MA-201

END-SEMESTER

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- 1901CS65

- COMPUTER SCIENCE

Ans 1:-

Given:

$x(\theta)$ and $y(\theta)$ are 2π periodic

$$x(\theta) = \sum_{n=-\infty}^{\infty} a_n e^{in\theta}$$

$$\gamma(\theta) = (x(\theta), y(\theta))$$

$$y(\theta) = \sum_{n=-\infty}^{\infty} b_n e^{in\theta}$$

$$\gamma[0,2\pi] \longrightarrow R^2$$

$$x'(\theta)^2 + y(\theta)^2 = 1$$

$\Rightarrow$ Now;

$$x'(\theta) = \sum_{n=-\infty}^{\infty} ina_n e^{in\theta}$$

$$\& \ y'(\theta) = \sum_{n=-\infty}^{\infty} inb_n e^{in\theta}$$

As given that for θ in range $[0,2\pi]$; $x'(\theta)^2 + y'(\theta)^2 = 1$,

and.

Perseval Identity:

If $f(t)$ is continous in range $(0,L)$, and its square is integrable and has fourrier coefficients A_n and B_n, then;

$$\frac{2}{L}\int_0^L (f(t))^2 dt = \frac{A_0^2}{2} + \sum_{n=-\infty}^{\infty} (A_n^2 + B_n^2)$$

This implies that :

$$x'(\theta)^2 + y'(\theta)^2 = 1$$

Integrating both sides

$$\int_0^{2\pi} x'(\theta)^2 + y'(\theta)^2 = \int_0^{2\pi} 1 \cdot d\theta$$

$$\int_0^{2\pi} x'(\theta)^2 + y'(\theta)^2 = 2\pi$$

$$\frac{1}{2\pi}\int_0^{2\pi} x'(\theta)^2 + y'(\theta)^2 = 1 \quad \text{—— (i)}$$

Using perseval identity:

$$\frac{2}{2\pi}\int_0^{2\pi} (x'(\theta))^2 d\theta = \frac{(0)^2}{2} + \sum_{n=-\infty}^{\infty} ((a_n i n)^2 + (a_n i n)^2)$$

$$\therefore \frac{1}{\pi}\int_0^{2\pi} x'(\theta)^2 d\theta = -2\sum_{n=\infty}^{\infty} a_n^2 n^2 \qquad \left[\begin{array}{l} x'(\theta) = \Sigma a_n i n \cos(n\theta) \\ \qquad + \\ \qquad i\,\Sigma a_n i n \sin(n\theta) \end{array}\right]$$

Taking mod...

$$\left|\int_0^{2\pi} (x'(\theta)^2) d\theta\right| = |2\pi| \sum_{n=-\infty}^{\infty} |n|^2 |a_n|^2 \quad \text{—— (ii)}$$

$$\left|\int_0^{2\pi} (y'(\theta))^2 d\theta\right| = |2\pi| \sum_{n=-\infty}^{\infty} |n|^2 |b_n|^2 \quad \text{—— (iii)}$$

Substituting (ii) and (iii) in (i)

$$\frac{1}{2\pi}\int_0^{2\pi}(x'(\theta)^2+y'(\theta)^2)\,d\theta = 1$$

$$\frac{1}{2\pi}\left(|2\pi|\left(\sum_{n=-\infty}^{\infty}|n|^2|a_n|^2+\sum_{n=-\infty}^{\infty}|n|^2|b_n|^2\right)\right) = 1$$

$$\boxed{\sum_{n=-\infty}^{\infty}\left(|n|^2\left(|a_n|^2+|b_n|^2\right)\right) = 1}$$

Hence Proved

Ans 2:- To find the general solution of the partial differential equations:

(a) $(x^3+3xy^2)p+(y^3+3x^2y)q = 2(x^2+y^2)z$; $p = z_x$, $q = z_y$.

$\rightarrow$ $$\frac{dx}{x^3+3xy^2} = \frac{dy}{y^3+3x^2y} = \frac{dz}{2(x^2+y^2)z}$$

$$\frac{dx}{x(x^2+3y^2)} = \frac{dz}{2(x^2+y^2)z} = \frac{dy}{y(y^2+3x^2)}$$

$$\frac{dx}{x} = \frac{(x^2+3y^2)\,dz}{2(x^2+y^2)z} \quad ; \quad \frac{dy}{y} = \frac{(y^2+3x^2)\,dz}{2(x^2+y^2)z}$$

(i) (ii)

(i) + (ii)

$$\frac{dx}{x} + \frac{dy}{y} = \frac{4(x^2+y^2)dz}{2(x^2+y^2)z}$$

$$\frac{dx}{x} + \frac{dy}{y} - \frac{2dz}{z} = 0$$

Integrating:

$$\int \frac{dx}{x} + \int \frac{dy}{y} - 2\int \frac{dz}{z} = 0$$

Taking log:

$$\ln(|x|) + \ln(|y|) - 2\ln(|z|) = c$$

$$\ln\left(\frac{|xy|}{z^2}\right) = c$$

$$\frac{xy}{z^2} = c_0 \quad \forall\ c_0 \text{ is a constant.}$$

$$\text{(i)} \rightarrow \underbrace{\frac{x\,dx}{x(x^3+3xy^2)} = \frac{dz}{2(x^2+y^2)z}} = \frac{y\,dy}{y(y^3+3x^2y)} \leftarrow \text{(ii)}$$

$$x\,dx = \frac{(x^4+3x^2y^2)dz}{2(x^2+y^2)z} \text{ — (1)} \qquad y\,dy = \frac{(y^4+3x^2y^2)dz}{2(x^2+y^2)z} \longrightarrow \text{(2)}$$

From (1) and (2)

$$x\,dx - y\,dy = \frac{x^4-y^4}{2(x^2+y^2)z}dz = \frac{(x^2-y^2)(x^2+y^2)dz}{2(x^2+y^2)z}$$

$$x\,dx - y\,dy = \frac{x^2-y^2}{2z}\,dz$$

$$\frac{2(x\,dx - y\,dy)}{x^2-y^2} = \frac{dz}{z}$$

Integrating

$$\int \frac{2x\,dx - 2y\,dy}{x^2-y^2} = \int \frac{dz}{z}$$

$$\ln|x^2-y^2| = \ln|z| + c$$

$$\left[\frac{x^2-y^2}{z}\right] = C_0$$

$$\boxed{\text{General solution is } F\left(\left[\frac{xy}{z^2}\right], \left[\frac{x^2-y^2}{z}\right]\right) = 0}$$

(b) $2x(y+z^2)p + y(2y+z^2)q = z^3$; $p = 2x,\ q = 2y$

$$\frac{dx}{2x(y+z^2)} = \frac{dy}{y(2y+z^2)} = \frac{dz}{z^3}$$

$$\frac{dx}{x} = \frac{2(y+z^2)\,dz}{z^3} \quad ; \quad \frac{dy}{y} = \frac{(2y+z^2)\,dz}{z^3}.$$

$$\frac{dx}{x} - \frac{dy}{y} = \frac{dz}{z}$$

Integrating

$$\int \frac{dx}{x} - \int \frac{dy}{y} = \int \frac{dz}{z}$$

$$\ln\left(\frac{|x|}{|y|}\right) = \ln|z| + c$$

$$\left[\frac{x}{yz}\right] = c$$

$$\Rightarrow \quad \frac{dy}{y(2y+z^2)} = \frac{dz}{z^3}$$

$$z^3\,dy = 2y^2\,dz + yz^2\,dz$$

$$z^3\,dy - yz^2\,dz = 2y^2\,dz$$

$$z^2(z\,dy - y\,dz) = 2y^2\,dz$$

$$\left(\frac{z\,dy - y\,dz}{y^2}\right) = \frac{-2\,dz}{z^2}$$

$$\frac{(y\,dz - z\,dy)}{y^2} = \frac{-2dz}{z^2} \qquad \left(d\frac{z}{y} = \frac{y\,dz - z\,dy}{y^2}\right)$$

$$d\left(\frac{z}{y}\right) = -2\frac{dz}{z^2}$$

$$\int d\left(\frac{z}{y}\right) = -2\int \frac{dz}{z^2} \qquad \left(\int \frac{dz}{z^2} = -\frac{1}{z}\right)$$

$$\frac{z}{y} = +\frac{2}{z} + c \quad ; \quad \frac{z^2 - 2y}{zy} = c$$

$$\boxed{\text{General solution is } F\left(\left[\frac{x}{yz}\right], \left[\frac{z^2 - 2y}{yz}\right]\right) = 0.}$$

Ans 3:

Given:

$U_{xx} + U_{yy} = 0$ on rectangle $= \left\{ \begin{matrix} 0 \le x \le \pi \\ 0 \le y \le 1 \end{matrix} \right\}$

$U(x,0) = f(x) = \sum_{k=1}^{\infty} a_k \sin(kx)$

$U(0,y) = U(\pi,y) = 0$

$U(x,1) = g(x) = \sum_{k=1}^{\infty} b_k \sin(kx)$

$\Rightarrow$ let us say

$U(x,y) = F(x) \cdot G(y)$ [using variable separation technique]

$\rightarrow U_{xx} = -U_{yy}$.

$$\frac{\partial^2}{\partial x^2}(F(x)G(y)) = -\frac{\partial^2}{\partial y^2}(F(x)G(y))$$

$$G(y)\frac{d^2F(x)}{dx^2} = -F(x)\frac{d^2G(y)}{dy^2}$$

$$\frac{1}{F}\frac{d^2F}{dx^2} = \frac{-1}{G}\frac{d^2G}{dy^2} = -k \quad [k>0]$$

$\Rightarrow$ A) $\frac{d^2F}{dx^2} + kF = 0$

given;

$U(0,y) = F(0)G(y) = 0$

$$U(\pi, y) = F(\pi)\,y(y) = 0$$

Assuming, $F(0) = F(\pi) = 0$

solving,

$$F_n(x) = F(x) = \sin\left(\frac{\pi n x}{L}\right) \qquad L = \pi$$

$$= \sin\left(\frac{\pi n x}{\pi}\right) \qquad \text{for } (k = -n^2)$$

$$\Rightarrow \frac{d^2 y}{dy^2} - dy = 0$$

$$y_n(y) = y(y) = A_n e^{\frac{n\pi y}{L}} + B_n e^{-\frac{n\pi y}{L}} \qquad L = \pi$$

$$= A_n e^{ny} + B_n e^{-ny}$$

$$\Rightarrow U_n(x, y) = F(x)\,y(y)$$

$$= \sin(nx)\left(A_n e^{ny} + B_n e^{-ny}\right)$$

<u>for boundary conditions:</u>

$$U(x, 0) = f(x) = \sum_{K=1}^{\infty} a_K \sin(Kx) = \sum_{K=1}^{\infty} \sin(Kx)\left(A_K e^{Ky} + B_K e^{-Ky}\right)$$

$$= \sum_{K=1}^{\infty} \sin(Kx)\left(A_K + B_K\right)$$

$$U(x, 1) = \sum_{K=1}^{\infty} \sin(Kx)\, b_K = \sum_{K=1}^{\infty} \sin(Kx)\left(A_K e^{K} + B_K e^{-K}\right)$$

$$a_k = A_k + B_k \quad \text{---} \ (1)$$

$$b_k = A_k e^k + B_k e^{-k} \quad \text{---} \ (2)$$

Solving,

$$a_k \times e^k = A_k e^k + B_k e^k$$

$$b_k = A_k e^k + B_k e^{-k}$$

$$B_k = \frac{a_k e^k - b_k}{e^k - e^{-k}} \quad ; \quad A_k = \frac{-a_k e^{-k} + b_k}{e^k - e^{-k}}$$

Substituting A_k and B_k.

$$U(x,y) = \sum_{k=1}^{\infty} \sin(kx)\left(A_k e^{ky} + B_k e^{-ky}\right)$$

$$= \sum_{k=1}^{\infty} \sin(kx)\left(\left(\frac{-a_k e^{-k} + b_k}{e^k - e^{-k}}\right)e^{ky} + \left(\frac{a_k e^k - b_k}{e^k - e^{-k}}\right)e^{-ky}\right)$$

$$= \sum_{k=1}^{\infty} \sin(kx)\left(\frac{a_k}{e^k - e^{-k}}\left(e^{k(1-y)} - e^{-k(1-y)}\right) + \frac{b_k}{e^k - e^{-k}}\left(e^{ky} - e^{-ky}\right)\right)$$

$$\sin(hk) = \frac{e^k - e^{-k}}{2}$$

$$U(x,y) = \sum_{k=1}^{\infty} \sin(kx)\left(\frac{\sinh(k(1-y))}{\sinh k} a_k + \frac{\sinh ky}{\sinh k} b_k\right)$$

$$U(x,y) = \sum_{k=1}^{\infty} \sin kx \left(\frac{\sinh k(1-y)}{\sinh k} a_k + \frac{\sinh ky}{\sinh k} b_k\right)$$

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