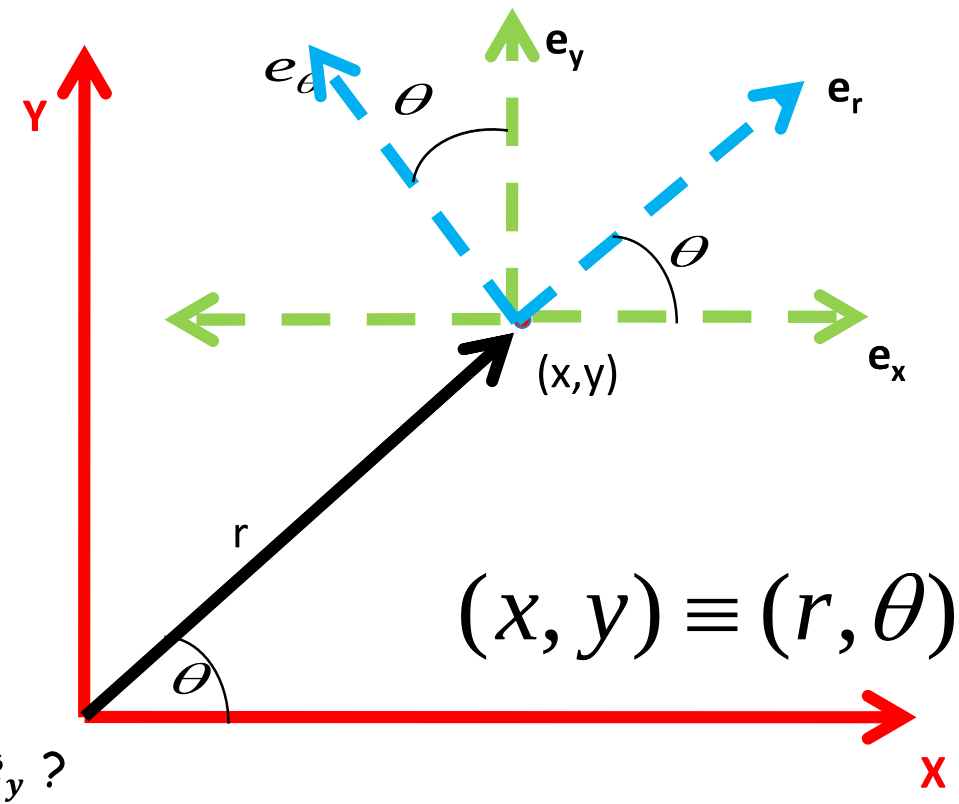
Coordinate Systems

Plane polar
Cylindrical
Spherical polar

Plane Polar Coordinates

$$x = r\cos(\theta)$$
$$y = r\sin(\theta)$$

$$r = \sqrt{x^2 + y^2}$$
$$\theta = \tan^{-1}\left(\frac{y}{x}\right)$$



What is $\hat{e}_r$ and $\hat{e}_\theta$ in terms of $\hat{e}_x$ and $\hat{e}_y$?

$$\hat{e}_r = \hat{e}_x\cos(\theta) + \hat{e}_y\sin(\theta)$$
$$\hat{e}_\theta = -\hat{e}_x\sin(\theta) + \hat{e}_y\cos(\theta)$$

What is $\hat{e}_x$ and $\hat{e}_y$ in terms of $\hat{e}_r$ and $\hat{e}_\theta$?

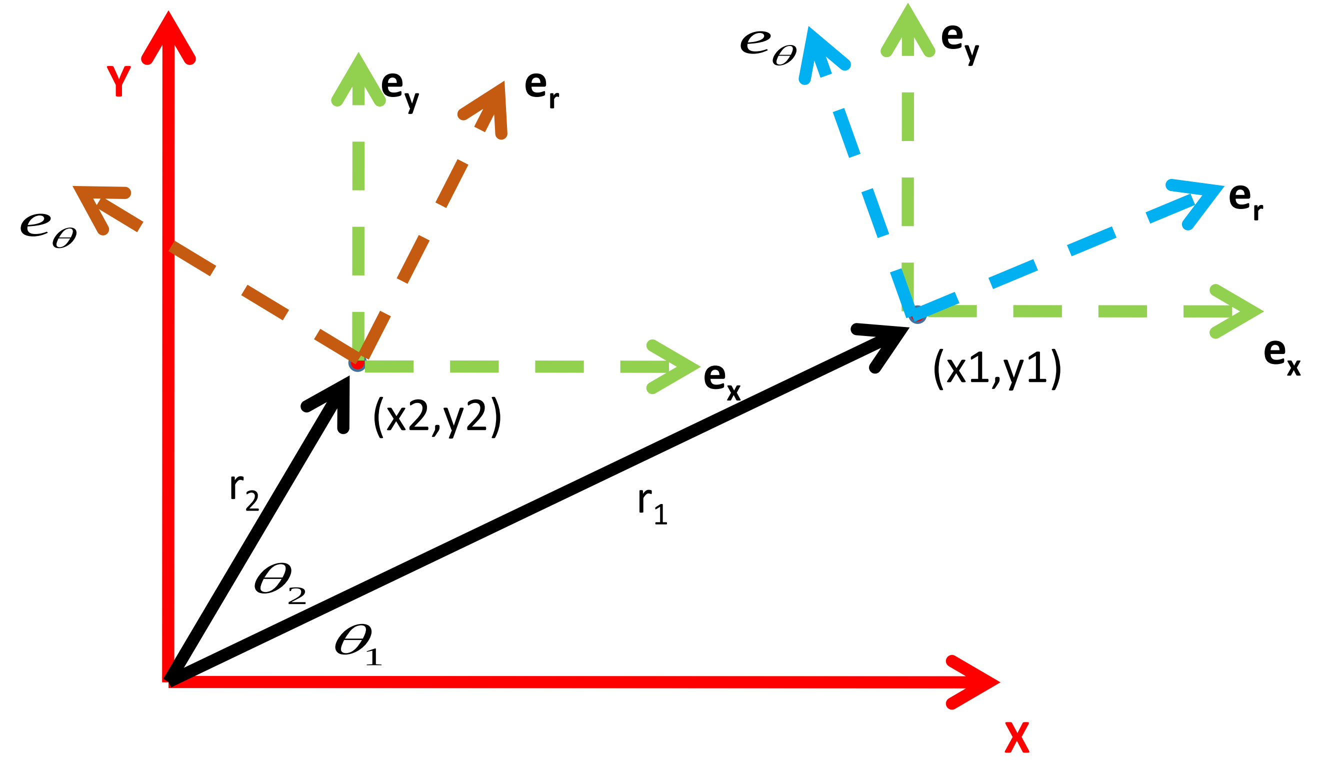
$$\hat{e}_x = \hat{e}_r\cos(\theta) - \hat{e}_\theta\sin(\theta)$$
$$\hat{e}_y = \hat{e}_r\sin(\theta) + \hat{e}_\theta\cos(\theta)$$

HW: Verify $\hat{e}_\theta . \hat{e}_r = 0$

Above vector in Polar Co-ordinates is represented as

$$\vec{r} = r\hat{e}_r$$

Motion in Plane Polar Coordinates



Cartesian coordinate system: Constant unit vectors

Plane polar coordinate system: Varying unit vectors

Change in unit vectors in Plane Polar Coordinates

$$\frac{d\hat{e}_\theta}{d\theta}$$

$$\frac{d\hat{e}_\theta}{dr} = 0$$

$$\hat{e}_r = \hat{e}_x \cos(\theta) + \hat{e}_y \sin(\theta)$$
$$\hat{e}_\theta = -\hat{e}_x \sin(\theta) + \hat{e}_y \cos(\theta)$$

Change in unit vectors

$$\frac{d\hat{e}_r}{d\theta} = \hat{e}_\theta \qquad \frac{d\hat{e}_\theta}{d\theta} = -\hat{e}_r$$

$$\frac{d\hat{e}_r}{dr} = 0 \qquad \frac{d\hat{e}_\theta}{dr} = 0$$

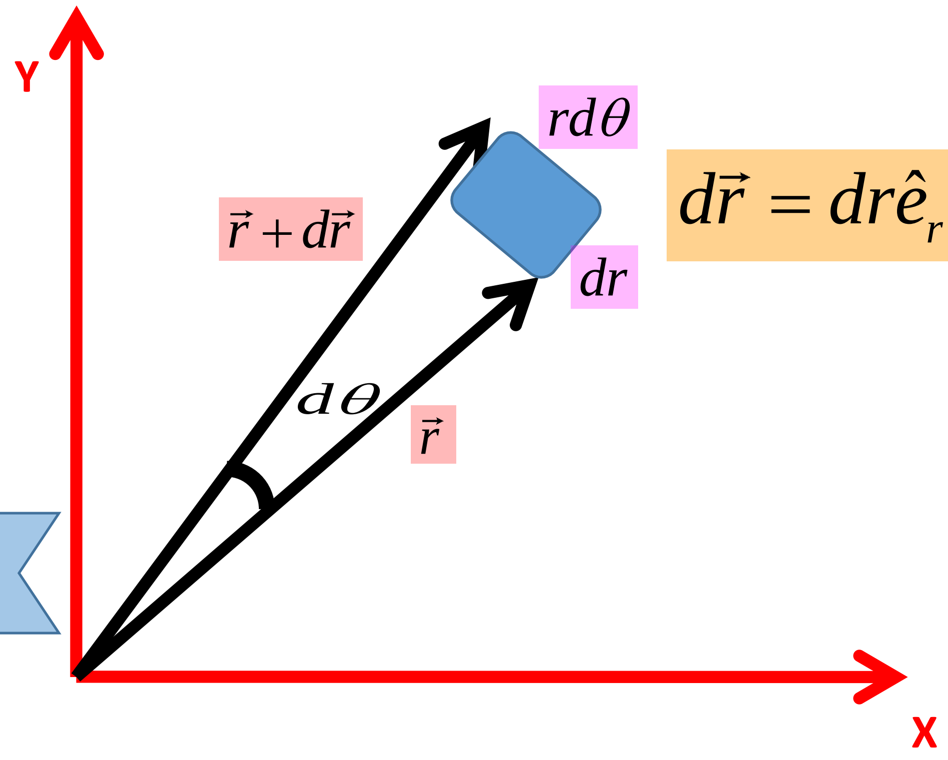
Elemental area in plane polar coordinates

Elemental area dA:

$rdrd\theta$

Area of Circle

Domain of integration



$$d\vec{r} = dr\hat{e}_r + rd\theta\hat{e}_\theta$$

$$\vec{r} = r\hat{e}_r$$

$$d\vec{r} = dr\hat{e}_r + rd\hat{e}_r$$

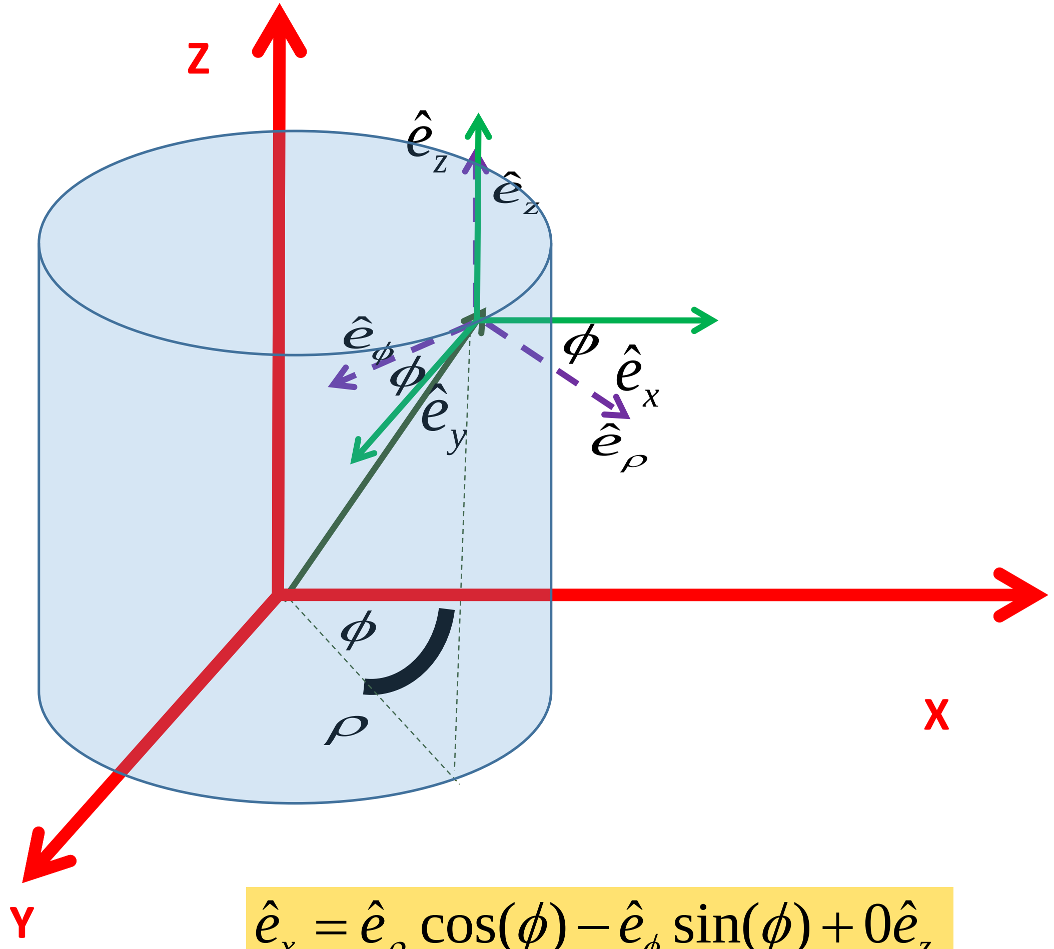
$$\frac{d\hat{e}_r}{d\theta} = \hat{e}_\theta$$

$$\frac{d\hat{e}_r}{dr} = 0$$

Cylindrical Polar Coordinate System

$$x = \rho\cos(\phi)$$
$$y = \rho\sin(\phi)$$
$$z = z$$

$$\rho = \sqrt{x^2 + y^2}$$
$$\phi = \tan^{-1}\left(\frac{y}{x}\right)$$
$$z = z$$



$$\hat{e}_\rho = \hat{e}_x\cos(\phi) + \hat{e}_y\sin(\phi) + 0\hat{e}_z$$
$$\hat{e}_\phi = -\hat{e}_x\sin(\phi) + \hat{e}_y\cos(\phi) + 0\hat{e}_z$$
$$\hat{e}_z = \hat{e}_z$$

$$\hat{e}_x = \hat{e}_\rho\cos(\phi) - \hat{e}_\phi\sin(\phi) + 0\hat{e}_z$$
$$\hat{e}_y = \hat{e}_\rho\sin(\phi) + \hat{e}_\phi\cos(\phi) + 0\hat{e}_z$$
$$\hat{e}_z = \hat{e}_z$$

Derivatives of unit vectors

$$\frac{d\hat{e}_\rho}{d\phi} = \hat{e}_\phi$$

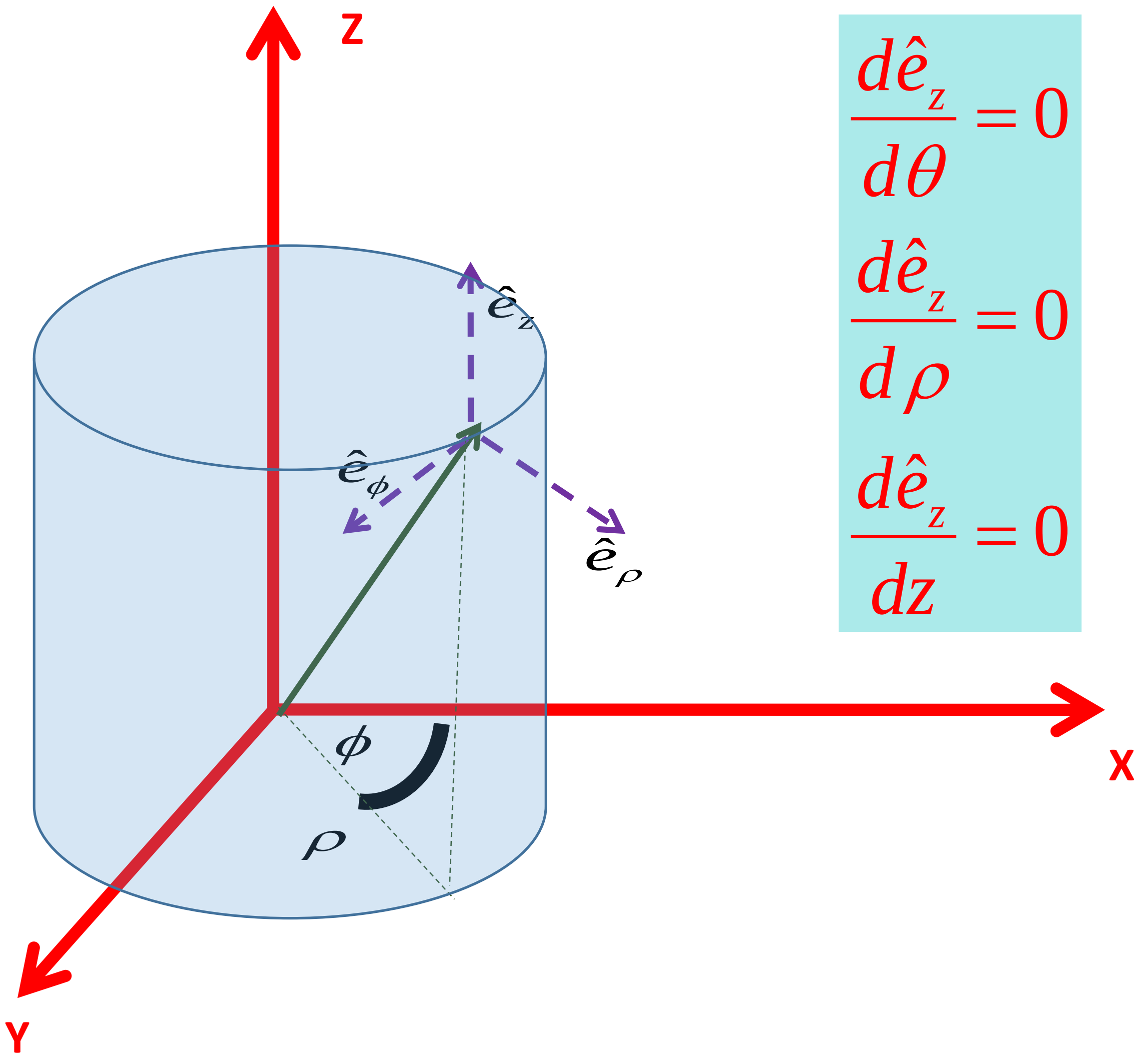
$$\frac{d\hat{e}_\rho}{dr} = 0$$

$$\frac{d\hat{e}_\rho}{dz} = 0$$

$$\frac{d\hat{e}_\phi}{d\phi} = -\hat{e}_\rho$$

$$\frac{d\hat{e}_\phi}{d\rho} = 0$$

$$\frac{d\hat{e}_\phi}{dz} = 0$$



$$\frac{d\hat{e}_z}{d\theta} = 0$$

$$\frac{d\hat{e}_z}{d\rho} = 0$$

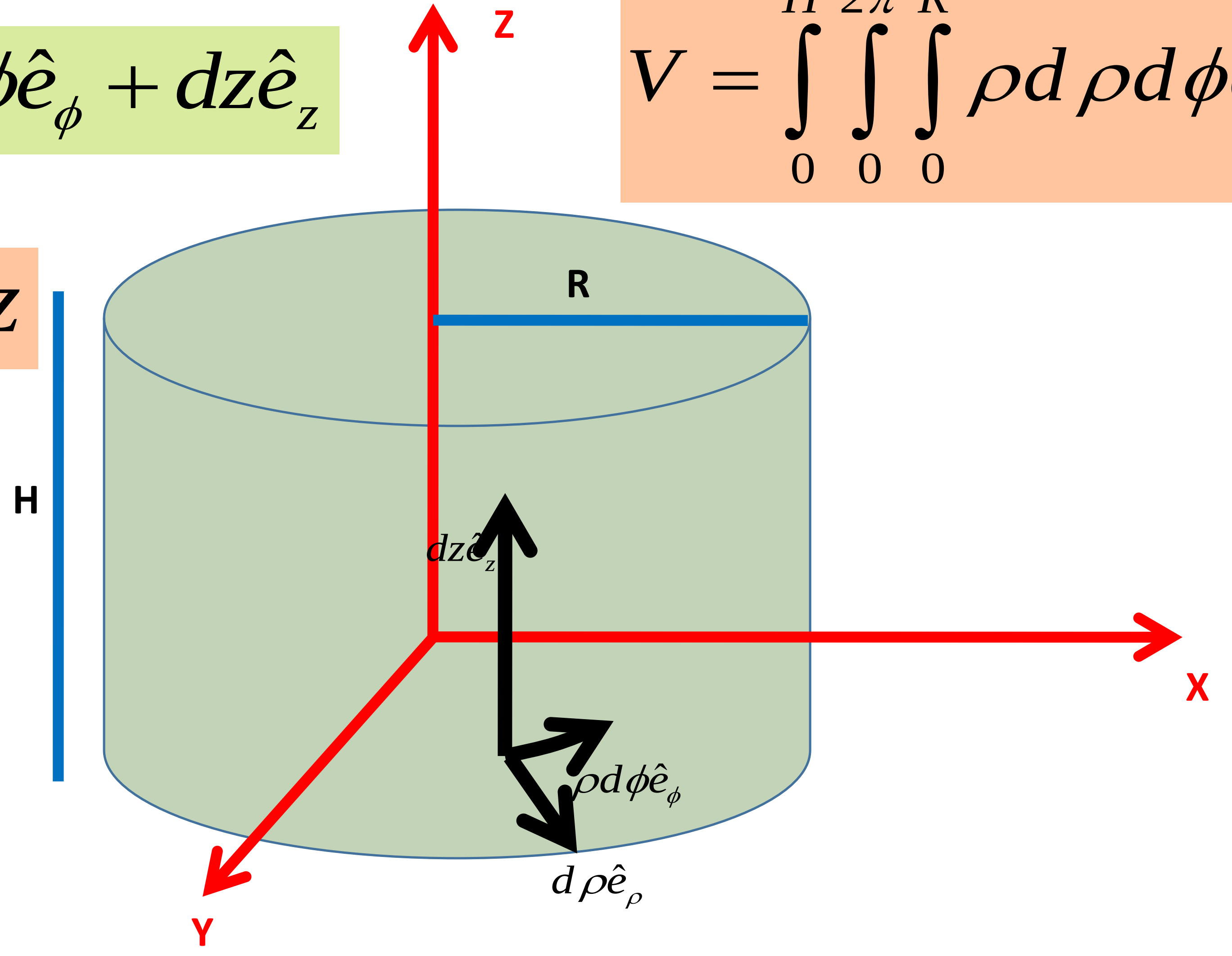
$$\frac{d\hat{e}_z}{dz} = 0$$

Domain of integration

$$d\vec{p} = d\rho\hat{e}_\rho + \rho d\phi\hat{e}_\phi + dz\hat{e}_z$$

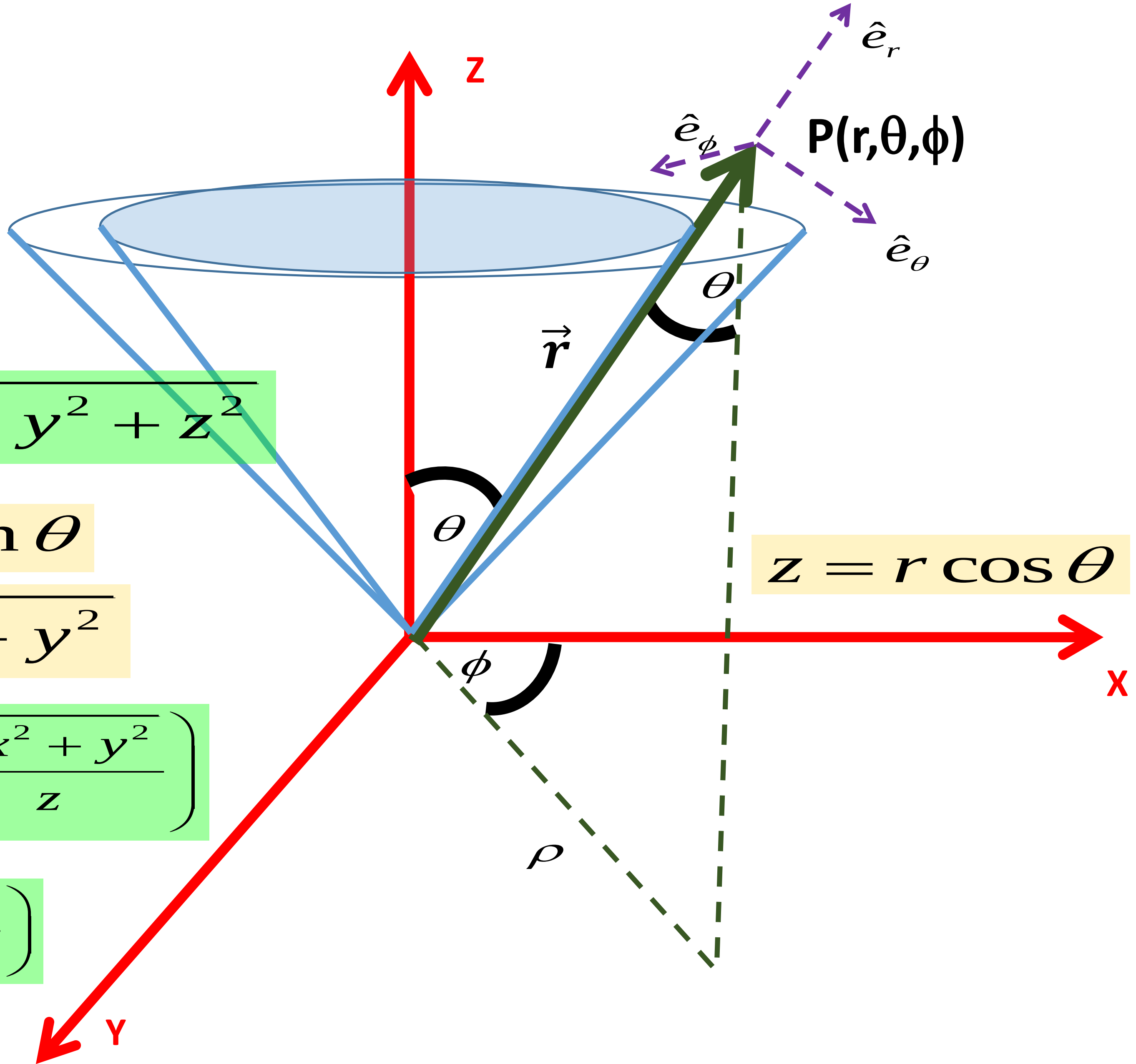
$$dV = \rho d\rho d\phi dz$$

$$dA = \rho d\phi dz$$



$$V = \int_0^H \int_0^{2\pi} \int_0^R \rho d\rho d\phi dz$$

Spherical Polar Coordinate System



$$r = \sqrt{x^2 + y^2 + z^2}$$

$$\rho = r \sin\theta$$

$$\rho = \sqrt{x^2 + y^2}$$

$$\theta = \tan^{-1}\left(\frac{\sqrt{x^2 + y^2}}{z}\right)$$

$$\phi = \tan^{-1}\left(\frac{y}{x}\right)$$

$$z = r\cos\theta$$

Transformation of Coordinates

$$r = \sqrt{x^2 + y^2 + z^2}$$

$$\theta = \tan^{-1}\left(\frac{\sqrt{x^2 + y^2}}{z}\right)$$

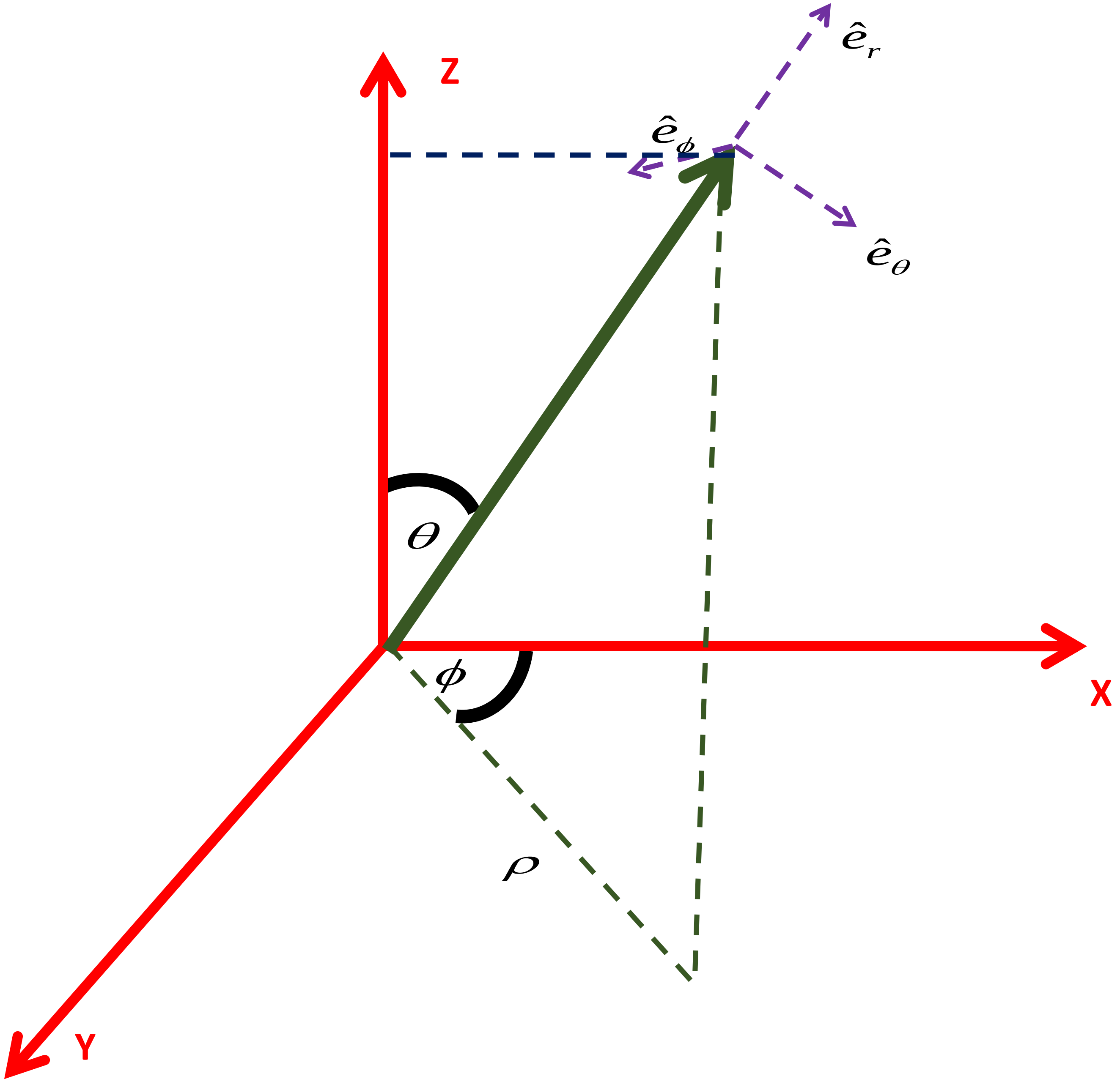
$$\phi = \tan^{-1}\left(\frac{y}{x}\right)$$

$$x = r\sin\theta\cos\phi$$

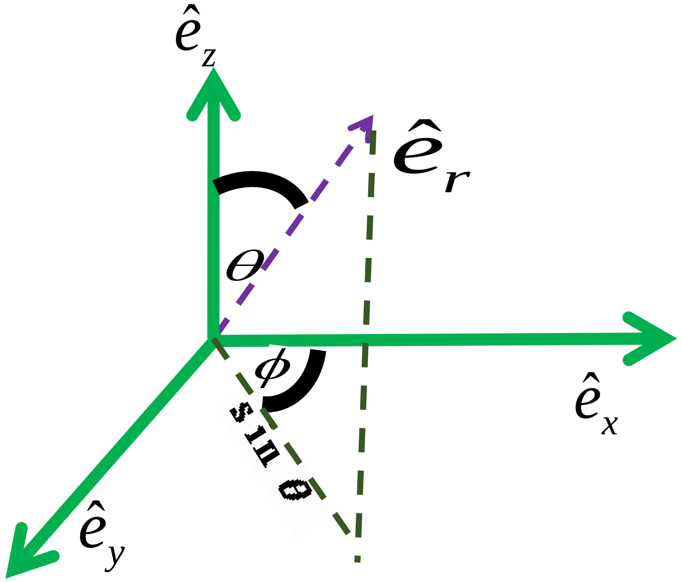
$$y = r\sin\theta\sin\phi$$

$$z = r\cos\theta$$

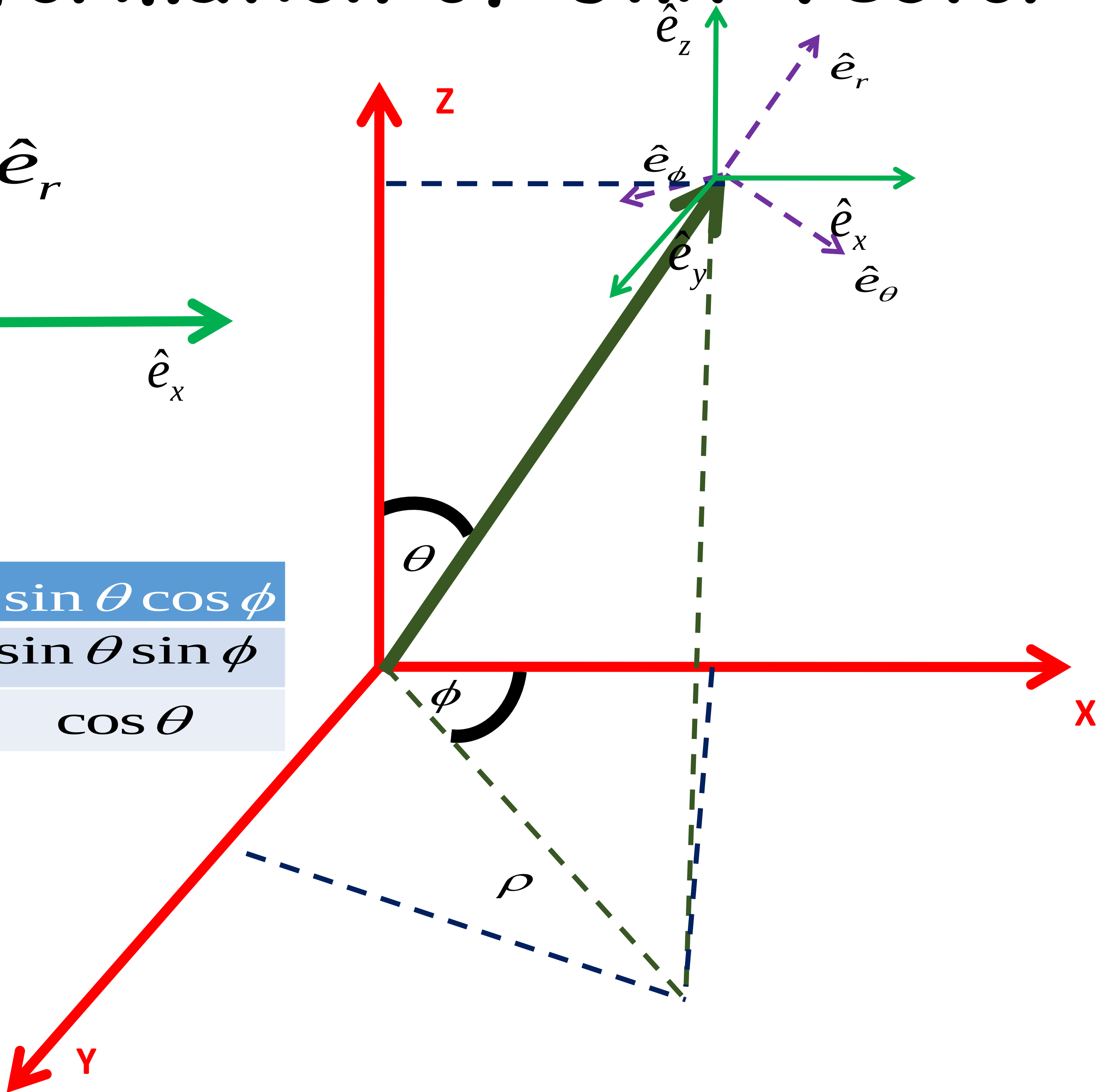
Transformation of Unit Vectors



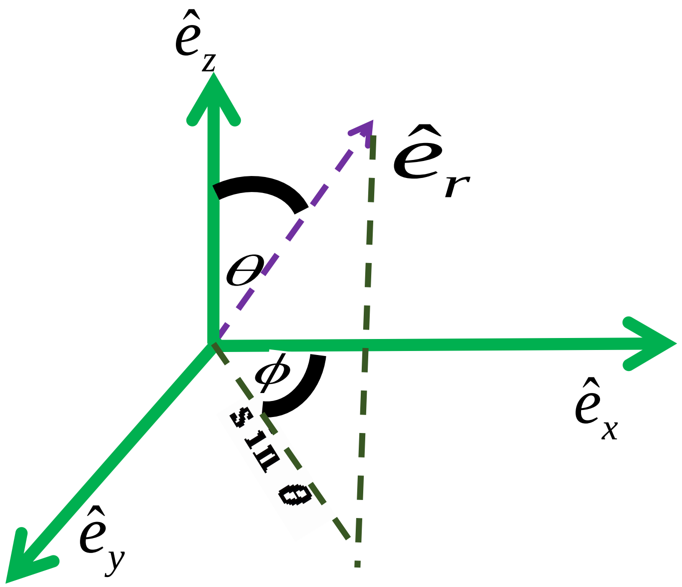
Transformation of Unit Vector ($\hat{e}_r$)



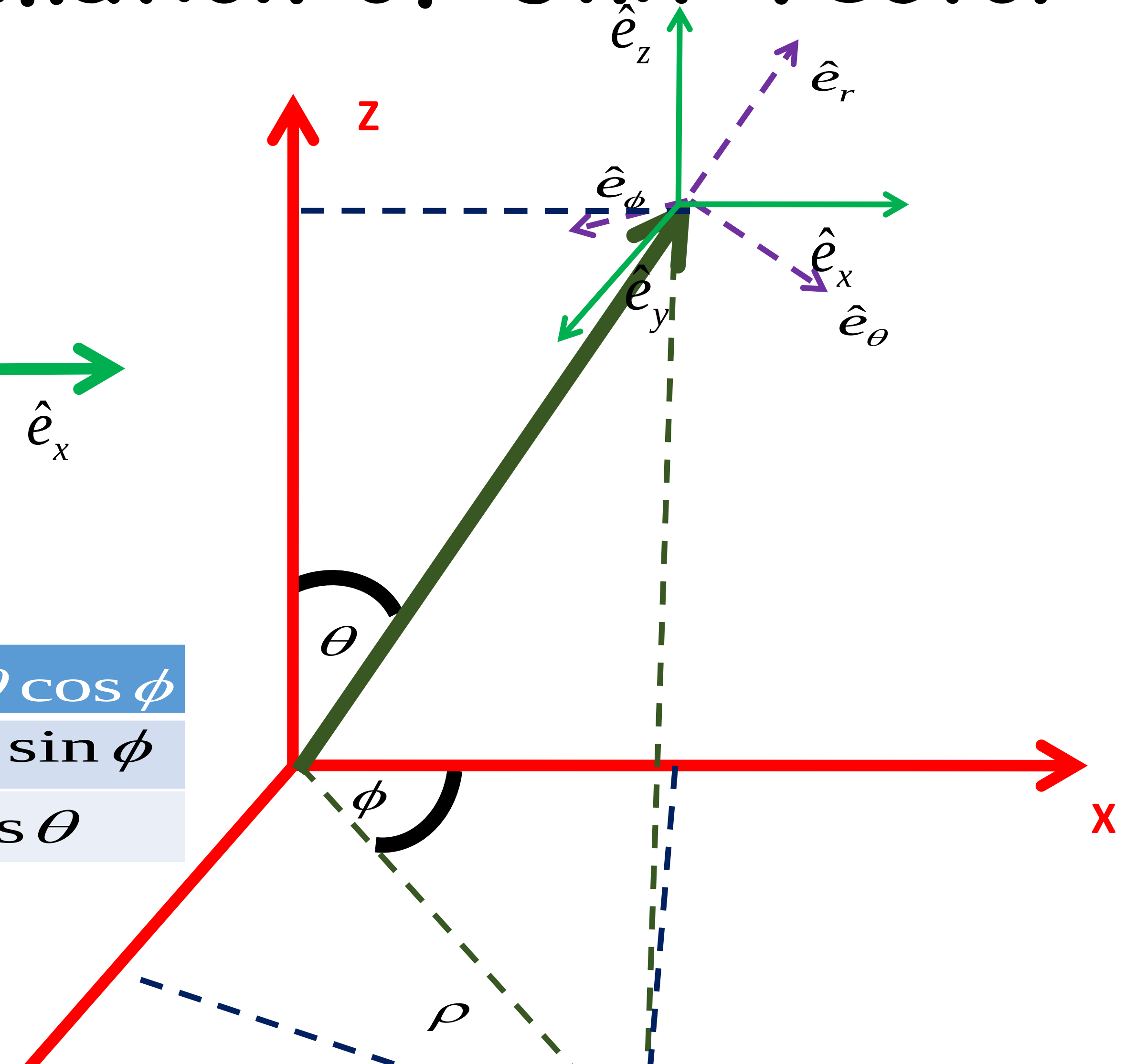
X component	$\sin\theta\cos\phi$
Y component	$\sin\theta\sin\phi$
Z component	$\cos\theta$



Transformation of Unit Vector ($\hat{e}_r$)

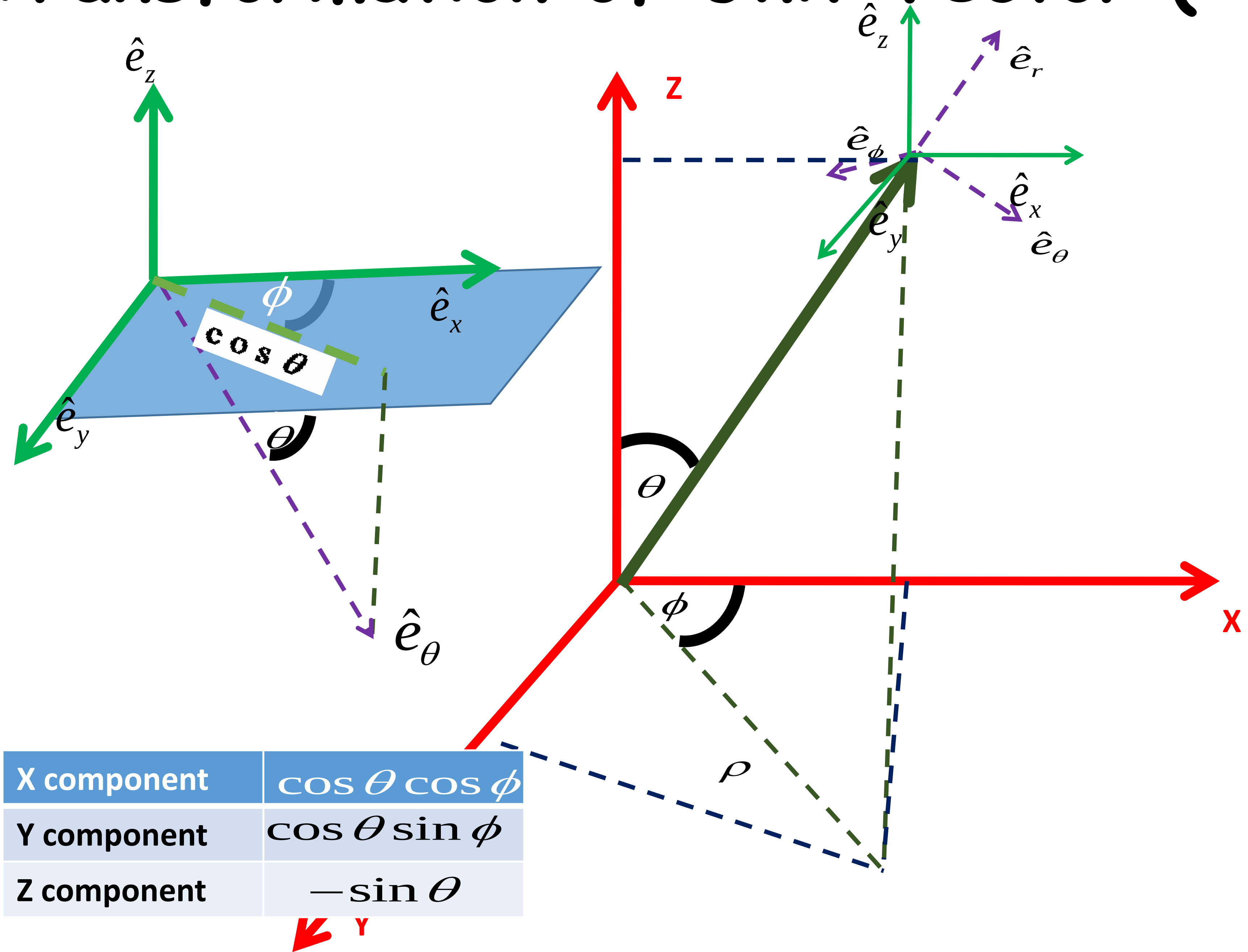


X component	$\sin\theta\cos\phi$
Y component	$\sin\theta\sin\phi$
Z component	$\cos\theta$



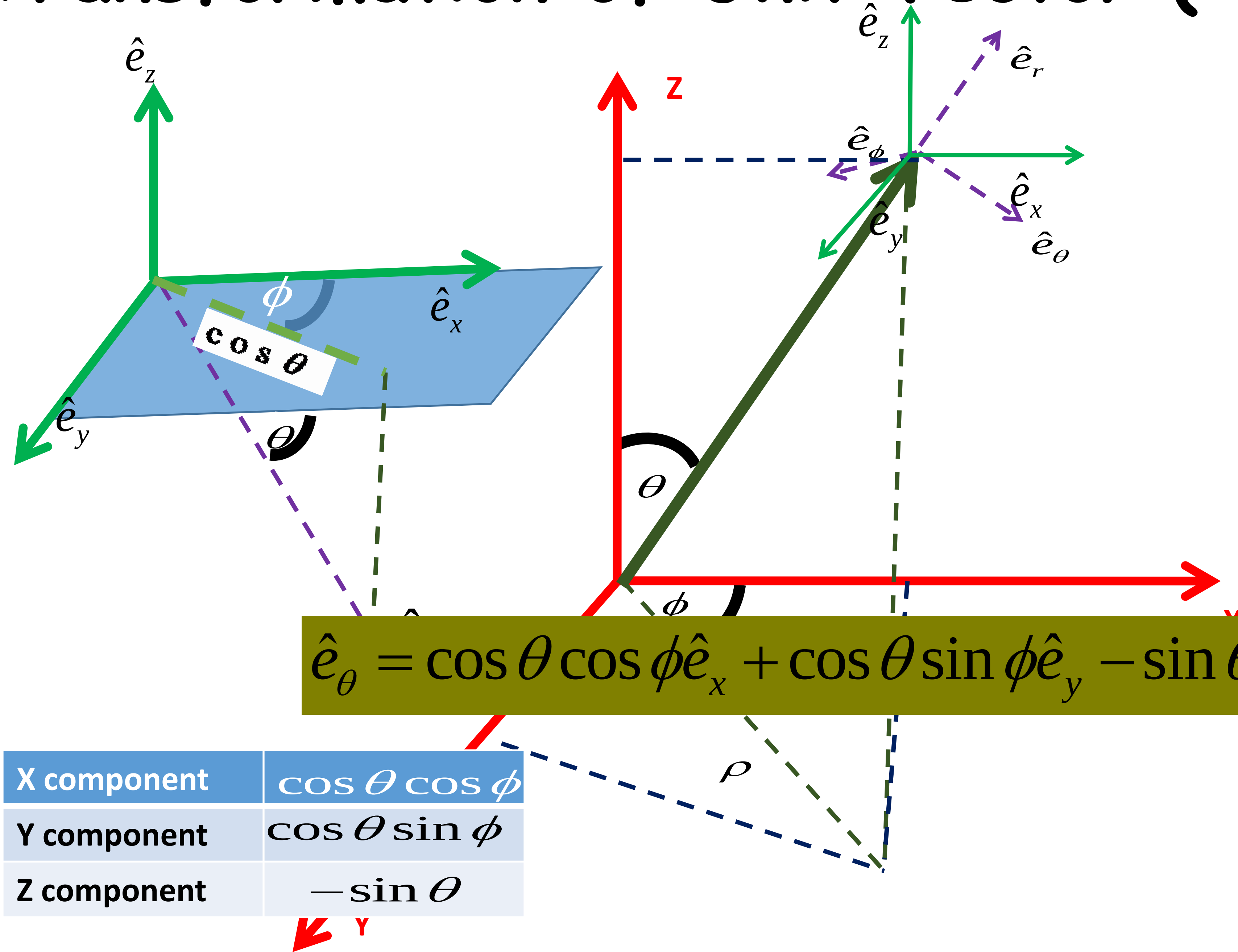
$$\hat{e}_r = \sin\theta\cos\phi\,\hat{e}_x + \sin\theta\sin\phi\,\hat{e}_y + \cos\theta\,\hat{e}_z$$

Transformation of Unit Vector ($\hat{e}_\theta$)



X component	$\cos\theta\cos\phi$
Y component	$\cos\theta\sin\phi$
Z component	$-\sin\theta$

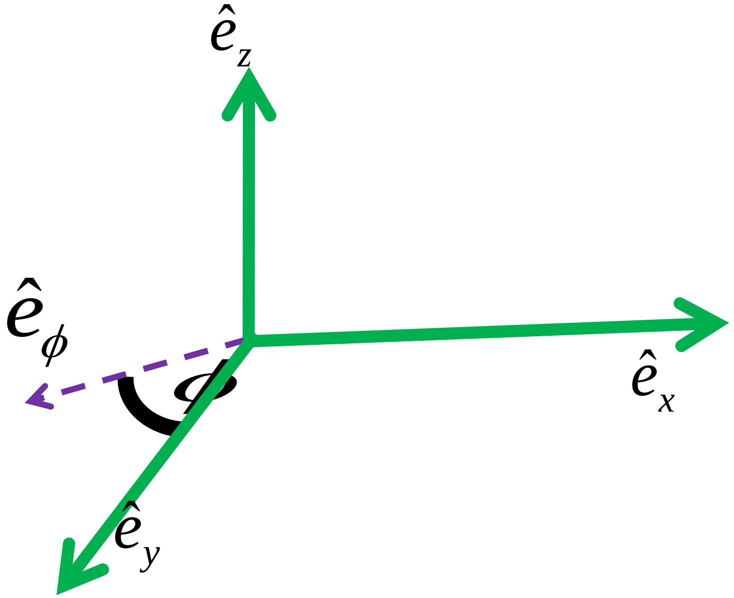
Transformation of Unit Vector ($\hat{e}_\theta$)

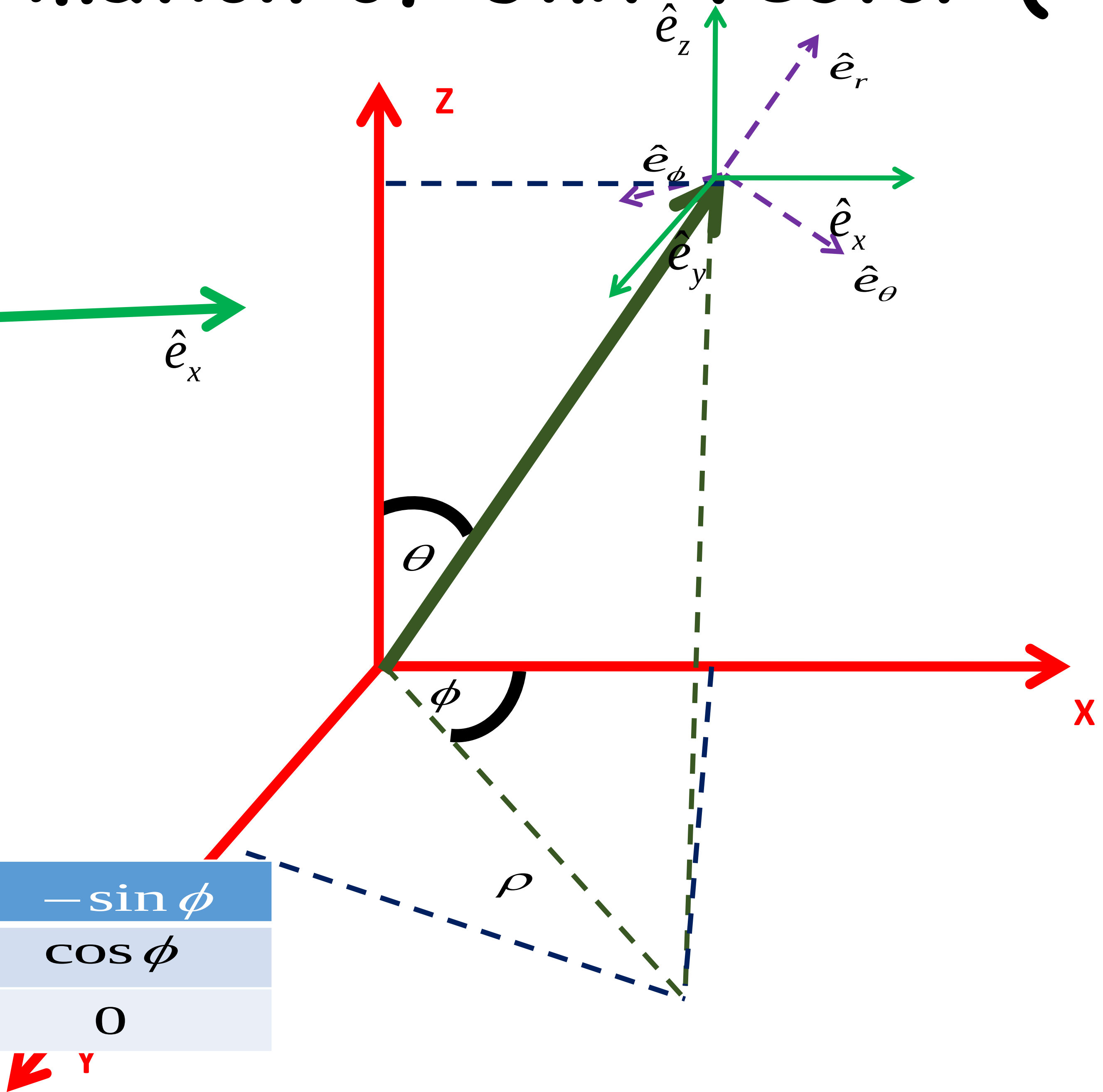


$$\hat{e}_\theta = \cos\theta\cos\phi\hat{e}_x + \cos\theta\sin\phi\hat{e}_y - \sin\theta\hat{e}_z$$

X component	$\cos\theta\cos\phi$
Y component	$\cos\theta\sin\phi$
Z component	$-\sin\theta$

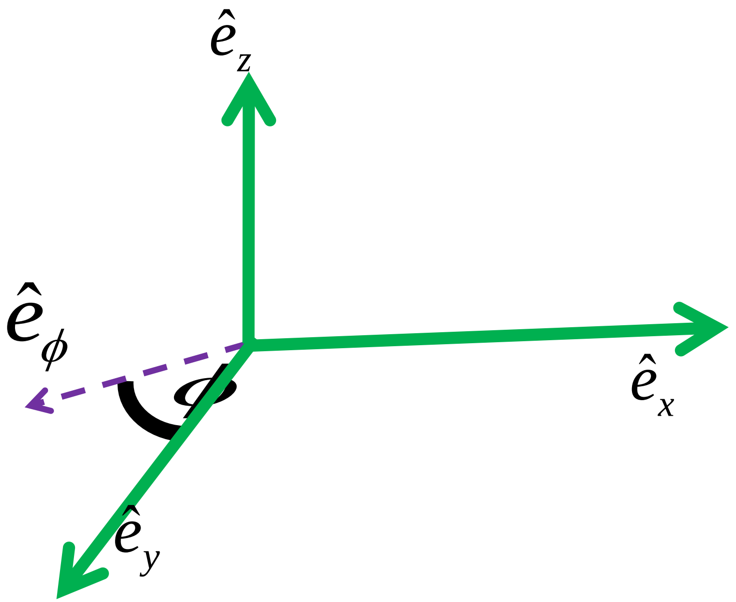
Transformation of Unit Vector ($\hat{e}_\phi$)

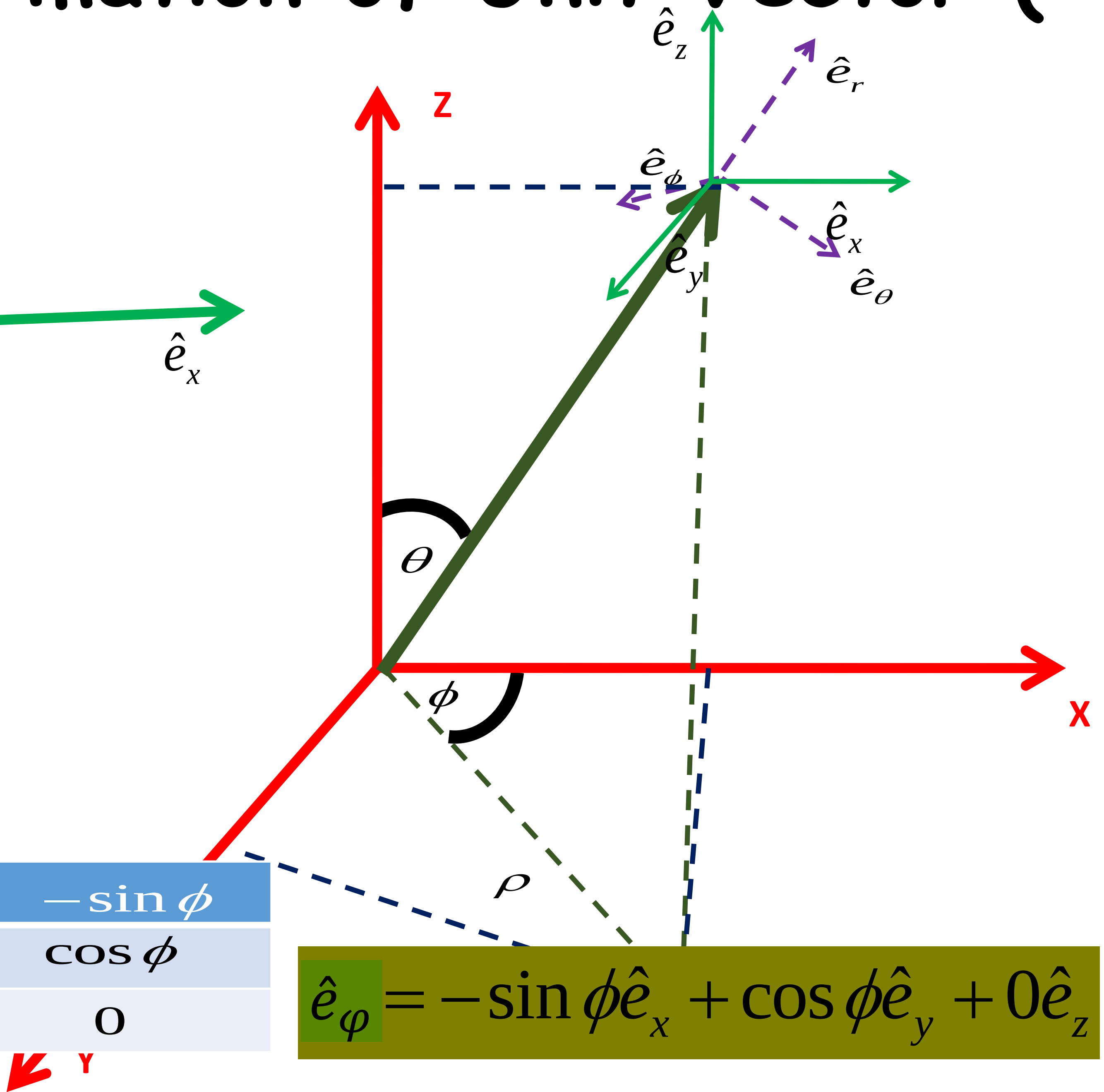




X component	$-\sin\phi$
Y component	$\cos\phi$
Z component	0

Transformation of Unit Vector ($\hat{e}_\phi$)





X component	$-\sin\phi$
Y component	$\cos\phi$
Z component	0

$$\hat{e}_\varphi = -\sin\phi\hat{e}_x + \cos\phi\hat{e}_y + 0\hat{e}_z$$

Transformation of Unit Vectors (Matrix notation)

$$\hat{e}_r = \sin\theta\cos\phi\hat{e}_x + \sin\theta\sin\phi\hat{e}_y + \cos\theta\hat{e}_z$$

$$\hat{e}_\theta = \cos\theta\cos\phi\hat{e}_x + \cos\theta\sin\phi\hat{e}_y - \sin\theta\hat{e}_z$$

$$\hat{e}_\varphi = -\sin\phi\hat{e}_x + \cos\phi\hat{e}_y + 0\hat{e}_z$$

$$\begin{bmatrix} \hat{e}_r \\ \hat{e}_\theta \\ \hat{e}_\phi \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\phi & \cos\phi & 0 \end{bmatrix} \begin{bmatrix} \hat{e}_x \\ \hat{e}_y \\ \hat{e}_z \end{bmatrix}$$

Transformation of Unit Vectors (Matrix notation)

$$\begin{bmatrix} \hat{e}_r \\ \hat{e}_\theta \\ \hat{e}_\phi \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\phi & \cos\phi & 0 \end{bmatrix} \begin{bmatrix} \hat{e}_x \\ \hat{e}_y \\ \hat{e}_z \end{bmatrix}$$

$$\begin{bmatrix} \hat{e}_x \\ \hat{e}_y \\ \hat{e}_z \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\phi & \cos\theta\cos\phi & -\sin\phi \\ \sin\theta\sin\phi & \cos\theta\sin\phi & \cos\phi \\ \cos\theta & -\sin\theta & 0 \end{bmatrix} \begin{bmatrix} \hat{e}_r \\ \hat{e}_\theta \\ \hat{e}_\phi \end{bmatrix}$$

Derivatives of Unit Vectors

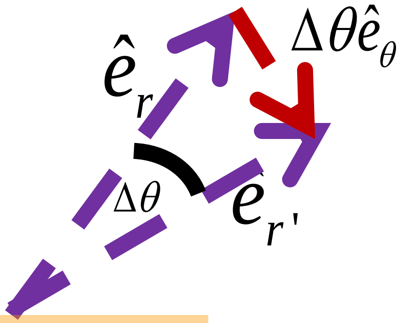
$$\begin{bmatrix} \hat{e}_r \\ \hat{e}_\theta \\ \hat{e}_\phi \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\phi & \cos\phi & 0 \end{bmatrix} \begin{bmatrix} \hat{e}_x \\ \hat{e}_y \\ \hat{e}_z \end{bmatrix}$$

$$\frac{d\hat{e}_r}{dr} = 0 \qquad \frac{d\hat{e}_r}{d\theta} = \hat{e}_\theta \qquad \frac{d\hat{e}_r}{d\phi} = \sin\theta\hat{e}_\phi$$

$$\frac{d\hat{e}_\theta}{dr} = 0 \qquad \frac{d\hat{e}_\theta}{d\theta} = -\hat{e}_r \qquad \frac{d\hat{e}_\theta}{d\phi} = \cos\theta\hat{e}_\phi$$

$$\frac{d\hat{e}_\phi}{dr} = 0 \qquad \frac{d\hat{e}_\phi}{d\theta} = 0 \qquad \frac{d\hat{e}_\phi}{d\phi} = -\cos\theta\hat{e}_\theta - \sin\theta\hat{e}_r$$

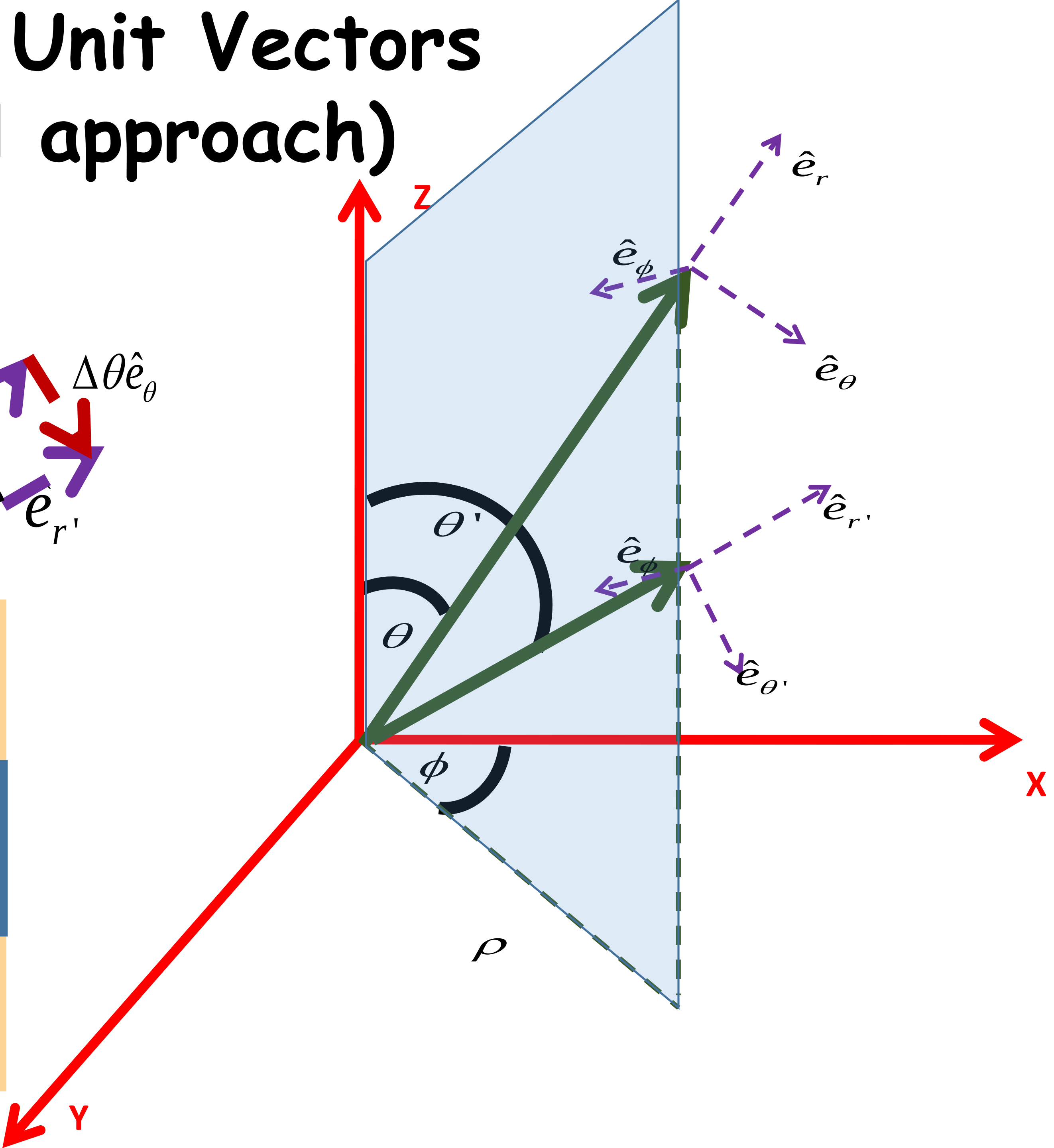
Derivatives of Unit Vectors (Geometrical approach)



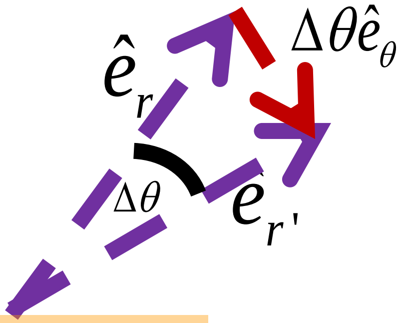
$$\frac{d\hat{e}_r}{dr} = 0$$

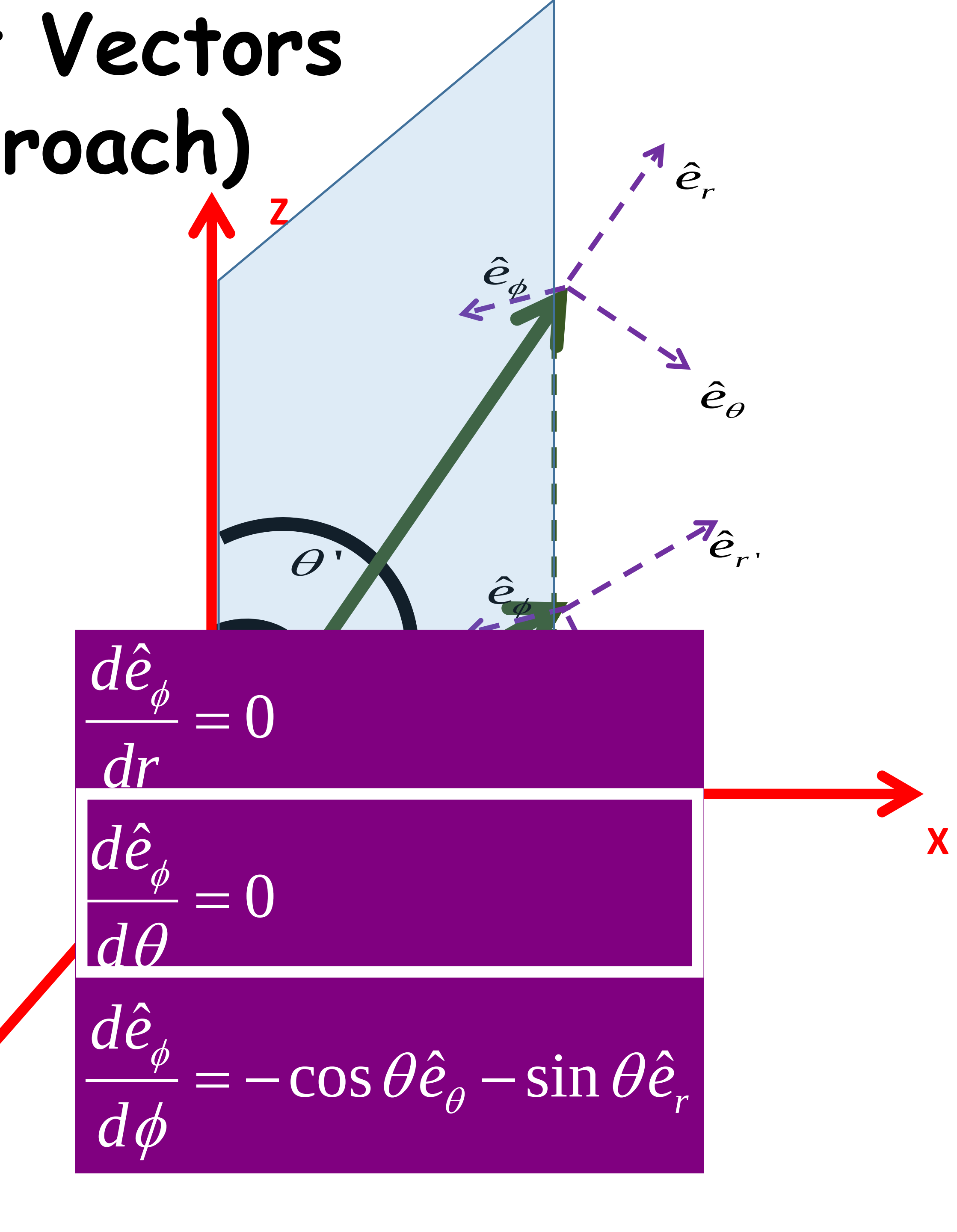
$$\boxed{\frac{d\hat{e}_r}{d\theta} = \hat{e}_\theta}$$

$$\frac{d\hat{e}_r}{d\phi} = \sin\theta\hat{e}_\phi$$



Derivatives of Unit Vectors (Geometrical approach)





$$\frac{d\hat{e}_r}{dr} = 0$$

$$\boxed{\frac{d\hat{e}_r}{d\theta} = \hat{e}_\theta}$$

$$\frac{d\hat{e}_r}{d\phi} = \sin\theta\hat{e}_\phi$$

$$\frac{d\hat{e}_\phi}{dr} = 0$$

$$\boxed{\frac{d\hat{e}_\phi}{d\theta} = 0}$$

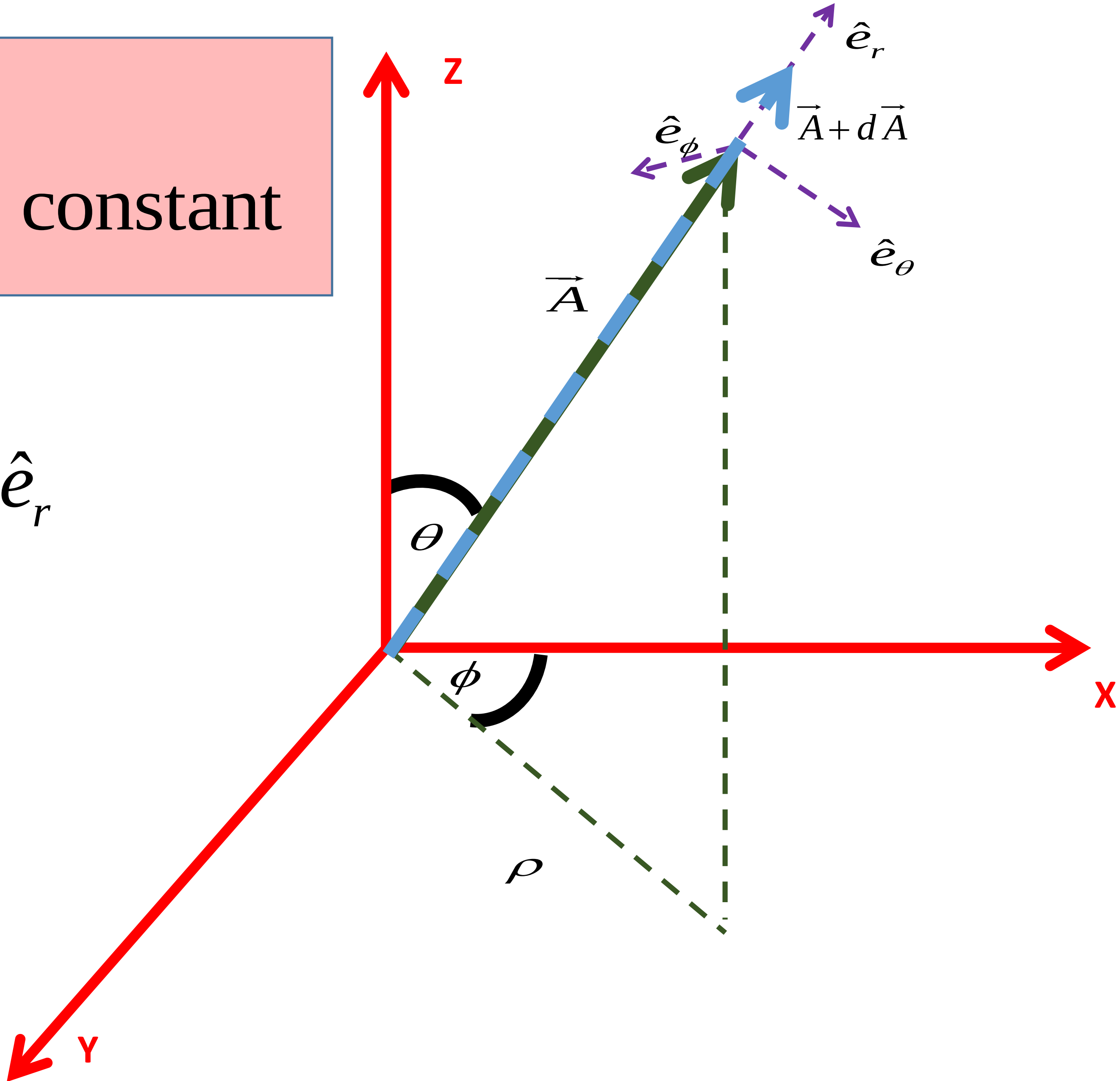
$$\frac{d\hat{e}_\phi}{d\phi} = -\cos\theta\hat{e}_\theta - \sin\theta\hat{e}_r$$

Infinitesimal line element

Case 1:

θ and ϕ is constant

$$d\vec{A} = dr\hat{e}_r$$

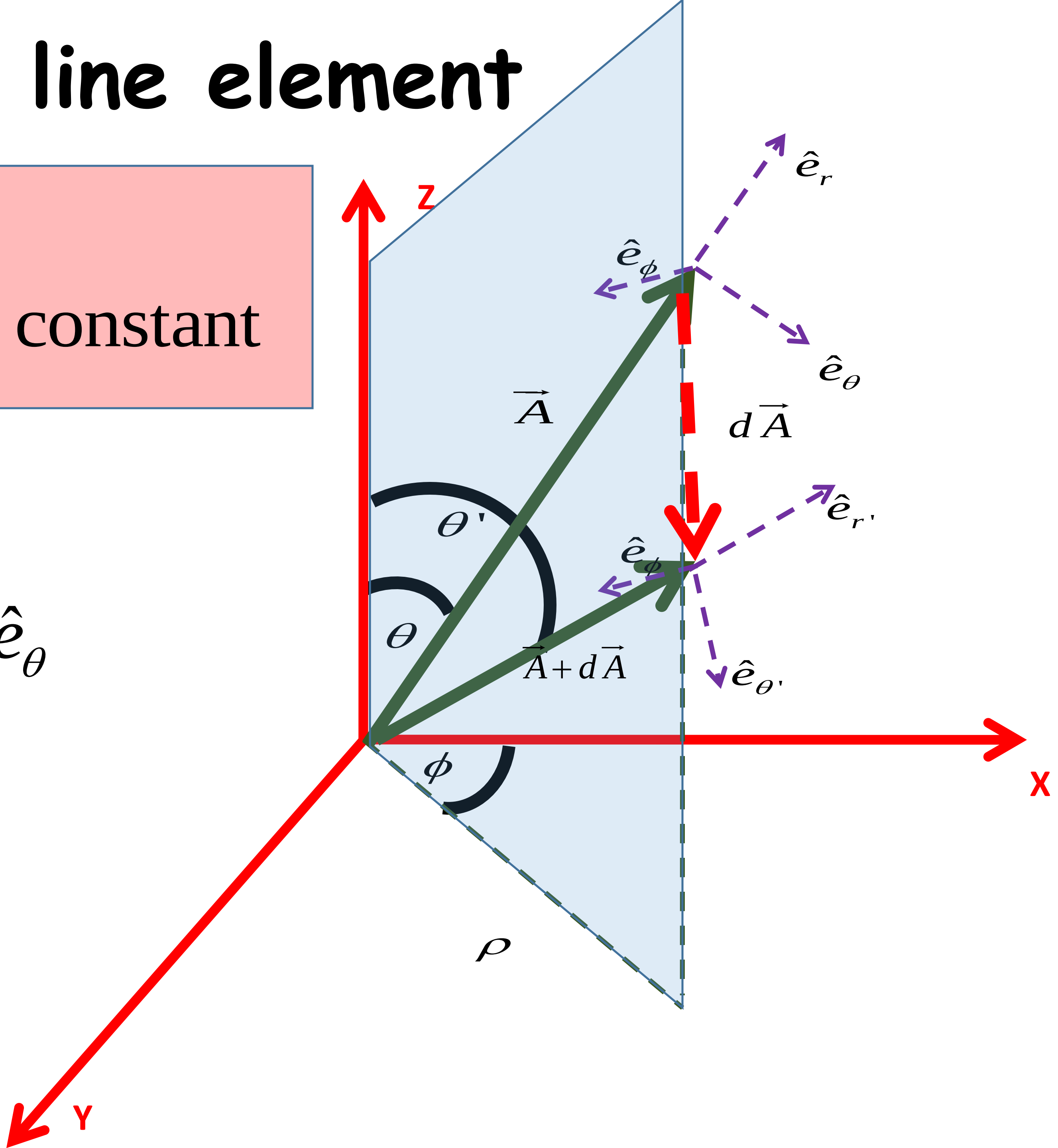


Infinitesimal line element

Case 2:

r and ϕ is constant

$$d\vec{A} = r d\theta \hat{e}_{\theta}$$

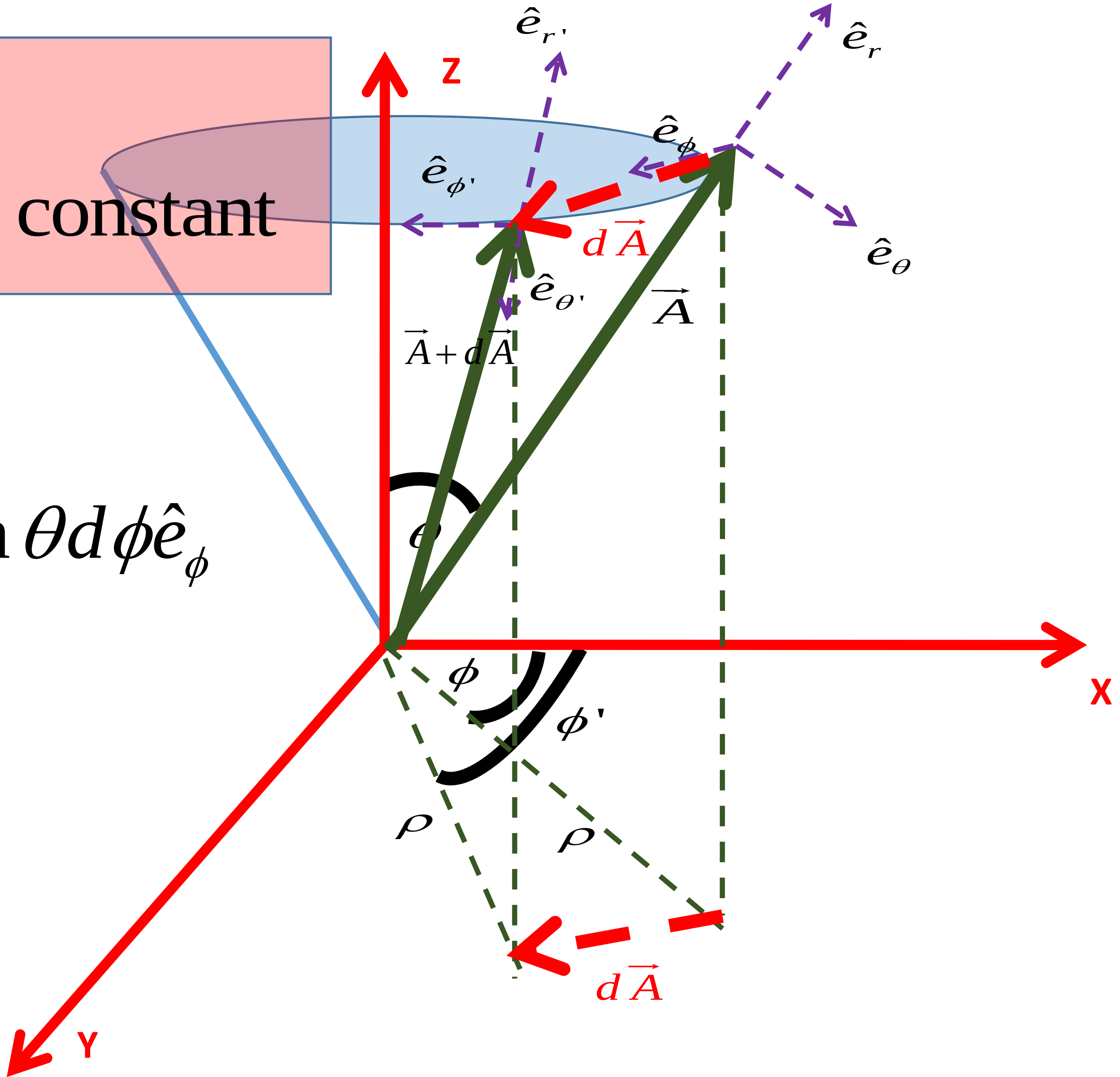


Infinitesimal line element

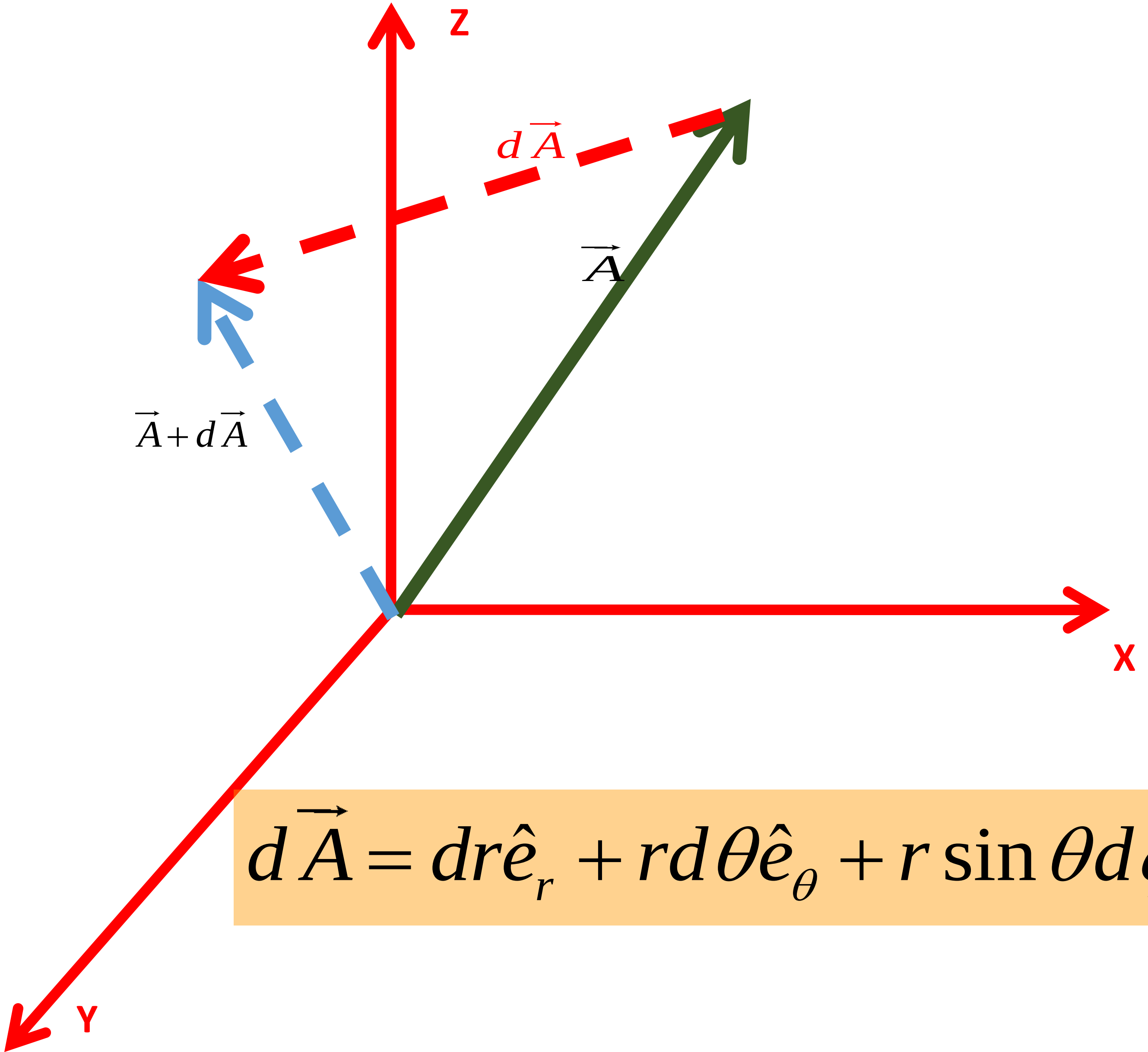
Case 3:

r and θ is constant

$$d\vec{A} = r\sin\theta d\phi \hat{e}_\phi$$

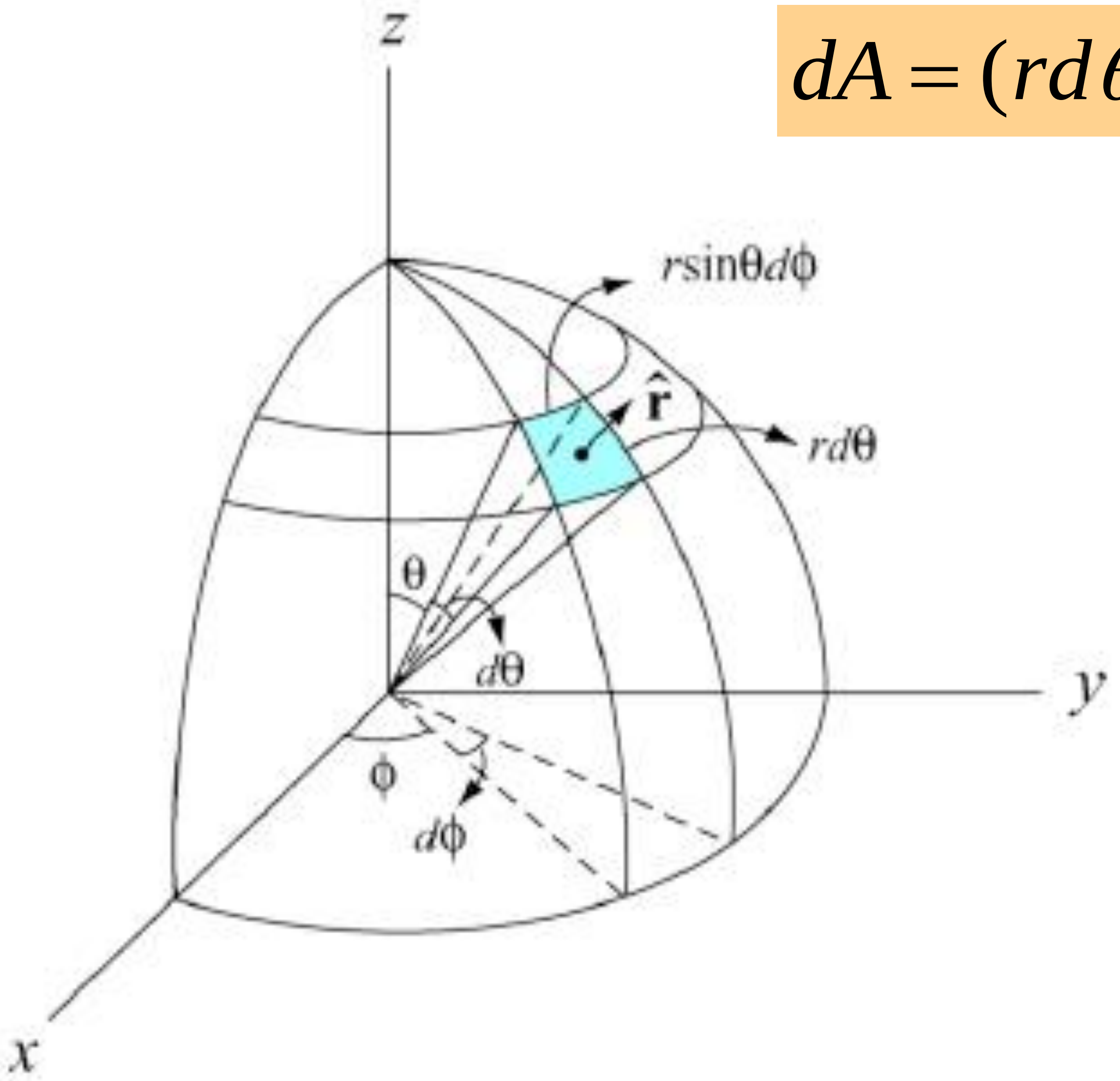


Infinitesimal line element



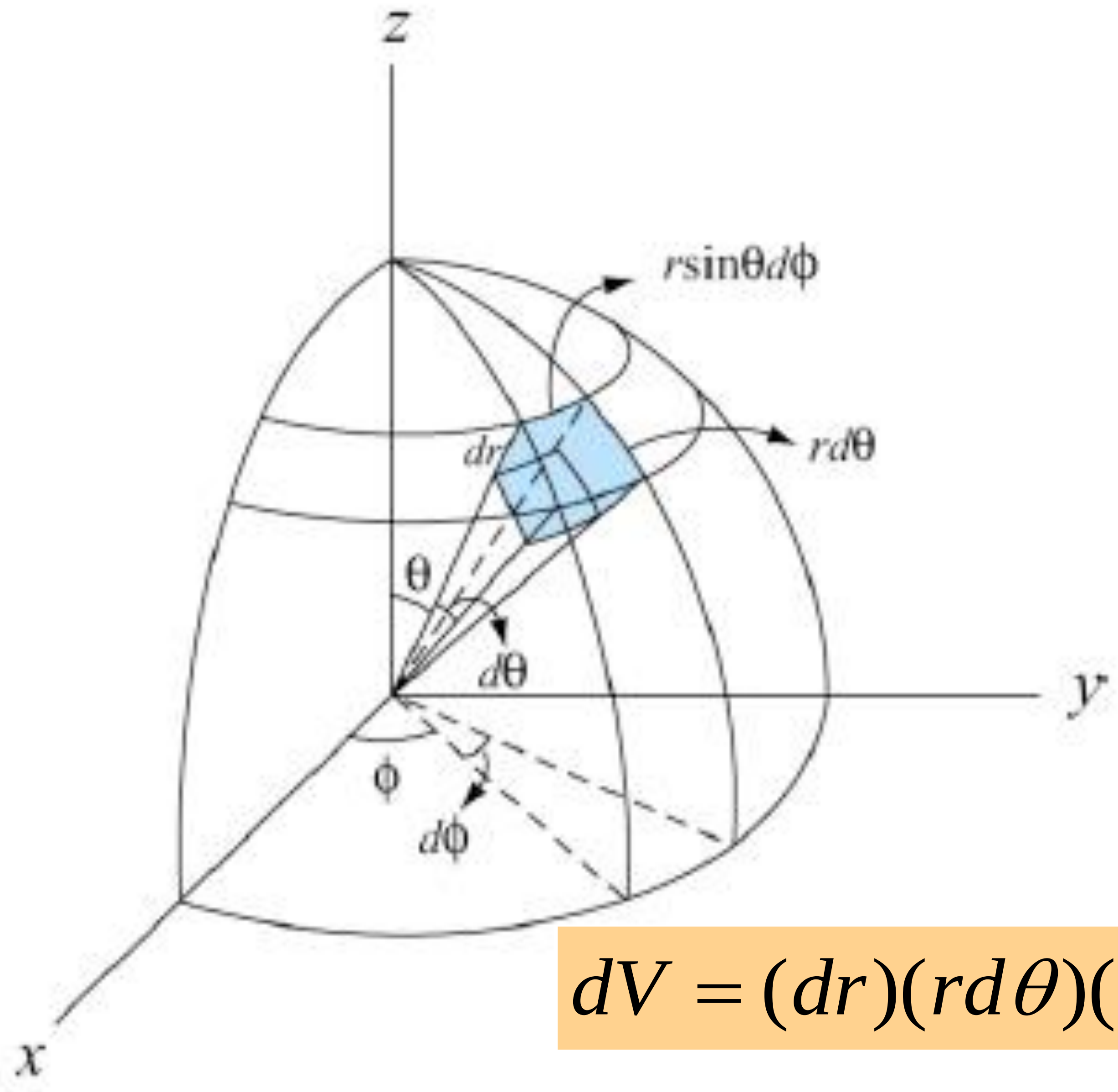
$$d\vec{A} = dr\hat{e}_r + rd\theta\hat{e}_\theta + r\sin\theta d\phi\hat{e}_\phi$$

Infinitesimal area element



$$dA = (rd\theta)(r\sin\theta d\phi) = r^2 \sin\theta d\theta d\phi$$

Infinitesimal volume element



$$dV = (dr)(rd\theta)(r\sin\theta d\phi) = r^2 \sin\theta drd\theta d\phi$$