MA201

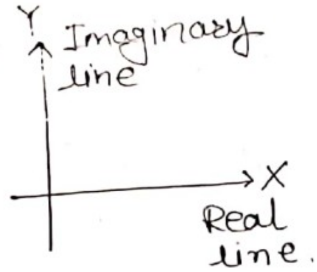
Complex Analysis (Euler 1707–1785)

$z = a + ib \Rightarrow$ complex number ; $i = \sqrt{-1}$;
$a, b \in \mathbb{R}$; $\mathbb{C} = \{ z \mid z = a + ib, a, b \in \mathbb{R} \}$;
$\mathbb{C}$ is a field.

$z_1 = x_1 + iy_1, z_2 = x_2 + iy_2 \in \mathbb{C}$
then $z_1 + z_2 = (x_1 + x_2) + i(y_1 + y_2) \in \mathbb{C}$.

$z_1 z_2 = (x_1 + iy_1)(x_2 + iy_2)$
$z_1 z_2 = (x_1 x_2 - y_1 y_2) + i(x_1 y_2 + x_2 y_1) \in \mathbb{C}$.

A complex number z can be represented by ordered pair (x, y) where x & y are known as real part and imaginary part resp. of z.



$x = Re(z)$; $y = Im(z)$;

* Guassian Plane (Argand Plane) :-

Plane where complex numbers are represented geometrically.

$i = (0, 1)$; $i^2 = (0,1).(0,1)$
$= (0 - 1) + i(0 + 0)$
$i^2 = -1 + i0 = (-1, 0)$
$\therefore \boxed{i^2 = (-1, 0)}$

$(\sqrt{-a})(\sqrt{-b})$
$= (\sqrt{-1})(\sqrt{a})(\sqrt{-1}).(\sqrt{b})$
$= (i\sqrt{a})(i\sqrt{b})$
$= i^2 \sqrt{ab}$
$= -\sqrt{ab}$

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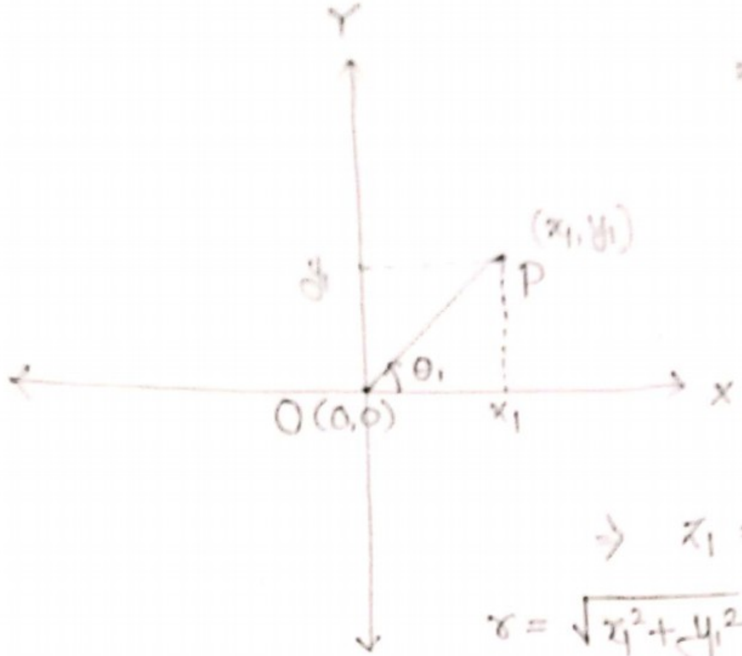
* Geometrically :

let, $z_1 = x_1 + iy_1 = (x_1, y_1)$; $z_2 = x_2 + iy_2 = (x_2, y_2)$,



$z_1 = \overrightarrow{OP}$, $|\overrightarrow{OP}| = OP$;

$z_1 = x_1 + iy_1 = (x_1, y_1)$

in polar form (r, θ)

where $r = |\overrightarrow{OP}|$

and $\theta = \theta_1$ in anticlock wise (positive) direction

$\rightarrow z_1 = x_1 + iy_1$

$r = \sqrt{x_1^2 + y_1^2}$; magnitude of the complex no. z_1 .

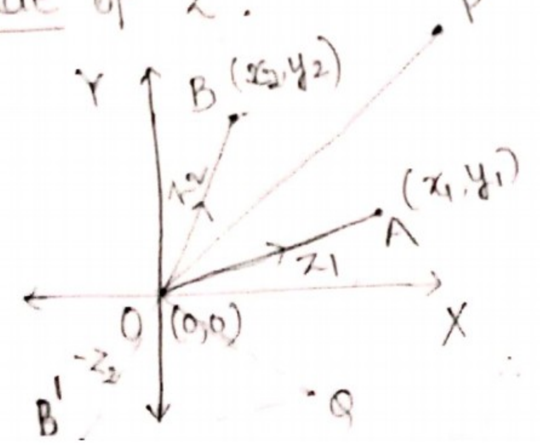
Amplitude of z_1 is θ_1 (-Argument); $amp(z_1) = \theta_1$.

* Modulus or Absolute value of a complex number z is denoted by $|z|$ and defined as $|z| = \sqrt{x^2 + y^2}$

* Amplitude of $z = x + iy$ is denoted by $amp(z)$ and defined as $amp(z) = \theta = \tan^{-1}(y/x)$.

Amplitude is not unique ($\because \tan(2n\pi + \theta) = \tan\theta$) .

Therefore for unique representation θ is restricted to $(-\pi \leq \theta \leq \pi)$, which is also called as principle amplitude of z .



$\overrightarrow{OB} = \overrightarrow{AP}$; $\overrightarrow{OA} = \overrightarrow{BP}$

$$z_1 + z_2 = \overrightarrow{OP} = \overrightarrow{OA} + \overrightarrow{OB}$$
$$= \overrightarrow{OA} + \overrightarrow{AP}$$
$$= \overrightarrow{OP}$$

$$\boxed{z_1 + z_2 = (x_1 + x_2, y_1 + y_2)}$$

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$$z_1 - z_2 = \overrightarrow{OA} - \overrightarrow{OB} = \overrightarrow{OA} + \overrightarrow{BO}$$
$$= \overrightarrow{OA} + \overrightarrow{OB'}$$
$$= \overrightarrow{OA} + \overrightarrow{AQ}$$
$$= \overrightarrow{OQ}.$$

$$\boxed{\therefore\ z_1 - z_2 = (x_1 - x_2, y_1 - y_2)};$$

$z = r \cdot e^{i\theta}$

magnitude direction

$z = r e^{i\theta} = r\cos\theta + i r \sin\theta.$

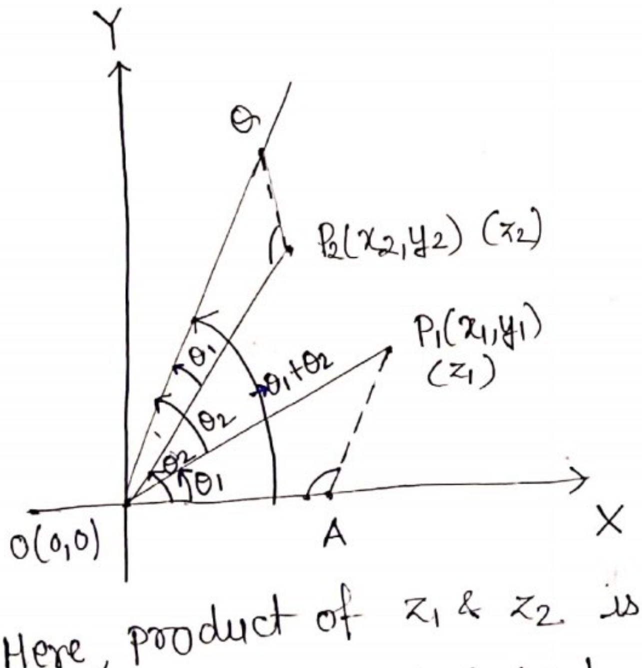
$|e^{i\theta}| = |\cos\theta + i\sin\theta| = 1.$

$z_1 = r_1 . e^{i\theta_1}, \; z_2 = r_2 . e^{i\theta_2} \Rightarrow z_1 . z_2 = r_1 . r_2 e^{i(\theta_1 + \theta_2)}.$

$z(x,y) = x + iy = x(t) + i\, y(t) = z(t).$

where, t is some parameter bounded by some given specifications. (e.g. $1 \le t \le n$; $0 \le t \le \pi$).

$f(z) = w \mid f : \mathbb{C} \to \mathbb{C}.$



OA is of unit magnitude

$\angle OAP_1 = \angle OP_2Q.$

These two triangles are similar, therefore we can write.

$$\frac{\overrightarrow{OP_1}}{\overrightarrow{OA}} = \frac{\overrightarrow{OQ}}{\overrightarrow{OP_2}}$$

$$\Rightarrow \overrightarrow{OQ} = \overrightarrow{OP_1} . \overrightarrow{OP_2}$$

Here, product of z_1 & z_2 is represented by $\overrightarrow{OQ}$.

$$|z_1 z_2| = |z_1| . |z_2|.$$

$$amp(z_1 z_2) = \theta_1 + \theta_2$$

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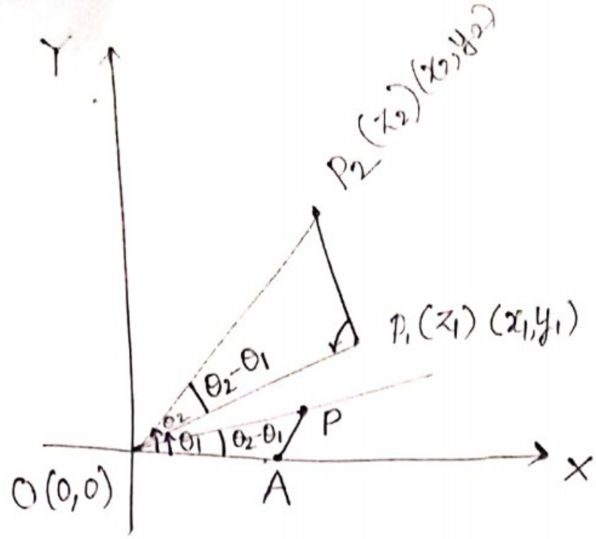

OA is of unit magnitude.

$\angle OAP = \angle OP_1P_2$

ΔOAP & ΔOP_1P_2 are similar triangles.

$$\frac{OP_2}{OP_1} = \frac{OP}{OA}.$$

$$\Rightarrow \quad \overline{OP} = \frac{OP_2}{OP_1} = \frac{z_2}{z_1}$$

$$z_1 = r_1 e^{i\theta_1}$$
$$z_2 = r_2 e^{i\theta_2} \Rightarrow \frac{z_2}{z_1} = \frac{r_2}{r_1} e^{i(\theta_2-\theta_1)}.$$

* <u>Powers of a complex number z</u> :-

$$\Rightarrow \quad z^2 = (x+iy)^2 = \alpha + i\beta \text{ (say)}.$$

$$\Rightarrow \quad x^2-y^2+2ixy = \alpha + i\beta.$$

$$\therefore \quad \alpha = x^2-y^2 \;;\; \beta = 2xy.$$

Square root of z. $\Rightarrow \sqrt{x+iy} = \alpha + i\beta$ (say.

$$\Rightarrow \quad x+iy = (\alpha+i\beta)^2$$

$$\Rightarrow \quad x+iy = \alpha^2-\beta^2+2i\alpha\beta$$

$$\Rightarrow \quad x = \alpha^2-\beta^2 \;;\; y = 2\alpha\beta.$$

$$(\alpha^2+\beta^2)^2 = (\alpha^2-\beta^2)^2 + 4\alpha^2\beta^2$$

$$(\alpha^2+\beta^2)^2 = x^2+y^2$$

$$\therefore \quad \alpha^2+\beta^2 = \pm\sqrt{x^2+y^2}$$

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$2\alpha^2 = x + \sqrt{x^2+y^2}$; $2\beta^2 = -x + \sqrt{x^2+y^2}$

$\Rightarrow \alpha = \pm\sqrt{\frac{1}{2}(x+\sqrt{x^2+y^2})}$; $\beta = \pm\sqrt{\frac{1}{2}(-x+\sqrt{x^2+y^2})}$

Note - Sign of y & $\alpha\beta$ should be same.

★ De-Moivre's Theorem :- $e^{i\theta} = \cos\theta + i\sin\theta$.

$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$.

for $n \in \mathbb{Z}$; for n of form $\frac{p}{q}$; $q \neq 0$

$\cos(n\theta) + i\sin(\theta n)$ is one of the value of $(\cos\theta + i\sin\theta)^n$.

* Some Properties of Complex Numbers :-

for $z = x+iy$; $x,y \in \mathbb{R}$; $|z| = \sqrt{x^2+y^2}$

$Re(z) = x \leq |z|$; $Im(z) = y \leq |z|$.

★ $-|z| \leq Re(z) \leq |z|$; $-|z| \leq Im(z) \leq |z|$.

★ $|z_1||z_2| = |z_1 z_2|$.

★ $|z_1 + z_2| \leq |z_1| + |z_2|$ (Triangle Inequality).

★ Conjugate of a complex number $\Rightarrow \bar{z} = x - iy$.

★ $|z| = |\bar{z}| = \sqrt{x^2+y^2}$.

★ $z\cdot\bar{z} = |z|^2$. ★ $\overline{z_1 + z_2} = \bar{z_1} + \bar{z_2}$.

★ Triangle inequality proof

$$|z_1+z_2|^2 = (z_1+z_2)\overline{(z_1+z_2)}$$
$$= |z_1|^2 + |z_2|^2 + 2Re(z_1\bar{z_2})$$
$$\leq |z_1|^2 + |z_2|^2 + 2|z_1||z_2|$$
$$= (|z_1| + |z_2|)^2$$

$z_1\bar{z_2} + z_2\bar{z_1} = 2Re(z_1\bar{z_2})$

$(\because z + \bar{z} = 2Re(z))$

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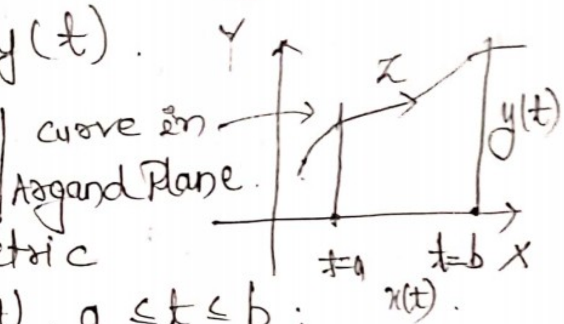
★ Cauchy's Inequality :- $\left|\sum_{i=1}^{n} x_i y_i\right|^2 =$

$|x_1y_1 + x_2y_2 + \ldots + x_ny_n|^2 \leq \sum_{i=1}^{n} |x_i|^2 + \sum_{i=1}^{n} |y_i|^2$

★ Z's parametric representation ⇒ $x(t), y(t)$

$a \leq t \leq b$; $z(t) = x(t) + y(t)$.



Curve :- A curve in argand plane $z = x + iy$ has a parametric representation as $x = x(t)$, $y(t)$, $a \leq t \leq b$; $(a, b \in \mathbb{R})$. i.e $z(t) = x(t) + iy(t)$.

If we represent this curve by Γ, then Γ is said to be closed if $z(a) = z(b)$.

Simple Curve :- A curve Γ for $a \leq t \leq b$ is said to be simple if $t_1 < t_2 \Rightarrow z(t_1) \neq z(t_2)$.

Simple curve is closed $z(a) = z(b)$; a & b are end points

simple Curve (Jordan arc) ; Closed Simple Curves (Closed Jordan arc).

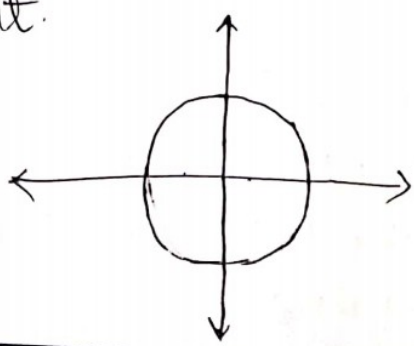
$\omega = \sqrt{z}$

① $z = t^2$; $-1 \leq t \leq 1$ $\quad z(-1) = 1$; $z(1) = 1$ → Closed curve
$z(-1/2) = 1/4$; $z(1/2) = 1/4$

② $z = \cos t + i \sin t$.

$z(0) = 1$

$z(2\pi) = 1$



→ Simple Closed curve.

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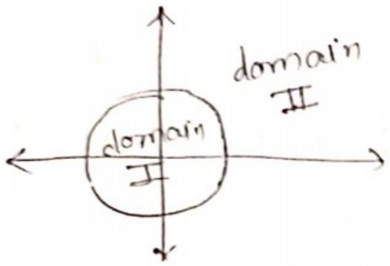
★ Jordan lemma :-

Every Simple Jordan curve divides the Argand plane into two open domains.

This Jordan curve will be the common portion (i.e boundary) of each of these two domains.



I ⇒ $|z| < a$.

II ⇒ $|z| > a$.

boundary ⇒ $|z| = a$.

★ complex Valued functions :-

A rule of set of rules for which for each $z \in D\ (\subseteq \mathbb{C})$; we get one or more than one complex number is said to be a complex valued function.

(OR) $f : D \to \mathbb{C}$ is said to be a complex valued function if for each $z \in D$ we have one $f(z)$ or more than one $f(z)$ in $\mathbb{C}$

If for each z we have only one value of $f(z)$, then function is said to be single valued.

If for some z we have more than one value of $f(z)$, then function is said to be multivalued or manyvalue-d.

If $(-\pi \leq \arg(z) < \pi)$ then f is single valued;

else f is multi or many valued;

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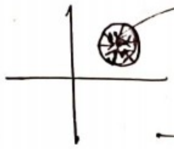
For limit, Continuity and Differentiability we have to consider single valued function.

$(\operatorname{Log} z = 2n\pi i + \log z)$.

Limit :- Let $w = f(z)$ be a complex valued function defined in a bounded domain D then we call $f(z)$ tends to a limit l as z tends to $a \in D$ if for any given $\epsilon > 0$ (very-very small). $\exists\ \delta > 0$ s.t. $|f(z) - l| < \epsilon$; for ~~each~~ all $z \in D$ and satisfying $|z-a| < \delta$.

(Deleted neighbourhood) $\Rightarrow$ very close to given point.

$$\Rightarrow \boxed{\lim_{z \to a} f(z) = l}$$



(a). functional value should be independent of path.

Continuity :- $\boxed{\lim_{z \to a} f(z) = f(a)}$

Let $w = f(z)$ be a complex valued function defined in a bounded domain D then we call $f(z)$ to be continuous if for any given $\epsilon > 0$, $\exists\ \delta > 0$ s.t. $|f(z) - f(a)| < \epsilon$ (as z tends to a) satisfying $|z - a| < \delta$.

Differentiability :-

$$\lim_{\delta z \to 0} \frac{f(z + \delta z) - f(z)}{\delta z}$$

exists and limit reduces to same value corresponding different path of approaching to z, then $f(z)$ is said to be differential at (z).

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TYPE	RANGE	DOMAIN
1	REAL	REAL
2	REAL	COMPLEX
3	COMPLEX	REAL
4	COMPLEX	COMPLEX

$\lim_{z \to a} \frac{f(z) - f(a)}{z - a}$ exists and has same value independent of path to a from z

(Differentiability at a for f)

★ Extended Domain C^* or $\hat{C} = C \cup \{-\infty, \infty\}$.

e.g. $f(z) = z \quad \left.\frac{df(z)}{dz}\right|_{z=a} \Rightarrow 1\Big|_{z=a} = 1$.

$f(z) = z^2 \quad \left.\frac{df(z)}{dz}\right|_{z=a} \Rightarrow 2z\Big|_{z=a} = 2a$.

★ $f(z)$ on D; $z \in D$; $\frac{df(z)}{dz}$ exists. then $f(z)$ is said to be differentiable over the domain D.

★ $f(z)$ is differentiable at $z = a$ but $f(z)$ is not differentiable at $a + \delta a$; $(\delta a > 0)$. ~~then it is said~~ ~~to be Any~~ (Irrespective of above example)

★ Analytic Function :-

A single valued function $\omega = f(z)$. defined on domain D ($\subseteq C^*$) is said to be analytic at $z = a \in D$ if $f(z)$ is differentiable at all points of some neighbourhoods of a.

Check ⇒ $|z|^2, |z|, \sqrt{|xy|}$; where $z = x + iy$.

★ are Analytic or not

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St. The necessary and sufficient condition for derivative of a singled valued function $w = f(z) = u + iv$ defined on D to be analytic on D if

(i) $\frac{\partial u}{\partial x}, \frac{\partial v}{\partial y}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}$ exist and are continuous

(ii) $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$.

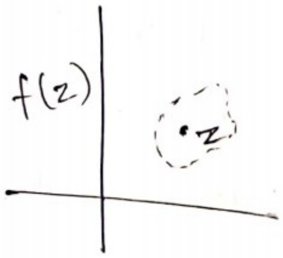
Proof:- Let, $w = f(z)$ be a complex valued function.

$\Rightarrow \quad w = u + iv \; ; \; z = x + iy \; ; \; (u, v, x, y) \in \mathbb{R}$

u & v are both dependent on x, y. $\left(\text{Domain} = D\right)$

$\longrightarrow$ Let, $w = f(z)$ is analytic (differentiable) on D.

$\Rightarrow \lim_{\partial z \to 0} \frac{f(z + \partial z) - f(z)}{\partial z}$ exists and has unique value, whatever the path of approaching dz to 0.



Let, $w + \partial w = f(z + \delta z) = [u(x,y) + \partial u(x,y)] + i[v(x,y) + \partial v(x,y)]$

Let ∂w be the changes coming due to the changes δu and δv in u & v at the neighbouring point $(x + \partial x) + i(y + \delta y) = z + \delta z$.

$$\lim_{\partial z \to 0} \frac{f(z + \partial z) - f(z)}{\partial z} = \lim_{\partial z \to 0} \frac{\cancel{u(x,y)} + \delta(u(x,y)) + \cancel{iv(x,y)} + i\delta(v(x,y)) - \cancel{u(x,y)} - \cancel{iv(x,y)}}{\delta z}$$

$$\frac{df(z)}{dz} = \lim_{\partial z \to 0} \frac{\delta u(x,y) + i\delta v(x,y)}{\delta z} \quad \text{--- (1)}$$

δz approaching to 0 along real axis.
then $\delta y = 0$; $\delta z = \delta x$.

$$\Rightarrow \frac{df(z)}{dz} = \lim_{\delta x \to 0} \left(\frac{\delta u(x,y)}{\delta x} + i \frac{\delta v(x,y)}{\delta x} \right)$$

$$\frac{df(z)}{dz} = \frac{\partial u}{\partial x} + i \cdot \frac{\partial v}{\partial x} \quad \text{------} \; (2)$$

δz approaching to 0 along img. axis
then $\delta x = 0$; $dz = i\delta y$

$$\Rightarrow \frac{df(z)}{dz} = \lim_{\partial y \to 0} \left(\frac{\delta u(x,y)}{i\delta y} + \frac{i \cdot \partial v(x,y)}{i \delta y} \right)$$

$$\frac{df(z)}{dz} = -i \frac{\partial u}{\partial y} + \frac{\cancel{i}}{\cancel{i}} \cdot \frac{\partial v}{\partial y} \quad \text{------} \; (3)$$

from (2) & (3) ;

$$\boxed{\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}} \quad \& \quad \boxed{\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}}$$

C–R equations (Cauchy-Riemann equations).

★ Taylor's Theorem :-

If $y = f(x)$ is continuously differentiable upto n^{th} order in the neighbourhood of a then

$$f(a+h) = f(a) + h.f'(a) + \frac{h^2}{2!} .f''(a) + \ldots + \underbrace{\frac{h^n}{n!} f^n(a+\theta h)}_{\text{remainder term}}$$

Taylor series :- $f(a+h) = f(a) + h.f'(a) + \frac{h^2}{2!} f''(a) + \ldots$
(i.e. finite terms)

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$z = f(x,y)$ on D ; $(a,b) \in D$ has derivative upto n^{th} order in the neighbourhood of (a,b),

$$f(a+h, b+k) = f(a,b) + \left(h.\frac{\partial f(a,b)}{\partial x} + k.\frac{\partial f(a,b)}{\partial y}\right) + \frac{\left(h.\frac{\partial}{\partial x} + k.\frac{\partial}{\partial y}\right)^2 f(a,b)}{2!}$$

$$+ \frac{1}{3!}\left(h.\frac{\partial}{\partial x} + k.\frac{\partial}{\partial y}\right)^3 . f(a,b) + \ldots\ldots + R_n$$

(Remainder term)

$$R_n = \frac{1}{n!}\left(h.\frac{\partial}{\partial x} + k.\frac{\partial}{\partial y}\right)^n . f(a+\theta h, b+\phi k).$$

* for Taylor series, infinte terms exist.

Neglecting higher order terms, we get

$$f(a+h, b+k) - f(a,b) = h.\frac{\partial f(a,b)}{\partial x} + k.\frac{\partial f(a,b)}{\partial y}.$$

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* **Sufficient Condition for Analyticity :-**

Let $w = f(z) = u + iv$ defined on a domain D such that $u(x,y)$ and $v(x,y)$ has continuous partial derivative of first order and $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ & $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

(i.e. $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ exist and are continuous).

Proof :- Then by Taylor's Theorem for two vari-ables function, we have;

$$w + \delta w = f(z + \delta z)$$

$$= u(x+\delta x, y+\delta y) + iv(x+\delta x, y+\delta y)$$

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(Since, u & v are real valued functions of real variables x, y).

$$= \left\{ u(x,y) + \left(\delta x \cdot \frac{\partial u}{\partial x} + \delta y \cdot \frac{\partial u}{\partial y} \right) + \cdots \right\}$$

$$+ i \left\{ v(x,y) + \left(\delta x \cdot \frac{\partial v}{\partial x} + \delta y \cdot \frac{\partial v}{\partial y} \right) + \cdots \right\}.$$

(with assumption that δx & δy are very small quantities, we are neglecting higher order terms in the Taylor's Series Expansion)

$$f(z+\delta z) = \left(u(x,y) + i v(x,y) \right) + \left(\frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} \right) \cdot \delta x + \left(\frac{\partial u}{\partial y} + i \cdot \frac{\partial v}{\partial y} \right) \cdot \delta y.$$

$$\Rightarrow f(z+\delta z) = f(z) + \left(\frac{\partial u}{\partial x} + i \cdot \frac{\partial v}{\partial x} \right) \cdot \delta x + \left(\frac{\partial u}{\partial y} + i \cdot \frac{\partial v}{\partial y} \right) \cdot \delta y$$

$$\Rightarrow f(z+\delta z) - f(z) = \left(\frac{\partial u}{\partial x} + i \cdot \frac{\partial v}{\partial x} \right) \cdot \delta x + \left(\frac{-\partial v}{\partial x} + i \cdot \frac{\partial u}{\partial x} \right) \cdot \delta y$$

$$= \left(\frac{\partial u}{\partial x} + i \cdot \frac{\partial v}{\partial x} \right) \cdot \delta x + \left(\frac{\partial u}{\partial x} + i \cdot \frac{\partial v}{\partial x} \right) i \cdot \delta y$$

$$= \left(\frac{\partial u}{\partial x} + i \cdot \frac{\partial v}{\partial x} \right) \cdot (\delta x + i \delta y)$$

$$= \left(\frac{\partial u}{\partial x} + i \cdot \frac{\partial v}{\partial x} \right) (\delta z)$$

$$\lim_{\delta z \to 0} \left(\frac{f(z+\delta z) - f(z)}{\delta z} \right) = \lim_{\delta z \to 0} \left(\frac{\partial u}{\partial x} + i \cdot \frac{\partial v}{\partial x} \right)$$

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$$\boxed{\frac{df(z)}{dz} = \frac{\partial}{\partial x}(u+iv) = \frac{\partial f(z)}{\partial x}}$$

$$\Rightarrow \frac{df(z)}{dz} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

$$= \frac{\partial v}{\partial y} + i\left(-\frac{\partial u}{\partial y}\right)$$

$$= -i\frac{\partial u}{\partial y} - i^2\frac{\partial v}{\partial y} \Rightarrow -i\left(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right).$$

$$= -i\frac{\partial}{\partial y}(u+iv)$$

$$\boxed{\frac{df(z)}{dz} = (-i)\frac{\partial f(z)}{\partial y}}$$

For any analytic function $w = f(z) = u(x,y) + iv(x,y)$, the functions $u(x,y)$ and $v(x,y)$ are said to be conjugate functions (or conjugate to each other.

Orthogonal curves

suppose $u(x,y) = c_1$ and $v(x,y) = c_2$ are the parts of an analytic function $f(z) = u(x,y) + iv(x,y)$.

$u(x,y) = c_1$

$$\Rightarrow \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial x} = 0$$

$$\Rightarrow \frac{dy}{dx} = -\frac{\partial u/\partial x}{\partial u/\partial y} \quad \text{--- (1)}$$

$v(x,y) = c_2$

$$\Rightarrow \frac{\partial v}{\partial x} + \frac{\partial v}{\partial y}\cdot\frac{dy}{dx} = 0$$

$$\Rightarrow \frac{dy}{dx} = \frac{-\partial v/\partial x}{\partial v/\partial y} \quad \text{--- (2)}$$

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$$-1 = \left(\frac{\left(\frac{\partial u}{\partial x}\right)}{\left(\frac{\partial u}{\partial y}\right)}\right) \cdot \left(\frac{\left(\frac{\partial v}{\partial x}\right)}{\left(\frac{\partial v}{\partial y}\right)}\right) = m_1 . m_2$$

$\Rightarrow$ (i.e) $\boxed{\frac{\partial u}{\partial x} \cdot \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \cdot \frac{\partial v}{\partial y} = 0}$

Every analytic function satisfies the C-R equations,

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \& \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

$$\Rightarrow \quad \frac{\partial u}{\partial x}\left(-\frac{\partial v}{\partial x}\right) = \frac{\partial u}{\partial y} \cdot \frac{\partial v}{\partial y}$$

$$\Rightarrow \quad \boxed{\frac{\partial u}{\partial x} \cdot \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \frac{\partial v}{\partial y} = 0}$$

$\therefore$ Conjugate parts of analytic function are orthogonal.

★ Polar form

$z = (x, y) = (r, \theta) \Rightarrow \quad x = r\cos\theta \; ; \; y = r\sin\theta$

$= r . e^{i\theta} \qquad r^2 = x^2 + y^2 \; ; \quad \theta = \tan^{-1}(y/x)$.

★ Representation of C-R equations in polar form :-

$$\frac{\partial u}{\partial r}, \frac{\partial u}{\partial \theta}, \frac{\partial v}{\partial r}, \frac{\partial v}{\partial \theta}$$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r} \cdot \frac{\partial r}{\partial x} + \frac{\partial u}{\partial \theta} \cdot \frac{\partial \theta}{\partial x}$$

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$z = r e^{i\theta} \qquad f(z) = f(r e^{i\theta})$.

$$\Rightarrow \quad \frac{\partial f(z)}{\partial r} = f'(r e^{i\theta}) . e^{i\theta}$$

$$\frac{\partial f(z)}{\partial \theta} = f'(r e^{i\theta}) r . e^{i\theta} . i = i . r \frac{\partial f(z)}{\partial r}$$

$$\Rightarrow \left(\frac{\partial u}{\partial \theta} + i . \frac{\partial v}{\partial \theta}\right) = i r \left(\frac{\partial u}{\partial r} + i . \frac{\partial v}{\partial r}\right)$$

$$= i r . \frac{\partial u}{\partial r} - r . \frac{\partial v}{\partial r}$$

$$\Rightarrow \quad \frac{\partial u}{\partial \theta} = -r . \frac{\partial v}{\partial r} \qquad \frac{\partial v}{\partial \theta} = r . \frac{\partial u}{\partial r}$$

$$\Rightarrow \quad \boxed{\frac{-1}{r} . \frac{\partial u}{\partial \theta} = \frac{\partial v}{\partial r} \quad ; \quad \frac{\partial u}{\partial r} = \frac{1}{r} . \frac{\partial v}{\partial \theta}}$$

Tutorial 1. complex Analysis.

1. Prove the following.

{ for $|z_1 - z_2|$ part replace z_2 by $-z_2$

① $|z_1 \pm z_2| \leq |z_1| + |z_2|$

$$\Rightarrow \quad |z_1 + z_2|^2 = (z_1 + z_2) . (\overline{z_1 + z_2})$$

$$= (z_1 + z_2) . (\overline{z_1} + \overline{z_2})$$

$$= z_1 . \overline{z_1} + z_2 \overline{z_1} + z_1 \overline{z_2} + z_2 . \overline{z_2}$$

$$= |z_1|^2 + z_2 . \overline{z_1} + \overline{\overline{z_1} . z_2} + |z_2|^2$$

$$= |z_1|^2 + 2 \mathrm{Re}(\overline{z_1} . z_2) + |z_2|^2 \qquad \left(\because \mathrm{Re}(\overline{z_1} z_2) \leq |z_1 \overline{z_2}| \right.$$

$$\leq |z_1|^2 + 2 . |z_1||z_2| + |z_2|^2$$

$$= (|z_1| + |z_2|)^2 \quad \Rightarrow \quad |z_1 \pm z_2| \leq |z_1| + |z_2|$$

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(iv) $\sqrt{2}|z| \geq |Re(z)| + |Im(z)|$

$\rightarrow \quad z = x + iy.$

$$(|x| + |y|)^2 = |x|^2 + |y|^2 + 2|x|.|y|.$$

$$\left\{ \begin{array}{l} (|x| - |y|)^2 \geq 0 \\ \Rightarrow |x|^2 + |y|^2 \geq 2|x||y|. \end{array} \right\}$$

$\rightarrow \quad (|x| + |y|)^2 \leq |x|^2 + |y|^2 + |x|^2 + |y|^2$

$$= 2(|x|^2 + |y|^2)$$

$$= 2|z|^2.$$

$\Rightarrow \quad |x| + |y| \leq \sqrt{2}(|z|)$

$\Rightarrow \quad Re(z) + Im(z) \leq \sqrt{2}(|z|).$

Q.3) $\quad Re(z_1\bar{z}_2) \leq |z_1\bar{z}_2|$

$z_1 = x_1 + iy_1 \quad ; \quad z_2 = x_2 + iy_2.$

$$Re(z_1.\bar{z}_2) = Re((x_1 + iy_1).(x_2 - iy_2))$$

$$= x_1x_2 + y_1y_2$$

$$= \sqrt{(x_1x_2 + y_1y_2)^2}$$

$$\leq \sqrt{(x_1x_2 + y_1y_2)^2 + (y_1x_2 - y_2x_1)^2}$$

$$= |z_1.\bar{z}_2|.$$

$\therefore \quad Re(z_1.\bar{z}_2) \leq |z_1.\bar{z}_2|$

$y_1x_2 - x_1y_2 = 0 \qquad \tan^{-1}\left(\frac{y_1}{x_1}\right) = \tan^{-1}\left(\frac{y_2}{x_2}\right)$

$\Rightarrow \frac{y_1}{x_1} = \frac{y_2}{x_2} \qquad \arg z_1 = \arg z_2.$

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$|z_1 + z_2| = |z_1| + |z_2|$ $\quad z_1 = r_1 e^{i\theta_1}$; $z_2 = r_2 e^{i\theta_2}$

$|z_1 - z_2| = \big||z_1| - |z_2|\big|$. $\quad \theta_1 = \theta_2$,

Q.4) $$P(z) = \sum_{i=0}^{n} a_i . z^i$$

As z_1 is root $\Rightarrow$ $P(z_1) = 0$.

To prove $\overline{z_1}$ is root we have to prove $P(\overline{z_1}) = 0$.

$$P(\overline{z_1}) = \sum_{i=0}^{n} a_i \overline{z_1}^{\,i}$$

$$= \sum_{i=0}^{n} a_i . \overline{z_1^i} = \overline{\sum_{i=0}^{n} a_i z_1^i} = \overline{P(z_1)} = \overline{0} = 0.$$

Q.5) (i) $\mathrm{Re}\left(\frac{1}{z}\right) = 2$; . $\bar{z} = x - iy$. $\frac{1}{z} = \frac{1}{(x-iy)}\frac{(x+iy)}{(x+iy)} = \frac{x+iy}{x^2+y^2}$.

$\mathrm{Re}(1/z) = 2$

$\Rightarrow \frac{x}{x^2+y^2} = 2$

$\Rightarrow \underline{x^2+y^2 - \frac{x}{2} = 0}$. circle

Q.6) . $$|\sin z| = \left| \frac{e^{iz} - e^{-iz}}{2i} \right|$$

$z = \pi + i\ln(2+\sqrt{5})$. $\sin(\pi$

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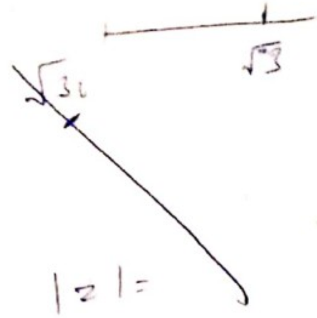
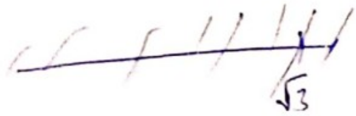
9.9. (ii) $\left|\frac{1}{z^4-4z^2+3}\right| \quad |z|=2.$



$\Rightarrow \quad |(z^4-4z^2+3)| = |(z^2-3)(z^2-1)|$

$= |z^2-3|\,|z^2-1|. \quad |z|=$

$\frac{1}{|z^2-3||z^2-1|}$



Q.10 $\quad 1+z+z^2+\dots+z^n = \frac{1-z^{n+1}}{1-z}, \quad z\neq 1.$ —— ①

$n=1; \quad 1+z = \frac{1-z^2}{1-z} = \frac{\cancel{(1-z)}(1+z)}{\cancel{(1-z)}} = 1+z.$

$n=k, \quad 1+z+\dots+z^k = \frac{1-z^{k+1}}{1-z}$

for $k+1$, $(1+z+\dots+z^k)+z^{k+1} = \frac{1-z^{k+1}}{1-z}+z^{k+1}$

$= \frac{1-z^{k+2}}{1-z}.$

(i) $\quad z = e^{i\theta}$ in ①.

$1+\cos\theta+\cos 2\theta+\dots+\cos n\theta + i\{\sin\theta+\sin 2\theta+\dots+\sin n\theta\}$

$= \frac{1-\cos(n+1)\theta - i\sin(n+1)\theta}{1-\cos\theta - i\sin\theta}.$

$= \frac{1-\cos(n+1)\theta - i\sin(n+1)\theta}{2\sin^2\theta/2 - i\,2\sin\theta/2\cdot\cos\theta/2}$

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$$= \frac{1}{2}\frac{(1-\cos(n+1)\theta - i\sin(n+1)\theta)}{\sin\frac{\theta}{2}\cdot(\sin\theta/2 - i\cos\theta/2)} \times \frac{(\sin\theta/2 + i\cos\theta/2)}{(\sin\theta/2 + i\cos\theta/2)}$$

$$= \frac{(1-\cos(n+1)\theta - i\sin(n+1)\theta)}{2\sin^2\theta/2}(\sin\theta/2 + i\cos\theta/2)$$

$$\Rightarrow 1+\cos\theta + \ldots + \cos n\theta = \frac{(1-\cos(n+1)\theta)\sin\theta/2 + \sin(n+1)\theta\cos\theta/2}{2\sin^2\theta/2}$$

$$= \frac{2\cdot\sin^2\left(\frac{n+1}{2}\right)\theta\ \sin\theta/2 + 2\cdot\sin\left(\frac{n+1}{2}\right)\theta\cdot\cos\frac{(n+1)\theta}{2}\cos\theta/2}{2\cdot\sin^2\theta/2}$$

$$= \frac{2\cdot\sin\left(\frac{n+1}{2}\right)\theta\left\{\sin\frac{\theta}{2}\cdot\sin\frac{(n+1)}{2}\theta + \cos\frac{(n+1)\theta}{2}\cdot\cos\frac{\theta}{2}\right)}{2\sin\theta/2}$$

$$= \frac{\sin(n+1)\theta/2}{\sin\theta/2}\left\{\cos\left(\frac{n+1}{2} - \frac{1}{2}\right)\theta\right\}$$

$$= \frac{\sin(n+1)\theta/2\cdot\cos\left(\frac{n}{2}\right)\theta}{\sin\theta/2}$$

$$= \frac{\left(\sin\frac{n\theta}{2}\cos\theta/2 + \cos\frac{n\theta}{2}\cdot\sin\frac{\theta}{2}\right)\cos\frac{n\theta}{2}}{\sin\theta/2}$$

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* Harmonic function :- A function $z = f(x,y)$ of two variables x and y is said to be a harmonic function if partial derivatives of first and second order of $f(x,y)$ exist and satisfies the Laplace Equation

i.e $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial^2 f}{\partial x^2}, \frac{\partial^2 f}{\partial y^2}, \frac{\partial^2 f}{\partial x \partial y}, \frac{\partial^2 f}{\partial y \partial x}$ exist

Laplace eqn for 2D plane.

$$\boxed{\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0}$$

for 3D plane

$$\boxed{\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = 0}$$

Let $f(z) = u(x,y) + iv(x,y)$ be an analytic function defined on the domain D. Then C-R equations exist

i.e. $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ —① ; $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ —② . We assume that u & v have second order partial derivatives.

Differentiating ① w.r.t. x partially, we get,

$$\Rightarrow \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x . \partial y} \quad \text{—③}$$

Again, Differentiating ② w.r.t y partially, we get

$$\Rightarrow \frac{\partial^2 u}{\partial y^2} = -\frac{\partial^2 v}{\partial y \partial x} \quad \text{—④}$$

Adding equations ③ & ④ we get,

$$\boxed{\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0}$$

$\Rightarrow u(x,y)$ is satisfying Laplace equation.

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Hence, $u(x,y)$ is a Harmonic function

→ Differentiating ① with y partially, we get,

$$\frac{\partial^2 u}{\partial y \cdot \partial x} = \frac{\partial^2 v}{\partial y^2} \quad \text{——⑤}$$

Differentiating ② with x partially, we get,

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{-\partial^2 v}{\partial x^2} \quad \text{——⑥}$$

$$⑤ - ⑥ \Rightarrow \boxed{\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0}$$

⇒ $v(x,y)$ is satisfying Laplace equation.

⇒ Hence, $v(x,y)$ is a Harmonic function.

★ <u>Conclusion</u> :-

→ Real and Imaginary parts of analytic function are harmonic functions. They are also said to be Harmonic Conjugates.

→ $z = x+iy$

$\bar{z} = x-iy$

$x = z+\bar{z}/2$ —①

$y = z-\bar{z}/2i$ —②

$f(z) = u(x,y) + i\,v(x,y)$

if x & y are replaced using z & $\bar{z}$ then $f(z)$ will become a function of z and $\bar{z}$.

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Ex. $f(z) = z^2$. Find $\frac{df(z)}{dz}$

Soln :- We have, $z^2 = (x+iy)^2$.

$$z^2 = x^2 - y^2 + 2ixy.$$

$$\frac{df(z)}{dz} = \frac{\partial f(z)}{\partial x} = 2x + 2iy = 2(x+iy) = 2z.$$

Ex. $f(z) = Im(z) = iy$ Determine the analyticity of $f(z)$.

Soln :- We have, $f(x) = iy = \underset{(u)}{0} + i \underset{(v)}{y}$.

Then, $\frac{\partial u}{\partial x} = 0$; $\frac{\partial u}{\partial y} = 0$; $\frac{\partial v}{\partial x} = 0$; $\frac{\partial v}{\partial y} = 1$.

$\therefore$ C-R equations are not satisfied.

$\therefore$ $f(z) = iy$ is not analytic.

Ex. The function $f(z) = e^z$ is analytic everywhere

Soln :- $f(z) = e^z = e^{x+iy} = e^x . e^{iy}$.

$$= e^x(\cos y + i\sin y) = \underset{(u)}{(e^x.\cos y)} + i \underset{(v)}{(e^x \sin y)}$$

Then, $\frac{\partial u}{\partial x} = e^x \cos y$; $\frac{\partial v}{\partial x} = e^x \sin y$

$\frac{\partial u}{\partial y} = -e^x \sin y$; $\frac{\partial v}{\partial y} = e^x \cos y$.

$\therefore$ $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ & $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$. C-R equations are satisfied.

$\therefore$ $f(z)$ is analytic everywhere.

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$$\frac{df(z)}{dz} = \frac{\partial f(z)}{\partial x} = u_x + iv_x = e^x \cos y + i e^x \sin y$$
$$= e^x e^{iy} = e^{x+iy} = e^z$$

$$= -i \cdot \frac{\partial f(z)}{\partial y} = -i(u_y + iv_y) = -i(-e^x \sin y) + e^x \cos y$$
$$= e^x(\cos y + i \sin y)$$
$$= e^x \cdot e^{iy} = e^{x+iy} = e^z$$

Theorem :- If $f'(z)=0$ in a domain D, then $f(z)$ is constant in D.

$$w = f(z) = \frac{dw}{dz} = \frac{\partial w}{\partial x} = \frac{\partial w}{\partial r} \cdot \frac{\partial r}{\partial x} + \frac{\partial w}{\partial \theta} \cdot \frac{\partial \theta}{\partial x}$$

$r^2 = x^2 + y^2$

$\theta = \tan^{-1}(y/x)$

$w = u(r,\theta) + iv(r,\theta)$

$$\boxed{\frac{dw}{dz} = e^{-i\theta} \cdot \frac{\partial w}{\partial r}}$$

$$\boxed{\frac{dw}{dz} = -\frac{i}{r} \cdot e^{-i\theta} \cdot \frac{\partial w}{\partial \theta}}$$

Ex. if $w = \log z$, find $\frac{dw}{dz}$ & determine where w is non-analytic.

$\Rightarrow$ $w = u + iv = \log z = \log(x+iy) = \log\left(\sqrt{x^2+y^2}\right) + i \cdot \tan^{-1}\left(\frac{y}{x}\right)$

$$= \frac{1}{2}\log(x^2+y^2) + i \cdot \tan^{-1}(y/x)$$

$\log(r \cdot e^{i\theta})$

$\Rightarrow \log(r) + i\theta \log e$

$\Rightarrow \log r + i\theta$

$$u = \frac{1}{2} \cdot \log(x^2+y^2) \; ; \quad v = \tan^{-1}(y/x)$$

$$\frac{\partial u}{\partial x} = \frac{1}{2} \cdot \frac{(2x)}{x^2+y^2} \cancel{\log(x^2+y^2)} \; ; \quad \frac{\partial u}{\partial y} = \frac{1}{2} \cdot \frac{2y}{x^2+y^2}$$

$$\frac{\partial v}{\partial x} = \frac{-1}{x^2}\left(\frac{x^2}{x^2+y^2}\right) y = \frac{-y}{x^2+y^2} \; ; \quad \frac{\partial v}{\partial y} = \frac{x}{x^2+y^2}$$

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$\therefore \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ & $\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$.

But ($x=0$ & $y=0$) i.e. $z=0$; partial derivatives donot exist. At $z=0$, ω is non-analytic ($z=0$ is a singular pt.)

★ Singular point :- A point $z=a$ is called a singular point of the complex function $w=f(z)$ if f is not analytic at $z=a$; but every neighbourhood of $z=a$ contains at least one point at which $f(z)$ is analytic.

⟶

★ Prove that $f(z)$ defined by $f(z) = \begin{cases} \frac{x^3(1+i) - y^3(1-i)}{x^2+y^2}, & z \neq 0 \\ 0, & z=0 \end{cases}$

is continuous and C-R equations are satisfied at the origin, yet $f'(0)$ does not exist.

Soln we have, $f(z)$

$\Rightarrow$ $\lim_{z\to 0} f(z) = \lim_{(x,y)\to(0,0)} \left[\frac{x^3(1+i) - y^3(1-i)}{x^2+y^2}\right]$

let $y = mx$;

$= \lim_{x\to 0} \frac{x^3(1+i) - m^3x^3(1-i)}{x^2+m^2x^2}$

$= \lim_{x\to 0} \frac{x[(1+i) - m^3(1-i)]}{[1+m^2]}$

$= 0$; $f(z)=0$

$\lim_{z\to 0} f(z) = f(z)$

$\therefore f(z)$ is continuous at origin.

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Q Show that a harmonic function $u(x,y)$ satisfies the formal diff. equation $\boxed{\dfrac{\partial^2 u}{\partial z \cdot \partial \bar{z}} = 0}$.

$\Rightarrow$

$$z = x + iy \; ; \; \bar{z} = x - iy \; ;$$

$$x = \frac{1}{2}(z + \bar{z}) \; ; \; y = \frac{1}{2i}(z - \bar{z}) \; ;$$

$$\frac{\partial u}{\partial \bar{z}} = \frac{\partial u}{\partial x} \cdot \underline{\frac{\partial x}{\partial \bar{z}}} + \frac{\partial u}{\partial y} \cdot \underline{\frac{\partial y}{\partial \bar{z}}}$$

$$= \frac{\partial u}{\partial x}\left(\frac{1}{2}\right) + \frac{\partial u}{\partial y} \cdot \left(\frac{-1}{2i}\right)$$

$$\frac{\partial u}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial u}{\partial x} + \frac{i\,\partial u}{\partial y}\right)$$

Partially w.r.t z, (Differentiating)

$$\Rightarrow \frac{\partial^2 u}{\partial z \cdot \partial \bar{z}} = \frac{1}{2} \quad \cancel{\frac{\partial^2 u}{\partial x^2}} \left[\frac{\partial w}{\partial x} \cdot \frac{\partial x}{\partial z} + \frac{\partial v}{\partial y} \cdot i \cdot \frac{\partial y}{\partial z} \right.$$

(where, $w = \partial u / \partial x$) $+ \dfrac{\partial w}{\partial y} \cdot \dfrac{\partial y}{\partial z} + \dfrac{\partial v}{\partial x} \cdot i \dfrac{\partial x}{\partial z}$ $\Big]$

$v = \partial u / \partial y$.

$$= \frac{1}{2}\left[\frac{\partial^2 u}{\partial x^2} \cdot \frac{\partial x}{\partial z} + \frac{\partial^2 u}{\partial y \partial x} \cdot \frac{\partial y}{\partial z} + \frac{\partial^2 u}{\partial y^2}\, i \cdot \frac{\partial y}{\partial z} + \frac{\partial^2 u}{\partial y \cdot \partial x}\, i \cdot \frac{\partial x}{\partial z}\right]$$

$$= \frac{1}{2} \frac{\partial^2 u}{\partial x^2} \cdot \frac{1}{2} + \frac{1}{2} \cdot \cancel{\frac{\partial^2 u}{\partial y \cdot \partial x}} \cdot \frac{1}{2i} - \frac{1}{2i} \cdot \frac{\partial^2 u}{\partial y^2} \cdot \frac{1}{2i}$$

$$- \frac{1}{2i} \cdot \cancel{\frac{\partial^2 u}{\partial x \cdot \partial y}} \frac{1}{2} = \frac{1}{4}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) = 0$$

($\because$ As u is harmonic ; $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$).

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★ Application of C-R equations to find the harmonic conjugate :-

Let, $\omega = f(z) = u(x,y) + iv(x,y)$ be the analytic function for which $u(x,y)$ is given and we have to obtain $v(x,y)$.

Since, v is a function of x & y. $(x, y \in \mathbb{R})$

$$dv = \frac{\partial v}{\partial x} \cdot dx + \frac{\partial v}{\partial y} \cdot dy .$$

$$\boxed{dv = -\frac{\partial u}{\partial y} \cdot dx + \frac{\partial u}{\partial x} \cdot dy} \quad \text{(By C-R equations)} - ①$$

The R.H.S. of the equation is of the form $Mdx + Ndy$, where; $M = -\frac{\partial u}{\partial y}$ and $N = \frac{\partial u}{\partial x}$.

$$\frac{\partial M}{\partial y} = \frac{-\partial^2 u}{\partial y^2} \text{ and } \frac{\partial N}{\partial x} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial^2 u}{\partial x^2} .$$

Since, u is a harmonic function; $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

$$\Rightarrow \frac{\partial^2 u}{\partial x^2} = -\frac{\partial^2 u}{\partial y^2} \Rightarrow \boxed{\frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}} \quad —②$$

from ②; ① is an exact differential equation and by integrating ① we get $v(x,y) + c$.

If $f(z) = u(x,y) + iv(x,y)$ is analytic, then v is called harmonic conjugate of u.

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$f(z) = u + iv \rightarrow$ analytic

$if(z) = i(u+iv) = -v + iu \rightarrow$ analytic } Antisymmetric property

v is a harmonic conjugate of u iff. u is a harmonic conjugate of $-v$.

★ Milne-Thompson's Method :-

I: Since, $f(z) = u(x,y) + i\,v(x,y)$, where $x = \frac{1}{2}(z+\bar{z})$ and $y = \frac{1}{2i}(z+\bar{z})$. Now we may write $f(z) =$

$$u\left[\frac{1}{2}(z+\bar{z}), \frac{1}{2i}(z-\bar{z})\right] + iv\left[\frac{1}{2}(z+\bar{z}), \frac{1}{2i}(z-\bar{z})\right];$$

II: On putting $z = \bar{z}$, we get $f(z) = u(z,0) + iv(z,0)$,

III: Therefore; $f'(z) = \frac{\partial u}{\partial x} + i.\frac{\partial v}{\partial x} = u_x + i\cancel{v_x}$

$$= u_x - i u_y.$$

$\left(\because \frac{df(z)}{dz} = \frac{\partial f}{\partial x} = -i.\frac{\partial f}{\partial y}\right)$ $\quad \left(\because \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}\right)$

$\Rightarrow f'(z) = \phi_1(x,y) - i\phi_2(x,y)$.

$u_x = \phi_1(x,y)$; $u_y = \phi_2(x,y)$ then.

$f'(z) = \phi_1(x,y) - i\phi_2(x,y) = \phi_1(z,0) - i\phi_2(z,0)$ —— ①

IV: Integrating ① ; we get

$$\boxed{f(z) = \int \phi_1(z,0)\,dz - i\int \phi_2(z,0)\,dz + C}$$

{ where, C is the constant of integration}

Similarly,

if $v(x,y)$ be given we have

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$$\boxed{f(z) = \int \psi_1(z,0)\,dz + i\int \psi_2(z,0)\,dz + c'}$$

where $\psi_1(x,y) = \frac{\partial v}{\partial y}$; $\psi_2(x,y) = \frac{\partial v}{\partial x}$

c' is the constant of Integration.

$\longrightarrow$

Q. If $f(z) = \frac{x^3y\,(y-ix)}{x^6+y^2}$; $(z \neq 0)$, $f(0)=0$. Prove that $\frac{f(z)-f(0)}{z} \to 0$ as $z \to 0$ along any radius vector but not as $z \to 0$ in any manner.

$\Rightarrow$ $f(z) = x^3y(y-ix)/x^6+y^2$; $x = r\cos\theta$, $y = r\sin\theta$.

$\to$ $f(z) = \frac{r^3\cos^3\theta\,.(r\sin\theta)\,(r\sin\theta - ir\cos\theta)}{r^6\cos^6\theta + r^2\sin^2\theta}$.

$= \frac{r^3\cos^3\theta\sin\theta(\sin\theta - i\cos\theta)}{r^4\cos^6\theta + \sin^2\theta}$.

If $z \to 0$, $r \to 0$;

$\lim_{z\to 0} \frac{f(z)-f(0)}{z} = \lim_{r\to 0} \frac{r^2\cos^3\theta\sin\theta(\sin\theta - i\cos\theta)}{(r^4\cos^6\theta+\sin\theta)\,(\cos\theta + i\sin\theta)} = 0$.

$f(z) = x^3y\,(y-ix)/x^6+y^2$ let $z \to 0$ along $y = mx$.

$\lim_{z\to 0} \frac{f(z)-f(0)}{z} = \lim_{z\to 0} \frac{x^3y\,(y-ix)}{(x^6+y^2)\,(x+iy)} = \lim_{x\to 0} \frac{x^3\,mx\,(mx-ix)}{(x^6+m^2x^2)\,(x+imx)}$.

$= \lim_{x\to 0} \frac{mx^2\,(m-i)}{(x^4+m^2)\,(1+im)} = 0$.

along $y = x^3$ $\lim_{z\to 0} \frac{f(z)-f(0)}{z} = \frac{x^6(x^3-ix)}{(x^6+x^6)(x+ix^3)} = \frac{-i}{2} \neq 0$.

Hence proved.

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Ex. Show that $u(x,y) = e^{-x}[x\sin y - y\cos y]$ is a harmonic and find its harmonic conjugate $v(x,y)$ such that $f(z) = u(x,y) + iv(x,y)$ is analytic.

Ans :- We have ; $u(x,y) = e^{-x}[x\sin y - y\cos y]$.

is continuous and differentiable.

$$\frac{\partial u}{\partial x} = e^{-x}\sin y - x.e^{-x}\sin y + y.e^{-x}\cos y \quad \text{— ①}$$

$$\frac{\partial u}{\partial y} = -x.e^{-x}\cos y + e^{-x}\cos y - y.e^{-x}\sin y \quad \text{— ②}$$

$$\frac{\partial^2 u}{\partial x^2} = -e^{-x}\sin y - e^{-x}\sin y + x e^{-x}\sin y \mp y.e^{-x}\cos y$$

$$\frac{\partial^2 u}{\partial y^2} = -\left(+x e^{-x}\sin y \mp e^{-x}\sin y - y e^{-x}\cos y - e^{-x}\sin y\right)$$

$$\Rightarrow \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

we know, $\left.\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x} \;\&\; \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}\right\}$ — ③

Integrating ① w.r.t. y, treating x as const. (using 3)

$$v(x,y) = y.e^{-x}\sin y + x.e^{-x}\cos y + g(x).$$

$$\frac{\partial v}{\partial x} = y\sin y.(-e^{-x}) + e^{-x}\cos y - x.e^{-x}\cos y + g'(x) \quad \text{— ④}$$

from ②, ③ & ④ $\therefore g'(x) = 0 \Rightarrow g(x) = c.$

$$\Rightarrow \boxed{v(x,y) = y.e^{-x}\sin y + x.e^{-x}\cos y + c}$$

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Examine the nature of the function $f(z) = \frac{x^2y^5(x+iy)}{x^4+y^{10}}$, $z \neq 0$ and $f(0)=0$ in a region including the origin.

Soln :- we have, $f(z) = \frac{x^2y^5(x+iy)}{x^4+y^{10}}$

$\lim_{z\to 0} f(z) = 0.$ $\qquad u(x,y) = \frac{x^3y^5}{x^4+y^{10}}$

$y^5 = x^2$; $= \lim_{x\to 0} \frac{x^2(x^2)(x+ix^{2/5})}{x^4+x^4}$ $\qquad v(x,y) = \frac{x^2y^6}{x^4+y^{10}}$.

$= \lim_{x\to 0} (x+ix^{2/5})/2 = 0.$

$\lim_{z\to 0} \frac{f(z)-f(0)}{z} = \frac{\left(\frac{x+ix^{2/5}}{2}\right)-0}{(x+ix^{2/5})} = \frac{1}{2}$ $\quad (y^5=x^2)$ — ①

$\lim_{z\to 0} \frac{f(z)-f(0)}{z} = \lim_{x\to 0} \frac{(x^2)(m^5x^5)(x+imx)}{x^4+m^{10}x^{10}} = \frac{\cancel{x^8}\,x^4 m^5(1+im)}{\cancel{x^4}(1+m^{10}x^6)}$ $(y=mx)$

$= \lim_{x\to 0} x^4 \frac{m^5(1+im)}{1+m^{10}x^6} = 0$ — ②

from ① & ② ;

∴ Derivative for f at $z=0$ doesnot exist. (though limit exists)

Q. If $u(x,y) = e^{+x}[x.\text{siny} - y\cos y]$, find the analytic function $u+iv$.
cosy siny

$u(x,y) = e^{+x}[x\sin y - y\cos y]$.
cosy siny

→

$\frac{\partial u}{\partial x} = e^x\cos y + x.e^x \cancel{\sin y}\cos y - e^x y\sin y$ — ① .

$= e^x(x\cancel{\sin y}\cos y - y\sin y + \cos y)$

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$$\frac{\partial u}{\partial y} = -xe^x\sin y - e^x\sin y - e^x y\cos y$$
$$= e^x(-x\sin y - \sin y - y\cos y) \quad \text{---} \ (2)$$

$$\frac{\partial^2 u}{\partial x^2} = e^x x\cos y + e^x\cos y - e^x y\sin y + e^x\cos y$$

$$\frac{\partial^2 u}{\partial y^2} = e^x(-x\cos y - \cos y + y\sin y - \cos y)$$

$$\Rightarrow \quad \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

Since, $v(x,y)$ is a function of (x,y).

$$dv = \frac{\partial v}{\partial x}.dx + \frac{\partial v}{\partial y}.dy.$$

$$= -\frac{\partial u}{\partial y}.dx + \frac{\partial u}{\partial x}.dy \quad [\because \text{ C-R equations}.$$

$$dv = e^x(x\sin y + \sin y + y\cos y).dx$$
$$+ e^x(x\cancel{\cos y} - y\sin y + \cos y).dy.$$

Integrating it, we get.

$$\int dv = \int e^x(x\sin y + \sin y + y\cos y).dx$$
$$+ \int e^x(\cancel{x\cos y} - y\sin y + \cos y)\,dy + c.$$

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$$\boxed{\int uv = u\int v - \int u'.\int v}$$

$= \sin y\left(e^x . x^2 x . e^x - \int e^x . dx\right) + e^x . \sin y + e^x y \cos y + e^x x \cancel{\sin y} + e^x \cancel{\sin y} + -e^x\left(y.(-(\cos y) - \int -\cos y\, dy\right) + c.$

$= \sin y\,(x.e^x - e^x) + e^x \sin y + e^x y \cos y + e^x x \cancel{\cos y} . \sin y.$
$+ e^x \sin y + e^x . y\cos y + e^x(-\sin y) + c.$

$= e^x \cancel{x} x \sin y + \sin y + 2y \cos y) - e^x . \sin y + c.$ ~~c = A(x) + B(y)~~

$V(x,y) = e^x (x\sin y + y\cos y) + c$. (on calculating c we get

$$\boxed{V(x,y) = e^x (x\sin y + y\cos y)}$$

$f(z) = u + iv$

$= e^x(x\cos y - y\sin y) + i\,(e^x(x\sin y + y\cos y) + c')$

$= e^x(x\cos y - y\sin y + ix\sin y + iy\cos y) + ic'$

$= e^x\left(x.(\cos y + i\sin y) + iy(\cos y + i\sin y)\right) + ic'$

$= e^x.\left(e^{iy} . (x + iy)\right) + ic'$

$= e^{x+iy} . (x+iy) + ic' = e^z . z + ic'.$

$$\Rightarrow \boxed{f(z) = e^z . z + c''}$$

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(II) If $f'(z)=0$ in D, then $f(z)$ is constant in D.

Proof :- $f'(z)=0$.

$$\frac{df(z)}{dz} = \frac{\partial}{\partial x} f(z) = \frac{\partial u}{\partial x} + i.\frac{\partial v}{\partial x} = 0$$

$$\Rightarrow \frac{\partial u}{\partial x} = 0 \;;\; \frac{\partial v}{\partial x} = 0 \;; \quad \text{——} (1)$$

$$\frac{df(z)}{dz} = -i\frac{\partial}{\partial y} f(z) = -i\left(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right) = 0$$

$$\Rightarrow \frac{\partial u}{\partial y} = 0 \;;\; \frac{\partial v}{\partial y} = 0 \;; \quad \text{——} (2)$$

from (1) & (2) ; $u(x,y)$ and $v(x,y)$ are both constant functions. Therefore, $f(z) = u(x,y) + iv(x,y)$ is also constant.

(III) Let $|f(z)|$ be constant in a region, where $f(z)$ is analytic. Then $f(z)$ is constant.

Proof :- Let, $f(z) = u + iv$ be an analytic function such that $|f(z)| = c$ (constant) (say).

$$\Rightarrow u^2 + v^2 = c^2.$$

Differentiating w.r.t. x. (partially)

$$2u\frac{\partial u}{\partial x} + 2v.\frac{\partial v}{\partial x} = 0 \Rightarrow u.\frac{\partial u}{\partial x} + v\frac{\partial v}{\partial x} = 0 \quad \text{——} (1)$$

Differentiating w.r.t. y (partially)

$$2u.\frac{\partial u}{\partial y} + 2v.\frac{\partial v}{\partial y} = 0 \Rightarrow u.\frac{\partial u}{\partial y} + v.\frac{\partial v}{\partial y} = 0 \quad (2)$$

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Use $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ & $\frac{\partial u}{\partial y} = \frac{-\partial v}{\partial x}$ in ① & ② ;

you will get $\frac{\partial u}{\partial x}$, $\frac{\partial v}{\partial y}$, $\frac{\partial u}{\partial y}$, $\frac{\partial v}{\partial x}$ all zero.

i.e. u and v as constant $\Rightarrow$ f is constant.

III) If $w = f(z) = u+iv$ and $u-v = e^x(\cos y - \sin y)$.

find w in terms of z. ~~$v = u - e^x(\cos y - \sin y)$~~

Soln :- $u - v = e^x(\cos y - \sin y)$.

$$\frac{\partial u}{\partial x} - \frac{\partial v}{\partial x} = e^x(\cos y - \sin y).$$

$$\frac{\partial u}{\partial y} - \frac{\partial v}{\partial y} = -e^x(\sin y + \cos y).$$

~~$-v_x = -u_x + e^x(\cos y - \sin y)$~~

~~$(u_x - v_x - u_y + v_y) = 2e^x \cos y$~~ ~~$v_y = u_y + e^x(\sin y + \cos y)$~~

~~$(v_x - u_y) + (u_y - u_x + 2e^x \cos y) = 2e^x \cos y)$~~

$\Rightarrow$ $v_y - v_x = e^x(\cos y - \sin y)$

$v_x + v_y = e^x(\sin y + \cos y)$.

$\Rightarrow$ $v_y = e^x \cos y$.

$\Rightarrow$ $v = e^x \sin y + g(x)$

$v_x = e^x \sin y$.

$\cancel{e^x \sin y} + g'(x) = \cancel{e^x \sin y}$

$g'(x) = 0$

$g(x) = 0$

$\Rightarrow$ $v = e^x \sin y + C$.

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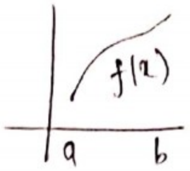
★ Complex Integration

$y = f(x)$



$\int_a^b f(x)\,dx$

Definite & Indefinite.

Improper Integral.

Integration as a Limit of sum

$\int_{-\infty}^{\infty} f(x)\,dx \equiv \int f(x).dx$.

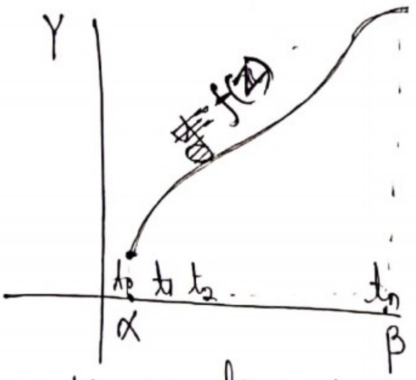
$\int_a^b f(x).dx$. $h = \frac{b-a}{n}$;

$= \lim_{n \to \infty} \sum_{r=0}^{n} f(a + rh)$

Reimann Integral.

★ Rectifiable Curve :- Let L be a continuous curve with equation $z = x(t) + iy(t)$, $\alpha \le t \le \beta$ ①. Suppose we divide $[\alpha, \beta]$ into n sub-intervals $[t_{k-1}, t_k]$; $k = 1, 2, \dots, n$. By Introducing n-1 intermediate points $t_1, t_2, \dots, t_{n-1}$ satisfying the inequality

$\alpha = t_0 < t_1 < t_2 < \dots < t_n = \beta$.



The set $P = \{t_0, t_1, \dots, t_n\}$ is called a partition and the largest length of the subintervals i.e. largest length among $t_1 - t_0$, $t_2 - t_1$, ..., $t_n - t_{n-1}$ is known as norm of the partition P, denoted by $\|P\|$.

Let $z_0, z_1, z_2, \dots, z_n$ be the points on the curve corresponding to the values $t_0, t_1, t_2, \dots, t_n$, i.e. $z(t_k) = z_k$. Clearly, the length of the polygon curve inscribed in L obtained by joining successively $z_0, z_1, z_2, \dots, z_n$ by straight line seg.

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~~1 sec 3x10^8 steps~~

Art By
Kunj Taneja

X

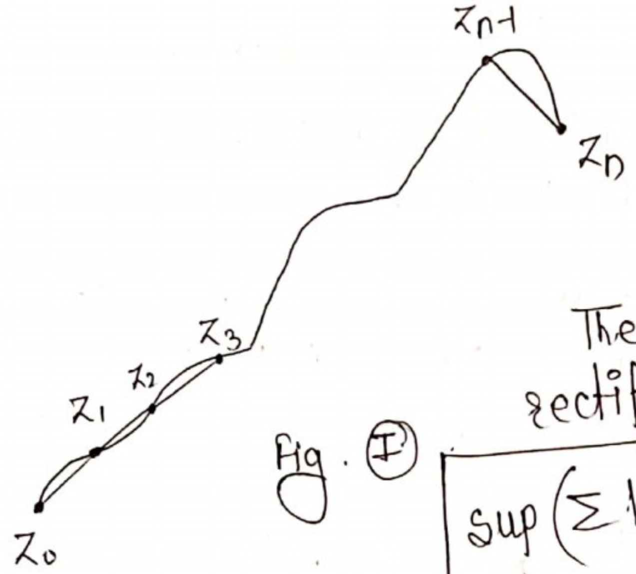

Fig. (I)

given in figure I is equal to

$$\sum_{k=1}^{n} |z_k - z_{k-1}| \quad \text{—— (2)}.$$

The curve L is said to be rectifiable if.

$$\boxed{\sup\left(\sum |z_k - z_{k-1}|\right) = l < \infty} \;;$$

where the least upper bound or supremum is taken over all possible partitions of $[\alpha, \beta]$.

Contour :- Let $z = x(t) + iy(t) \longrightarrow (1)$ where t runs through the interval $\alpha \leq t \leq \beta$. and $x(t)$, $y(t)$ are continuous functions of t represent a continuous arc L in the complex plane.

If equation (1) satisfied by more that one value of t in given range, then the point z or (x, y) is a multiple point of the arc.

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A continuous arc without multiple point is called ~~multiple~~ arc. (Jordan arc)

If for a point z on a Jordan arc, z as expressed in ① is single valued and $x(t), y(t)$ are continuous and if $x'(t), y'(t)$ are continuous in the range $\alpha \le t \le \beta$, then arc is known as regular arc of the Jordan curve.

A Jordan curve consisting of continuous chain of a finite number of regular arcs is called a Contour.

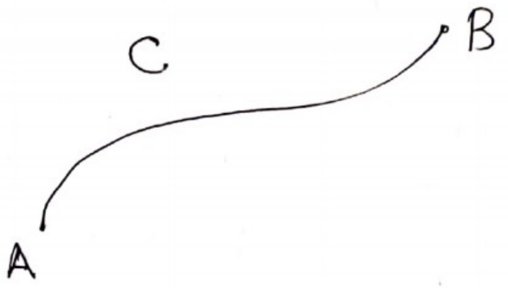
* **Integral as a limit of sum :-**

$$z = x(t) + i y(t) \; ; \; \alpha \le t \le \beta \quad [\alpha, \beta].$$

If $f(z)$ is a continuous function of the complex variable $z = x + iy$ defined at all points of a curve C having end points A and B. Divide C into n parts at the points.

$(A = A_0(z_0)), A_1(z_1), A_2(z_2), \ldots ; (A_n(z_n) = B)$.

Let, $\delta z_i = z_i - z_{i-1}$ and ξ_i be any point on the arc $A_{i-1} A_i$.



Then the integration as a limit of sum

$$\boxed{\sum_{i=1}^{n} f(\xi_i)\,\delta z_i}$$

as $n \to \infty$ in such a way that the length of the chord z_i approaches to zero, is called the line integral of $f(z)$ take along curve C. i.e. $\int_C f(z).dz$

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$w = f(z) = u(x,y) + iv(x,y)$

$z = x+iy$

$dz = dx + i\,dy$

$$\int_C f(z)\,dz = \int_C (u(x,y) + iv(x,y)).(dx+idy).$$

$$= \int_C (u(x,y)\,dx - v(x,y).dy) + i\,(v(x,y).dx + u(x,y)\,dy)$$

$$= \int_C (u(x,y)\,dx - v(x,y)\,dy) + i\int_C (v(x,y).dx + u(x,y).dy).$$

$$\boxed{\therefore \int_C f(z)\,dz = \int_C (u(x,y)\,dx - v(x,y).dy) + i\int_C (v(x,y).dx + u(x,y)\,dy)}$$

Line integral of $f(z)$ over C.

Ex. Using the def^n of an integral as the limit of sum evaluate the following integrals. (i) $\int_C .dz$ (ii) $\int_C |dz|$ (iii) $\int_C z.dz$ where C is any rectifiable curve joining the points a and b.

(i) By using the def^n of integral as a limit of sum

$$\int_C .dz = \cancel{\lim_{n\to\infty} f(\xi_r)(z_r - z_{r-1})} = \lim_{n\to\infty} \sum_{r=1}^{n} 1\,(z_r - z_{r-1})$$

$$(f(z) = f(\xi_r) = 1)$$

$$= \lim_{n\to\infty} [(z_1 - z_0) + (z_2 - z_1) + \dots + (z_n - z_{n-1})]$$

$$= \lim_{n\to\infty} [z_n - z_0] = \lim_{n\to\infty} [b-a] = b-a \text{ (Ans.)}$$

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ii) $\int_C |dz| = \lim_{n\to\infty} \sum_{r=1}^{n} |z_r - z_{r-1}|$

$= \lim_{n\to\infty} \left(|z_1 - z_0| + |z_2 - z_1| + \ldots + |z_n - z_{n-1}| \right).$

$= \lim_{n\to\infty} \left(\text{arc } z_0 z_1 + \text{arc } z_1 z_2 + \ldots + \text{arc } z_{n-1} z_n \right).$

$=$ arc. length of C.

iii) $\int_C z.dz = \lim_{n\to\infty} \sum_{r=1}^{n} f(\xi_r)(z_r - z_{r-1})$

$f(z) = z.$ $\Rightarrow$ Since, ξ_r is arbitrary point in the r^{th} arc joining $z_{r-1} z_r$.

Thus, Taking $\xi_r = z_{r-1}$ and $\xi_r = z_r$; then

$\int_C z.dz = \lim_{n\to\infty} \sum_{r=1}^{n} z_{r-1}(z_r - z_{r-1})$ —— ①

Also, $\int_C z.dz = \lim_{n\to\infty} \sum_{r=1}^{n} z_r(z_r - z_{r-1})$ —— ②

Adding ① & ②, we get

$2 \int_C z.dz = \lim_{n\to\infty} \sum_{r=1}^{n} (z_r - z_{r-1})(z_r + z_{r-1}).$

$= \lim_{n\to\infty} \sum_{r=1}^{n} (z_r^2 - z_{r-1}^2).$

$= \lim_{n\to\infty} (z_n^2 - z_0^2)$

$\Rightarrow 2\int_C z.dz = \lim_{n\to\infty} (b^2 - a^2).$

$\Rightarrow \int_C z.dz = \dfrac{b^2 - a^2}{2}$

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Ex. Prove that

(i) $\displaystyle\int_C \frac{dz}{z-a} = 2\pi i$.

(ii) $\displaystyle\int_C (z-a)^n dz = 0$ [n, any integer $\neq -1$] ;

where C is the circle $|z-a| = r$.

Sol^n:- (i) $\displaystyle\int_C \frac{dz}{z-a} = \int_0^{2\pi} \frac{1}{\cancel{r.e^{i\theta}}} . \cancel{re^{i\theta}} . i d\theta$

$z = r.e^{i\theta} + a$

$dz = re^{i\theta} . i d\theta$.

$\displaystyle = i. \int_0^{2\pi} d\theta = 2\pi i$.

$|z-a| = r$

$\therefore z - a = r.e^{i\theta}$

(ii) $\displaystyle\int_C (z-a)^n dz = \int_0^{2\pi} r e^{in\theta} . re^{i\theta} . i d\theta$

$\displaystyle = ir^2 \int_0^{2\pi} e^{i\theta(n+1)} d\theta$

$\displaystyle = ir^2 \int_0^{2\pi} [\cos(n+1)\theta + i \sin(n+1)\theta.] . d\theta$.

$\displaystyle = ir^2 \left[\sin(n+1)\theta \Big|_0^{2\pi} (n+1) - i\cos(n+1)\theta \Big|_0^{2\pi} (n+1) \right]$

$= ir^2 (n+1) [(0-0) - i(1-1)]$

$= 0$.

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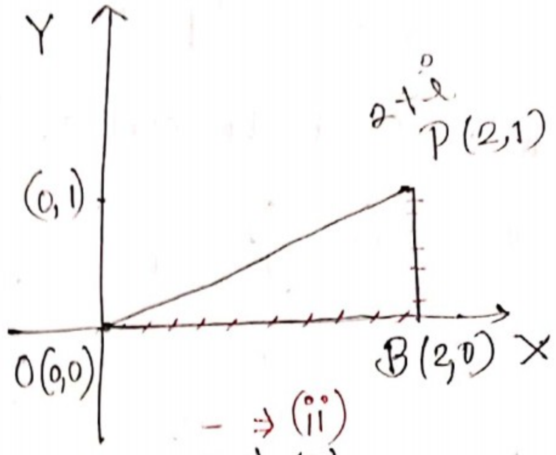
Ex. Evaluate $\int_0^{2+i} (\bar{z})^2 . dz$ along

(i) the line $y = x/2$

(ii) the real axis to 2 and then vertically to $2+i$



Ans :- let, $I = \int_0^{2+i} (\bar{z})^2 . dz$

(i) We have to evalute I along the line $y = x/2 \Rightarrow x = 2y$.

$z = x + iy = 2y + iy = (2+i)y$

$\bar{z} = x - iy = 2y - iy = (2-i)y$.

$dz = (2+i) . dy$

$$I = \int_0^1 (2-i)^2 . y^2 (2+i)\, dy = (2-i)^2 (2+i) \int_0^1 y^2 . dy$$

$$= (2-i)(5) \frac{y^3}{3}\Big|_0^1 = \frac{5(2-i)}{3}.$$

(ii) $$I = \int_0^{2+i} (\bar{z})^2 dz = \int_{OB} (\bar{z})^2 . dz + \int_{BP} (\bar{z})^2 . dz.$$

Now, at OB, $y = 0 \Rightarrow z = x \Rightarrow dz = dx$.

$\bar{z} = x$

$$\Rightarrow \int_{OB} (\bar{z})^2 dz = \int_0^2 x^2 . dx = \frac{x^3}{3}\Big|_0^2 = 8/3$$

Now, at BP, $(x = 2)$ $z = x + iy = 2 + iy \quad dz = i . dy$

$\bar{z} = 2 - iy$

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$$\Rightarrow \int_{BP} (\bar{z})^2 dz = \int_0^1 (2-iy)^2 \, i \, dy = i\int_0^1 (4 - y^2 - 4iy).dy$$

$$= i.\ 4(y)_0^1 - \frac{(y^3)_0^1}{3} - 4i\frac{(y^2)_0^1}{2}$$

$$= 4i - \frac{i}{3} - \frac{4i^2}{2} = 2 + \frac{11i}{3}$$

$$I = \int_{OB} (\bar{z})^2 dz + \int_{BP} (\bar{z})^2 dz = \frac{8}{3} + 2 + \frac{11i}{3} = \frac{14+11i}{3}$$

5. $f(z) = (\bar{z})^2$ in $[0, 2+i]$. Check analyticity.

$$= (x-iy)^2 = x^2 - y^2 - i\,2xy$$

$$= (x^2-y^2) + i(-2xy)$$

$u = x^2 - y^2$ $\qquad v = -2xy$

$u_x = 2x$ — (1) $\qquad v_x = -2y$ — (3)

$u_y = -2y$ — (2) $\qquad v_y = -2x$ — (4)

$u_x \neq v_y$ & $v_x \neq u_y$

$\therefore$ CR eqns not satisfied (Not analytic)

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(I) Cauchy - Goursat Theorem :-

Cauchy's Fundamental Theorem :-

St. if $f(z)$ is analytic, with a continuous derivative in a simply connected domain G and C is a closed contour lying in G, then $\int_C f(z).dz = 0$.

Simply connected domain ~~cannot~~ can be shrunk into a point.

Multi connected domain cannot be shrunk into a point.

Green's Theorem :- Let C be a positively oriented piecewise smooth, simple closed curve in a plane and let D be the region bounded by C. If L and M are functions of x & y defined on an open region containining D and have continuous partial derivatives there, then.

$$\boxed{\oint_C (Ldx + Mdy) = \iint_D \left(\frac{\partial M}{\partial x} - \frac{\partial L}{\partial y}\right) dx.dy}$$

Proof of Cauchy's F.T. :- Let, $f(z) = u(x,y) + iv(x,y)$ ——(1) be an analytic function with a continuous derivative in a simply connected domain G and C is a closed contour lying in G.

Then by C-R eqns for analytic function ;

$f'(z) = u_x + iv_x = v_y - iu_y$ ——(2) ; for all points in the domain

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Since, $f'(z)$ is continuous, the four partial derivatives u_x, u_y, v_x, v_y must also be continuous in Domain G.

Also, $z = x+iy \Rightarrow dz = dx + i\,dy$.

$$\int_C f(z)\,dz = \int_C (u+iv).(dx+i\,dy)$$

$$\int_C f(z)\,dz = \int_C (u\,dx - v\,dy) + i\int_C (v\,dx + u\,dy) \quad \text{——— (3)}$$

Therefore By Green's theorem, we have

$$\int_C f(z).dz = \iint_G -\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) dx\,dy + i\iint_G \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) dx\,dy \quad \text{——— (4)}$$

$$\Rightarrow \left(\because \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \;\&\; \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} \right) \quad \text{——— (5)}$$

$$\Rightarrow \boxed{\int_C f(z)\,dz = 0} \quad (\text{from (4) \& (5)})$$

★ <u>The Absolute value of a complex integral</u> :-

<u>Theorem</u>: If $f(z)$ is continuous on a closed contour C of length l and $|f(z)| \le M$ for every z on C, then

$$\left|\int_C f(z)\,dz\right| \le \int_C |f(z)|.|dz| = Ml$$

<u>Proof</u> :-

$$\int_C f(z)\,dz = \lim_{n\to\infty} \sum_{r=1}^{n} f(\xi_r)\,(z_r - z_{r-1}).$$

where ξ_r is any point on the arc $z_r z_{r-1}$.

Now, $$\left|\sum_{r=1}^{n} f(\xi_r)(z_r - z_{r-1})\right| \le \sum_{r=1}^{n} |f(\xi_r)||(z_r - z_{r-1})|$$

$$\le M.\sum_{r=1}^{n} |z_r - z_{r-1}|$$

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$$\lim_{n\to\infty} \left| \sum f(\xi_r)(z_r - z_{r-1}) \right| \le \lim_{n\to\infty} M \sum_{r=1}^{n} |z_r - z_{r-1}|$$

$$\le Ml = \int_C |f(z)|\,|dz| .$$

$$\lim_{n\to\infty} \left| \sum f(\xi_r)(z_r - z_{r-1}) \right| \le \int_C |f(z)|\,|dz| .$$

$$\Rightarrow \left| \int_C f(z)\,dz \right| \le \int_C |f(z)|\,|dz|$$ Hence proved.

* Circle $x^2+y^2+2gx+2fy+c=0$

$(-g,-f)$, $r=\sqrt{g^2+f^2-c}$

$|z-z_0|^2 = r^2 \Rightarrow (z-z_0)(\overline{z-z_0}) = r^2$.

Center at $(0,0)$ $x^2+y^2=r^2$; $|z|=r$; $z\cdot\bar{z}=r^2$

The general equation of a circle in argand plane represented by $az\bar{z}+\alpha z+\bar{\alpha}\bar{z}+c=0$ —— (1)

where a, c are real constants, α is a complex constant and z is a complex variable.

let $\alpha = a_1+ia_2$ $z=x+iy$.

$\bar{\alpha} = a_1-ia_2$ $\bar{z}=x-iy$.

$a(x^2+y^2)+(a_1+ia_2)(x+iy)+(a_1-ia_2)(x-iy)+c=0$.

$a(x^2+y^2)+2a_1x-2a_2y+c=0$ —— (2)

Centre is $\left(\frac{-a_1}{a}, \frac{a_2}{a}\right)$

radius $\Rightarrow \sqrt{\left(\frac{+a_1}{a}\right)^2+\left(\frac{-a_2}{a}\right)^2-\frac{c}{a}} = \sqrt{\frac{a_1^2+a_2^2-ac}{a^2}}$

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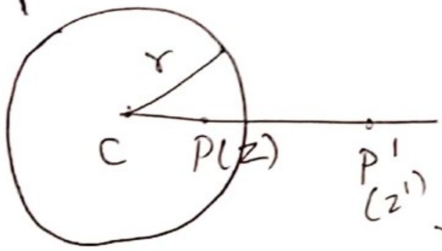
$$r = \sqrt{\frac{a_1^2 + a_2^2 - ac}{a^2}} = \sqrt{\frac{\alpha\bar{\alpha} - ac}{a^2}}$$

$$\frac{-a_1}{a} + i\frac{a_2}{a} = -\left(\frac{a_1 - ia_2}{a}\right) = \frac{-\bar{\alpha}}{a} = \text{Centre}.$$

Inverse point of a circle :-



$\boxed{CP.CP' = r^2}$

p' is inverse of p w.r.t circle

The relation w.r.t. the circle $azz̄ + \alpha z + \bar{\alpha}\bar{z} + c = 0$ ① . Then let p' & p be the inverse pts. w.r.t. circle ①.

$$CP.CP' = r^2$$

$$\left|z + \frac{\bar{\alpha}}{a}\right| \left|z' + \frac{\bar{\alpha}}{a}\right| = r^2 \quad \text{—— ⑤}$$

Since, C, P, P' are collinear, then

$$\arg\left(z + \frac{\bar{\alpha}}{a}\right) = \arg\left(z' + \frac{\bar{\alpha}}{a}\right) \quad \text{—— ③}$$

which may be written as

$$\arg\left(z' + \frac{\bar{\alpha}}{a}\right) = -\arg\left(\bar{z} + \frac{\alpha}{a}\right) \quad \text{—— ④} \qquad \left(\because \arg(z) = -\arg(\bar{z})\right)$$

$$\Rightarrow \arg\left(z' + \frac{\bar{\alpha}}{a}\right) + \arg\left(\bar{z} + \frac{\alpha}{a}\right) = 0.$$

from ④ $\left(z' + \frac{\bar{\alpha}}{a}\right)\left(\bar{z} + \frac{\alpha}{a}\right)$ is a positive real number.

$$\left|z' + \frac{\bar{\alpha}}{a}\right| \left|\bar{z} + \frac{\alpha}{a}\right| = r^2. \quad \left(\text{from ⑤ } |z| = |\bar{z}|\right) \quad \text{—— ⑥}$$

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$z \cdot \bar{z} = r^2 \quad |z||z'| = r^2 \quad \therefore \quad |z| = \frac{r^2}{|z'|}$

if center of circle is at origin

$z' \bar{z} = r^2 \Rightarrow z' = r^2/\bar{z}$ ..

★ **connected region :-**

A region is said to be connected region G if any two points in the region G then curve joining these two points lies entirely within the region.

★ **Simply connected & Multiply connected :-**

A connected region is simply connected if every closed curve in the region can be shrunk to a point without passing out of the region otherwise region is said to be multiply connected.

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx \ ; \ a \leq c \leq b$$

① $$\int_L (f(z)+g(z))\,dz = \int_L f(z)\,dz + \int_L g(z)\,dz$$

② $$\int_L f(z)\,dz = -\int_{-L} f(z)\,dz$$

③ $$\int_{L_1+L_2} f(z)\,dz = \int_{L_1} f(z)\,dz + \int_{L_2} f(z)\,dz,$$ where final pt of L_1 is initial for L_2.

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③ can be generalized for $L = L_1 + L_2 + \dots + L_n$

④ $\int_L c \cdot f(z)\,dz = c \int_L f(z) \cdot dz$; $\{c_1 f_1(z) + c_2 f_2(z) + \dots + c_n f_n(z)\}$

⑤ $\left| \int_L f(z)\,dz \right| \leq \int_L |f(z)|\,|dz|$.

★ Cauchy's Integral Formula :-

St. Let $f(z)$ be an analytic function in a simply connected domain G bounded by a rectifiable Jordon curve C and continuous on C. Then.

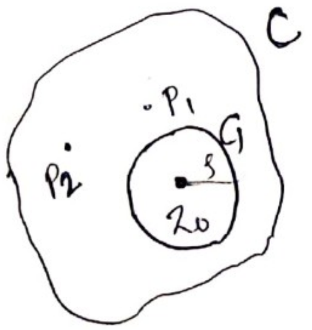
$$f(z_0) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z - z_0} \cdot dz.$$

OR

If $f(z)$ is analytic within and on closed contour C and z_0 is any point within C. Then

$$\boxed{f(z_0) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z - z_0} \cdot dz}$$

Proof :- Given, $f(z)$ is analytic f^n within and on closed contour C and z_0 is any pt. within C.



We describe a circle C_1 defined by the eq^n $|z - z_0| = \rho < d$.

where d is distance of z_0 from C.

Then the f^n $\phi(z) = \dfrac{f(z)}{z - z_0}$

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is analytic in the doubly connected region bounded by C and C_1

Hence, we have $\int_C \phi(z)\,dz = \int_{C_1} \phi(z)\,dz$ —— ① where C & C_1 both are traversed in counterclockwise dirn

It is clear that the integral on R.H.S. of ① is independent of ρ and so we may choose ρ as small as we please.

Now, $\frac{1}{2\pi i}\int_{C_1} \frac{f(z)}{z-z_0}\,dz = \frac{1}{2\pi i}\int_{C_1}\left(\frac{f(z)-f(z_0)}{z-z_0} + \frac{f(z_0)}{z-z_0}\right)dz$

$$= \frac{1}{2\pi i}\int_{C_1} \frac{f(z)-f(z_0)}{z-z_0}\,dz + \frac{1}{2\pi i}\int_{C_1} \frac{f(z_0)}{z-z_0}\,dz \quad \text{—— ②}$$

Writing $z - z_0 = \rho e^{i\theta} \Rightarrow dz = \rho . i e^{i\theta} . d\theta$

$$\int_{C_1} \frac{f(z_0)}{z-z_0}\,dz = f(z_0)\int_0^{2\pi} \frac{\rho . e^{i\theta} . d\theta}{\rho e^{i\theta}} = (2\pi i) f(z_0)$$

$$\Rightarrow \quad f(z_0) = \frac{1}{2\pi i}\int_{C_1} \frac{f(z_0)}{z-z_0}\,dz \quad \text{—— ③}$$

Hence from ② we can write

$$\frac{1}{2\pi i}\int_{C_1} \frac{f(z)}{z-z_0}\,dz = \frac{1}{2\pi i}\int_{C_1} \frac{f(z)-f(z_0)}{z-z_0}\,dz + f(z_0).$$

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$$\frac{1}{2\pi i}\int_{C_1}\frac{f(z)}{z-z_0}\,dz - f(z_0) = \frac{1}{2\pi i}\int_{C_1}\frac{f(z)-f(z_0)}{z-z_0}\,dz \quad \text{—— (4)}$$

for given $\epsilon>0$, $\exists$ a $\delta>0$ s.t. $|f(z)-f(z_0)|<\epsilon$ —— (5)

for all z satisfying the inequality $|z-z_0|<\delta$.

we can choose δ in such a way that $\rho<\delta$ i.e. $\rho<\delta$ so that the inequality (5) satisfied for all points on C_1.

Hence $\left|\frac{1}{2\pi i}\int_{C_1}\left(\frac{f(z)-f(z_0)}{z-z_0}\right).dz\right|$

$$= \left|\frac{1}{2\pi i}\int_0^{2\pi}\frac{f(z)-f(z_0)}{\cancel{\rho e^{i\theta}}}.\cancel{\rho e^{i\theta}}.d\theta\right|$$

$$\leq \frac{1}{2\pi}\int_0^{2\pi}|f(z)-f(z_0)|\,.d\theta = \frac{1}{\cancel{2\pi}}.\epsilon\,\cancel{(2\pi)}$$

Thus, $\left|\frac{1}{2\pi i}\int_{C_1}\frac{f(z)-f(z_0)}{z-z_0}.dz\right| < \epsilon$. —— (6).

from (4) & (6) $\left|\frac{1}{2\pi i}\int_{C_1}\frac{f(z)}{z-z_0}dz - f(z_0)\right| < \epsilon$.

$$\Rightarrow \boxed{f(z_0) = \frac{1}{2\pi i}\int_{C_1}\frac{f(z)}{z-z_0}.dz}$$ Hence proved.

$f'(z_0) = \quad 1$

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The derivative of an analytic function

① Theorem :- If a function $f(z)$ is analytic within and on a simple closed contour C, then its derivative at any point z_0 inside C is given by

$$\boxed{f'(z_0) = \frac{1}{2\pi i}\int_C \frac{f(z)}{(z-z_0)^2}\,dz}$$

② Theorem :- Derivative of an Analytic function is itself an analytic function.

③ If a function is analytic within and on simple closed contour C then $f(z)$ has derivatives of all orders at each point z_0 inside C with

$$\boxed{f^n(z_0) = \frac{\lfloor n}{2\pi i}\int_C \frac{f(z)}{(z-z_0)^{n+1}}\,dz}$$

Proof for ① : point $(z_0+h) \xrightarrow{nbd} z_0$

$$f(z_0+h) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z-z_0-h}\,dz \quad \text{—} ②$$

$$f(z_0) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z-z_0}\,dz \quad \text{—} ①$$

$$\lim_{h\to 0}\frac{f(z_0+h)-f(z_0)}{h} = \frac{1}{2\pi i}\int_C \frac{f(z)}{(z-z_0)^2}\,dz$$

$$+ \frac{h}{2\pi i}\int_C \frac{f(z)\,dz}{(z-z_0-h)(z-z_0)^2}$$

★ Integral Function :- (Entire function)

A function which is analytic in every finite region of the z-plane is called an Integral function or Entire function.

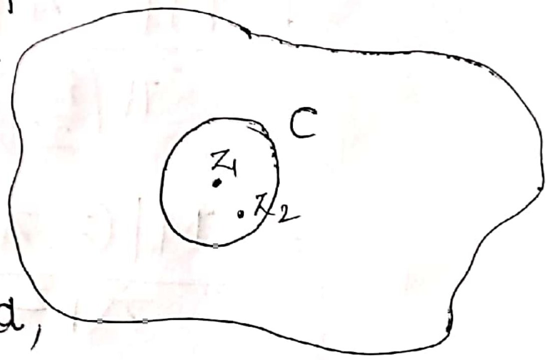
★ Liouville's Theorem :-

If a function $f(z)$ is analytic for all finite values of z and is bounded then it is a constant function.

OR

Every bounded Entire function is constant.

Proof :- Let z_1, z_2 be two points in the z-plane. C be a circle with centre at z_1 and radius R s.t. the point z_2 is interior to C.



Then by Cauchy's Integral formula,

$$f(z_1) = \frac{1}{2\pi i}\int_C \frac{f(z)}{(z-z_1)}\,dz. \quad \text{--- (I)}$$

$$f(z_2) = \frac{1}{2\pi i}\int_C \frac{f(z)}{(z-z_2)}\,dz. \quad \text{--- (II)}$$

(II) − (I) $\Rightarrow$ $f(z_2)-f(z_1) = \frac{1}{2\pi i}(z_2-z_1)\int_C \frac{f(z)}{(z-z_1)(z-z_2)}\,dz.$ --- (1)

Now, we choose R so large s.t. $|z_2-z_1| < R/2$;

Then since $|z-z_1| = R$, we have

$$|z-z_2| = |z-z_1+z_1-z_2| = |(z-z_1)-(z_2-z_1)|$$

$\geq |z - z_1| - |z_2 - z_1| \geq R - \frac{R}{2} = R/2$.

Also $f(z)$ is bounded, say $|f(z)| \leq M$.

Hence from eqⁿ ① we get,

$$|f(z_2) - f(z_1)| = \left| \frac{(z_2 - z_1)}{2\pi i} \int_C \frac{f(z)\, dz}{(z - z_1)(z - z_2)} \right|$$

$$\leq \frac{|(z_2 - z_1)|}{2\pi} \int_C \frac{|f(z)| \cdot |dz|}{|z - z_1|\,|z - z_2|}$$

$$\leq \frac{|(z_2 - z_1)|}{2\pi} \int_C \frac{M |dz|}{R \cdot R/2}$$

$$= \frac{M |(z_2 - z_1)|}{2\pi \left(\frac{R^2}{2}\right)} \int_C |dz|$$

$$= \frac{M |(z_2 - z_1)|}{\pi R^2} (2\pi R) = \frac{2 |z_2 - z_1| M}{R} \quad ②$$

Let, $R \to \infty$ we see that the R.H.S. of eqⁿ ② $\to 0$ and consequently, $|f(z_2) - f(z_1)| = 0$.

$$\Rightarrow f(z_2) - f(z_1) = 0$$

$$\Rightarrow \boxed{f(z_2) = f(z_1)}$$

∴ Hence proved.

Ex. find the value of $\frac{1}{2\pi i}\int_C \frac{x^n . e^{xz}}{\lfloor n . z^{n+1}} dz$, where C is any closed contour surrounding the origin.

(OR)

Using the Integral representation of $f^n(a)$; where a is any pt of the domain. Prove that

$$\left(\frac{x^n}{\lfloor n}\right)^2 = \frac{1}{2\pi i}\int_C \frac{x^n e^{xz}}{\lfloor n . z^{n+1}} dz \text{ ; where C is any closed contour surrounding origin}$$

and hence show that $\sum_{n=0}^{\infty}\left(\frac{x^n}{\lfloor n}\right)^2 = \frac{1}{2\pi}\int_0^{2\pi} e^{2x\cos\theta} d\theta$.

$$\Rightarrow \quad \frac{n!}{2\pi i}\int_C \frac{\frac{x^n . e^{xz}}{(n!)^2}}{z^{n+1}} = f^n(0)$$

$$f(z) = \frac{x^n . e^{xz}}{(n!)^2}$$

$$\Rightarrow f^n(z) = \frac{x^n . (x \ldots n \text{ times}) . e^{xz}}{(n!)^2}$$

$$\Rightarrow f^n(0) = \left(\frac{x^n}{n!}\right)^2 = \frac{1}{2\pi i}\int_C \frac{x^n . e^{xz}}{n! (z^{n+1})} dz$$

$$\sum_{n=0}^{\infty}\left(\frac{x^n}{n!}\right)^2 = \sum_{n=0}^{\infty}\frac{1}{2\pi i}\int_C \frac{x^n . e^{xz}}{n!\, z^{n+1}} . dz$$

$$= \frac{1}{2\pi i}\int_C \sum_{n=0}^{\infty}\frac{x^n . e^{xz}}{n!\, z^{n+1}} dz \quad \left(\text{Since } \frac{(x/z)^n}{n!} \text{ is converging}\right)$$

$$= \frac{1}{2\pi i}\int_C \frac{1}{z} . e^{x/z} . e^{xz} . dz .$$

$$= \frac{1}{2\pi \cancel{i}} \int_C \frac{1}{\cancel{e^{i\theta}}} e^{x(e^{i\theta}+e^{-i\theta})} \, d\theta \, . \cancel{i e^{i\theta}}$$

$$= \frac{1}{2\pi} \int e^{x \cdot \left(\frac{e^{i\theta}+e^{-i\theta}}{2}\right) \cdot 2} \, d\theta \, .$$

$$= \frac{1}{2\pi} \int e^{2x\cos\theta} \, d\theta \, .$$

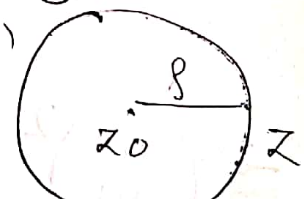
(Assuming $z = e^{i\theta}$. ~~and~~ as C is any closed curve around origin)

<u>Guass Mean Value Theorem</u> :-

<u>Corollary</u>: If $f(z)$ is an analytic function on a domain D and if the circular region $|z-z_0| \leq \rho$ is contained in D, then

$$f(z_0) = \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + \rho e^{i\theta}) \, d\theta \, .$$

The value of $f(z)$ at $z = z_0$ is equal to the average of its value on the boundary of the circle $|z - z_0| = \rho$.



<u>Proof</u> :- $z - z_0 = \rho e^{i\theta}$. By cauchy's Integral formula;

$\Rightarrow z = z_0 + \rho e^{i\theta}$

$dz = i\rho e^{i\theta} \, d\theta$.

$$f(z_0) = \frac{1}{2\pi i} \int_0^{2\pi} \frac{f(z)}{\underbrace{z - z_0}_{\cancel{\rho e^{i\theta}}}} \cancel{i\rho e^{i\theta}} \, d\theta \, .$$

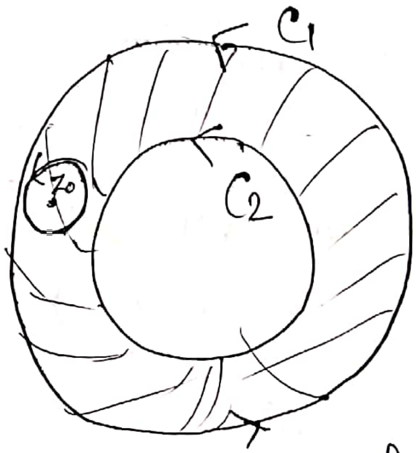
$$\Rightarrow f(z_0) = \frac{1}{2\pi}\int_0^{2\pi} f(z)\,d\theta$$

$$\boxed{f(z_0) = \frac{1}{2\pi}\int_0^{2\pi} f(z_0 + \rho e^{i\theta})\,d\theta}$$

<u>Corollary</u>: If $f(z)$ is analytic in the region bounded by two closed curves C_1 and C_2 and z_0 is any point in the region, then

$$f(z_0) = \frac{1}{2\pi i}\int_{C_1} \frac{f(z)}{z-z_0}\,dz - \frac{1}{2\pi i}\int_{C_2} \frac{f(z)}{z-z_0}\,dz.$$

<u>Proof</u> :- Let, ~~C_1 & C_2~~ the region is bounded by closed curves C_1 & C_2 over which $f(z)$ is analytic.



Draw a circle C around pt. z_0 in the given region and consider $\frac{f(z)}{z-z_0}$; which is analytic in the region bounded by three curves C_1, C_2 and C ($\because$ $z-z_0$ is not zero for any value of z in this region). Then by Cauchy's Theorem for multiply connected regions;

$$\frac{1}{2\pi i}\int_{C_1} \frac{f(z)}{z-z_0}\,dz - \frac{1}{2\pi i}\int_{C_2} \frac{f(z)}{z-z_0}\,dz - \frac{1}{2\pi i}\int_{C} \frac{f(z)}{z-z_0}\,dz = 0.$$

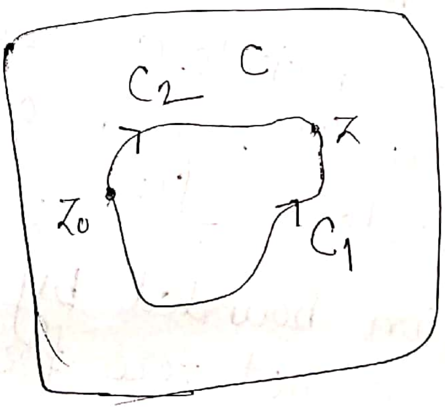
where integration is taken along each curve in anticlockwise direction.

$$\frac{1}{2\pi i}\int_C \frac{f(z)}{z-z_0}\,dz = \frac{1}{2\pi i}\int_{C_1}\frac{f(z)}{z-z_0}\,dz - \frac{1}{2\pi i}\int_{C_2}\frac{f(z)}{z-z_0}\,dz$$

$$\Rightarrow \boxed{f(z_0) = \frac{1}{2\pi i}\int_{C_1}\frac{f(z)}{z-z_0}\,dz - \frac{1}{2\pi i}\int_{C_2}\frac{f(z)}{z-z_0}\,dz}$$

★ <u>Morera's</u> <u>Theorem</u> (Converse of Cauchy's Fundamental Theorem) :-

Let $f(z)$ be a continuous function in a simply connected domain G; if $\int_C f(z)\,dz = 0$ along a simple closed contour C in G, then $f(z)$ is analytic in G.



<u>Proof.</u> Let z_0 be a fixed pt. and z be a varible pt. in G and C_1, C_2 be any two continuous rectifiable curves in G joining z to z_0.

Let C denotes the closed contour consisting of two curves C_1 & $-C_2$.

$$\therefore \int_C f(z)\,dz = \int_{C_1} f(z)\,dz + \int_{-C_2} f(z)\,dz = 0.$$

$$\Rightarrow \int_{C_1} f(z).dz = \int_{C_2} f(z).dz .$$

This shows that the Integral along every cur-ve in G joining z_0 to z is the same. Hence, Taking variable point ξ as the variable of integration, we have.

$$F(z) = \int_{z_0}^{z} f(\xi).d\xi \quad \text{——— (1)}$$

As the integral depends only on z_0 & z (independent of path).

Let $z+h$ be any pt. in G near to pt. z.

$$\text{Then, } F(z+h) - F(z) = \int_{z_0}^{z+h} f(\xi)d\xi - \int_{z_0}^{z} f(\xi).d\xi .$$

$$\Rightarrow F(z+h) - F(z) = \int_{z}^{z+h} f(\xi).d\xi .$$

Independent of path joining z to $z+h$.

In particular, we may choose the path as the straig-ht line segment provided you choose $|h|$ small enough so that this path lies inside the dom-ain G.

$$\text{Thus; } \frac{F(z+h) - F(z)}{h} - f(z) = \frac{1}{h} \int_{z}^{z+h} f(\xi).d\xi - \frac{1}{h}. f(z)h .$$

$$= \frac{1}{h} \int_{z}^{z+h} f(\xi).d\xi - \frac{1}{h} f(z) \int_{z}^{z+h} d\xi .$$

$$\frac{F(z+h)-F(z)}{h} - f(z) = \frac{1}{h}\int_{z}^{z+h} [f(\xi)-f(z)]\cdot d\xi$$

As $f(z)$ is continuous ; $\epsilon > 0$; $|f(\xi)-f(z)| < \epsilon$.

$|\xi - z| < \delta \Rightarrow |h| < \delta$.

$$\leq \frac{1}{h}\cdot \epsilon \cdot h = \epsilon .$$

$$\Rightarrow \frac{F(z+h)-F(z)}{h} - f(z) \leq \epsilon .$$

$$F'(z) = \lim_{h\to 0} \frac{F(z+h)-F(z)}{h}$$

$$\Rightarrow \quad F'(z) - f(z) \leq \epsilon .$$

$$\Rightarrow \quad F'(z) = f(z) .$$

$\Rightarrow$ Since, $F'(z)$ exists $\Rightarrow$ $F(z)$ is analytic

$\Rightarrow$ $F'(z)$ is analytic.

$\Rightarrow$ $f(z)$ is analytic. Hence proved.

★ **~~Cauchy~~ Poisson Integral formula for a circle :-**

Let $f(z)$ be analytic in the region $|z| < \rho$ and let $z = r\cdot e^{j\theta}$ be any pt. of this region

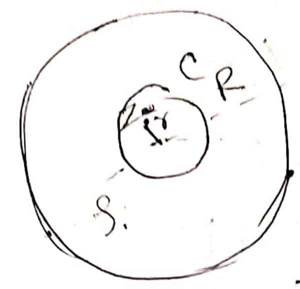
Then

$$f(r\cdot e^{i\theta}) = \frac{1}{2\pi}\int_0^{2\pi} \frac{(R^2-r^2) f(R\cdot e^{i\phi})}{R^2 - 2Rr\cos(\theta-\phi) + r^2} \cdot d\phi .$$

where R is any number such that

$$0 < r < R < \rho .$$

$|z| < \rho$.



Let C be the circle $|z| = R$ such that $r < R < \rho$ as given $z = re^{i\theta}$ is any point of the region $|z| < \rho$. Hence By Cauchy's Integral formula,

$$f(z) = \frac{1}{2\pi i} \int_C \frac{f(w)}{w - z}\, dw \quad \text{——— (1)}$$

Now, the inverse of the point z w.r.t. C is $\frac{R^2}{\bar{z}}$ and lies outside of the circle C so that the function $\frac{f(w)}{w - \left(\frac{R^2}{\bar{z}}\right)}$ is analytic within the circle C. Therefore, by Cauchy-Goursat theorem, we have,

$$\int_C \frac{f(w)}{w - \frac{R^2}{\bar{z}}}\, dw = 0 \quad \text{——— (2)}$$

$$f(z) = \frac{1}{2\pi i} \int_C \left(\frac{1}{w - z} - \frac{1}{w - R^2/\bar{z}} \right) f(w)\, dw.$$

$$f(z) = \frac{1}{2\pi i} \int_C \frac{z - R^2/\bar{z}}{(w - z)(w - R^2/\bar{z})} f(w)\, dw \quad \text{——— (3)}$$

Now, we write $z = re^{i\theta}$, $w = Re^{i\phi}$, Then $\bar{z} = re^{-i\theta}$

$dw = iRe^{i\phi}\, d\phi$.

from (3),

$$f(re^{i\theta}) = \frac{1}{2\pi \cancel{i}} \int_0^{2\pi} \frac{\left(re^{i\theta} - \frac{R^2}{r} e^{i\theta}\right) f(Re^{i\phi})\, \cancel{i}Re^{i\phi}\, d\phi}{(Re^{i\phi} - re^{i\theta})(Re^{i\phi} - \frac{R^2}{r} e^{i\theta})}$$

$$f(re^{i\theta}) = \frac{1}{2\pi}\int_0^{2\pi} \frac{\cancel{R}e^{i\phi}\cdot(r^2-R^2)e^{i\theta}\, f(Re^{i\phi})\cdot d\phi}{(Re^{i\phi}-re^{i\theta})(\cancel{R}re^{i\phi}-R^{\cancel{2}}e^{i\theta})}$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \frac{(r^2-R^2)\cdot e^{i\theta}\cdot e^{i\phi}\cdot d\phi\ f(Re^{i\phi})}{Rr\,e^{2i\phi} - R^2e^{i\phi}e^{i\theta} - r^2e^{i\theta}\cdot e^{i\phi} + Rr\cdot e^{2i\theta}}$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \frac{(R^2-r^2)\ e^{i\theta}\cdot e^{i\phi}\cdot d\phi\ f(Re^{i\phi})}{R^2\cdot e^{i\theta}\cdot e^{i\phi} + r^2\cdot e^{i\theta}\cdot e^{i\phi} - \cancel{2}Rr(e^{2i\theta}+e^{2i\phi})}$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \frac{(R^2-r^2)\ f(Re^{i\phi})\ d\phi}{R^2+r^2-2Rr\left(\frac{e^{i\theta-i\phi}+e^{i\phi-i\theta}}{2}\right)}$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \frac{(R^2-r^2)\ f(Re^{i\phi})\cdot d\phi}{R^2+r^2-2Rr\cos(\theta-\phi)}$$

$$\boxed{f(re^{i\theta}) = \frac{1}{2\pi}\int_0^{2\pi} \frac{(R^2-r^2)\cdot f(Re^{i\phi})}{R^2-2Rr\cdot\cos(\theta-\phi)+r^2}\, d\phi}$$

Hence, proved.

$f(re^{i\theta}) = u(r,\theta) + iv(r,\theta)$; $f(Re^{i\phi}) = u(R,\phi)+iv(R,\phi)$

Q. Find the value of $I = \int_{|z|=1} \frac{\sin^6 z}{(z-\pi/6)^3}\, dz$

$\Rightarrow$ $f^{n}(z_0) = \frac{n!}{2\pi i}\int_C \frac{f(z)}{(z-z_0)^{n+1}}\, dz$

$n=2$; $f(z) = \sin^6 z$; $z_0 = \pi/6$

$$I = \frac{f''(z_0)}{\cancel{2\pi i}}\,\frac{2\pi i}{2!} = \frac{f''(\pi/6)\,\cancel{2}\pi i}{\cancel{2}} = \frac{i\,21\pi}{16}$$

$f'(z) = 6\cdot\sin^5 z\cdot\cos z \Rightarrow f''(z) = 30\cdot\sin^4 z\cos^2 z - 6\sin^5 z\cdot\sin z$

$\Rightarrow f''(z) = 30 \sin^4 z . \cos^2 z - 6 \sin^5 z . \sin z$

$$\Rightarrow f''\left(\frac{\pi}{6}\right) = 30 \left(\frac{1}{2}\right)^4 \left(\frac{\sqrt{3}}{2}\right)^2 - 6 . \left(\frac{1}{2}\right)^5 \left(\frac{1}{2}\right)$$

$$= \frac{30 (3)}{64} - \frac{6}{64} = \frac{84}{64} = \frac{21}{16}$$

$\therefore \boxed{I = \frac{21}{16} \pi i}$

* Maximum Modulus Principle :-

Let $f(z)$ is analytic within and on a simple closed contour C. Then $|f(z)|$ reaches its maximum value on C (and not inside C), unless $f(z)$ is constant.

OR

If M is the maximum value of $|f(z)|$ on and within C, then unless $f(z)$ is constant, $|f(z)| < M$ for every point z within C.

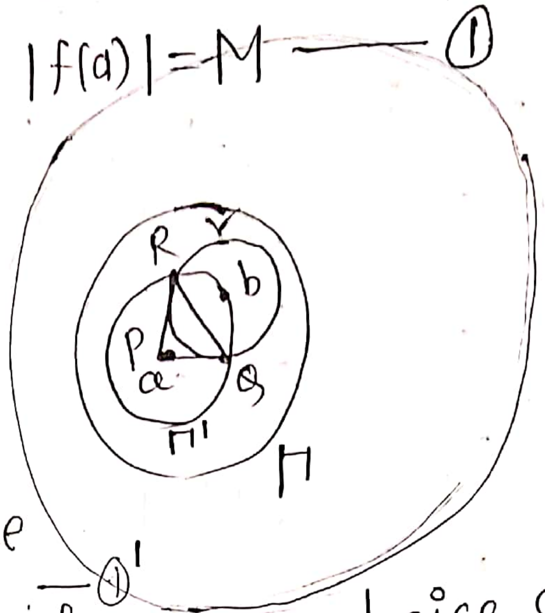
Proof :- Since $f(z)$ is analytic within and on C, it follows that $|f(z)|$ must reach its max. value M at some points on or within C.

We consider $f(z)$ is not constant in C.

Then we wish to prove that $|f(z)|$ takes the value M at some point on C.

Suppose, $|f(z)|$ attains its max value at point ~~A~~ a within C. $\Rightarrow |f(a)| = M$ —— (1)

If $|f(z)| = M$ is the maximum value of $|f(z)|$ and $f(z)$ is const., there exists a point, say b inside Γ s.t. $|f(b)| < M$. Since



$|f(b)| = M - \epsilon$ —— (1)'

$|f(z)|$ is continuous at b, for one choice of $\epsilon > 0$

$|f(z) - f(b)| < \epsilon/2$; whenever $|z - b| < \delta$.

$\Rightarrow \quad |f(z)| - |f(b)| \leq \epsilon/2$

$\Rightarrow \quad |f(z)| \leq |f(b)| + \epsilon/2$

$\left(\because \big| |f(z)| - |f(b)| \big| \geq |f(z)| - |f(b)| \right)$

$\Rightarrow \quad |f(z)| < M - \epsilon + \epsilon/2 = M - \epsilon/2$ ~~from~~ (1)

$\Rightarrow \quad |f(z)| < M - \epsilon/2$ —— (2)

for all pts. z satisfying $|z-b| < \delta$ i.e. for all points z inside a circle γ with center at b at radius δ.

Now, draw a circle Γ' with centre at a and passing through the point b. The arc QR of the circle Γ' lies within the circle γ, so that on this arc, we have

$|f(z)| < M - \epsilon/2$

On the remaining portion (or arc) of Γ'; $|f(z)| \le M$

The radius of $\Gamma' = |b-a| = R$ (say).

Then By Cauchy's Integral Formula,

$$f(a) = \frac{1}{2\pi i} \int_{\Gamma'} \frac{f(z)}{z-a}\, dz.$$

Now on circle Γ'; we have $z - a = R.e^{i\theta}$.

$\Rightarrow$ $dz = iRe^{i\theta}.d\theta$. $\Rightarrow z = a + Re^{i\theta}$.

$$f(a) = \frac{1}{2\pi \cancel{i}} \int_0^{2\pi} \frac{f(a+Re^{i\theta})}{\cancel{Re^{i\theta}}} . \cancel{iRe^{i\theta}} . d\theta$$

$$f(a) = \frac{1}{2\pi} \int_0^{2\pi} f(a+Re^{i\theta}).d\theta.$$

If we measure θ from PQ in ACW dirn; if $\angle QPR = \alpha$; then.

$$f(a) = \frac{1}{2\pi} \int_0^{\alpha} f(a+Re^{i\theta}).d\theta + \frac{1}{2\pi}\int_{\alpha}^{2\pi} f(a+Re^{i\theta}).d\theta.$$

$$\Rightarrow |f(a)| \le \frac{1}{2\pi} \int_0^{\alpha} |f(a+Re^{i\theta})|\,|d\theta| + \frac{1}{2\pi} \int_{\alpha}^{2\pi} |f(a+Re^{i\theta})|\,|d\theta|$$

$$\le \frac{\alpha}{2\pi}(M - \epsilon/2) + \frac{M}{2\pi}(2\pi - \alpha)$$

$$= M - \frac{\alpha\epsilon}{4\pi}.$$

$$\Rightarrow \boxed{|f(a)| \le M - \frac{\alpha\epsilon}{4\pi}} \quad \text{———} \quad (3)$$

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This is an absurd result since M can not be less than $M - \frac{\cancel{\alpha}\epsilon}{4\pi}$. (from ① & ③)

$\Rightarrow$ Contradiction occurs.

$\therefore$ Our Assumption that $|f(z)|$ attains max value at pt. a within C was wrong.

$\therefore$ a must lies on C. for $|f(z)|$ to attain max value.

☆ <u>Minimum Modulus Principle</u> :-

If f(z) is analytic within and on a closed contour C. Then $|f(z)|$ reaches its minimum value on C (and not within C); unless f(z) is constant.

☆ <u>Power Series</u> :-

$\{x_n\}$ x_1 x_2 --- $x_m, x_{m+1},$ --, $x_n,$ -- ,∞

for given, $\epsilon > 0$; $\exists$ a positive integer s.t.

if. $|x_n - x| < \epsilon$; $n \geq m$

x ↓ limit.

then $\{x_n\}$ is said to be converging sequence and x is said to be converging point.

if $\sum_{n=1}^{\infty} x_n = y \Rightarrow$ (finite), then given series is converging.

$S_n = \sum_{i=1}^{n} x_i \Rightarrow \{S_n\}$. if $\{S_n\}$ is converging

i.e. $\lim_{n\to\infty} S_n = S$ then $\sum_{n=1}^{\infty} x_n$ is converging.

Cauchy Sequence :- Any sequence $\{x_n\}$ is said to be Cauchy if $|x_m - x_n| < \epsilon$; $m, n \geq p$.

Cauchy sequence is always convergent (Converse is also true).

$\{z_n\} \Rightarrow$ Sequence of complex numbers is Cauchy if $|z_m - z_n| < \epsilon$ for all $m, n \geq n_0$.

★ **Weierstrass M-test** :- The series $\sum f_n(x)$ of functions each defined on the same set A converges uniformly on A if

(i) $|f_n(x)| \leq M_n$, for every positive integer n and every $x \in A$; where M_n is a +ve constant independent of x. and

(ii) the series $\sum M_n$ is convergent.

★ **D'Alembert's Ratio Test** :-

$\sum u_n$, let $L = \left|\frac{u_{n+1}}{u_n}\right|$ then

(i) If $L < 1$; then series is absolutely convergent (Absolute convergence $\Rightarrow$ Convergence).

(ii) If $L > 1$; then series is divergent.

(iii) If $L = 1$; then it may be convergent, divergent or oscillatory.

Theorem :- The power series $\sum a_n z^n$ either
(i) converges for all values of z.
(ii) converges only for $z = 0$
(iii) converges for z in some region in complex plane

Proof :- It is sufficient to construct some examples which proves these conditions.

(i) $\sum_{n=1}^{\infty} \frac{z^n}{n!}$

$u_n = \frac{z^n}{n!}$

$u_{n+1} = \frac{z^{n+1}}{(n+1)!}$

$L = \left|\frac{u_{n+1}}{u_n}\right| = \left|\frac{z}{(n+1)}\right|$

$\lim_{n\to\infty} |L| = \lim_{n\to\infty} \left|\frac{z}{(n+1)}\right|$

$= 0 < 1$.

$\therefore$ Given, series is absolutely convergent for all values of z

(ii) $\sum_{n=1}^{\infty} \lfloor n \, . z^n$

$u_n = z^n . n!$

$u_{n+1} = z^{n+1} . (n+1)!$

$L = \left|\frac{u_{n+1}}{u_n}\right| = |z(n+1)|$

$\lim_{n\to\infty} |z(n+1)|$

$= \infty$ if $z \neq 0$.
> 1

$\therefore$ Given series is convergent only for $z = 0$.

(iii) $\sum_{n=0}^{\infty} z^n$.

$u_n = z^n$

$u_{n+1} = z^{n+1}$

$L = \left|\frac{z^{n+1}}{z^n}\right| = |z|$.

$\lim_{n\to\infty} |z| = |z|$

for convergence

$|z| < 1$.

$\therefore$ Given, series is convergent inside unit circle.

★ **Abel's Theorem :-** If the power series $\sum_{n=0}^{\infty} a_n z^n$ converges for a particular value z_0 of z, then it converges absolutely for all values of z for which $|z| < |z_0|$

★ Radius of Convergence of Power Series :-

The circle $|z| = R$, which includes in its interior $|z| < R$, all the values of z for which power series in z_n converges is called the circle of convergence. Radius of this circle is known as Radius of convergence of Power Series.

★ Cauchy-Hadamard's Theorem :-

The radius of convergence R of a power series is given by $\frac{1}{R} = \lim_{n\to\infty} |a_n|^{1/n}$

In practice, there is a simpler formular for finding R is given by $\boxed{\frac{1}{R} = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|}$, provided this limit exists.

Theorem :- Let $\sum_{n=0}^{\infty} a_n z^n$ be a power series and let $\sum_{n=1}^{\infty} n a_n z^{n-1}$ be the power series obtained by differentiating the given series term by terms. Then the derived series has the same radius of convergence as the original series.

Hint Proof

$$\frac{1}{R} = \lim_{n\to\infty} |n.a_n|^{1/n}$$

$$= \lim_{n\to\infty} n^{1/n}.|a_n|^{1/n}$$

$$n^{1/n} = 1 + h_n$$

$$\frac{1}{n}.\log n = \log(1+h_n)$$

$$n = (1+h_n)^n$$

Q. Find the R (radius of convergence) of the following power series.

(i) $\sum \frac{z^n}{n!}$; (ii) $\sum \left(1+\frac{1}{n}\right)^{n^2} \cdot z_n$; (iii) $\sum \frac{(n+1)}{(n+2)(n+3)} z^n$

(iv) $\sum \frac{(n!)^2}{(2n!)} \cdot z^n$; (v) $\sum (\log n)^n \cdot z^n$; (vi) $\sum \frac{z^n}{2^n+1}$

(i) $\Rightarrow$

$$\frac{a_{n+1}}{a_n} = \frac{\cancel{z^{n+1}}\,(n)!}{(n+1)!} = \frac{1}{(n+1)}$$

$$\Rightarrow \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \frac{1}{(n+1)} = 0.$$

$$\Rightarrow \frac{1}{R} = 0 \Rightarrow \boxed{R = \infty}.$$

This series is convergent for all z.

(ii) $\Rightarrow$ $a_n = \left(1+\frac{1}{n}\right)^{n^2}$.

$$\frac{1}{R} = \lim_{n\to\infty} |a_n|^{1/n} = \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^{n^{\cancel{2}}\cdot\frac{1}{\cancel{n}}}$$

$$= e \Rightarrow \boxed{R = 1/e}$$

(iii) $a_n = \frac{(n+1)}{(n+2)(n+3)}$ $a_{n+1} = \frac{(n+2)}{(n+3)(n+4)}$

$$\Rightarrow \frac{(n+2)}{\cancel{(n+3)}(n+4)} \times \frac{(n+2)\cancel{(n+3)}}{(n+1)} \Rightarrow \frac{(n+2)(n+2)}{(n+1)(n+4)} = L$$

$$\lim_{n\to\infty} L = 1 \Rightarrow \frac{1}{R} = 1 \Rightarrow \boxed{R=1}.$$

* Relation betn analyticity and circle/radius of convergence of power series :-

Q. Find R for following power series,

(i) $\sum (3+4i)^n . z^n$, (ii) $\sum \frac{n\sqrt{2}+i}{1+2in} . z^n$

(iii) $\sum \frac{(-1)^n}{n} . (z-2i)^n$.

(ii) → $a_n = \frac{n\sqrt{2}+i}{(1+2in)} \times \frac{1-2in}{(1-2in)} = \frac{\sqrt{2}n - i\, 2\sqrt{2}n^2 + i + 2n}{1+4n^2}$

$$|a_n| = \left| \frac{(2n+\sqrt{2}n) + i(1-2\sqrt{2}n^2)}{1+4n^2} \right|$$

$$= \frac{1}{1+4n^2} \sqrt{(2n+\sqrt{2}n)^2 + (1-2\sqrt{2}n^2)^2}$$

$$|a_n| = \frac{\sqrt{8n^4+6n^2+1}}{1+4n^2} .$$

$$|a_{n+1}| = \sqrt{8(n+1)^4+6(n+1)^2+1} \Big/ 1+4(n+1)^2 .$$

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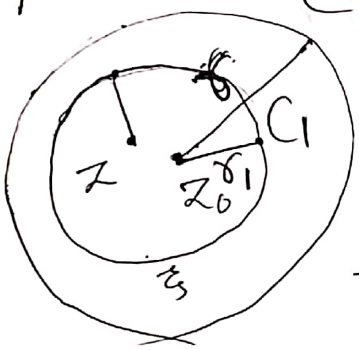
★ <u>Taylor's Theorem</u> :- If a function $f(z)$ is analytic at all pts. within a circle C with centre z_0 and radius R, then at each pt. z within the circle C., expansion of $f(z)$ is.

$$f(z) = f(z_0) + (z-z_0).f'(z_0) + \frac{(z-z_0)^2}{\lfloor 2}.f''(z_0) + \dots + \frac{(z-z_0)^n}{\lfloor n} f^n(z_0) + \dots$$

i.e. $$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n \; ; \quad a_n = \frac{f^n(z_0)}{\lfloor n}.$$

Proof :-



$$f(z) = \frac{1}{2\pi i}\int_{C_1} \frac{f(\xi)}{\xi - z}.d\xi, \quad \text{--- ①}$$

$$|\xi - z_0| = r_1 \; ;$$

$$|z - z_0| = \rho < r_1 < r \; ;$$

$$\frac{1}{\xi - z} = \frac{1}{\xi - z_0 + z_0 - z}.$$

$$= \frac{1}{(\xi-z_0)\left(1 - \frac{z-z_0}{\xi-z_0}\right)} = \frac{1}{(\xi - z_0)}.\left(1 - \frac{z-z_0}{\xi-z_0}\right)^{-1}$$

$$= \frac{1}{\xi - z_0}\left\{1 + \frac{z-z_0}{\xi-z_0} + \left(\frac{z-z_0}{\xi-z_0}\right)^2 + \dots + \left(\frac{z-z_0}{\xi-z_0}\right)^{n-1} + \left(\frac{z-z_0}{\xi-z_0}\right)^n \left(\frac{\xi-z_0}{\xi-z}\right)\right\}$$

$$= \frac{1}{\xi-z_0} + \frac{z-z_0}{(\xi-z_0)^2} + \frac{(z-z_0)^2}{(\xi-z_0)^3} + \dots + \frac{(z-z_0)^{n-1}}{(\xi-z_0)^n} + \frac{(z-z_0)^n}{(\xi-z_0)^n}.\frac{1}{\xi-z} \quad \text{--- ②}$$

$$(1-\alpha)^{-1} = 1 + \alpha + \alpha^2 + \dots + \alpha^n + \alpha^{n+1} + \dots$$
$$= 1 + \alpha + \alpha^2 + \dots + \alpha^n(1 + \alpha + \alpha^2 + \dots)$$

Multiply ② on both sides with $\frac{1}{2\pi i} \cdot f(\xi)$.

$$\frac{1}{2\pi i} \frac{f(\xi)}{\xi - z} = \frac{1}{2\pi i} \frac{f(\xi)}{\xi - z_0} + \frac{1}{2\pi i} \cdot \frac{f(\xi).(z-z_0)}{(\xi - z_0)^2} + \cdots + \frac{1}{2\pi i} \frac{f(\xi)(z-z_0)^{n-1}}{(\xi - z_0)^n} + \frac{1}{2\pi i} \frac{(z-z_0)^n}{(\xi - z_0)^n} \frac{f(\xi)}{\xi - z}$$

$$\frac{1}{2\pi i} \int_{C_1} \frac{f(\xi)}{\xi - z} d\xi = \frac{1}{2\pi i} \int_{C_1} \frac{f(\xi)}{\xi - z_0} d\xi + \frac{1}{2\pi i} \int_{C_1} \frac{(z-z_0) \cdot f(\xi)}{(\xi - z_0)^2} d\xi + \cdots + \frac{1}{2\pi i} \int_{C_1} \frac{(z-z_0)^{n-1} f(\xi) . d\xi}{(\xi - z_0)^n} + R_n .$$

where, $$R_n = \frac{1}{2\pi i} \int_{C_1} \frac{(z-z_0)^n . f(\xi) \, d\xi}{(\xi - z_0)^n . (\xi - z)} .$$

$$f(z) = f(z_0) + (z-z_0) f'(z_0) + \frac{(z-z_0)^2}{\lfloor 2} . f''(z_0) + \cdots + \frac{(z-z_0)^{n-1}}{\lfloor n-1} . f^{(n-1)}(z_0) + R_n .$$

Since, $|z - z_0| = \rho$; $|\xi - z_0| = r_1$

$\therefore |\xi - z| = |(\xi - z_0) - (z - z_0)| \geq |\xi - z_0| - |z - z_0|$

$= r_1 - \rho$.

$$\Rightarrow \quad |R_n| \le \frac{1}{2\pi}\int_{C_1} \left|\frac{z-z_0}{\xi-z_0}\right|^n \left|\frac{f(\xi)}{\xi-z}\right| \cdot |d\xi|$$

As $\Rightarrow (|f(\xi)| < M)$ $\quad \therefore \quad |R_n| < \frac{M}{2\pi}\left(\frac{\rho}{r_1}\right)^n \frac{1}{r_1-\rho} 2\pi r_1$

$$= M\left(\frac{\rho}{r_1}\right)^n \left(\frac{1}{1-\rho/r_1}\right).$$

$$\Rightarrow \lim_{n\to\infty} |R_n| = \lim_{n\to\infty} M\left(\frac{\rho}{r_1}\right)^n \left(\frac{1}{1-\rho/r_1}\right)$$

$$= 0 \qquad (\text{as } \rho < r_1).$$

$$\therefore f(z) = f(z_0) + (z-z_0)f'(z_0) + \frac{(z-z_0)^2}{\lfloor 2}.f''(z_0) + \cdots + \frac{(z-z_0)^{n-1}}{\lfloor n-1}.f^{(n-1)}(z_0) + 0.$$

$$\Rightarrow \boxed{f(z) = f(z_0) + (z-z_0)f'(z_0) + \frac{(z-z_0)^2}{\lfloor 2}f''(z_0) + \cdots + \frac{(z-z_0)^{n-1}}{\lfloor n-1}.f^{(n-1)}(z_0).}$$

eg. Expand $\log(1+z)$ in a Taylor's series about $z=0$ and determine the region of convergence for the resulting series.

$$f(z) = \log(1+z) \qquad (z \ne -1).$$

$$\Rightarrow \quad f'(z) = \frac{1}{(1+z)} \; ; \quad f''(z) = \frac{-1}{(1+z)^2}$$

$$f^n(z) = (-1)^{n-1}\frac{\lfloor n-1}{(1+z)^n}$$

$$\log(1+z) = \log(1+z_0) + (z-z_0)\left(\frac{1}{1+z_0}\right) + \frac{(z-z_0)^2}{2!}\frac{(-1)}{(1+z_0)^2}$$

$$+ \frac{(z-z_0)^{n-1}}{\lfloor n-1} f$$

$$\log(1+z) = \log 1 + \frac{z}{\lfloor 1} - \frac{z^2}{\lfloor 2} + \frac{z^3}{\lfloor 3}\lfloor 2 - \frac{z^4}{\lfloor 4}\lfloor 3$$

$(z_0 = 0)$

$$= z - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4} + \ldots + (-1)^{n-1}\cdot\frac{z^n}{n} + \ldots$$

$$u_n = \frac{(-1)^{n-1}\cdot z^n}{n} \; ; \quad u_{n+1} = \frac{(-1)^n\cdot z^{n+1}}{n+1}.$$

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{z^{n+1}(n)}{n+1\; z^n}\right|$$

$$= \lim_{n\to\infty}\left|\frac{z\,(n)}{(n+1)}\right|$$

$$= |z|.$$

for convergence ; $|z| < 1$.

also convergent at $|z| = 1$ except at $z = -1$ (singular point).

☆

★ Laurent's Theorem :-

St. Let $f(z)$ be analytic in the annulus (ring shaped region) between two circles C_1 and C_2 with the centre z_0 and radii R_1 and R_2 ($R_1 > R_2$), respectively then at any point z of the annulus;

$$\boxed{f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \sum_{n=1}^{\infty} b_n (z-z_0)^{-n}}$$

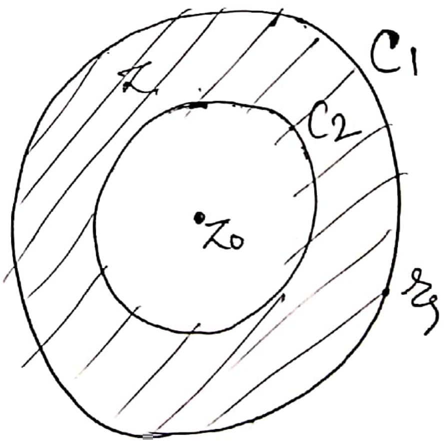
» Principle part of laurent series expansion.

where
$$a_n = \frac{1}{2\pi i}\int_{C_1} \frac{f(\xi)}{(\xi-z_0)^{n+1}}\,d\xi \;;\; n = 0, 1, 2, \ldots$$

$$b_n = \frac{1}{2\pi i}\int_{C_2} \frac{f(\xi)}{(\xi-z_0)^{-n+1}}\,d\xi \;;\; n = +1, 2, \ldots$$

Proof :-

$$f(z) = \frac{1}{2\pi i}\int_{C_1} \frac{f(\xi)\,d\xi}{\xi - z} - \frac{1}{2\pi i}\int_{C_2} \frac{f(\xi)\,d\xi}{\xi - z}.$$



For C_1; (ξ on C_1)

$$\frac{1}{\xi - z} = \frac{1}{\xi - z_0 + z_0 - z}$$

$$= \frac{1}{\xi - z_0}\left(1 - \frac{z-z_0}{\xi - z_0}\right)^{-1}$$

If $b_n = a_{-n} \Rightarrow \boxed{f(z) = \sum_{n=-\infty}^{\infty} a_n (z-z_0)^n}$

Q. Obtain the Taylor's and Laurent's expansion of the function $\frac{z^2-1}{(z+2)(z+3)}$ in the regions.

(i) $|z|<2$ (ii) $2<|z|<3$ (iii) $|z|>3$.

→ We have,

$$f(z) = \frac{z^2-1}{(z+2)(z+3)} = 1 - \frac{5z+7}{(z+2)(z+3)}$$

$$f(z) = 1 + \frac{3}{z+2} - \frac{8}{z+3}$$

(i) for $|z|<2$, we have

$$f(z) = 1 + \frac{3}{2(1+\frac{z}{2})} - \frac{8}{3(1+z/3)}$$

$$= 1 + \frac{3}{2}\left(1+\frac{z}{2}\right)^{-1} - \frac{8}{3}\left(1+\frac{z}{3}\right)^{-1}$$

$$= 1 + \sum\left[(-1)^n \left\{\frac{3}{2^{n+1}} - \frac{8}{3^{n+1}}\right\} z^n\right\}.$$

(ii) for $2<|z|<3$, we have.

$$f(z) = 1 + \frac{3}{2}\left(\frac{1}{(1+\frac{z}{2})}\right) - \frac{8}{3(1+\frac{z}{3})}$$

$$= 1 + \frac{3}{z(1+\frac{2}{z})} - \frac{8}{3(1+\frac{z}{3})}$$

$$= 1 + \frac{3}{z}\left(1+\frac{2}{z}\right)^{-1} - \frac{8}{3}\left(1+\frac{z}{3}\right)^{-1}$$

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Q. Obtain the Taylor's or Laurent's Expansion of

$$f(z) = \frac{1}{(1+z^2)(z+2)}$$

(i) $|z|<1$ (ii) $1<|z|<2$ (iii) $|z|>2$.

Q. If the function $f(z)$ is analytic when $|z|<R$ and has the Taylor's Expansion $\sum_{n=0}^{\infty} a_n z^n$, then show that for $r<R$, we have

$$\frac{1}{2\pi}\int_0^{2\pi} |f(r.e^{i\theta})|^2 d\theta = \sum_{n=0}^{\infty} |a_n|^2 r^{2n}$$

Hence prove that if $|f(z)| \leq M$, where $|z|<R$,

then $\sum_{n=0}^{\infty} |a_n|^2 r^{2n} \leq M^2$.

Solution :- $z = re^{i\theta}$, $f(z)$ $|z|<R$.

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

$|re^{i\theta}| = r < R$ $\quad f(r.e^{i\theta}) = \sum_{n=0}^{\infty} a_n . r^n e^{ni\theta}$ —(1)

$$\overline{f(r.e^{i\theta})} = \sum_{m=0}^{\infty} \bar{a}_m.(r^m.e^{-mi\theta}) \quad \text{---} ② \qquad \text{Let } \bar{a}_m = \text{congugate of } a_m..$$

$$|f(re^{i\theta})|^2 = \sum_{n=0}^{\infty} \cdots \sum_{m=0}^{\infty} \cdots$$

$$\Rightarrow \int_0^{2\pi} |f(r.e^{i\theta})|^2 . d\theta = \int_0^{2\pi} e^{i\theta(m-n)} . d\theta = \begin{cases} 0 \ ; m \neq n \\ 2\pi \ ; m = n . \end{cases}$$

★ Singular Points :-

If any given function is not analytic at point z_0 of the given domain D, then z_0 is called singular point of $f(z)$

A function $f(z)$ which is analytic at all points of a bounded domain expt at a finite numbe of points. Then these exceptional points are called singular points.

When in the neighbourhood of z_0 we do not have other singular point. for $f(z)$ then it is called isolated singular pt. and property is called isolated singularity.

Ex. $f(z) = \dfrac{z+3}{z^2(z^2+2)}$; $z=0$; $z = \pm\sqrt{2}i$
all are isolated singularities.

$f(z) = \dfrac{1}{\sin\left(\frac{\pi}{z}\right)}$ for $z = \pm\frac{1}{n}$ (isolated)

$f(z) = \cot\left(\frac{\pi}{z}\right) = \dfrac{\cos(\pi/z)}{\sin(\pi/z)}$ both are non-analytic.

$z=0$; (non-isolated singularity).

ex. $f(z) = \dfrac{z+2}{(z+1)(z-3)}$ for ($z=-1, 3$ are singular points)

since in the deleted neighbourhood of these pts there cannot be other singularities. therefore these are isolated singularities.

let $z = z_0$ be an isolated singularity of a function $f(z)$. Since, singularity is isolated, there exists a deleted nbd. (ex $0 < |z-z_0| < \delta$) in which $f(z)$ is analytic. Then $f(z)$ has a Laurent's Expansion of form $f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \underbrace{\sum_{n=1}^{\infty} b_n (z-z_0)^{-n}}_{\text{principle part}}$.

(A) Pole for the isolated singularity.

(B) Essential Isolated Singularity.

If principle part of $f(z)$ at $z=z_0$ consists of a finite no. of terms say m, we say z_0 is a pole of order m.

If principle part of $f(z)$ at $z=z_0$ consists of infinite no. of terms, then z_0 is called essent-ial isolated singularity.

Removable isolated singularity.

$$f(z) = \frac{\sin z}{z} \quad ; \quad \lim_{z \to 0} \frac{\sin z}{z} =$$

Non-removable isolated singularity is called.

Ex. find the kind of singularities of the functions.

(i) $\frac{\cot \pi z}{(z-a)^2}$; $z=a$, $z=\infty$.

(ii) $\frac{z^2+4}{e^z}$; $z=\infty$

(iii) Consider the singularities of the function represented by the series. $\sum_{n=0}^{\infty} \frac{1}{\lfloor n} \frac{1}{(1+2^n z)^2}$ and obtain expansion by Laurent's Theorem at $z=a$, $z=\infty$.

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MA201 (PDE) : Pratibhamoy Das.

Books : T. Amarnath (PDE).

K. Sankara Rao (PDE).

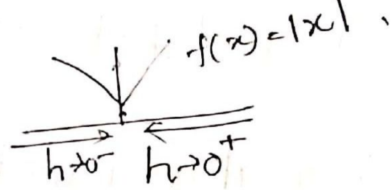
Notes (By Shivaji [illegible] (IITB)).

Notes (Hunter UC Davis).

172.16.1.3/~pratibhamoy.

Derivative $f: D \to R$ is diff at x $(f: \mathbb{R} \to \mathbb{R})$.

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}.$$



Eqⁿ. $f(x) = |x|$ at $x = 0$.

$$\lim_{h\to 0^+} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0^+} \frac{h-0}{h} = 1.$$

$$\lim_{h\to 0^-} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0^-} \frac{-h-0}{h} = -1.$$

f is not diff. at $x = 0$.

$$C(\Omega) = \{ f \mid f \text{ is cont. in } \Omega \ f: \Omega \to \mathbb{R} \}.$$

Ω ↓ Domain

$$C^1(\Omega) = \{ f' \text{ is cont. in } \Omega \ f: \Omega \to \mathbb{R} \}.$$

Define similarly $C^p(\Omega)$.

e.g. $f(x) = x|x| = \begin{cases} x^2 ; & x \geq 0 \\ -x^2 ; & x \leq 0 \end{cases}$

$$f'(x) = \begin{cases} 2x ; & x > 0 \\ -2x ; & x < 0 \end{cases} = 2|x|. \quad f \in C^1(\mathbb{R})$$

$f(x) = 1$, $x \in [a,b] \Rightarrow f$ is integrable

$f(x) = 1$; $x \in (-\infty, \infty) \Rightarrow f$ is not integrable

$$L^1(\Omega) = \left\{ f : \Omega \to \mathbb{R} \; \int_\Omega |f|\, dx < \infty \right\}.$$

$\Omega = [a,b]$; $f(x) = 1 \in L^1(\Omega)$

$\Omega = (-\infty, \infty)$; $f(x) = 1 \notin L^1(\mathbb{R})$.

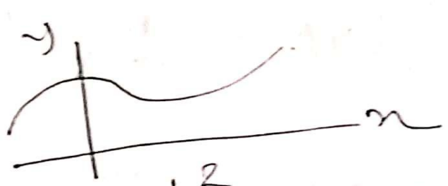
$$L^p(\Omega) = \left\{ f : \Omega \to \mathbb{R} \; \int_\Omega |f(x)|^p\, dx < \infty \right\}$$

$f(x) = |x|$, $x \in \mathbb{R}$; $f : \mathbb{R} \to \mathbb{R}$.

Obs:- Two directions (One direction) in 1 D.

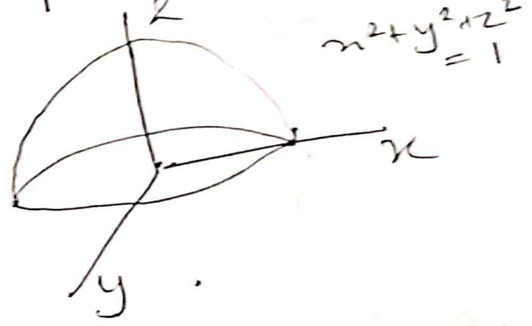
$f : \mathbb{R} \to \mathbb{R}$.

Graph will be in $\mathbb{R}^2$

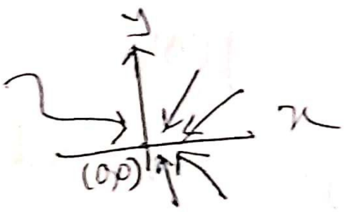


$f : \mathbb{R}^2 \to \mathbb{R}$.

Graph will be in $\mathbb{R}^3$.



$f : \mathbb{R}^2 \to \mathbb{R}$



Obs. Infinite direction.

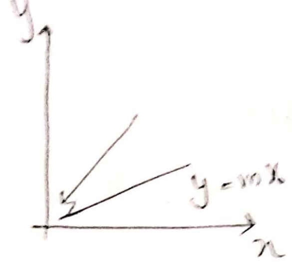
$u : \mathbb{R}^2 \to \mathbb{R}$

$$u_x(0,0) = \lim_{h \to 0} \frac{u(h,0) - u(0,0)}{h}$$

$$u_y(0,0) = \lim_{k \to 0} \frac{u(0,k) - u(0,0)}{k}.$$

In 1D ; Partial Derivative = Derivative .

<u>Example</u> $f(x,y) = \begin{cases} \frac{xy}{x^2+y^2} & ; xy \neq 0 \\ 0 & ; xy = 0 \end{cases}$



$$\lim_{(x,y)\to(0,0)} \frac{x \cdot mx}{x^2 + m^2x^2}$$
$$= m/1+m^2$$

will be different for different m. f is not cont./diff. at (0,0) .

$$f_x(0,0) = \lim_{h\to 0} \frac{\frac{h.0}{h^2} - 0}{h} = 0 .$$

$$f_y(0,0) = \lim_{k\to 0} \frac{\frac{k.0}{k^2} - 0}{k} = 0 .$$

Partial Derivative $\not\Rightarrow$ Continuity .

f_x, f_y exist at $(0,0)$; $f: \mathbb{R}^2 \to \mathbb{R}$.

<u>PDE</u> :- $2f_x + 3f_y = g(x,y) \to$ PDE .

<u>ODE</u> :- $y' = f(x,y)$ f is cont. ⇒ f is diff.

Sol^n of PDE may not be even continuous .

<u>PDE</u> :

$$\alpha = (\alpha_1, \ldots, \alpha_n) \in (\mathbb{N} \cup \{0\})^n$$
$$|\alpha| = \alpha_1 + \ldots + \alpha_n .$$

$$D^\alpha u = \frac{\partial^\alpha u}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \ldots \partial x_n^{\alpha_n}}$$

Let $\Omega \subseteq \mathbb{R}^k$; $F : \Omega \times \mathbb{R}^p \times \mathbb{R}^{np} \times \dots \times \mathbb{R}^{n^m p} \to \mathbb{R}^q$

PDE of order m is of the form

$F(x, u, Du, D^2u, \dots, D^m u) = 0$ is called PDE of m^{th} order.

$n = 2$; $u_x \; u_y$; $u_{xx} \; u_{xy} \; u_{yy} \; u_{yx}$

$f : \mathbb{R} \to \mathbb{R}$

$f' : \mathbb{R} \to \mathbb{R}$.

$f' = \nabla f$

(f_x, f_y)

$f : \mathbb{R}^2 \to \mathbb{R}$

$(x,y) \to f(x,y)$

$\Uparrow \qquad \Uparrow$

$\mathbb{R}^2 \qquad \mathbb{R}$.

$f' : \mathbb{R}^2 \to \mathbb{R}^2$.

<u>Partial Derivative</u>

$f : \mathbb{R}^2 \to \mathbb{R}$

$$\frac{\partial f(x,y)}{\partial x} = \lim_{h \to 0} \frac{f(x_0+h, y_0) - f(x_0, y_0)}{h}$$

$$\frac{\partial f(x,y)}{\partial y} = \lim_{k \to 0} \frac{f(x_0, y_0+k) - f(x_0, y_0)}{k}$$

$f : \mathbb{R}^n \to \mathbb{R}$ $\quad e_i = (0, 0, \dots, \underset{i^{th} \text{ place}}{1}, \dots, 0)$

$$\frac{\partial f(x_0)}{\partial x_i} = \lim_{h \to 0} \frac{f(x_0 + h e_i) - f(x_0)}{h}.$$

Partial Der. $\not\Rightarrow$ continuity.

$\not\Leftarrow$

Date / 201

★ Directional Derivative :-

$f : \mathbb{R}^n \to \mathbb{R}$; $a \in \mathbb{R}^n$

Let $u \in \mathbb{R}^n$ s.t. $\|u\| = 1$ Then the limit exists when

$$\underset{\text{direction}}{D_u} f(\underset{\text{point}}{a}) = \lim_{t \to 0} \frac{f(a + tu) - f(a)}{t}$$

$$= \frac{d}{dt} f(a+tu) \Big|_{t=0}.$$

= Rate of change of f at a in the direction u.

Ex $f : \mathbb{R}^2 \to \mathbb{R}$ $f(x,y) = \sqrt{|xy|}$ then directional der. $D_u f(0,0)$ does not exist for $u_1, u_2 \neq 0$; $u = (u_1, u_2)$.
(Try yourself).

Results:

Part. der. $\not\Rightarrow$ Direct. der.
$\not\Downarrow$
Direc. der $\not\Leftarrow$ continuity

Thm :- If f is differentiable at x_0 ; then $D_{u_0} f(x_0)$ $(\mathbb{R}^3)$. exist for all $u \in \mathbb{R}^n$

& $D_{u_0} f(x_0) = \left(f_x(x_0), f_y(x_0), f_z(x_0) \right) . u$

$f : \mathbb{R}^3 \to \mathbb{R}$.

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Gradient $= (f_x(x_0), f_y(x_0), f_z(x_0)) = \nabla f(x_0)$.

(Direction where f increases (or decreases) most rapidly along x_0).

$f: \mathbb{R} \to \mathbb{R}$. $\quad f: \mathbb{R}^3 \to \mathbb{R}$

$f': \mathbb{R} \to \mathbb{R}$. $\quad Df: \mathbb{R}^3 \to \mathbb{R}^3$.

$f: \mathbb{R} \to \mathbb{R}$. $\quad f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$.

f is diff at x iff $\exists\ \alpha \in \mathbb{R}$. s.t. (to be derivative of f at x)

$$\frac{|f(x+h) - f(x) - \alpha \cdot h|}{h} \to 0 \text{ as } h \to 0.$$

If f is diff. at x, then $\alpha = f'(x)$.

(α, h are vectors)

Suppose. $f: \mathbb{R}^3 \to \mathbb{R}$, f is diff. at x; if $\exists$ $\alpha = (\alpha_1, \alpha_2, \alpha_3) \to \mathbb{R}^3$ s.t. the error function

$$\epsilon(H) = \frac{f(x+H) - f(x) - \alpha \cdot H}{\|H\|} \longrightarrow 0$$

as $\|H\| \to 0$ $(H \to 0)$.

Theorem: $f: \mathbb{R}^n \to \mathbb{R}$; If f is diff at x

$\Rightarrow$ f is cont. at x

$\Rightarrow$ f has all its direc. der.

$\Rightarrow$ f has partial der.

Theorem :- f is diff at x.

Then $f'(x) = (\alpha_1, \alpha_2, \alpha_3)$

$$= \left(\frac{\partial f}{\partial x}\Big|_X , \frac{\partial f}{\partial y}\Big|_X , \frac{\partial f}{\partial z}\Big|_X \right).$$

★ { Norm ; $x = (x_1, x_2, x_3)$; $\|x\| = \sqrt{x_1^2 + x_2^2 + x_3^3}$ }

Theorem :- $f : \mathbb{R}^n \to \mathbb{R}$ s.t. all part. der. exist in a neighbourhood of x_0 and f is cont. at x_0 then f is diff. at x_0.

$$f : \mathbb{R} \to \mathbb{R} ; \lim_{x \to a} \frac{f(x) - f(a)}{x - a} = f'(a).$$

$$\lim_{x \to a} \frac{f(x) - f(a) - f'(a).(x-a)}{x-a} = 0.$$

Let, $h(x) = f'(a).(x-a) + f(a)$. (Linear function)

$h(x) \to f(x)$ faster than $x \to a$.

Def^n. f is diff if $\exists$ a linear function $L(x)$. s.t.

$$\lim_{x \to a} \frac{f(x) - f(a) - L(x)}{x-a} = 0$$

where $L(x) = f'(a).(x-a)$.

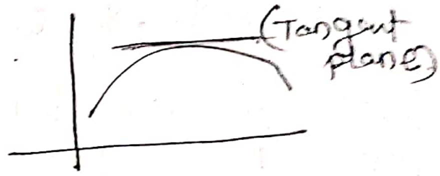
(Tangent Plane)

$f : \mathbb{R}^2 \to \mathbb{R}$

$h(x,y) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$.

is a good linear approximation of $f(x,y)$ near (a,b)

$$\lim_{(x,y)\to(a,b)} \frac{f(x,y) - h(x,y)}{\sqrt{(x-a)^2+(y-b)^2}} = 0 .$$



Difference between ODE & PDE :-

Linear ODE. (2nd order)

↳ 2 LI solns (Two arbitrary constants).

$y'' + ay' + by = f \Rightarrow CF = c_1y_1 + c_2y_2$

$y_1, y_2 \to$ LI.

$y' = Ay \; ; \; y \in \mathbb{R}^n$

$y(0) = y_0 \in \mathbb{R}^n$.

PDE $\Rightarrow$ arbitrary const. / arbitrary func.

$u = u(x,y)$ arbitrary

$u_x = 0 \Rightarrow u = g(y)$

$= c_1 y + c_2$

linear PDE

$u_t + a u_x = 0 \Rightarrow$ Burgers' Equation

$x \in \mathbb{R}, t > 0$

$u = u(x,y)$.

$u(x,0) = f(x)$.

$u = f(x - at)$.

$f(x)$ should lie in function space

(Function space

$C'(0) = \{ f : D \to \mathbb{R}.$
f' is cont$\}$.

Well-posed

a PDE is well-posed if

1. it has a sol^n
2. Sol^n is unique
3. Sol^n is stable

eg. $u_x^2 + u_y^2 + 1 = 0$

Does not have any sol^n in $\mathbb{R}$.

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Order and Degree of PDE similar to ODE.

Linear PDE $\subsetneq$ Semi Linear PDE $\subsetneq$ Quasilinear PDE $\subseteq$ Fully Nonlinear PDE.

1st order PDE $\Rightarrow f(x, y, u, u_x, u_y) = 0$.

Linear PDE $\Rightarrow a(x,y)u_x + b(x,y)u_y = c(x,y)u + d(x,y)$

Semilinear PDE $\Rightarrow a(x,y)u_x + b(x,y)u_y = c(x,y,u)$

$[u_x + u_y = e^u]$ Expected to be Nonlinear.

Quasilinear PDE $\rightarrow a(x,y,u)u_x + b(x,y,u)u_y = c(x,y,u)$.

Fully Nonlinear $\Rightarrow u_x^2 + u_y^2 = 1$.

$a(x,y,u,u_x,u_y).u_x + b(x,y,u,u_x,u_y)\,u_y$

$= c(x,y,u,u_x,u_y)$.

Geometry
- $\rightarrow$ Lagrange's Method (quasilinear PDE).
- $\rightarrow$ Charpit Method (Fully Nonlinear PDE).

Q If two arbitrary functions/constants are in the soln. Can I say that the corresponding PDE will be of more than first order.

Eg. $u = f(x^2+y^2) = f(g)$; arbitrary; where $g = x^2+y^2$

$u_x = f'(g)\, 2x$ $\boxed{y u_x - x u_y = 0}$

$u_y = f'(g) \cdot 2y$ 1st order PDE.

Can I say $u = f(x^2+y^2)$ is a soln of

$\frac{\partial}{\partial}\,[y u_x - x u_y] = 0$.

$x^2+y^2+(u-c)^2 = a^2$ ← Two arbitrary const. c, a

Diff. w.r.t x $2x + 2(u-c)u_x = 0 \Rightarrow x + (u-c)u_x = 0$

w.r.t y. $2y + 2(u-c)u_y = 0 \Rightarrow y + (u-c)u_y = 0$

Eliminate c ; $\frac{-x}{u_x} = \frac{-y}{u_y}$

$\Rightarrow y u_x - x u_y = 0$.

Quasilinear PDE

$a(x,y,u)\, u_x + b(x,y,u) \cdot u_y = c(x,y,u)$; $a, b, c \in C^1(\Omega)$

Cauchy Problem

Consider $x = f(s)$

$y = g(s)$

$z = h(s)$. $s \in I$

→ Assume curve to be differentiable.

curve:

$\nu : [a,b] \Rightarrow \mathbb{R}^2$ (or $\mathbb{R}^3$).

which is continuous.

(we will take ν to be differentiable).

Eg. $t \to (\cos t, \sin t)$ (Image is circle)

$[0, 2\pi]$; $t \to (t, \sqrt{1-t^2})$ Is it possible to draw a curve.

$[0,1]$ DRs Gradient → $\left(\frac{dx}{dt}, \frac{dy}{dt}\right)$

Assume $h(s) = u(f(s), g(s))$ Initial curve / Data curve.

Cauchy Problem means finding solution in a nbd. of this curve.

Consider a Surface $F(x,y,z)=0$

$C = \{ x(s), y(s), z(s) \}$ ⇐ curve on the surface

$F(x(s), y(s), z(s)) = 0$ {Assume F is diff. w.r.t s}

$$\frac{\partial F}{\partial x}\cdot\frac{dx}{ds} + \frac{\partial F}{\partial y}\cdot\frac{dy}{ds} + \frac{\partial F}{\partial z}\cdot\frac{dz}{ds} = 0.$$

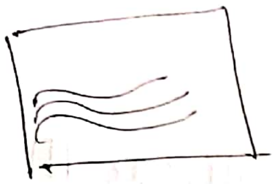
$$(F_x, F_y, F_z)\cdot\left(\frac{dx}{ds}, \frac{dy}{ds}, \frac{dz}{ds}\right) = 0.$$

$$a(x,y,z)\,z_x + b(x,y,z)\,z_y = c(x,y,z).$$

quasilinear 1st order. PDE $\rightarrow (a,b,c).(z_x, z_y, -1) = 0$. —— (**)

Solution Surface ($\mathbb{R}^3$).

Assume Solution Surface $F(x,y,z) = 0$.



Curve: $C: (x(s), y(s), z(s))$ ⇐ diff. w.r.t. 's'.

$$F(x(s), y(s), z(s)) = 0 \Rightarrow \frac{\partial F}{\partial x}\cdot\frac{dx}{ds} + \frac{\partial F}{\partial y}\cdot\frac{dy}{ds} + \frac{\partial F}{\partial z}\cdot\frac{dz}{ds} = 0.$$

Choose $F = \underbrace{z(x,y)}_{} - z$. (expression of x & y)

$$\Rightarrow \underbrace{(z_x, z_y, -1)}_{\text{DRs of normal}}\underbrace{\left(\frac{dx}{ds}, \frac{dy}{ds}, \frac{dz}{ds}\right)}_{\text{DRs of tangent of } (x(s),y(s),z(s))} = 0.$$ —— (*)

cond^n at $t=0$ $z(0,s) = \ldots$

From (*) & (**) ;

$$\left.\begin{aligned} \frac{dx}{ds} &= a(x(s), y(s), z(s)) \\ \frac{dy}{ds} &= b(x(s), y(s), z(s)) \\ \frac{dz}{ds} &= c(x(s), y(s), z(s)) \end{aligned}\right\}$$ Characteristic Equation. system of ODE

$\Rightarrow$ Curve [Picard's Theorem
a, b, c are Lipschitz & cont. w.r.t. all the variables.

$$\left.\frac{dx}{a} = \frac{dy}{b} = \frac{dz}{c}\right\}$$ Characteristic Equation.

$(x(s), y(s), z(s)) \longrightarrow$ Characteristic Curve.

Theorem :-

$a z_x + b z_y = c(x, y, z)$ ——— (**)

where $a, b, c \in C^1$, $a^2 + b^2 + c^2 \neq 0$.

The general solution of (**) $F(u, v) = 0$ where F is arbitrary function and $u(x, y, z) = c_1$; $v(x, y, z) = c_2$ are two LI sol^{ns} of

$$\frac{dx}{a} = \frac{dy}{b} = \frac{dz}{c} = \frac{l_1 dx + l_2 dy + l_3 dz}{l_1 a + l_2 a + l_3 c}.$$

l_1, l_2, l_3 are called multiples.

★ Picard's Theorem (for ODE) $y' = f(x,y)$

If $|f(x,y_1) - f(x,y_2)| \le M|y_1 - y_2|$ & f is cont. w.r.t x & y then $\exists$ unique solution in a nbd of x_0.

Solution surface is a union of characteristic curve.

Ex. $yz \cdot z_x + xz \cdot z_y = xy$ $\Rightarrow$ Solution Surface.

$\boxed{G.S}$ $F(x^2-y^2, y^2-z^2) = 0$ Implicit form

Ans:- $\dfrac{dx}{yz} = \dfrac{dy}{xz} = \dfrac{dz}{xy}$.

$\boxed{S.S}$ $y^2-z^2 = c_2(x^2-y^2)$ ψ arbitrary

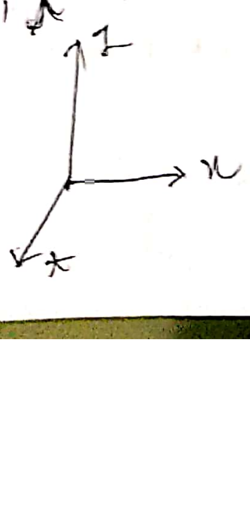
$\Rightarrow \dfrac{dx}{y} = \dfrac{dy}{x} \Rightarrow x\,dx - y\,dy = 0$

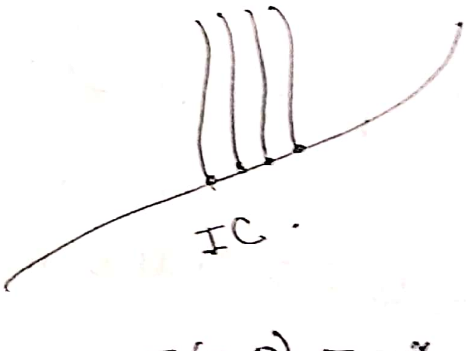
$\Rightarrow x^2 - y^2 = c_1$.

$\Rightarrow \dfrac{dy}{z} = \dfrac{dz}{y} \Rightarrow y^2 - z^2 = c_2$

Ex. $z_t + z \cdot z_x = 0$ (Burgers's Equation) when the soln passes through $\boxed{z(x,0) = -x}$ $\rightarrow$ curve.

$x = s$, $t = 0$, $z = -s$.





IC.

$z(x,0) = -x$

Ans $\Rightarrow$ $\dfrac{dt}{1} = \dfrac{dx}{z} = \dfrac{dz}{0}$

$\boxed{\begin{array}{l}\text{General sol}^n \\ F(z, x-zt) = 0.\end{array}}$

$\dfrac{dt}{1} = \dfrac{dz}{0} \Rightarrow z = c_1 \quad (u = z = c_1)$

$\dfrac{dt}{1} = \dfrac{dx}{c_1} \Rightarrow x = c_1 t = c_2.$

$\Rightarrow x - zt = c_2. \Rightarrow (v = x - zt = c_2).$

$\Rightarrow z(x,t) = \varphi(x - zt) \qquad z(x,0) = -x.$

$\Rightarrow -x = \phi(x).$

Sol^n surface.

$z(x,t) = zt - x.$

$\Rightarrow z = zt - x \Rightarrow \boxed{z = \dfrac{x}{t-1}}$

Ex. $u_x = cu + d(x,y) \leftarrow$ PDE
cont diff.

By Integrating factors (y as parameters).

$$u(x,y) = e^{cx}\left(\int_0^x e^{-c\xi} d(\xi, y)\, d\xi + u(0,y)\right)$$

unique sol^n.

$u_x = cu + d(x,y), \quad u(0,y) = y.$
Initial curve.

$\Rightarrow$ Sol^n is unique

$u_x = cu, \quad u(x,0) = e^{cx} \qquad u(x,y) = e^{cx} T(y).$

$\Rightarrow$ Infinite sol

$e^{cx} = e^{cx}.T(0)$

$\Rightarrow T(0) = 1.$

Check for $u_x = cu$, $u(x,0) = \sin(x)$.
~~$u_x - cu = 0$~~

$u(x,y) = e^{cx} T(y) \Rightarrow \sin(x) = e^{cx}.T(0)$

$\Rightarrow \boxed{T(0) = e^{-cx}.\sin x} \Rightarrow$ No solution.

$P(x,y,z).z_x + Q(x,y,z) z_y = R(x,y,z)$ —— (1).

$\rightarrow$ Quasilinear 1st order PDE.

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} \quad (= dt).$$

$\hookrightarrow$ Curve ($\mathbb{R}^3$)

→ Three arbitrary parameters (in general).

L.I. $\begin{cases} u(x,y,z) = c_1 \\ v(x,y,z) = c_2 \end{cases}$ > Arbitrary const.

Arbitrary $\Rightarrow F(u,v) = 0 \Rightarrow$ Solution of (1).

Solution surface will be generated by one ~~particular~~ parameter family of curves.

Thm :- Existance & Uniqueness Theorem.

$P(x,y,z) z_x + Q(x,y,z) z_y = R(x,y,z)$

$P, Q, R \in C'$ and $P^2 + Q^2 + R^2 \neq 0$.

Consider $x = x_0(s)$, $y = y_0(s)$, $z = z_0(s)$ is the Initial data curve (continuously differentiable in $s \in [a,b]$)

If

$$\frac{dy_0}{ds} P(x_0(s), y_0(s), z_0(s)) - \frac{dx_0}{ds} Q(x_0(s), y_0(s), z_0(s)) \neq 0 .$$

$\{ (x,y) \to (s,t)$; $\underset{(\text{Jacobian})}{J \neq 0} \}$

Then

$\exists$ a unique solution in a nbd. of $x = x_0(s)$, $y = y_0(s)$ $z = z_0(s)$ which satisfy $z(x_0(s), y_0(s)) = z_0(s)$.

---

① $u_x = cu + d(x,y)$; $u(0,y) = y$ $J \neq 0$ Sol^n unique

② $u_x = cu$; $u(x,0) = e^{cx}$ $J = 0$ Sol^n infinite

③ $u_x = cu$; $u(x,0) = \sin(x)$ $J = 0$, Sol^n does not exist

Singular Solution :-

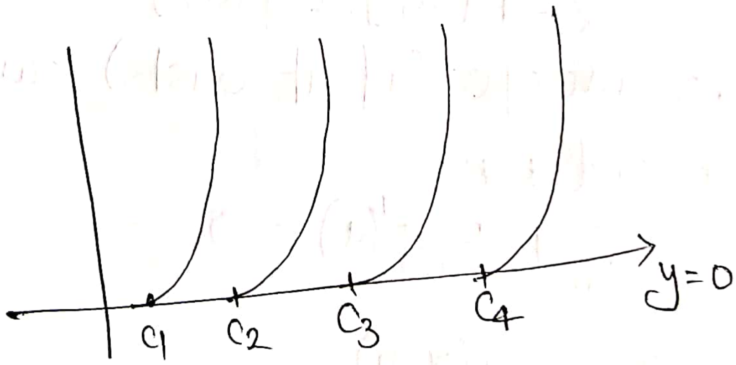
$$\left.\begin{aligned} y' &= 2\sqrt{y} \\ y(0) &= 0 \end{aligned}\right\} \quad y = 0 \to \text{Singular Sol}^n$$

$$y(x) = \begin{cases} (x-c)^2 ; & x \geq c > 0 \\ 0 ; & x < c \end{cases} \quad \text{differentiable}$$

Arbitrary (one parameter) $\longrightarrow$ Infinitely many solutions.

$$\frac{dy}{dx} = 2\sqrt{y}$$

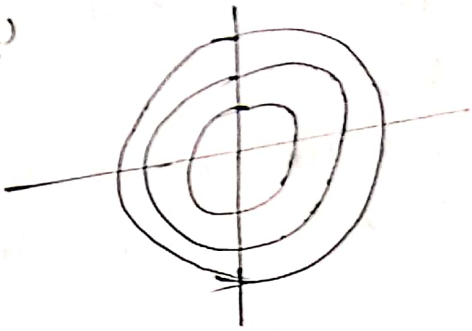
$$\frac{dy}{2\sqrt{y}} = dx$$



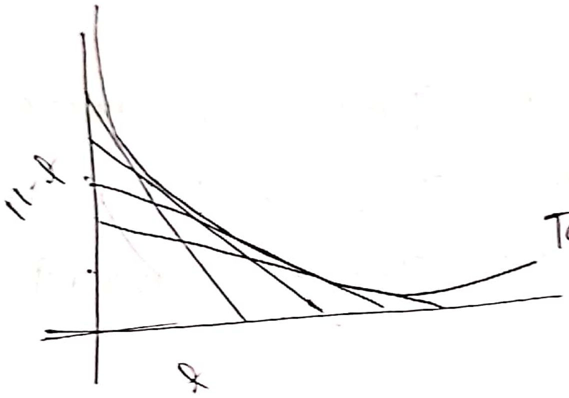
Envelope of a family of curves on a plane is a curve that in tangent to each member of the family at some point.

$$F(x, y, t) = 0$$
$$\frac{\partial F}{\partial t}(x, y, t) = 0$$
} Remove 't' parameter 't'

$x^2+y^2=c^2$



e.g. $\dfrac{x}{t} + \dfrac{y}{11-t} = 1$



Tangent curve (envelope) of $\dfrac{x}{t} + \dfrac{y}{11-t} = 1$

(a) Complete Integral :- A two parameter family of $z = F(x, y, a, b)$ is called Complete Integral.

(b) General Integral :-
Consider $b = \varphi(a)$ ← Specific Assumption.

$$z = F(x, y, a, \varphi(a)) \quad \text{———} \; (*)$$

Then the envelope (if it exists) can be obtained from (*) and (**)

$$F_b + F_a \varphi'(a) = 0 \quad \text{———} \; (**)$$

Say, $a = a(x, y)$.
Then $z = F(x, y, a(x,y), \varphi(a(x,y)))$ is general integral when φ is arbitrary.

If ϕ is specific then it is called Particular Integral.

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Singular Integral :- Find the envelope of

$$\begin{cases} z = F(x,y,a,b). \\ F_a = 0 \\ F_b = 0 \end{cases}$$

Singular Solution can be obtained by

$$\left. \begin{array}{l} f(x,y,z,p,q) = 0 \\ f_p(x,y,z,p,q) = 0 \\ f_q(x,y,z,p,q) = 0 \end{array} \right\} \text{Eliminate } p,q.$$

$Pu_x + Qu_y = R$ ④ Initial Data Curve.

Monge direction. (P,Q,R)

Monge curve → Characteristic Curve $\left(\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}.\right)$

$f(x,y,z,p,q) = 0$; $p = u_x$; $q = u_y$.

Monge Curve

Assume $f_p^2 + f_q^2 \neq 0$.

$f_q \neq 0 \Rightarrow q = q(x,y,z,\phi)$

Consider (x_0,y_0,z_0) on the surface, Normal dir^n $(z_x, z_y, -1)$
$\simeq (p,q,-1)$

Consider a plane passing through (x_0,y_0,z_0) →

$$z - z_0 = p(x-x_0) + q(y-y_0) \quad \text{—①}$$

Again $q = q(x_0, y_0, z_0, p)$ —②

Find the envelope of ① by eliminating p & q.

This is called Monge Curve.

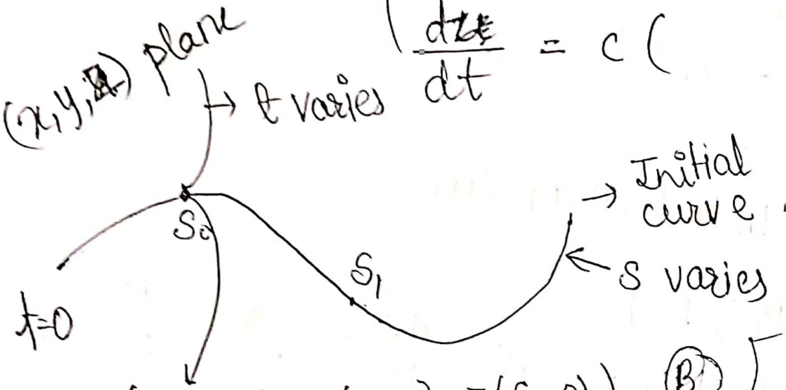
$$a(x,y,z)\,u_x + b(x,y,z)\,.\,u_y = c(x,y,z).$$

I.C. $u(f(s), g(s)) = h(s)$, $s \in I \rightarrow$ parameter form

$\left[\ x = f(s),\ y = g(s),\ z = h(s), s \in I;\ f, g, h \in C'(I)\ \right]$

<u>Lagrange's Equation</u> :-

$$\text{Ⓐ}\begin{cases} \dfrac{dx}{dt} = a(x(t), y(t), u(t)) \\ \dfrac{dy}{dt} = b(\qquad\qquad) \quad t \in D_t \\ \dfrac{dz}{dt} = c(\qquad\qquad) \quad (x,y) \rightarrow (s,t) \end{cases}$$



<u>I.C.</u> at $t = 0$

$$\text{Ⓑ}\begin{cases} x(s_0,t)\big|_{t=0} = x(0) = f(s_0) \\ y(s_0,t)\big|_{t=0} = y(0) = g(s_0) \\ z(s_0,t)\big|_{t=0} = z(0) = h(s_0) \end{cases}$$

$(x(s_0,0), y(s_0,0), z(s_0,0))$
$= (f(s_0), g(s_0), h(s_0))$.

Solving Ⓐ & Ⓑ,

$(s,t) = (s_0, 0)$

$x = X(s_0, t)$ $\qquad x = X(s,t)$
$y = Y(s_0, t) \rightarrow y = Y(s,t)$
$z = Z(s_0, t)$ $\qquad z = Z(s,t)$

$$J = \frac{\partial(X,Y)}{\partial(s,t)}\Bigg| \neq 0.$$

$$\Rightarrow \begin{vmatrix} X_s(s_0,0) & X_t(s_0,0) \\ Y_s(s_0,0) & Y_t(s_0,0) \end{vmatrix} \neq 0$$

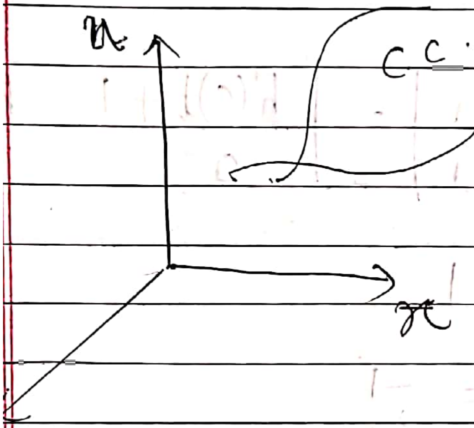
$$\Rightarrow \begin{vmatrix} f'(s_0) & a(f(s_0), g(s_0), h(s_0)) \\ g'(s_0) & b(f(s_0), g(s_0), h(s_0)) \end{vmatrix} \neq 0$$

$\hookrightarrow (s,t) \rightarrow (x,y)$ if $J \neq 0$.
uniquely

MA201 PDE

Transversality Condition at $D(s_0, 0)$ $(f(s), g(s), h(s)) \in \Gamma$, $(s \in I)$ if the base characteristics corresponding to the characteristic curve passing through $(f(s), g(s), h(s))$ intersects the projection of Γ non tangentially.

Base Characteristic :- The projection of characteristic curve in the xy plane.



What if Transversality cond^n fails.

i.e. If $J(s_0) = 0$.

① Γ has characteristic dir^n at s_0

i.e. $\frac{f'}{a} = \frac{g'}{b} = \frac{h'}{c}$; at $s = s_0$

{ direction of initial curve and characteristic curve are same at a point ; tangetially }

② $J = 0$ identically along Γ, then Γ is a characteristic curve.

e.g. Burger's Eq^n $u_y + u.u_x = 0$; $u(x,0) = h(x)$; $x \in \mathbb{R}$.

Γ : $x = s, y = 0, z = h(s)$; $s \in I$

Lagrages Eq^n : $\frac{dx}{dt} = u$; $\frac{dy}{dt} = 1$; $\frac{du}{dt} = 0$.

initial cond^n at $t=0$:

$X(0,s) = s$ $\quad$ $x = X(s,t) = h(s)t + s$

$y(0,s) = 0$ $\quad$ $y = Y(s,t) = t$

$Z(0,s) = h(s)$ $\quad$ $z = Z(s,t) = h(s)$

Base characteristic :- Parametric form $(h(s)t+s,\ t),\ t \in \mathbb{R}$

Eqn of base characteristic in cartesian coordinate

$$x = h(s)y + s \quad ; \quad y = \frac{x}{h(s)} - \frac{s}{h(s)}.$$

$$\text{slope} = \frac{1}{h(s)}$$

$$J = \frac{\partial(x,y)}{\partial(s,t)} = \begin{vmatrix} x_s & x_t \\ y_s & y_t \end{vmatrix} = \begin{vmatrix} h'(s)t+1 & h(s) \\ 0 & 1 \end{vmatrix}$$

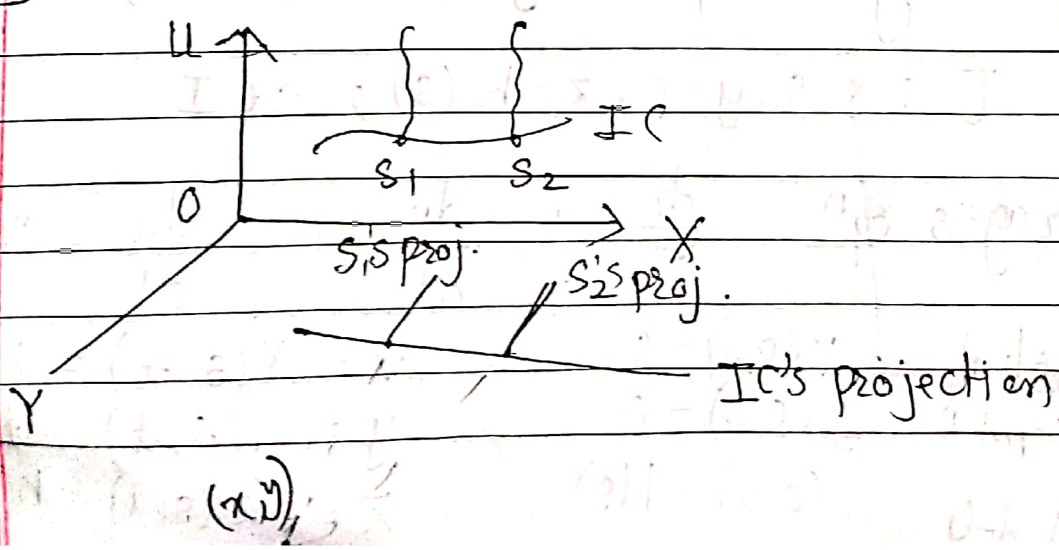
$$= h'(s)t + 1$$

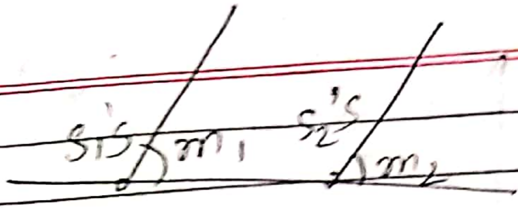
$$J = 0 \Rightarrow h'(s) = \frac{-1}{t}$$

$$\Rightarrow t = \frac{-1}{h'(s)}$$

So,
for the solution to exist.
Soln :-

[Check] $z = h(x - yz)$ or $u = h(x - yu)$





For non-intersection with base characteristic $m_1 > m_2$.

$$m_1 = \frac{1}{h(s_1)} \; ; \; m_2 = \frac{1}{h(s_2)}$$

$s_1 < s_2$

$$\frac{1}{h(s_1)} > \frac{1}{h(s_2)}$$

$$h(s_1) < h(s_2)$$

h is an increasing function.

If $m_1 < m_2$, then base characteristic intersects (h is monotonically decreasing).

$y>0$, assume h' is decreasing function ($h'<0$).

$$u = h(x-yu)$$

$$u_x = h'(x-yu)(1-yu_x)$$

$$u_y = h'(x-yu)(-u-yu_y)$$

$$u_x = \frac{h'(x-yu)}{1+yh'(x-yu)}.$$

u_x will become infinite if, $1+yh'(x-yu)=0$.

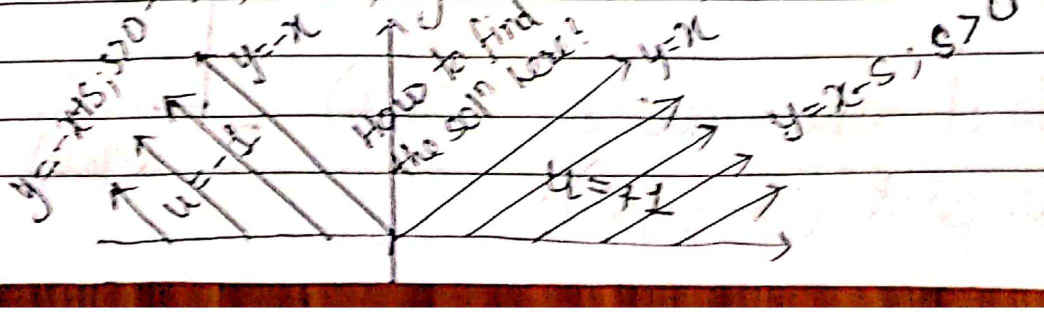
→ Gradient Catastrophe.

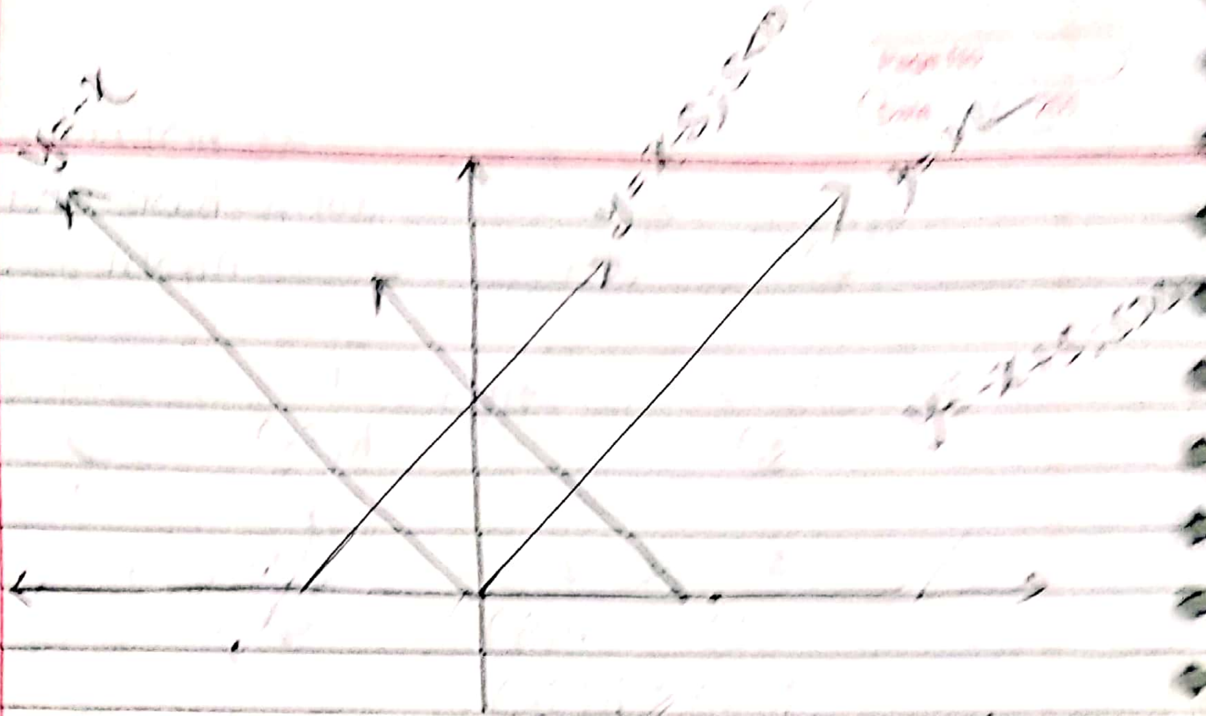
$$h(x) = \begin{cases} -1 \; ; & x<0 \\ 1 \; ; & x \geq 0 \end{cases}$$

Base characteristic: $y = \frac{x}{h(s)} - \frac{s}{h(s)}$.

$s<0$, $h(s) = -1$, $y = -x + s$, $s<0$.

$s>0$, $h(s) = 1$, $y = x - s$, $s>0$.





Domain of Dependence ; Range / Domain of Influence

$u_x + u_y = 0$; $u(x,0) = \sin x$, $x \geq 0$ }

Sol : $u = \sin(x-y)$ where $x - y \geq 0$ or $y \leq x$.

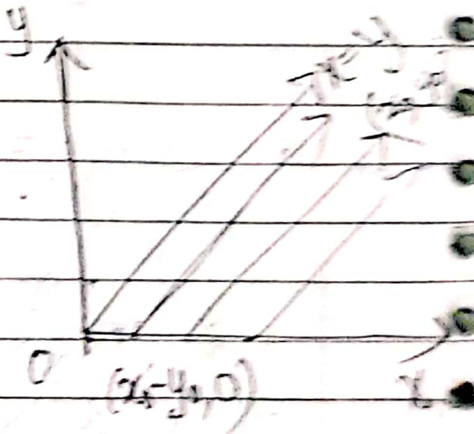
$x = s$, $y = 0$, $u = \sin(s)$.

$\frac{dx}{dt} = 1$; $x = t + c_1$ $\quad x(s,t)\big|_{t=0} = s \Rightarrow c_1 = s$

$\frac{dy}{dt} = 1$; $y = t + c_2$ $\quad y(s,t)\big|_{t=0} = 0 \rightarrow c_2 = 0$

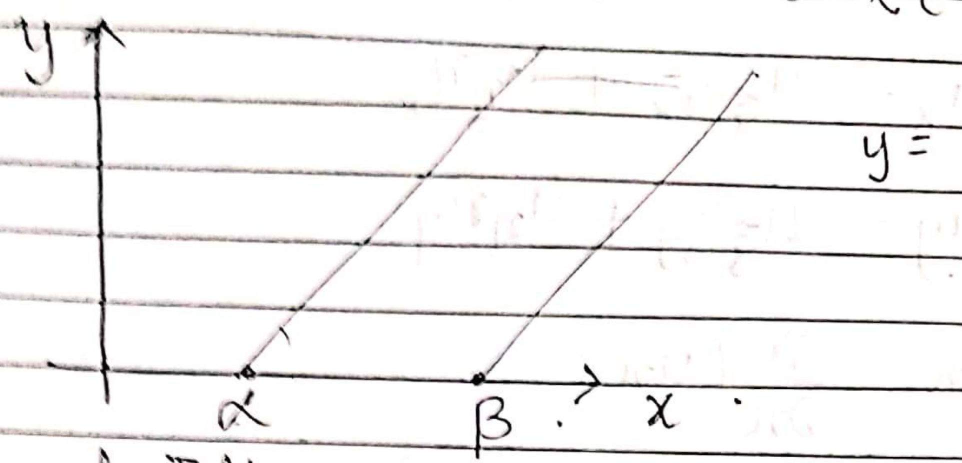
$\frac{du}{dt} = 0$; $u = c_3$ $\quad z(s,t)\big|_{t=0} = \sin(s) \Rightarrow c_3 = \sin(s)$

$$\left.\begin{array}{l} x = s + t \\ y = t \\ u = \sin(s) \end{array}\right\} \begin{array}{l} s = x - y \\ y = x - s \\ = \sin(x-y),\ x - y \geq 0 . \end{array}$$



~~Domain of dependence of P(x_0, y_0)~~ is the point $(x_0 - y_0, 0)$.

Range of Influence, Consider $x \in [\alpha, \beta]$.



Range of Influence

$$(x+\alpha \leq y \leq x+\beta).$$

172.16.1.3/~pratibhamoy.

MA201 — Shivaji Notes
chapter (1A, 2A, 2B, 2C)

Literature text.

Second Order PDEs :-

Semilinear / Linear Second-Order PDE

Principal part of PDE.

$$a(x,y)\,u_{xx} + 2b(x,y)\,u_{xy} + c(x,y)\,u_{yy} +$$
$$d(x,y)\,u_x + e(x,y).u_y + f(x,y).u = g(x,y)$$

We may take d, e, f as function of (x, y, u_x, u_y).

Transformation $(x,y) \rightarrow (\xi, \eta)$

$$J = \frac{\partial(\xi,\eta)}{\partial(x,y)} \neq 0, \infty.$$

$\xi(x,y)\ ;\ \eta(x,y).$

$$u_x = u_\xi \xi_x + u_\eta \eta_x$$

$$u_y = u_\xi \xi_y + u_\eta \eta_y$$

$$u_{xx} = \frac{\partial}{\partial x}(u_x)$$

$$= \frac{\partial}{\partial x}\left(u_\xi \xi_x + u_\eta . \eta_x\right)$$

$$= u_\xi . \xi_{xx} + \xi_x . \frac{\partial}{\partial x}(u_\xi) + u_\eta . \eta_{xx} + \eta_x \frac{\partial}{\partial x}(u_\eta)$$

$$= u_\xi . \xi_{xx} + \xi_x \left[u_{\xi\xi} . \xi_x + u_{\xi\eta}\, \eta_x\right] + u_\eta . \eta_{xx}$$

$$+ \eta_x . \left[\eta_{\xi\eta} . \xi_x + \eta_{\eta\eta}\, \eta_x\right]$$

$$u_{xx} = u_\xi . \xi_{xx} + \xi_x^2 . u_{\xi\xi} + 2\, \xi_x . \eta_x . u_{\xi\eta} + \eta_x^2 . \eta_{\eta\eta}$$

$$+ u_\eta . \eta_{xx}.$$

$$u_{yy} = u_{\xi\xi} . \xi_y^2 + 2 u_{\xi\eta}\, \xi_x . \eta_y + u_{\eta\eta} . \eta_y^2 + u_\xi\, \xi_{yy}$$

$$+ u_\eta . \eta_{yy}.$$

$$u_{xy} = u_{\xi\xi} . \xi_x . \xi_y + u_{\xi\eta}\left(\xi_x\, \eta_y + \xi_y . \eta_x\right) + u_{\eta\eta} . \eta_x .$$

$$\eta_y + u_\xi\, \xi_{xy} + u_\eta\, \eta_{xy}.$$

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$$D_\xi = \frac{\xi_x}{\xi_y} \quad ; \quad D_\eta = \frac{\eta_x}{\eta_y}$$

$$\bar{A} = a.\xi_x^2 + 2b\xi_x\xi_y + c.\xi_y^2 = [aD_\xi^2 + 2bD_\xi + c]\,\xi_y^2$$

$$\bar{B} = a.\xi_x\eta_x + b[\xi_x\eta_y + \xi_y\eta_x] + c\xi_y\eta_y$$

$$= [aD_\xi D_\eta + b(D_\xi + D_\eta) + c]\,\xi_y\eta_y.$$

$$\bar{C} = a\eta_x^2 + 2b\eta_x\eta_y + c\eta_y^2 = [aD_\eta^2 + 2bD_\eta + c]\,\eta_y^2$$

$$(\bar{B}^2 - \bar{A}.\bar{C}) = (b^2 - ac)J^2.$$

Quadratic Equation $\rightarrow$ $aD^2 + 2bD + C = 0$

$$\Rightarrow \quad D = \frac{-b \pm \sqrt{b^2 - ac}}{a}$$

Let, $b^2 - ac \neq 0 \Rightarrow$ Two Distinct roots.

$$D_\xi\left(= \frac{\xi_x}{\xi_y}\right) = \frac{-b - \sqrt{b^2 - ac}}{a} \quad \text{or} \quad \bar{A} = 0.$$

$$D_\eta\left(= \frac{\eta_x}{\eta_y}\right) = \frac{-b + \sqrt{b^2 - ac}}{a} \quad \text{or} \quad \bar{C} = 0$$

Show that $\bar{B} = 0$, Consider $\xi(x, y) = c_1$

$$\Rightarrow d\xi(= \xi_x.dx + \xi_y.dy) = 0$$

$$\Rightarrow \frac{dy}{dx} = \frac{-\xi_x}{\xi_y} = \frac{b + \sqrt{b^2 - ac}}{a}$$

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$$\boxed{\frac{dy}{dx} = \frac{b+\sqrt{b^2-ac}}{a}} \longrightarrow \xi(x,y) = c_1$$

$$\eta(x,y) = c_2 \Rightarrow \boxed{\frac{dy}{dx} = \frac{b-\sqrt{b^2-ac}}{a}}.$$

$$\hookrightarrow \eta(x,y) = c_2$$

If $b^2-ac = 0$;

$$\frac{dy}{dx} = \frac{b}{a} \rightarrow \xi(x,y) = c_1.$$

Find η s.t. $\frac{\partial(\xi,\eta)}{\partial(x,y)} \neq 0$.

$\Rightarrow \bar{A} = 0$ (ξ, η are LI) also $B = 0$

ξ, η are called characteristic curves

$b^2-ac < 0 \rightarrow$ $\left.\begin{array}{l}\xi = c_1 \\ \eta = c_2\end{array}\right\}$ Complex Transformation

$\alpha = \xi + \eta$; $\beta = i(\xi - \eta)$ $\}$ Real Transformation

(i) $b^2-ac > 0 \longrightarrow$ Hyperbolic.

$(x,y) \rightarrow (\xi, \eta)$

$u_{\xi\eta}$ + lower order terms $= 0$.

(ii) $b^2-ac = 0$, ξ; Find η at $\xi \neq 0$ Parabolic

$u_{\eta\eta}$ + lower order terms $= 0$

(iii) $b^2-ac < 0$ Elliptic

$u_{\alpha\alpha} + u_{\beta\beta}$ + lower order terms $= 0$

eg. $u_{tt} - u_{xx} = 0$ (eg. of (i))
→ wave eqⁿ

$u_t - u_{xx} = 0$ (eg. of (ii))
→ Heat eqⁿ

$u_{xx} + u_{yy} = 0$ (eg. of (iii))
Harmonic fn.

Poisson eqⁿ : $u_{xx} + u_{yy} = f(x,y)$

Tricomi Equation.

$$y.u_{xx} + u_{yy} = 0$$

$b^2 - ac = -y < 0$ if $y > 0$ { elliptic.

$b^2 - ac = -y > 0$ if $y < 0$ { Hyperbolic.

Elliptic or Hyperbolic PDE does not surely has elliptic or hyperbolic solution.

$a(x,y)u_{xx} + 2b(x,y)u_{xy} + c(x,y)u_{yy} + du_x + eu_y + fu = g$. $(x,y) \rightleftarrows (\xi, \eta)$; $J \neq 0$.

$b^2 - ac \neq 0$; $\frac{dy}{dx} = \frac{b + \sqrt{b^2 - ac}}{a} \Rightarrow \xi = \text{const}$

$\frac{dy}{dx} = \frac{b - \sqrt{b^2 - ac}}{a} \Rightarrow \eta = \text{const}$.

$b^2 - ac > 0$; $b^2 - ac = 0$; $b^2 - ac < 0$
Hyperbolic Parabolic Elliptic.

$u_{xx} + u_{yy} = 0$

$b = 0,\ a = 1,\ c = 1 \Rightarrow b^2 - ac < 0$.

Elliptic $\rightarrow$	Imaginary Characteristics	(1),	Max Principle,	Global Solution.
Parabolic $\rightarrow$	Real Characteristics	(1),	Max Principle,	Global Solution
Hyperbolic $\rightarrow$	Real Characteristic	(2),	May not satisfy Max principle,	Local Solution.

$\hookrightarrow$ sol^n may be discont. if initial ~~derivative~~ data curve is non-smooth.

Most of the first order PDEs are hyperbolic.

$f(x) = x + y$ (1D elliptic).

$u_x + u u_y = 0$; $u(x,0) = h(x)$.

* Reduce it to canonical form and solve.

① $y^2 u_{xx} - 2xy u_{xy} + x^2 u_{yy} = \frac{1}{xy}\left[y^3 u_x + x^3 u_y\right]$

Ans. $a = y^2$; $b = -xy$; $c = x^2$

$b^2 - ac = 0 \Rightarrow$ Parabolic.

$$\frac{dy}{dx} = \frac{b}{a} = \frac{-xy}{y^2} = \frac{-x}{y^2}.$$

$\Rightarrow x^2 + y^2 = c = \xi$; Let $\eta = y$ (or choose $\eta = x$)

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$\rightarrow \quad J = \begin{vmatrix} 2x & 2y \\ 0 & 1 \end{vmatrix} = 2x \neq 0$ if $x \neq 0$.

$u_x = u_\xi \xi_x + u_\eta . \eta_x$

$u_x = u_\xi . 2x + u_\eta . 0 = 2x u_\xi$.

$u_y = 2y u_\xi + u_\eta$.

$u_{xx} = 2u_\xi + u_{\xi\xi} . 4x^2$

$u_{xy} = 4xy . u_{\xi\xi} + 2x u_{\xi\eta}$.

$u_{yy} = 2u_\xi + 4y^2 . u_{\xi\xi} + 4y u_{\xi\eta} + u_{\eta\eta}$

PDE Reduced to $\Rightarrow \quad u_{\eta\eta} - \frac{1}{\eta} u_\eta = 0$.

let, $v = u_\eta$ (Roughly)

$\rightarrow \quad v_\eta - \frac{1}{\eta} . v = 0$.

$\Rightarrow \quad \frac{dv}{v} = \frac{d\eta}{\eta}$.

$\Rightarrow \quad v = c_1(\xi) . \eta$.

$\Rightarrow \quad \frac{du}{d\eta} = c_1(\xi) . \eta$

$\Rightarrow \quad u = c_1(\xi) . \eta^2 + c_2(\xi)$

$\Rightarrow \boxed{u = y^2 . f(x^2+y^2) + g(x^2+y^2)}$ {arbitrary}

(2) $u_{xx} + u_{xy} + u_{yy} = 0$.

$\rightarrow$ $b = \frac{1}{2}$; $c = 1$; $a = 1$ $\Big\}$ $b^2 - ac < 0$
$= \frac{1}{4} - 1 = -\frac{3}{4}$.

$$\frac{dy}{dx} = \frac{1 + i\sqrt{3}}{2} \Rightarrow \xi = y - \frac{1}{2}(1 + i\sqrt{3})x.$$

$$\frac{dy}{dx} = \frac{1 - i\sqrt{3}}{2} \Rightarrow \eta = y - \frac{1}{2}(1 - i\sqrt{3})x.$$

$$\left.\begin{aligned} \alpha &= \xi + \eta = 2y - x \\ \beta &= i(\xi - \eta) = \sqrt{3}x \end{aligned}\right\} \text{Real Transformation}$$

$\bigstar$ $$\sum_{i,j=1}^{n} a_{ij} u_{x_i x_j} + \sum_{i=1}^{n} b_i u_{x_i} + cu = f.$$

Assume a_{ij}, b_i are all constants (locally).
let $u \in C^2$ ($u_{xy} = u_{yx}$; $u_{x_i x_j} = u_{x_j x_i}$)

At least two times continuously differentiable.

$\Rightarrow$ a_{ij} is symmetric ; λ_k is the eigen value of (a_{ij})

(a) $\rightarrow$ Elliptic iff all λ_k are non zero and have same sign.

e.g. $u_{xx} + u_{xy} + u_{yy} = 0$; $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

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$u = u(x,y,z)$; $\begin{bmatrix} u_{xx} & u_{xy} & u_{xz} \\ u_{yx} & u_{yy} & u_{yz} \\ u_{zx} & u_{zy} & u_{zz} \end{bmatrix}$ = ~~Coefficient matrix.~~ (a_{ij})

Hyperbolic iff. λ_k are non-zero and have the same sign except precisely one.

e.g. $u_{tt} - c^2[u_{xx} + u_{yy} + u_{zz}]$. $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -c^2 & 0 & 0 \\ 0 & 0 & -c^2 & 0 \\ 0 & 0 & 0 & -c^2 \end{bmatrix}$

Ultrahyperbolic :- When all the λ_k are non-zero and there are at least two of each sign.

eg. $u_{x_1x_1} + u_{x_2x_2} = u_{x_3x_3} + u_{x_4x_4}$. $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$

Parabolic :- If any of the λ_k vanishes.

$u_t - k[u_{xx} + u_{yy} + u_{zz}] = 0$. $\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & -k & 0 & 0 \\ 0 & 0 & -k & 0 \\ 0 & 0 & 0 & -k \end{bmatrix}$

Cauchy Problem :-

$a u_{xx} + 2b u_{xy} + c u_{yy} + F(x, y, u_x, u_y, u) = 0$

Parameterize the curve $x = \phi(s)$; $y = \theta(s)$
$u = f(s)$ $s_0 \le s \le s_1$

Can we specify u, u_x, u_y,

Assume on Γ : u, u_x, u_y, are given for $s \in [s_0, s_1]$

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$$\phi'(s) = \frac{\partial u}{\partial s} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial s}$$

$$= u_x \phi'(s) + u_y \phi'(s)$$

has to be ← Initial data can't be specified arbitrarily.

let $p = u_x$, $q = u_y$ are given on the curve

$$\frac{dp}{ds} = \frac{d}{ds}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial^2 u}{\partial x^2}\cdot\frac{\partial x}{\partial s} + \frac{\partial^2 u}{\partial x \cdot \partial y}\cdot\frac{\partial y}{\partial s}.$$

$$\frac{dq}{ds} = \frac{d}{ds}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial^2 u}{\partial x \cdot \partial y}\cdot\frac{\partial x}{\partial s} + \frac{\partial^2 u}{\partial y^2}\cdot\frac{\partial y}{\partial s}.$$

$$\{ u \in C^2 ;\ u_{xy} = u_{yx} \}.$$

$$\underbrace{\begin{bmatrix} a & 2b & c \\ dx/ds & dy/ds & 0 \\ 0 & dx/ds & dy/ds \end{bmatrix}}_{M} \begin{bmatrix} u_{xx} \\ u_{xy} \\ u_{yy} \end{bmatrix} = \begin{bmatrix} -F \\ dp/ds \\ dq/ds \end{bmatrix}$$

If $\det(M) \neq 0$, then we can find u_{xx}, u_{xy} and u_{yy} uniquely on Γ.

$u, u_x, u_y, u_{xx}, u_{xy}, u_{yy}$ ⇒ Known uniquely on the curve.

Hence I can find the sol uniquely on the nbd of curve.

$$f(x,y) = f(0,0) + x\,f_x(0,0) + y\,f_y(0,0) + \frac{x^2}{2} f_{xx}(0,0) + xy\,f_{xy}(0,0) + \frac{y^2}{2} f_{yy}(0,0) + \cdots$$

Suppose det $M = 0$; Expanding the determinant.

$$a\left(\frac{dy}{ds}\right)^2 - 2b\left(\frac{dx}{ds}\right)\left(\frac{dy}{ds}\right) + c\left(\frac{dx}{ds}\right)^2 = 0$$

Divide by $\left(\frac{dx}{ds}\right)^2$ and note that $\frac{dy}{dx} = \frac{dy/ds}{dx/ds}$.

$$\Rightarrow \quad a\left(\frac{dy}{dx}\right)^2 - 2b\left(\frac{dy}{dx}\right) + c = 0$$

$$\boxed{\frac{dy}{dx} = \frac{b \pm \sqrt{b^2 - ac}}{a}} \leftarrow \text{Characteristic Curve}$$

Characteristic Equation of PDE; a, b, c are function of x and y only.

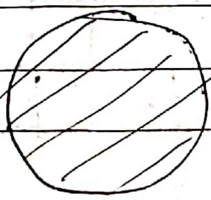
So, if we specify the data on the characteristic curve, we can't except unique sol^n

① $-\Delta u = 0 \Rightarrow$ Laplace Eq^n.

② $-\Delta u = f \to$ Poisson Eq^n.

③ $-\Delta u = f \quad u \in \Omega \leftarrow$ Domain
$\quad u|_{\partial\Omega} = g$
} Dirichlet Problem.

$u \in C^2(\Omega) \cap C(\bar{\Omega})$



$$\Omega = \{(x,y) \in \mathbb{R}^2;\ x^2 + y^2 < 1\}$$

$$\partial\Omega = \{(x,y) \in \mathbb{R}^2;\ x^2 + y^2 = 1\}$$

$u \in C^2(\Omega) \cap C^1(\bar{\Omega})$

④ $\left.\begin{aligned} -\Delta u &= f \\ \left.\frac{\partial u}{\partial n}\right|_{\partial\Omega} &= g \end{aligned}\right\}$ Neumann Problem.

☆ Normal Derivative $\rightarrow \frac{\partial u}{\partial n} = \left(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right) \cdot (n_1, n_2)$

$$= n_1 u_x + n_2 u_y .$$

Normal Direction $= (n_1, n_2)$

⑤. $\left.\begin{aligned} -\Delta u &= f . \\ \alpha u + \beta . \left.\frac{\partial u}{\partial n}\right|_{\partial\Omega} &= g . \end{aligned}\right\}$ Mixed BVP.

We are interested for sol^n : $u \in C^2(\Omega) \cap C^1(\bar{\Omega})$

$$\bar{\Omega} = \Omega \cup \partial\Omega .$$

☆ Compatibility Condition :-

Let $f \in C(\bar{\Omega})$; If $u \in C^2(\bar{\Omega})$,
is a sol^n of $\Delta u = f$ on Ω.

$$\left.\frac{\partial u}{\partial n}\right|_{\partial\Omega} = g$$

then $\int_\Omega f(x)\, dx = \int_{\partial\Omega} g(x) . d\sigma(y)$.

Divergence theorem $\rightarrow \int_\Omega \nabla F . dx = \int_{\partial\Omega} \cancel{F(x)}\, d\sigma(y)$.
F_n

$\left(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right)(n_1, n_2)$
$n_1 u_x + n_2 u_y$

$F = \nabla u$ then $\int_{\Omega} \Delta u \, du = \int_{\partial\Omega} \nabla u \cdot n \, \partial\sigma(y)$

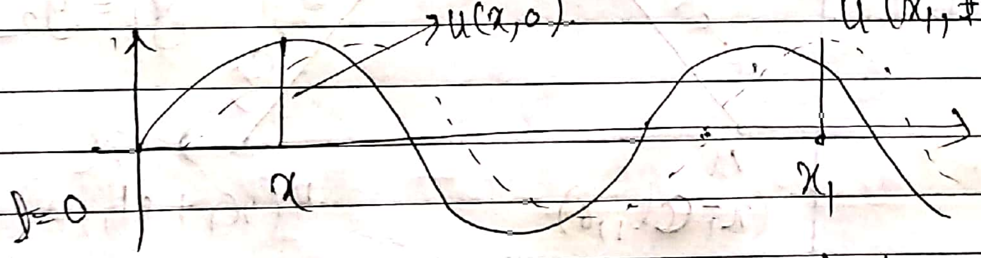
$\Rightarrow \int_{\Omega} f \cdot \partial x = \int_{\partial\Omega} \frac{\partial u}{\partial n} \cdot d\sigma(y) = \int_{\partial\Omega} g(y) \, d\sigma(y).$

★ **D. Alembert Solⁿ:-** (Wave Eqⁿ)

external force

$u_{tt} - c^2 u_{xx} = F(x,t)$; $-\infty < x < \infty$. $t > 0$ wave Eqⁿ.

(To solve with IC)



c represents the velocity of the wave.

To solve. u_x
$a=1, b=0$
$c = -c^2$
Solve
$\frac{dy}{dx} = \frac{b + \sqrt{b^2 - ac}}{a}$; $\xi = x - ct$
$\frac{dy}{dx} = \frac{b - \sqrt{b^2 - ac}}{a}$; $\eta = x + ct$

$u_{\xi\eta} = 0 \Leftarrow$ Canonical form

$u = F(\xi) + G(\eta)$,
F, G are arbitrary.
$= F(x-ct) + G(x+ct)$

(1) $\begin{cases} u_{tt} - c^2 u_{xx} = 0 \\ u(x,0) = f(x) \\ u_t(x,0) = g(x) \end{cases}$ $-\infty < x < \infty$, $t > 0$.

Now solve for f and g

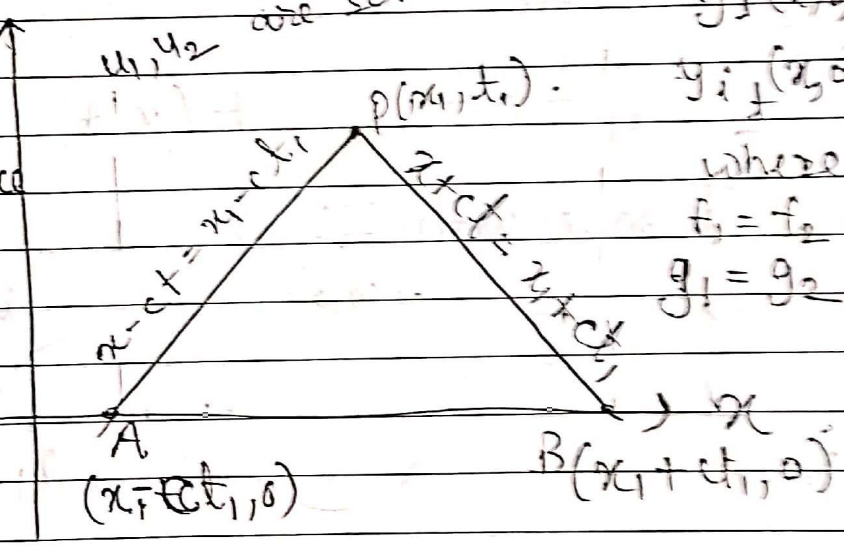
$$u(x,t) = \frac{1}{2}\left[f(x+ct) + f(x-ct) \right] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\,ds .$$

$\rightarrow$ D. Alembert Sol^n (Domain is Infinite)

- What if the domain is
  - $\rightarrow$ Semi Infinite $(0,\infty)$
  - $\rightarrow$ Finite : $(-l, l)$.

u_1, u_2 are sol^n of ① with $y_i(x,0) = f_i$, $y_{i_t}(x,0) = g_i$ where $f_1 = f_2$ in AB, $g_1 = g_2$.

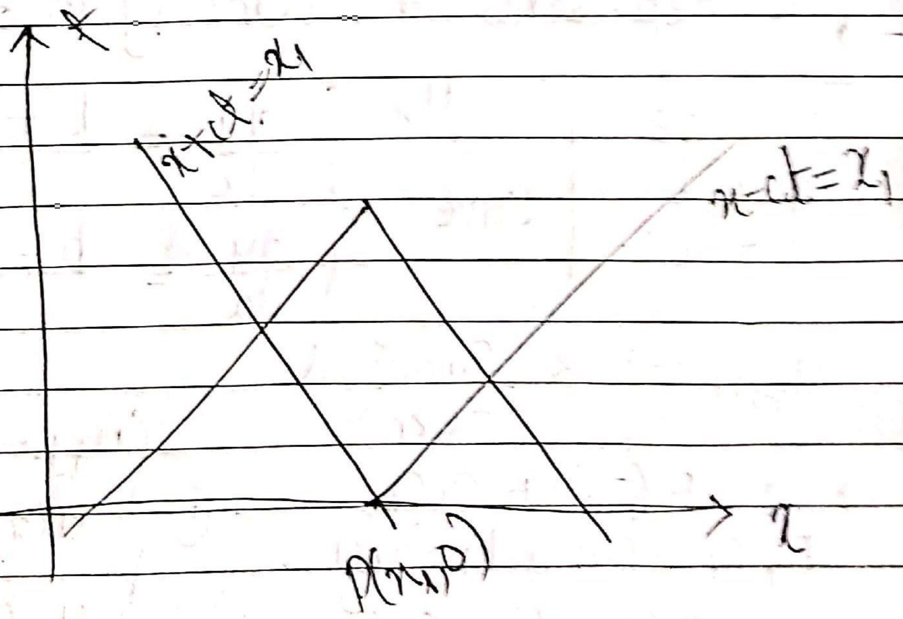
Domain of dependance for the soln of $p(x_1, t_1)$ in AB interval



Region of Influence of $P(x_1, 0)$ is inside the line segments:

$x + ct = x_1$

$x - ct = x_1$.



☆ Vibration of semi-infinite string :-

$y_{tt} - c^2 y_{xx} = 0$, $x \in (0,\infty)$; $t > 0$

$c \Rightarrow$ speed of the wave

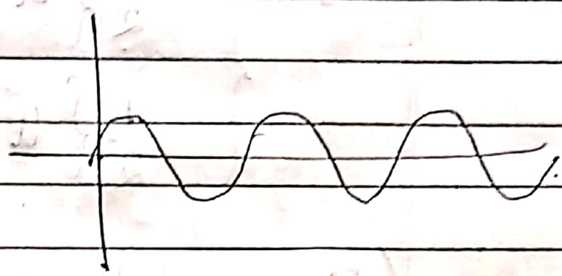
IC. $y(x,0) = f(x)$, $y_t(x,0) = g(x)$.

BC $y(0,t) = 0$ ($\Rightarrow y_t(0,t) = 0$).

Extend f and g by odd expansion

Odd Expansion

$$F(x) = \begin{cases} f(x) ; & x > 0 \\ -f(-x) ; & x \le 0 . \end{cases}$$



$$G(x) = \begin{cases} g(x) ; & x > 0 \\ -g(-x) ; & x \le 0 . \end{cases}$$

So, now the domain is $(-\infty,\infty)$.

$$y(x,t) = \frac{1}{2}\left\{F(x+ct) + F(x-ct)\right\} + \frac{1}{2c}\int_{x-ct}^{x+ct} G(s)\,ds . \quad \text{(DAlembert Sol.)}$$

$$y(x,0) = \frac{1}{2}\left\{F(x) + F(x)\right\} + 0 = \frac{1}{2}\left[f(x) + f(x)\right]$$

$$= f(x)$$

$$y(0,t) = \frac{1}{2}\left[F(ct) + F(-ct)\right] + \frac{1}{2c}\int_{-ct}^{ct} G(s)\,ds .$$

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$$y(0,t) = \frac{1}{2}\left[f(ct) - f(ct)\right] + 0$$

$$y(0,t) = 0$$

similarly show that $y_t(x,0) = g(x)$.

If $g=0$, then $y(x,t)$

$$= \begin{cases} \frac{1}{2}\{u(x-ct) + u(x+ct)\}, & x \geq ct \\ \frac{1}{2}\{u(x+ct) - u(ct-c)\}; & x < ct \end{cases}$$

(If we take even expansion for this problem IC, BC does not satisfy).

★ Duhamel's Principle :- (Wave Eqn)

$$\left.\begin{array}{l} y_{tt} - c^2 y_{xx} = \underline{h(x,t)} \\ -\infty < x < \infty \;;\; t > 0 \\ \text{IC} \begin{cases} y(x,0) = 0 \\ y_t(x,0) = 0 \end{cases} \end{array}\right\} \text{acceleration} \quad (2)$$

At $t = s - \Delta s$ $h(x,s)$ is applied on the string at $t = s$; stop $h(x,s)$.

Due to Δs +ve, the string will acquire a velocity $= (\Delta s)\, h(x,s)$ ($v = u + at$
$= 0 + h(x,s).\Delta s$.

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String position will be changed to

$\frac{1}{2} h(x,s)\Delta s^2$ (can be neglected as Δs is small)

New PDE

$$\left\{\begin{array}{l} \omega_{tt} - c^2 \omega_{xx} = 0, \quad t \geq s. \\ \omega(x,s,s) = 0 \leftarrow \text{Displacement} \\ \omega_t(x,s,s) = h(x,s) \leftarrow \text{velocity.} \\ \text{IC at } t = s. \end{array}\right\} \text{——} (3)$$

Change of variable

$$\left.\begin{array}{l} \omega(x,t,s) = \tilde{\omega}(x,t-s,s) \\ \rightarrow \tilde{\omega}_{tt} = c^2 \tilde{\omega}_{xx} \; ; \; t \geq 0 \\ \tilde{\omega}(x,0,s) = 0 \\ \tilde{\omega}_t(x,0,s) = h(x,s) \end{array}\right\} \text{——} (4).$$

By D. Alembert Sol (4).

$$\tilde{\omega}(x,t,s) = \frac{1}{2c} \int_{x-ct}^{x+ct} h(r,s)\,dr.$$

Sol of (3)

$$\omega(x,t,s) = \tilde{\omega}(x,t-s,s).$$

$$= \frac{1}{2c} \int_{x-c(t-s)}^{x+c(t-s)} h(r,s)\,dr.$$

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Sol^n of ②

$$y(x,t) = \int_0^t w(x,t,s).ds.$$

$$= \int_0^t \tilde{w}(x,t-s,s)\,ds.$$

$$= \frac{1}{2c}\int_0^t \int_{x-c(t-s)}^{x+c(t-s)} h(z,s).dz.ds. \quad (**)$$

Check by differentiating that it solves $y_{tt} - c^2 y_{xx} = h(x,t)$.

Use Leibnitz Rule.

$$\frac{d}{dx}\left[\int_{a(x)}^{b(x)} f(x,t)\,dt\right] = f(x,b(x)).b'(x) - f(x,a(x)).a'(x) + \int_{a(x)}^{b(x)} \frac{\partial}{\partial x} f(x,s)\,ds.$$

Eg. Solve $u_{tt} - u_{xx} = x - t \qquad -\infty < x < \infty$, $t > 0$.

$u(x,0) = x^4, \quad u_t(x,0) = \sin(x).$

Prob 1

$$\underbrace{\left.\begin{aligned} &u_{tt} - u_{xx} = 0 \\ &u(x,0) = x^4 \\ &u_t(x,0) = \sin x \end{aligned}\right\}}_{\text{DAlembert Sol}}$$

$\oplus$

Prob 2

$$\underbrace{\left\{\begin{aligned} &u_{tt} - u_{xx} = x-t \\ &u(x,0) = 0 \\ &u_t(x,0) = 0 \end{aligned}\right.}_{\text{Duhamel Principle}}$$

$$u_{tt} - c^2 u_{xx} = h(x,t).$$
$$u(x,0) = f(x).$$
$$u_t(x,0) = g(x)$$

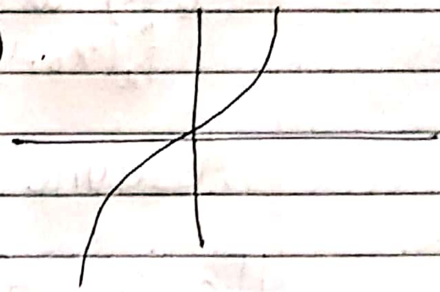
$$u(x,t) = \frac{1}{2}\left[f(x-ct) + f(x+ct) \right] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\,ds + \frac{1}{2c}\int_0^t \int_{x-c(t-s)}^{x+c(t-s)} h(r,s)\,dr\,ds.$$

★ Maximum Principle :-

$f: \mathbb{R} \to \mathbb{R}$ has a local max. at $x = x_0$ if $\exists\ \delta > 0$ and nbd of x_0, say N_δ s.t. $f(x_0) \geq f(x)\ \forall\ x \in N_\delta$

Suppose f is differentiable. Necessary Condn
$f'(x_0) = 0$ (Not sufficient) $f(x) = x^3$.
↳ Interior point

Sufficient Condition : $x_0 \in (a,b)$.
$f(a,b) \to \mathbb{R}$.
$f'(x_0) = 0$; $f''(x_0) < 0$.
$\Rightarrow$ x_0 is max.



What about convergence?

$f'(x_0) = 0$; $f''(x_0) > 0$
$\to$ x_0 is a point of min.

eg. $f(x) = -x^4$; $f: (-1,1) \longrightarrow \mathbb{R}$.
$f'(0) = 0$; $f''(0) = 0$.

If f has max at x_0 then $f''(x_0) \leq 0$
& $f'(x_0) = 0$.

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$f : \mathbb{R}^2 \to \mathbb{R}$. Existance : $D \to$ closed + bounded
$f : D \to \mathbb{R}$ cont $\Rightarrow$ f has a max / min.

<u>Necessary Cond^n</u> :- let (x_0, y_0) is an interior point in D.

$f : D \to \mathbb{R}$ & f_x, f_y exist in D. Then.

$f : (0,1) \to \mathbb{R}$
$f(x) = x$; $f'(x) = 1$
$f'(x) = 1$, $x \in (0,1)$.
$f(0) = 0$
$f(1) = 1$.

$f_x(x_0, y_0) = f_y(x_0, y_0) = 0$

Then condition is not sufficient for min/max.

$f : \mathbb{R}^2 \to \mathbb{R}$; $f(x,y) = xy$.
$f_x(0,0) = f_y(0,0) = 0$ but $(0,0)$ is neither max nor min.

<u>Second Derivative Test</u> :-
(x_0, y_0) is an interior pt. of D
$f_x(x_0, y_0) = f_y(x_0, y_0) = 0$.

Assume $(f_{xx} f_{yy} - f_{xy}^2)(x_0, y_0) > 0$.

Then ① $f_{xx}(x_0, y_0) > 0 \Rightarrow f$ has min at (x_0, y_0)

② $f_{xx}(x_0, y_0) < 0 \Rightarrow f$ has min at (x_0, y_0)

<u>Suppose</u> $(f_{xx} f_{yy} - f_{xy}^2)(x_0, y_0) < 0$.

$(x_0, y_0) \Rightarrow$ Saddle point (In a nbd. of (x_0, y_0) there will be two points s.t. (x_1, y_1) & (x_2, y_2)

$f(x_0, y_0) > f(x_1, y_1)$ $f(x_0, y_0) < f(x_2, y_2)$.

$(f_{xx}f_{yy} - f_{xy}^2)(x_0, y_0) = 0 \Rightarrow$ No conclusion of (x_0, y_0)

★ Let $\Omega \subseteq \mathbb{R}^2$ bounded domain $u : \Omega \to \mathbb{R}$ is cont. let u can be extended to the boundary of Ω i.e. $\partial\Omega$ by continuity such a func. is called $C(\overline{\Omega})$

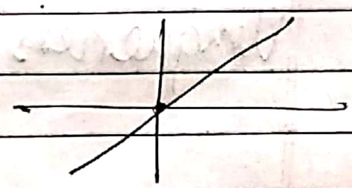
Suppose $u \in C(\overline{\Omega}) \Rightarrow u$ has max & min at Ω.

★ Weak Maximum Principle

Let $\Omega \subseteq \mathbb{R}^2$ bounded. Let $u \in C^2(\Omega) \cap C(\overline{\Omega})$

$\Delta u = u_{xx} + u_{yy} = 0$. Then max value of u in $\overline{\Omega}$ is acheived at boundary $\partial\Omega$.

$u = ax + b$; $u_{xx} = 0$, C



e.g. $\log(x^2+y^2) = \log(|z|^2)$ is harmonic except $(0,0)$. (Think!)

★ At an interior point max.

$u_{xx}(x_0, y_0) \le 0$, $u_{yy}(x_0, y_0) \le 0$.

$\Rightarrow \Delta u(x_0, y_0) \le 0$.

Then if v is a function s.t. $\Delta v > 0 \Rightarrow$ Max of v will be at bdy.

Existance of v : $v = u + \epsilon(x^2+y^2)$ — ① $\epsilon > 0$.

$\Rightarrow v \in C^2(\Omega) \cap C(\overline{\Omega})$ as $(u \in C^2(\Omega) \cap C(\overline{\Omega}))$.

$\Delta v > 0$ in Ω, let $\underset{(\partial\Omega)}{\text{Max}}\, u = M$ and $\text{Max}(x^2+y^2) = m$

$\Rightarrow v \le M + \epsilon m \quad \forall (x, y) \in \Omega$.

Since $u \le v$ in Ω from ①.

$\Rightarrow u \leq M + \epsilon m$ $\Rightarrow$ As $\epsilon \to 0$, $u \leq M = \underset{\partial\Omega}{\text{Max of}}$

☆ Weak Minimum Principle :-

Ω bdd; $u \in C^2(\Omega) \cap C(\bar{\Omega})$ $\Delta u = 0$ $\Rightarrow$
$\text{Min}\, u(\bar{\Omega}) = \underset{(\partial\Omega)}{\text{Min}\, u}$

$\boxed{v = -u}$

eg. $\Delta u = 0$ in $\Omega = \{x \in \mathbb{R}, y > 0\}$.
$u = 0$ as $y = 0$.

$u = 0 \Rightarrow u = xy$ So the solⁿ is not unique.

Uniqueness Theorem :-

Uniqueness of the sol of $\left.\begin{array}{l}\Delta u = f \text{ in } \Omega \\ u = g \text{ on } \partial\Omega\end{array}\right] u \in C^2(\Omega) \cap C(\bar{\Omega})$

Pf: Assume u_1 & u_2 are two solns.
$v = u_1 - u_2$; $\Delta v = 0$ on Ω
$v = 0$ on $\partial\Omega$.

$\Rightarrow$ $v = 0$ (By Max/Min Principle)

$\Rightarrow$ $u_1 = u_2$

☆ Stability of Solⁿ :-

$$\left.\begin{array}{l}\Delta u_i = f \\ u|_{\partial\Omega} = g_i\end{array}\right] u_i \in C(\bar{\Omega}) \cap C^2(\Omega).$$

Then $\underset{\bar{\Omega}}{\max} |u_1(x) - u_2(x)| \leq \underset{\partial\Omega}{\text{Max}} |g_1 - g_2|$

Proof :- $\omega = u_1 - u_2 \rightarrow \Delta\omega = 0$

$\omega|_{\partial\Omega} = g_1 - g_2$

$\Rightarrow$ Max-min principle.

$\min_{\partial\Omega}|g_1 - g_2| \leq |\omega_1 - \omega_2| \leq \max_{\partial\Omega}|g_1 - g_2|$.

★ **Strong Maximum Principle** :-

$\Omega \subseteq \mathbb{R}^3$ (not necessarily bounded).

$\Delta u = 0$, If u attains max in Ω then u is constant.

$\Rightarrow$ does not talk about where the max will be attained.

~~Proof~~ Eg. $\Delta u = 0 \rightarrow u = \log(x^2 + y^2)$; $(x, y) \neq (0,0)$

As $(x, y) \rightarrow (0,0)$, $|u| \rightarrow \infty$.

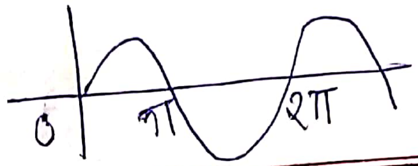
Similarly strong minimum principle will hold.

★ **Weak Max Principle** :-

Consider $Lu = \sum_{i,j=1}^{n} a_{ij}(x) \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i(x) \frac{\partial u}{\partial x_i} + cu$; $\Omega \subset \mathbb{R}^n$ bounded.

Let $u \in C^2(\Omega) \cap C(\bar{\Omega})$, let (a_{ij}) symmetric and strictly positive definite ($\Rightarrow$ L is elliptic).

Theorem :- If $Lu \geq 0$ (or $Lu \leq 0$) in a bounded domain & $c(x) = 0$ in Ω. Then max (or min) of u will be achieved at $\partial\Omega$.



$u_{xx} + 2u_{xy} + 3y^2 u_{yy} + 0\, u_{yx} = 0$.

$$\begin{bmatrix} 1 & 1 \\ 1 & 3y^2 \end{bmatrix} : \begin{bmatrix} u_{xx} & u_{xy} \\ u_{xy} & u_{yy} \end{bmatrix}$$

$u \in C^2(\Omega) \Rightarrow u_{xy} = u_{yx}$.

$u_{xx} + 2u_{xy} + 3u_{yx} + u_{yy} = 0$.

$$\frac{2+3}{2} = \frac{5}{2}$$

$Lu = u'' + u = 0$ $\qquad\qquad$ $Lu = u'' - u = 0$.

$a = 1,\ b = 0,\ c = 1$ $\qquad\qquad$ $\Rightarrow u = e^x$

1D $\Rightarrow$ $u = \sin x, \cos x$

$x \in (0, 2\pi)$ $\quad u(0) = u(2\pi) = 0$.

Strong Max Principle :-

Assume $Lu \geq 0$ (or $Lu \leq 0$) in Ω (not necessarily bounded) and assume u is not constant. If $c=0$, then u does not achieve its max (or min) in the interior of Ω. If $c \leq 0$, u cannot achieve a non-negative max (or non-negative min) in the interior of Ω.

Regardless of sign of c, u cannot be zero at Lu interior max (min)

$Lu = u'' - u = \sin x \geq 0$.

Parabolic Problem :-

$$Lu = -\frac{\partial u}{\partial t} + \sum_{i=1}^{n}\sum_{j=1}^{n} \underbrace{a_{ij}(x,t)}_{\text{const.}} \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{n} \underbrace{b_i(x,t)}_{\text{const.}} \frac{\partial u}{\partial x_i}$$

$$+ \underbrace{c}_{\text{const}} u$$

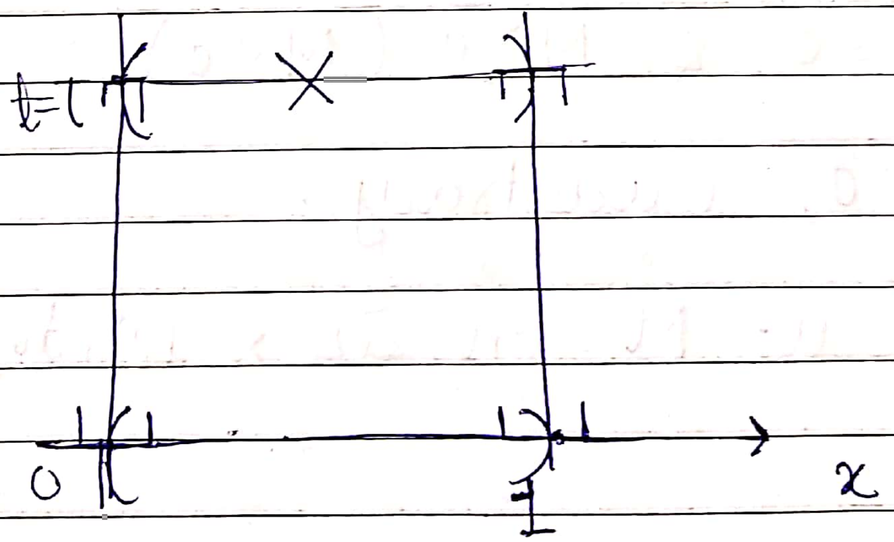
$\Omega \times (0,T)$ (open).

$D = \Omega \times (0,T]$
$\mathcal{Q} = \Omega \times (0,T)$
$\Sigma = (\partial\Omega \times [0,T]) \cup (\Omega \times \{0\})$.

1D $\quad x \in (0,1) = \Omega$
$\quad t \in (0,1]$

$$-u_t + u_{xx} + a u_x + bu = f(x,t).$$

$D = (0,1) \times (0,T]$.
$\mathcal{Q} = (0,1) \times (0,T]$.
$\Sigma = [\{0,1\} \times [0,T]] \cup [(0,1) \times \{0\}]$



☆. Weak Max Principle

(a_{ij}) Positive def $u \in C^2(\Omega) \cap C(\bar{\Omega})$.
Ω bounded $Lu \geq 0$ ($Lu \leq 0$).
with $c(x,t) = 0$, Then max (min) of u acheived at Σ

$$u_t = u_{xx} + x^3 \qquad (0,1)$$
$$u_t - u_{xx} \geq 0.$$

$Lu = -u_t + u_{xx} \leq 0 \rightarrow$ Min will attain at Σ.

Strong Max Principle.

Assume $Lu \geq 0$ ($Lu \leq 0$) Let $M = \sup_D u$ ($M = \inf_D u$) let $u = M$ at a point $(x_0, t_0) \in D$.

~~Let u = t~~ & one of the following hold

① $c = 0$, M is arbitrary.

② $c \leq 0$ & $M \geq 0$ ($M \leq 0$)

③ $M = 0$, c arbitrary.

Then $u = M$ on $\bar{\Omega} \times [0, t_0]$.

Fourier Series :-

Power series / Taylor series $f : \mathbb{R} \to \mathbb{R}$.

$$f(x) = f(0) + x.f'(0) + \frac{x^2}{2}.f''(0) +$$

$\Rightarrow$ $f \in C^\infty$ (infinite time diff)

$\Rightarrow$ Power series will converge to $f(x)$.

$\Rightarrow$ We know the coefficient.

$\rightarrow$ Basis $= \{1, x, x^2, \ldots\}$.

$\rightarrow$ Radius of Convergence

$$f(x) = \frac{1}{1-x} = 1 + x + x^2 + \ldots + \; ; \quad |x| < 1 .$$

$$f : [-\pi, \pi] \to \mathbb{R} \quad f(x) \simeq a_0 + \sum_{n=1}^{\infty} [a_n \cos(nx) + b_n \sin(nx)]$$

Basis $: \{1, \cos(nx), \sin(nx)\}$ $\leftarrow$ Periodic in $[-\pi, \pi]$

$\leftarrow$ Infinite element.

Inner element $\langle f, g \rangle = \int_{-\pi}^{\pi} f(x).g(x).dx$.

$$= \int_{-\pi}^{\pi} \sin mx \cos nx \,.dx = 0 .$$

$$\int_{-\pi}^{\pi} \sin mx . \sin nx \, dx = 0 \; ; \; m \neq n .$$

$$\int_{-\pi}^{\pi} \cos mx . \cos nx \, dx = 0 \; ; \; m \neq n .$$

Orthogonalization

f, g are orthogonal $\Rightarrow \langle f, g \rangle = 0$

$$\frac{f}{\|f\|}$$

Application :- To write non-smooth function in terms of fourier series.

- Series Convergent
- Solving PDE (specially if the domain is bounded).

More-General :-

u be an inner product space

orthonormal set $\{g_0, g_1, - - \}$

given f, whether f can be represented as

$$f = \sum_{n=0}^{\infty} a_n b_n \ ?$$

* Periodic function :-

$f : R \to R$. p (least positive number).
Consider $p > 0$; $\leftarrow$ period of f.
if $f(x) = f(x+p)$

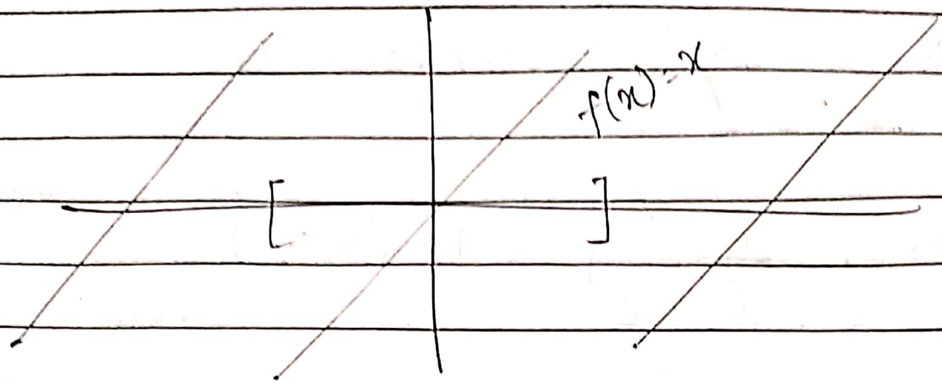
$$\sin(x) = \underbrace{\sin(x+2\pi)}_{\checkmark \text{period}} = \sin(x+4\pi)$$

Periodic extension of f

$f(a,b) \to R$. To define $f : R \to R$, with period $(b-a)$.

We define $f(x) = f(x + n(b-a))$

$\Rightarrow$ where n s.t. $a \le x + n(b-a) \le b$



★ Change of variable :-

$[-\pi, \pi] \longleftrightarrow [-L, L]$. $f : [-L, L] \to \mathbb{R}$

$$f(x) \simeq \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right].$$

$$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cdot \cos \frac{n\pi x}{L} \cdot dx$$

$$b_n = \frac{1}{L} \int_{-L}^{L} f(x) \cdot \sin \frac{n\pi x}{L} dx.$$

fourier coefficients $\{a_0, a_n, b_n\}$.

Q :-

① For which x, $f(x)$ will converge to fourie series.

② If Fourier series of f converges at x, will it converge to $f(x)$??

~~Find Fourier Series~~

eg.

$$f(x) = \begin{cases} 0, & x \in [\pi, 0] \\ 1, & x \in [0, \pi] \end{cases}$$

$\Rightarrow$

$$a_n = \frac{1}{\pi} \int_0^{\pi} \cos nx \, dx = 0$$

$$b_n = \frac{1}{\pi} \int_0^{\pi} \sin nx \, dx = \begin{cases} 0, & n\%2 = 0 \\ 2/n\pi ; & n\%2 = 1 \end{cases}$$

$$a_0 = \frac{1}{\pi} \int_0^{\pi} f(x) \, dx = 1.$$

$$f(x) \approx \frac{1}{2} + \sum_{1}^{\infty} \frac{-2}{\pi(2n-1)} \cdot \sin((2n-1)x).$$

Note :-

F.S. of f converges $\forall x \in [-\pi, \pi]$.

F.S. at $x = 0, \pm\pi$ converges to $1/2$

F.S. converge to $f(x)$ for $x \in (-\pi, \pi)$ $x \neq 0$.

Def :- $\left(\lim_{x \to x_0^+} f(x)\right) \Rightarrow f(x_0^+) = f(x_0^-) \Rightarrow f$ is cont. at x_0.

If $f(x_0^+)$, $f(x_0^-)$ exists but not equal, we say f has jump discontinuity.

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★ Piecewise continuous function :-

$f : [a,b] \Rightarrow R$. f is continuous at all points except possibly at finite no. of points where it has a jump discontinuity.

e.g. $f(x) = \begin{cases} 1 ; & x > 0 \\ 2 ; & x \le 0 \end{cases}$ piecewise continuous fn.

$g(x) = 1/x ; x \neq 0$ Not piecewise continuous

★ $C'[a,b] = \{ f \mid f' \text{ is cont in } [a,b] \}$.

piecewise $C' = \{ f \mid f' \text{ is piecewise cont} \}$.

★ Adjusted function.

f is piecewise cont. in $[-L, L]$.

$$f^*(x) = \begin{cases} \frac{1}{2}[f(x_-) + f(x_+)] ; & x \in (-L, L). \\ \frac{1}{2}[f(-L_+) + f(L^-)] ; & x = L, -L. \end{cases}$$

★ Pointwise Convergence :-

$f_n : A \to R$.
$f : A \to R$.

$f_n(x) \longrightarrow f(x), \forall x \in A$, ptwise if given $\epsilon > 0, \exists N > 0, N = N(x, \epsilon)$ s.t.

$$|f_n(x) - f(x)| < \epsilon \quad \forall n \geq N.$$

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★ Uniform Convergence

Given $\epsilon > 0$, $\exists N > 0$.
s.t. $\forall x \in A$.

$$|f_n(x) - f(x)| < \epsilon \;\forall n > N(\epsilon).$$

★ Fourier Convergence for piecewise ~~continuo~~ C' ~~fn~~ function :-

let f be periodic with period $2L$,
f is piecewise. $C'[-L,L]$.

The fourier series of f converges to f^* on $[-L,L]$.

★ Fourier series convergence for C' function:-

Let $f \in C'[-l,l]$, Assume $f(l) = f(-L)$
$f'(L) = f'(-L)$.

Then fourier series of f converges pointwise to $f(x)$ $\forall$ $x \in [-L,L]$.

★ Fourier series convergence (Uniform Convergence).

let $f \in C^2[-L,L]$ Assume $f(L) = f(-L)$.
$f'(L) = f'(-L)$.

Then FS. of f converges uniformly to $f(x)$ $\forall x$ $[-L,L]$

In particular,

$$\left| f(x) - \left[\frac{a_0}{2} + \sum_{n=1}^{N} \left(a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right) \right] \right|$$

$$\leq \frac{4L^2 M}{\pi^2 N}, \quad \forall x \in [-L, L] \quad \text{where } M = \max_{[L,L]} |f'(x)|.$$

Ex. Find the F.S. of $f(x) = x$.

$\Rightarrow$ An $f(x) \sim 2 \cdot \sum_{n=1}^{\infty} (-1)^{n+1} \frac{\sin nx}{n}$.

Diff. both sides,

$1 \sim 2 \sum_{n=1}^{\infty} (-1)^{n+1} \cdot \cos nx$. ✗

$f : \mathbb{R} \to \mathbb{R}$

Theorem :- Differentiation of Fourier Series; f is cont. $f(x+2L) = f(x)$, f', f'' piecewise cont. $[-L, L]$ Then F.S. of f' can be obtained by diff. F.S. of f by term by term diff.

If $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right)$

$\Rightarrow f'(x) = \sum_{n=1}^{\infty} \frac{n\pi}{L} \left[-a_n \sin \frac{n\pi x}{L} + b_n \cdot \cos \frac{n\pi x}{L} \right]$.

Q. (Infinite sum of periodic function $\Rightarrow$ will it be periodic!)

Theorem :- Integration of Fourier Series

$f(x) : [-L, L] \to \mathbb{R}$ piecewise cont. with

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}$$

Then for any $x \in [-L, L]$.

$$\int_{-L}^{x} f(t)\, dt = \int_{-L}^{x} \frac{a_0}{2}\, dt + \sum_{n=1}^{\infty} \int_{-L}^{x} \left[a_n \cos\frac{n\pi t}{L} + b_n \sin\frac{n\pi t}{L} \right] dt$$

★ **Bessel's Inequality :-**

Suppose $\int_{-L}^{L} |f(x)|^2 < \infty$ Then

$$\frac{a_0^2}{2} + \sum_{n=1}^{\infty} (a_n^2 + b_n^2) \leq \frac{1}{L} \int_{-L}^{L} |f(x)|^2 \, dx .$$

★ If F. series is convergent then

$\lim_{n\to\infty} a_n = 0$ & $\lim_{n\to\infty} b_n = 0$.

Converse need not be true.

$\sum_{n=1}^{\infty} \frac{1}{n} \to$ divergent

★ Paraseval Identity :-

If f is cont. $[-L, L]$ then

$$\frac{a_0^2}{2} + \sum_{n=1}^{\infty} (a_n^2 + b_n^2) = \frac{1}{L} \int_{-L}^{L} |f(x)|^2 \, dx$$

$$f(x) = \begin{cases} 1 & ; \; x \in [-1, 0] \\ 0 & ; \; x \in [0, 1] \end{cases}$$

★ Even Extension :-

$f : [0, L] \to R$ to $[-L, L]$

$f_e : [-L, L] \to \mathbb{R}$

$$f_e = \begin{cases} f(x) & ; \; x \in [0, L] \\ f(-x) & ; \; x \in [-L, 0] \end{cases}$$

★ Odd Extension :-

$f : [0, L] \to \mathbb{R}$ to $[-L, L]$

$f_0 \; [-L, L] \to \mathbb{R}$ with $f(0) = 0$.

$$f_0(x) = \begin{cases} f(x) & ; \; x \in [0, L] \\ -f(-x) & ; \; x \in [-L, 0] \end{cases}$$

$$\|f - g\|_{\infty} = \max_{x \in [a, b]} |f(x) - g(x)|$$

$$\|f - g\|_{L^2[a, b]} = \left[\int_a^b |f(x) - g(x)|^2 \, dx \right]^{1/2}$$

$g =$ F.S. of f.

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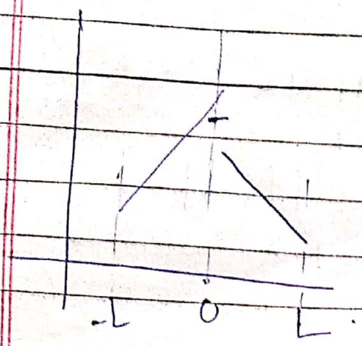

f is discont at $x=0$.

$$\|f-g\|_{L^2[-L,L]}$$

$$= \left[\int_{-L}^{L} |f(x)-g(x)|^2 \, dx\right]^{1/2}$$

$$= \int_{-L}^{0} + \int_{0}^{L}$$

Q. ~~& Evaluate~~ Determine the fourier series expansion of $H(x)$ in $[-\pi,\pi]$.

$1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\ldots$

Ans. $(\pi/4)$

$$H(x) = \begin{cases} 1; & x>0 \\ 0; & x<0 \end{cases}$$

$$H(x) = \frac{1}{2} + \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{\sin(2n-1)(x)}{2n-1}.$$

Q. <u>Heat conduction in thin rod</u> :-

$u_t = \alpha u_{xx}$; $0<x<L$; $t>0$.

$u(x,0) = f(x)$.

① Dirichlet cond^n $u(0,t) = u_0$; $u(L,t) = u_L$; $t>0$.

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