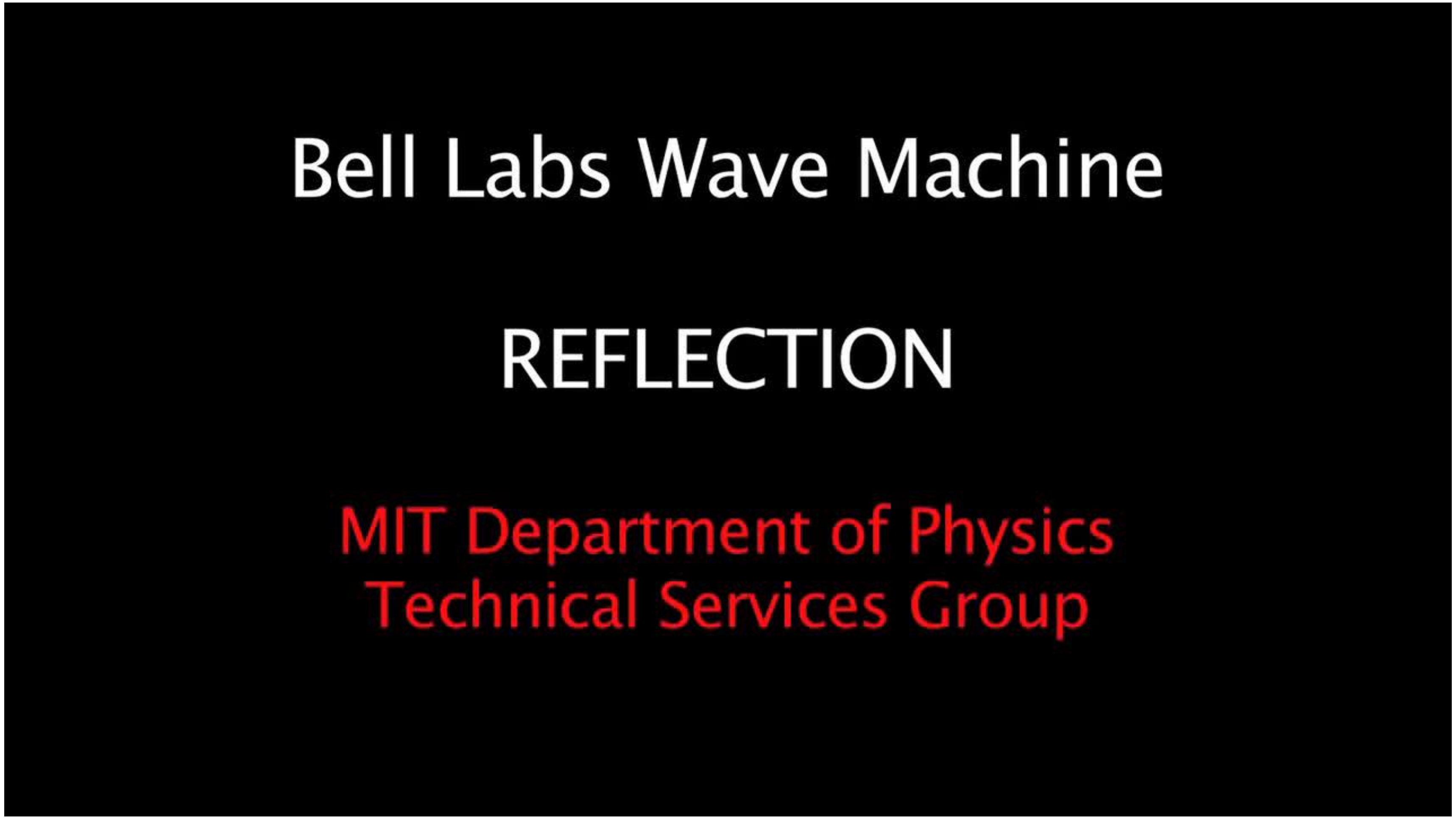
Waves and Oscillations

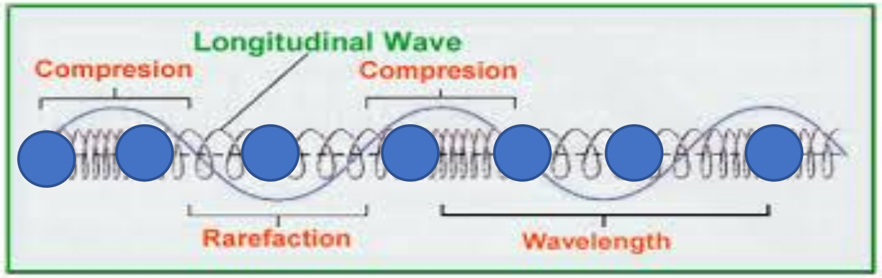

Bell Labs Wave Machine
REFLECTION
MIT Department of Physics
Technical Services Group

Oscillations to Waves: How are they Related ?

Vibrating or Oscillating objects are sources of waves, that travels through space-time.

It can be a pulse or continuous.

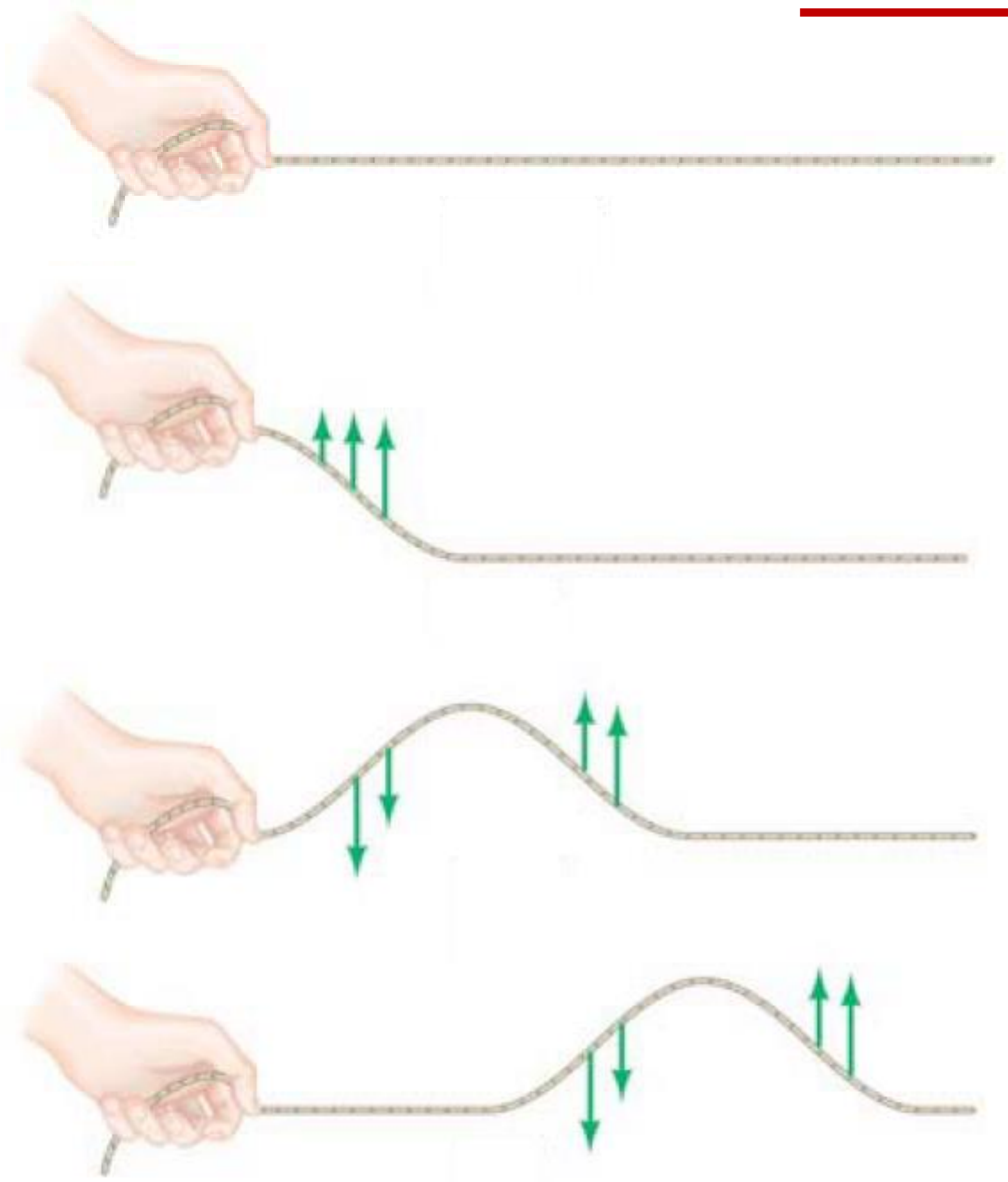
Can be compared to motion of large number of coupled oscillator.



If wave propagation is parallel to the oscillation: Longitudinal.
If it is perpendicular to the oscillation: Transverse Waves.

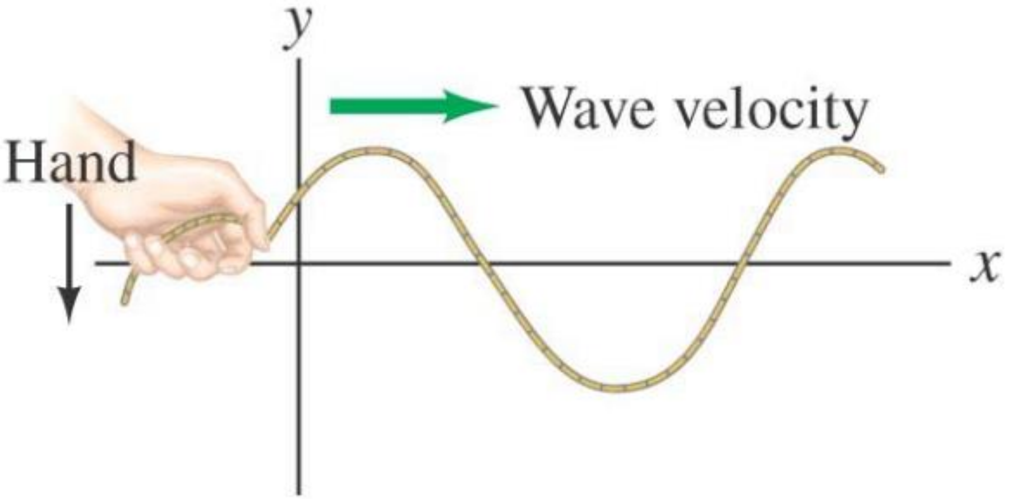
A wave does not move mass in the direction of propagation; it only transfers energy.

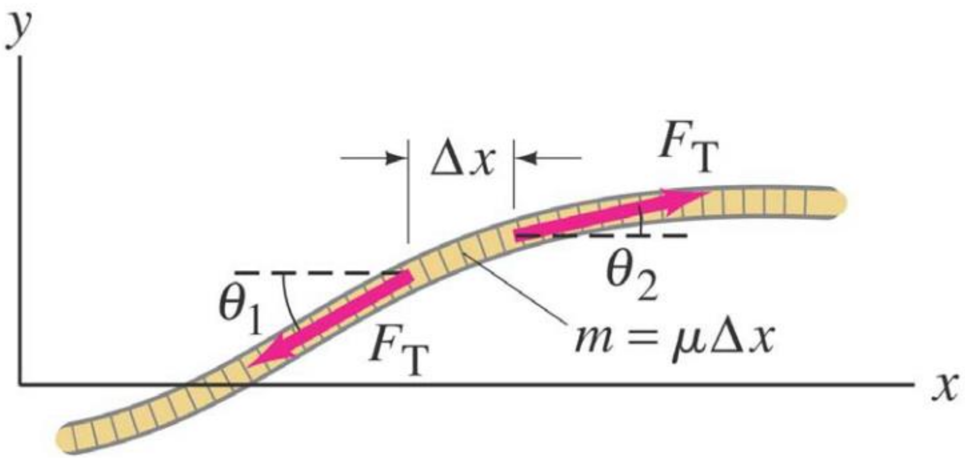
Travelling waves



- Travelling waves transport energy
- Study of a single wave pulse shows that, it becomes with a vibration and is transmitted through the medium

Derivation of 1D-Wave Equations

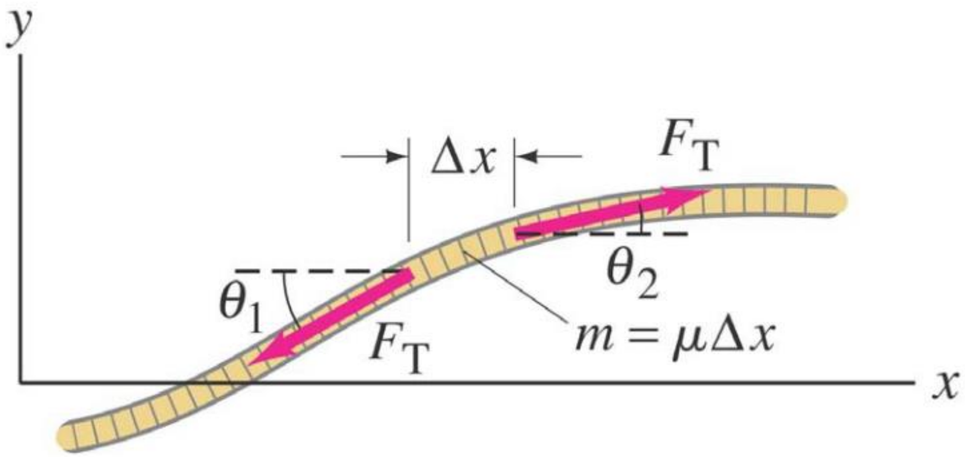




Approximations:

- The string is homogeneous (mass per unit length is constant)
- The string is elastic and doesn't offer resistance to bend
- Every particle shows small transverse motion (strictly vertical)
- Deflection and slope are small

Derivation of 1D-Wave Equations



We ll only consider the motion of the string in y-direction for the section "Δx". Applying Newton's law,

$$F_y = -T\sin\theta_1 + T\sin\theta_2$$

Considering the angles to be small, we have

$$F_y = -T\theta_1 + T\theta_2 = T(\theta_2 - \theta_1) = T\Delta\theta$$

$$F_y = (dm)\ddot{y} = T\Delta\theta$$

$$(\mu\Delta x)\ddot{y} = T\Delta\theta$$

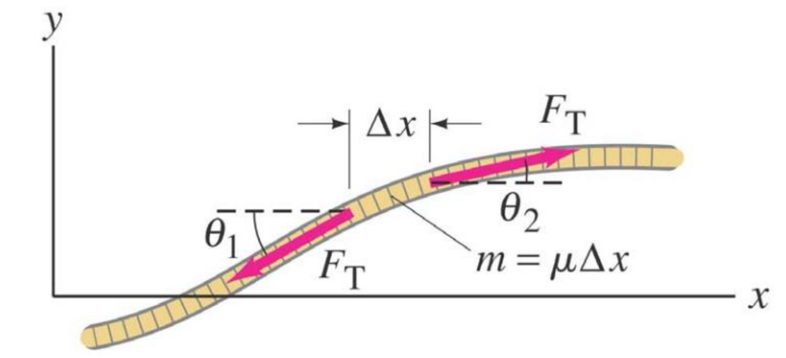
Lets consider; $\tan\theta = \dfrac{\partial y}{\partial x}$

Taking the derivatives, we have $\dfrac{1}{\cos^2\Theta}\dfrac{\partial\theta}{\partial x} = \dfrac{\partial^2 y}{\partial x^2}$

Derivation of 1D-Wave Equations

For small angle approximation cosine =1,

$$\frac{\partial\theta}{\partial x} = \frac{\partial^2 y}{\partial x^2} \qquad \partial\theta = \frac{\partial^2 y}{\partial x^2}\partial x$$



Putting these values in the parent equation

$$(\mu\Delta x)\ddot{y} = T\Delta\theta \qquad \boldsymbol{\mu\Delta x\frac{\partial^2 y}{\partial t^2} = T\frac{\partial^2 y}{\partial x^2}\partial x}$$

$$\boldsymbol{\frac{\mu}{T}\frac{\partial^2 y}{\partial t^2} = \frac{\partial^2 y}{\partial x^2}}$$

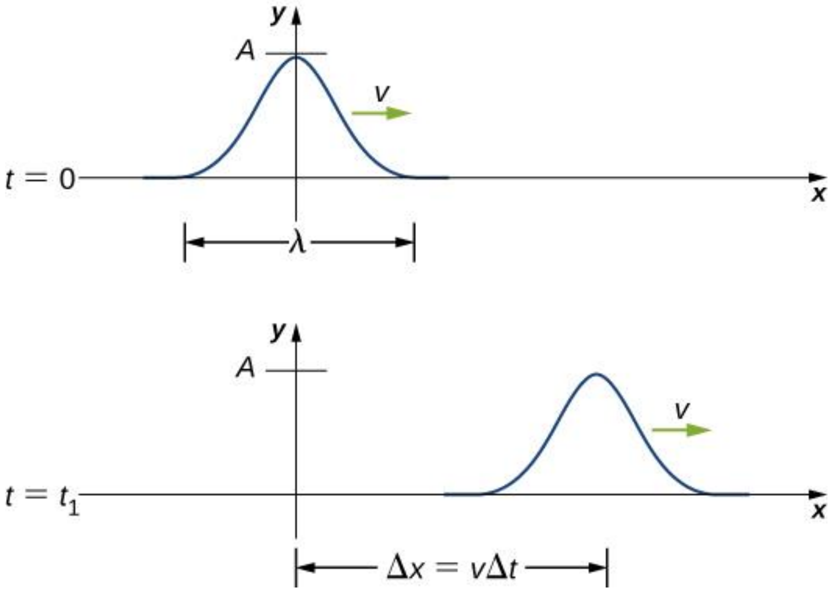
$$\boxed{\boldsymbol{\frac{1}{v_p{}^2}\frac{\partial^2 y(x,t)}{\partial t^2} = \frac{\partial^2 y(x,t)}{\partial x^2}}} \qquad \boldsymbol{v_p = \sqrt{\frac{T}{\mu}}}$$

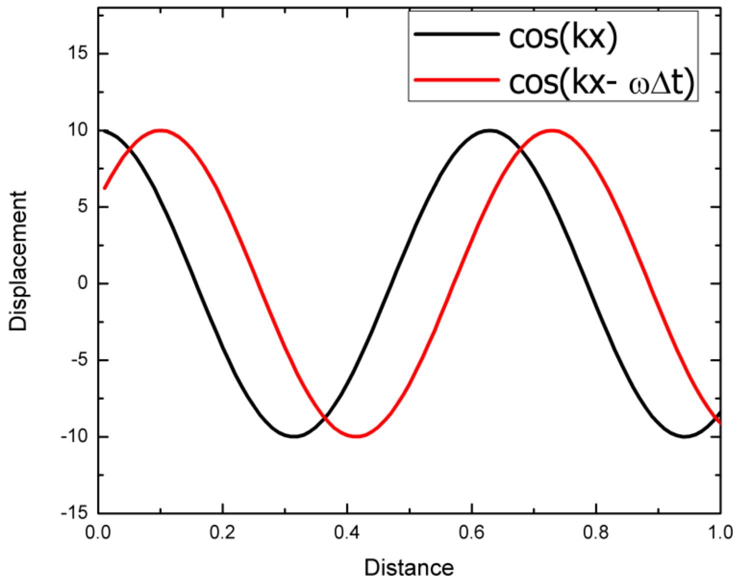
This is known as the wave equation

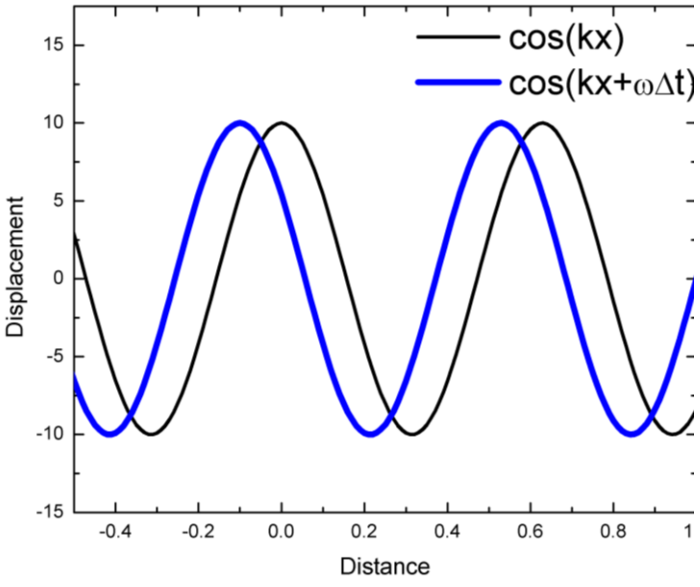
Trial solutions of the Wave Equations

$$\frac{1}{v_p^2}\frac{\partial^2 y(x,t)}{\partial t^2} = \frac{\partial^2 y(x,t)}{\partial x^2}$$

$$y(x,t) = A\sin(kx \pm \omega t)$$







Trial solutions of the Wave Equations

We get an infinite number of coupled equations of motion,
What are the normal modes;

$$\frac{1}{\boldsymbol{v_p}^2}\frac{\partial^2 \boldsymbol{y}(\boldsymbol{x},\boldsymbol{t})}{\partial \boldsymbol{t}^2} = \frac{\partial^2 \boldsymbol{y}(\boldsymbol{x},\boldsymbol{t})}{\partial \boldsymbol{x}^2}$$

$$y(x,t) = A(x)B(t)$$

The wave equation becomes;

$$\frac{1}{v_p^2}\,A(x)\frac{\partial^2 B(t)}{\partial t^2} = B(t)\frac{\partial^2 A(x)}{\partial x^2}$$

$$\frac{1}{v_p^2 B(t)}\frac{\partial^2 B(t)}{\partial t^2} = \frac{1}{A(x)}\frac{\partial A(x)}{\partial x^2}$$

This equation must be satisfied for all x and t, so both sides must be a constant

$$\frac{1}{v_p^2 B(t)}\frac{\partial^2 B(t)}{\partial t^2} = \frac{1}{A(x)}\frac{\partial A(x)}{\partial x^2} = -\,k_m^2$$

Let's find out A(x) and B(t)

$$\frac{1}{v_p^2 B(t)}\frac{\partial^2 B(t)}{\partial t^2} = \frac{1}{A(x)}\frac{\partial A(x)}{\partial x^2} = -k_m^2$$

$$\frac{1}{v_p^2 B(t)}\frac{\partial^2 B(t)}{\partial t^2} = -k_m^2$$

$$\frac{\partial^2}{\partial t^2}B(t) = -k_m^2 v_p^2\, B(t)$$

$$B(t) = B_m \sin(\omega_m t + \beta_m)$$

$$\omega_m = k_m v_p$$

$$\frac{1}{A(x)}\frac{\partial^2 A(x)}{\partial x^2} = -k_m^2$$

$$\frac{\partial^2}{\partial x^2}A(x) = -k_m^2\, \mathrm{A(x)}$$

$$A(x) = C_m \sin(k_m x + \alpha_m)$$

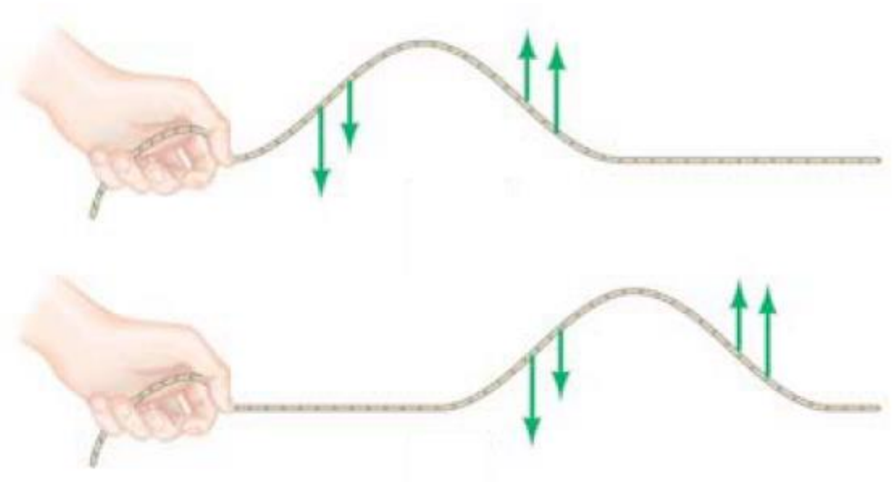
$$y(x,t) = A(x)B(t) = A_m \sin(\omega_m t + \beta_m)\sin(k_m x + \alpha_M)$$

Superposition and Standing Waves

Forward moving wave,

$$y_1(x,t) = A\sin(kx - \omega t)$$

$$k = \frac{2\pi}{\lambda}, \omega = kv$$



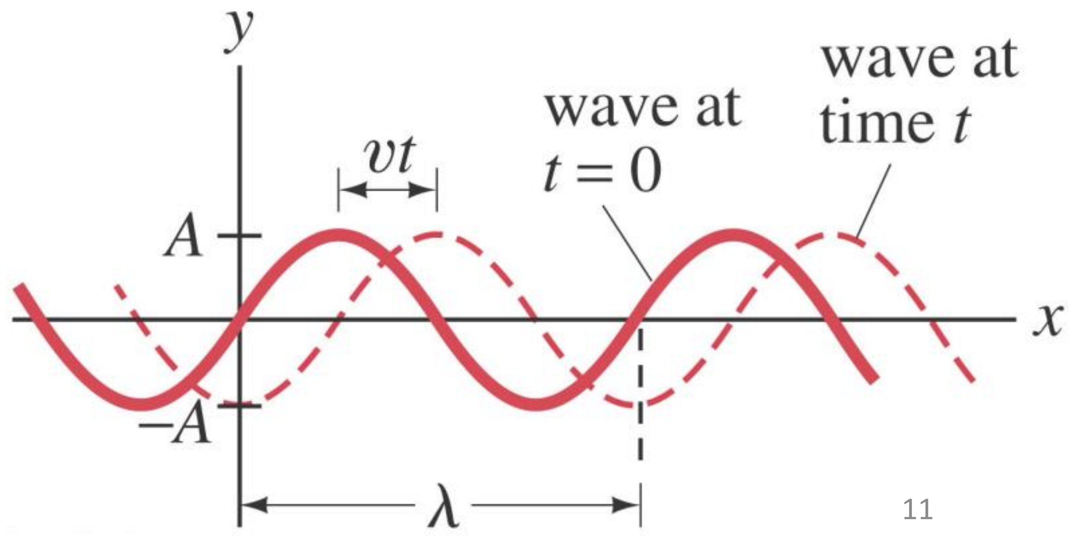
Wave moving backward,

$$y_2(x,t) = A\sin(kx + \omega t)$$

The superposition of the two waves will lead to

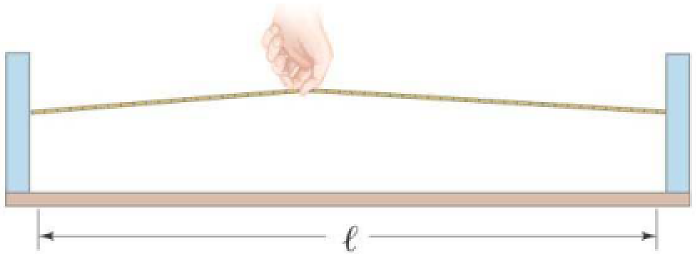
$$y = y_1 + y_2 = 2\,A\sin(kx)\cos(\omega t)$$

Caution: If the string is fixed at both ends, then y becomes 0 for x = 0 and x=L



Boundary Condition and Standing Waves

$$y = y_1 + y_2 = 2\,A\,\sin(kx)\,\cos(\omega t)$$



Let's analyse the values of y, the boundary conditions restricts that, y must vanish at x=0 and x=l. Now y can also be zero for condition $k_n = \frac{n\pi}{L}$, n =1, 2.. This leads to formation of nodes in the string.

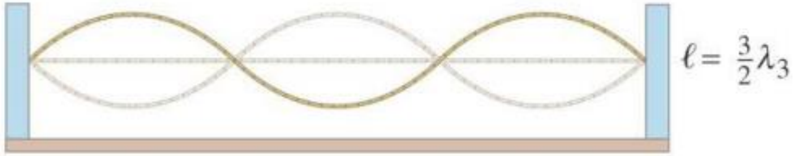
$$\lambda_n = \frac{2\pi}{k_n} = \frac{2L}{n} \qquad \omega_n = k_n v = \frac{n\pi}{L} v$$

Most generalized solution will be

$$y = \sum_n y_n(x,t) = A_n \sin\left(\frac{n\pi x}{L}\right) \cos(\omega_n t)$$

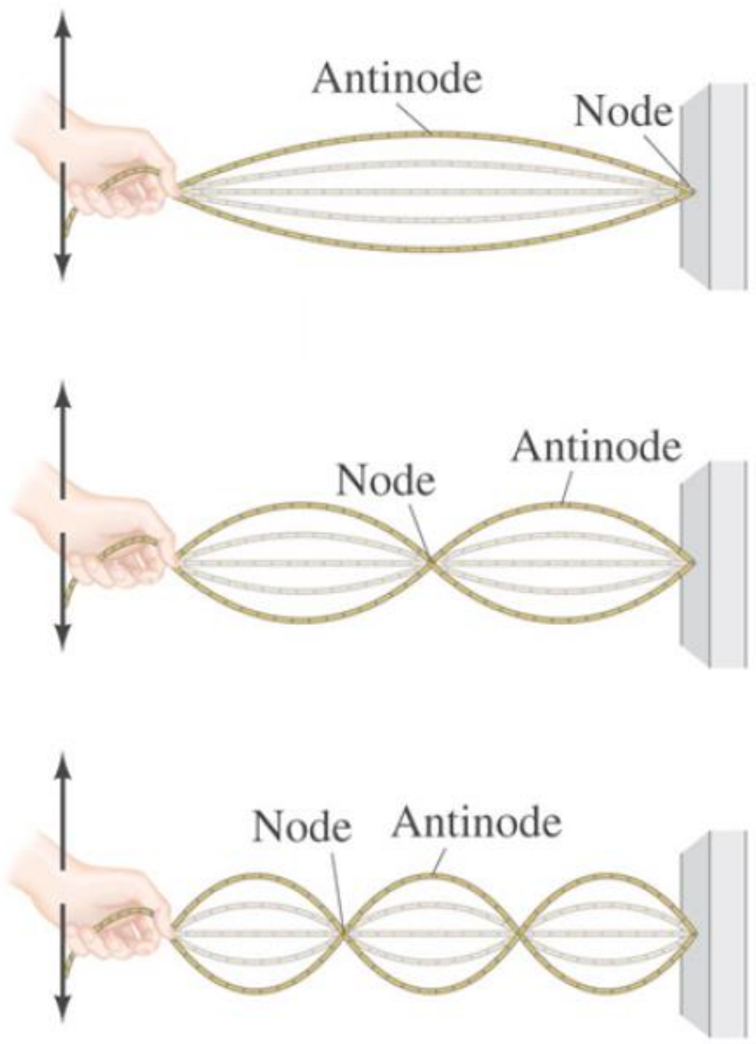
Fundamental or first harmonic, f_1

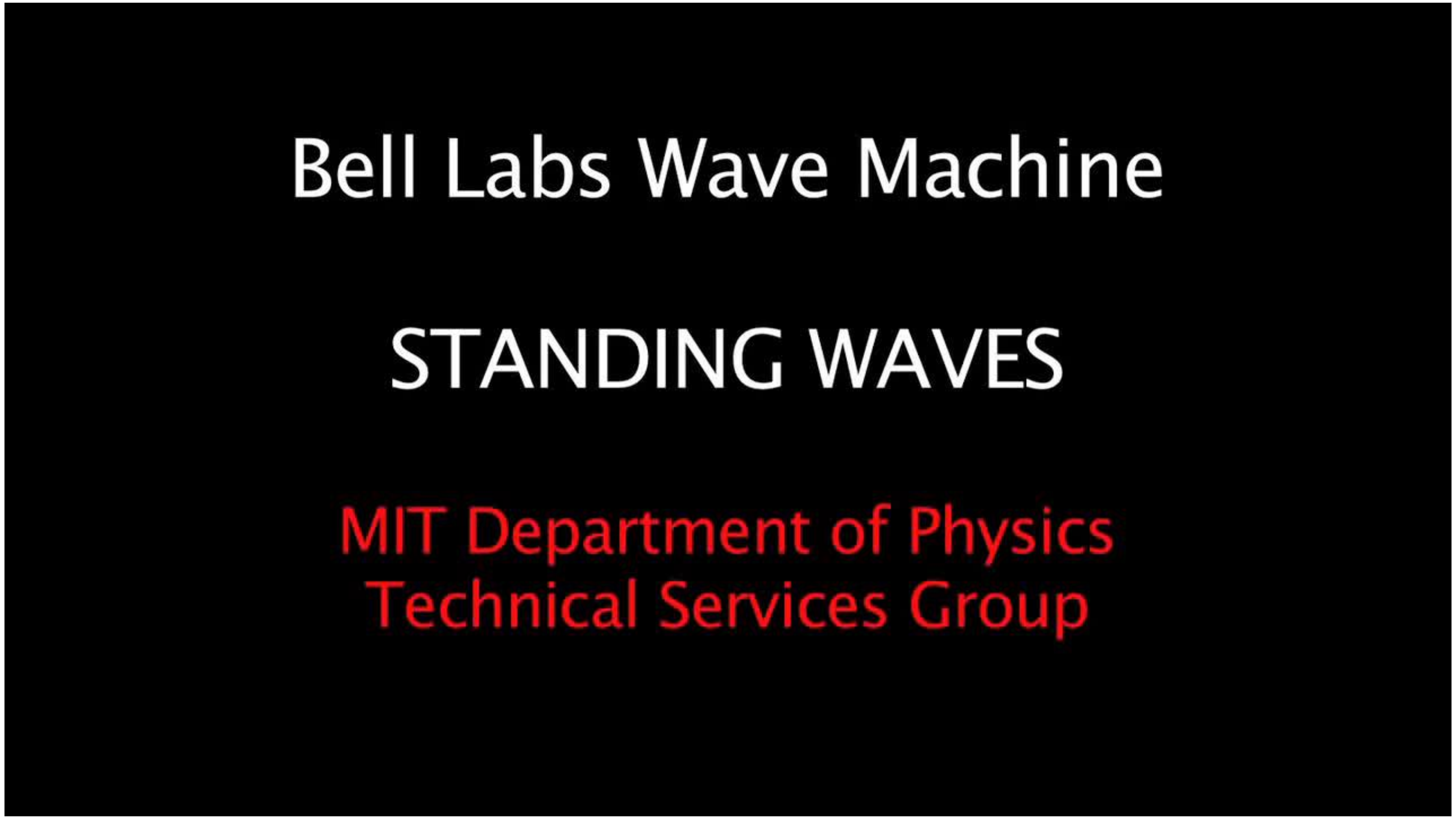
First overtone or second harmonic, $f_2 = 2f_1$



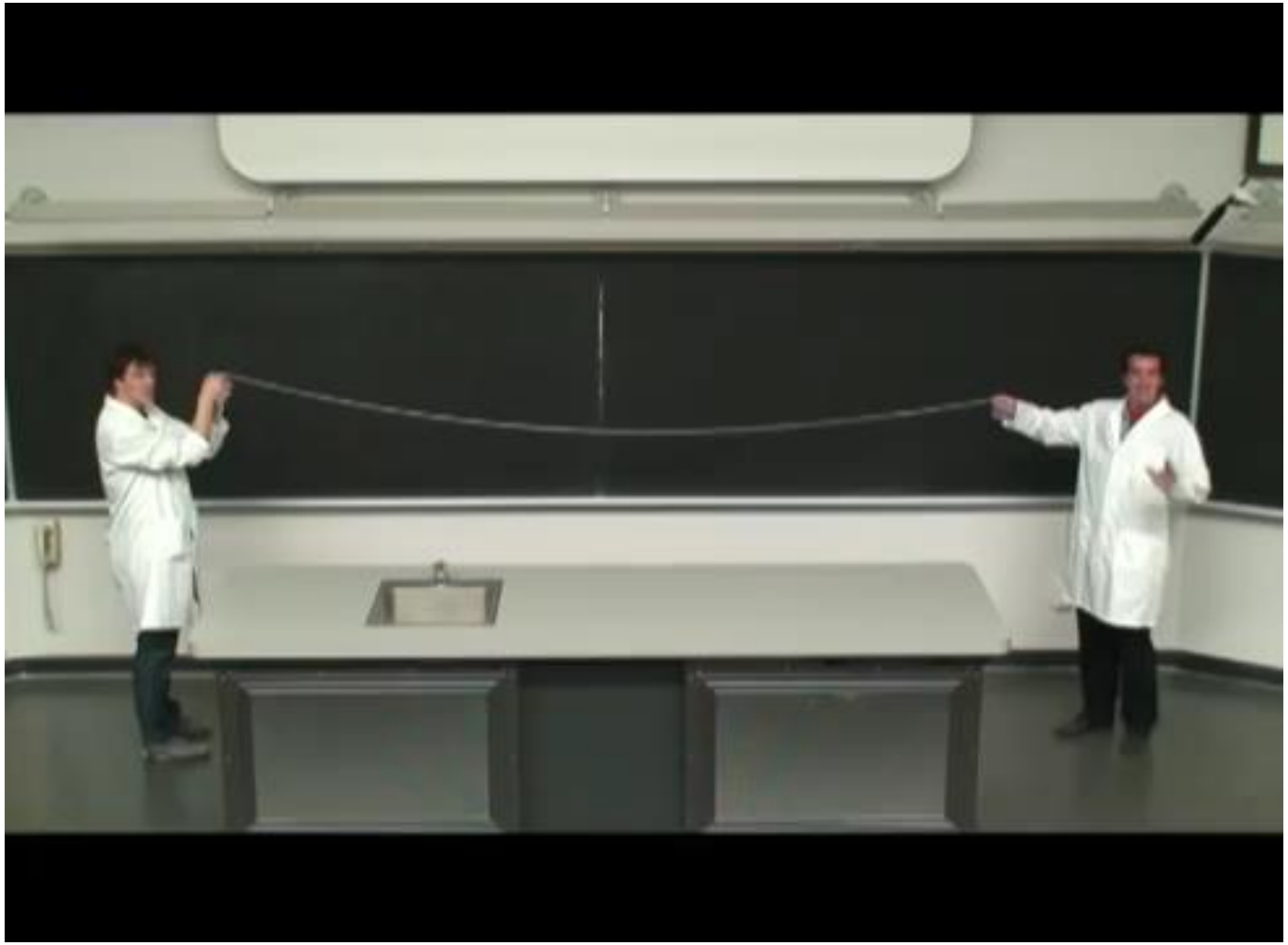
Second overtone or third harmonic, $f_3 = 3f_1$

Standing waves

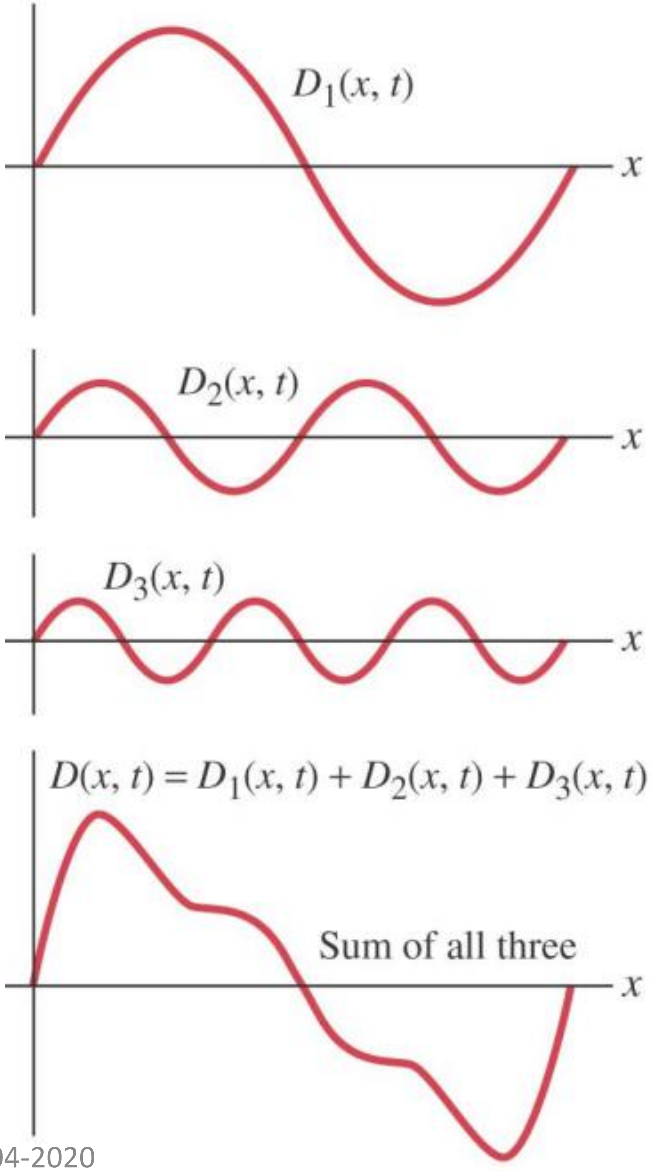


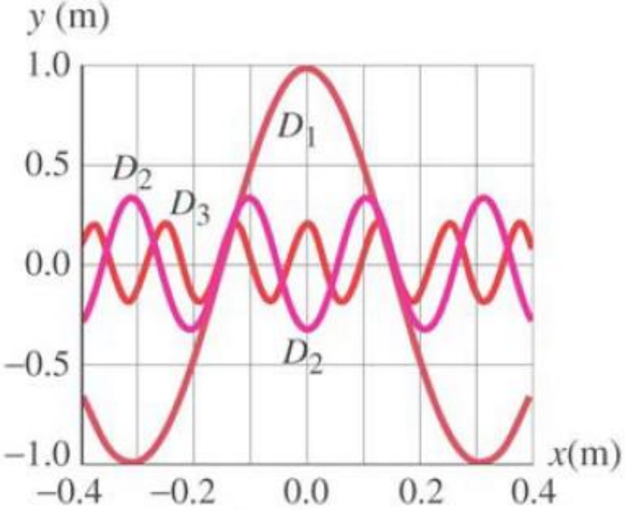


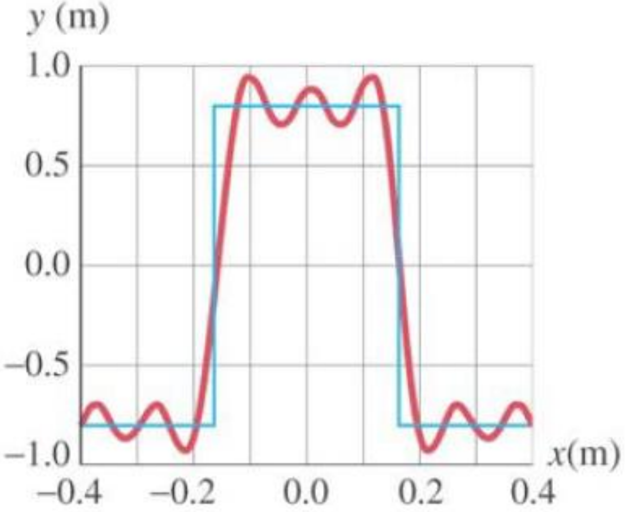
Summary of superposition and standing waves



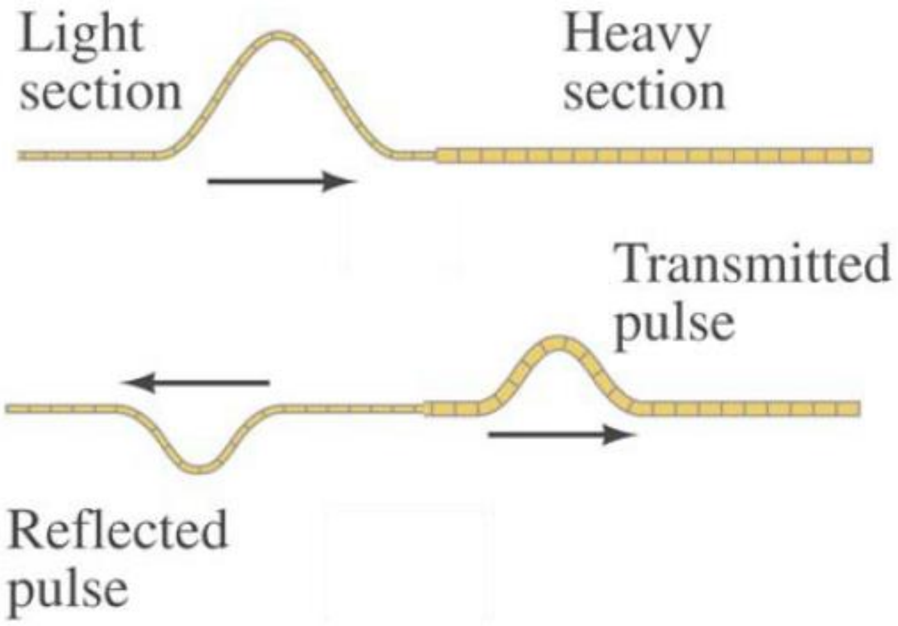
Superposition and Fourier Theorems







Boundary condition at the Interface of two medium



Boundary conditions:

At $x = 0, y_1 = y_2$ (otherwise the string will break)

Differentials at the boundary must be continuous

$$\frac{\partial y_1}{\partial x} = \frac{\partial y_2}{\partial x}$$

The frequency remains same across the boundary

$$\omega = v_1 k_1 = v_2 k_2$$

If there is no loss of energy at the boundary

$$A_i = A_r + A_t$$

Lets assume part of the incident wave is transmitted and part is reflected

$$y_i = A_i \sin(k_1 x - \omega t) \qquad y_t = A_t \sin(k_2 x - \omega t)$$

$$y_r = A_r \sin(k_1 x + \omega t)$$

Reflectance and Transmittance

$$r = \frac{A_r}{A_i} = \frac{v_2 - v_1}{v_1 + v_2} \qquad t = \frac{A_t}{A_i} = \frac{2v_2}{v_1 + v_2}$$

(a)Fixed wall/end, v_2=0,

t = 0 and r = -1

A wave hitting a fixed point will be reflected, and the amplitude will be inverted

A mountain will be a valley and a valley will be a mountain