MA - 102: B.Tech. I year; Spring Semester: 2019-20 (Tutorial Sheet)

(LU/PLU - Square (invertible) system)

1. Solve the following systems by Gauss elimination method:

$$\begin{array}{ccccccc} x & + & y & + & z & = & 4 \\ 2x & + & 5y & - & 2z & = & 3 \\ x & + & 7y & - & 7z & = & 5 \end{array}$$

2. Use Gauss elimination method to show that following system has no solution:

$$\begin{array}{ccccccc} 2\sin x & - & \cos y & + & 3\tan z & = & 3 \\ 4\sin x & + & 2\cos y & - & 2\tan z & = & 10 \\ 6\sin x & - & 3\cos y & + & \tan z & = & 9 \end{array}$$

3. Find Cholesky decomposition for following matrices.

- $A = \begin{bmatrix} 1 & -1 & 2 \\ -1 & 5 & -4 \\ 2 & -4 & 6 \end{bmatrix}$

- $A = \begin{bmatrix} 25 & 15 & -5 \\ 15 & 18 & 0 \\ -5 & 0 & 11 \end{bmatrix}$

4. Find LU/PLU for following matrices and hence find solution for $Ax = b$ for given vector b:

- $A = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 2 & 2 \\ 3 & 4 & 5 \end{bmatrix}$ $\qquad b = \begin{bmatrix} 1 \\ 4 \\ 7 \end{bmatrix}$

- $A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$ $\qquad b = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

- $A = \begin{bmatrix} 1 & 4 & 2 \\ -2 & -8 & 3 \\ 0 & 1 & 1 \end{bmatrix}$ $\qquad b = \begin{bmatrix} -2 \\ 32 \\ 1 \end{bmatrix}$

5. Use Gauss Jordan method to find the solution of following system:

$$\begin{array}{ccccccc} 2x & + & y & + & z & = & 1 \\ 4x & - & 6y & & & = & 1 \\ -2x & + & 7y & + & 2z & = & 1 \end{array}$$

(Vector Spaces, Subspaces and Linear Span)

1(i). Suppose we define addition on $\mathbb{R}^2$ by the rule $(a_1, a_2) + (b_1, b_2) = (a_1 + b_1, 0)$. Show that additive identity does not exist in $\mathbb{R}^2$ w.r.t. above rule.

1(ii). Suppose we define addition on $\mathbb{R}^3$ by the rule $(a_1, a_2, a_3) + (b_1, b_2, b_3) = (a_1 b_1, a_2 b_2, a_3 b_3)$. Show that we have an additive identity for this operation in $\mathbb{R}^3$ but inverse may not exist for some elements.

2. Let $\mathbb{R}^+$ be the set of all positive real numbers. Define operations of addition $\bigoplus$ and the scalar multiplication $\bigotimes$ as follows: $u \bigoplus v = uv$ for all $u, v \in \mathbb{R}^+$ and $\alpha \bigotimes u = u^\alpha$ for all $u \in \mathbb{R}^+$ and $\alpha \in \mathbb{R}$ (here $\mathbb{R}$ is the field of scalars). Prove that $(\mathbb{R}^+, \bigoplus, \bigotimes)$ is a real vector space.

3. Let $\mathrm{V} = \mathbb{R}^2$. Define operations of addition $\bigoplus$ and the scalar multiplication $\bigotimes$ as follows: $(a_1, a_2) \bigoplus (b_1, b_2) = (a_1 + b_2, a_2 + b_1)$ and $\alpha \bigotimes (a_1, a_2) = (\alpha a_1, \alpha a_2), \alpha \in \mathbb{R}$ (here $\mathbb{R}$ is the field of scalars). Does $(\mathrm{V}, \bigoplus, \bigotimes)$ form a real vector space? Give reasons for your assertion.

4. Elaborate: In any real vector space $(\mathrm{V}, \bigoplus, \bigotimes)$, we have
(i) $\alpha \bigotimes \mathbf{0} = \mathbf{0}$ for every scalar α.
(ii) $0 \bigotimes u = \mathbf{0}$ for every $u \in V$.
(iii) $(-1) \bigotimes u = -u$ for every $u \in V$.
(iv) $\alpha \bigotimes u = \mathbf{0} \Rightarrow \alpha = 0 \,\text{or}\, u = \mathbf{0}$, where u is vector and α is scalar.

5. Prove that a nonempty subset S of a vector space $(\mathrm{V}, \bigoplus, \bigotimes)$ is a subspace iff $(\alpha \bigotimes u) \bigoplus v \in \mathrm{S}$ for all scalars α and $u, v \in \mathrm{S}$.

6. Let $\mathrm{V} = C[0,1]$ be the set of all real valued function defined and continuous on the closed interval $[0,1]$. Prove that V is a real vector space with respect to pointwise addition and multiplication. Further, determine that which of the following subsets of V are subspaces
(i) $\{f \in \mathrm{V} : f(1/2) = 0\}$
(ii) $\{f \in \mathrm{V} : f(3/4) = 1\}$
(iii) $\{f \in \mathrm{V} : f(0) = f(1)\}$
(iv) $\{f \in \mathrm{V} : f(x) = 0$ only at a finite number of points$\}$

7. Determine whether each of the following set S form a subspace of $\mathbb{R}^4$, if addition and multiplication rules are defined in the usual way.
(i) $S = \{(a, b, c, d) : a = c + d\}$.
(ii) $S = \{(a, b, c, d) : b = c - d$ and $a = c + d\}$.
(iii) $S = \{(a, b, c, d) : c = d\}$.
(iv) $S = \{(-a + c, a - b, b + c, a + b) : a, b, c \in \mathbb{R}\}$.
(v) $S = \{(a, b, c, d) : a = 1\}$.
(vi) $S = \{(a, b, c, d) : a \leq b\}$.
(vii) $S = \{(a, b, c, d) : a = b = c = d\}$.
(viii) $S = \{(a, b, c, d) : a$ is an integer$\}$.
(ix) $S = \{(a, b, c, d) : a^2 - b^2 = 0\}$.

8. Which of the following subsets of $\mathcal{P}$ are subspaces. Where, $\mathcal{P}$ is the real vector space of all polynomials w.r.t. usual vector addition and scalar multiplication rules.
(i) $\{p \in \mathcal{P} : \deg . p \leq 4\}$ (ii) $\{p \in \mathcal{P} : \deg . p = 4\}$
(iii) $\{p \in \mathcal{P} : \deg . p \geq 4\}$ (iv) $\{p \in \mathcal{P} : p(1) = 0\}$
(v) $\{p \in \mathcal{P} : p(2) = 1\}$ (vi) $\{p \in \mathcal{P} : p'(1) = 0\}$

9. Which of the following subsets of $\mathbb{R}^{2\times 2}$ are subspaces. Note that, $\mathbb{R}^{m\times n}$ is the vector space over real field of all matrices of order $m \times n$ under usual definitions of addition and scalar multiplication of matrices.
(i) All diagonal matrices. (ii) All upper triangular matrices.

(iii) All symmetric matrices. (iv) All invertible matrices.

(v) All matrices which commute with a given matrix T.

(vi) All matrices with zero determinant.

10. Let W_1 and W_2 be subspaces of a vector space V such that $\mathrm{W}_1 \bigcup \mathrm{W}_2$ is also a subspace. Show that $\mathrm{W}_1 \subseteq \mathrm{W}_2$ or $\mathrm{W}_2 \subseteq \mathrm{W}_1$.

11. Let W_1 and W_2 be subspaces of a vector space V such that $\mathrm{W}_1 + \mathrm{W}_2 = \mathrm{V}$ and $\mathrm{W}_1 \bigcap \mathrm{W}_2 = \{0\}$. Show that for each vector u in V there are *unique* vectors $u_1 \in \mathrm{W}_1$ and $u_2 \in \mathrm{W}_2$ such that $u = u_1 + u_2$.

12. Let $\mathrm{S} = \{(1,2,3),(1,1,-1),(3,5,5)\}$. Determine which of the following are in $L[\mathrm{S}]$
(i) $(0,0,0)$ (ii) $(1,1,0)$ (iii) $(4,5,0)$ (iv) $(1,-3,8)$.

13. In the complex vector space $\mathbb{C}^2$, determine whether or not $(1+i, 1-i) \in L[(1+i,1),(1,1-i)]$.

14. Let M and N be subsets of the vector space $(\mathrm{V},+,.)$. Define $M+N=\{m+n : m \in M \text{ and } n \in N\}$. Then
(i) $M \subset N \Rightarrow L[M] \subset L[N]$ (ii) M is a subspace of $V \Leftrightarrow L[M]=M$ (iii) $L[L[M]] = L[M]$.

Answers

3. Not a vector space. 6. (i) Yes (ii) No (iii) Yes (iv) No
7. (i) Yes (ii) Yes (iii) Yes (iv) Yes (v) No (vi) No (vii) Yes (viii) No (ix) No
8. (i) Yes (ii) No (iii) No (iv) Yes (v) No (vi) Yes
9. (i) Yes (ii) Yes (iii) Yes (iv) No (v) Yes (vi) No
12. (i) and (iii) are in $L[S]$. 13. Yes

(RREF/Four Fundamental Subspaces/Solution of $Ax = b$)

1. Find the row-reduced echelon forms and hence rank of following matrices:

(i) $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ (ii) $\begin{bmatrix} 1 & 2 & 3 & 0 \\ 2 & 4 & 3 & 2 \\ 3 & 2 & 1 & 3 \\ 6 & 8 & 7 & 5 \end{bmatrix}$

2(i). Obtain for what values of λ and μ the equations

$$\begin{array}{ccccccc} x & + & y & + & z & = & 6 \\ x & + & 2y & + & 3z & = & 10 \\ x & + & 2y & + & \lambda z & = & \mu \end{array}$$

have (i) no solution (ii) a unique solution (iii) infinitely many solutions.

2(ii). Obtain for what values of λ the equations

$$\begin{array}{ccccccc} x & + & y & - & z & = & 1 \\ 2x & + & 3y & + & \lambda z & = & 3 \\ x & + & \lambda y & + & 3z & = & 2 \end{array}$$

have (i) no solution (ii) a unique solution (iii) infinitely many solutions.

2(iii). In the following system of linear equations

$$\begin{aligned} ax_1 + x_2 + x_3 &= p \\ x_1 + ax_2 + x_3 &= q \\ x_1 + x_2 + ax_3 &= r \end{aligned}$$

determine all values of a, p, q, r for which the resulting linear system has (i) unique solution (ii) infinitely many solutions (iii) no solution.

3. Does the system:

$$\begin{array}{ccccccc} x & + & y & + & z & = & 1 \\ 2x & + & 2y & + & z & = & 3 \end{array}$$

has a solution for $z = 7$? Find the general solution of system by Gauss elimination.

4. Show that the rank of matrix AB is less than or equal to rank of A as well as rank of B. Further prove that rank of AB is equal to rank of A, if B is invertible.

5. Suppose that $A_{m\times n}$ has rank k. Show that $\exists\, B_{m\times k}$, $C_{k\times n}$ such that rank $(A) =$ rank $(B) = k$ and $A = BC$.

6. Find Row reduced Echelon form of the following matrices and hence find all four fundamental spaces:

- $A_1 = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 2 \end{bmatrix}$ $A_2 = \begin{bmatrix} 1 & 2 & 2 \\ 1 & 3 & 4 \\ 1 & 2 & 3 \end{bmatrix}$
- $A_3 = \begin{bmatrix} 1 & 2 \\ 1 & 3 \\ 1 & 2 \end{bmatrix}$ $A_4 = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 1 \\ 2 & 5 & 2 \end{bmatrix}$

7. Use Gauss elimination method to find all polynomials $f \in \mathcal{P}_2 : f(1) = 2$ and $f(-1) = 6$.

(LI, LD, Basis and Dimension)

1(i). Check the linear dependence or linear independence of the following sets in respective real vector spaces
(a) $\{e^x, e^{2x}\}$ in $\mathcal{C}^\infty(\mathbb{R})$.
(b) $\{x, |x|\}$ in $\mathcal{C}[-1,1]$.
(c) $\{(\frac{1}{2}, \frac{1}{3}, 1), (-3,1,0), (1,2,-3)\}$ in $\mathbb{R}^3$.
(d) $\{(1,1,1,0), (3,2,2,1), (1,1,3,-2), (1,2,6,-5)\}$ in $\mathbb{R}^4$.
(e) $\{(x, x^3-x, x^4+x^2, x+x^2+x^4+\frac{1}{2}\}$ in $\mathcal{P}_4$.

1(ii). Show that the set $\mathrm{S} = \{\sin x, \sin 2x, \cdots, \sin nx\}$ is a LI subset of $\mathcal{C}[-\pi,\pi]$ for every positive integer n.

2(i). If u, v and w are LI vectors os a vector space V, then prove that $u+v, v+w,$ and $w+u$ are also LI.

2(ii). Let $\mathrm{S}_1, \mathrm{S}_2$ be subsets of a vector space V such that $\mathrm{S}_1 \subset \mathrm{S}_2$. Then prove that
(a) S_1 is LD $\Rightarrow$ S_2 is LD.
(b) S_2 is LI $\Rightarrow$ S_1 is LI.

2(iii). Let S be a LI subset of a vector space V. Let $v \in L[\mathrm{S}]$. Prove that $\{v\} \cup \mathrm{S}$ is a LD set.

2(iv). Let S be a LI subset of a vector space V. Let v does not belong in $L[\mathrm{S}]$. Prove that $\{v\} \cup \mathrm{S}$ is a LI set also.

3(i). In a vector space V, if a **ordered** set $\mathrm{S} = \{v_1, v_2, v_3, \cdots, v_n\}$ is LD **with** $v_1 \neq 0$ then prove that $\exists$ a vector v_k, $2 \leq k \leq n$ such that $v_k \in L[\{v_1, v_2, v_3, \cdots, v_{\mathbf{k-1}}\}]$.

3(ii). In a vector space V, if a set $\mathrm{S} = \{v_1, v_2, v_3, \cdots, v_n\}$ is LI and $\mathrm{S}_1 = \{w_1, w_2, w_3, \cdots, w_m\}$ generates the space V then prove that $n \leq m$.

4. Determine whether the following sets are bases for given vector spaces V over field F
(i) $\{(2,4,0), (0,2,-2)\}$; $\mathrm{V} = \mathbb{R}^3$ and $F = \mathbb{R}$.
(ii) $\{(6,4,4), (-2,4,2), (0,7,0)\}$; $\mathrm{V} = \mathbb{R}^3$ and $F = \mathbb{R}$.
(iii) $\left\{\begin{pmatrix} -1 & -1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 2 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ -1 & -1 \end{pmatrix}\right\}$; $\mathrm{V} = \mathcal{M}_{2\times 2}$ and $F = \mathbb{R}$.
(iv) $\{1, x-2, (x-2)^2, (x-2)^3\}$; $\mathrm{V} = \mathcal{P}_3$ and $F = \mathbb{R}$.
(v) $\{x-1, x^2+x-1, x^2-x+1\}$; $\mathrm{V} = \mathcal{P}_2$ and $F = \mathbb{R}$.
(vi) $\{(1,i,1+i), (1,i,1-i), (i,-i,1)\}$; $\mathrm{V} = \mathbb{C}^3$ and $F = \mathbb{C}$.

5(i). Find the co-ordinates of the following vector of $\mathbb{R}^3$ relative to the ordered basis $B = \{(2,1,0), (2,1,1), (2,2,1)\}$
(i) $(1,2,-1)$ (ii)$(2,0,-1)$ (iii) $(-1,3,1)$

5(ii). Find the relation between the co-ordinates of the vector $(1,5)$ with respect to the ordered bases $B_1 = \{(1,1), (0,1)\}$ and $B_2 = \{(-1,4), (7,6)\}$

6. Find a basis for the plane $P: x - 2\,y + 3\,z = 0$ in $\mathbb{R}^3$. Find a basis for the intersection of P with with the xy-plane. Also, find a basis for the space of vectors perpendicular to the plane P.

7(i). Let $\mathrm{S} = \{(4,5,6), (a,2,4), (4,3,2)\}$ be a set in $\mathbb{R}^3$. Find the values for a such that $L[\mathrm{S}] \neq \mathbb{R}^3$.

7(ii). For what values of k vectors $\mathrm{S} = \{(k+1, -k, k), (2k, 2k-1, k+2), (-2k, k, -k)\}$ form a basis of $\mathbb{R}^3$.

8. For each of followings, find a basis (here all vector spaces are real)
(i) $\{(x_1, x_2, x_3)$ in $\mathbb{R}^3 : x_1 - x_3 = 0\}$.
(ii) $\{(x_1, x_2, x_3)$ in $\mathbb{R}^3 : 2x_1 + x_2 + x_3 = 0\}$.
(iii) $\{(x_1, x_2, x_3, x_4)$ in $\mathbb{R}^4 : x_1 + x_2 + 2x_3 = 0, 2x_2 + x_3 = 0$ and $x_1 - x_2 + x_3 = 0\}$.
(iv) $\{a + bx + cx^3$ in $\mathcal{P}_3 : a - 2b + c = 0\}$.

(v) $\{p \text{ in } \mathcal{P}_4 : p(7) = 0 \text{ and } p'(1) = 0\}$.

(vi) $\{\begin{bmatrix} a & b \\ c & d \end{bmatrix} \text{ in } \mathbb{R}^{2\times 2} : a - d + c = 0\}$.

(vii) $\{A \text{ in } \mathbb{R}^{4\times 4} : A \text{ is a real symmetric martix}\}$.

(viii) $\{A \text{ in } \mathbb{R}^{5\times 5} : \text{Trace } A = 0\}$.

(ix) $\{A \text{ in } \mathbb{R}^{2\times 2} : A \text{ is a complex Hermitian martix}\}$.

(x) $\{A \text{ in } \mathbb{R}^{m\times n} : \text{ sum of each row of } A = 0\}$.

9(i). Write two bases of $\mathbb{R}^4$ that have no common elements.

9(ii). Write two different bases of $\mathbb{R}^4$ that have the vectors $(0,0,1,0)$ and $(0,0,0,1)$ in common.

9(iii). Find a basis of $L[\{(1,-1,2,3),(1,0,1,0),(3,-2,5,2)\}]$ which includes the vectors $(1,1,0,-1)$.

9(iv). Extend the set $\{(1,1,-1,0),(1,0,1,1),(1,2,1,1)\}$ to a basis of $\mathbb{R}^4$.

10. Find a basis for U, W, $U \bigcap W$ and $U + W$ in the following cases for a vector space V.
(i) $U = \{(x_1,x_2,x_3) : x_1 + x_2 - x_3 = 0\},\ W = \{(x_1,x_2,x_3) : 2x_1 + x_2 = 0\},\ \text{V} = \mathbb{R}^3$.
(ii) $U = \{a_0 + a_1x + a_2x^2 : a_1 + a_2 = 0\},\ W = \{a_0 + a_1x + a_2x^2 : 2a_0 + a_1 = 0\},\ \text{V} = \mathcal{P}_2$.
(iii) $U = \{p : p(2) = 0\},\ W = \{p : p'(2) = 0\},\ \text{V} = \mathcal{P}_4$.

11. Find the subspaces $S \bigcap T$, $S + T$ of vector space V. Further, find $\dim(S)$, $\dim(T)$, $\dim(S \bigcap T)$ $\dim(S+T)$ if
(i) $S = L[\{(1,-1,0),(1,0,2)\}],\ T = L[\{(0,1,0),(0,1,2)\}],\ \text{V} = \mathbb{R}^3$.
(ii) $S = L[\{(2,2,-1,2),(1,1,1,-2),(0,0,2,-4)\}],\ T = L[\{(2,-1,1,1),(-2,1,3,3),(3,-6,0,0)\}],\ \text{V} = \mathbb{R}^4$.

Answers

1(i). (a) LI (b) LI (c) LI (d) LD (e) LI
4. (i) No (ii) Yes (iii) Yes (iv) Yes (v) No (vi) Yes

(Linear Transformation)

1(i). Find a LT $T : \mathbb{R}^2 \to \mathbb{R}^2$ such that $T(1,0) = (1,1)$ and $T(1,1) = (-1,2)$. Also prove that T maps square with vertices at $(0,0)$, $(1,0)$, $(1,1)$, $(0,1)$ into a parallelogram.

1(ii). If possible, find a LT $T : A \to B$ such that
(a) $T(2,3) = (4,5)$, $T(1,0) = (0,0)$, where $A = \mathbb{R}^2$ and $B = \mathbb{R}^2$.

(b) $T(1,1) = (1,0,1)$, $T(0,1) = (1,0,0)$, $T(1,2) = (2,1,1)$ where $A = \mathbb{R}^2$ and $B = \mathbb{R}^3$.

(c) $T(1,0,0) = (2,3)$, $T(0,1,0) = (1,2)$, $T(0,0,1) = (-1,-4)$ where $A = \mathbb{R}^3$ and $B = \mathbb{R}^2$.

(d) $T(1,1,0) = (0,1,1)$, $T(0,0,0) = (0,0,1)$, $T(1,0,1) = (0,0,0)$ where $A = B = \mathbb{R}^3$.

2(i). Find a LT $T : \mathbb{R}^3 \to \mathbb{R}^3$, whose range is spanned by the vectors $(1,0,-1)$ and $(1,2,2)$.

2(ii). Find a nonzero LT $T : \mathbb{R}^2 \to \mathbb{R}^2$, which maps all the vectors on the line $y = x$ onto the origin.

3. Find the range and null space of followings LTs. Also find the rank and nullity wherever applicable:
(i) $T : \mathbb{R}^2 \to \mathbb{R}^3$ defined by $T(x_1, x_2) = (3x_1 + x_2, 0, 0)$.

(ii) $T : \mathbb{R}^4 \to \mathbb{R}^3$ defined by $T(x_1, x_2, x_3, x_4) = (x_1 - x_4, x_2 + x_3, x_3 - x_4)$.

(iii) $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T(x_1, x_2) = (x_1 + x_2, x_1 + x_2)$.

(iv) $T : \mathcal{P}_3 \to \mathbb{R}^3$ defined by $T(a_0 + a_1x + a_2x^2 + a_3x^3) = (a_0 + a_1 + 2a_3, 2a_1 + a_2, a_3 + a_1)$.

(v) $T : \mathcal{C}(0,1) \to \mathcal{C}(0,1)$ defined by $T(f)x = f(x)\sin x$.

4. Examine whether the following transformations are linear or not. In case of LT, find their matrix representation with respect to given bases B_1 and B_2.
(i) $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T(x_1, x_2) = (x_1 + x_2, x_2)$; B_1 and B_2 are standard bases.

(ii) $T : \mathbb{R}^2 \to \mathbb{R}^3$ defined by $T(x_1, x_2) = (x_1, x_1 + x_2, x_2)$; B_1 and B_2 are standard bases.

(iii) $T : \mathbb{C}^2 \to \mathbb{C}^2$ defined by $T(x_1 + ix_2, x_3 + ix_4) = (x_1, x_2)$; $B_1 = \{(0,1), (1,1)\}$ and B_2 is standard bases.

(iv) $T : \mathcal{P}_2 \to \mathcal{P}_2$ defined by $T(a_0 + a_1x + a_2x^2) = -a_0 + 2a_1x + (a_2 + a_0)x^2$; B_1 and B_2 are standard bases.

(v) $T : \mathcal{P}_3 \to \mathcal{P}_3$ defined by $T(a_0 + a_1x + a_2x^2 + a_3x^3) = a_0 + a_1(x+1) + a_2(x+1)^2 + a_3(x+1)^3$; $B_2 = \{1,\ 1+x,\ 1+x^2,\ 1+x^3\}$ and B_1 is standard basis.

(vi) $T : \mathcal{P}_2 \to \mathcal{P}_3$ defined by $T(p(x)) = xp(x) + \int_0^x p(t)$; B_1 and B_2 are standard bases.

(vii) $T : \mathcal{P}_2 \to \mathbb{R}^4$ defined by $T(a_0 + a_1x + a_2x^2) = (a_0 + a_2, a_1 - a_0, a_2 - a_1, a_0)$; $B_1 = \{1; 1+x; x+x^2\}$ and $B_2 = \{(1,0,1,0); (1,0,0,0); (0,1,-1,0); (0,0,1,1)\}$.

(viii) $T : \mathbb{R}^{2\times 2} \to \mathbb{R}^{2\times 2}$ defined by $T(A) = AM, \forall A \in \mathbb{R}^{2\times 2}$, where $M = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$ is a fixed matrix in $\mathbb{R}^{2\times 2}$; B_1 and B_2 are standard bases.

(ix) Repeat part (viii), when $T : \mathbb{R}^{2\times 2} \to \mathbb{R}^{2\times 2}$ is defined by $T(A) = A + M$.

5. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be defined by $T(x_1, x_2, x_3) = (x_1 + x_2, x_1 + 2x_2, 3x_3 + x_2)$. Show that T is invertible and further, find a formula for T^{-1}. Match the result by matrix representation also.

6(i). Find a LT $T : \mathbb{R}^3 \to \mathbb{R}^3$, whose matrix representation is $\begin{bmatrix} 2 & 0 & 0 \\ 2 & -5 & 0 \\ 0 & 2 & 1 \end{bmatrix}$, with respect to standard bases. Find its inverse matrix also.

6(ii). Find a LT $T : \mathbb{R}^3 \to \mathbb{R}^3$, whose matrix representation is $\begin{bmatrix} 1 & 0 & -1 \\ 0 & 2 & 1 \\ 1 & 0 & -1 \end{bmatrix}$, with respect to standard bases. Find the matrix of T with respect to basis $\{(1,1,-1), (1,2,0), (1,0,1)\}$.

6(iii). Find a LT $T : \mathcal{P}_3 \to \mathbb{R}^3$, whose matrix representation is $\begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 1 & 1 & 1 \\ 5 & 4 & 1 & -1 \end{bmatrix}$, with respect to $\{1;\ 1+x^2;\ x+x^3;\ 1+x+x^2\}$ and $\{(1,0,1), (2,4,5), (0,0,1)\}$.

(Eigenvalues and Eigenvectors)

1. For each matrix, find all eigenvalues and eigenvectors;

(i) $\begin{bmatrix} 3 & 1 & 1 \\ 2 & 4 & 2 \\ 1 & 1 & 3 \end{bmatrix}$ (ii) $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

2. (i) If λ is an eigenvalue of a nonsingular matrix $A_{n\times n}$, then verify that λ^{-1} is an eigenvalue of A^{-1}.

(ii) If A and P be both $n \times n$ matrices and P be nonsingular, then verify that A and $P^{-1}AP$ have the same eigen values.

(iii) Prove that eigen values of a real symmetric matrices are all real.

(iv) Prove that eigen values of a real skew symmetric matrix are purely imaginary or zero.

(v) Prove that eigen vectors corresponding to two distinct eigen values of a real symmetric matrix are orthogonal.

(vi) Prove that eigen value of a real orthogonal matrix has unit modulus.

(vii) Prove that any skew-symmetric Matrix of odd order has zero determinant.

$(viii)$ Let A and B be matrices of order n. Show that AB and BA have same eigenvalues.

3. Find a matrix P such that $P^{-1}AP$ is a diagonal matrix where A is

(i) $\begin{bmatrix} 4 & -2 & 0 \\ -2 & 2 & -2 \\ 0 & -2 & 4 \end{bmatrix}$ (ii) $\begin{bmatrix} 0 & 2 & -1 \\ 2 & 3 & -2 \\ -1 & -2 & 0 \end{bmatrix}$

4. Let $A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$. Find A^{-1} and A^4 by Cayley-Hamilton theorem.

5. Find e^{2A} and A^{50} when (i) $A = \begin{bmatrix} -4 & 1 & 0 \\ 0 & -3 & 1 \\ 0 & 0 & -2 \end{bmatrix}$ (ii) $A = \begin{bmatrix} -2 & 4 & 3 \\ 0 & 4 & 2 \\ 0 & 0 & 4 \end{bmatrix}$