Key word to learn — REF/RREF → Row Reduced Echelon form (1)

staircase pattern

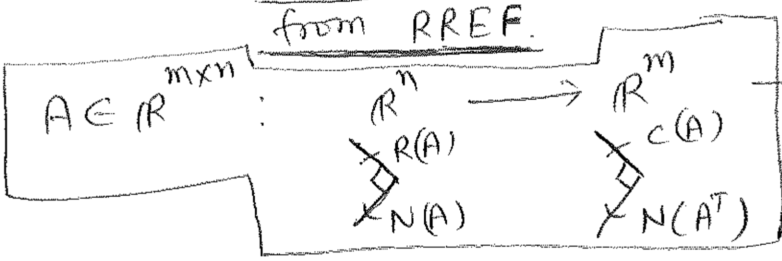
Any matrix $A \in F^{m\times n}$ can be reduced in the following form:—
(by applying elementary row operations of Type I, II, and III)

1. The pivots are the first nonzero entries in ~~their~~ nonzero rows
2. Below each pivot, all column entries are zero
3. Each pivot lies to the right of the pivot in the row above
4. All zero rows are ~~at the bottom~~ below the nonzero rows of matrix
5. All pivots are one
6. Above each pivot, all column entries are zero

A form that has properties 1–4 is called REF of A.
———————————— 1–6 ———————— RREF of A.

Important Points

(1) Remember — How we can find basis of $C(A)$, $N(A)$, $R(A)$, and $N(A^T)$ from RREF.



→ This picture IS important

→ $\dim R(A) + \dim N(A) = n$

→ $\dim C(A) + \dim N(A^T) = m$

(2) Two spaces $X_1 \perp X_2 \Leftrightarrow x_1 \perp x_2$ for any $x_1 \in X_1$ and $x_2 \in X_2$, i.e. $x_1^T x_2 = 0$ dot product

(3) Remember — $\text{Rank}(A) = \#$ pivots in REF/RREF.

In School, we have read:—

A number r is said to be the rank of $A \in F^{m\times n}$ if

(i) there is at least one square submatrix of A of order r whose determinant is **not** equal to ~~zero~~ 0 (zero).

(II) If A contains any square submatrix of order $r+1$, then its determinant is zero.

(4) $r = \dim C(A) = \dim R(A)$

(5) rank + nullity $= n$ (always) → Rank-Nullity Theorem

Here nullity $= \dim N(A)$.

Ques Find RREF and rank of following matrices

(i) $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ (ii) $\begin{bmatrix} 1 & 2 & 3 & 0 \\ 2 & 4 & 3 & 2 \\ 3 & 2 & 1 & 3 \\ 6 & 8 & 7 & 5 \end{bmatrix}$

Solution

$$\begin{bmatrix} \boxed{1} & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & -1 \\ 0 & -1 & -1 \end{bmatrix} \sim \begin{bmatrix} \boxed{1} & 1 & 1 \\ 0 & \boxed{-1} & -1 \\ 0 & 0 & \boxed{-1} \end{bmatrix} = \text{REF}$$

rank = 3

$$\sim \begin{bmatrix} \boxed{1} & 1 & 1 \\ 0 & \boxed{1} & 1 \\ 0 & 0 & \boxed{1} \end{bmatrix} \sim \begin{array}{c} \begin{bmatrix} \boxed{1} & 1 & 0 \\ 0 & \boxed{1} & 0 \\ 0 & 0 & \boxed{1} \end{bmatrix} \end{array} \begin{array}{l} r_1 - r_3 \\ r_2 - r_3 \\ r_3 \end{array} \sim$$

$$\text{RREF} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{array}{l} r_1 - r_2 \\ r_2 \\ r_3 \end{array}$$

(ii)

$$\begin{bmatrix} \boxed{1} & 2 & 3 & 0 \\ 2 & 4 & 3 & 2 \\ 3 & 2 & 1 & 3 \\ 6 & 8 & 7 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 0 & -3 & 2 \\ 0 & -4 & -8 & 3 \\ 0 & -4 & -11 & 5 \end{bmatrix} \sim \begin{bmatrix} \boxed{1} & 2 & 3 & 0 \\ 0 & \boxed{-4} & -8 & 3 \\ 0 & 0 & -3 & 2 \\ 0 & -4 & -11 & 5 \end{bmatrix}$$

$$\sim \begin{bmatrix} \boxed{1} & 2 & 3 & 0 \\ 0 & \boxed{-4} & -8 & 3 \\ 0 & 0 & \boxed{-3} & 2 \\ 0 & 0 & -3 & 2 \end{bmatrix} \begin{array}{l} r_1 \\ r_2 \\ r_3 - 0r_2 \\ r_4 - r_2 \end{array}$$

REF
rank = 3

$$= \begin{bmatrix} \boxed{1} & 2 & 3 & 0 \\ 0 & \boxed{-4} & -8 & 3 \\ 0 & 0 & \boxed{-3} & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} \boxed{1} & 2 & 3 & 0 \\ 0 & \boxed{1} & 2 & -3/4 \\ 0 & 0 & \boxed{1} & -2/3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} \boxed{1} & 2 & 0 & 2 \\ 0 & \boxed{1} & 0 & 7/12 \\ 0 & 0 & \boxed{1} & -2/3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\cancel{-\frac{3}{4}+\frac{4}{3}=\frac{7}{12}}$

$2-\frac{7}{6}$

$$\sim \begin{bmatrix} \boxed{1} & 0 & 0 & 5/6 \\ 0 & \boxed{1} & 0 & 7/12 \\ 0 & 0 & \boxed{1} & -2/3 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \text{RREF}$$

Ques Find RREF and basis for all four fundamental subspaces

(i) $\begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 2 \end{bmatrix}$ (ii) $\begin{bmatrix} 1 & 2 & 2 \\ 1 & 3 & 4 \\ 1 & 2 & 3 \end{bmatrix}$ (iii) $\begin{bmatrix} 1 & 2 \\ 1 & 3 \\ 1 & 2 \end{bmatrix}$ (iv) $\begin{bmatrix} 1 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 1 \\ 2 & 5 & 2 \end{bmatrix}$

Solution (iv)

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 1 \\ 2 & 5 & 2 \end{bmatrix} \sim \begin{bmatrix} \boxed{1} & 2 & 1 \\ 0 & \boxed{1} & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} \boxed{1} & 2 & 1 \\ 0 & \boxed{1} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} \boxed{1} & 0 & 1 \\ 0 & \boxed{1} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = RREF$$

Row space basis $= \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$ Remember! pivot rows in RREF

Column space basis $= \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 2 \\ 5 \end{bmatrix} \right\}$ Remember! pivot columns in original A.

Seperate pivot subsystem and free subsystem

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = -\begin{bmatrix} 1 \\ 0 \end{bmatrix} x_3 \xrightarrow[\text{put } x_3 = 1]{\text{Solve}} \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

Hence basis for $N(A) = \left\{ \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$

To find basis of $N(A^T)$ capture the matrix B s.t. $BA = RREF$

and basis for $N(A^T) =$ rows in B corresponding to zero rows in RREF

We know

$$E_3 E_2 E_1 A = RREF$$

Hence
$$B = E_3 E_2 E_1 = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ -2 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ -2 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ -1 & -1 & 0 & 1 \end{bmatrix}$$

Hence Basis for $N(A^T) = \left\{ \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ 0 \\ 1 \end{bmatrix} \right\}$

You can capture this matrix B by finding RREF of A and by augmenting A with I

$$[A \mid I] \longrightarrow [RREF(A) \mid B]$$

when $BA = RREF(A)$.

Solution of $Ax=b$

Here $A \equiv m\times n$ matrix $\in \mathbb{R}^{m\times n}$
$b \in \mathbb{R}^m$
$x \in \mathbb{R}^n$ (unknown vector)

$Ax=b$ has a solution $\Leftrightarrow b \in C(A)$
$\Leftrightarrow \text{Rank}(A) = \text{Rank}(A|b)$

No solution when $\text{Rank}(A) \neq \text{Rank}(A|b)$

Solution of $Ax=b$, if exists, is unique $\Leftrightarrow N(A) = \{0\}$ only.
$\Leftrightarrow \text{Rank}(A) = n$
$= \#$ columns in A.

Hence solution exists uniquely $\Leftrightarrow \text{Rank}(A) = \text{Rank}(A|b) = n$

Remember
- $\text{Rank}(A) = m = \#$ rows in $A \Rightarrow$ Solution exists
- If $n > m$, then (if solution exists) there are ∞-many solutions always.

→ Because If $A \equiv m\times n$, then $\text{rank}(A) \leq \min.\{m,n\}$.

If A^{-1} exists then following statements equivalent:
- A^{-1} exists OR A is invertible OR A is nonsingular
- $\det(A) \neq 0$
- $Ax=0$ has only 0 solution
- $Ax=b$ has unique solution, i.e. $x = A^{-1}b$
  ↳ [But Remember - never compute inverse - use LU/PLU/RREF for solution
- $\text{rank}(A) = n$ and nullity $= 0$
- $\text{Col}(A) = \mathbb{R}^n$ and $N(A) = \{0\}$
- pivot $= n$ all rows and cols are LI.

→ Remember! • If $A \equiv m\times n$ and $n > m$ then always there is a nontrivial null space $N(A)$.
- $C(AB) \subseteq C(A)$
- $R(AB) \subseteq R(B)$

→ easy to learn, if you know matrix-matrix product

$Ax=b$ → If ∞-many solutions — then solutions are

general solution: $\boxed{x = x_p + x_n}$ (x_p: particular solution
$x_n \in N(A)$, i.e. null space of A)

→ Imp. Always remember - how we compute x_p and Null space.

Lecture-picture

- If A is $m \times n$ matrix, then $\text{Rank}(A) \le \min\{m, n\}$ $\dim C(A) = \dim R(A) = \#\text{pivots}$
- ____________, and $m < n$ then $N(A)$ is always nontrivial. (because $\dim N(A) = n - r$)
- $C(AB) \subseteq C(A) \Rightarrow \text{rank}(AB) \le \text{rank}(A)$
  &
  $R(AB) \subseteq R(B) \Rightarrow \text{rank}(AB) \le \text{rank}(B)$
  $\Rightarrow \text{rank}(AB) \le \min\{\text{rank}(A), \text{rank}(B)\}$

- $Ax = b$, $A \in \mathbb{R}^{m \times n}$, $x \in \mathbb{R}^n$, $b \in \mathbb{R}^m$

Solution exists $\Leftrightarrow \text{Rank}(A) = \text{Rank}(A|b)$
$\Leftrightarrow b \in C(A)$

Solution (if exists) is unique $\Leftrightarrow N(A) = \{0\}$ only

Solution not unique $\Leftrightarrow N(A) \neq \{0\}$

proof:- ($\Leftarrow$ part) Let x_p be a sol. i.e. $Ax_p = b$; Let $x_n \in N(A)$, i.e. $Ax_n = 0$ $\Rightarrow A(x_p + x_n) = b$ $\Downarrow$ $x_p + x_n$ is a sol.

($\Rightarrow$ part) Let $Ax = b$ has two solutions, say x_1 and x_2, then $x_1 - x_2 \in N(A)$.

Hence general solution $x = x_p + x_n$ (If exists)
- x_p → particular solution
- x_n → vector in null space

- No Solution $\Leftrightarrow \text{Rank}(A) \neq \text{Rank}(A|b)$
  $\Leftrightarrow b \notin C(A)$.

- Solution exists uniquely $\Leftrightarrow$
  $R(A) = \text{Rank}(A|b) = n$
- ∞ many solutions $\Leftrightarrow R(A) = \text{Ran}(A|b) < n$

- Suppose $Ax = b$ solution exists then
  $\#$ LI Solutions (If $b = 0$) $= n - r$ (homogeneous system)
  $\#$ LI Solutions (If $b \neq 0$) $= n - r + 1$ (Non-homogeneous System).

Ques
Obtain for what values of λ and μ the following system has (i) no solution (ii) a unique solution (iii) infinitely many solutions

$$\begin{aligned} x+y+z &= 6 \\ x+2y+3z &= 10 \\ x+2y+\lambda z &= \mu \end{aligned} \quad \equiv \quad AX = b \quad A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 2 & \lambda \end{bmatrix} \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad b = \begin{bmatrix} 6 \\ 10 \\ \mu \end{bmatrix}$$

$$[A|b] = \left[\begin{array}{ccc|c} \boxed{1} & 1 & 1 & 6 \\ 1 & 2 & 3 & 10 \\ 1 & 2 & \lambda & \mu \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 1 & \lambda-1 & \mu-6 \end{array}\right] \sim \left[\begin{array}{ccc|c} \boxed{1} & 1 & 1 & 6 \\ 0 & \boxed{1} & 2 & 4 \\ 0 & 0 & \boxed{\lambda-3} & \mu-10 \end{array}\right] = \underline{REF}$$

(i) Recall! system has no solution Rank $(A) \neq$ Rank $(A\,b)$
i.e. $b \notin C(A)$

This situation will arise if $\boxed{\lambda = 3 \text{ and } \mu \neq 10}$

(ii) Recall! system has unique solution Rank(A) = Rank $(A|b)$ = # columns = 3 (here)

OR

$b \in C(A)$ and $N(A) = \{0\}$.

Hence it will be when $\lambda \neq 3$ (otherwise there will be no 3rd pivot & Rank $(A) < 3$).

Here No constraint on μ.

(iii) Recall! system has ∞-many solution if Rank (A) = Rank $(A|b) < n$

OR

$\underline{b \in C(A) \text{ and } N(A) \text{ has nonzero element}}$.

for this $\boxed{\lambda = 3 \text{ and } \mu = 10}$

Ques

Obtain for what values of λ the follog equations have (i) no solution (II) a unique sol. (III) ∞-many solutions.

$$x + y - z = 1$$
$$2x + 3y + \lambda z = 3$$
$$x + \lambda y + 3z = 2$$

$$(A|b) = \left[\begin{array}{ccc|c} \boxed{1} & 1 & -1 & 1 \\ 2 & 3 & \lambda & 3 \\ 1 & \lambda & 3 & 2 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & -1 & 1 \\ 0 & 1 & \lambda+2 & 1 \\ 0 & \lambda-1 & 4 & 1 \end{array}\right] \sim \cancel{\left[\begin{array}{ccc|c} 1 & 1 & -1 & 1 \\ 0 & 1 & \lambda+2 & 1 \\ 0 & & & \end{array}\right]}$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & -1 & 1 \\ 0 & 1 & \lambda+2 & 1 \\ 0 & 0 & 4-(\lambda-1)\cdot(\lambda+2) & 1-(\lambda-1) \end{array}\right] = \left[\begin{array}{ccc|c} 1 & 1 & -1 & 1 \\ 0 & 1 & \lambda+2 & 1 \\ 0 & 0 & -\lambda^2-\lambda+6 & -\lambda+2 \end{array}\right]$$

$\cancel{\lambda^2-\lambda+2\lambda-2}$

$\cancel{4-\lambda^2-\lambda+2}$

$\cancel{(\lambda^2+\lambda-6)}$

$\cancel{(\lambda+3)(\lambda-2)}$

$$\parallel$$

$$\left[\begin{array}{ccc|c} \boxed{1} & 1 & -1 & 1 \\ 0 & \boxed{1} & \lambda+2 & 1 \\ 0 & 0 & -(\lambda-2)(\lambda+3) & -(\lambda-2) \end{array}\right]$$

(i) <u>No solution</u> $\quad \lambda = -3 \quad$ then $\underset{2}{\underset{\parallel}{\text{Rank}(A)}} \neq \underset{3}{\underset{\parallel}{\text{Rank}[A|b]}}$

(II) <u>∞-many solutions</u> $\quad \lambda = 2 \quad$ then Rank $(A) =$ Rank $[A|b] = 2 < 3 =$ # cols.

(III) <u>Unique solution</u> $\quad \lambda \neq 2,\ \lambda \neq -3$ then Rank $(A) =$ Rank $(A|b) = 3 =$ # cols.

Ques Solve following systems by Gauss-elimination

(i)
$$x + y + z = 4$$
$$2x + 5y - 2z = 3$$
$$x + 7y - 7z = 5$$

Solution given: $Ax = b$ $\quad A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 5 & -2 \\ 1 & 7 & -7 \end{bmatrix}$ $\quad x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ $\quad b = \begin{bmatrix} 4 \\ 3 \\ 5 \end{bmatrix}$

$$[A|b] = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 2 & 5 & -2 & 3 \\ 1 & 7 & -7 & 5 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 0 & 3 & -4 & -5 \\ 0 & 6 & -8 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 0 & 3 & -4 & -5 \\ 0 & 0 & 0 & 11 \end{array}\right]$$

$\text{Rank}(A) = 2 \qquad \text{Rank}(A|b) = 3$

$\Downarrow$

$b \notin C(A)$ = col space of A

Hence no solution exists.

(II)
$$\left.\begin{array}{r} x + 2y + z = 2 \\ 3x + y - 2z = 1 \\ 4x - 3y - z = 3 \\ 2x + 4y + 2z = 4 \end{array}\right] \equiv \; AX = b \quad A = \begin{bmatrix} 1 & 2 & 1 \\ 3 & 1 & -2 \\ 4 & -3 & -1 \\ 2 & 4 & 2 \end{bmatrix} \quad X = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \quad b = \begin{pmatrix} 2 \\ 1 \\ 3 \\ 4 \end{pmatrix}$$

$$A = \left[\begin{array}{ccc|c} \boxed{1} & 2 & 1 & 2 \\ 3 & 1 & -2 & 1 \\ 4 & -3 & -1 & 3 \\ 2 & 4 & 2 & 4 \end{array}\right] \sim \left[\begin{array}{ccc|c} \boxed{1} & 2 & 1 & 2 \\ 0 & \boxed{-5} & -5 & -5 \\ 0 & -11 & -5 & -5 \\ 0 & 0 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} \boxed{1} & 2 & 1 & 2 \\ 0 & \boxed{-5} & -5 & -5 \\ 0 & 0 & \boxed{6} & 6 \\ 0 & 0 & 0 & 0 \end{array}\right] = \text{REF}$$

~~R(A)=P~~

$\text{Rank}(A) = \text{Rank}(A|b) = n$

Solution exists uniquely.

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & 1 & 2 \\ 0 & \boxed{1} & 1 & 1 \\ 0 & 0 & \boxed{1} & 1 \\ 0 & 0 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 2 & 0 & 1 \\ 0 & \boxed{1} & 0 & 0 \\ 0 & 0 & \boxed{1} & 1 \\ 0 & 0 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} \boxed{1} & 0 & 0 & 1 \\ 0 & \boxed{1} & 0 & 0 \\ 0 & 0 & \boxed{1} & 1 \\ 0 & 0 & 0 & 0 \end{array}\right] = \text{RREF}$$

Solution is $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

$\Downarrow$

Even write solution from REF directly because here $N(A) = \{0\}$, i.e. solution is unique.

$\Downarrow$

Apply backward substitution here

Computational Remarks

- Let $\{x_1, x_2, \cdots, x_n\}$ be n- vectors in a VS $(V, +, \cdot)$.

How can we check whether these n- vectors are LI OR LD.

Make a matrix A whose columns are $x_1, x_2 \cdots, x_n$

i.e.

$$A = \left[\begin{array}{c|c|c|c|c} x_1 & x_2 & x_3 & \cdots & x_n \\ \downarrow & \downarrow & \downarrow & & \downarrow \end{array} \right]$$

All vectors are LI $\iff$ Rank$(A) = n$

$\iff N(A) = \{0\}$ only $\iff$ nullity $= 0$

otherwise If Rank$(A) < n$, vectors are LD.

- Let $\{x_1, x_2, \cdots, x_n\}$ be n- vectors in a VS $(V, +, \cdot)$

Then following problems are same –

(i) Is y a linear combination of x_i's.

(II) Find $\alpha_1, \alpha_2, \cdots, \alpha_n$ s.t. $y = \alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n$

(III) Solve $Ax = b$ where $A = \left[\begin{array}{c|c|c|c} x_1 & x_2 & \cdots & x_n \\ \downarrow & \downarrow & & \downarrow \end{array} \right]$

$b = y$ and $x = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{bmatrix}$

(iv) check whether $y \in L[S]$, where $S = \{x_1, x_2, \cdots, x_n\}$.

Thus If you want answer to any of above, always convert the problem into (III)rd form & then solve.

- Let $S = \{x_1, x_2, \cdots, x_n\} \subset V$. Then following problems are exactly same

(i) S generates/spans V

(II) $Ax = b$ has a solution for any $b \in V$, i.e. we have to show that any $b \in V$ is also in $C(A)$.

Lecture picture

Q1 Are vectors $x_1, x_2, \ldots, x_n$ LI / LD $\equiv$ S is LI/LD

Q2 Is y a lc of vectors $x_1, x_2, \ldots x_n$ $\equiv$ Is $y \in L(S)$ where set $S = \{x_1, \ldots x_n\}$

Q3 Is set $\{x_1, x_2, \ldots, x_n\}$ a basis of vector space V.

Q4 How can we convert any subspace from mathematical form to computational form and visa versa.

To give answer we form a matrix

$$A = \left[\begin{array}{c|c|c|c} x_1 & x_2 & \cdots & x_n \\ \downarrow & \downarrow & & \downarrow \end{array} \right]$$

- vectors $x_1, x_2, \ldots, x_n$ are LI $\iff$ Rank $(A) = n$
  $\iff N(A) = \{0\}$ only
  $\iff$ nullity $= 0$

- y is a lc of vectors $x_1, x_2, \ldots x_n \iff \exists$ scalars $\alpha_1, \alpha_2, \ldots \alpha_n \in F$ s.t. $\alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n = y$
  $\iff$ linear system $Az = y$ has a solution $z = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{bmatrix}$.
  $\iff$ Rank$(A) =$ Rank $[A|y]$

- Sets $\{x_1, x_2, \ldots, x_n\}$ is a basis of vector space V
  $\Updownarrow$
  (i) S is LI $\iff$ Rank $(A) = n$
  (ii) S spans V $\iff$ Rank$(A) =$ Rank $[A|X]$ where X is any arbitary vector of V.

Rank$(A) = n =$ Rank $[A|X]$

Ques Check LI/LD

(i) $\left\{ \begin{bmatrix}1\\1\\1\\0\end{bmatrix}, \begin{bmatrix}3\\2\\2\\1\end{bmatrix}, \begin{bmatrix}1\\1\\3\\-2\end{bmatrix}, \begin{bmatrix}1\\2\\6\\-5\end{bmatrix} \right\}$ in $\mathbb{R}^4$

$$A = \begin{bmatrix} \boxed{1} & 3 & 1 & 1 \\ 1 & 2 & 1 & 2 \\ 1 & 2 & 3 & 6 \\ 0 & 1 & -2 & -5 \end{bmatrix} \sim \begin{bmatrix} \boxed{1} & 3 & 1 & 1 \\ 0 & \boxed{-1} & 0 & 1 \\ 0 & -1 & 2 & 5 \\ 0 & 1 & -2 & -5 \end{bmatrix} \sim \begin{bmatrix} \boxed{1} & 3 & 1 & 1 \\ 0 & \boxed{-1} & 0 & 1 \\ 0 & 0 & \boxed{2} & 4 \\ 0 & 0 & -2 & -4 \end{bmatrix} \sim \begin{bmatrix} \boxed{1} & 3 & 1 & 1 \\ 0 & \boxed{-1} & 0 & 1 \\ 0 & 0 & \boxed{2} & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Hence rank$(A) = 3 \neq 4$ hence given 4 vectors are LD.

(ii) $\{x,\ x^3 - x,\ x^4 + x^2,\ x + x^2 + x^4 + \frac{1}{2}\}$ in P_4

$$A = \begin{bmatrix} 0 & 0 & 0 & 1/2 \\ 1 & -1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} \boxed{1} & -1 & 0 & 1 \\ 0 & 0 & 0 & 1/2 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} \boxed{1} & -1 & 0 & 1 \\ 0 & \boxed{1} & 0 & 0 \\ 0 & 0 & \boxed{1} & 1 \\ 0 & 0 & 0 & 1/2 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} \boxed{1} & -1 & 0 & 1 \\ 0 & \boxed{1} & 0 & 0 \\ 0 & 0 & \boxed{1} & 1 \\ 0 & 0 & 0 & \boxed{1/2} \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

rank = 4 = # vectors
Hence given vectors in P_4 are LI.

Learn! In case of P_n how we write any vector in P_n as a column of <u>matrix</u>.

(iii) $\left\{ \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 2 & 2 \\ 0 & 2 \end{bmatrix} \right\}$ in $\mathbb{R}^{2\times 2}$

$$A = \begin{pmatrix} \boxed{1} & 1 & 2 \\ 1 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 1 & 2 \end{pmatrix} \sim \begin{bmatrix} \boxed{1} & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 2 \end{bmatrix} \sim \begin{bmatrix} \boxed{1} & 1 & 2 \\ 0 & \boxed{1} & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Rank $(A) = 2 \neq 3$. Hence given set is LD.

<u>Learn!</u> how we write vectors as column of A in case vector space is $\mathbb{R}^{m\times n}$.

Topic to learn here:

How we check LI/LD in $C[a,b]$

↳ space of functions.

Method 1: (by definition)

n functions, $f_1, f_2, \dots, f_n$, are LI iff

$(\alpha_1 f_1 + \alpha_2 f_2 + \alpha_3 f_3 + \dots + \alpha_n f_n)(x) = 0 \ \forall x \in [a,b]$

$\Rightarrow \alpha_i$'s are 0

Otherwise, f_i's are LD iff $\exists \ \alpha_1, \alpha_2, \dots, \alpha_n$ (not all zero simultaneously) s.t.

$\alpha_1 f_1(x) + \alpha_2 f_2(x) + \dots + \alpha_n f_n(x) = 0 \ \forall x \in [a,b]$

Method 2: (by Wronskian)

Definition Wronskian of $f_1, f_2, \dots f_n$ at any point $x \in [a,b]$ is

determinant
$$\begin{vmatrix} f_1(x) & f_2(x) & \dots & f_n(x) \\ f_1'(x) & f_2'(x) & \dots & f_n'(x) \\ f_1''(x) & f_2''(x) & \dots & f_n''(x) \\ f_1^{(n-1)}(x) & f_2^{(n-1)}(x) & \dots & f_n^{(n-1)}(x) \end{vmatrix}.$$

$\boxed{\text{Result}}$ → If Wronskian of n functions, $f_1, f_2, \dots f_n$, is non-zero for at least one point of $[a,b]$, then functions are LI on $[a,b]$.

Proof: (in case of 2-functions)

To show $\begin{vmatrix} f_1(x) & f_2(x) \\ f_1'(x) & f_2'(x) \end{vmatrix} \neq 0$ for at least one $x \in [a,b] \Rightarrow f_1, f_2$ are LI on $[a,b]$

Let $\begin{vmatrix} f_1(x_0) & f_2(x_0) \\ f_1'(x_0) & f_2'(x_0) \end{vmatrix} \neq 0$ —(1) Assume f_1 and f_2 are LD. Therefore there exist c_1 and c_2 (not both zero) s.t.

$(c_1 f_1 + c_2 f_2)(x) = 0 \ \forall x \Rightarrow$ $\left.\begin{array}{l} c_1 f_1(x) + c_2 f_2(x) = 0 \\ c_1 f_1'(x) + c_2 f_2'(x) = 0 \end{array}\right] \forall x \in [a,b].$

$$\Rightarrow \quad \begin{array}{l} c_1 f_1(x_0) + c_2 f_2(x_0) = 0 \\ c_1 f_1'(x_0) + c_2 f_2'(x_0) = 0 \end{array} \Rightarrow \begin{bmatrix} f_1(x_0) & f_2(x_0) \\ f_1'(x_0) & f_2'(x_0) \end{bmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Downarrow$$

$$\left.\begin{array}{l} c_1 = 0 \\ \& \ c_2 = 0 \end{array}\right) \text{ by } (1)$$

This contradicts the fact that c_1 and c_2 both are not zero.

Hence, f_1 and f_2 are L.I.

Ques check LI / LD.

(i) $\{e^x, e^{x+1}\}$ in $C(\mathbb{R})$ Here $e^{x+1} = e\, e^x$ is scalar multiple of e^x so L.D.

(ii) $\{e^x, e^{2x}\}$ in $C(\mathbb{R})$

$$\begin{vmatrix} e^x & e^{2x} \\ e^x & 2e^{2x} \end{vmatrix} = e^{3x} \neq 0 \text{ for any } x \in \mathbb{R} \Rightarrow e^x, e^{2x} \text{ are LI.}$$

(iii) $\{x, |x|\}$ in $C[-1,1]$. $|x| = \begin{cases} x & x \in [0,1] \\ -x & x \in [-1,0) \end{cases}$ is not a scalar multiple of x in $[-1,1]$

Hence L.I.

(iv) $\{\sin x, \sin 2x, \ldots \sin nx\}$ in $C[-\pi,\pi]$, $n \in \mathbb{N}$.

Let $c_1 \sin x + c_2 \sin 2x + \cdots + c_n \sin nx = 0$ — (1)

(1) $\times \sin x$ and then integrate from $-\pi$ to π, we obtain $c_1 = 0$

(1) $\times \sin 2x$ ———————— $c_2 = 0$

Similarly, we obtain $c_n = 0$

Thus (1) $\Rightarrow$ all c_i's $= 0$ hence given set is LI.

Here Remember!
$$\int_{-\pi}^{\pi} \sin mx \sin nx = 0 \quad \text{if } m \neq n$$
$$= 1 \quad \text{if } m = n$$

Ques Let $S = \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 3 \\ 5 \\ 5 \end{bmatrix} \right\}$. Determine which of the following are in $L[S]$.

(i) $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ (ii) $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ (iii) $\begin{bmatrix} 4 \\ 5 \\ 0 \end{bmatrix}$ (iv) $\begin{bmatrix} 1 \\ -3 \\ 8 \end{bmatrix}$

Solution • Since $L[S]$ is a subspace and $0 \in$ subspace always.
Let us check by computation

make matrix $A = \begin{bmatrix} 1 & 1 & 3 \\ 2 & -1 & 5 \\ 3 & -1 & 5 \end{bmatrix}$

(i) check $Ax = b$ where $b = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ has a solution (always – 0 – solution)

$$\left[\begin{array}{ccc|c} 1 & 1 & 3 & 0 \\ 2 & +1 & 5 & 0 \\ 3 & -1 & 5 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 3 & 0 \\ 0 & -1 & -1 & 0 \\ 0 & -4 & -4 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 3 & 0 \\ 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Hence $Ax = b$ has ∞-many solution.

(ii)
$$\left[\begin{array}{ccc|c} 1 & 1 & 3 & 1 \\ 2 & 1 & 5 & 1 \\ 3 & -1 & 5 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 3 & 1 \\ 0 & -1 & -1 & -1 \\ 0 & -4 & -4 & -3 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 3 & 1 \\ 0 & -1 & -1 & -1 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

$\text{Rank}(A) \neq \text{Rank}(A|b)$ Hence No solution. $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \notin L(S)$.

Similarly you can check easily that $\begin{bmatrix} 4 \\ 5 \\ 0 \end{bmatrix} \in L[S]$ and $\begin{bmatrix} 1 \\ -3 \\ 8 \end{bmatrix} \notin L(S)$.

Ques In $\mathbb{C}^2$, determine whether or not $\begin{bmatrix} 1+i \\ 1-i \end{bmatrix} \in L\left[\left\{ \begin{bmatrix} 1+i \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1-i \end{bmatrix} \right\}\right]$

$$\left[\begin{array}{cc|c} 1+i & 1 & 1+i \\ 1 & 1-i & 1-i \end{array}\right] \sim \left[\begin{array}{cc|c} 1+i & 1 & 1+i \\ 0 & \frac{1}{2}(1-i) & -i \end{array}\right] \longrightarrow \boxed{\text{multiplier} = -\frac{1}{1+i} = -\frac{1-i}{1-i^2} = -\frac{1}{2}(1-i)}$$

↳ solution exists. Hence given point $\in$ span of given set.

More ÷ Find Solution above

$$x_2 = \frac{-i}{(1-i)/2} = -2i\,\frac{1+i}{1-i^2} = -\frac{2i}{2}(1+i) = -(i+i^2) = 1-i$$

$$x_1 = \frac{(1+i) - x_2}{(1+i)} = 1 - \frac{x_2}{1+i} = 1 - \frac{1-i}{1+i} = 1 - \frac{1}{2}(1-i)^2 = 1 - \frac{1}{2}[1+i^2-2i] = 1+i$$

Hence check $x_1 \begin{bmatrix} 1+i \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 1-i \end{bmatrix} = \begin{bmatrix} (1+i)^2 + 1-i \\ 1+i+(1-i)^2 \end{bmatrix} = \begin{bmatrix} 1+i^2+2i+1-i \\ 1+i+1+i^2-2i \end{bmatrix}$

$= \begin{bmatrix} 1+i \\ 1-i \end{bmatrix} \rightarrow$ given vector.

Alternative way to solve previous questions

$$L[S] = \alpha_1 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + \alpha_2 \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} + \alpha_3 \begin{bmatrix} 3 \\ 5 \\ 5 \end{bmatrix} = \alpha_1 X_1 + \alpha_2 X_2 + \alpha_3 X_3$$

$$\cancel{= \begin{bmatrix} \alpha_1 + \alpha_2 + 3\alpha_3 \\ 2\alpha_1 + \alpha_2 + 5\alpha_3 \\ 3\alpha_1 - \alpha_2 + 5\alpha_3 \end{bmatrix}}$$

We have to find the col. space of vector X_1, X_2, X_3.

~~Assume~~ $\cancel{\alpha_1 + \alpha_2 + 3\alpha_3 = x_1}$

$$\left[\begin{array}{ccc|c} 1 & 1 & 3 & x_1 \\ 2 & 1 & 5 & x_2 \\ 3 & -1 & 5 & x_3 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 3 & x_1 \\ 0 & -1 & -1 & x_2 - 2x_1 \\ 0 & -4 & -4 & x_3 - 3x_1 \end{array}\right] \sim \left[\begin{array}{ccc|c} \boxed{1} & 1 & 3 & x_1 \\ 0 & \boxed{-1} & -1 & x_2 - 2x_1 \\ 0 & 0 & 0 & x_3 - 3x_1 - 4(x_2 - 2x_1) \end{array}\right]$$

Hence for the solution $x_3 - 3x_1 - 4x_2 + 8x_1 = 0$

$$\Rightarrow 5x_1 - 4x_2 + x_3 = 0$$

Hence $L[S] = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} ; \ 5x_1 - 4x_2 + x_3 = 0 \right\}$

<u>Check</u> the condition $5x_1 - 4x_2 + x_3 = 0$ is satisfied by given vectors

(i) $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ Yes ✓

(ii) $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ NO

(iii) $\begin{bmatrix} 4 \\ 5 \\ 0 \end{bmatrix}$ Yes ✓

(iv) $\begin{bmatrix} 1 \\ -3 \\ 8 \end{bmatrix}$ NO ✓

This is the procedure that how we can write L(S) in the form of subspace

<u>Ques</u> Determine whether the following sets are basis for given VS V.

(i) $\left\{ \begin{bmatrix} 2 \\ 4 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ -2 \end{bmatrix} \right\}$; $V = \mathbb{R}^3$ over $\mathbb{R}$

<u>Sol</u> $\left[\begin{array}{cc|c} \boxed{2} & 0 & x \\ 4 & 2 & y \\ 0 & -2 & z \end{array}\right] \sim \left[\begin{array}{cc|c} \boxed{2} & 0 & x \\ 0 & \boxed{2} & y-2x \\ 0 & -2 & z \end{array}\right] \sim \left[\begin{array}{cc|c} \boxed{2} & 0 & x \\ 0 & \boxed{2} & y-2x \\ 0 & 0 & z+y-2x \end{array}\right]$

See Rank (A) = 2, Hence given set is LI but arb $\begin{bmatrix} x \\ y \\ z \end{bmatrix} \notin C(A)$ Hence given set is not a generator of $\mathbb{R}^3$. Actually the given two vectors span the following subspace of $\mathbb{R}^3$

$\left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} : z+y-2x = 0 \right\}$

→ Remember! columns of REF (A) does not span this subspace. It is spanned by the pivot cols of original matrix

(II) $\{x-1, x^2+x-1, x^2-x+1\}$; $V = P_2$ over $\mathbb{R}$.

Recall arb. element of P_2 is $a+bx+cx^2$

$\left[\begin{array}{ccc|c} -1 & -1 & 1 & a \\ 1 & 1 & -1 & b \\ 0 & 1 & 1 & c \end{array}\right]$

$\sim \left[\begin{array}{ccc|c} -1 & -1 & 1 & a \\ 0 & 0 & 0 & b+a \\ 0 & 1 & 1 & c \end{array}\right] \sim \left[\begin{array}{ccc|c} \boxed{-1} & -1 & 1 & a \\ 0 & \boxed{1} & 1 & c \\ 0 & 0 & 0 & b+a \end{array}\right]$

See Rank (A) = 2. Hence given set is not LI as well as it does not generate full P_2 also.

Now topic to learn here is :- how can we convert a subspace from

Mathematical form $\longleftrightarrow$ computation form,

$\downarrow$

write subspace interms of mathematical expressions

$\{x \in V :$ some constraint/condition on $x\}$

$\downarrow$

write basis of the subspace.

you feed basis in memory of CPU and play with the whole subspace

example ~~Let~~ Consider a subspace

$$\boxed{S = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \mathbb{R}^3 : -2x_1 + x_2 + x_3 = 0 \right\}.}$$ $\longrightarrow$ mathematical form

Clearly S is a subspace.

Its basis is null space of matrix $\begin{bmatrix} -2 & 1 & 1 \end{bmatrix}$

$\begin{bmatrix} 1 & -1/2 & -1/2 \end{bmatrix}$

$$B = \left\{ \begin{bmatrix} 1/2 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} 1/2 \\ 0 \\ 1 \end{bmatrix} \right\}$$

Hence S is a subspace of $\mathbb{R}^3$ with basis

$$B : \left\{ \begin{bmatrix} 1/2 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} 1/2 \\ 0 \\ 1 \end{bmatrix} \right\}$$ $\longrightarrow$ Computational form.

Hence procedure from mathematical $\rightarrow$ computational form

↳ Find basis of Null space of matrix obtained from constraints.

form computation form $\rightarrow$ mathematical form

... we find condition for

solution of $Ax = b$

where $b \in V$ (arb. element of vector Space V)

and $A = \left[\begin{array}{c|c|c|c} a_1 & a_2 & \cdots & a_n \\ \downarrow & \downarrow & & \end{array} \right]$, where

a_i's are basis elements of subspace.

$$\left[\begin{array}{cc|c} 1/2 & 1/2 & x_1 \\ 1 & 0 & x_2 \\ 0 & 1 & x_3 \end{array}\right] \sim \left[\begin{array}{cc|c} 1/2 & 1/2 & x_1 \\ 0 & -1 & x_2 - 2x_1 \\ 0 & 1 & x_3 \end{array}\right] \sim \left[\begin{array}{cc|c} 1/2 & 1/2 & x_1 \\ 0 & -1 & x_2 - 2x_1 \\ 0 & 0 & x_3 + x_2 - 2x_1 \end{array}\right]$$

Hence required subspace is

$$\left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \mathbb{R}^3 : -2x_1 + x_2 + x_3 \right\}$$

<u>Ques</u> For each of the following subspaces find a basis

(i) $S = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \in \mathbb{R}^4 : \begin{matrix} x_1 + x_2 + 2x_3 = 0 \\ 2x_2 + x_3 = 0 \\ x_1 - x_2 + x_3 = 0 \end{matrix} \right\}$

Constraints are
$$\begin{aligned} x_1 + x_2 + 2x_3 + 0x_4 &= 0 \\ 0x_1 + 2x_2 + x_3 + 0x_4 &= 0 \\ x_1 - x_2 + 1x_3 + 0x_4 &= 0 \end{aligned}$$

<u>Solution</u> Actually, problem is to find the basis of Null space of

$$A = \begin{bmatrix} 1 & 1 & 2 & 0 \\ 0 & 2 & 1 & 0 \\ 1 & -1 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 2 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & -2 & -1 & 0 \end{bmatrix} \sim \begin{bmatrix} \boxed{1} & 1 & 2 & 0 \\ 0 & \boxed{2} & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} \boxed{1} & 1 & 2 & 0 \\ 0 & \boxed{1} & 1/2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} \boxed{1} & 0 & 3/2 & 0 \\ 0 & \boxed{1} & 1/2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Hence Basis of $S = \left\{ \begin{bmatrix} -3/2 \\ -1/2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$

(ii) $\{ p \in P_3 : a - 2b + c = 0 \}$ where $p = a + bx + cx^3$
$= a + bx + 0x^2 + cx^3$

matrix $A = [\boxed{1} \; -2 \; 0 \; 1]$

Basis of $N(A)$ $= \left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\} \equiv \boxed{2 + x,\ x^2,\ -1 + x^3}$

Topic to learn: how can we find subspaces $S_1 \cap S_2$ and $S_1 + S_2$ when subspaces S_1 and S_2 are given.

Remember! for $\cap \rightarrow$
- Convert S_1 and S_2 in mathematical form
- Collect all constraints together from S_1 and S_2 (It is $S_1 \cap S_2$)
- If required find basis of $S_1 \cap S_2$, i.e. basis of Null space of a matrix of all constraints.

for $+ \rightarrow$
- convert S_1 and S_2 in computational form, i.e. find basis of S_1 and S_2
- Construct a matrix A where columns are vectors of Bases of S_1 & S_2.
- find basis of $C(A)$. $\rightarrow$ this is basis of $S_1 + S_2$
- Convert subspace in mathematical form if required.

Remember!

- $\dim S_1 + \dim S_2 - \dim(S_1 \cap S_2) = \dim(S_1 + S_2)$
- If $S_1 + S_2 = V$ (full space) and $S_1 \cap S_2 = \{0\}$ $\Big\}$ Then V is called direct sum of S_1 and S_2.

<u>Ques:</u> Find a basis for U, W, $U \cap W$, and $U+W$, where

$$U = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \mathbb{R}^3 : x_1 + x_2 - x_3 = 0 \right\}$$

$$W = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \mathbb{R}^3 ; 2x_1 + x_2 = 0 \right\}$$

<u>Solution</u> Basis of U = basis of null space of $\begin{bmatrix} 1 & 1 & -1 \end{bmatrix}$

$\begin{bmatrix} \boxed{1} & 1 & -1 \end{bmatrix}$ = RREF

So Basis of U: $\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$

<u>Similarly</u> Basis of W is basis of null space of $\begin{bmatrix} 2 & 1 & 0 \end{bmatrix}$

$\begin{bmatrix} \boxed{2} & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} \boxed{1} & 1/2 & 0 \end{bmatrix}$

So Basis of W = $\left\{ \begin{bmatrix} -1/2 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$

$$U \cap W = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} : \begin{matrix} x_1 + x_2 - x_3 = 0 \\ 2x_1 + x_2 = 0 \end{matrix} \right\}$$

So Basis of $U \cap W$ is Null space of $\begin{bmatrix} 1 & 1 & -1 \\ 2 & 1 & 0 \end{bmatrix}$

$$\begin{bmatrix} \boxed{1} & 1 & -1 \\ 2 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} \boxed{1} & 1 & -1 \\ 0 & \boxed{-1} & 2 \end{bmatrix} \sim \begin{bmatrix} \boxed{1} & 0 & 1 \\ 0 & \boxed{-1} & 2 \end{bmatrix}$$

So its basis is $\left\{ \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} \right\}$

Bases of $U+W$ is basis of column space of $\begin{bmatrix} \overbrace{\begin{matrix} -1 & 1 \\ 1 & 0 \\ 0 & 1 \end{matrix}}^{\text{Basis of } U} & \overbrace{\begin{matrix} -1/2 & 0 \\ 1 & 0 \\ 0 & 1 \end{matrix}}^{\text{Basis of } W} \end{bmatrix}$

$$\begin{bmatrix} \boxed{-1} & 1 & -1/2 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} \boxed{-1} & 1 & -1/2 & 0 \\ 0 & \boxed{1} & 1/2 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} \boxed{-1} & 1 & -1/2 & 0 \\ 0 & \boxed{1} & 1/2 & 0 \\ 0 & 0 & \boxed{-1/2} & 1 \end{bmatrix}$$

So basis of $U + W = \left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1/2 \\ 1 \\ 0 \end{bmatrix} \right\}$

<u>Check</u>:- $\dim(U+W) = \dim U + \dim W - \dim(U \cap W)$

$$\overset{\shortparallel}{3} = 2 + \overset{\shortparallel}{2} - 1$$

Ques Find $S\cap T$, $S+T$ and basis also when

$$S = L\left[\left\{\begin{bmatrix}1\\-1\\0\end{bmatrix}\begin{bmatrix}1\\0\\2\end{bmatrix}\right\}\right] \quad T = L\left[\left\{\begin{bmatrix}0\\1\\0\end{bmatrix}\begin{bmatrix}0\\1\\2\end{bmatrix}\right\}\right]$$

Solution First write S & T in mathematical form

$$\left[\begin{array}{cc|c}\boxed{1} & 1 & x_1\\ -1 & 0 & x_2\\ 0 & 2 & x_3\end{array}\right] \sim \left[\begin{array}{cc|c}\boxed{1} & 1 & x_1\\ 0 & \boxed{1} & x_2+x_1\\ 0 & 2 & x_3\end{array}\right] \sim \left[\begin{array}{cc|c}1 & 1 & x_1\\ 0 & 1 & x_2+x_1\\ 0 & 0 & x_3-2x_2-2x_1\end{array}\right]$$

Thus, $S = \left\{\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} \in \mathbb{R}^3 : -2x_1 - 2x_2 + x_3 = 0\right\}$

For T, we obtain

$$\left[\begin{array}{cc|c}0 & 0 & x_1\\ 1 & 1 & x_2\\ 0 & 2 & x_3\end{array}\right] \sim \left[\begin{array}{cc|c}1 & 1 & x_2\\ 0 & 0 & x_1\\ 0 & 2 & x_3\end{array}\right] \sim \left[\begin{array}{cc|c}1 & 1 & x_2\\ 0 & 2 & x_3\\ 0 & 0 & x_1\end{array}\right]$$

Thus $T = \left\{\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} : x_1 = 0\right\}$

$$S\cap T = \left\{\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} : \begin{array}{c} x_1 = 0 \\ -2x_1 - 2x_2 + x_3 = 0\end{array}\right\}$$

For its basis find null space of

$$\begin{bmatrix}1 & 0 & 0\\ -2 & -2 & 1\end{bmatrix} \sim \begin{bmatrix}\boxed{1} & 0 & 0\\ 0 & \boxed{-2} & 1\end{bmatrix} \sim \begin{bmatrix}\boxed{1} & 0 & 0\\ 0 & \boxed{1} & -1/2\end{bmatrix}$$

So its basis is $\left\{\begin{bmatrix}0\\1/2\\1\end{bmatrix}\right\}$

Basis of $S+T$ is basis of col. space of

$$\begin{bmatrix}\boxed{1} & 1 & 0 & 0\\ -1 & 0 & 1 & 1\\ 0 & 2 & 0 & 2\end{bmatrix} \sim \begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 1 & 1\\ 0 & 2 & 0 & 2\end{bmatrix} \sim \begin{bmatrix}\boxed{1} & 0 & 0 & 0\\ 0 & \boxed{1} & 1 & 1\\ 0 & 0 & \boxed{2} & 0\end{bmatrix}$$

Thus basis of $S+T$ is $\left\{\begin{bmatrix}1\\-1\\0\end{bmatrix}, \begin{bmatrix}1\\0\\2\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}\right\} = \underline{\underline{\mathbb{R}^3}}$

Ques Let V be a VS. Let $B_1 = \{v_1, v_2, \cdots, v_n\}$ and $B_2 = \{w_1, w_2, \cdots, w_m\}$ be two bases of V.
Then $n = m$.

Solution Let $A = [v_1 | v_2 | \cdots | v_n]$

$B = [w_1 | w_2 | \cdots | w_m]$

Let x_j be a vector such that

$w_j = A x_j$ for $j = 1, 2, \cdots, m$

Since cols of A spans V, it is clear that x_j exist for each j and note that $x_j \in \mathbb{R}^n \ \forall j$

Let $C = [x_1 | x_2 | \cdots | x_m]$.

Then, by our construction, C is a matrix of order $n \times m$ such that $B = AC$.

We now show the claim by contradiction.

Part 1: Assume $n < m$.

Then C has a non-trivial nullspace, i.e. $\exists\, y \in \mathbb{R}^m$ s.t. $y \neq 0$ and $Cy = 0$. Thus, we obtain

$By = ACy = 0$

which implies that B has non-trivial null space, i.e. all columns of B are not L.I. This contradicts the fact that columns of B is a basis of V.

Hence $n \not< m$.

Part 2 – Assume $m < n$

Then, by changing the role of A & B in Part 1, show that $m \not< n$.

But, here see another proof ÷

If $m < n$, then # rows in C is more that # cols in C, i.e. we can find a non-zero $z \in \mathbb{R}^n$ s.t.

$Cp \neq z$ for any $p \in \mathbb{R}^m$

$\Rightarrow ACp \neq Az \rightarrow$ [$\because$ If $x_1 \neq x_2$ then $Ax_1 \neq Ax_2$ if $N(A) = \{0\}$ only]

$\Rightarrow Bp \neq Az$

$\Rightarrow \text{Rank}(B) \neq \text{Rank}[B | Az] \Rightarrow Az \notin C(B)$ This contradicts the fact that cols of B is a basis.

Thus $m \not< n$.

From Part 1 and Part 2, we obtain $m = n$

See carefully :

Proof of Part 1 also shows that

If $m > n$ and $\dim(V) = n$ then B_2 is L.D.

Proof of Part 2 also shows that

If $m < n$ and $\dim(V) = n$ then B_2 does not span V.

Learn :- how co-ordinate changes if we change basis.

Let $B_1 = \{v_1, v_2, \cdots, v_n\}$
$B_2 = \{\omega_1, \omega_2, \cdots, \omega_n\}$
be two bases of space V.

Let $\omega_j = [\text{Matrix of } B_1 \text{ - column write}]\, x_j \quad (j = 1, 2, \cdots, n)$

Let $\alpha = \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix}$ be coordinate of a pt $x \in V$ wrt B_1

Let $\beta = \begin{pmatrix} \beta_1 \\ \vdots \\ \beta_n \end{pmatrix}$ be ——— of same $x \in V$. wrt B_2

Then $\alpha = A\beta$ where $A = [x_1 | x_2 | \cdots | x_n]$

Ques Find relation between co-ordinates of vector $\begin{pmatrix} 1 \\ 5 \end{pmatrix}$ with respect to bases $B_1 = \left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right\}$
& $B_2 = \left\{\begin{bmatrix} -1 \\ 4 \end{bmatrix}, \begin{bmatrix} 7 \\ 6 \end{bmatrix}\right\}$

$$\left(\begin{array}{cc|c} 1 & 0 & 1 \\ 1 & 1 & 5 \end{array}\right) \sim \left[\begin{array}{cc|c} \boxed{1} & 0 & 1 \\ 0 & \boxed{1} & 4 \end{array}\right] \Rightarrow \alpha = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} -1 & 7 & 1 \\ 4 & 6 & 5 \end{array}\right] \sim \left[\begin{array}{cc|c} -1 & 7 & 1 \\ 0 & 34 & 9 \end{array}\right] \Rightarrow \beta = \begin{bmatrix} -(1 - 63/34) = 29/34 \\ 9/34 \end{bmatrix} = \begin{bmatrix} 29/34 \\ 9/34 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} 1 & 0 & -1 \\ 1 & 1 & 4 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 0 & -1 \\ 0 & 1 & 5 \end{array}\right] \Rightarrow x_1 = \begin{bmatrix} -1 \\ 5 \end{bmatrix}, \quad \left[\begin{array}{cc|c} 1 & 0 & 7 \\ 1 & 1 & 6 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 0 & 7 \\ 0 & 1 & -1 \end{array}\right]; \; x_2 = \begin{bmatrix} 7 \\ -1 \end{bmatrix}$$

Check

$$\begin{bmatrix} 1 \\ 4 \end{bmatrix} = \begin{bmatrix} -1 & 7 \\ 5 & -1 \end{bmatrix} \begin{bmatrix} 29/34 \\ 9/34 \end{bmatrix} \rightarrow \underline{\underline{\text{OK}}}$$

Lecture

Suppose B_2 is known

for some pts β_i is known (all pts is large!)

Now we got new Basis B_1

find coordinates of each vector of B_2 in terms

of B_1 and construct A. matrix

then $\alpha_i = A\beta_i \;\forall i$