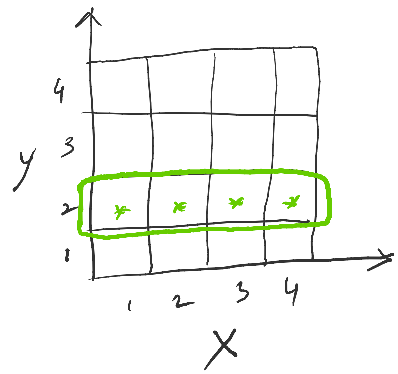
# Conditional Expectation and Independence

Tuesday, February 9, 2021 8:56 AM

Defi. $E[X/A] = \sum_x x \cdot p_{X/A}(x)$

$E[g(X)/A] = \sum_x g(x) \cdot p_{X/A}(x)$




Let $X$ and $Y$ be two r.v.s $A \equiv \{Y = y\}$

$$\underline{E[X|Y=y] = \sum_x x \cdot p_{X|Y}(x|y)}.$$

$Y = 2$

$$p_X(x) = \sum_y p_Y(y) \cdot p_{X|Y}(x|y)$$

$$\sum_x x\, p_X(x) = \sum_x x \cdot \underbrace{\sum_y p_Y(y)} \cdot p_{X|Y}(x|y)$$

$\Downarrow$

$$E[X] = \sum_y p_Y(y) \cdot \underbrace{\sum_x x \cdot p_{X|Y}(x|y)}$$

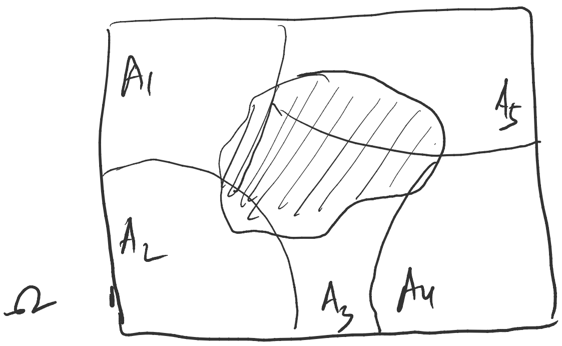
$$\boxed{E[X] = \sum_y p_Y(y) \cdot E[X|Y=y]}$$

Total expection th.

$A_i \equiv \{Y = y_i\}$ $Y \in \{y_1, y_2 \cdots y_n\}$

$$E[X] = \sum^n P(A_i) \cdot E[X|A_i]$$

$$\boxed{E[X] = \sum_{i=1}^{n} \underline{P(A_i)} \cdot \underline{E[X \mid A_i]}}$$

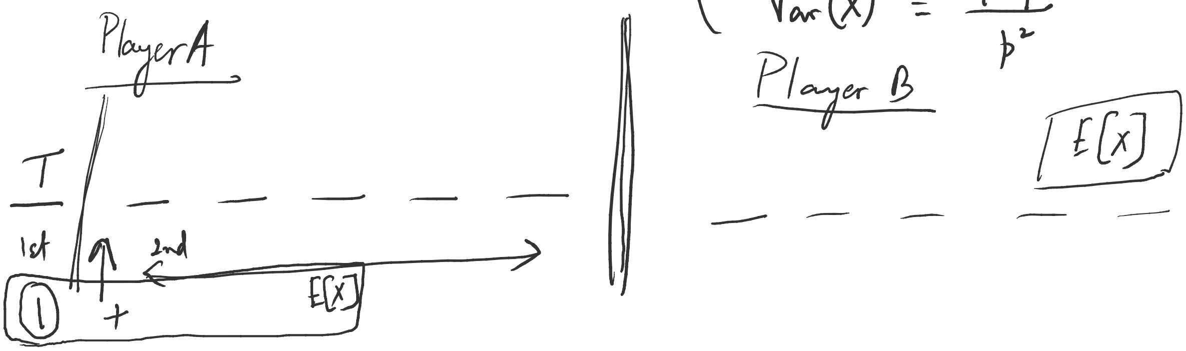



Expectation of Geometric r.v. $(X)$ and Variance

$$p_X(k) = (1-p)^{k-1} p \qquad k = 1, 2, 3, \ldots$$

$$E[X] = \frac{1}{p}$$

$$Var(X) = \frac{1-p}{p^2}$$




Case I Given 1st toss is H

$$E[X] = 1$$

$$\boxed{E[X \mid X=1] = 1}$$

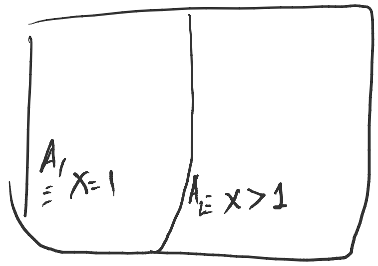
Case II Given 1st toss is T

$$E[X] = 1 + E(X)$$

$$\boxed{E[X \mid X > 1] = 1 + E[X]}$$

$$E[1+X] = 1 + E(X)$$

$E[1+X] = 1 + E[X]$




$$E[X] = P(A_1) \cdot E[X \mid A_1] + P(A_2) \cdot E[X \mid A_2]$$

$$= p \cdot 1 + (1-p) \cdot (1 + E[X])$$

$$\Rightarrow \boxed{E[X] = \frac{1}{p}}$$

$$\mathrm{Var}(X) = \frac{1-p}{p^2} \qquad E[X^2 \mid X=1]$$

$$E[X^2 \mid X>1] = E[(1+X)^2]$$

(Ex)

## Independence of random variables

A random variable $X$ is ind of $A$ if

$$\boxed{P(\{X=x\} \text{ and } A) = P(X=x) \cdot P(A)} \quad \text{for all } x.$$

$$P(\{X=x\} \text{ and } A) = P(A) \cdot p_{X|A}(x) \quad \text{for all } x.$$

$$\boxed{p_{X|A}(x) = p_X(x)}$$

$X$ and $Y$ are independent if

Def: $\boxed{p_{X,Y}(x,y) = p_X(x) \cdot p_Y(y)}$ for all $x, y$

$p_{X,Y}(x,y) = p_Y(y) \cdot p_{X|Y}(x|y)$ for all $x, y$

$$\boxed{p_{X|Y}(x|y) = p_X(x)}$$

$X$ and $Y$ are ind.

$$E[XY] = E[X] \cdot E[Y]$$

$$= \sum_x \sum_y xy \, \underline{p_{X,Y}(x,y)}$$

$$= \sum_x \sum_y xy \, p_X(x) \cdot p_Y(y)$$

$$= \sum_x x \, p_X(x) \cdot \sum_y y \, p_Y(y)$$

$$= E[X] \cdot E[Y]$$

$$E[g(X) \cdot h(Y)] = E[g(X)] \cdot E[h(Y)]$$

$Z = X + Y$ where $X$ and $Y$ are ind

$$Var(Z) = E\left[ (X + Y - E[X+Y])^2 \right]$$

$$= E\left[\left\{(X - E[X]) + (Y - E[Y])\right\}^2\right]$$

$$= E\left[(X - E[X])^2\right] + E\left[(Y - E[Y])^2\right]$$
$$+ 2\,\underline{E\left[(X - E[X]).(Y - E[Y])\right]}$$

$$= Var(X) + Var(Y)$$

$$E\left[(X - E[X]).(Y - E[Y])\right]$$

$$= E\left[X - E[X]\right] . E\left[Y - E[Y]\right]$$

$$= 0$$

$$\boxed{Var(X+Y) = Var(X) + Var(Y)}$$ when X and Y ind

$$Var(X_1 + X_2 + X_3 + \cdots + X_n) = Var(X_1) + Var(X_2) + \cdots + Var(X_n)$$

$X_i$'s are ind.

<u>Variance of Binomial</u> $X(n, p)$

$$X = X_1 + X_2 + \cdots + X_n$$

$$Var(X) = n.\, Var(X_i) = np(1-p)$$

<u>Variance of Neg. Binomial</u>

$$X = X_1 + X_2 + \cdots + X_r$$

$$Var(X) = r.\, Var(X_i) = \frac{r.(1-p)}{\phantom{p}}$$

$$\mathrm{Var}(X) = r \cdot \mathrm{Var}(X_i) = \frac{r \cdot (1-p)}{p^2}$$