Department of Mathematics
Indian Institute of Technology Patna
B.Tech. II year (Autumn Semester 2020-21)

Tutorial Sheet-1: MA201 (Complex Analysis)

1. Show that the field of complex numbers $\mathbb{C}$ is not ordered.

2. Prove the followings:
(i) $|z_1 \pm z_2| \leq |z_1| + |z_2|$, (ii) $||z_1| - |z_2|| \leq |z_1 \pm z_2|$,
(iii) $|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2(|z_1|^2 + |z_2|^2)$, and then write the simplified expression for $|z_1 - z_2|^2$.
(iv) $\sqrt{2}|z| \geq |Re(z)| + |Im(z)|$.

3. Use principle of mathematical induction to prove $|\sum_{i=1}^{n} z_i| \leq \sum_{i=1}^{n} |z_i|$, where $z_1, z_2, \dots, z_n$ are some complex numbers.

4. Show that $Re(z_1\overline{z}_2) \leq |z_1\overline{z}_2|$, for any two complex numbers z_1 and z_2. Under what conditions these two quantities will be equal. Show that in that case, $|z_1 + z_2| = |z_1| + |z_2|$ and $|z_1 - z_2| = ||z_1| - |z_2||$.

5. Let $p(z)$ be a polynomial of degree n, where $p(z) = a_n z^n + a_{n-1}z^{n-1} + \ldots + a_1 z + a_0$ with all coefficients being real. Show that if z_1 is a root of $p(z)$, then so is $\overline{z}_1$.

6. Find the locus of the followings:
(i) $Re(\frac{1}{\overline{z}}) = 2$, (ii) $Re(z^2) \leq 1$, (iii) $|z - 4i| + |z + 4i| = 10$,
(iv) $|z - z_0| = k|z - z_1|$, $k \neq 1$.

7. Find $|\sin z|$ at $z = \pi + i\ln(2 + \sqrt{5})$, where ln is the natural logarithm.

8. If α is a complex number such that $|\alpha| < 1$, prove that $|\dfrac{z-\alpha}{1-\bar{\alpha}z}| < 1$, if $|z| < 1$ and $|\dfrac{z-\alpha}{1-\bar{\alpha}z}| = 1$, if $|z| = 1$.

9. Show that $z + \frac{1}{z}$ is real iff $Im(z) = 0$ or $|z| = 1$.

10. If $|z| = 1$, prove that $|z^2 - z + 1| \leq 3$ and $|z^2 - 2| \geq 1$.

11. Find the upper bounds for the followings:
(i) $|\frac{1}{z^4 - 4z^2 + 3}|$ (ii) $|\frac{-1}{z^4 - 5z + 1}|$ (ii) $|\frac{1}{z^4 - 5z^2 + 6}|$ where $|z| = 2$.

12. Establish the identity $1 + z + z^2 + \cdots + z^n = \dfrac{1 - z^{n+1}}{1 - z}$, $z \neq 1$, and hence prove that:
(i) $1 + \cos\theta + \cdots + \cos n\theta = \dfrac{1}{2} + \dfrac{\sin((n+\frac{1}{2})\theta)}{2\sin\frac{\theta}{2}}$,
(ii) $\sin\theta + \cdots + \sin n\theta = \dfrac{\cos\frac{\theta}{2} - \cos((n+\frac{1}{2})\theta)}{2\sin\frac{\theta}{2}}$. Here n is any positive integer and $0 < \theta < 2\pi$.

13. Solve the followings:
(i) $x^8 - 16 = 0$ (ii) $x^6 + i + 1 = 0$ (iii) $z^4 - 4z^3 + 6z^2 - 4z + 5 = 0$ given that i is a root.
(iv) $z^{3/2} = 4\sqrt{2} + i4\sqrt{2}$.

14. Find the four zeros of the polynomial $z^4 + 4$, and represent it into quadratic factors with real coefficients.

15. Evaluate the followings:
(i) $(-\sqrt{3} - i)^{-6}$ (ii) Find polar form of $\frac{\sqrt{2}+i\sqrt{6}}{-1+i\sqrt{3}}$, and then write in the form of $x + iy$,

(iii) Compute $(2-2i)^5$ $\quad(iv)$ $(0.5+0.5i)^{10}$.

16. Find the sum of the p^{th} powers of the roots of the equation $z^n = 1$, where p is a positive integer.

17. Let the equation $z^n = 1$ have the roots $1, z_1, z_2, \ldots, z_{n-1}$, then show that $(1-z_1)(1-z_2)\ldots(1-z_{n-1}) = n$.

18. Prove the identity: $\sin(\pi/n)\sin(2\pi/n)\ldots\sin(\pi(n-1)/n) = \frac{n}{2^{n-1}}$.

19. Prove the identity: $z^{2n} - 1 = (z^2-1)\Pi_{k=1}^{n-1}(z^2 - 2z\cos(k\pi/n) + 1)$.
Hence show that $\sin(\pi/2n)\sin(2\pi/2n)\ldots\sin(\pi(n-1)/2n) = \frac{\sqrt{n}}{2^{n-1}}$.