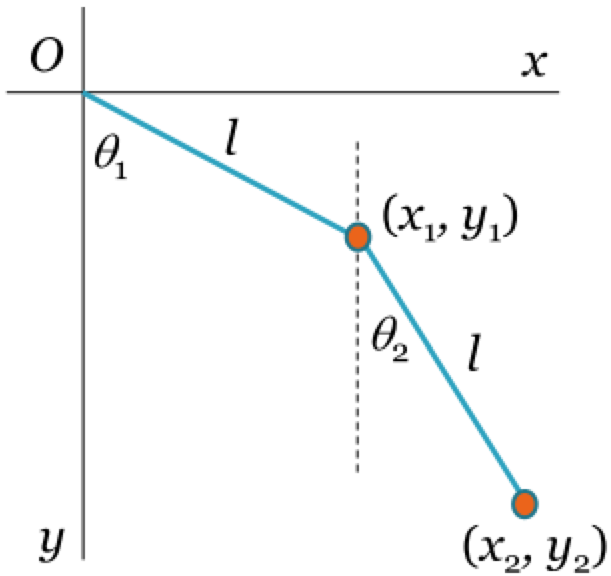
➢ Generalized coordinates can be anything: angles, energy units, momentum units, or even amplitudes in the Fourier expansion of $\mathbf{r}_i$

➢ But, they must completely specify the state of a given system

➢The choice of a particular set of generalized coordinates is not unique.

➢No specific rule in finding the most "suitable" (resulting in simplest EOM)

Example: Generalized Co-ordinates



(Double Plane Pendulum)

In regular Cartesian coord $\{\mathbf{r}_i\}$:

$$(x_1, y_1, x_2, y_2) \qquad 4 \text{ dof}$$

2 constraints: $\begin{cases} x_1^2 + y_1^2 - l^2 = 0 \\ (x_2 - x_1)^2 + (y_2 - y_1)^2 - l^2 = 0 \end{cases}$

But, there are only 2 indep dof...

In generalized coord $\{q_j\}$:

$$(\theta_1, \theta_2) \qquad 2 \ \textit{indep} \text{ dof}$$

Coord Transformation:

(constraints are implicitly encoded here)

$$\theta_1 = \tan^{-1}(x_1/y_1)$$

$$\theta_2 = \tan^{-1}((x_2 - x_1)/(y_2 - y_1))$$

Difficulties involving Constraints and Solutions

1. Through $f\left(\mathbf{r}_1,\mathbf{r}_2,\cdots;t\right)=0$, the individual coordinates $\boldsymbol{r}_i$ are *no* longer *independent*

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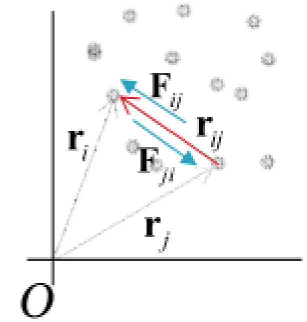

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Dealing with Constraints: Principle of Virtual Work

Consider a system in *equilibrium* first,

- The net force on each particle vanishes: $\mathbf{F}_i = 0$ (note: i labels the particles)

Consider an arbitrary "virtual" infinitesimal change in the coordinates, $\delta \mathbf{r}_i$

- Virtual means that it is done with *no change in time* during which forces and constraints do *not* change.

Since all the $\mathbf{F}_i$ are zero (equilibrium), obviously we have $\sum_i \mathbf{F}_i \cdot \delta \mathbf{r}_i = 0$

(virtual work)

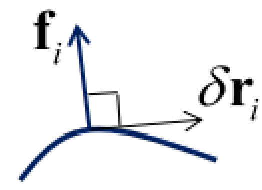
Separating the forces into applied $\mathbf{F}_i^{(a)}$ and constraint forces $\mathbf{f}_i$,

$$\mathbf{F}_i = \mathbf{F}_i^{(a)} + \mathbf{f}_i$$

Then, $$\sum_i \mathbf{F}_i^{(a)} \cdot \delta \mathbf{r}_i + \sum_i \mathbf{f}_i \cdot \delta \mathbf{r}_i = 0$$

Dealing with Constraints: Principle of Virtual Work

→ the *virtual work done by the constraint forces along the virtual displacement must be zero*.



$$\mathbf{f}_i \perp \delta\mathbf{r}_i \quad \text{or} \quad \mathbf{f}_i \cdot \delta\mathbf{r}_i = 0$$

This leaves us with the statement,

$$\sum_i \mathbf{F}_i^{(a)} \cdot \delta\mathbf{r}_i = 0$$

→ The virtual work of the applied forces must also vanish!

This is called the Principle of Virtual Work.

D'Alembert's Principle

"Principle of virtual work" is good to deal with systems at equilibrium;

"What about system in dynamics"

$\mathbf{F}_i = \dot{\mathbf{p}}_i$ or $\mathbf{F}_i - \dot{\mathbf{p}}_i = 0$ for the i^{th} particle in the system.

We again consider a virtual infinitesimal displacement $\delta\mathbf{r}_i$ consistent with the given constraint. Since we have $\mathbf{F}_i - \dot{\mathbf{p}}_i = 0$ for all the particles,

We have, $$\sum_i \left(\mathbf{F}_i - \dot{\mathbf{p}}_i\right) \cdot \delta\mathbf{r}_i = 0$$

Again, we separate out the applied and constraint forces, $\mathbf{F}_i = \mathbf{F}_i^{(a)} + \mathbf{f}_i$

This gives, $$\sum_i \left(\mathbf{F}_i^{(a)} - \dot{\mathbf{p}}_i\right) \cdot \delta\mathbf{r}_i + \sum_i \mathbf{f}_i \cdot \delta\mathbf{r}_i = 0$$

We can write down, $$\sum_i \left(\mathbf{F}_i^{(a)} - \dot{\mathbf{p}}_i\right) \cdot \delta\mathbf{r}_i = 0$$

This is the D'Alembert's Principle.

Derivation of Lagrange's equation from D'Alembert's Principle

Break $\sum_i \left(\mathbf{F}_i^{(a)} - \dot{\mathbf{p}}_i\right) \cdot \delta \mathbf{r}_i = 0$ into two pieces:

1. $\sum_i \mathbf{F}_i^{(a)} \cdot \delta \mathbf{r}_i \quad (1)$

Assume that we have a set of n=3N-K independent generalized coordinates q_j and the coordinate transformation,

$$\mathbf{r}_i = \mathbf{r}_i\left(q_1, q_2, \cdots, q_n, t\right)$$

From chain rule, we have

$$\delta \mathbf{r}_i = \sum_j \frac{\partial \mathbf{r}_i}{\partial q_j} \delta q_j$$

(note: $\frac{\partial \mathbf{r}_i}{\partial t} \delta t = 0$ since it is a virtual disp)

(Index convention: i goes over # particles and j over generalized coords)

Derivation of Lagrange's equation from D'Alembert's Principle

This links the variations in $\mathbf{r}_i$ to q_j, substituting it into expression (1), we have,

$$\sum_i \mathbf{F}_i^{(a)} \cdot \delta \mathbf{r}_i = \sum_i \sum_j \left(\mathbf{F}_i^{(a)} \cdot \frac{\partial \mathbf{r}_i}{\partial q_j} \delta q_j \right) = \sum_j \left[\sum_i \mathbf{F}_i^{(a)} \cdot \frac{\partial \mathbf{r}_i}{\partial q_j} \right] \delta q_j$$

Defining

$$\boxed{Q_j \equiv \sum_i \mathbf{F}_i^{(a)} \cdot \frac{\partial \mathbf{r}_i}{\partial q_j}}$$

as the "generalized forces"

We can then write,

$$\sum_i \mathbf{F}_i^{(a)} \cdot \delta \mathbf{r}_i = \sum_j Q_j \, \delta q_j \quad (1')$$

(Note: Q_j needs not have the dimensions of force but $Q_j \, \delta q_j$ must have dimensions of work.)

Derivation of Lagrange's equation from D'Alembert's Principle

Now, we look at the second piece involving $\dot{\mathbf{p}}_i$:

2. $\sum_i \dot{\mathbf{p}}_i \cdot \delta \mathbf{r}_i$ (2) (don't forget the "-" sign in the original Eq)

$$= \sum_i m_i \ddot{\mathbf{r}}_i \cdot \delta \mathbf{r}_i \qquad \text{(mass is assumed to be constant)}$$

$$= \sum_i m_i \ddot{\mathbf{r}}_i \cdot \left(\sum_j \frac{\partial \mathbf{r}_i}{\partial q_j} \delta q_j \right)$$

$$= \sum_i \sum_j \left(m_i \ddot{\mathbf{r}}_i \cdot \frac{\partial \mathbf{r}_i}{\partial q_j} \right) \delta q_j \qquad (2a)$$

Derivation of Lagrange's equation from D'Alembert's Principle

Let, go backward a bit. Consider the following time derivative:

$$\frac{d}{dt}\left(m_i\dot{\mathbf{r}}_i\cdot\frac{\partial\mathbf{r}_i}{\partial q_j}\right)=m_i\dot{\mathbf{r}}_i\cdot\frac{d}{dt}\left(\frac{\partial\mathbf{r}_i}{\partial q_j}\right)+m_i\ddot{\mathbf{r}}_i\cdot\frac{\partial\mathbf{r}_i}{\partial q_j}$$

Rearranging, the last term (from the previous page) can be written as,

$$m_i\ddot{\mathbf{r}}_i\cdot\frac{\partial\mathbf{r}_i}{\partial q_j}=\frac{d}{dt}\left(m_i\mathbf{v}_i\cdot\frac{\partial\mathbf{r}_i}{\partial q_j}\right)-m_i\mathbf{v}_i\cdot\frac{d}{dt}\left(\frac{\partial\mathbf{r}_i}{\partial q_j}\right)\qquad(2b)\qquad\text{where }\ \mathbf{v}_i=\frac{d\mathbf{r}_i}{dt}$$

Now, consider the blue and red terms in detail,

Derivation of Lagrange's equation from D'Alembert's Principle

blue term: $\dfrac{\partial \mathbf{r}_i}{\partial q_j}$

Since we have $\mathbf{r}_i = \mathbf{r}_i\left(q_1, q_2, \cdots, q_n, t\right)$, applying chain rule, we have

$$\mathbf{v}_i = \frac{d\mathbf{r}_i}{dt} = \sum_k \frac{\partial \mathbf{r}_i}{\partial q_k}\dot{q}_k + \frac{\partial \mathbf{r}_i}{\partial t}$$

Taking the partial of above expression with respect to $\dot{q}_j$, we have

$$\frac{\partial \mathbf{v}_i}{\partial \dot{q}_j} = \frac{\partial \mathbf{r}_i}{\partial q_j}$$

(note: $\mathbf{r}_i$ does not depend on $\dot{q}_j$)

red term: $\dfrac{d}{dt}\left(\dfrac{\partial \mathbf{r}_i}{\partial q_j}\right) = \dfrac{\partial}{\partial q_j}\left(\dfrac{d\mathbf{r}_i}{dt}\right) = \dfrac{\partial \mathbf{v}_i}{\partial q_j}$

(switching derivative order)

Is it ok? Check ...

Derivation of Lagrange's equation from D'Alembert's Principle

Putting these two terms back into Eq. (2b):

$$\boxed{m_i\ddot{\mathbf{r}}_i\cdot\frac{\partial \mathbf{r}_i}{\partial q_j}=\frac{d}{dt}\left(m_i\mathbf{v}_i\cdot\frac{\partial \mathbf{r}_i}{\partial q_j}\right)-m_i\mathbf{v}_i\cdot\frac{d}{dt}\left(\frac{\partial \mathbf{r}_i}{\partial q_j}\right)}$$

$$m_i\ddot{\mathbf{r}}_i\cdot\frac{\partial \mathbf{r}_i}{\partial q_j}=\frac{d}{dt}\left(m_i\mathbf{v}_i\cdot\frac{\partial \mathbf{v}_i}{\partial \dot{q}_j}\right)-m_i\mathbf{v}_i\cdot\frac{\partial \mathbf{v}_i}{\partial q_j}$$

With this, we finally have the following for expression (2):

$$\sum_i \dot{\mathbf{p}}_i\cdot\delta\mathbf{r}_i=\sum_i\sum_j\left(m_i\ddot{\mathbf{r}}_i\cdot\frac{\partial \mathbf{r}_i}{\partial q_j}\right)\delta q_j$$

$$=\sum_i\sum_j\left[\frac{d}{dt}\left(m_i\mathbf{v}_i\cdot\frac{\partial \mathbf{v}_i}{\partial \dot{q}_j}\right)-m_i\mathbf{v}_i\cdot\frac{\partial \mathbf{v}_i}{\partial q_j}\right]\delta q_j \qquad (2c)$$

(reminder: i sums over # particles and j sums over generalized coords)

Derivation of Lagrange's equation from D'Alembert's Principle

We are almost there but not quite done yet. Consider taking the q_j derivative of the Kinetic Energy,

$$\frac{\partial T}{\partial q_j} = \frac{\partial}{\partial q_j}\left(\sum_i \frac{1}{2} m_i \mathbf{v}_i^2\right) = \frac{\partial}{\partial q_j}\left(\sum_i \frac{1}{2} m_i \mathbf{v}_i \cdot \mathbf{v}_i\right)$$

$$= \frac{1}{2}\sum_i m_i \left[\left(\mathbf{v}_i \cdot \frac{\partial \mathbf{v}_i}{\partial q_j}\right) + \left(\frac{\partial \mathbf{v}_i}{\partial q_j} \cdot \mathbf{v}_i\right)\right]$$

$$= \sum_i m_i \mathbf{v}_i \cdot \frac{\partial \mathbf{v}_i}{\partial q_j}$$

Similarly, we can do the same manipulations on T wrt to $\dot{q}_j$,

$$\frac{\partial T}{\partial \dot{q}_j} = \frac{\partial}{\partial \dot{q}_j}\left(\sum_i \frac{1}{2} m_i \mathbf{v}_i^2\right) = \sum_i m_i \mathbf{v}_i \cdot \frac{\partial \mathbf{v}_i}{\partial \dot{q}_j}$$

Derivation of Lagrange's equation from D'Alembert's Principle

Substituting these two expressions into Eq. (2c), we have:

$$\sum_i \dot{\mathbf{p}}_i \cdot \delta \mathbf{r}_i = \sum_j \left[\frac{d}{dt}\left(\sum_i m_i \mathbf{v}_i \cdot \frac{\partial \mathbf{v}_i}{\partial \dot{q}_j} \right) - \sum_i m_i \mathbf{v}_i \cdot \frac{\partial \mathbf{v}_i}{\partial q_j} \right] \delta q_j$$

$$= \sum_j \left[\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j} \right) - \frac{\partial T}{\partial q_j} \right] \delta q_j$$

Finally, reconstructing the two terms in the D'Alembert's Principle, we have:

$$\left[\sum_i \left(\mathbf{F}_i^{(a)} - \dot{\mathbf{p}}_i \right) \cdot \delta \mathbf{r}_i = 0 \right]$$

$$\sum_j \left[Q_j - \left\{ \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j} \right) - \frac{\partial T}{\partial q_j} \right\} \right] \delta q_j = 0$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j} \right) - \frac{\partial T}{\partial q_j} = Q_j \qquad (3)$$

Euler-Lagrange's equation for Conservative forces

$$\mathbf{F}_i^{(a)} = -\nabla_i U\left(\mathbf{r}_1, \mathbf{r}_2, \cdots, \mathbf{r}_N, t\right) \quad \text{(note: } U \text{ not depend on velocities)}$$

$$Q_j = \sum_i \mathbf{F}_i^{(a)} \cdot \frac{\partial \mathbf{r}_i}{\partial q_j} = -\sum_i \nabla_i U \cdot \frac{\partial \mathbf{r}_i}{\partial q_j}$$

$$= -\sum_i \left(\left[\frac{\partial}{\partial x_i}\hat{\mathbf{i}} + \frac{\partial}{\partial y_i}\hat{\mathbf{j}} + \frac{\partial}{\partial z_i}\hat{\mathbf{k}} \right] U \cdot \frac{\partial}{\partial q_j}\left[x_i\hat{\mathbf{i}} + y_i\hat{\mathbf{j}} + z_i\hat{\mathbf{k}} \right] \right)$$

$$= -\sum_i \left(\frac{\partial U}{\partial x_i}\frac{\partial x_i}{\partial q_j} + \frac{\partial U}{\partial y_i}\frac{\partial y_i}{\partial q_j} + \frac{\partial U}{\partial z_i}\frac{\partial z_i}{\partial q_j} \right)$$

$$\boxed{Q_j = -\frac{\partial U}{\partial q_j}}$$

Euler-Lagrange's equation for Conservative forces

Putting this expression into the RHS of Eq. (3), we have,

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} = Q_j = -\frac{\partial U}{\partial q_j}$$

Notice that since U does not depends on the generalized velocity $\dot{q}_j$, we are free to subtract U from T in the first term,

$$\frac{d}{dt}\left(\frac{\partial (T-U)}{\partial \dot{q}_j}\right) - \frac{\partial (T-U)}{\partial q_j} = 0$$

We now define the **Lagrangian** function $L = T - U$ and the desired Euler-Lagrange's Equation is:

$$\boxed{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = 0}$$