ICS141: Discrete Mathematics for Computer Science I

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Provided by McGraw-Hill

Lecture 11

Chapter 2. Basic Structures

2.3 Functions

2.4 Sequences and Summations

The Identity Function

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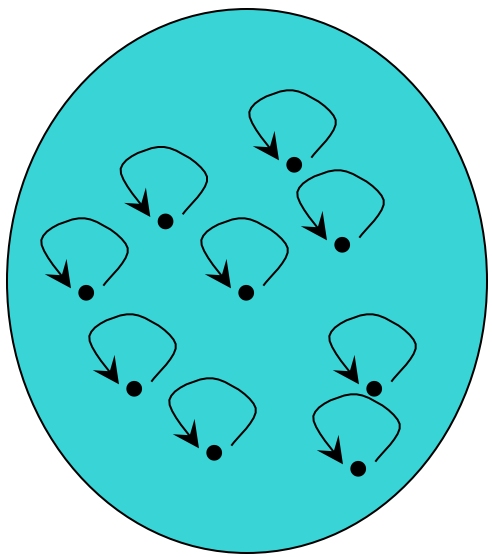
- For any domain A, the ***identity function*** $I: A \to A$ (also written as I_A, **1**, $\mathbf{1}_A$) is the unique function such that $\forall a \in A: I(a) = a$.
- Note that the identity function is always both one-to-one and onto (i.e., bijective).
- For a bijection $f : A \to B$ and its inverse function $f^{-1}: B \to A$,

$$f^{-1} \circ f = I_A$$

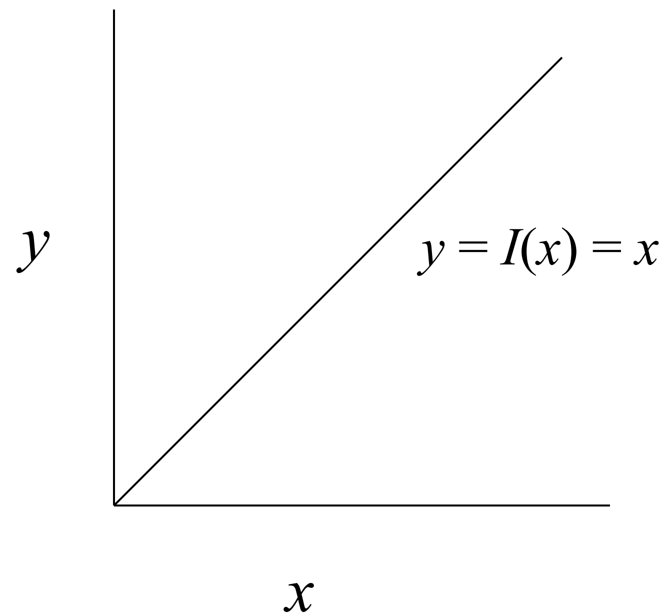
- Some identity functions you've seen:
  - $+\ 0$, $\times\ 1$, $\wedge$ **T**, $\vee$ **F**, $\cup\ \varnothing$, $\cap\ U$.

Identity Function Illustrations

- The identity function:



Domain and range



Graphs of Functions

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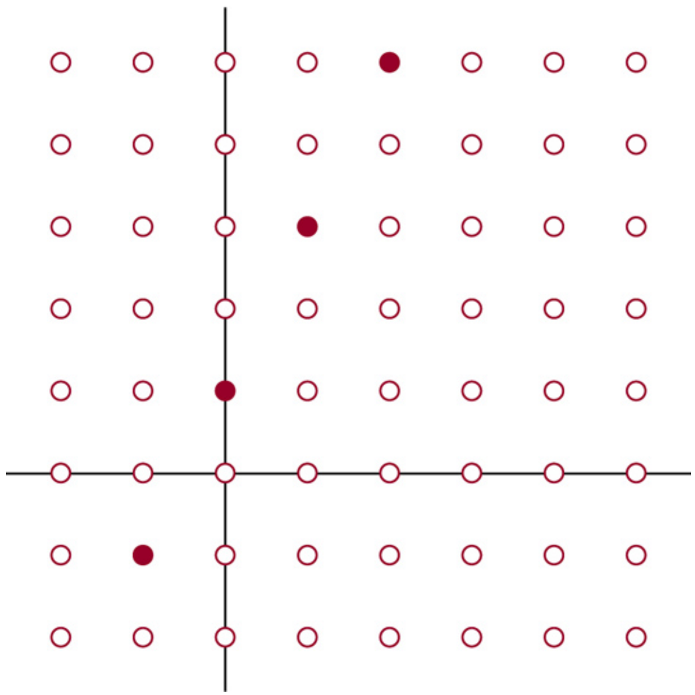
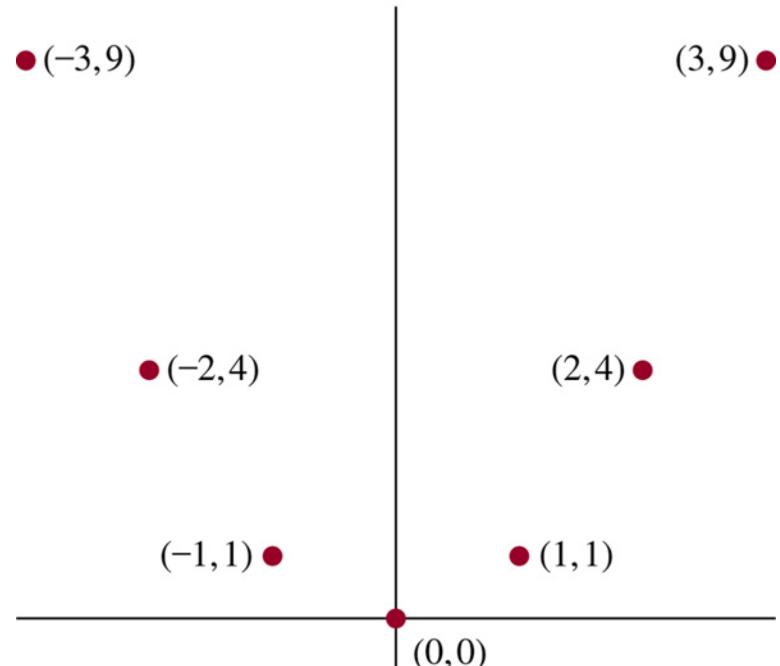
- We can represent a function $f : A \rightarrow B$ as a set of ordered pairs $\{(a, f(a)) \mid a \in A\}$. ← The function's *graph*.
- Note that $\forall a \in A$, there is only 1 pair (a, b).
  - Later (ch.8): *relations* loosen this restriction.
- For functions over numbers, we can represent an ordered pair (x, y) as a point on a plane.
  - A function is then drawn as a curve (set of points), with only one y for each x.

Graphs of Functions: Examples



The graph of $f(n) = 2n + 1$ from **Z** to **Z**



The graph of $f(x) = x^2$ from **Z** to **Z**

Floor & Ceiling Functions

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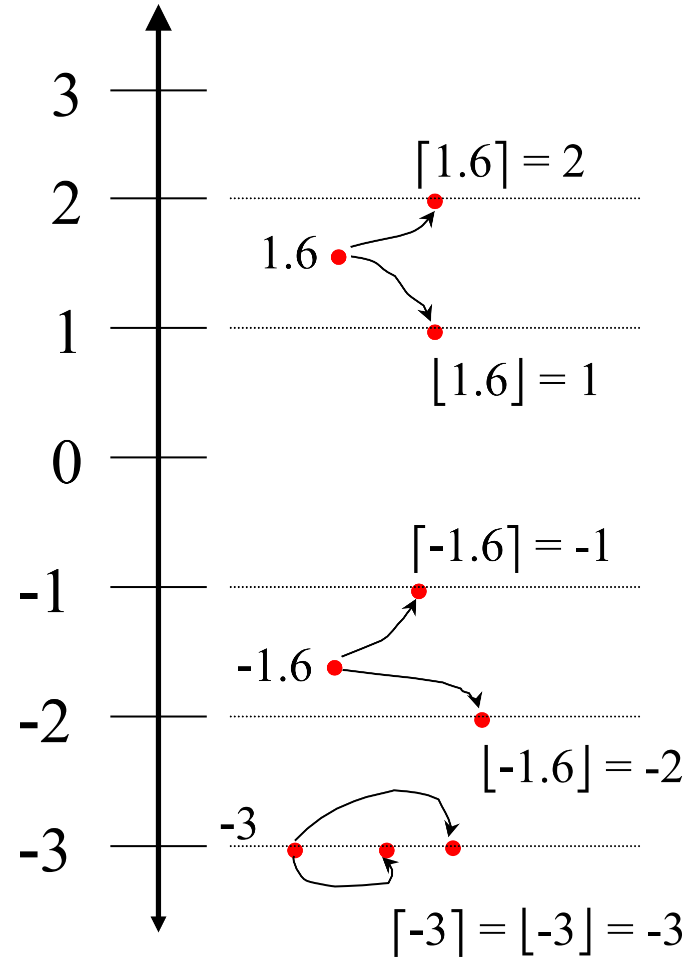
- In discrete math, we frequently use the following two functions over real numbers:
  - The ***floor*** function $\lfloor \cdot \rfloor : \mathbf{R} \to \mathbf{Z}$, where $\lfloor x \rfloor$ ("floor of x") means the **largest integer ≤ *x***, i.e., $\lfloor x \rfloor = \max(\{i \in \mathbf{Z} \mid i \leq x\})$.
    - E.g. $\lfloor 2.3 \rfloor =$, $\lfloor 5 \rfloor =$, $\lfloor -1.2 \rfloor =$
  - The ***ceiling*** function $\lceil \cdot \rceil : \mathbf{R} \to \mathbf{Z}$, where $\lceil x \rceil$ ("ceiling of x") means the **smallest integer ≥ *x***, i.e., $\lceil x \rceil = \min(\{i \in \mathbf{Z} \mid i \geq x\})$
    - E.g. $\lceil 2.3 \rceil =$, $\lceil 5 \rceil =$, $\lceil -1.2 \rceil =$

Visualizing Floor & Ceiling

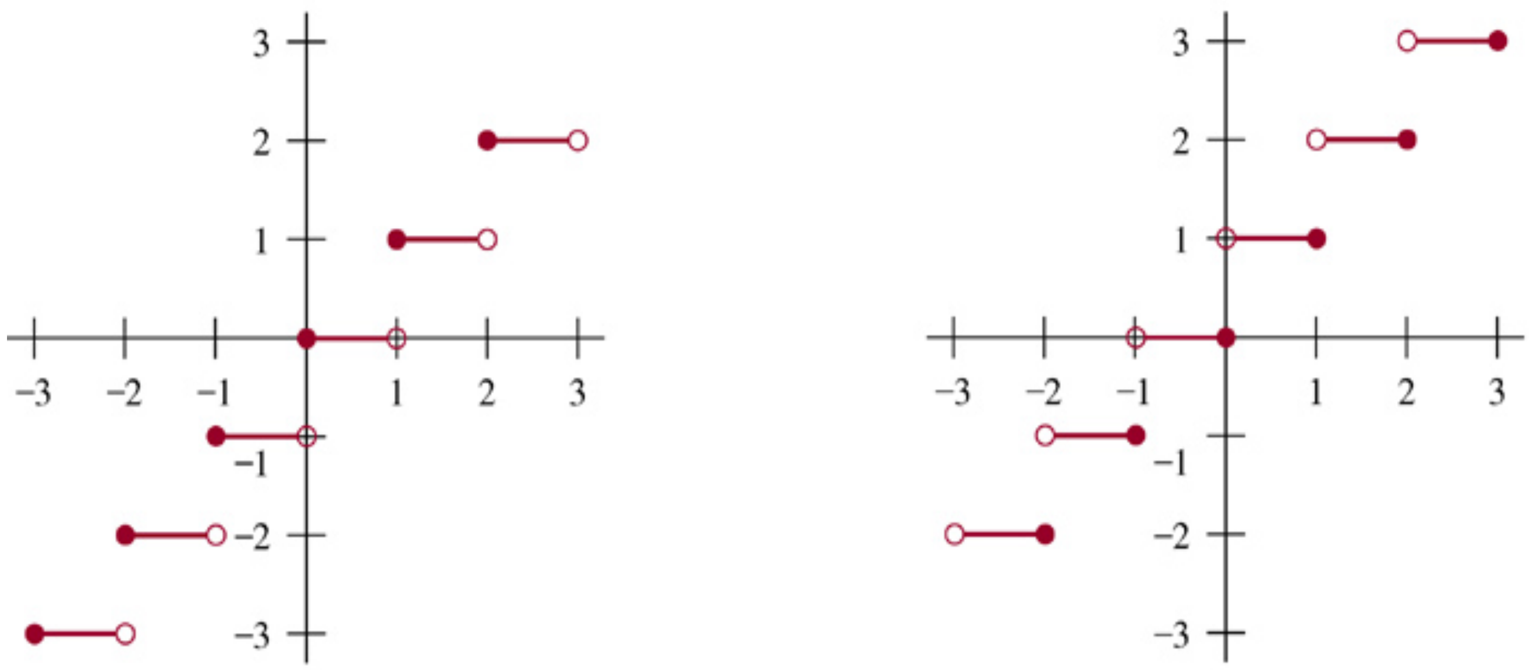
- Real numbers "fall to their floor" or "rise to their ceiling."



- Note that if $x \notin \mathbf{Z}$,
  $\lfloor -x \rfloor \neq - \lfloor x \rfloor$ &
  $\lceil -x \rceil \neq - \lceil x \rceil$

- Note that if $x \in \mathbf{Z}$,
  $\lfloor x \rfloor = \lceil x \rceil = x$.

Plots with Floor/Ceiling: Example



$y = \lfloor x \rfloor$

$y = \lceil x \rceil$

Plots with Floor/Ceiling

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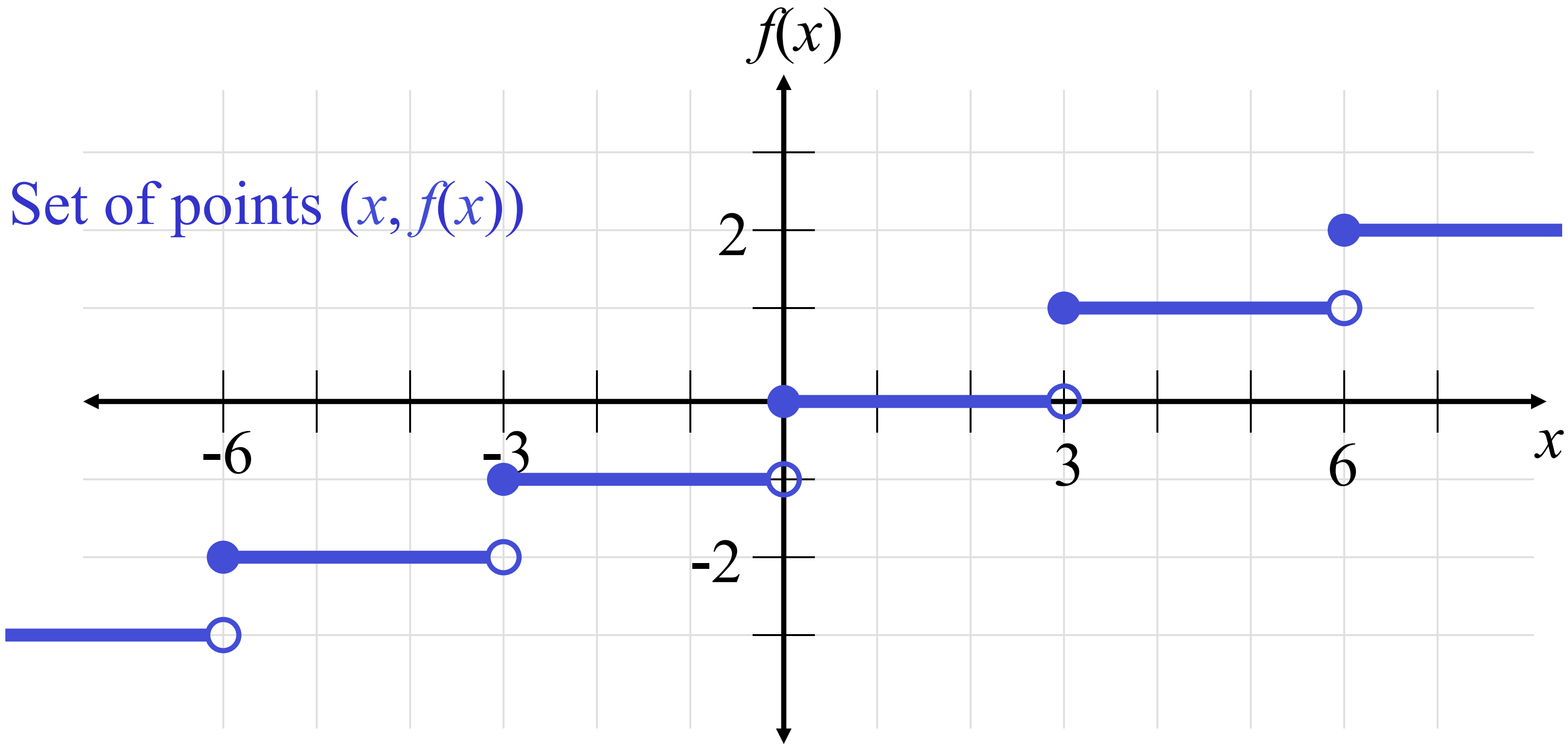
- Note that for $f(x) = \lfloor x \rfloor$, the graph of f includes the point $(a, 0)$ for all values of a such that $0 \leq a < 1$, but not for the value $a = 1$.
- We say that the set of points $(a, 0)$ that is in f does not include its *limit* or *boundary* point $(a,1)$.
  - Sets that do not include all of their limit points are called *open sets*.
- In a plot, we draw a limit point of a curve using an open dot (circle) if the limit point is not on the curve, and with a closed (solid) dot if it is on the curve.

Plots with Floor/Ceiling: Another Example

- Plot of graph of function $f(x) = \lfloor x/3 \rfloor$:



2.4 Sequences and Summations

- A ***sequence*** or *series* is just like an ordered *n*-tuple, except:
  - Each element in the sequence has an associated *index* number.
  - A sequence or series may be infinite.
- A ***summation*** is a compact notation for the sum of the terms in a (possibly infinite) sequence.

Sequences

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- A ***sequence*** or ***series*** $\{a_n\}$ is identified with a ***generating function*** $f : I \rightarrow S$ for some subset $I \subseteq \mathbf{N}$ and for some set S.
  - Often we have $I = \mathbf{N}$ or $I = \mathbf{Z}^+ = \mathbf{N} - \{0\}$.
- If f is a generating function for a sequence $\{a_n\}$, then for $n \in I$, the symbol a_n denotes $f(n)$, also called ***term*** n of the sequence.
  - The ***index*** of a_n is n. (Or, often i is used.)
- A sequence is sometimes denoted by listing its first and/or last few elements, and using ellipsis (…) notation.
  - *E.g.*, "$\{a_n\} = 0, 1, 4, 9, 16, 25, \ldots$" is taken to mean $\forall n \in \mathbf{N}, a_n = n^2$.

Sequence Examples

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- Some authors write "the sequence $a_1, a_2, \ldots$" instead of $\{a_n\}$, to ensure that the set of indices is clear.
  - Be careful: Our book often leaves the indices ambiguous.
- An example of an infinite sequence:
  - Consider the sequence $\{a_n\} = a_1, a_2, \ldots,$ where $(\forall n \geq 1)\ a_n = f(n) = 1/n$.
  - Then, we have $\{a_n\} = 1, 1/2, 1/3, \ldots$
    - Called "harmonic series"

Example with Repetitions

- Like tuples, but unlike sets, a sequence may contain *repeated* instances of an element.
- Consider the sequence $\{b_n\} = b_0, b_1, \ldots$ (note that 0 is an index) where $b_n = (-1)^n$.
  - Thus, $\{b_n\} = 1, -1, 1, -1, \ldots$
    - Note repetitions!
  - This $\{b_n\}$ denotes an infinite sequence of 1's and -1's, *not* the 2-element set {1, -1}.

Geometric Progression

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- A geometric progression is a sequence of the form

  $$a, ar, ar^2, \ldots, ar^n, \ldots$$

  where the ***initial term a*** and the ***common ratio r*** are real numbers.
- A geometric progression is a discrete analogue of the exponential function $f(x) = ar^x$
- Examples

  Assuming $n = 0, 1, 2, \ldots$

  - $\{b_n\}$ with $b_n = (-1)^n$ — initial term 1, common ratio –1
  - $\{c_n\}$ with $c_n = 2 \cdot 5^n$ — initial term 2, common ratio 5
  - $\{d_n\}$ with $d_n = 6 \cdot (1/3)^n$ — initial term 6, common ratio 1/3

Arithmetic Progression

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- An arithmetic progression is a sequence of the form

  $$a, a+d, a+2d, \ldots, a+nd, \ldots$$

  where the ***initial term*** a and the ***common difference*** d are real numbers.
- An arithmetic progression is a discrete analogue of the linear function $f(x) = a + dx$
- Examples

  Assuming $n = 0, 1, 2, \ldots$
  - $\{s_n\}$ with $s_n = -1 + 4n$ initial term -1, common diff. 4
  - $\{t_n\}$ with $t_n = 7 - 3n$ initial term 7, common diff. -3

Recognizing Sequences (I)

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- Sometimes, you're given the first few terms of a sequence,
  - and you are asked to find the sequence's generating function,
  - or a procedure to enumerate the sequence.
- Examples: What's the next number?
  - 1, 2, 3, 4,… 5 (the 5th smallest number > 0)
  - 1, 3, 5, 7, 9,… 11 (the 6th smallest odd number > 0)
  - 2, 3, 5, 7, 11,... 13 (the 6th smallest prime number)

Recognizing Sequences (II)

- General problems
  - Given a sequence, find a formula or a general rule that produced it

- <u>Examples</u>: How can we produce the terms of a sequence if the first 10 terms are
  - 1, 2, 2, 3, 3, 3, 4, 4, 4, 4?

    Possible match: next five terms would all be 5, the following six terms would all be 6, and so on.

  - 5, 11, 17, 23, 29, 35, 41, 47, 53, 59?

    Possible match: nth term is $5 + 6(n - 1) = 6n - 1$

    (assuming $n = 1, 2, 3, \ldots$)

Special Integer Sequences

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- A useful technique for finding a rule for generating the terms of a sequence is to compare the terms of a sequence of interest with the terms of a well-known integer sequences (e.g. arithmetic/geometric progressions, perfect squares, perfect cubes, etc.)

TABLE 1 **Some Useful Sequences.**

nth Term	*First 10 Terms*
n^2	1, 4, 9, 16, 25, 36, 49, 64, 81, 100, . . .
n^3	1, 8, 27, 64, 125, 216, 343, 512, 729, 1000, . . .
n^4	1, 16, 81, 256, 625, 1296, 2401, 4096, 6561, 10000, . . .
2^n	2, 4, 8, 16, 32, 64, 128, 256, 512, 1024, . . .
3^n	3, 9, 27, 81, 243, 729, 2187, 6561, 19683, 59049, . . .
$n!$	1, 2, 6, 24, 120, 720, 5040, 40320, 362880, 3628800, . . .

Coding: Fibbonaci Series

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- Series $\{a_n\} = \{1, 1, 2, 3, 5, 8, 13, 21, \ldots\}$
- Generating function (recursive definition!):
  - $a_0 = a_1 = 1$ and
  - $a_n = a_{n-1} + a_{n-2}$ for all $n > 1$
- Now let's find the entire series $\{a_n\}$:

```
int [] a = new int [n];
a[0] = 1;
a[1] = 1;
for (int i = 2; i < n; i++) {
  a[i] = a[i-1] + a[i-2];
}
return a;
```

Coding: Factorial Series

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- Factorial series $\{a_n\} = \{1, 2, 6, 24, 120, \ldots\}$

- Generating function:
  - $a_n = n! = 1 \times 2 \times 3 \times \ldots \times n$

- This time, let's just find the term a_n:
  - ```
    int an = 1;
    for (int i = 1; i <= n; i++) {
      an = an * i;
    }
    return an;
    ```
    ```