De Moivre's Theorem :

$$e^{i\theta} = \cos\theta + i\sin\theta, \quad -\pi \le \theta \le \pi.$$

$$\text{or } 0 \le \theta \le 2\pi.$$

Statement : (i) If n is an integer (positive or negative), then

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta.$$

(ii) If n is a fraction (positive or negative), then one of the values of $(\cos\theta + i\sin\theta)^n$ is $\cos n\theta + i\sin n\theta$.

Hint: Case I :

$$(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2)$$

$$= \cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)$$

Repeating above, we have

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$$

where $\theta_1 = \theta_2 = \cdots = \theta_n = \theta$.

Case II : When n is a negative integer. Let $n = -m$, where m is a +ve integer.

$$\therefore (\cos\theta + i\sin\theta)^n = (\cos\theta + i\sin\theta)^{-m} = \frac{1}{(\cos\theta + i\sin\theta)^m}$$

$$= \frac{\cos m\theta - i\sin m\theta}{(\cos m\theta + i\sin m\theta)(\cos m\theta - i\sin m\theta)}$$

$$= \cos m\theta - i\sin m\theta.$$

$$= \cos n\theta + i\sin n\theta \quad \left[\because \cos(-m)\theta = \cos m\theta \text{ and } \sin(-m\theta) = -\sin m\theta\right]$$

Case III : When n is a fraction, positive or negative :

Let $n = \frac{p}{q}$, where q is a +ve integer and p is any integer, +ve or -ve.

Now, $\left(\cos\theta/q + i\sin\theta/q\right)^q = \cos q\cdot\frac{\theta}{q} + i\sin q\cdot\frac{\theta}{q}$

$= \cos\theta + i\sin\theta$

$\therefore$ Taking qth root of both sides, we have one of the qth roots of $(\cos\theta + i\sin\theta)$ i.e. $(\cos\theta + i\sin\theta)^{1/q}$ is $\cos\frac{\theta}{q} + i\sin\frac{\theta}{q}$.

Raise both sides to power p, one of the values of $(\cos\theta + i\sin\theta)^{p/q}$ is $\cos(p/q)\theta + i\sin(p/q)\theta$.

Questions;

(1) Prove that

$$(1+\cos\theta + i\sin\theta)^n + (1+\cos\theta - i\sin\theta)^n = 2^{n+1}\cos^n(\theta/2)\cdot(\cos\theta/2).$$

(2) If $2\cos\theta = x + \frac{1}{x}$, prove that $2\cos r\theta = x^r + \frac{1}{x^r}$.

(3) If α, β are the roots of $x^2 - 2x + 4 = 0$, prove that

$$\alpha^n + \beta^n = 2^{n+1}\cos n\pi/3.$$

(4) Find all the values of $\left(\frac{1}{2} + \sqrt{3}\, i/2\right)^{3/4}$.

(5) Use De Moivre's theorem to solve the equation

$$x^4 - x^3 + x^2 - x + 1 = 0.$$

6. Find the roots common to the equations

$$x^4 + 1 = 0 \text{ and } x^6 - i = 0.$$

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}, \quad \sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i} \longrightarrow (1)$$

Similarly, for complex variable z, the circular functions

$$\cos z = \frac{e^{iz} + e^{-iz}}{2} \quad \text{and} \quad \sin z = \frac{e^{iz} - e^{-iz}}{2}$$

$$\tan z = \frac{\sin z}{\cos z}.$$

Similarly, we can define e^z.

Hyperbolic Functions:

If x is real or complex,

$\frac{e^x + e^{-x}}{2}$ is defined as hyperbolic cosine of x

and written as $\cosh x$.

i.e. $$\cosh x = \frac{e^x + e^{-x}}{2}.$$

Similarly, $$\sinh x = \frac{e^x - e^{-x}}{2}.$$

$$\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$ and others can be defined.

Relation: From (1), put $\theta = ix$, then

$$\cos\theta = \frac{e^{i \cdot ix} + e^{-i \cdot ix}}{2} = \frac{e^x + e^{-x}}{2} = \cosh x$$

i.e. $\underline{\cos ix = \cosh x}$.

Also, $$\sin\theta = \frac{e^{i \cdot ix} - e^{-i \cdot ix}}{2i} = -\frac{e^x - e^{-x}}{2i} = i^2\frac{(e^x - e^{-x})}{2i}$$

$$= i\,\frac{e^x - e^{-x}}{2} = i\sinh x.$$

$\therefore$ $\underline{\sin ix = i\sinh x}$.

Thus,
$$\sin ix = i\sinh x$$
$$\cos ix = \cosh x$$
$$\tan ix = i\tanh x$$
$$\sinh ix = i\sin x$$
$$\cosh ix = \cos x$$
$$\tanh ix = i\tan x.$$

Also,
$$\cosh^2 x - \sinh^2 x = 1$$
$$\operatorname{sech}^2 x + \tanh^2 x = 1$$
$$\coth^2 x - \operatorname{cosech}^2 x = 1.$$

$$\sinh(x \pm y) = \sinh x \cosh y \pm \cosh x \sinh y$$
$$\cosh(x \pm y) = \cosh x \cosh y \pm \sinh x \sinh y.$$

Inverse hyperbolic Functions: If $\sinh u = z$, then $\sinh^{-1} z = u$ is hyperbolic sine inverse.

Q: Prove that

(i) $\sinh^{-1} z = \log\left(z + \sqrt{z^2+1}\right)$

(ii) $\cosh^{-1} z = \log\left[z + \sqrt{z^2-1}\right]$

(iii) $\tanh^{-1} z = \frac{1}{2}\log\frac{1+z}{1-z}$.

Functions of a Complex Variable

A curve is a particular kind of geometrical configuration. For complex function theory, it is important to consider a curve as having an addition structure, viz. a specific parametric representation.

$x = x(t)$, and $y = y(t)$ in a plane.

$z = z(t) = x(t) + i\,y(t)$, where $z = x+iy$.

If for each value of the complex variable $z = x+iy$ in a given region D, we have one or more values of $w = u+iv$, then w is said to be a function of z and we write

$$w = u(x,y) + i\,v(x,y) = f(z)$$

where u, v are real valued functions of x and y.

Partial Derivatives: (Recalled):

$z = f(x,y)$ defined on $D \subseteq \mathbb{R}^2$

We have a point $(a,b) \in D$, then

Limits:

$$\lim_{(x,y) \to (a,b)} f(x,y) = \lim_{\substack{x \to a \\ y \to b}} f(x,y)$$

$$\lim_{x \to a} \lim_{y \to b} f(x,y)$$

$$\lim_{y \to b} \lim_{x \to a} f(x,y)$$

20.

$$\lim_{(x,y) \to (a,b)} f(x,y) = f(a,b)$$

(a,b), a neighbouring point $(a+h, b+K)$, then

$$\left.\frac{\partial f(x,y)}{\partial x}\right)_{(a,b)} = \lim_{h \to 0} \frac{f(a+h,b) - f(a,b)}{h}$$

$$\left.\frac{\partial f(x,y)}{\partial y}\right)_{(a,b)} = \lim_{K \to 0} \frac{f(a,b+K) - f(a,b)}{K}.$$

Derivative of f(z) :-

Let $w = f(z)$ be a single-valued function of the variable $z = x+iy$. Then the derivative of $w = f(z)$ is defined to be

$$\frac{dw}{dz} = f'(z) = \lim_{\delta z \to 0} \frac{f(z+\delta z) - f(z)}{\delta z}$$

provided the limit exists and has the same value for all the different ways in which δz approaches zero.

Analytic or Holomorphic or Regular Function :-

A single valued function is said to be analytic at a point if it is differentiable everywhere in some neighbourhood of the point.

A function $f(z)$ which is single-valued and possesses a unique derivative with respect to z at all points of a region D, is called an analytic or a regular function of z in that region.

A point at which an analytic function ceases to possess a derivative is called a singular point of the function.

or, If $f(z)$ is not analytic at a point z_0, then z_0 is called the singular point of $f(z)$.

The real & imaginary parts of an analytic function are called conjugate functions. The relation between two conjugate functions is given by C-R equations.

Th :- The necessary and sufficient conditions for the derivative of the function $w = u(x,y) + iv(x,y) = f(z)$ to exist for all values of z in a region D, are

(i) $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ are continuous functions of x and y in D

(ii) $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$.

The relation (ii) are known as Cauchy-Riemann equations or briefly C-R equations.

Proof :- (a) Condition is Necessary :-

Let δu and δv be the increments of u and v respectively corresponding to the increment δx and δy of x and y, so that $\delta z = \delta x + i\delta y$.

If $f(z)$ possesses a unique derivative at $P(z)$, then

$$f'(z) = \lim_{\delta z \to 0} \frac{f(z+\delta z) - f(z)}{\delta z}$$

$$= \lim_{\delta z \to 0} \frac{(u+\delta u) + i(v+\delta v) - (u+iv)}{\delta z}$$

$$= \lim_{\delta z \to 0} \left(\frac{\delta u}{\delta z} + i\frac{\delta v}{\delta z} \right)$$

Since δz can approach zero in any manner, we can first assume δz to be wholly real and then wholly imaginary.

When δz is wholly real, then, $\delta y = 0$ and $\delta z = \delta x$

$$\therefore f'(z) = \lim_{\delta x \to 0} \left(\frac{\delta u}{\delta x} + i\frac{\delta v}{\delta x} \right)$$

$$= \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} \longrightarrow (1)$$

When δz is wholly imaginary, then $\delta x = 0$ and $\delta z = i\delta y$

$$\therefore f'(z) = \lim_{\delta y \to 0} \left(\frac{\delta u}{i\delta y} + i\frac{\delta v}{i\delta y} \right) = \frac{1}{i}\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$$

$$= \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y} \longrightarrow (2)$$

Now the existence of $f'(z)$ requires the equality of (1) and (2).

$$\therefore \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}$$

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \longrightarrow (3)$$

Thus the necessary conditions for the existence of the derivative of $f(z)$ is that the C-R equations should be satisfied.

(b) Condition is sufficient :-

Suppose $f(z)$ is a single valued function possessing partial derivatives $\frac{\partial u}{\partial x}$, u_y, v_x, v_y at each point of the region & that the C-R equations are satisfied.

Then by Taylor's theorem for functions of two variables

$$f(z+\delta z) = u(x+\delta x, y+\delta y) + iv(x+\delta x, y+\delta y)$$

$$= u(x,y) + \left(\frac{\partial u}{\partial x}\delta x + \frac{\partial u}{\partial y}\delta y\right) + \cdots$$

$$+ i\left[v(x,y) + \left(\frac{\partial v}{\partial x}\delta x + \frac{\partial v}{\partial y}\delta y\right) + \cdots\right]$$