Dr. Salom

Lecture - 6 -

23.09.20

Power Series : series of the form

$$\sum_{n=0}^{\infty} a_n (z - z_0)^n = a_0 + a_1(z - z_0) + a_2(z - z_0)^2$$

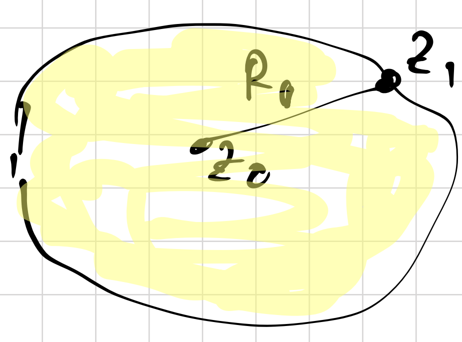
where $z_0 \in \mathbb{C}$, $a_n \in \mathbb{C}$ $\forall n$.

Eg: $\sum z^n$ (with $z_0 = 0$ & $a_n = 1$)

$\sum \frac{z^n}{n!}$ (with $z_0 = 0$ & $a_n = \frac{1}{n!}$)

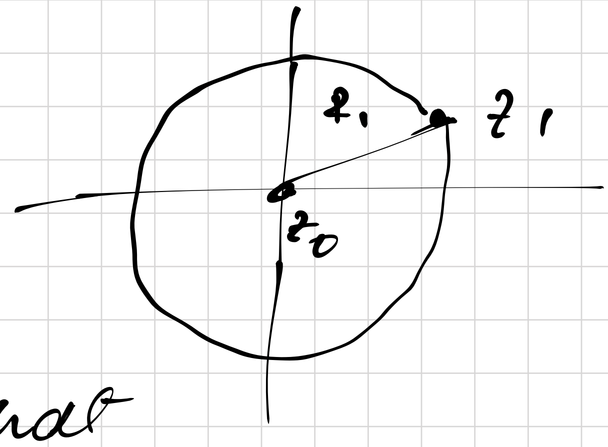
Def^n: If the series $\sum_{n=0}^{\infty} a_n (z_1 - z_0)^n$ converges for some number $z_1 \in \mathbb{C}$, then we say that the power series $\sum_{n=0}^{\infty} a_n (z - z_0)^n$ is convergent at $z = z_1$.

Theorem: Suppose the power series $\sum_{n=0}^{\infty} a_n (z - z_0)^n$ converges at a point $z = z_1$; $z_1 \neq z_0$. Then $\sum_{n=0}^{\infty} a_n (z - z_0)^n$ converges absolutely at each point z in the open disk $|z - z_0| < R_1$ where $R_1 = |z_1 - z_0|$



Proof : Case-I when $z_0 = 0$ and $z_1 \neq 0$.

So, $R_1 = |z_1|$



It is given to us that

$$\sum_{n=0}^{\infty} a_n z_1^n \text{ converges.}$$

$\Rightarrow$ $a_n z_1^n \longrightarrow 0$ as $n \to \infty$ [from an earlier result]

$\Rightarrow$ the sequence $\{a_n z_1^n\}_{n \in \mathbb{N}}$ is bounded.

$\Rightarrow$ there exists a (real) constant $M \geq 0$ s.t.

$$|a_n z_1^n| \leq M \quad \forall n \geq 0$$

Now

$$|a_n z^n| = |a_n||z|^n = |a_n| \left| z_1 \cdot \frac{z}{z_1} \right|^n \quad [\because z_1 \neq 0]$$

$$= |a_n| \cdot |z_1|^n \left| \frac{z}{z_1} \right|^n$$

$$\leq M \cdot \left| \frac{z}{z_1} \right|^n \quad \forall n \geq 0$$

Now, note that the real series $\sum_{n=0}^{\infty} M \cdot \left| \frac{z}{z_1} \right|^n$ converges for all z with $\left| \frac{z}{z_1} \right| < 1$

[Recall geometric (real) series $\sum \rho^n$ converges when $\rho < 1$]

This implies that $\sum_{n=0}^{\infty} |a_n z|^n$ converges when

(real series)

$\frac{|z|}{|z_1|} < 1$ i.e. $|z| < |z_1|$

In other words, $\sum_{n=0}^{\infty} a_n z^n$ converges absolutely for all z with $|z| < |z_1| = R_1$

<u>Case. II</u> : $z_0 \neq 0$ & $z_1 \neq z_0$.

Given that $\sum_{n=0}^{\infty} a_n (z - z_0)^n$ converges at z_1.

[Idea : translate it to case. 1]

Let $w := z - z_0$ & $w_1 := z_1 - z_0$ } then we have $\sum a_n w^n$ converges at w_1

Now by case. I, $\sum_{n=0}^{\infty} a_n w^n$ converges absolutely in the open disk $|w| < R_1$ where $R_1 = |w_1|$.

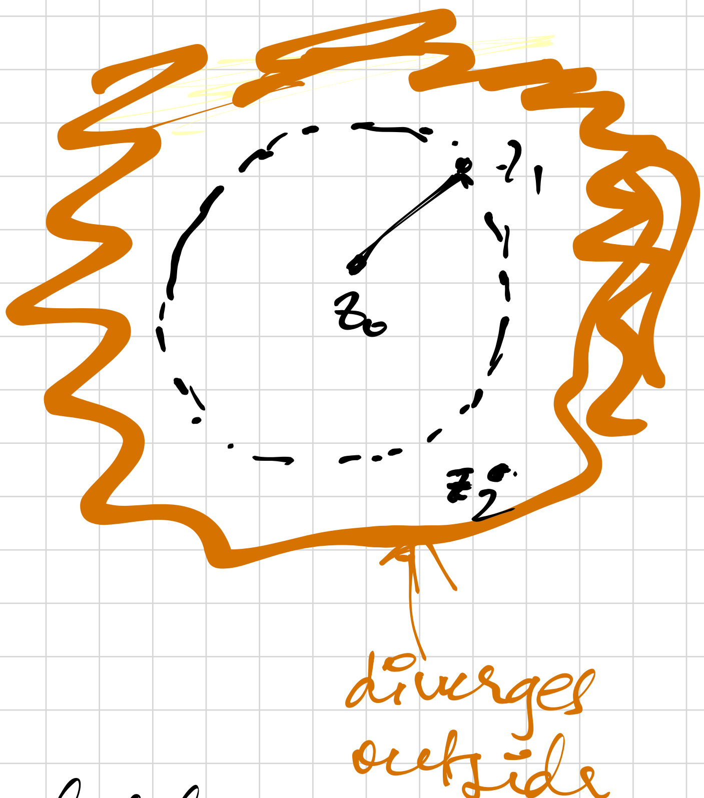
In other words (get back to z) : $\sum a_n (z - z_0)^n$ conv. absolutely in open disk $|z - z_0| < R_1$ where $R_1 = |z_1 - z_0|$

Remarks

(1) The above theorem tells that, if the power series diverges at a point $z = z_1$, then it diverges for all z with $|z - z_0| > |z_1 - z_0|$

Proof: Suppose z_2 is a point with $|z_2 - z_0| > |z_1 - z_0|$

If possible, the series converges at z_2.



- Then by the theorem,

$$\sum_{n=0}^{\infty} a_n (z - z_0)^n \text{ converges absolutely } \forall\, z \text{ with } |z - z_0| < |z_2 - z_0|$$

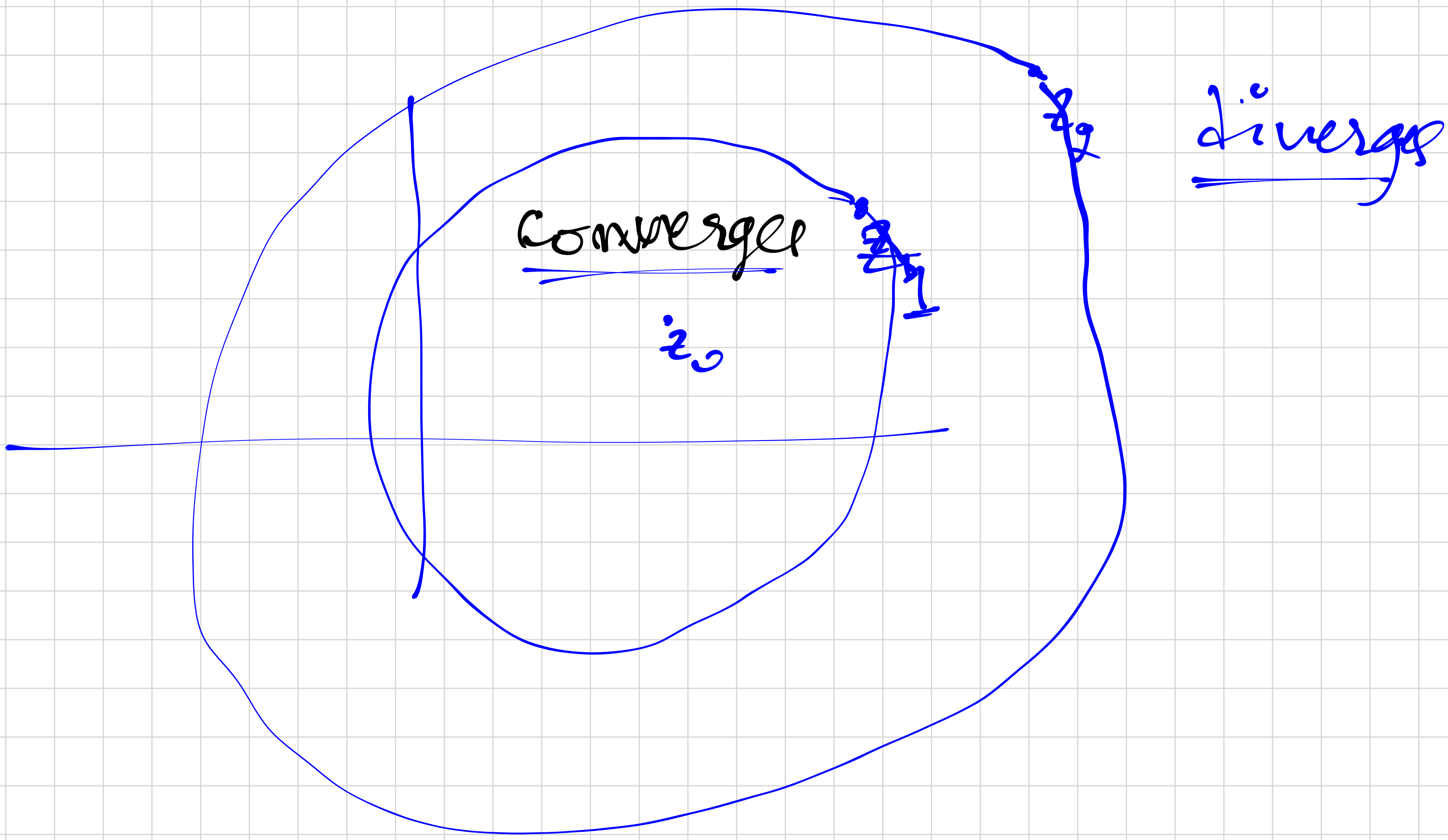
- The point z_1 satisfies $\rightarrow$

$\Rightarrow$ the series converges at $z = z_1$ which is a contradiction.

Definition: The greatest circle centered at z_0 such that the series $\sum_{n=0}^{\infty} a_n (z - z_0)^n$ converges at each point inside of the circle, is called the circle of convergence of the series $\sum_{n=0}^{\infty} a_n (z - z_0)^n$.

Remark - (2) Suppose the series is convergent at some pt. $z_1 \neq z_0$, then the circle of convergence has radius $\geq |z_1 - z_0|$.

(3) If the series diverges at some pt. $z_2 \neq z_0$, then circle of convergence has radius $< |z_1 - z_2|$.



———— x ———— x ————

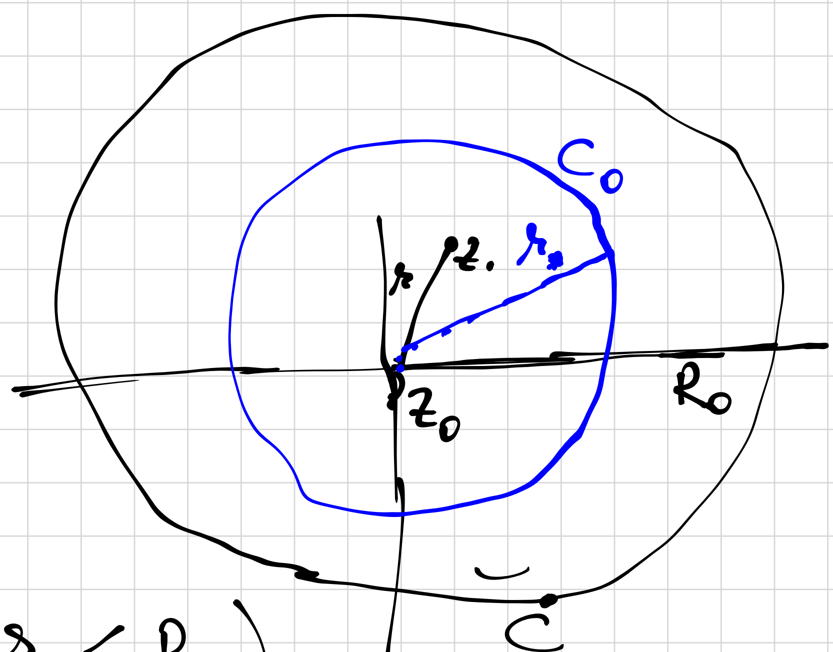
A very important application of Cauchy's integral formula is Taylor's theorem for complex analytic function.

Taylor's theorem : Suppose $f(z)$ is analytic throughout a disk $|z-z_0|<R_0$. Then $f(z)$ has the power series representation

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n \qquad \text{for } |z-z_0|<R_0$$

where
$$a_n = \frac{f^{(n)}(z_0)}{n!} \qquad n = 0, 1, 2, \ldots$$

Proof: **Case-I** : $z_0 = 0$



Take $C : |z| = R_0$

Let z_1 be a point with $|z_1| < R_0$.

Say $|z_1| = r$. $(0 < r < R_0)$.

Now choose r_0 with $r < r_0 < R_0$

and $C_0 : |z| = r_0$

then z_1 is inside of C_0 and by Cauchy's integral formula

$$f(z_1) = \frac{1}{2\pi i}\int_{C_0} \frac{f(z)\,dz}{(z-z_1)} \qquad \text{—— I}$$

Now write $\dfrac{1}{z-z_1} = \dfrac{1}{z\cdot\left(1-\frac{z_1}{z}\right)}$

Use $\displaystyle\sum_{n=0}^{\infty} z^n = \frac{1}{1-z}$

$$= \frac{1}{z}\left(\sum_{n=0}^{\infty}\left(\frac{z_1}{z}\right)^n\right)$$

$$= \frac{1}{z}\left(\sum_{n=0}^{N-1}\left(\frac{z_1}{z}\right)^n + \frac{(z_1/z)^N}{(1-z_1/z)}\right)$$

$$= \sum_{n=0}^{N-1}\frac{z_1^n}{z^{n+1}} + \frac{z_1^N}{(z-z_1)\,z^N}$$

put this value in I to get

$$2\pi i f(z_1) = \int_{C_0}\sum_{n=0}^{N-1}\frac{f(z)\cdot z_1^n}{z^{n+1}}\,dz + \int_{C_0}\frac{f(z)\cdot z_1^N\,dz}{(z-z_1)\,z^N}$$

$$= \sum_{n=0}^{\infty} z_1^n\int_{C_0}\frac{f(z)\,dz}{(z-0)^{n+1}} + z_1^N\int_{C_0}\frac{f(z)}{(z-z_1)\,z^N}$$

$$= \sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}\,z_1^n + \rho_N(z).$$

The result follows once we show that

$$\lim_{N\to\infty} \frac{f_N(z)}{2\pi i} = 0.$$

$$\frac{f_N(z)}{2\pi i} = \frac{z_1^N}{2\pi i}\int_{C_0} \frac{f(z)}{(z-z_1)\,z^N}\,dz$$

Here ① $|z-z_1| \geq \big|\,|z|-|z_1|\,\big| = r_0 - r$

[z is on C_0, so $|z| = r_0$]

② Let $|f(z)| \leq M \;\forall z$ on C_0

③ .

So $0 \leq \left|\frac{f_N(z)}{2\pi i}\right| \leq \frac{|z_1|^N}{|2\pi i|}\left|\int_{C_0}\frac{f(z)}{(z-z_1)z^N}\,dz\right|$

$$\leq \frac{|z_1|^N}{2\pi}\cdot\frac{M}{(r_0-r)\cdot r_0^N}\cdot \text{length of } C$$

$$= \frac{r^N}{\cancel{2\pi}}\cdot\frac{M}{(r_0-r)\cdot r_0^N}\cdot \cancel{2\pi} r_0$$

$$= \frac{M r_0}{(r_0-r)}\left(\frac{r}{r_0}\right)^N$$

$r < r_0 \Rightarrow \lim_{N\to\infty}\left(\frac{r}{r_0}\right)^N = 0$

so $\lim_{N\to\infty}\frac{f_N(z)}{2\pi i} = 0 \;\forall\; |z| = r_0$

i.e. z is on C_0

Case-II when $z_0 \neq 0$. $z_1 \neq z_0$.

Given that $f(z)$ is analytic in the disk $|z-z_0| < R_0$.

Let $w := z - z_0$ (then $z = w + z_0$)

so $f(z) = f(w + z_0)$ is analytic in $|w| < R_0$

Rename, & say $g(w) := f(w+z_0)$

Apply Case-I to $g(w)$, &

so $g(w) = \sum \frac{g^{(n)}(0)}{n!} w^n$ for $|w| < R_0$

(get back to z: $g^{(n)}(w) = f^{(n)}(w+z_0)$

$g^{(n)}(0) = f^{(n)}(z_0)$

Finally,

$$\underset{\displaystyle f(z)}{\overset{}{f(w+z_0)}} \;//\; = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!} (z-z_0)^n \quad \forall\ |z-z_0| < R_0$$

■

Comment :- (1) For a function $f(z)$, the series expansion $\sum_{n=0}^{\infty} a_n(z-z_0)$ (a_n as given in theorem) of $f(z)$ in a disk around z_0 is called Taylor series expansion of $f(z)$ around z_0.

(2) If $z_0 = 0$ in Taylor series, the series is known as Maclaurin series

(3) Question :- If $f(z)$ is analytic at z_0, does it have Taylor series expansion around z_0 ?