§ Zeros of analytic functions

Def: – Suppose $f(z)$ is analytic at z_0. If $f(z_0)=0$ and there is an integer m s.t. $f^{(m)}(z_0) \neq 0$ but $f^{(1)}(z_0) = \cdots = f^{(m-1)}(z_0) = 0$

then z_0 is called a zero of $f(z)$ of order m.

(Note that if $f(z)$ is analytic at z_0, then $f^{(n)}(z_0)$ exists $\forall n$)

Theorem: – Suppose $f(z)$ is analytic at z_0. Then z_0 is a zero of $f(z)$ of order m if and only if

$$f(z) = (z-z_0)^m g(z)$$

where $g(z)$ is a function analytic at z_0 and $g(z_0) \neq 0$

Proof: – (key: use Taylor series)

(Sufficient cond^n:) Suppose $f(z) = (z-z_0)^m g(z)$

Since $g(z)$ is analytic at z_0, so it has Taylor series expansion around z_0,

$$g(z) = g(z_0) + g'(z_0)(z-z_0) + \frac{g''(z_0)}{2!}(z-z_0)^2 + \ldots$$

$|z-z_0| < \varepsilon$ for some $\varepsilon > 0$

Then

$$f(z) = (z-z_0)^m g(z)$$

$$= (z-z_0)^m g(z_0) + g'(z_0)(z-z_0)^{m+1} + \frac{g''(z_0)}{2!}(z-z_0)^{m+2} + \dots$$

(this makes sense $\forall\ |z-z_0|<\varepsilon$ since $f(z)$ is analytic at z_0)

which is the Taylor series of $f(z)$ around z_0

so, coeff of $(z-z_0)^n$, $n<m$ are all zeroes

$$\Rightarrow \frac{f^{(n)}(z_0)}{n!} = 0 \quad \forall\ n<m$$

i.e. $\boxed{f(z_0) = f'(z_0) = \dots = f^{(m-1)}(z_0) = 0.}$

and coeff. of $(z-z_0)^m = \frac{f^{(m)}(z_0)}{m!}$ (in Taylor series)

$= g(z_0) \neq 0$ given.

$$\Rightarrow \frac{f^{(m)}(z_0)}{m!} \neq 0. \Rightarrow \boxed{f^{(m)}(z_0) \neq 0.}$$

Therefore z_0 is a zero of order m.

Necessary cond. : Assume that $f(z)$ has a zero at z_0 of order m.

Since $f(z)$ is analytic at z_0, it has Taylor series expansion, so in $|z-z_0|<\epsilon$,

$$f(z) = \underset{\overset{\shortparallel}{0}}{f(z_0)} + (z-z_0)\underset{\overset{\shortparallel}{0}}{f'(z_0)} + (z-z_0)^2 \frac{f''(z_0)}{2!} + \cdots$$

$$= (z-z_0)^m \frac{f^{(m)}(z_0)}{m!} + (z-z_0)^{m+1} \frac{f^{(m+1)}(z_0)}{(m+1)!} + \cdots$$

$$= (z-z_0)^m \underbrace{\left[\frac{f^{(m)}(z_0)}{m!} + (z-z_0) \frac{f^{(m+1)}(z_0)}{(m+1)!} + \cdots \right]}_{\text{define } g(z)}, \quad |z-z_0|<\epsilon$$

So, $\boxed{f(z) = (z-z_0)^m g(z)}$

The convergence of the above series in $|z-z_0|<\epsilon$ implies that $g(z)$ is analytic in $|z-z_0|<\epsilon$

$\Rightarrow$ $g(z)$ is analytic at z_0.

& $g(z_0) = \frac{f^{(m)}(z_0)}{m!} \neq 0$

Eg

$$f(z) = z^2(e^z - 1) = z^3 \cdot g(z) = z^3 \cdot \frac{(e^z-1)}{z}.$$

$$f(0) = 0$$

$$f'(z) = 2z(e^z-1) + z^2 e^z \Rightarrow f'(0) = 0$$

$$f''(z) = 2(e^z-1) + 4z e^z + z^2 e^z \Rightarrow f''(0) = 0$$

$$f'''(0) \neq 0$$

So $z=0$ is a zero of order $m=3$. In this case,

$$f(z) = z^3 \cdot \frac{(e^z-1)}{z}, \quad z \neq 0$$

Take
$$g(z) = \begin{cases} \dfrac{(e^z-1)}{z} & z \neq 0 \\ 1 & z = 0. \end{cases}$$

must be

$g(z)$ is cont^s at $z=0$,

$$g(0) = \lim_{z \to 0} g(z) = \lim_{z \to 0} \frac{(e^z-1)}{z} = 1$$

Then clearly $f(z) = z^3 \cdot g(z) \quad \forall z$

Ex Is $g(z)$ analytic at $z_0 = 0$? Is $g(0) \neq 0$?

Yes. Yes.

[Use: The series $1 + z/2! + z^2/3! + \cdots$
converges to $g(z)$ $\forall z$.
Thus $g(z)$ is analytic everywhere.]

§69. Zeros and poles of analytic (rational) functions are related.

Theorem 1: Suppose that $f(z) = \dfrac{p(z)}{q(z)}$

(i) $p(z)$ and $q(z)$ are analytic at z_0.

(ii) $p(z_0) \neq 0$ and $q(z)$ has zero of order m at z_0.

Then $f(z)$ has a pole of order m at z_0.

Theorem 2 Let $f(z) = \dfrac{p(z)}{q(z)}$,

(i) as above
(ii) as above with $m = 1$, (i.e. $q'(z_0) \neq 0$)

Then $f(z)$ has a pole of order 1 at z_0
and

$$\operatorname*{Res}_{z=z_0} f(z) = \frac{p(z_0)}{q'(z_0)}$$

Eg. $f(z) = \cot z = \dfrac{\cos z}{\sin z} = \dfrac{p(z)}{q(z)}$

zeros of $q(z)$ are $z = n\pi$, $n \in \mathbb{Z}$

- $q(n\pi) = 0$ & $q'(n\pi) = (\cos z)\big|_{n\pi}$

$= (-1)^n \neq 0.$

- $\cos z$ and $\sin z$ are entire.

$\Rightarrow$ $z = n\pi$ is a pole of order one of $f(z)$ with residue $= \dfrac{p(n\pi)}{q'(n\pi)} = \dfrac{(-1)^n}{(-1)^n} = 1$

Eg. $f(z) = \dfrac{1}{z(e^z - 1)} = \dfrac{p(z)}{q(z)}$

with $p(z) = 1$ & $q(z) = z(e^z - 1)$

(i) p & q are analytic at $z = 0$.

(ii) $p(0) \neq 0$. and $q(0) = 0$.

$q'(0) = (e^z - 1) + z \cdot e^z\big|_{z=0}$

$= 0$

$q''(0) = ?$

Proof :- (1) $q(z)$ has a zero of order m at z_0
$\Rightarrow q^{(m)}(z_0) \neq 0$, and z_0 is a singular pt of $\frac{p(z)}{q(z)}$.

(First we show that z_0 is an isolated sing. pt)

We have $q(z) = (z-z_0)^m g(z)$ by earlier result.

where $g(z)$ is analytic at z_0 & $g(z_0) \neq 0$

$\Downarrow$

$g(z)$ is cont^s at z_0 & $g(z_0) \neq 0$

$\Downarrow$

$\lim_{z \to z_0} g(z) = g(z_0) \neq 0$

$\Downarrow \longrightarrow$ Ex.

$\exists$ a nbd of z_0 where $g(z) \neq 0$ say $|z-z_0| < \varepsilon$

Then for $z \neq z_0$, $|z-z_0| < \varepsilon$

$q(z) = (z-z_0)^m g(z) \neq 0$.

So, in $0 < |z-z_0| < \varepsilon$, $f(z) = \frac{p(z)}{q(z)}$ is analytic $\Rightarrow$ $f(z)$ has isolated singularity at z_0.

Now $f(z) = \dfrac{p(z)/g(z)}{(z-z_0)^m} = \dfrac{\phi(z)}{(z-z_0)^m}$

where $\phi = p(z)/g(z)$ is analytic at z_0

& $\dfrac{p(z_0)}{g(z_0)} \neq 0$.

$\Rightarrow$ z_0 is pole of order m of $f(z)$

(2) By (1), z_0 is pole of order 1

and $f(z) = \dfrac{p(z)/g(z)}{(z-z_0)} = \dfrac{\phi(z)}{(z-z_0)}$

& by an earlier result,

$$\operatorname{Res}_{z=z_0} f(z) = \phi(z_0) = \frac{p(z_0)}{g(z_0)}$$

$$= \frac{p(z_0)}{q'(z_0)}$$

Note $q(z) = (z-z_0)g(z)$

$\Rightarrow$ $q'(z) = (z-z_0)g'(z) + g(z)$

$\Rightarrow$ $q'(z_0) = g(z_0)$

∎

Argument principle and Rouche's theorem

Theorem: If $f(z)$ is analytic inside and on a simple closed contour C with no zeros on C, then

$$\frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)}\,dz = \text{no. of zeros of } f(z) \text{ inside of } C$$

(when a zero of order α is counted α times)

We further generalize the above theorem

As above C: simple closed contour.

Theorem (Argument principle):

let $f(z)$ be analytic inside and on C except for a finite no. of poles inside C

- $f(z) \neq 0$ on C. Then

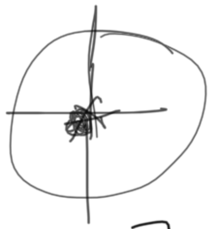
$$\frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)}\,dz = N_f - P_f \quad \text{where}$$

N_f : no. of zeros inside C

P_f : no. of poles insides C

(both counted with multiplicity)

Let's see some applications



(0) $\displaystyle\int_{|z|=3} \frac{1}{z}\, dz = 2\pi i \cdot (1)$. $\left[f(z) = z \right]$

(i) $\displaystyle I = \int_{|z|=2} \frac{dz}{3z+4} = ?$ say $f(z) = 3z+4$

$f'(z) = 3$

we can calculate $\displaystyle I' = \int_{|z|=2} \frac{3 \quad dz}{3z+4} = N_f - P_f$

$f(z) = 3z+4 \begin{cases} \text{has only one zero } \frac{-4}{3} \text{ of order } 1 \\ \text{has no pole} \end{cases}$

inside $|z|=2$

so $I' = 2\pi i \cdot (1 - 0)$

$\Rightarrow I = \frac{1}{3} I' = \frac{2\pi i}{3}$

(iii) Suppose

$$f(z) = \frac{(z+1)(z+7)^5(z-i)^2}{(z^2-2z+2)^4(z+i)^8(z-5i)^3}$$

$C : \{z : |z| = 2\}$

Then zeros of f : $-1, -7, i$

poles of f : $-i, 5i, 1 \pm i$.

Inside of C : zeros : -1 of order 1

and i of order 2.

So, $N_f =$ no. of zeros inside of $C = 1+2=3$

poles : $-i$ of order 8

$1+i$ of order 4

$1-i$ of order 4

So, $P_f =$ no. of poles inside of $C = 8+4+4 = 16$

$N_f - P_f = 3 - 16 = -13$.

By Argument principle,

$$\int_{|z|=2} \frac{f'(z)}{f(z)}\,dz = 2\pi i(-13) = -26\pi i$$

Ex Use Argument principle to find

(i) $I = \int_C \frac{z+i}{z^2+2iz-4}$ $C: |z+1+i| = 2$

(ii) $I = \int_{|z|=2} \frac{z+2}{z(z+1)}\, dz$

Ans: (i) $I = \pi i$

(ii) $I = 2\pi i$

One surprising fact, one can observe from argument principle is that,

$$\frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)}\, dz \text{ is always an integer.}$$

This is actually based on the properties of logarithm function. We explain it below.

(Read only if interested)

(If you are curious to know why it is called argument priniple.)

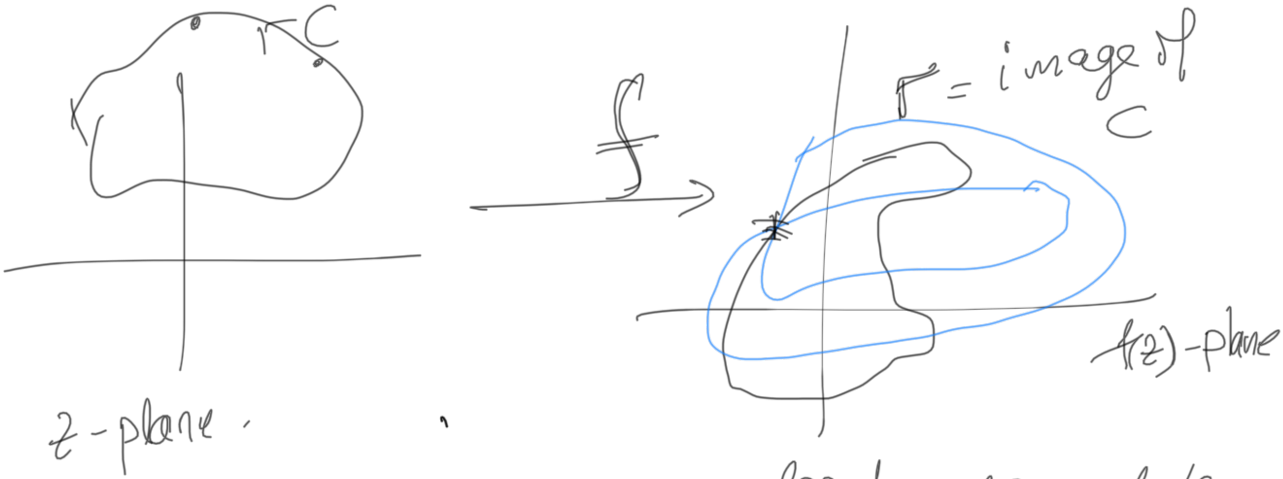
This integration can be performed in an alternative way.

Let C : $z(t)$: parametrization of C.
$a \le t \le b$

C : closed, $z(a) = z(b)$

Then $\displaystyle\int_C \frac{f'(z)}{f(z)}\,dz = \int_a^b \frac{f'(z(t))}{f(z(t))} \cdot z'(t)\,dt$ ——— $(*)$

$\underset{w}{\parallel} = -$



$z(a) = z(b)$

$f(z(a)) = f(z(b))$

Γ : closed, may not be simple

: may or may not contain zero (origin).

: does not pass through zero.

($f(z)$ does not have zero on C)

each point on Γ has a representation $\rho e^{i\phi}$

More precisely, $\left\{ \begin{array}{l} f(z(t)) = \rho(t)\, e^{i\phi(t)} \\ a \le t \le b \end{array} \right.$

(ρ & ϕ are cont^s)

$$\frac{d}{dt} f(z(t)) = f'(z(t)) \cdot z'(t) \quad \left(= \frac{d f(z)}{dz} \frac{d z(t)}{dt} \right)$$

$$\frac{d}{dt}\left(\rho(t) e^{i\phi(t)}\right) = \rho'(t) e^{i\phi(t)} + i \rho(t) e^{i\phi(t)} \phi'(t)$$

Then : (if $\rho'(t)$ & $\phi'(t)$ are piecewise cont^s on $a \le t \le b$, then integrat^n makes sense)

$$\int_a^b \frac{f'(z(t))\, z'(t)}{f(z(t))} dt = \int_a^b \frac{\rho'(t) e^{i\phi(t)} + i\rho(t) e^{i\phi(t)} \phi'(t)}{\rho(t) e^{i\phi(t)}} dt$$

$$= \int_a^b \frac{\rho'(t)}{\rho(t)} dt + i \int_a^b \phi'(t) dt \quad (\text{real int}^n)$$

$$= \ln \rho(t) \Big]_a^b + i\, \phi(t) \Big]_a^b$$

$$\underset{[\text{as } \rho(a) = \rho(b)]}{0}$$

$$\int_C \frac{f'(z)}{f(z)} dz = i\,(\phi(b) - \phi(a))$$

where $\phi(t)$: one of the arguments of $f(z(t))$.

$$= i\,(\arg f(z(b)) - \arg f(z(a)))$$

$$=: i\, \Delta_C \arg f(z).$$

This integration only depends on the change in the argument of $f(z)$ as z moves along C: (written as $\Delta_C \arg f(z)$).

So, $$\frac{\Delta_C \arg f(z)}{2\pi} = N_f - P_f.$$

Rouche's theorem C: simple closed contour

Let $f(z)$ and $g(z)$ are two functions, analytic inside and on C. Suppose

$*$ $|f(z)| > |g(z)|$ $\forall z \in C$.

Then $f(z)$ and $f(z)+g(z)$ have the same no. of zeroes, counting multiplicities, inside C.