MA201, TA-1

(1)

Def^n :- A field F is ordered if there is a non-empty set $P \subset F$ (called the positive subset) for which

(1) $a, b \in P \Rightarrow a+b \in P$

(2) $a, b \in P \Rightarrow ab \in P$

(3) $a \in F \Rightarrow a \in P, -a \in P$ or $a = 0$.

Suppose that $\mathbb{C}$ is ordered then take $P = \overline{\mathbb{C}}^+ :=$ Right half complex plane

Clearly $0 \in \overline{\mathbb{C}}^+$ i.e. $\overline{\mathbb{C}}^+ \neq \phi$

$i \in \mathbb{C}$ then atleast one of the following hold
$i \in \overline{\mathbb{C}}^+$ or $-i \in \overline{\mathbb{C}}^+$ as $i \neq 0$

Suppose that $i \in \overline{\mathbb{C}}^+$ then
$(i)(i) = -1 < 0 \Rightarrow i \notin \overline{\mathbb{C}}^+$

Suppose $-i \in \overline{\mathbb{C}}^+$ then
$(-i)(-i) = -1 < 0 \Rightarrow -i \notin \overline{\mathbb{C}}^+$

And hence $\mathbb{C}$ is not ordered.

(2) (i) ~~If~~ To show that

$|z_1 \pm z_2| \le |z_1| + |z_2|$

If $z_1 + z_2 = 0$ then ~~the~~ $|z_1 + z_2| = 0 \le |z_1| + |z_2| > 0$

If $z_1 + z_2 \ne 0$ then

$$|z_1 + z_2|^2 = (z_1 + z_2)\overline{(z_1 + z_2)}$$
$$= (z_1 + z_2)(\bar{z}_1 + \bar{z}_2)$$
$$= z_1\bar{z}_1 + z_1\bar{z}_2 + z_2\bar{z}_1 + z_2\bar{z}_2$$
$$= |z_1|^2 + z_1\bar{z}_2 + \overline{z_1\bar{z}_2} + |z_2|^2$$
$$= |z_1|^2 + |z_2|^2 + 2Re(z_1\bar{z}_2)$$
$$\le |z_1|^2 + |z_2|^2 + 2|z_1\bar{z}_2|$$
$$= |z_1|^2 + |z_2|^2 + 2|z_1||z_2|$$
$$= (|z_1| + |z_2|)^2$$
$$\Rightarrow |z_1 + z_2| \le |z_1| + |z_2|$$

& $|z_1 - z_2| = |z_1 + (-z_2)| \le |z_1| + |-z_2| = |z_1| + |z_2|$

Thus $|z_1 \pm z_2| \le |z_1| + |z_2|$ #.

(ii) To show $|z_1 \pm z_2| \ge ||z_1| - |z_2||$.

Now, $|z_1| = |z_1 + z_2 - z_2|$

$\le |z_1 + z_2| + |z_2|$

$\Rightarrow |z_1| - |z_2| \le |z_1 + z_2|$ — (1)

Similarly, $|z_2| - |z_1| \le |z_2 + z_1| = |z_1 + z_2|$

$\Rightarrow -(|z_1| - |z_2|) \le |z_1 + z_2|$ — (2)

From (1) & (2), we have

$||z_1| - |z_2|| \le |z_1 + z_2|$ #

(iii) $|z_1+z_2|^2 + |z_1-z_2|^2$

$= |z_1|^2+|z_2|^2+2Re(z_1\bar{z}_2)+|z_1|^2+|z_2|^2-2Re(z_1\bar{z}_2)$

$= 2(|z_1|^2+|z_2|^2)$

& $|z_1-z_2|^2 = 2(|z_1|^2+|z_2|^2) - |z_1+z_2|^2$ #

(iv) Let us write $z = x+iy$ then

$Re(z) = x$ & $Im(z) = y$

Now $(x \pm y)^2 \geq 0$ ~~for x≠0 ∀y~~

$\Rightarrow x^2+y^2 \pm 2xy \geq 0$

$\Rightarrow x^2+y^2 \geq \mp 2xy$

$\Rightarrow |z|^2 \geq 2\,Re(z)\,Im(z)$ (taking (+)ve sign)

Also $|z|^2 = x^2+y^2 = Re(z)^2 + Im(z)^2$

So,

$2|z|^2 \geq Re(z)^2 + Im(z)^2 + 2\,Re(z)\,Im(z)$

$= \{Re(z)+Im(z)\}^2$

$\Rightarrow \sqrt{2}|z| \geq Re(z) + Im(z)$ #

(3) To show $|\sum_{i=1}^{n} z_i| \leq \sum_{i=1}^{n} |z_i|$

for $n=1$, trivial

for $n=2$, $|z_1+z_2| \leq |z_1|+|z_2|$

(prove above)

Suppose it is true for $n=k$.

$$\therefore \left|\sum_{i=1}^{k+1} z_i\right| \le \left|\sum_{i=1}^{k} z_i\right| + |z_{k+1}|$$

$$\le \sum_{i=1}^{k+1} |z_i|$$

Hence, it is true for arbitrary n. #

(4) $Re(z_1\bar{z}_2) = \frac{1}{2}(z_1\bar{z}_2 + \overline{z_1\bar{z}_2})$

Now, $Re(z_1\bar{z}_2) \le |Re(z_1\bar{z}_2)|$

$$\le \frac{1}{2}\{|z_1\bar{z}_2| + |\overline{z_1\bar{z}_2}|\}$$

$$= \frac{1}{2} \times 2|z_1\bar{z}_2| = |z_1\bar{z}_2|$$

$\Rightarrow Re(z_1\bar{z}_2) \le |z_1\bar{z}_2|$ #

Denoting $z_1 = x_1 + iy_1$

$z_2 = x_2 + iy_2$

then $z_1\bar{z}_2 = (x_1+iy_1)(x_2-iy_2)$

$$= (x_1x_2 + y_1y_2) + i(x_2y_1 - x_1y_2)$$

$\therefore Re(z_1\bar{z}_2) = |z_1\bar{z}_2|$ iff

$x_2y_1 - x_1y_2 = 0$ i.e. $y_1/y_2 = x_1/x_2$

i.e. z_1 is a scalar multiple of z_2. #

(ii) Now, $|z_1+z_2| = |(x_1+x_2)+i(y_1+y_2)|$
$$= \sqrt{(x_1+x_2)^2+(y_1+y_2)^2}$$

& $|z_1| = \sqrt{x_1^2+y_1^2}$, $|z_2| = \sqrt{x_2^2+y_2^2}$

$\therefore$ $|z_1+z_2| = |z_1|+|z_2|$

$\Leftrightarrow \{(x_1+x_2)^2+(y_1+y_2)^2\}^{1/2} = (x_1^2+y_1^2)^{1/2}+(x_2^2+y_2^2)^{1/2}$

$\Leftrightarrow (x_1+x_2)^2+(y_1+y_2)^2 = (x_1^2+y_1^2)+(x_2^2+y_2^2)+$
$$2\sqrt{(x_1^2+y_1^2)(x_2^2+y_2^2)}$$

$\Leftrightarrow \cancel{2}(x_1x_2+y_1y_2) = \cancel{2}\sqrt{(x_1^2+y_1^2)(x_2^2+y_2^2)}$

$\Leftrightarrow \operatorname{Re}(z_1\bar{z}_2) = |z_1||z_2| = |z_1||\bar{z}_2| = |z_1\bar{z}_2|$

$\Leftrightarrow$ z_1 is scalar multiple of z_2.

#

(iii) To show
$$|z_1-z_2| = ||z_1|-|z_2||$$

L.H.S. $= |z_1-z_2| = \sqrt{(x_1-x_2)^2+(y_1-y_2)^2}$

R.H.S. $= ||z_1|-|z_2||$
$$= |\sqrt{x_1^2+y_1^2}-\sqrt{x_2^2+y_2^2}|$$

$\therefore$ L.H.S. $=$ RHS $\Leftrightarrow$

$$(x_1-x_2)^2+(y_1-y_2)^2 = \left(\sqrt{x_1^2+y_1^2}-\sqrt{x_2^2+y_2^2}\right)^2$$

$$\Leftrightarrow (x_1-x_2)^2+(y_1-y_2)^2 = x_1^2+y_1^2+x_2^2+y_2^2 - 2\sqrt{(x_1^2+y_1^2)(x_2^2+y_2^2)}$$

$$\Leftrightarrow -\cancel{2}(x_1x_2+y_1y_2) = -\cancel{2}\sqrt{(x_1^2+y_1^2)(x_2^2+y_2^2)}$$

$$\Leftrightarrow \quad Re(z_1\overline{z_2}) = |z_1\overline{z_2}|$$

#

(5) $p(z) = a_n z^n + a_{n-1}z^{n-1} + \cdots + a_1 z_1 + a_0$

$a_i \in \mathbb{R},\ \forall i$

Given that $p(z_1)=0$

i.e. $a_n z_1^n + a_{n-1}z_1^{n-1} + \cdots + a_1 z_1 + a_0 = 0$

Taking conjugate of above eqn, we have

$$\overline{a_n z_1^n + a_{n-1}z_1^{n-1} + \cdots + a_1 z_1 + a_0} = \overline{0} = 0$$

or, $0 = a_n \overline{z_1}^n + a_{n-1}\overline{z_1}^{n-1} + \cdots + a_1\overline{z_1} + a_0$

$= p(\overline{z_1})$

$\Rightarrow \overline{z_1}$ is also a root of $p(z)$. #

(7) $|\sin z|$ at $z = \pi + i\ln(2+\sqrt{5})$

$$|\sin z| = |\sin(\pi + i\ln(2+\sqrt{5}))|$$
$$= |-\sin(i\ln(2+\sqrt{5}))|$$
$$= |\sin(i\ln(2+\sqrt{5}))|$$

$$e^{i\theta} = \cos\theta + i\sin\theta$$

$$e^{-\theta} = e^{i(i\theta)} = \cos(i\theta) + i\sin(i\theta)$$

$$= \left| \frac{e^{-(\ln(2+\sqrt{5}))} \quad e^{\ln(2+\sqrt{5})}}{2i} \right|$$

$$= \left| \frac{e^{\ln(1/2+\sqrt{5})} - e^{\ln(2+\sqrt{5})}}{2i} \right|$$

$$= \left| \frac{\frac{1}{2+\sqrt{5}} - 2+\sqrt{5}}{2} \right|$$

$$= \left| \frac{1-(2+\sqrt{5})^2}{2(2+\sqrt{5})} \right|$$

$$= \left| \frac{1-(4+5-4\sqrt{5})}{2(2+\sqrt{5})} \right|$$

$$= \left| \frac{-4(2-\sqrt{5})}{2(2+\sqrt{5})} \right|$$

$$= \left| (-2)\left(\frac{2-\sqrt{5}}{2+\sqrt{5}}\right) \right|$$

$$= \left| (-2)\frac{(2-\sqrt{5})^2}{4-5} \right| = \left| \frac{-2(4+5-4\sqrt{5})}{-1} \right|$$

$$= \left| 2(9-4\sqrt{5}) \right| \#$$

(8) Given that $|\alpha| < 1$.

Then $|z-\alpha|^2 - |1-\bar{\alpha}z|^2 = (z-\alpha)\overline{(z-\alpha)} - (1-\bar{\alpha}z)\overline{(1-\bar{\alpha}z)}$

$= (z-\alpha)(\bar{z}-\bar{\alpha}) - (1-\bar{\alpha}z)(1-\alpha\bar{z})$

$= z\bar{z} + \alpha\bar{\alpha} - 1 - (\bar{\alpha}z)(\alpha\bar{z})$

$= |z|^2 + |\alpha|^2 - 1 - |\bar{\alpha}z|^2$

$= |z|^2 + |\alpha|^2 - 1 - |\alpha|^2|z|^2$

$= |z|^2(1-|\alpha|^2) - (1-|\alpha|^2)$

$= (|z|^2-1)(1-|\alpha|^2) < 0$ for $|z|<1$

$\Rightarrow |z-\alpha|^2 < |1-\bar{\alpha}z|^2$

$\Rightarrow \frac{|z-\alpha|}{|1-\bar{\alpha}z|} < 1$

And for $|z|=1$

$|z-\alpha|^2 - |1-\bar{\alpha}z|^2 = 0$

$\Rightarrow |z-\alpha| = |1-\bar{\alpha}z|$

$\Rightarrow \frac{|z-\alpha|}{|1-\bar{\alpha}z|} = 1$. #

(9) Since $z = x+iy$

$\therefore z + \frac{1}{z} = x+iy + \frac{1}{x+iy}$

$= x+iy + \frac{x-iy}{x^2+y^2}$

$= x\left(1+\frac{1}{x^2+y^2}\right) + iy\left(1-\frac{1}{x^2+y^2}\right)$

Therefore $z+\frac{1}{z}$ is real iff $y\left(1-\frac{1}{x^2+y^2}\right)=0$

i.e. either $y=0$ or $|z|=1$.

(10) $|z^2-z+1| \le |z|^2+|z|+1 = 3$

& $|z^2-2| \geq \left||z|^2-2\right| = 1$ $\Big\}$ as $|z|=1$.

(11) (i) $|z^4-4z^2+3| = |(z^2-1)(z^2-3)|$

$= |z^2-1|\,|z^2-3|$

$\geq (|z|^2-1)(|z|^2-3)$

$= (4-1)(4-3) = 3$

$\therefore \frac{1}{|z^4-4z^2+3|} \le 1/3$

(iii) $|z^4-5z^2+6| = |(z^2-2)(z^2-3)|$

$\geq (|z|^2-2)(|z|^2-3)$

$= (4-2)(4-3) = 2$

$\Rightarrow \left|\frac{1}{z^4-5z^2+6}\right| \le 1/2$

(ii) z^4-5z^2+1

Do yourself.

(19) $1+z+z^2+ \cdots + z^n = \frac{1-z^{n+1}}{1-z}$ — (1)

Putting $z = e^{i\theta}$ & using Euler's formula $e^{ik\theta} = \cos(k\theta) + i\sin(k\theta)$ in (1), we have

$$(1+\cos\theta+\cos 2\theta + \cdots + \cos n\theta) + i(\sin\theta + \sin 2\theta + \cdots + \sin n\theta) = \frac{1-\cos(n+1)\theta - i\sin(n+1)\theta}{1-\cos\theta - i\sin\theta}$$

$$= \frac{1-\cos(n+1)\theta - i\sin(n+1)\theta}{1-\cos\theta - i\sin\theta} \times \frac{1-\cos\theta + i\sin\theta}{1-\cos\theta+i\sin\theta}$$

Comparing real & imaginary parts of above equations, we have

$$1+\cos\theta+\cos 2\theta + \cdots + \cos(n\theta)$$

$$= \frac{\{1-\cos(n+1)\theta\}\{1-\cos\theta\} + \sin(n+1)\theta\sin\theta}{(1-\cos\theta)^2+\sin^2\theta}$$

$$= \frac{1-\cos(n+1)\theta - \cos\theta + \cos(n+1)\theta\cos\theta + \sin(n+1)\theta\sin\theta}{1+\cos^2\theta - 2\cos\theta + \sin^2\theta}$$

$$= \frac{1-\cos\theta - \cos(n+1)\theta + \cos n\theta}{2(1-\cos\theta)}$$

$$= \frac{1}{2} + \frac{\cos n\theta - \cos(n+1)\theta}{2(1-\cos\theta)}$$

$$\Rightarrow \frac{1}{2} + \frac{2\sin\theta/_2 \, \sin(n+1/_2)\theta}{2(2\sin^2\theta/_2)}$$

$$= \frac{1}{2} + \frac{\sin(n+1/_2)\theta}{2\sin\theta/_2}$$

#

Similarly from imaginary part, we have

$$\sin\theta + \sin 2\theta + \text{———} + \sin(n\theta)$$

$$= \frac{\cos\theta/_2 - \cos(n+1/_2)\theta}{2\sin\theta/_2}$$

#

(13) (1) $z^8 - 16 = 0$

$\Rightarrow z^8 = 16 = (\sqrt{2})^8 = 16\, e^{i(2k\pi)}, k \in \mathbb{Z}$

$\Rightarrow z = (16\, e^{i2k\pi})^{1/8}$

$= \sqrt{2}\, e^{i2k\pi/8} \quad ; k \in \mathbb{Z}$

Therefore roots of $z^8 - 16 = 0$ are given by $\sqrt{2}e^{i(2k\pi/8)}$, $k = 0, 1, \text{——}, 7$.

#

Write yourself in the form of $a+ib$.

(15) (i) $(-\sqrt{3}-i)^{-6}$

We know that

$x+iy = re^{i\theta}$

where $r = \sqrt{x^2+y^2}$

& $\theta = \tan^{-1}(y/x)$

For $(-\sqrt{3}-i)$

$r = \sqrt{3+1} = \sqrt{4} = 2$

$\theta = \tan^{-1}\left(\frac{-1}{-\sqrt{3}}\right) = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = \pi/6$

$\therefore -\sqrt{3}-i = 2e^{i\pi/6}$

$\Rightarrow (-\sqrt{3}-i)^{-6} = (2e^{i\pi/6})^{-6} = 2^{-6}e^{-i\pi}$

$= 2^{-6}\{\cos(-\pi) + i\sin(-\pi)\}$

$= 2^{-6}(-1) = -2^{-6}$ #

Remaining parts do yourself.

(16) Roots of the eqn $z^n = 1$ are

$1, \omega, \omega^2, ---, \omega^{n-1}$, where $\omega = e^{i\left(\frac{2\pi}{n}\right)}$

We have to find sum of p^{th} powers of the roots.

i.e. $1 + \omega^p + \omega^{2p} + --- + \omega^{(n-1)p} = ??$

Case 1:- If $p \neq kn$, for some $k \in \mathbb{Z}$

then,

$1+\omega^{p}+\omega^{2p}+ \_\_\_\_ +\omega^{(n-1)p}$

$$= \frac{1-(\omega^{p})^{n}}{1-\omega} = \frac{1-(\omega^{n})^{p}}{1-\omega} = \frac{1-1}{1-\omega} = 0$$

Case 2:-

If $p = kn$, for some $k \in \mathbb{Z}$ then

$1+\omega^{p}+\omega^{2p}+ \_\_ +\omega^{p(n-1)}$

$$= 1+1+ \_\_\_\_ +1 = n$$

\#

(7) $z^n = 1$

But $z^n - 1 = (z-1)(1+z+z^2+ \_\_\_\_ +z^{n-1})$ — (1)

Also, if $1, z_1, \_\_\_ z_{n-1}$ are roots of $z^n = 1$ then

$z^n - 1 = (z-1)(z-z_1) \_\_\_ (z-z_{n-1})$ — (2)

from (1) & (2), we have

$f(z) := 1+z+z^2+ \_\_\_\_ +z^{n-1}$

& $g(z) := (z-z_1)(z-z_2) \_\_\_\_ (z-z_{n-1})$

$f(z) = g(z)$ for $z \neq 1$

But $f(z)$ & $g(z)$ are polynomials and therefore they are continuous and so

$f(1) = g(1)$

$\Rightarrow (1-z_1)(1-z_2) \_\_\_\_ (1-z_{n-1}) = 1+1+ \_\_ +1$

$= n$ #

(18) To prove that

$$\sin(\pi/n)\sin(2\pi/n)\cdots\sin\left(\frac{(n-1)\pi}{n}\right) = \frac{n}{2^{n-1}}$$

Since, $e^{i\theta} = \cos\theta + i\sin\theta$

$e^{-i\theta} = \cos\theta - i\sin\theta$

$$\therefore \sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i} = \frac{e^{-i\theta}(e^{2i\theta} - 1)}{2i}$$

So,

$\sin\pi/n \, \sin(2\pi/n) \cdots \sin((n-1)\pi/n)$

$$= \left\{\frac{e^{-i\pi/n}(e^{2i\pi/n} - 1)}{2i}\right\}\left\{\frac{e^{-i2\pi/n}}{2i}(e^{i4\pi/n} - 1)\right\}\cdots\left\{\frac{e^{-i\frac{(n-1)\pi}{n}}}{2i}(e^{\frac{i2\pi(n-1)}{n}} - 1)\right\}$$

$$= \frac{(-1)^{n-1}}{(2i)^{n-1}} e^{-\frac{i\pi}{n}(1+2+\cdots+(n-1))} (1 - e^{2\pi i/n})(1 - e^{4\pi i/n})\cdots(1 - e^{\frac{i(n-1)2\pi}{n}})$$

$$= \frac{(-1)^{n-1} e^{-\frac{i\pi}{n}\frac{n(n-1)}{2}}(1-\omega)(1-\omega^2)\cdots(1-\omega^{n-1})}{(2i)^{n-1}}$$

where $\omega = e^{i2\pi/n}$ is the n^{th} root of unity and from (17)th soln we have

$(1-\omega)(1-\omega^2)\cdots(1-\omega^{n-1}) = n$

$$\text{So, L.H.S.} = \frac{(-1)^{n-1} e^{-\frac{i(n-1)\pi}{2}} n}{2^{n-1}(i)^{n-1}}$$

$$= \frac{(-1)^{n-1}(-i)^{n-1}}{(i)^{n-1}} \frac{n}{2^{n-1}} = \frac{n}{2^{n-1}}$$ #

(19) $z^{2n}-1 = \prod_{k=0}^{2n-1}\left(z - e^{\frac{i2k\pi}{2n}}\right)$

$= \prod_{k=0}^{2n-1}\left(z-\left(\cos\frac{k\pi}{n}+i\sin\frac{k\pi}{n}\right)\right)$

$= (z-1)\prod_{k=1}^{n-1}\left\{z-\left(\cos\frac{k\pi}{n}+i\sin\frac{k\pi}{n}\right)\right\}$

$\cdot (z+1)\prod_{k=n+1}^{2n-1}\left\{z-\left(\cos\frac{k\pi}{n}+i\sin\frac{k\pi}{n}\right)\right\}$

$= (z-1)\prod_{k=1}^{n-1}\left\{z-\left(\cos\frac{k\pi}{n}+i\sin\frac{k\pi}{n}\right)\right\}$

$\cdot (z+1)\prod_{k=1}^{n-1}\left\{z-\left(\cos\left(\frac{k\pi}{n}\right)-i\sin\left(\frac{k\pi}{n}\right)\right)\right\}$

$= (z^2-1)\prod_{k=1}^{n-1}\left\{z-\left(\cos\frac{k\pi}{n}+i\sin\frac{k\pi}{n}\right)\right\}\left\{z-\left(\cos\frac{k\pi}{n}-i\sin\frac{k\pi}{n}\right)\right\}$

$= (z^2-1)\prod_{k=1}^{n-1}\left\{z^2-2z\cos\left(\frac{k\pi}{n}\right)+1\right\}$

#

Thus

$\prod_{k=1}^{n-1}\left(z^2-2z\cos\frac{k\pi}{n}+1\right) = 1+z^2+(z^2)^2+\text{—}+(z^2)^{n-1}$

Taking $z=1$, we have

$\prod_{k=1}^{n-1} 2(1-\cos k\pi/n) = 1+1+-\text{—}+1 = n$

$\prod_{k=1}^{n-1} 2\left(2\sin^2 \frac{k\pi}{2n}\right) = n$

$$\Rightarrow \prod_{k=1}^{n-1} \sin\left(\frac{k\pi}{2n}\right) = \frac{\sqrt{n}}{2^{n-1}} \quad \#$$