classmate

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Chapter-1
Logic

Basic Set Theory:

i) $C \subseteq D$ $\forall x, x \in C \Rightarrow x \in D$

ii) $C \subset D$ $\forall x, \exists x \in D$ s.t. $x \notin C$

iii) $C = D$

iv) $P(C)$

v) Null set $= \{\} \phi$

vi) $C \cup D$

vii) $C \cap D$

viii) C/D

ix) $C \Delta D = (C/D) \cup (D/C)$

x) $\bar{C}$

xi) $|C \cup D| = |C| + |D| - |C \cap D|$

xii) $|C \cup D \cup E| = |C| + |D| + |E| - |C \cap D| - |C \cap E| - |D \cap E| + |C \cap D \cap E|$

Que: 100 students can speak French, 50 can speak Russian & 20 both. If this includes all students what are the total number of students?

Ans: $|F| = 100$

$|R| = 50$

$|F \cap R| = 20$

$|F \cup R| = 100 + 50 - 20$

$= 130$

Ans

PTO

→ FUNDAMENTALS OF LOGIC

- **Propositions** – Any declarative sentence to which it is meaningful to assign one and only one truth value : TRUE or FALSE

p: A week has 7 days → propositions
q: A week has more days than a month
r: What is your name? → not a proposition

NOTE: p, q, r are propositional variables

- **Connectives**

- Connectives are used to combine or relate propositions
- The types of connectives are:

1. Conjunction – 'and' '$\wedge$'
2. Disjunction – 'or' '$\vee$'
3. Negation – 'not' '$-$ / $\sim$ / $\neg$'
4. Conditional – 'if –, then –', '$\rightarrow$ / $\Rightarrow$'
5. Biconditional – 'iff', '$\leftrightarrow$ / $\Leftrightarrow$'
(Equivalence)

P.T.O

1. CONJUNCTION

'p and q' '$p \wedge q$'

p	q	$p \wedge q$
F	F	F
F	T	F
T	F	F
T	T	T

2. DISJUNCTION

'p or q' '$p \vee q$'

p	q	$p \vee q$
F	F	F
F	T	T
T	F	T
T	T	T

3. NEGATION

'not p' $\bar{p}, \sim p, \neg p$

(Truth table – Duh! 😑)

4. Conditional

'p implies q' , $p \to q$

if ____ , then ____

antecedent consequent

p	q	$p \to q$
F	F	T
F	T	T
T	F	F
T	T	T

5. BICONDITIONAL

'p iff q' $p \leftrightarrow q$

p	q	$p \to q$	$q \to p$	$p \leftrightarrow q$
F	F	T	T	T
F	T	T	F	F
T	F	F	T	F
T	T	T	T	T

→ true only if $p \to q$ & $q \to p$

Que: Draw TT for

$$f(p, q, r) = (p \wedge q) \vee \bar{r}$$

Ans:

p	q	r	$(p \wedge q) \vee \bar{r}$
F	F	F	T
F	F	T	F
F	T	F	T
F	T	T	F
T	F	F	T
T	F	T	F
T	T	F	T
T	T	T	T

- <u>Equivalence of propositions</u>

Consider $p \to q$ & $\bar{p} \vee q$

p	q	$f(x)$: $p \to q$	$f(y)$: $\bar{p} \vee q$	$f(x) \to f(y)$	$f(y) \to f(x)$	$f(x) \leftrightarrow f(y)$
F	F	T	T	T	T	T
F	T	T	T	T	T	T
T	F	F	F	T	T	T
T	T	T	T	T	T	T

$\therefore (p \rightarrow q) \equiv (\bar{p} \vee q)$ since all $(p \rightarrow q) \leftrightarrow (\bar{p} \vee q)$ are true.

- Tautology : All true

Contradiction/Absurdity: All false

Contingency : both true and false

→ LOGICAL INFERENCE

$$p_1 \wedge p_2 \wedge p_3 \wedge p_4 \wedge \dots \wedge p_n \longrightarrow q$$

argument form/ inference

$$\left.\begin{array}{l} p_1 \\ p_2 \\ p_3 \\ p_4 \\ \vdots \\ p_n \end{array}\right] \text{premises}$$

$$\therefore q \;] \text{ conclusion}$$

- A statement is an inference only if it is a tautology.

PTO

- RULES

Rule of inferend	Tautological form	name
i) p $\therefore p \vee q$	$p \rightarrow (p \vee q)$	Addition
ii) $p \wedge q$ $\therefore p$	$(p \wedge q) \rightarrow p$	Simplification
iii) $p \rightarrow q$ p $\therefore q$	$(p \rightarrow q) \wedge p \rightarrow q$ eg- If you have pw, you can log in you have pw ∴ you can log in	modus ponens
iv) $p \rightarrow q$ $\sim q$ $\therefore \sim p$	$(p \rightarrow q) \wedge \sim q \rightarrow \sim p$ eg- If you have pw, you can login you can't log in ∴ you don't have pw	modus tollens
v) $\sim p$ $p \vee q$ $\therefore q$	$\sim p \wedge (p \vee q) \rightarrow q$ eg- The ice cream is not vanilla The ice cream is either van. or choc ∴ The ice cream is choc.	Disjunctive syllogism

i) $p \to q$ $q \to r$ $\therefore p \to r$	$(p \to q) \wedge (q \to r) \to (p \to r)$ eg- If it rains, I won't go to school If I don't go to school, I don't need to do HW $\therefore$ If it rains, I don't need to do HW.	Hypothetical syllogism
ii) $(p \to q) \wedge (r \to s)$ $p \vee r$ $\therefore q \vee s$	$(p \to q) \wedge (r \to s) \wedge (p \vee r) \to q \vee s$ eg- If it rains, I will take a leave If it is hot outside, I will go for a shower Either it will rain or it is hot outside I will take a leave or I will go for a shower	constructive dilemma
iii) $(p \to q) \wedge (r \to s)$ $\sim q \vee \sim s$ $\therefore \sim p \vee \sim r$	$(p \to q) \wedge (r \to s) \wedge (\sim q \vee \sim s) \to (\sim p \vee \sim r)$ eg- If it rains, I take a leave If it's hot outside, I'll go for a shower Either I will not take a leave, or I won't go for a shower	Destructive dilemma

9. $\overline{p \cap q} \equiv \bar{p} \cup \bar{q}$ $\longrightarrow$ De morgan's Law

$\overline{p \cup q} \equiv \bar{p} \cap \bar{q}$

10. $p \to q \equiv \sim q \to \sim p$ $\longrightarrow$ Contrapositive law

- # FALLACIES

i) $p \to q$ q $\therefore p$	$(p \to q) \wedge q \to p$ TT $\to$	Affirming the consequent
ii) $p \to q$ $\bar{p}$ $\therefore \bar{q}$	$(p \to q) \wedge \bar{p} \to \bar{q}$	Denying the antecedent
iii) p $\therefore q$	$p \to q$ eg- Socrates is a man Socrates is mortal	non-sequitter

Truth table for i):

p	q	$p \to q$	$(p \to q) \wedge q$	$(p \to q) \wedge q \to p$
F	F	T	F	T
F	T	T	T	F
T	F	F	F	T
T	T	T	T	T

Since we get a false $\therefore$ not a valid inference

→ Methods of Proof

1. Trivial proof of $p \to q$

Establish 'q' to be true, regardless of the truth value of 'p', then $p \to q$ is true.

p	q	$p \to q$
F	F	T
F	T ✓	T ✓
T	F	F
T	T ✓	T ✓

2. Vacuous proof of $p \to q$

If 'p' is false, regardless of truth value of 'q', $p \to q$ is true.

3. Direct proof of $p \to q$

Assume p to be true, then based on assumption show 'q' to be true.

eg- Show that square of an even number is even.

Ans: Rephrasing-

$$\text{If } \underbrace{n \text{ is even}}_{p}, \text{ then } \underbrace{n^2 \text{ is even}}_{q}$$

Assume

$n = 2k$

$n^2 = 4k^2 = 2(2k^2)$

$\therefore$ q is true.

4. Indirect proof of $p \rightarrow q$

$\bar{q} \rightarrow \bar{p}$ This is what we need to do for indirect proof, since, we know that

$$p \rightarrow q \equiv \bar{q} \rightarrow \bar{p}$$

For this,

i) Assume q to be false

ii) Prove p to be false from (i)

eg- Prove If n^2 is an odd integer, then n is an odd integer.

$$p \rightarrow q$$

$\therefore$

$\bar{q} \rightarrow \bar{p}$ is

If n is an even integer, then n^2 is even

prove

Assume n as even

$n = 2k$

$n^2 = 2(2k^2)$

$\therefore \bar{q} \rightarrow \bar{p}$ is true.

$\therefore p \rightarrow q$ is true.

imp

5. Proof of $p \rightarrow q$ by contradiction

$p \rightarrow q$ is true if $p \wedge \bar{q}$ is false.

$$p \rightarrow q \equiv \bar{p} \vee q$$

$$\overline{p \rightarrow q} \equiv \bar{\bar{p}} \wedge \bar{q}$$

$$\equiv p \wedge \bar{q}$$

i) Assume $p \wedge \bar{q}$ is true
ii) Discover based on assumption some conclusion which shows (i) to be false
iii) Contradiction

eg. If $\underbrace{n^3+5 \text{ is odd}}_{p}$, then $\underbrace{n \text{ is even}}_{q}$.

Ans: i) If $(\underbrace{n^3+5 \text{ is odd}}_{p} \underbrace{\text{ and }}_{\wedge} \underbrace{n \text{ is odd}}_{\bar{q}})$ is true

ii) ~~$n^3+5 = 2k+1$~~ ~~$\Rightarrow n^3 = 2k-4$~~ $\quad n = 2k+1$

$n^3+5 = (2k+1)^3+5$

$\Rightarrow n^3+5 = 8k^3+6k^2+6k+1+5$

$= 2(4k^3+3k^2+3k+3)$

$\therefore n^3+5$ is even

iii) Hence contradiction.

$\therefore$ $p \wedge \bar{q}$ is false. Thus $p \to q$ is true

6. Proof of $p \to q$ by cases

p is of the form $p_1 \vee p_2 \vee \ldots \vee p_n$

$p_1 \vee p_2 \vee \ldots \vee p_n \longrightarrow q$

$\Rightarrow \quad p_1 \to q \wedge p_2 \to q \wedge p_3 \to q \wedge \ldots \wedge p_n \to q$

PTO (Proving this on next page)

$$p \to q \equiv \bar{p} \vee q$$

$$(p_1 \vee p_2 \vee \cdots \vee p_n) \to q \equiv \overline{(p_1 \vee p_2 \vee \cdots \vee p_n)} \vee q$$

$$\equiv (\bar{p}_1 \wedge \bar{p}_2 \wedge \cdots \wedge \bar{p}_n) \vee q$$

using distributive law

$$\equiv (\bar{p}_1 \vee q) \wedge (\bar{p}_2 \vee q) \wedge \cdots \wedge (\bar{p}_n \vee q)$$

$$\equiv (p_1 \to q) \wedge (p_2 \to q) \wedge \cdots \wedge (p_n \to q)$$

Q→ Let '$\sqcup$' denote minimum operation. $a \sqcup b$ gives min of a & b, $a, b \in \mathbb{Z}$

Prove that

$$a \sqcup (b \sqcup c) = (a \sqcup b) \sqcup c \quad \forall\, a, b, c \in \mathbb{Z}$$

(Hint: show this for all ~~cases~~ possibilities of a, b and c)

7. Proof by elimination of cases.

i) p has to be true or q has to be true
ii) If we verify 'p' is false
iii) then q is true.

$\underbrace{\qquad\qquad\qquad\qquad}_{\text{disjunctive syllogism}}$

$$(p \wedge q) \wedge \bar{p} \longrightarrow q$$

8. conditional proof

$$p \to (q \to r) \quad \& \quad (p \wedge q) \to r$$

9. Proof of equivalence

$$p \leftrightarrow \; \equiv (p \to q) \wedge (q \to p)$$

TUTORIAL

Que: Find set of all nos. $1 \le x \le 100$ which are not divisible by ~~2 or 3 or 5~~. 2 or 3 or 5

Ans: Let

U = set of integers x, s.t. $1 \le x \le 100$

A_1 = Set of elements of U divisible by 2
A_2 = " " " " " " 3
A_3 = " " " " " " 5

then $\bar{A}_1$ = set of all elements not div by 2.

$\bar{A}_1 \cap \bar{A}_2 \cap \bar{A}_3$ = required set

$\equiv \overline{A_1 \cup A_2 \cup A_3}$

Again cardinality

$$|\bar{A}_1 \cap \bar{A}_2 \cap \bar{A}_3| = |\overline{A_1 \cup A_2 \cup A_3}|$$

$$= |U| - |A_1 \cup A_2 \cup A_3|$$

$|U| = 100$

$$|A_1 \cup A_2 \cup A_3| = \frac{100}{2} + \left|\frac{100}{3}\right| + \left|\frac{100}{5}\right| - \left|\frac{100}{6}\right| - \left|\frac{100}{15}\right|$$

$$- \left|\frac{100}{10}\right| - \left|\frac{100}{15}\right|$$

$$= 74$$

$$\therefore |A_1 \cup A_2 \cup A_3| = 74$$

$$\therefore \bar{A}_1 \cap \bar{A}_2 \cap \bar{A}_3 = 26 \quad \underline{\text{Ans}}$$

2Que: Which of the following is not equivalent to $p \leftrightarrow q$

a) $(\neg p \vee q) \wedge (p \vee \neg q)$

b) $(\neg p \vee q) \wedge (q \rightarrow p)$

c) $(\neg p \wedge q) \vee (p \wedge \neg q)$

d) $(\neg p \wedge \neg q) \vee (p \wedge q)$

p	q	$\neg p \vee q$	$p \vee \neg q$	E	$p \rightarrow q$	$q \rightarrow p$	
F	F	T	T	T	T	T	T
F	T	T	F	F	T	F	F
T	F	F	T	F	F	T	F
T	T	T	T	T	T	T	T

p	q
F	F
F	T
T	F
T	T

Ans: ~~[illegible]~~ Make TT in exams but for shorter verification, convert to boolean

eg for (a) boolean version is →

$(\bar{p} + q) \cdot (p + q)$

remember

$$\boxed{\begin{array}{lll} \wedge & \longrightarrow & \cdot \\ \vee & \longrightarrow & + \\ p \rightarrow q & \longrightarrow & (\bar{p} \vee q) \end{array}}$$

Q→ Find whether $(a \wedge b \wedge c) \rightarrow (c \vee a)$ is a tautology

Ans:

a	b	c	$a \wedge b \wedge c$	$(c \vee a)$	$(a \wedge b \wedge c) \rightarrow (c \vee a)$
F	F	F	F	F	T
F	F	T	F	T	T
F	T	F	F	F	T
F	T	T	F	T	T
T	F	F	F	T	T
T	F	T	F	T	T
T	T	F	F	T	T
T	T	T	T	T	T

∴ Tautology. (Shortcut method se MCQ me help milegi ☺)

Q→ Given the premises

p

$p \rightarrow q$

$s \vee r$

$r \rightarrow \neg q$

arrive at the conclusion

$s \vee t$ ⟶ (yes! It's t!)

Ans. ~~This means p is true & p→q is true.~~

Step	Reason
i) p	Premise
ii) $p \rightarrow q$	Premise
iii) q	modus ponens (using 1&2)
iv) $r \rightarrow \neg q$	Premise
v) $q \rightarrow \neg r$	contrapositive statement
vi) $\neg r$	modus ponens (using

vii) $s \vee r$ — Premise

viii) s — Disjunctive syllogism (using 6, 7)

ix) $s \vee t$ — Disjunctive amplification (using 8)

Que: If I get my christmas bonus and my friends are free, then I will take a road trip with them. If my friends don't find a job after christmas, then they will be free. I got my christmas bonus & my friends did not find a job after christmas. ∴ I will take a road trip with my friends.

Ans: "Christmas bonus" – p
"friends free" – q
"I'll take a road trip" – r
"Friends find a job" – s

Don't write short phrases in exam

The arguments, symbolically, are as follows

$(p \wedge q) \longrightarrow r$

$\neg s \longrightarrow q$

~~p~~ $\neg s$

Conclusion: r

Steps	Reason
i) $\neg s \longrightarrow q$	Premise
ii) $\neg s$	Premise
iii) q	Modus ponens (using 1 & 2)

iv)	p	premise
v)	$p \wedge q$	conjunction
vi)	$(p \wedge q) \to r$	premise
vii)	r	modus ponens (using 5&6)

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PREDICATE AND PREDICATE LOGIC

- $3/4$ is rational. T
  $1/2$ is rational. T
  $\sqrt{2}$ is rational. F

Let u : set of real numbers
$x \in u$

$P(x)$: x is a rational number
↓
PREDICATE (since x has no fixed value, ∴ assigning T or F to the statement is meaningless due to which it is not a proposition. Until & unless, x is assigned some value, it is not a proposition. Such statements are called predicate)

- $u \xrightarrow{P(\cdot)} \{T, F\}$
  A predicate maps values from universal set to a set of true or false.

eg- $S(x, y)$: $x + y = 5$

$x, y \in u$

- u: universe of discourse

$$\left.\begin{array}{l} R(x) : x \text{ is rational} \\ G(y) : y > 5 \\ S(x,y) : x + y = 5 \end{array}\right\} x, y \in u : \text{set of real nos.}$$

$$\left.\begin{array}{l} E(x) : x \text{ climbed Mt. Everest} \\ C(y) : y \text{ is a lawyer} \end{array}\right\} x, y \in u : \text{set of human beings.}$$

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→ QUANTIFIERS

i) Universal quantifier '$\forall$'

u

$x \in u$, $\forall x, P(x)$

for all x (in u), $P(x)$ is true.

ii) Existential quantifier '$\exists$'

$x \in u$ $\exists x, P(x)$

there exists x (in u) such that $P(x)$ is true.

• Proposition	Abbreviation
a) $\forall x, F(x)$	All true
b) $\exists x, F(x)$	Atleast one true
c) $\sim[\exists x, F(x)]$	~~All false~~ none true
d) $\forall x, [\sim F(x)]$	All false
e) $\exists x, [\sim F(x)]$	Atleast one false
f) $\sim[\exists x, [\sim F(x)]]$	none false
g) $\sim[\forall x, F(x)]$	not all true
h) $\sim[\forall x, [\sim F(x)]]$	not all false

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- Some of the above show equality.

1. All true : $\forall x, F(x) \equiv$ none false : $\sim[\exists x, [\sim F(x)]]$

2. All false : $\forall x, [\sim F(x)] \equiv$ none true : $\sim[\exists x, F(x)]$

3. not all true : $\sim[\forall x, F(x)] \equiv$ At least one false $\exists x, [\sim F(x)]$

4. not all false : $\sim[\forall x, [\sim F(x)]] \equiv$ atleast one true $\exists x, F(x)$

$$\boxed{\begin{array}{l} \sim(p \wedge q) \equiv \sim p \vee \sim q \\ \sim(p \vee q) \equiv \sim p \wedge \sim q \end{array}}$$

→ we have to use De morgans Theorem to prove above results.

$u = \{a, b, c, d\}, \ x \in u$

$\forall x, F(x) : F(a) \wedge F(b) \wedge F(c) \wedge F(d)$
$\exists x, F(x) : F(a) \vee F(b) \vee F(c) \vee F(d)$

- now to prove, we bring RHS to LHS

1. $\sim[\exists x, [\sim F(x)]]$

$\equiv \sim(\sim F(a) \vee \sim F(b) \vee \sim F(c) \vee \sim F(d))$

$\equiv F(a) \wedge F(b) \wedge F(c) \wedge F(d)$

$= \forall x, F(x)$

Hence, proved

2. $\sim [\exists x, F(x)]$

$\equiv \sim (F(a) \vee F(b) \vee F(c) \vee F(d))$

$\equiv \sim F(a) \wedge \sim F(b) \wedge \sim F(c) \wedge \sim F(d)$

$\equiv \forall x, [\sim F(x)]$

Hence, proved.

3. $\exists x, [\sim F(x)]$

$\equiv \overline{F(a)} \vee \overline{F(b)} \vee \overline{F(c)} \vee \overline{F(d)}$

$\equiv \sim [F(a) \wedge F(b) \wedge F(c) \wedge F(d)]$

$\equiv \sim [\forall x, F(x)]$

Hence, proved.

4. $\exists x, F(x)$

$\equiv F(a) \vee F(b) \vee F(c) \vee F(d)$

$\equiv \sim [\overline{F(a)} \wedge \overline{F(b)} \wedge \overline{F(c)} \wedge \overline{F(d)}]$

$\equiv \sim [\forall x, \sim [F(x)]]$

Hence proved.

- Proof by ~~eg~~ example

$\exists x$, ~~F~~ $F(x)$ is true, $F(c)$ for some 'c' in u

eg- $\exists x$, x is prime $x \in Z$

- Proof by exhaustion

$\forall x$, $\sim F(x)$ choose all $c \in u$ and show false

- Proof ~~of~~ by contradiction / counter-eg

$\forall x, F(x)$ is false, one c is false.

eg- $P(x)$: $x + y = 8$

TUTORIAL

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Proof by example

Q: Show that "some prime numbers are integers" is true.

Ans: Let $F(x) = x$ is prime
Rewriting the above statement, we get

$\exists x, F(x)$ where $U \in Z$
↳ universal set

Since $F(x)$ holds for $x = 3, 5, 7, \ldots$ etc.

Hence from proof by eg., we can say that $\exists x, F(x)$ is true.

Proof by exhaustion

Q: Show that diff. of two consecutive square numbers b/w 25 and 100 are odd.

Ans: $6^2 = 36$, $7^2 = 49$, $8^2 = 64$, $9^2 = 81$

$7^2 - 6^2 = 13$
$8^2 - 7^2 = 15$
$9^2 - 8^2 = 17$

So by proof of exhaustion, we can say that the difference of ....

Proof by contradiction

Q: Show that $x+y=8$ is not true for all integers.

Ans: Let $P(x,y): x+y=8$

Assume $\forall x, y\ P(x,y)$ is true.
when $x=2$, $y=3$
$P(x,y): \neq 8$

∴ using proof by contradiction, it is proved that $x+y=8$ is not true for all integers.

Q: Show that the premises: "Everyone in this Discrete maths class has taken a course in CS".
" Marla is a student in this class".
Imply the conclusion:
"Marla has taken a course in CS"

Ans:

Let $D(x)$ = "x is in this Discrete Mathematics class"
$C(x)$ = "x has taken a course in CS"

Then the premises are
i) $\forall x\ (D(x) \rightarrow C(x))$
ii) $D(\text{marla})$
The conclusion $C(\text{marla})$

Steps	Reason
1. $\forall x\,(D(x) \to C(x))$	Premise
2. $D(\text{marla}) \to C(\text{marla})$	Universal Instantiation
3. $D(\text{marla})$	Premise
4. $C(\text{marla})$	Modus Ponens from 2-3

Q: Represent the following argument symbolically & decide if they are valid:

i) "If x is a lion, then x is carnivorous"
ii) "moo is not carnivorous"

Then prove that "moo is not a lion"

Ans: $L(x) \to$ x is a lion
$C(x) \to$ x is carnivorous

Symbolically

$$\forall x\ L(x) \to C(x)$$
$$\underline{\neg C(\text{moo})}$$
$$\neg L(\text{moo})$$

Assertion	Reason
1. $\forall x\ L(x) \to C(x)$	Premise
2. $L(\text{moo}) \to C(\text{moo})$	Universal Instantiation from (i)
3. $\sim C(\text{moo})$	Premise
4. $\sim L(\text{moo})$	Modus tollens (i) & (ii)

Q: "All lions are fierce"
"Some lions don't drink coffee"
∴ "Some fierce creatures don't drink coffee.

Ans: $L(x)$ = "x is a lion"
$F(x)$ = "x is fierce"
$C(x)$ = "x drinks coffee"
Symbolic representation

$\forall x \ L(x) \rightarrow F(x)$
$\exists x \ L(x) \wedge \neg C(x)$
$\exists x (F(x) \wedge \neg C(x))$

Assertion	Reason
i) $\exists x (L(x) \wedge \neg C(x))$	Premise
ii) $L(Zoo) \wedge \neg C(Zoo)$	Existential instantiation from (i)
iii) $L(Zoo)$	Simplification from (ii)
iv) $\sim C(Zoo)$	Simplification from (ii)
v) $\forall x (L(x) \rightarrow F(x))$	premise
vi) $L(Zoo) \rightarrow F(Zoo)$	Universal instantiation from (v)
vii) $F(Zoo)$	modus ponens from (iii) & (vi)
viii) $F(Zoo) \wedge \neg C(Zoo)$	Conjunction from (iv) & (vii)
ix) $\exists x (F(x) \wedge \sim C(x))$	Existential generalisation from viii)

MATHEMATICAL INDUCTION

- It is a method of proof

- P(n) is a statement which can be either true or false, for all +ve integers 'n'. To prove P(n) to be true for $n \geq 1$,

1. P(1) is true
2. For all $k \geq 1$ $P(k) \rightarrow P(k+1)$

- Solution:

1. Basis of Induction : P(1) is true.
2. Induction hypothesis : Assume P(k) to be true.
3. Inductive step : Based on 'step 2' show that P(k+1) is true.

NOTE: It is not necessary that we take 1 as initial point. For different cases it might be possible that condition might be true only after a certain integer n_0. In that case initial point is 'n_0'.

Que: Show that if 'n' is a +ve integer, then

$$1+2+\ldots\ldots+n = \frac{n(n+1)}{2}$$

Ans: Let $P(n)$: Sum of all integers up to n is $\frac{n(n+1)}{2}$

Step 1: $P(1) = \frac{1(1+1)}{2} = 1$

Step 2: For all $k \geq 1$, $P(k) = \frac{k(k+1)}{2}$

Step 3: $P(k+1) = \underbrace{1+2+\ldots\ldots+k}_{P(k)} + (k+1)$

$$= \frac{k(k+1)}{2} + (k+1)$$

$$= \frac{(k+1)(k+2)}{2} = \frac{(k+1)(k+1+1)}{2}$$

Hence, proved.

Que: Conjecture a formula for the sum of first n positive odd ~~nos.~~ integers. Then prove your conjecture using MI.

Ans: $n = 1, 3, 5, 7, 9, \ldots$

$n=1$ $\quad 1 = 1^2$

$n=2$ $\quad 1+3 = 4 = 2^2$

$n=3$ $\quad 1+3+5 = 9 = 3^2$

$n=4$ $\quad 1+3+5+7 = 16 = 4^2$

$\vdots$

Let $P(n)$: sum of first 'n' positive odd integers is n^2

Step 1: $P(1) = 1 = 1^2$

Step 2: For all $k \geq 1$ $P(k) = k^2$

Step 3. $P(k+1) = \underbrace{1 + 3 + \cdots\cdots}_{P(k)} + (2k+1)$

$= k^2 + 2k + 1$

$= (k+1)^2$

Hence, proved.

→ STRONG MATHEMATICAL INDUCTION

- Let $P(n)$ be a statement which is either true or false for each integer 'n'. Then $P(n)$ is true for all +ve integers 'n', if there is an integer $q \geq 1$ s.t.

1. $P(1), P(2), \ldots\ldots, P(q)$ are all true.
2. $k \geq q$, the assumption that $P(i)$ is true, where $1 \leq i \leq k$, implies $P(k+1)$ is true.

Step 1: Basis of induction : $P(1), \ldots\ldots, P(q)$ are true.

Step 2: Strong inductive hypothesis : Assume $P(i)$ to be true, $1 \leq i \leq k$, for $k \geq q$

Step 3: Inductive step: P(k+1) is true.

Que: Suppose we have stamps of denominations ₹3 and ₹5. Show that it is possible to make postages of ₹8 or more by using stamps of these denominations only.

Ans:
$8 = 5+3$
$9 = 3+3+3$
$10 = 5+5$
$11 = 5+3+3$
$12 = 3+3+3+3$

Step 1: $P(8), P(9), \ldots, P(12)$ is true.

Step 2: $P(i)$ is true, $8 \le i \le k$, $k \ge 12$

Step 3: $P(k-4)$ is true.
$8 \le k-4 \le k$
$\therefore P(k-4)$ is true.
Adding ₹5 to ₹$(k-4)$ we get ₹$(k+1)$

$\therefore$ $P(k+1)$ is true.
Hence proved.

We can also prove this using simple induction.

Step 1: $P(8)$ is true

Step 2: Let $P(k)$ be true

Step 3: For $P(k+1)$, If $P(k)$ has ₹5, we take it away

& add $(\mathbb{Z}_3 + \mathbb{Z}_3)$ $\therefore$ $\underline{k+1}$

In case it does not have a $\mathbb{Z}_5$, then we subtract $-(\mathbb{Z}_3 + \mathbb{Z}_3 + \mathbb{Z}_3)$ & add $(\mathbb{Z}_5 + \mathbb{Z}_5)$

$\therefore$ $\underline{k+1}$.

Hence proved.

~~Que~~

TUTORIAL

1 Que: Use MI to prove $1^2 + 2^2 + \dots + n^2 = \frac{n}{6}(n+1)(2n+1)$

Ans: **Step 1:** Basic step

for $p(1)$

$LHS = 1^2 = 1$

$RHS = \frac{1}{6}(1+1)(3) = 1$

$LHS = RHS$

$\Rightarrow p(1)$ is true.

Step 2: Inductive step

Assume $p(k)$ is true.

$\therefore 1^2 + 2^2 + \dots + k^2 = \frac{k}{6}(k+1)(2k+1)$ —(i)

we have to prove that $p(k+1)$ is also true when $p(k)$ is true.

Step 3:

For $p(k+1)$

$LHS = 1^2 + 2^2 + \dots + k^2 + (k+1)^2$

$$= \frac{k}{6}(k+1)(2k+1) + (k+1)^2$$

$$= (k+1)\left[\frac{k}{6}(2k+1) + (k+1)\right]$$

$$= (k+1)\left[\frac{2k^2+k+6k+6}{6}\right]$$

$$= \frac{(k+1)}{6}\left[2k^2+7k+6\right]$$

$$= \frac{(k+1)}{6}(k+2)(2k+3)$$

$$= \frac{(k+1)}{6}((k+1)+1)(2(k+1)+1)$$

$$= \frac{(k+1)}{6}(k+2)(2k+3)$$

$\therefore$ LHS = RHS

Using MI it is proved that ....

Que: n^3-n is divisible by 3 whenever n is a positive integer.

Ans:

Here $p(n)$: n^3-n is divisible by 3 $\forall n>0$, $n \in N$

Step 1: For $p(1)$ (Basic step)

$1^3-1 = 1-1=0$ is divisible by 3

Step 2: Inductive Step

Assume $p(k)$ is true

k^3-k is divisible by 3

ie $k^3-k = 3m$

Step 3: we have to prove $p(k+1)$ is also true when $p(k)$ is true.

For $p(k+1)$

$$(k+1)^3 - (k+1) = k^3 + \cancel{1} + 3k^2 + 3k - k - \cancel{1}$$

$$\cancel{= k^3 + 3k^2 + 2k}$$

$$\cancel{= k(k^2 + 3k + 2)}$$

$$\cancel{= k(k+1)(k+2)}$$

$$= k^3 - k + 3k^3 + 3k$$

$$= 3m + 3k^3 + 3k$$

$$= 3(m + k^3 + k)$$

$\Rightarrow (k+1)^3 - (k+1)$ is divisible by 3

$\Rightarrow p(k+1)$ is also true.

Hence using MI it is proved that $n^3 - n$ is divisible by 3.

Que: Using MI, prove that for every +ve integer n with $n \geq 4$, $2^n < n!$

Ans: **Step 1:** Basic step

For $p(4)$

$2^4 = 16 < 24 = 4!$ $\Rightarrow 2^4 < 4!$

Step 2: Inductive step

Assume $p(k)$ is true

$\Rightarrow 2^k < k!$

Step 3: Now we need to prove $p(k+1)$ is also true

For $p(k+1)$

$2^{k+1} = 2^k \cdot 2$

$< k! \cdot 2$

As $k \geq 4$

$\Rightarrow k+1 \geq 4 > 2$

$\therefore 2^{k+1} < k!\,(k+1)$

$\therefore 2^{k+1} < (k+1)!$

$\Rightarrow p(k+1)$ is true.

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4 Que: using MI prove $1+2+2^2+\dots+2^n=2^{n+1}-1$ for all non-negative integers.

Ans: Step 1: Basic step.

For p(0) (since we need to prove for non-negative integers)

LHS = 1
RHS = $2^{0+1}-1 = 2^1 - 1 = 1$

$\therefore$ p(0) is true.

Step 2: Inductive step.

Assume p(k) is true.

$\Rightarrow 1+2+2^2+\dots+2^k = 2^{k+1}-1$ — (i)

Step 3:
Now we have to prove:
For p(k+1)

LHS $= 1+2+\dots\dots+2^k+2^{k+1}$

$= 2^{k+1}-1+2^{k+1}$

$= 2^{k+2}-1$

RHS $= 2^{(k+1)+1}-1$
$= 2^{k+2}-1$

$\therefore$ LHS = RHS
$\Rightarrow$ p(k+1) is true)
$\therefore$ using MI it is proved that -

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5 Que: Let a_n be the sequence defined by $a_1 = 1$, $a_2 = 8$ and $a_n = a_{n-1} + 2a_{n-2}$ for $n \geq 3$,

prove that $a_n = 3 \cdot 2^{n-1} + 2(-1)^n$ for all $n \in N$

Ans:

Step 1: Basic step

For a_1

$LHS = 1$

$RHS = 3 \cdot 2^0 + 2(-1) \quad = 1$

$LHS = RHS$

$\therefore a_1$ is true

For a_2

$LHS = 8$

$RHS = 3 \cdot 2^{2-1} + 2(-1)^2$

$= 8$

$\therefore LHS = RHS$

$\therefore a_2$ is true.

Step 2: Inductive step

Assume a_k is true & a_{k-1} is also true

$a_{k-1} = 3 \cdot 2^{k-2} + 2(-1)^{k-1}$

$a_k = 3 \cdot 2^{k-1} + 2(-1)^k$

Step 3: using strong MI we have to prove

$a_{k+1} = 3 \cdot 2^k + 2(-1)^{k+1}$

$a_{k+1} = a_k + 2a_{k-1}$

$$LHS = 3\cdot 2^{k-1} + 2(-1)^{k} + 2\left[3\cdot 2^{k-2} + 2(-1)^{k-1}\right]$$

$$= 3\left[2^{k-1} + 2^{k-1}\right] + 2\left[(-1)^{k} + 2(-1)^{k-1}\right]$$

$$= 3\cdot 2^{k} + 2(-1)^{k+1}$$

$$RHS = 3\cdot 2^{(k+1)-1} + 2(-1)^{k+1}$$

$$= 3\cdot 2^{k} + 2(-1)^{k+1}$$

$\therefore LHS = RHS$

$\therefore a_{k+1}$ is true

$\therefore$ using strong MI it is proved that ......

6 Que: Using MI prove $3 + 7 + 11 + \dots + (4n-1) = n(2n+1)$

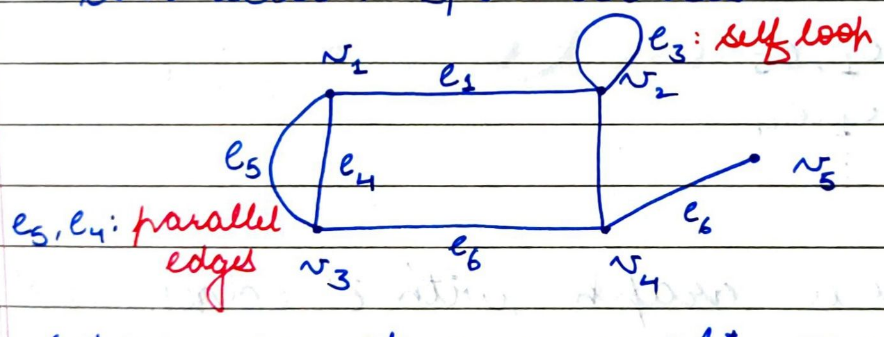
Chapter - 2
Graph Theory

→ GRAPHS

- Def: A graph consists of 2 set of objects
  $V = \{v_1, v_2 \ldots\ldots, v_n\}$ and $E = \{e_1, e_2, \ldots\ldots, e_n\}$
  $V \downarrow$ vertices, $E \downarrow$ edges

- A graph can be represented as $G = (V, E)$

- $e_k = \{v_i, v_j\}$ An edge is formed by connection b/w vertices.

- 

  ⇝ a graph

edges are not necessarily ~~xxxx~~ straight.

• SIMPLE GRAPH

- It is a graph with no parallel edges or self-loops.

NOTE: Incidence: If a v_i is the endpoint of an edge 'e_j', then v_i and e_j are incident on each other.

- Degree of a vertex: The no. of edges incident on a vertex v_i with self loops counted as 2 is the degree of v_i, $d(v_i)$

eg- above, $d(v_1) = 3$ and $d(v_2) = 4$

- Adjacency: • 2 vertices are adjacent to each other if they have a common edge v_1, v_2 ....

  • 2 edges are said to be adjacent if they are incident on the same vertex.

  eg- e_1, e_5
  e_1, e_4
  ⋮

Que: If G is a graph with 'e' edges and 'n' vertices $v = \{v_1 \ldots\ldots v_n\}$ find $\sum_{i=1}^{n} d(v_i)$

Ans: every edge has 2 ends and both of them contribute to the degree of some vertex. ∴

$$\boxed{\sum_{i=1}^{n} d(v_i) = 2e}$$

THEOREM 1 (HANDSHAKING THEOREM)

- The no. of vertices, of odd degree in a graph is always even

- PROOF:

$G(V,E)$ where $V = \{v_1, \ldots\ldots, v_n\}$, $|E| = e$

Let

V_{odd} be set of all odd degree vertices.

&

V_{even} be set of all even degree vertices.

From observation, we know

$$\sum_{i=1}^{n} d(v_i) = 2e$$

$$\Rightarrow \sum_{v_i \in V_{odd}} d(v_i) + \sum_{v_i \in V_{even}} d(v_i) = 2e$$

$$\Rightarrow \sum_{v_i \in V_{odd}} d(v_i) = \underbrace{2e - \sum_{v_i \in V_{even}} d(v_i)}_{even} = 2m$$

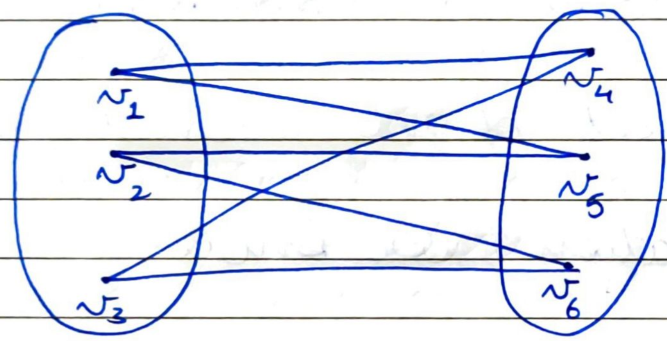
This is only possible if V_{odd} has even numbers of elements.

Hence, proved.

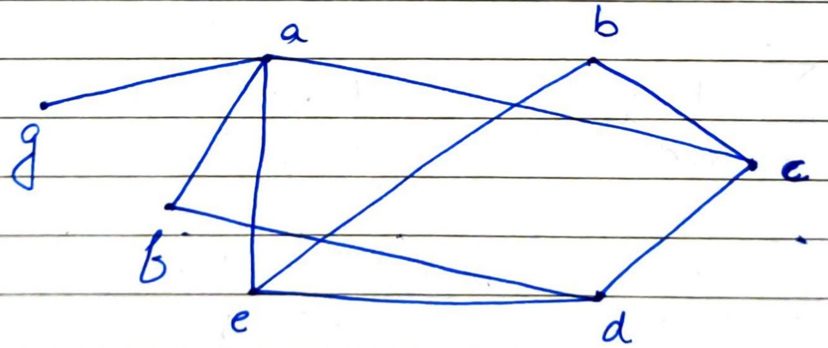
- BIPARTITE GRAPH

- A simple graph whose vertex set 'V' can be partitioned into 2 ~~diagonal~~ disjoint subsets 'V_1' and 'V_2' s.t. every edge in the graph connects a vertex in V_1 and a vertex in V_2 with no edge connecting either 2 vertices in V_1 or V_2.

- eg -



Que:



Split in sets

Ans:

e	a
f	b
g	d
c	

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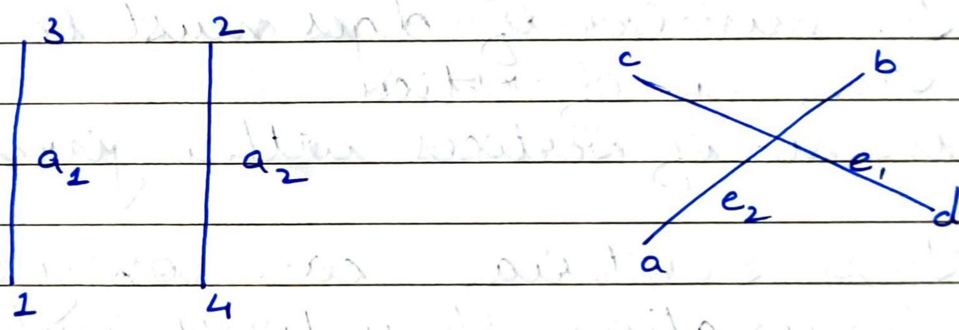
→ ISOMORPHISM

- $G = (V, E)$
  V: set of vertices
  E: set of edges

- Def: 2 graphs G & G′ are said to be isomorphic if there is a one to one correspondence b/w the set of vertices and the set of edges such that the incidence /adjacency criteria is maintained.

eg-



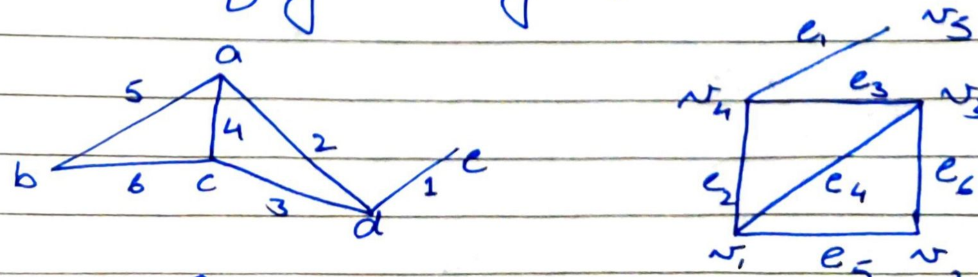
$1 \rightarrow a$ $\quad a_1 \rightarrow e_2$

$2 \rightarrow c$ $\quad a_2 \rightarrow e_1$

$3 \rightarrow b$

$4 \rightarrow d$

Que: Find if given graphs are isomorphic



G

Ans: The given graphs are isomorphic

PTO

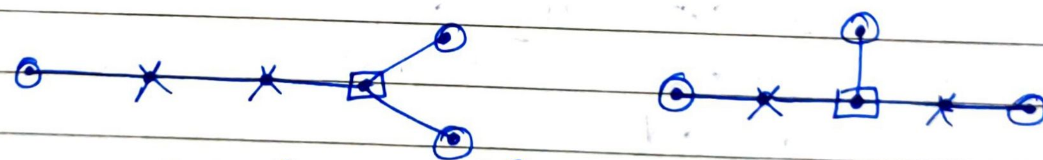
$a \to v_1$ $\quad$ $1 \to e_1$
$b \to v_2$ $\quad$ $2 \to e_2$
$c \to v_3$ $\quad$ $3 \to e_3$
$d \to v_4$ $\quad$ $4 \to e_4$
$e \to v_5$ $\quad$ $5 \to e_5$

- establishing ~~a~~ isomorphism is a practically difficult problem, but eliminating it is easy and can be done by using these 3 steps.

a) The number of edges must be same
b) Same no of vertices
c) the no of vertices with a given degree

NOTE: These 3 criteria can only be used for elimination. They don't imply isomorphism

eg-



not isomorphic

PTO

→ SUBGRAPH

- given $G=(V,E)$

$V' \subseteq V$, $E' \subseteq E$ with all pairs in E' are in V
, then
$G' = (V', E')$ is a subgraph of G.

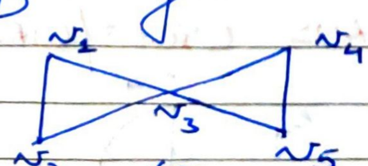
- PROPERTIES

1. Every graph is its own subgraph. 😑
2. The subgraph of a subgraph, of a graph G, is a subgraph of G.
3. A single vertex.
4. A single edge with its end vertices.

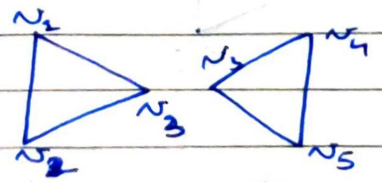
- TYPES

1. Edge disjoint subgraphs

G_1 and G_2 are said to be edge disjoint subgraphs if they have no edge in common.



(G)



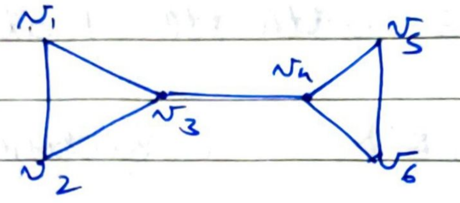
(g_1) (g_2)

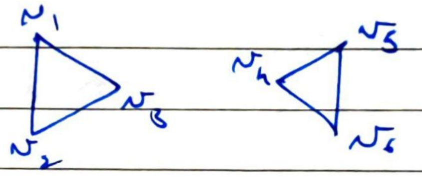
g_1 & g_2 are edge disjoint subgraphs

2. Vertex disjoint subgraphs

2 edge disjoint subgraphs with no common vertex.

eg-





NOTE: • Isolated vertex: vertex with degree '0'.

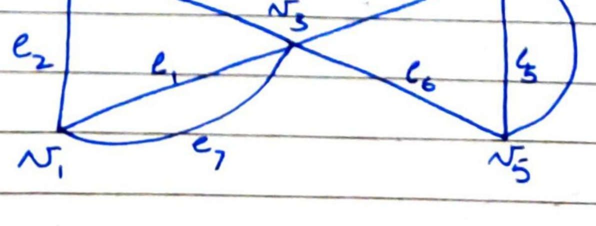
• Pendant vertex: vertex with degree 1

• Null graph: A graph with no edges, just vertices.

• Regular graph: A graph with all vertices of same degree. eg- 2-regular graph has all vertices of degree 2.

imp →

WALK, TRAIL, PATH AND CYCLE/CIRCUIT



- Walk: A walk in a graph is a finite sequence of the form

$$v_{i_0}, e_{j_1}, v_{i_1}, e_{j_2}, \ldots\ldots, e_{j_k}, v_{j_k}$$

eg- $v_1, e_1, v_3, e_3, v_2, e_2, v_1, e_1, v_3, e_6, v_5$

NOTE. - Edges and vertices may repeat in a walk.

imp. A walk with the same terminal vertices is called a closed walk else an open walk

- Trail: A walk with no repetition of edges
- Path: An open walk with no vertex repeated.
- Cycle/Circuit: A closed walk with no repetition of vertex, except the terminals

NOTE: An undirected graph is one in which there is no direction associated with the edges

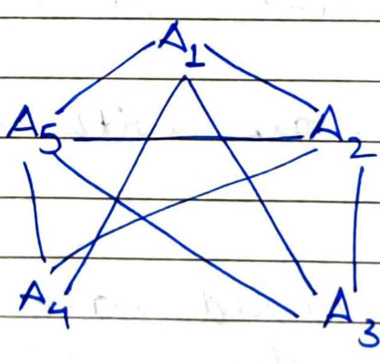
TUTORIAL

- An intersection graph or a collection of sets $s_1, s_2, \ldots\ldots, s_n$ is a graph of n vertices (one for every set) and there exists an edge b/w two vertices s_i and s_j if s_i and s_j is a non-empty intersection

1Que: $A_1 = \{\ldots\ldots, -4, -3, -2, -1, 0\}$
$A_2 = \{\ldots\ldots, -2, -1, 0, 1, 2\}$
$A_3 = \{\ldots\ldots, -6, -4, -2, 0, 2, 4, \ldots\ldots\}$
$A_4 = \{-5, -3, -1, 1, 3, 5, \ldots\ldots\}$
$A_5 = \{\ldots\ldots -6, -3, 0, 3, 6, \ldots\ldots\}$

Draw an intersection graph.

Ans:



2Que: What is the total no. of vertices in an undirected connected graph with 27 edges, 6 vertices of degree 2, and 3 vertices of degree 4 and remaining vertices are of degree 3.

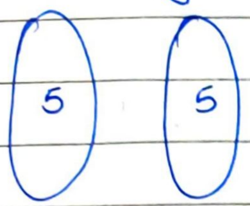
Ans: Sum of degrees of vertices $= 2E$

$\therefore\ 6\times 2 + 3\times 4 + 3(n-9) = 27\times 2$

$\Rightarrow n = 19$

3 Que: What is the max no. of edges in a bipartite graph with 10 vertices?

Ans:



& max edges $= n/2 \times n/2$

$= 25$

- **Degree sequence:** It is the sequence of degrees of vertices, written in non-decreasing order. It is said to be valid or graphic if we can construct a graph out of that degree sequence.

NOTE: Whenever we talk about degree sequence, we always refer to a simple graph (no ‖ edges or self loops)

- $S = d_1 \geq d_2 \geq \dots\dots \geq d_n$

necessary conditions

i) $\boxed{\Sigma d_i = 2e}$

ii) $\boxed{\forall i \quad d_i \leq (n-1)}$

4 Que: Find degree sequences which are valid

a) 3,3,3,3,2
× b) 5,4,3,2,1
c) 4,4,3,2,1
× d) 4,4,3,3,3
e) 3,2,2,1,0
× f) 1,1,1,1,1
g) 3,3,3,1

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• Havel & Hakimi

$S = d_1 \; d_2 \; \ldots\ldots d_n$

such that $d_1 \geq d_2 \geq \ldots\ldots \geq d_n$ is a valid degree sequence if.

$$S' = \underbrace{d_2 - 1 \;,\; d_3 - 1 \;,\; d_{d_1+1} - 1}_{\text{decrement until here}} \;,\; \underbrace{d_{d_1+2} \ldots\ldots d_n}_{\text{leave as is}}$$

is a valid degree sequence.

Repeat until $\exists i$ such that d_i is negative or $\forall i \; d_i = 0$

if $\exists i \; d_i < 0$, then "not a valid DS"

if $\forall i \; d_i = 0$ "valid degree sequence"

∴ In previous que.

Ans: a) 3, 3, 3, 3, 2

$d_1 + 1 = 4$ $\qquad S_1 = 2, 2, 2, 2$

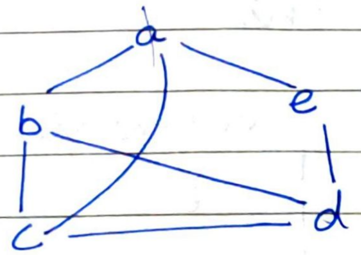
$d_1 + 1 = 3$ $\qquad S_2 = 1, 1, 2$

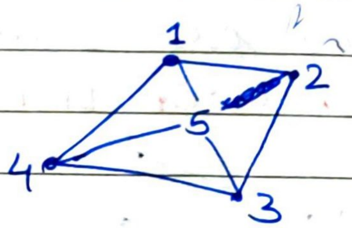
$d_1 + 1 = 3$ $\qquad S_3 = 0, 0$

∴ valid degree seq.

solve rest yourself

5 Que: Find if following graphs are isomorphic





Ans:

$|V| = 5$ $\qquad$ $|V| = 5$

$|E| = 7$ $\qquad$ $|E| = 7$

3, 3, 3, 3, 2 $\qquad$ 3, 3, 3, 3, 2

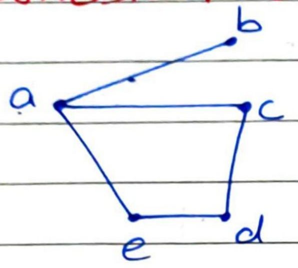
no of cycles of length (3) = 2 $\qquad$ (3) = 2

(4) = 3 $\qquad$ (4) = 3

→ REPRESENTING GRAPHS

- There are 2 ways to do this

1. ADJACENCY LIST



Vertex	Adj vertex
a	b, c, e
b	a
c	d, d, e
d	e, c
e	a, c, d

2. MATRIX METHOD a) (ADJACENT MATRIX)

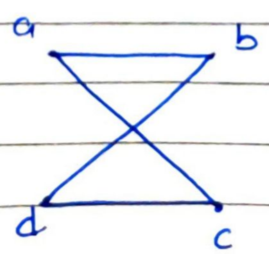
$$
\begin{array}{c|ccccc}
 & a & b & c & d & e \\
\hline
a & 0 & 1 & 1 & 0 & 1 \\
b & 1 & 0 & 0 & 0 & 0 \\
c & 1 & 0 & 0 & 1 & 1 \\
d & 0 & 0 & 1 & 0 & 1 \\
e & 1 & 0 & 1 & 1 & 0
\end{array}
$$

For above graph

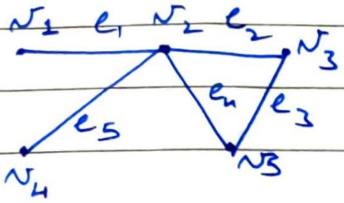
Que. Build graph from

$$
\begin{array}{cc}
 & \begin{array}{cccc} a & b & c & d \end{array} \\
\begin{array}{c} a \\ b \\ c \\ d \end{array} & \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}
\end{array}
$$

Ans:



b) INCIDENCE MATRIX



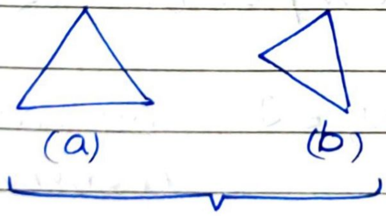
	e_1	e_2	e_3	e_4	e_5
v_1	1	0	0	0	0
v_2	1	1	0	1	1
v_3	0	1	1	0	0
v_4	0	0	0	0	1
v_5	0	0	1	1	0

→ CONNECTED GRAPH

- A graph G is connected if there exists atleast one path b/w every pair of vertices.

→ DISCONNECTED GRAPH

- consists of 2 or more connected subgraphs, also called connected component.



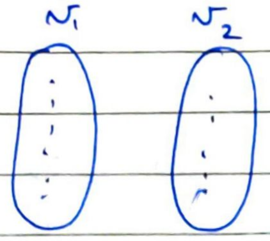
(a) (b)

(c)

a, b → connected component
c → disconnected graph

→ THEOREM 2

- A graph 'G' is a disconnected graph iff its vertex set 'v' can be partitioned into 2 ~~connected components~~ non-empty disjoint subsets v_1 & v_2 such that there is no edge in G whose 1 end vertex lies in 'v_1' & other in 'v_2'.



G

→ THEOREM 3

- If a graph (connected or disconnected) has exactly 2 vertices of odd degree, then there must be a path joining these 2 vertices.

For connected graph, it is obvious

Let G be a graph with all even degree vertices except 'v_1' & 'v_2'

'J' a connected component of 'G'

If $v_1 \in J$, then $v_2 \in J$ (by handshaking theorem)

Hence, proved

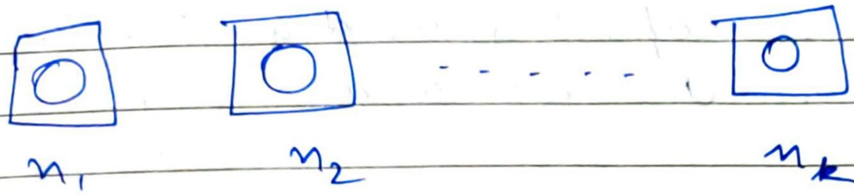
→ THEOREM 4

- A simple graph with 'n' vertices and 'k' connected components has at most $\frac{(n-k)(n-k+1)}{2}$ edges.

PROOF:

$$n_1 + n_2 + \dots + n_k = n$$

$$\sum_{i=1}^{k} n_i = n$$



no. of ways of selecting 2 out of n_1 = max edges ~~tot~~ in n_1

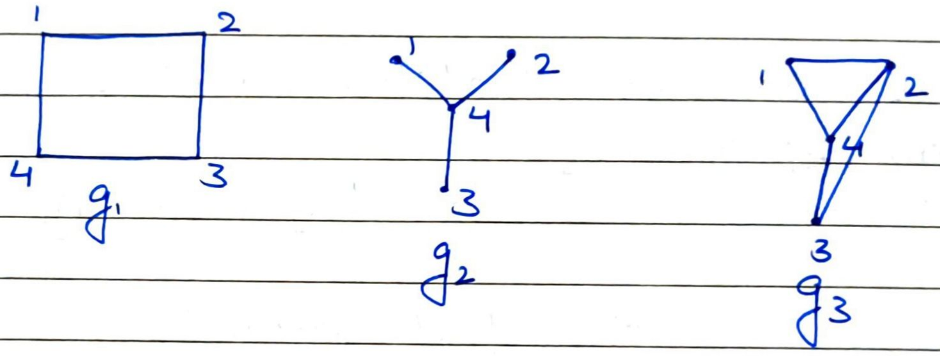
$$= {}^{n_1}C_2 = \frac{n_1(n_1-1)}{2}$$

rest complete yourself

→ UNION, INTERSECTION, RINGSUM, DECOMPOSITION, DELETION

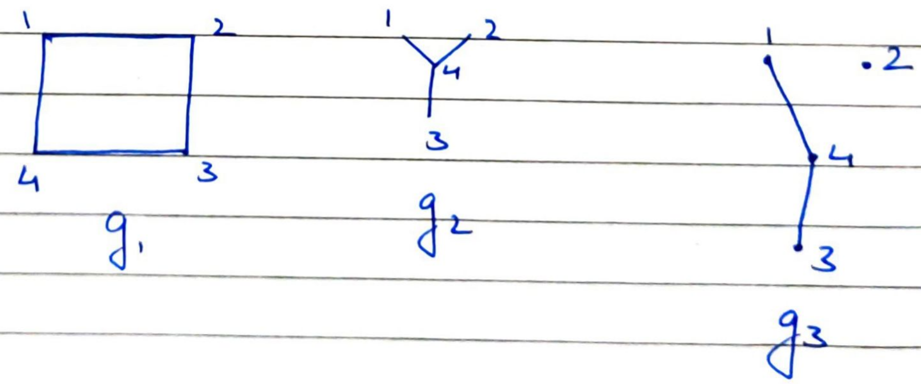
- $G(v, E)$
  $V(G)$, $E(G)$

union of 2 graphs g_1 and g_2 is g_3 s.t.
$v(g_3) = v(g_1) \cup v(g_2)$ and $E(g_3) = E(g_1) \cup E(g_2)$



- **Intersection** of 2 graphs

$v(g_3) = v(g_1) \cup v(g_2)$ & $E(g_3) = E(g_1) \cap E(g_2)$



• **Ringsum** of g_1 and g_2 ($g_1 \oplus g_2$) is a graph g_3 s.t. $V(g_1) \oplus V(g_2) = V(g_1) \cup V(g_2)$

$$e \in E(g_1) \oplus E(g_2)$$

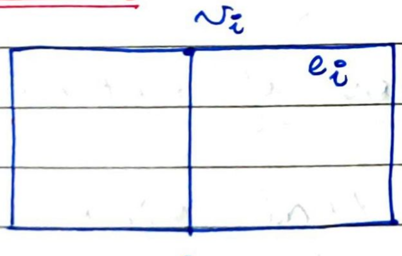
$$[E(g_1) \cup E(g_2)] - [E(g_1) \cap E(g_2)]$$

• **Decomposition** : Any G can be decomposed into g_1 and g_2 s.t

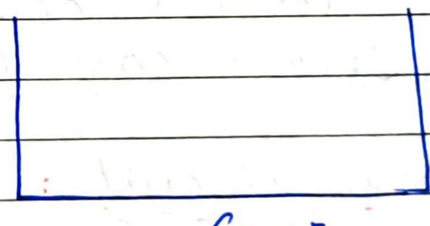
$$g_1 \cup g_2 = G$$

$$g_1 \cap g_2 = \phi$$

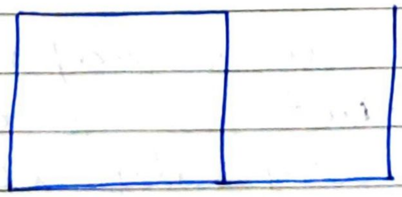
• **Deletion**



G

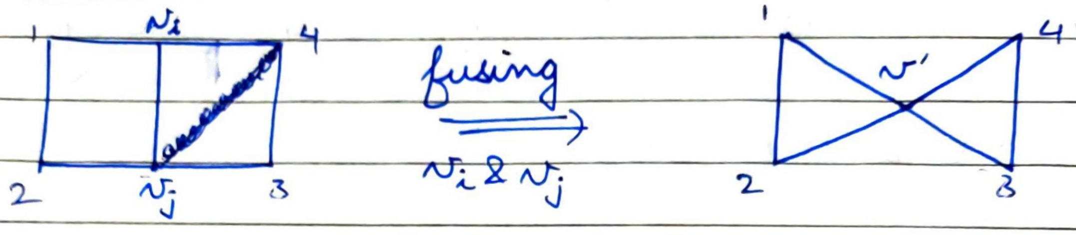


$G - v_i$



$G - e_i$

• <u>Fusion</u>



G

(The edge connecting 2 vertices disappears)

→ EULER GRAPH

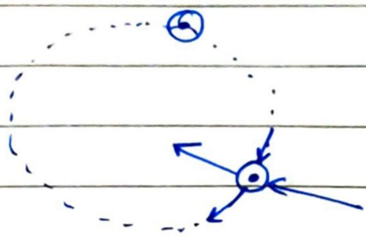
1. Path/Trail : An Euler trail is a trail which visits every edge of the graph exactly once.
[There can be repetition of vertices]

2. Euler circuit : An Euler trail whose terminal vertices are the same.

3. Euler graph : An Euler graph must have an Euler circuit, i.e., a closed trail which visits every edge of the graph exactly once.

→ THEOREM 5

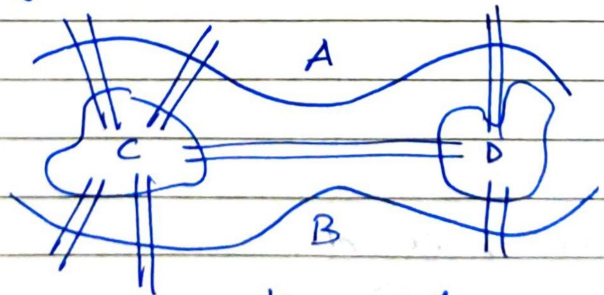
• Graph G (connected) is an euler graph iff all its vertices are of even degree



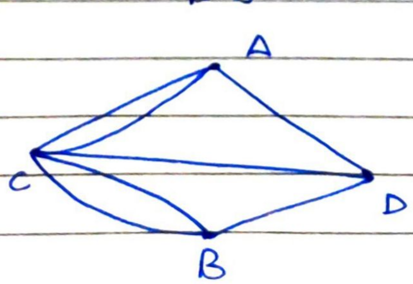
* you can visualise this as, for every entry into a vertex, there must be a new & separate exit.

eg- This theorem gives rise to a solution of an interesting problem

A, B, C, D are landmasses (C & D are islands in a river) There are 7 bridges as shown. Can someone traverse all these bridges exactly once & come back to the same land-mass



⇓ graph



A, B, C & D have odd degree thus this is not possible due to absence of Euler circuit

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Date
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→ THEOREM 6

- A connected graph is an Euler graph iff it can be decomposed into circuits.

TUT missed.

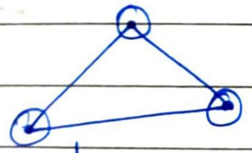
→ HAMILTONIAN PATHS AND CIRCUITS

- Path: A path which traverses every vertex of a graph exactly once is called Hamiltonian path.

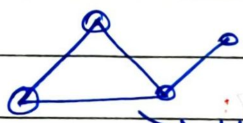
- Circuit: A Hamiltonian path whose terminal vertices are same

- Graph: An H. graph will have an H. circuit ie a path which will traverse every vertex of the graph exactly once except terminal vertices.

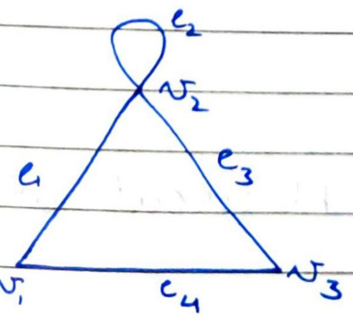
eg-



H. circuit can be formed.



H. path can be formed



$v_1 e_1 v_2 e_2 v_2 e_3 v_3 e_4 v_1$ — E.C.

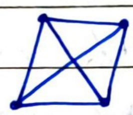
$v_1 e_1 v_2 e_3 v_3 e_4 v_1$ — H.C. { circuit

↓

both H. graph and E. graph

→ COMPLETE GRAPH

- A simple graph where there exists a unique edge b/w every pair of vertices.
- Denoted by K_n
- eg- K_2 :  K_3 : 

  K_4 : 

- Total edges in complete graph $= \frac{n(n-1)}{2}$

→ THEOREM 7:

- In a complete graph with 'n' vertices, there are $\frac{n-1}{2}$ edges disjoint H-circuits

if n is odd and $n \geq 3$

PROOF: HW

Hint: An n vertex H-circuit will have 'n' edges

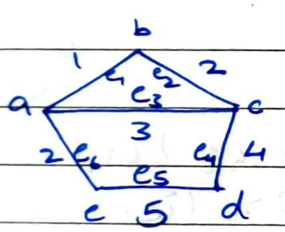
→ WEIGHTED GRAPH

- $G = (V, E)$

  $w : E \longrightarrow \mathbb{R}^+$

  $e_1 \in E$

  $w(e_1) \longrightarrow \mathbb{R}^+$

- 

  $w(e_1) = w(a, b) = 1$

  $w(e_2) = w(b, c) = 2$

  $\vdots$

  $G(\{a, b, \ldots\ldots, e\}, \{e_1, e_2, \ldots\ldots, e_6\})$

  $w : \{e_1, e_2, \ldots\ldots, e_5\} \longrightarrow \mathbb{R}^+$

- consider that we have to find shortest path b/w a and c

  $a \xrightarrow{1} b,\ b \xrightarrow{2} c \longrightarrow 3$

  $a \xrightarrow{3} c \longrightarrow 3$

  $a \xrightarrow{2} e,\ e \xrightarrow{5} d,\ d \xrightarrow{4} c \longrightarrow 11$

TUTORIAL

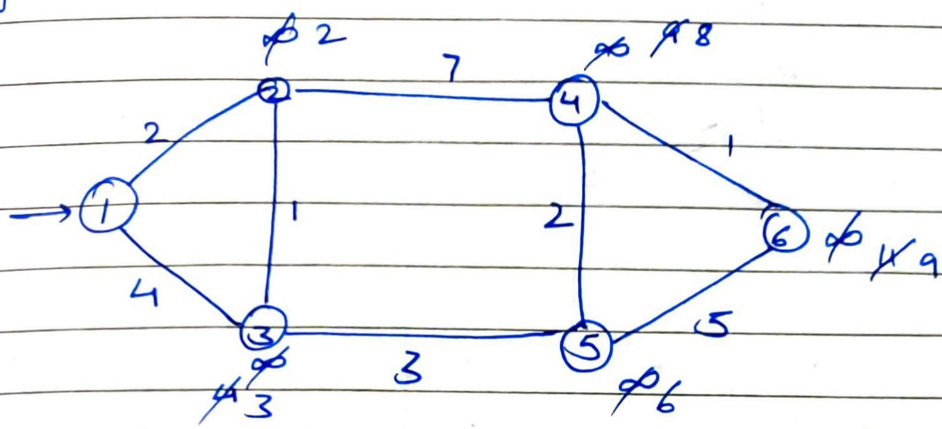
v imp → DIJKSTRA'S ALGORITHM

- Algorithm to traverse each node with shortest distance possible. This algorithm is used to find such a short path.

- dist [s] ← 0 → start vertex
  for all v ∈ V − {S}
  do dist[v] ← ∞
  S ← ~~φ~~ φ
  Q ← V
  while Q ≠ φ
  do u ← mindistance (Q, dist)
  S ← S ∪ {u}
  for all v ∈ neighbours [u]
  do if dist [v] > dist [v] + w [u,v]
  then dist [v] ← dist[v] + w[u,v]
  weight
  return dist;

PTO

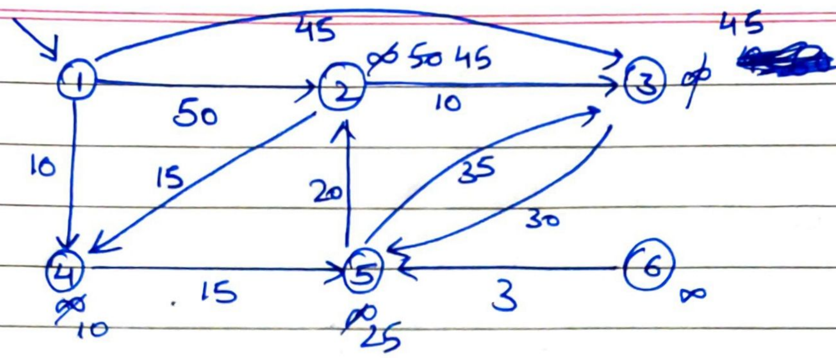
• eg-



Selected nodes	1	2	3	4	5	6
1	0	∞	∞	∞	∞	∞
2	~~0~~	2	4	∞	∞	~~∞~~ ∞
3			3	9	∞	∞
5				9	6	∞
4				8		11
6						9

nodes[v]	dist [v]
1	0
2	2
3	3
4	8
5	6
6	9

2 Que:

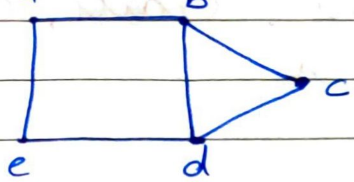


Ans:

Selected vertex	1	2	3	4	5	6
1	0	∞	∞	∞	∞	∞
4		50	45	10	∞	∞
5		50	45		25	∞
2		45	~~60~~ 45			∞
3			45			∞
6						∞

6 is unreachable

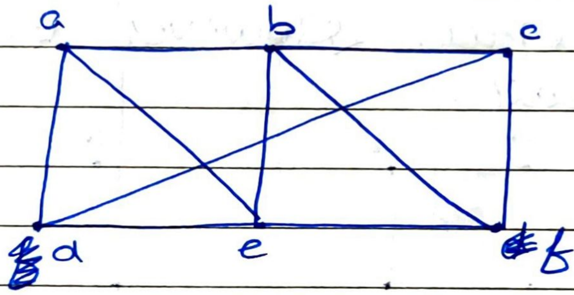
Que:



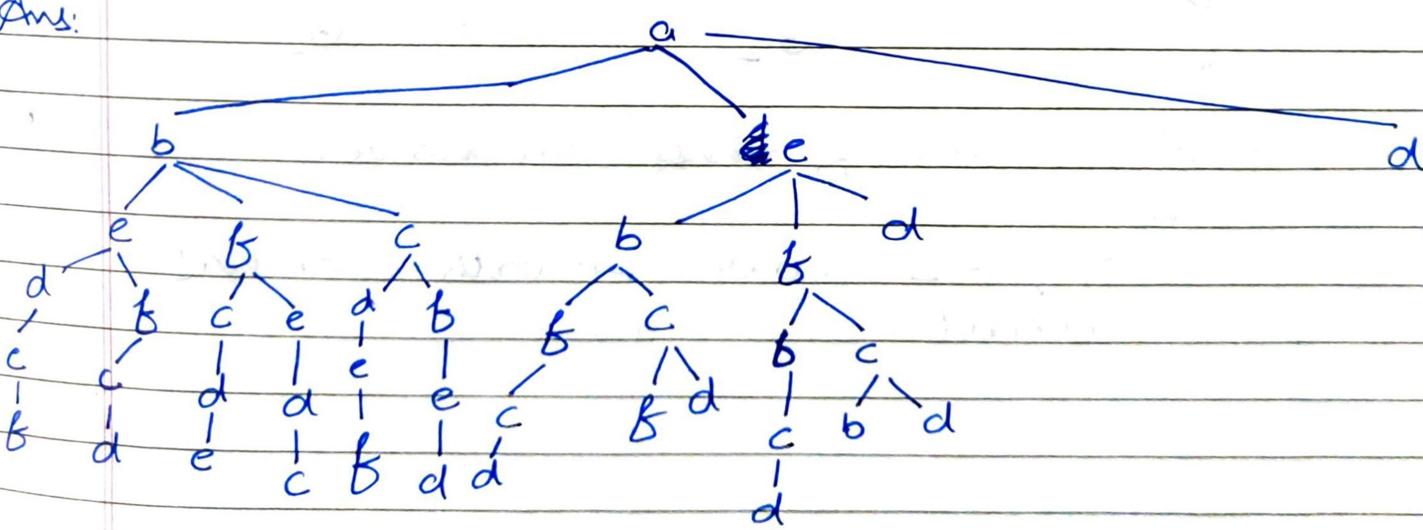
count the number of cycles.

Ans: 3 a,b,d,e,a.
b,c,d,b
a,b,c,d,e,a

Que: count no. of possible paths from vertex a



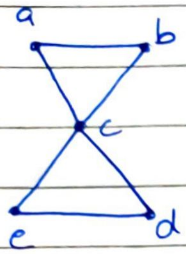
Ans:



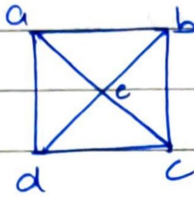
* Design an algo which will take as input: A simple graph $G=(V,E)$ and a starting vertex

Output no. of possible paths, trails, circuits from starting vertex

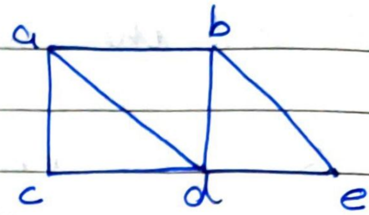
Que:



G_1



G_2

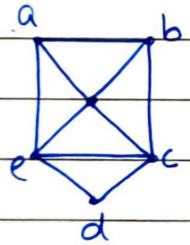


G_3

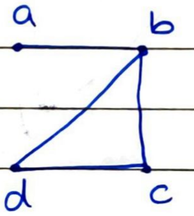
which of these ~~G~~ have Euler circuits?

Ans: Only G_1. Since G_2 & G_3 have vert. of odd degree. However G_3 has an Euler trail.

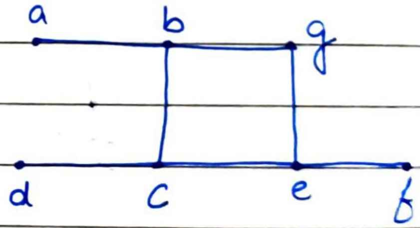
Que:



G_1



G_2



G_3

Find which of these can have a Hamil. Path.

Ans: G_1, G_2 have H. Path. G_1 has H. circuit.

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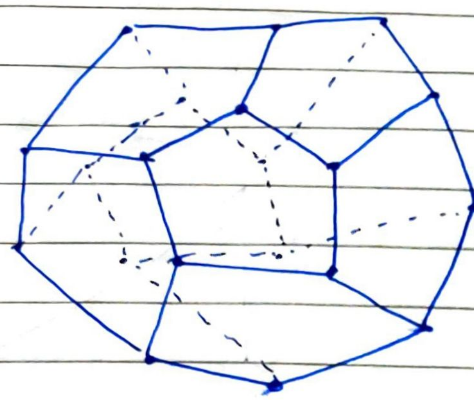
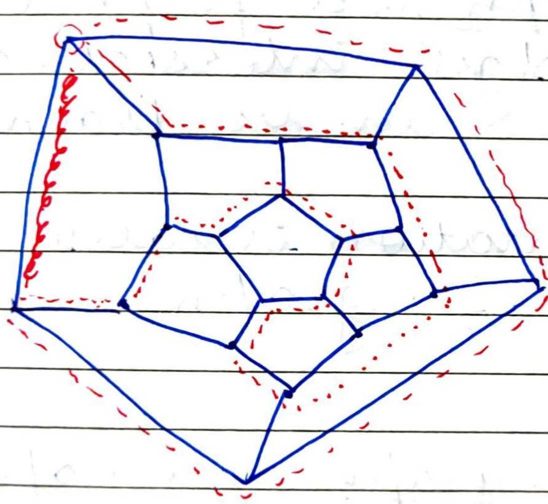
Que:



Dodecahedron.
(Ache se nhi banaya 😅)

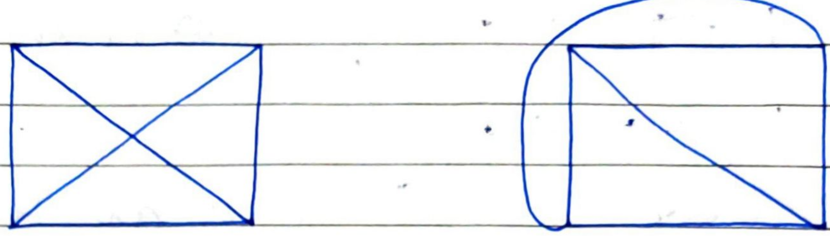
Q Is an H. circuit possible?

Ans: Draw in 2D



→ PLANAR GRAPHS

- $G(V, E)$



G   G′

- G′ is a representation without edge intersection.

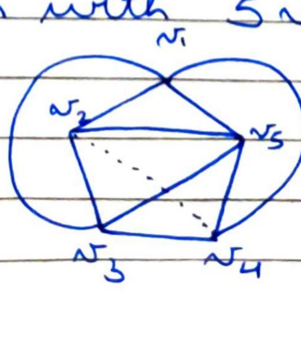
- A planar graph is a graph without any of its edges intersecting (ie a graph which can be drawn this way)

Such representation is called the planar representation of a graph

- A non-planar graph is a graph which can't be represented without any of its edges intersecting.

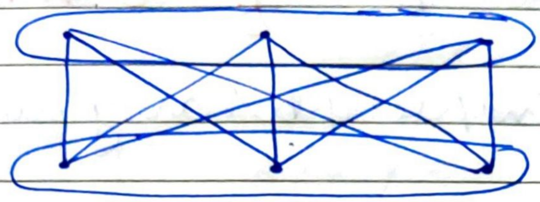
→ KURATOWSKI'S GRAPH (NON-PLANAR GRAPHS)

1. Kuratowski's first graph: A complete graph with 5 vertices K_5



→ non-planar

2. Kuratowski's second graph: complete bipartite graph $K_{3,3}$



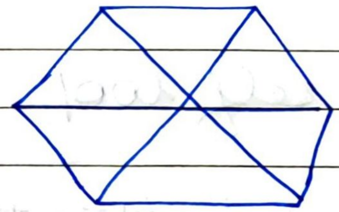
V_1

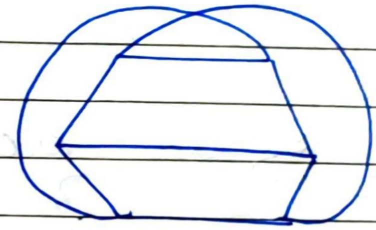
$V = V_1 \cup V_2$

V_2

$V_1 \cap V_2 = \phi$

Its isomorphic representation is

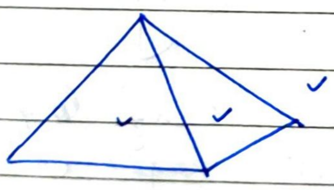




non-planar

→ DEFINITIONS

1. FACE / REGION (f)



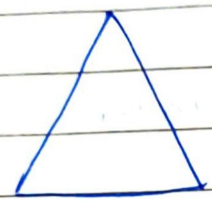
we have 3 regions/faces, 2 are finite, one is infinite.

- Every graph has an ∞ face/region.
- Degree of faces - The ~~no.~~ no. of edges encountered while traversing the boundaries of a face is called its degree

imp
- The sum of degree of faces in a graph is twice its no. of edges.

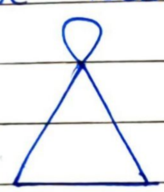
- Euler's Theorem: A connected planar graph with 'n' vertices and 'e' edges will have ~~f~~ $f = e - n + 2$

PROOF: consider simple planar graph



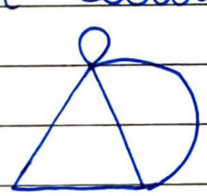
$f = 3 - 3 + 2 = 2$
$\therefore$ this holds true

If we add a self loop



$f = 4 - 3 + 2 = 3$
$\therefore$ Again true

If we add a // edge



$f = 5 - 3 + 2 = 4$
$\therefore$ Again true

$\therefore$ Adding any no. of self loops or // edges will still be governed by given eqn.

Hence proved.

Corollary (extension) to Euler's Theorem

In any simple (planar) graph with 'f' regions, 'n' vertices & 'e' edges ($e > 2$)

$$e \geq 3/2 f \quad , \quad e \leq 3n - 6$$

PROOF:

$$\sum Deg(faces) = 2e$$

$$\Rightarrow 2e \geq 3f$$ (since every face must at least have 3 edges)

$$\Rightarrow \boxed{e \geq \frac{3f}{2}}$$

$$\therefore e \geq \frac{3}{2}(e - n + 2)$$

$$\Rightarrow 2e \geq 3e - 3n + 6$$

$$\Rightarrow \boxed{e \leq 3n - 6}$$

Hence, proved.

- eg – For Kuratowski's graphs

K_5, $\longrightarrow$ $e = 10$ $v = 5$

$e \leq 3n - 6$

$10 \leq 9$ not true

$\therefore$ it is not planar

For $K_{3,3}$ → ~~...~~ $e = 9, v = 6$

~~........~~

$e \leq 3n - 6$

$9 \leq 12$ true.

but for condition

$$e \geq 3/2 \, f$$

$f = 5$ $\quad 18 \geq 15$

Both conditions are satisfied. But this does not mean graph is planar.

* These 2 conditions are not sufficient for graph to be planar.

• Another corollary

If a connected planar simple graph has 'e' edges and 'n' vertices ($n \geq 3$) and no circuits of length 3, then $e \leq 2n - 4$

PROOF:

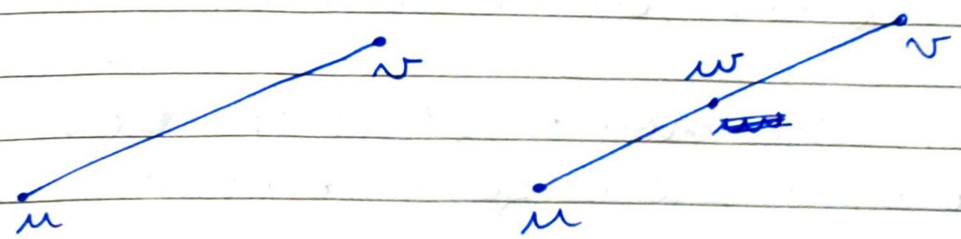
$\sum$ Degree (faces) $\leq 4f$ (since every face must now at least have 4 edges)

$\Rightarrow 2e \leq 4f$

$\Rightarrow 2e \leq 4(e - n + 2)$

$\Rightarrow e \leq 2n - 4$

→ ELEMENTARY SUBDIVISION (HOMEOMORPHIC GRAPHS)

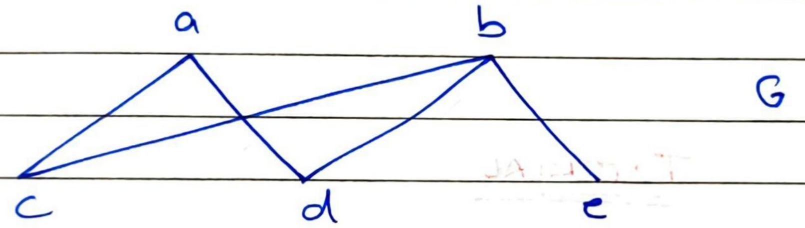


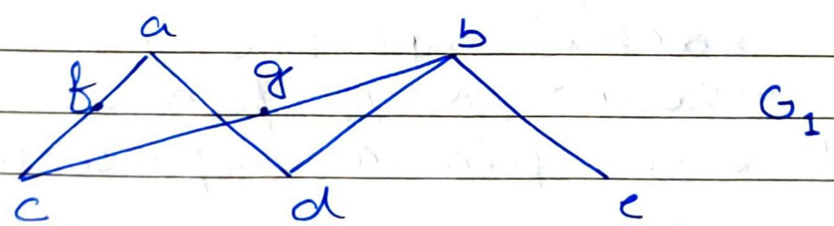
$G(V, E)$

$G_1 \equiv (V_1, E_1)$ $\quad G_2 \equiv (V_2, E_2)$

as homeomorphic graphs, if G_1 and G_2 can be derived from G using a sequence of elementary subdivision.

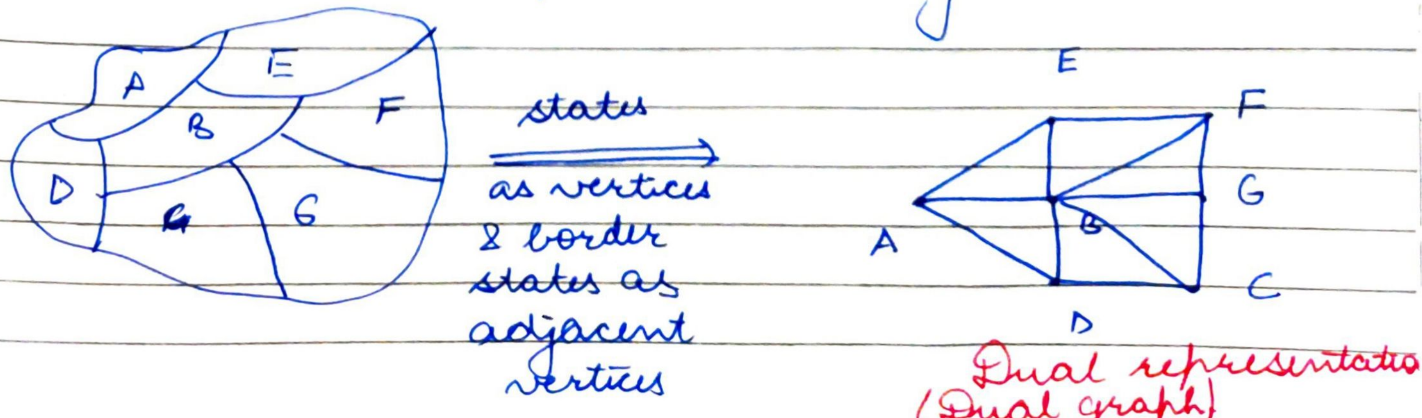
eg-





→ GRAPH COLOURING

- consider states in a country



Dual representation
(Dual graph)

- coloring - no 2 adjacent vertices have the same colour.

- chromatic no. - It is the least number of colours used for colouring $\chi(G)$

- FOUR COLOUR THEOREM

Least number of colours for colouring a planar graph is 4.
(ie for any planar graph, it can be coloured in 4 colours)

Que: Find chromatic no. of K_n

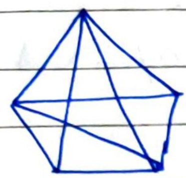
Ans: n

TUTORIAL

1 Que: A planar graph (connected) has 6 vertices, each of deg = 4. Into how many regions is the plane divided by a planar representation of graph

PTO

2Que: Is given graph planar



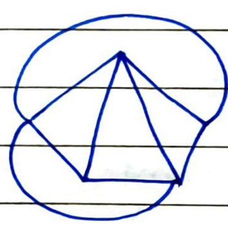
Ans: yes.

$$e \le 3n - 6$$

$$9 \le 15 - 6$$

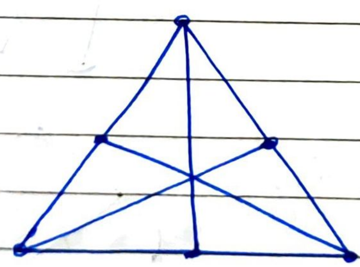
$$9 \le 9$$

$\therefore$ it holds



imp

3Que: planar or not?



Ans: Since, we have no face with 3 edges,
$\therefore$ our inequality is

$$e \le 2v - 4$$

$$6 \le 12 - 4$$

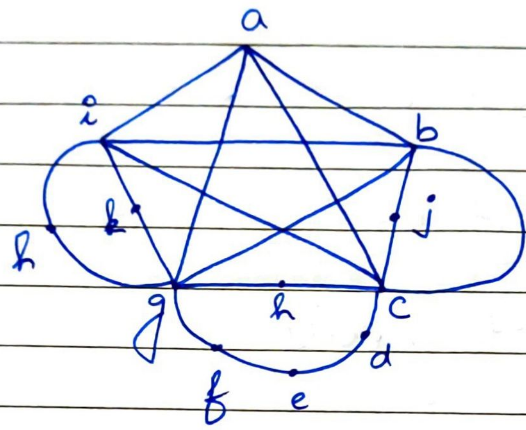
$$6 \le 8$$

Does not hold

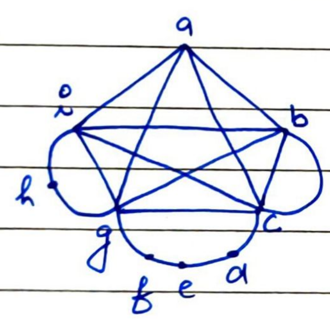
KURATOWSKI THEOREM

- A graph is non planar if and only if it contains a subgraph homeomorphic to $K_{3,3}$ or K_5

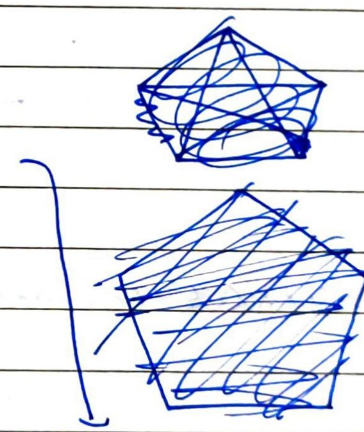
eg-

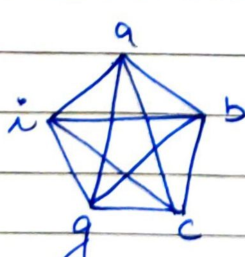


removing $\cong$ vertices



removing $\cong$ edges

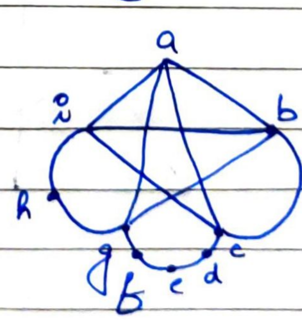




ie

K_5

removing $\cong$ vertices

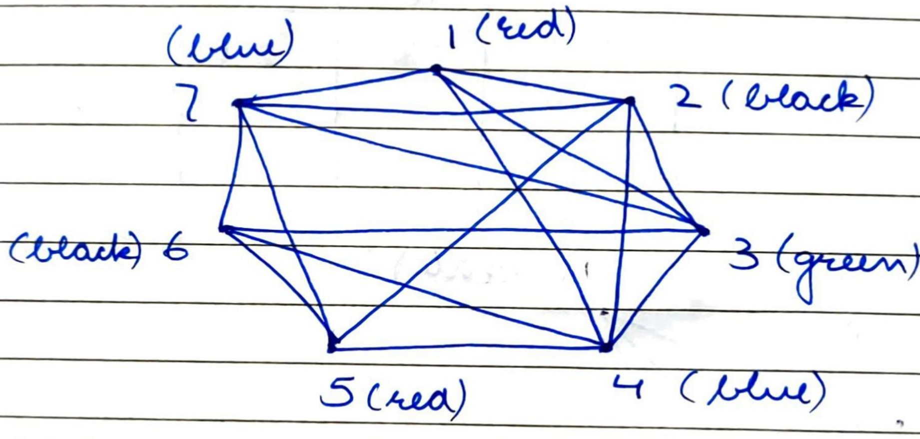


4 Que: Suppose there are 7 finals to be scheduled. These 7 courses have common students. These pairs of courses have common students

1 and 2, 1 and 3, (1,4), (1,7), (2,3), (2,4), (2,5), (2,7), (3,4), (3,6), (3,7), (4,5), (4,6), (5,6), (5,7), (6,7)

How can the exams be conducted so that there are no clashes?

Ans:



Using graph colouring, we can colour using 4 colours.

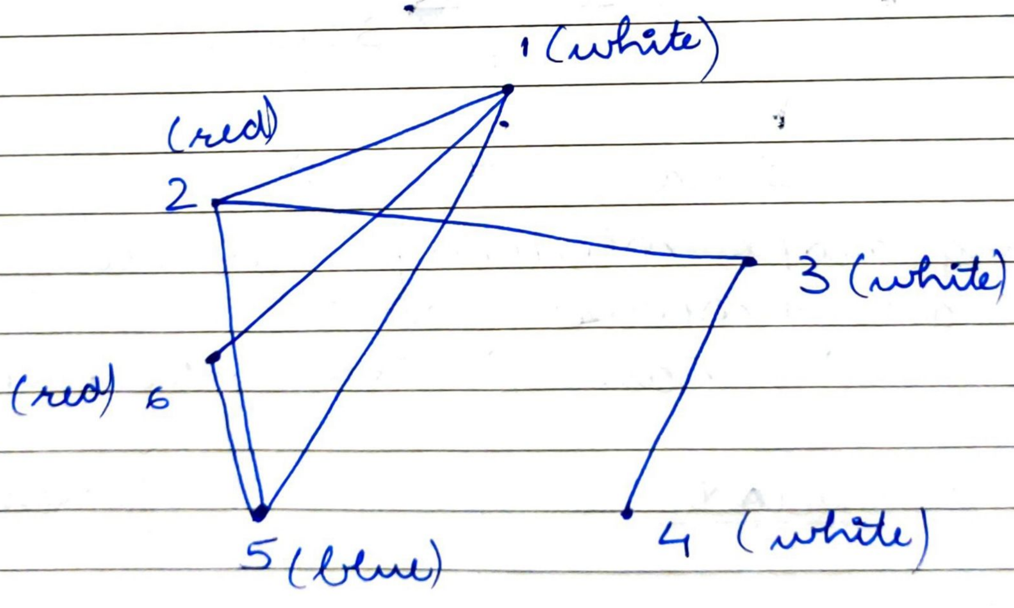
$\therefore$ We can finish within 4 days

Time period	Courses
I	1, 5
II	2, 6
III	3
IV	4, 7

5 Que: ~~How~~ many channels are needed for 6 stations located at the distances shown in table? 2 stations can't ~~be connect~~ use same channel if they are within 150 miles of each other

Ans:

	1	2	3	4	5	6
1	-	85	175	200	50	100
2		—	125	175	100	160
3			—	100	200	250
4				-	210	220
5					-	100
6						—



Channel	Station
1	1, 3, 4
2	2, 6
3	5

NOTE :

$$|E(G)| + |E(\bar{G})| = \frac{n(n-1)}{2}$$

TUTORIAL

1 Que: A simple graph G has 10 vertices & ~~28~~ 21 edges. Find no. of edges in $\bar{G}$.

2 Que: A simple graph ~~G~~ has ~~28~~ 30 edges. $\bar{G}$ has 36 edges. Find no of vertices in G.

1 Ans: $^{10}C_2 = \frac{\cancel{10}^{5} \times 9}{\cancel{2}} = 45$

$\therefore$ no. of edges in $\bar{G} = 45 - \cancel{36}\,21 = \cancel{9}\,24$

2 Ans $30 + 36 = {}^{n}C_2 = \frac{n(n-1)}{2}$

$66 = \frac{n^2 - n}{2}$

$132 = n^2 - n$

solving $\rightarrow$ $\underline{n = 12}$

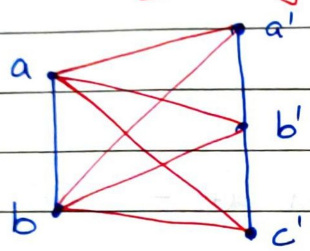
$n^2 - n - 132 = 0$

$(n-12)(n+11) = 0$

NOTE: G^k can be defined as a graph in which we introduce an edge within G between all pairs of vertices which have a dist k (at most)

ie k^{th} power of an undirected graph G ie (G^k) is another graph with same set of vertices but in which 2 vertices are adjacent when their distance in G is at most k.

- Sum of 2 graphs:

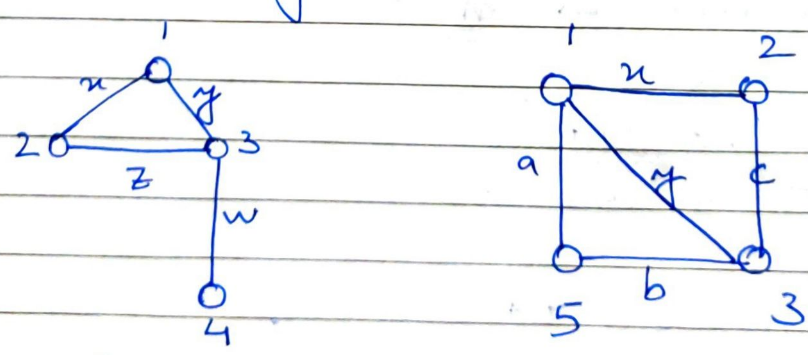


$G_1 + G_2 \equiv v(G_1 + G_2) = v(G_1) + v(G_2)$

$E(G_1 + G_2) = E(G_1) \cup E(G_2)$

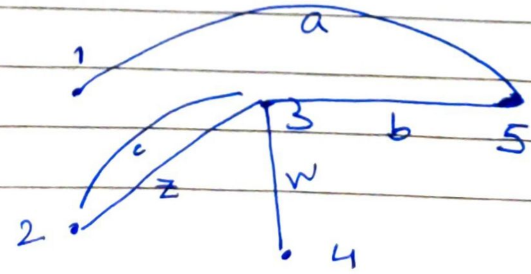
ie there exists an edge b/w every pair of vertices (u, v) where $u \in G$ and $v \in G$

3Que: Find ringsum of these 2 graphs



G_1 G_2

Ans:



- Product of 2 graphs

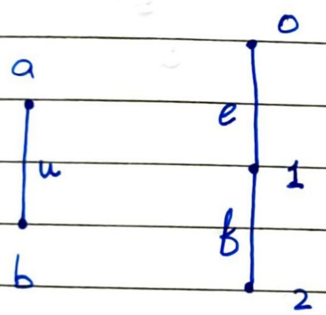
- $G_1 \times G_2$

$$V(G_1 \times G_2) = V(G_1) \times V(G_2)$$

$$E(G_1 \times G_2) = V(G_1) \times E(G_2) \cup V(G_2) \times E(G_1)$$

$V(G_1) = \{a, b\}$ $\quad$ $V(G_2) = \{0, 1, 2\}$

$E(G_1) = \{u\}$ $\quad$ $E(G_2) = \{e, f\}$



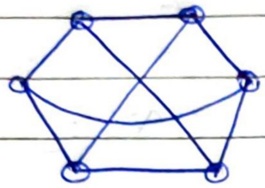
$$V(G_1 \times G_2) = \{(a,0), (a,1), (a,2), (b,0), (b,1), (b,2)\}$$

$$V(G_1) \times E(G_2) = \{(a,e), (a,f), (b,e), (b,f)\}$$

$$V(G_2) \times E(G_1) = \{(0,u), (1,u), (2,u)\}$$

TUTORIAL

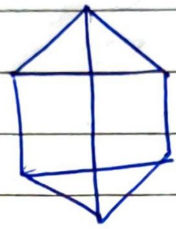
1. Que: Check for planarity



$e \leq 2v - 4$

$9 \leq 8$

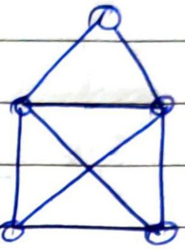
✗



$e \leq 3v - 6$

$9 \leq 12$

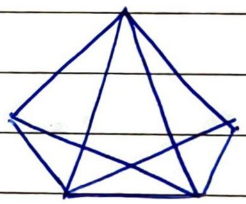
✓



$8 \leq 3(5) - 6$

$8 \leq 9$

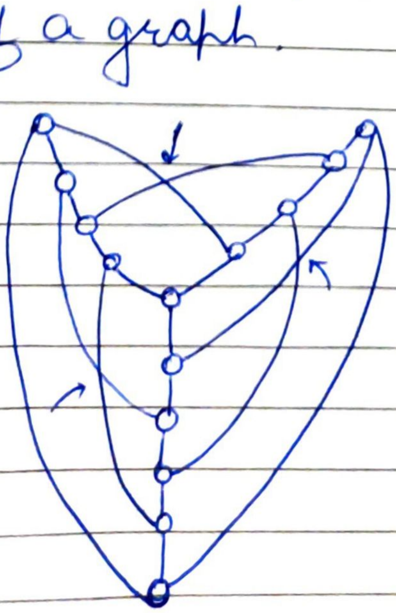
✓



✗

~~2Que~~

Crossing number of a graph: Minimum number of edges which are crossing each other in a planar representation of a graph.



In this eg crossing no. = 3

classmate

Date

Page

NOTE $C_r(K_p) \le \frac{1}{4} \lfloor p/2 \rfloor \lfloor \frac{p-1}{2} \rfloor \lfloor \frac{p-2}{2} \rfloor \lfloor \frac{p-3}{2} \rfloor$

$C_r(K_{m,n}) \le \lfloor \frac{m}{2} \rfloor \lfloor \frac{m-1}{2} \rfloor \lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor$

Que: Find upper bound of crossing pt for :-

i) K_5 $\Rightarrow C_r(K_5) \le \frac{1}{4} \lfloor \frac{5}{2} \rfloor \lfloor \frac{4}{2} \rfloor \lfloor \frac{3}{2} \rfloor \lfloor \frac{2}{2} \rfloor$

≤ 1

ii) K_6 $\Rightarrow C_r(K_6) \le \frac{1}{4} \lfloor \frac{6}{2} \rfloor \lfloor \frac{5}{2} \rfloor \lfloor \frac{4}{2} \rfloor \lfloor \frac{3}{2} \rfloor$

≤ 3

iii) K_7 $\Rightarrow C_r(K_7) \le \frac{1}{4} (3) \times 3 \times 2 \times 2 \le 9$

iv) $K_{3,4}$ $\Rightarrow C_r(K_{3,4}) \le \lfloor \frac{3}{2} \rfloor \lfloor \frac{2}{2} \rfloor \lfloor \frac{4}{2} \rfloor \lfloor \frac{3}{2} \rfloor$

≤ 2

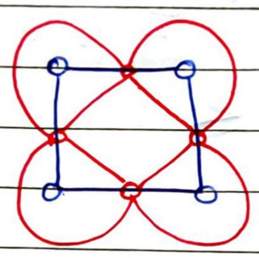
v) $K_{4,4}$ $\Rightarrow C_r(4,4) \le 4$

vi) $K_{5,5}$ $\Rightarrow C_r(5,5) \le$

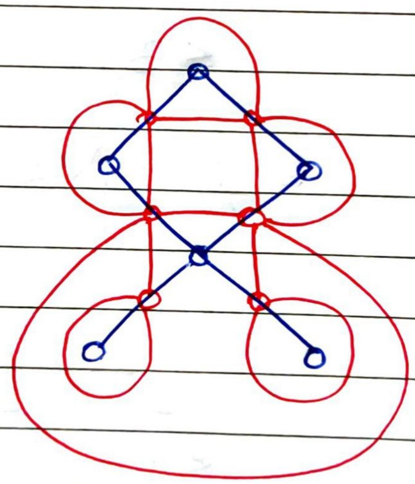
medial graph: medial graph of a planar graph G is a graph denoted by M(G) that represents adjacency b/w edges in the faces of G

Basically in a medial graph, we take edges as vertices & vertices as edges. (We join the edges ~~both~~ from all faces common to them

eg -



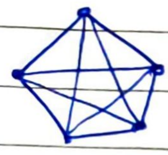
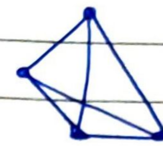
NOTE: Pendant vertices (Degree 1) have their edges joined as self loops in medial graph



Que Which of the non-planar graph has the property that removal of any one vertex produces a planar graph

a) K_5 b) K_6 c) $K_{3,3}$ d) $K_{3,4}$

i) K_5

 $\xrightarrow{\text{removal}}$ 

↓

planar

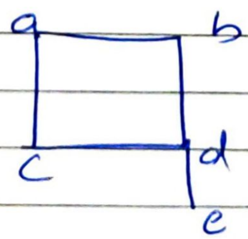
ii)

\# **Independent Set:** Set of vertices S which itself is a subset of v s.t no two vertices are adjacent

\# **Maximal Indep. Set:** If no more vertices can be added to it without breaking the max. independent set property.

\# **Independence no. :** no. of elements present in maximal independent set.

eg-



{a,d} and {c,b}

{a,d} , $\underbrace{\{c,b,e\}}_{max}$

IN = 3

Chapter - 3
Relations

- Consider 2 sets A and B. Then any relation R is a subset of Cartesian product of A and B

- A, B $\quad A \times B \rightsquigarrow$ Cartesian product (cross product)

- $R \subseteq A \times B = \{ (a,b) \mid a \in A, b \in B \}$
  ordered pair of sets

- eg- $A = \{1, 2, 3\}, \quad B = \{4, 5\}$

  $A \times B = \{(1,4), (1,5), (2,4), (2,5), (3,4), (3,5)\}$

  $B \times A = \{(4,1), (4,2), (4,3), (5,1), (5,2), (5,3)\}$

- For any two sets A, B not necessarily distinct any subset of $A \times B$ is said to be a <u>relation</u> from A to B

- $a, b \in R$
  $\Rightarrow a R b \quad , a \in A \; \& \; b \in B$
  $a \not R b \quad , (a,b) \notin R$

- eg- $A = \{1, 2, 3, 4\}$

  $R = \{ (a,b) \mid a \text{ divides } b \; \& \; a, b \in A \}$

  $R = \{ (1,1), (1,2), (1,3), (1,4), (2,2), (2,4), (3,3), (4,4) \}$

- $A = \{1,2,3,4\}$, $R_1 = \{(1,1),(1,2),(2,1),(2,2),(3,4),(4,1),(4,4)\}$
$R_2 = \{(1,1),(1,2),(2,1)\}$, $R_3 = \{(1,1),(1,2),(1,4),(2,1),(2,2),(3,3),(4,1),(4,4)\}$
$R_4\ \{(1,1),(2,1),(2,2),(3,1),(4,1),(4,2),(4,4)\}$, $R_5 = \{(3,4)\}$

$R_6 = \{(2,1),(3,1),(3,2),(4,1),(4,2),(4,3)\}$

- **Reflexive:** A relation R on a set 'A' is said to be reflexive if $(a,a) \in R$ for all $a \in A$. eg- R_3

- **Irreflexive:** $(a,a) \notin R$ for any $a \in A$. eg- R_5, R_6

- **Symmetric:** $(b,a) \in R$ for all $(a,b) \in R$ $\forall a,b \in A$
eg- R_2 and R_3

- **Anti symmetric:** $(a,b) \in R$ & $(b,a) \in R$, if & only if $a = b$
$\forall a,b \in A$ & if $(a,b) \in R$ & $a \neq b$, $(b,a) \notin R$
eg- R_4, R_5, R_6

- **Assymetric:** If $(a,b) \in R$, then $(b,a) \notin R$
$\forall (a,b) \in R$ $\forall a,b \in A$
eg- R_6 and R_5

- **Transitive:** $(a,b) \in R$ & $(b,c) \in R \Rightarrow (a,c) \in R$
$\forall a,b,c \in A$ (a & c are not equal)
eg- R_4, R_5, R_6

→ OPERATIONS ON RELATIONS

- $A = \{1,2,3\}$, $B = \{1,2,3,4\}$

$R_1 = \{(1,1),(2,2),(3,3)\}$
$R_2 = \{(1,1),(1,2),(1,3),(1,4)\}$

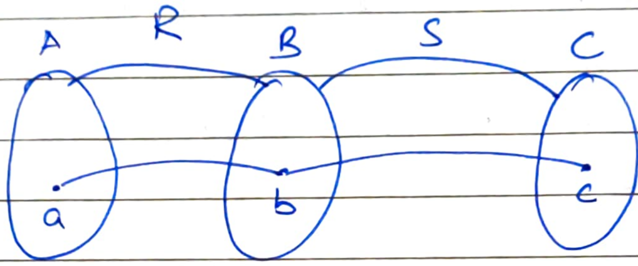
$R_1 \cup R_2 = \{(1,1),(2,2),(3,3),(1,2),(1,3),(1,4)\}$

$R_1 \cap R_2 = \{(1,1)\}$

$R_1 - R_2 = \{(2,2),(3,3)\}$

$R_2 - R_1 = \{(1,2),(1,3),(1,4)\}$

→ COMPOSITION OF RELATIONS



$R \subseteq A \times B$

$S \subseteq B \times C$

$\underline{a\,(R \circ S)\,c}$ if $\exists\, b \in B$ s.t. aRb and bSc

↳ R composition S

eg- Let $A = \{1, 2, 3, 4\}$

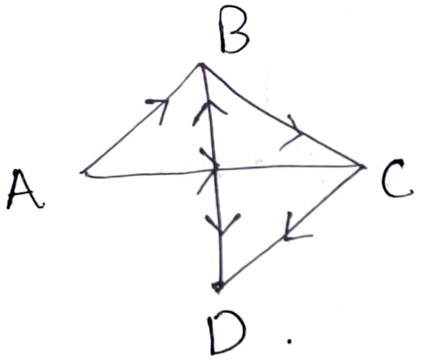
$R = \{(1,2),(1,3),(2,4)\}$: $1R2$, $1R3$, $2R4$

$S = \{(1,2),(2,3),(4,4)\}$: $1S2$, $2S3$, $4S4$

$R \circ S = \{(1,3),(2,4)\}$

REPRESENTING RELATIONS

$G = \langle V, E \rangle$; $E \subseteq V \times V$.

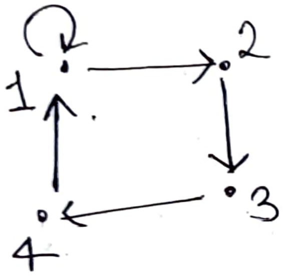


$V = \{A, B, C, D\}$

$E = \{(A,B), (B,C), (C,D), (A,C), (B,D), (D,B)\}$.

$G = \langle V, E \rangle$.

$A = \{1,2,3,4\}$; $R = \{(1,1), (1,2), (2,3), (3,4), (4,1)\}$.



DIGRAPHS

$A = \{a_1, a_2, \text{——}, a_n\}$

$B = \{b_1, b_2, \text{——}, b_m\}$.

R	a_1	a_2	a_3	...	a_n
b_1	m_{11}	m_{12}		—	m_{1n}
b_2	m_{21}			—	m_{2n}
⋮					
b_m	m_{m1}			—	m_{mn}

$$m_{ij} = \begin{cases} 1 & (a_i, a_j) \in R \\ 0 & (a_i, a_j) \notin R \end{cases}$$

Relation $\leq$ on the set $\{1,2,3,4\}$.

R	1	2	3	4
1	1	1	1	1
2	0	1	1	1
3	0	0	1	1
4	0	0	0	1

$R_1 = \{(a,b) \mid a \leq b\}$

$R_2 = \{(a,b) \mid a > b\}$

$R_3 = \{(a,b) \mid a = b \text{ or } a = -b\}$.

$R_4 = \{(a,b) \mid a = b\}$.

$R_5 = \{(a,b) \mid a = b+1\}$

$R_6 = \{(a,b) \mid a + b \leq 3\}$

Which of the relations contain each of the pair $(1,1), (1,2), (2,1), (1,-1)$ and $(2,2)$.

$(1,1) \downarrow R_1, R_3, R_4, R_6$

$(1,2) \downarrow R_1, R_6$

$(2,1) \downarrow R_2, R_5, R_6$

$(1,-1) \downarrow R_2, R_3, R_6$

$(2,2) \downarrow R_1, R_3, R_4$,

N-ARY Relation :

$A_1, A_2, \_\_\_, A_n$ Domain of the relation

$R \subseteq A_1 \times A_2 \times A_3 \ldots \times A_n$

degree = n

Let R be the relation on $N \times N \times N$ consisting of triples where $a, b, c \in \mathbb{Z}$ & $a < b < c$.

(a, b, c)

$(1,2,3) \in R$; $(2,4,3) \notin R$.

CLOSURE OF RELATIONS :- $R \subseteq A \times A$

P: reflexive, symmetric & transitive

Def: The closure of a relation R on a set w.r.t. a given P will be a set R_p s.t. $R \subseteq R_p \subseteq A \times A$.

Reflexive:- Closure $R = \{(1,1), (1,2), (2,1), (3,2)\}$ on $A = \{1,2,3\}$.

$R_p = \{(1,1), (1,2), (2,1), (3,2), (2,2), (3,3)\}$.

$R \subseteq R_p \subseteq A \times A$.

Symmetric closure :- $R = \{(1,1), (1,2), (2,2), (2,3), (3,1), (3,2)\}$ on $\{1,2,3\}$.

$R_p = \{(1,1), (1,2), \underline{(2,1)}, (2,2), (2,3), (3,1), \underline{(1,3)}, (3,2)\}$

Transitive Closure :- $R = \{(1,3), (1,4), (2,1), (3,2)\}$ on $A = \{1,2,3,4\}$

$R_p' = \{(1,3), (1,4), (2,1), (3,2), (1,2), (2,3), (2,4), (3,1)\}$ ~~(1,1)~~

~~(3,3), (3,4)~~

$R_p = \{(1,3), (1,4), (2,1), (3,2), (1,2), (2,3), (2,4), (3,1), \cancel{(3,3)}$
$(3,4), \cancel{(1,1)}, \cancel{(2,2)},$

R_1 & R_2 $\quad M_{R_1}$ $\quad M_{R_2}$ $\quad M_{R_1 \cup R_2}$ $\quad M_{R_1 \cap R_2}$ ..

- **Path:** A path from a to b is a sequence of edges $(x_0, x_1), (x_1, x_2), (x_2, x_3), \ldots, (x_{n-1}, x_n)$

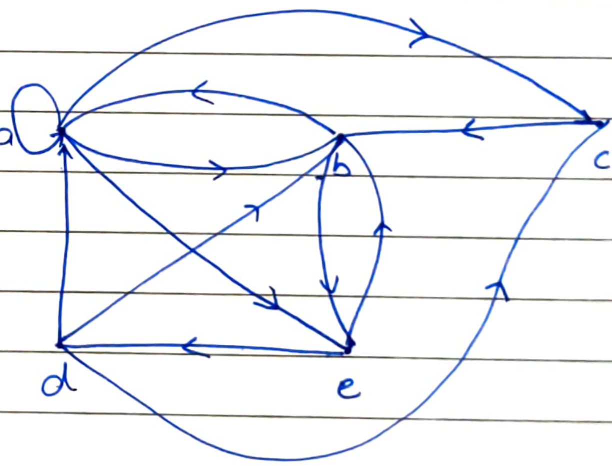
where n = non-negative $x_0 = a$ and $x_n = b$

$x_0, x_1, \ldots, x_{n-1}, x_n$

length = n

- **Cycle:** $n \geq 1$ and terminal vertices are same.

Que:



Find lengths (if exists) for:

i) a, b, e, d
(ii) a, e, c, d, b
(iii) b, a, c, b, a, a, b

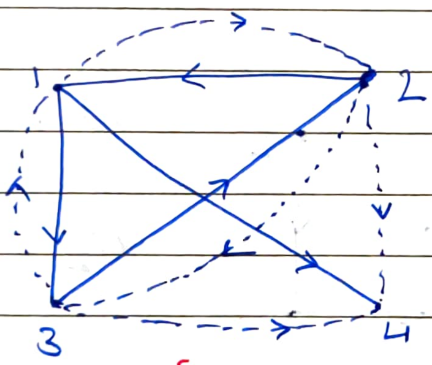
Ans: L=3 → for (i)
(e, c) $\notin$ E → for (ii)
L=6 → for (iii)

→ DRAWING CLOSURE GRAPHS

- In a relation's representation of a graph, we say that a and b are in a ~~rel~~ relation R if there is a sequence of elements
  $a, x_1, x_2, \ldots\ldots x_{n-1}, b$
  $(a, x_1) \in R, \ldots\ldots, (x_{n-1}, b) \in R$

- For a relation R to be transitive closure the terminal vertices of all possible paths of a graph must have a common edge. Cycles are not considered.

$R = \{(1,3), (1,4), (2,1), (3,2)\}$ on $\{1, 2, 3, 4\}$

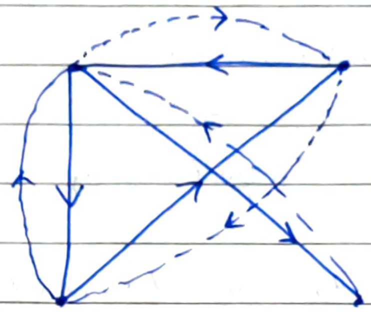


$v_1 : (1,3), (3,2) \rightarrow (1,2)$
$v_2 : (2,1), (1,3) \rightarrow (2,3)$
$(2,1), (1,4) \rightarrow (2,4)$
$v_3 : (3,2), (2,1) \rightarrow (3,1)$
$(3,2), (2,1), (1,4) \rightarrow (3,4)$

transitive closure graph of R

PTO

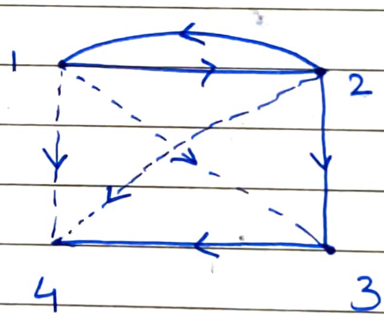
- Symmetric closure for above relation



Que: Draw transitive closure graph for

$R = \{(1,2), (2,3), (3,4), (2,1)\}$ on $\{1,2,3,4\}$

s:



v_1 : $(1,2), (2,3) \rightarrow (1,3)$

$(1,3), (3,4) \rightarrow (1,4)$

v_2 : $(2,3), (3,4) \rightarrow (2,4)$

- Equivalence relation

A relation R in a set A is equivalence relation, if it is reflexive, symmetric and transitive.

eg- $a, b, c \in A$

'=' is equivalence if

$a = a$ reflexive

$a = b \rightarrow b = a$ symmetric

$a = b$ & $b = c \rightarrow a = c$ transitive

→ PARTIAL ORDERINGS

- A relation R on a set A is said to be a partial ordered relation if it is reflexive, anti-symmetric and transitive.

$(A, R) \longrightarrow$ partial ordered set or poset

- eg- $\geq$ on the set of all +ve integers Z

$a \geq a$
if $a \geq b$ & $b \geq a$ when $a = b$
if $a \geq b$ & $b \geq c$ $\rightarrow$ $a \geq c$

$(Z, \geq)$ is a poset.

- notation of poset : $\leq$ and denoted as $(A, \leq)$ for set A

- $\leq \in \{ \geq, \leq, \subseteq, \ldots \}$

- $a, b \in A$
Comparable: if either $a \leq b$ or $b \leq a$
Incomparable: if neither $a \leq b$ nor $b \leq a$

- If 2 elements of a poset are comparable, then it will be a total order / linear / simple order. In that case $(A, \leq)$ will be a chain.

- eg- $(Z, \leq)$ is a chain $Z = \{1, 2, 3, 4, \ldots\}$

TUTORIAL

1 Que: Let R be a relation on set $\{0, 1, 2, 3\}$ containing the ordered pairs

$(0,1), (1,1), (1,2), (2,0), (2,2), (3,0)$

i) Find reflexive closure of R

ii) Find symmetric closure of R

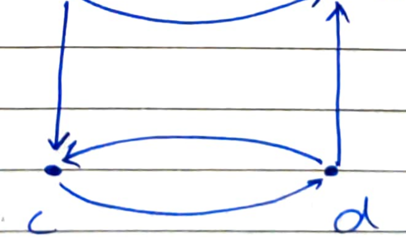
Ans:

i) ~~(0,1), (1,1)~~

$R^{+} = \{(0,1), (1,1), (1,2), (2,0), (2,2), (3,0), (0,0), (3,3)\}$

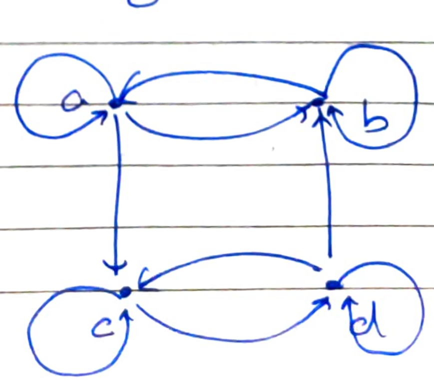
ii) $R^{s} = \{(0,1), (1,1), (1,2), (2,0), (2,2), (3,0), (1,0), (2,1), (0,2), (0,3)\}$

2 Que:



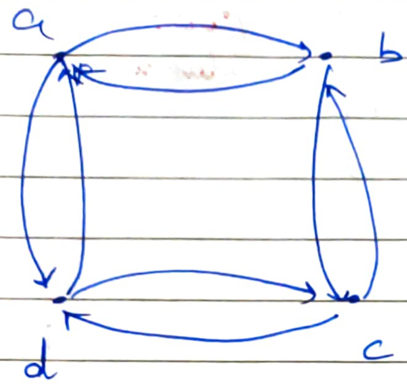
Find its reflexive closure of given directed graph

Ans:



3 Que: Find symmetric closure of above graph.

Ans:



HW
4 Que: Find transitive closure of above graph (Q 2).

imp
5 Que: Use Warshall's algorithm to find the transitive closure of these relation on the set $\{1,2,3,4\}$

a) $\{(1,2),(2,1),(2,3),(3,4),(4,1)\}$

Ans: First we will form adjacency matrix.

$$W_0 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

We will form 4 matrices W_1, W_2, W_3 and W_4.

For W_1 we consider 1^{st} column & 1^{st} row for previous matrix and so on uptil W_4.

PTO

for W_0

p_i	q_j
2	2
4	

position of '1' in first column

position of '1' in first row

$$W_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \end{bmatrix}$$

we take cartesian product of these two

ie $(2,2), (4,2)$

& we check if they are present in matrix. If not, we add them.

for W_1

p_i	q_j
1	1
2	2
4	3

$\Rightarrow (1,1)(1,2)(1,3)(2,1)(2,2)(2,3)$
$(4,1)(4,2)\ (4,3)$

$$W_2 = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

for W_2

p_i	q_j
1	
2	4
4	

$\Rightarrow (1,4)\ (2,4)\ (4,4)$

$$W_3 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

p_i	q_i
1	1
2	2
3	3
4	4

$$W_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

$\therefore$ Closure

$R^+ = \{ \quad \}$

write whole set

6 Que: Use Warshall's algorithm to find transitive closure for relation on $\{a, b, c, d\}$ for relation

$\{(a, d), (b, a), (b, c), (c, a), (c, d), (d, c)\}$

Ans:

$$W_0 = \begin{array}{c} \\ a \\ b \\ c \\ d \end{array}\begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

p_i	q_j
b	d
c	

$\Rightarrow (b, d), (c, d)$

$$W_1 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

p_i	q_j
ϕ	a
	c
	d

$\Rightarrow$ no pairs

$$W_2 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

p_i	q_j
b	a
d	d

$\Rightarrow (b, d), (b, a), (d, a), (d, d$

$$W_3 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix}$$

p_i	q_j
a	a
b	c
c	d
d	

$\Rightarrow (a, a),$
$(a, c), (a, d$
$(b, a), (b, c), (b, d$
$(c, a), (c, c), (c, d$

$$W_4 = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix}$$

$\therefore$ closure $R^T = \{(a,a), (a,c), (a,d), (b,a), (b,c), (b,d)$
$(c,a), (c,c), (c,d), (d,a), (d,c), (d,d)\}$

7Que: Find equivalence relations among these

$R_1 = \{(a,b) \in Z \mid a-b$ is an integer$\}$ ✓

$R_2 = \{(a,b) \in Z \mid a-b$ is divisible by $3\}$ ✓

$R_3 = \{(a,b) \in Z \mid a-b$ is an odd number$\}$ not reflexive

$R_4 = \{(a,b) \in Z \mid a-b$ is an even number$\}$ ✓

Ans: give reasons for equivalence.

for not equivalent, just mention why.

8Que: For set $\{0, 1, 2, 3\}$ find if following relations are equivalence or not

$R_1 = \{(0,0), (1,1), (2,2), (3,3)\}$ ✓

$R_2 = \{(0,0), (0,2), (2,0), (2,2), (3,3), (2,3), (3,2)\}$ not reflexive

$R_3 = \{(0,0), (1,1), (1,2), (2,1), (2,2), (3,3)\}$ ✓

$R_4 = \{(0,0), (1,1), (1,3), (2,2), (2,3), (3,1), (3,2), (3,3)\}$ not transitive

Que: check equivalence

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

Ans: reflexive & symmetric for sure.

For transitive, use Warshall algorithm

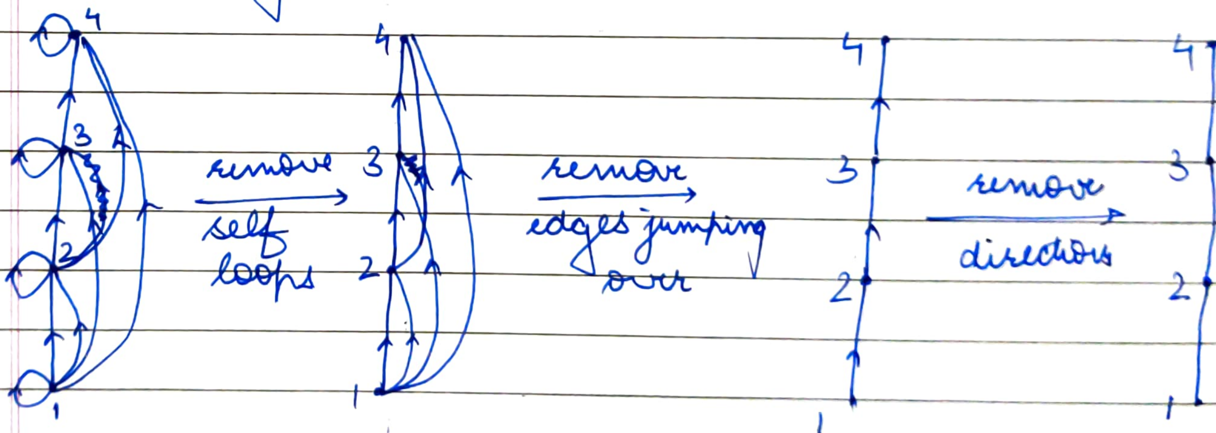
→ HASSE DIAGRAMS

- $\{(1, 2, 3, 4), \leq\}$ ⇝ poset

$\leq = \{(1,1), (1,2), (1,3), (1,4), (2,2), (2,3), (2,4), (3,3), (3,4), (4,4)\}$

- Hasse diagrams are drawn for posets (reflexive, anti-symmetric and transitive)

- Consider we have to draw Hasse diagram for poset mentioned above, first, we draw starting vertex (1) at bottom and terminal vertex (4) at top such that all edges are directed in one direction



Hasse diagram

Que: Construct Hasse diagram of

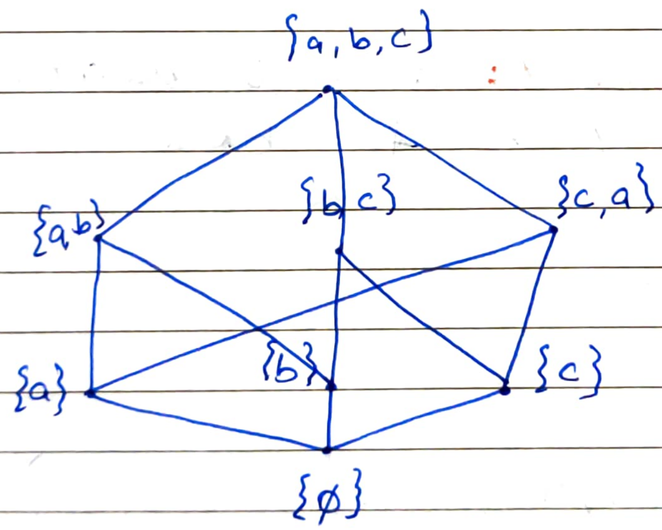
$(P(\{a, b, c\}), \subseteq)$

Ans:

$$P(\{a,b,c\}) = \{\phi, \{a\}, \{b\}, \{c\}, \{a,b\}, \{b,c\}, \{c,a\}, \{a,b,c\}\}$$

$$R = \{(\{\phi\},\{a\}), (\phi,\{b\}), (\phi,\{c\}), (\phi,\{a,b\}), (\phi,\{b,c\}), (\phi,\{c,a\}),$$
$$(\phi,\{a,b,c\}), (\phi,\phi), (\{a\},\{a,b\}), (\{a\},\{c,a\}), (\{a\},\{a,b,c\}),$$
$$(\{b\},\{a,b\}), (\{b\},\{b,c\}), (\{b\},\{a,b,c\}), (\{c\},\{b,c\}), (\{c\},\{c,a\}),$$
$$(\{c\},\{a,b,c\}), (\{a,b\},\{a,b,c\}), (\{b,c\},\{a,b,c\}), \ldots$$

..... baki khud likho 😑 }



classmate
Date
Page

→ SOME DEFINITIONS

- maximal element: $a \in A$ is called maximal if there does not exist $b \in A$ such that aRb

  eg- In above que, $\{a, b, c\}$ is maximal

- minimal element: $a \in A$ $\nexists$ $b \in A$ st. bRa

  eg- In above que, ϕ

- Least element: $a \in A$ is a least element if for every $b \in A$ aRb

  eg- $\{\phi\}$

- Greatest element: $a \in A$ is greatest element if for every $b \in A$ bRa

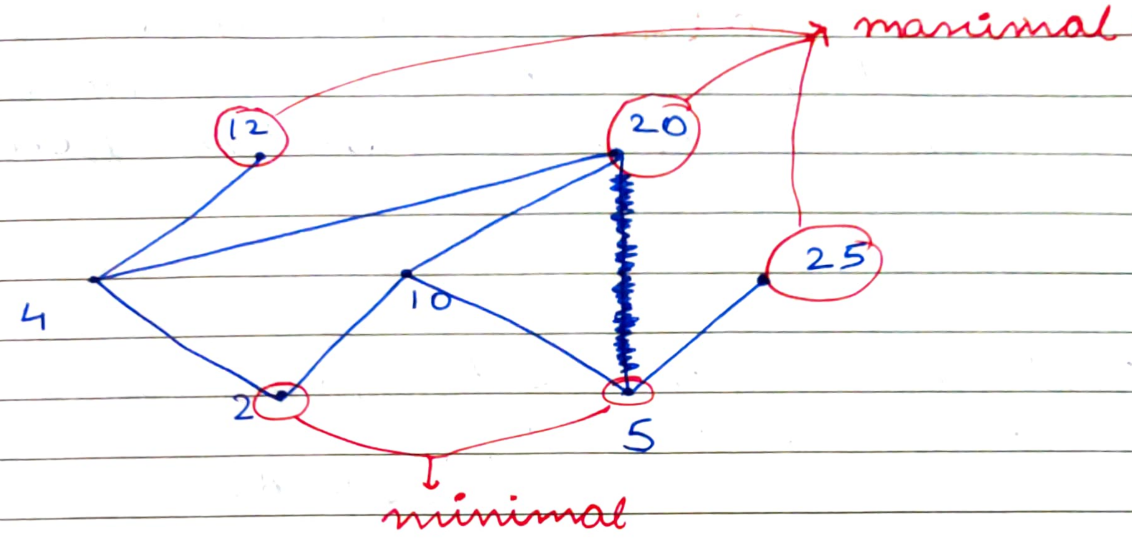
  eg- $\{a, b, c\}$

- Example

$$(\{2, 4, 5, 10, 12, 20, 25\}, /)$$

$R \rightarrow a$ divides b

$R = \{(2,2), (2,4), (2,10), (2,12), (2,20), (4,4), (4,12), (4,20), (5,5), (5,10), (5,20), (5,25), (10,10), (10,20), (12,12), (20,20), (25,25)\}$

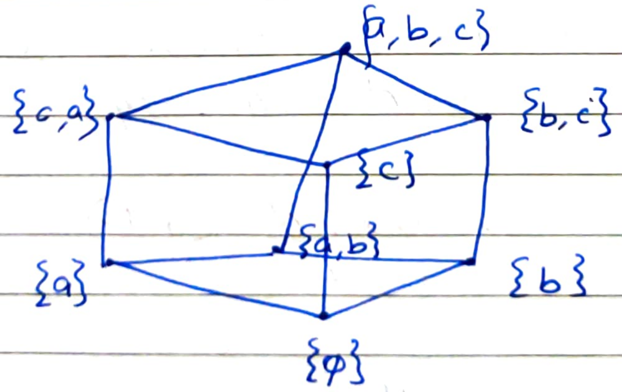


no least or greatest elements.

- upper bound: consider a subset S in (A, R) in poset

If there is an element a ∈ A such that sRa where s ∈ S, then a will be called an upper bound.

eg- for



{a, b, c} is upper bound.

- Least upper ~~Classes~~ bound(lub) : Let a be an upper bound of subset S. a will be lub if aRz where z is an upper bound for S.

Que: Find lub of the set (3, 9, 12) if they exist on poset $(Z^+, |)$ divides

$a \in Z^+$ and sRa where $s \in \{3, 9, 12\}$: s divides a

Ans: $a \in \{36, 72, \ldots\}$ LCM of 3, 9, 12

lub(s)
$l \in S$ s.t. lRz where $z \in \{36, 72, \ldots\}$
lub(s) : 36

- Lower bound: Consider a subset S in (A, R) If there is an element $a \in A$ such that aRs where $s \in S$, then a will be called a lower bound. eg - $\{\phi\}$

- greatest lower bound (glb) : l is a lower bound for subset S. l will be glb if z R l whenever 'z' is the lower bound of 'S'

Que: In prev. que,

$a \in Z^+$ s.t. a divides $s \in \{3, 9, 12\}$ aRs

Ans: glb : 3.

Que: glb and lub of $\{1, 2, 4, 5, 10\}$ on the poset $(Z^+, |)$

means divides.

Ans. UB : $\{20, 40, 60\}$
lub: 20 → LCM
LB : $\{1\}$
glb : 1 → HCF

→ LATTICE

- A partially ordered set in which every pair of elements has a lub and glb
- $(L, \leq)$ where (a, b)
- lub(a, b) by $a \vee b$ (the join of a & b)

  glb (a, b) $a \wedge b$ (the meet of a & b)

- eg- poset $(Z^+, |)$ is a lattice.

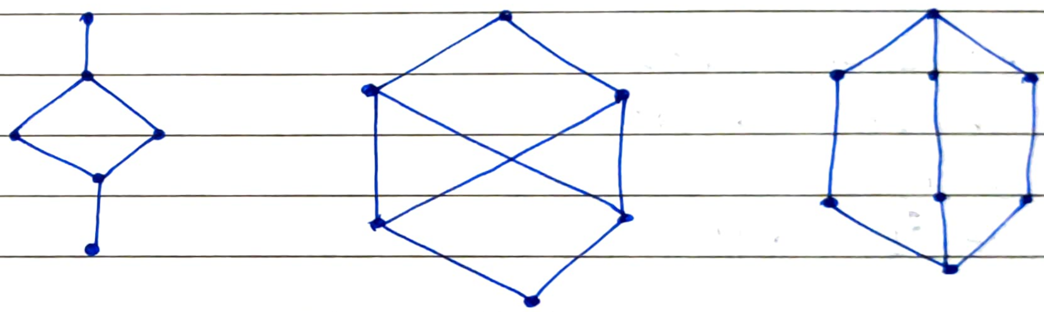
- eg- $(\{1, 2, 4, 8, 16\}, |)$

for any 2 elements for glb, HCF is within the set and LCM too for lub. ∴ its a lattice

- eg $(\{1, 2, 3, 4, 5\}, |)$

LCM does not exist for any 2 elements ie lub does not exist ∴ its not a lattice.

Que : Which of these is a lattice?



(a) (b) (c)

Ans:

TUTORIAL

1 Que: given $A = \{1, 2, 3\}$

$R_1 = \{\ \}$

$R_2 = \{(1,1), (2,2), (3,3)\}$

$R_3 = \{(1,1), (2,2), (3,3), (1,2), (2,1)\}$

$R_4 = \{(1,1), (2,2), (3,3), (1,3), (2,3)\}$

$R_5 = \{(1,1), (1,2), (2,3), (1,3)\}$

$R_6 = A \times A$

which of them is a poset?

Ans: $\cancel{R_4}$

R_1 — not a poset. not reflexive
R_2 — poset
R_3 — not a poset. not anti-symmetric
R_4 — poset
R_5 — not a poset. not reflexive
R_6 — not a poset. not anti-symm.

2 Que: $R_1 = \{(a,b) \mid a, b \in Z, a < b\}$ ×

$R_2 = \{(a,b) \mid a, b \in Z, a \leq b\}$ ✓

$R_3 = \{(A,B) \mid A, B \in P(x), A \subseteq B\}$ ✓

$R_4 = \{(a,b) \mid a, b \in Z\ b/a$ ✓

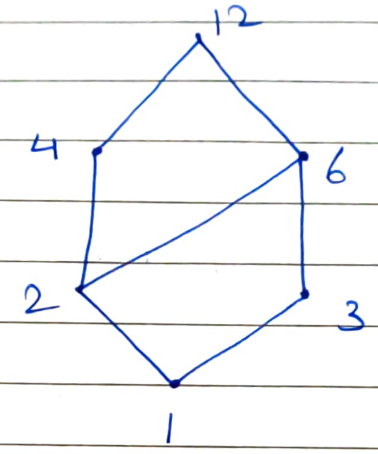
NOTE: To make Hasse dia,
a) remove self loop
b) remove transitive connections
c) remove direction

3 Que:

$[\{1, 2, 3, 4, 6, 12\}, |]$

Draw Hasse Diagram.

Ans

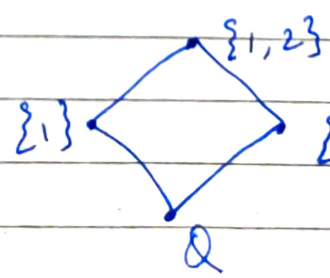


NOTE: While ~~constructing~~ adding a new element to Hasse Diagram, ~~see~~ look for its relation with top most level. In case relation is there, keep it just above that level else move to the next lower level & look for relations there and so on.

4 Que: Draw Hasse dia. for

$[\{\phi, \{1\}, \{2\}, \{1,2\}\}, \subseteq]$

Ans

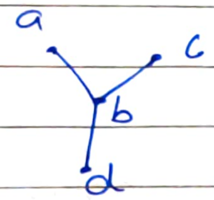


NOTE: For maximal - There should be no ~~element at~~ edge going upwards. Only 1 maximal makes it maximum

For minimal - There should be no edge going downwards. Only 1 minimal makes it minimum

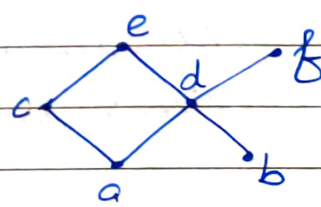
Ques: Find maximal & minimal elements

i)



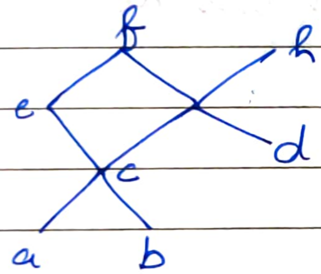
maximal - a, c
minimal - d
minimum - d

ii)



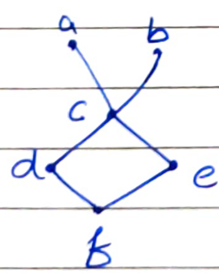
maximal - e, f
minimal - a, b

iii)



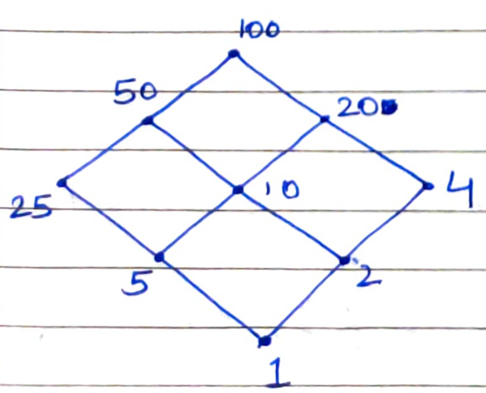
maximal - f, h
minimal - a, b, d

iv)



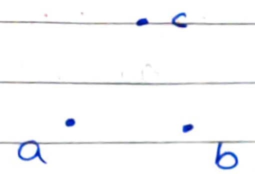
maximal - a, b
minimal - f
minimum - f

v)



maximal - 100
minimal - 1
maximum - 100
minimum - 1

vi)

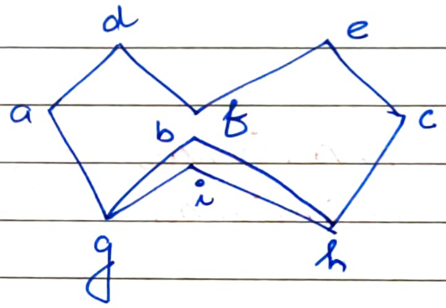


Maximal - a, b, c
Minimal - a, b, c

vii)

maximal - d
minimal - d
maximum - d
minimum - d

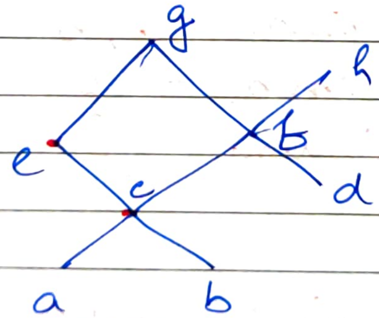
viii)



maximal - d, e, f, i
minimal - g, h

Que: Find upper & lower bound

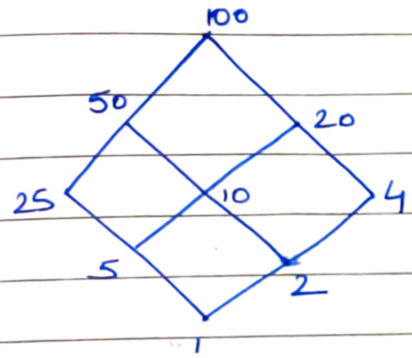
a)



i) $B = \{c, e\} \rightarrow UB = \{g, e\}$, $LB = \{a, b, c\}$

ii) $B = \{e, f, d) \rightarrow UB = \{h, g, f\}$, $LB = \phi$

b)

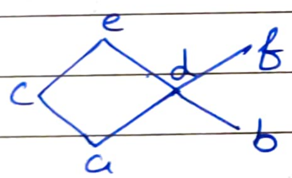


i) $B = \{50, 10\}$, $UB = \{50, 100\}$, $LB = \{10, 5, 1, 2\}$

ii)

7 Que. Find LUB & GLB

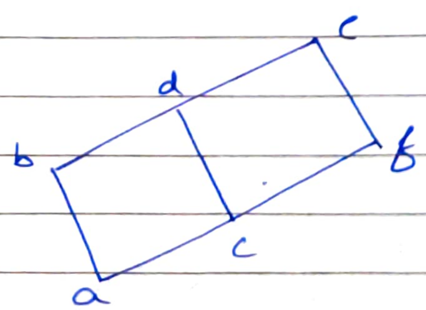
a)



i) $B = \{c, d\}$, $UB = \{e\}$, $LB = \{a\}$, $LUB = e$, $GLB = a$

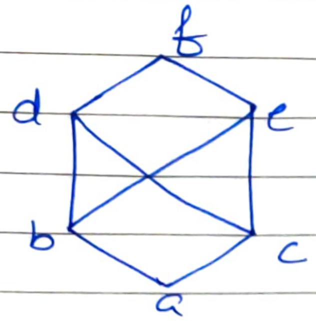
ii) $B = \{a, b\}$, $UB = \{d, e, f\}$, $LB = \phi$, $LUB = d$,
$GLB = \phi$

b)



i) $B = \{a, c, f\}$, $UB = \{f, e\}$, $LB = \{a\}$ $LUB = f$
$GLB = a$

c)



i) $B = \{d, e\}$, $UB = \{f\}$, $LB = \{a, c, b\}$, $LUB = \{f\}$,

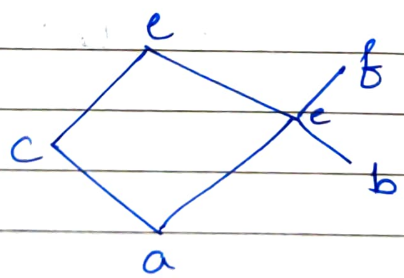
$GLB = \phi$

since b & c both are lower bounds on same level

ii) $B = \{b, c\}$, $UB = \{d, e, f\}$, $LB = \{a\}$, $LUB = \phi$, $GLB = a$

Que: Identify join semi lattice (if LUB exists for all pairs then Hasse dia is), meet semi lattice (if GLB exists for all pairs) and lattice

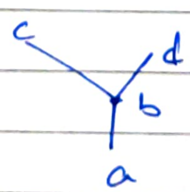
i)



$e \vee f = \phi$ JSL ✗

$a \wedge b = \phi$ MSL ✗

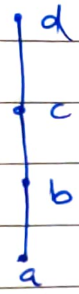
ii)



$c \vee d = \phi$ JSL ✗

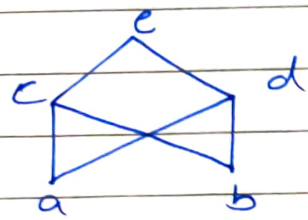
MSL ✓

iii)



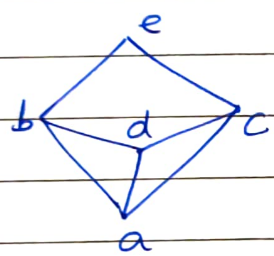
JSL ✓
MSL ✓
Lattice ✓

iv)



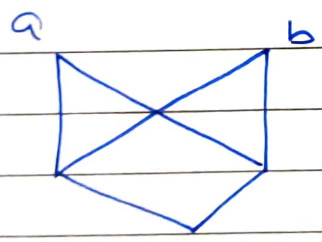
$a \vee b = \phi$
$a \wedge b = \phi$
JSL = ✗
MSL = ✗

v)

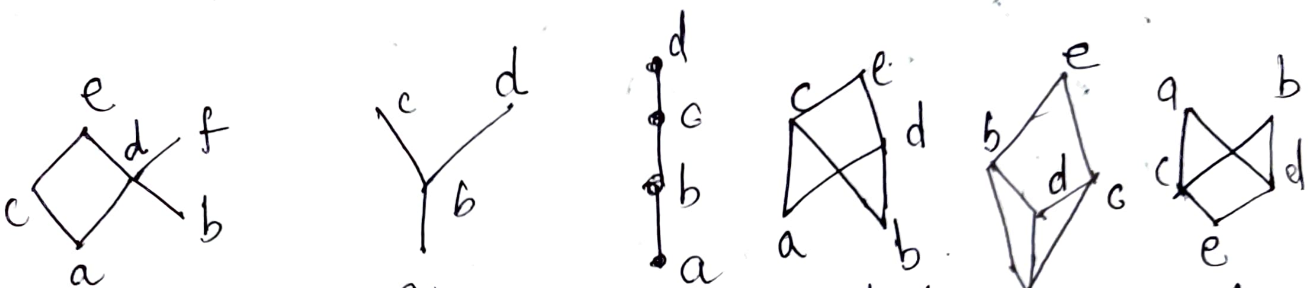


JSL = ✓
MSL = ✓
Lattice = ✓

vi)

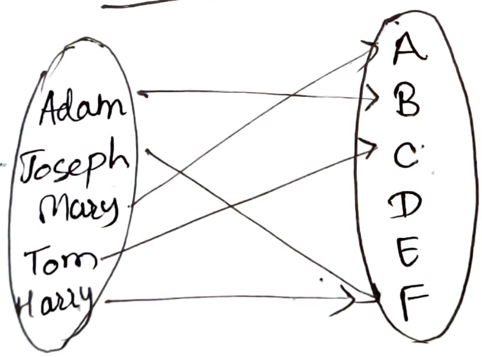


Q. Join Semi-lattice, Meet Semi-lattice.
(∨) (∧)
(lub) (glb)



$e \vee f = \phi$	$c \vee d = \phi$		$a \vee b = \phi$		$a \wedge b = \phi$
$a \wedge b = \phi$			$a \wedge b = \phi$		$a \vee b = \phi$
JSL ✗	JSL ✗	JSL ✓	JSL ✗	JSL ✓	MSL → ✗
MSL ✗	MSL ✓	MSL ✓	MSL ✗	MSL ✓	JSL → ✗
		Lattice ✓		Lattice ✓	

FUNCTIONS



A B

A function from A to B is an assignment of exactly one element of B to each element of A.

$f: A \to B$

$f(a) = b$ if b is the unique element of set B which is assigned to $a \in A$.

$f: A \to B$

$f(\text{Adam}) = B$.

$f(\text{Joseph}) = F$.

A ⇒ Domain of 'f' B ⇒ Co-domain of 'f'

$f(a) = b$; b: image of 'a' ; a: preimage of 'b'

The range of f will be the set of all images that are assigned to $a \in A$.

{Adam, Joseph, ..., } is the domain.

{A, B, C, D, E, F} is the co-domain.

Range: {A, B, C, F}.

Eg. Let $f: \mathbb{Z} \to \mathbb{Z}$ assign the square of an integer to that integer.

$f(x) = x^2$ Domain: $\mathbb{Z}$ Co-domain: $\mathbb{Z}$. Range $\{0, 1, 4, 9, \dots\}$.

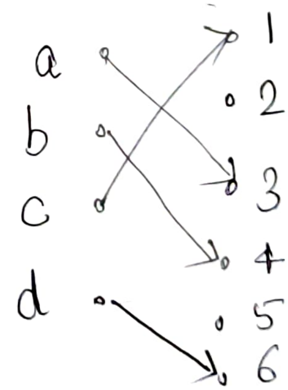
Two Real valued functions can be added and multiplied. Let f_1 and f_2 are two fns. from $A \to R$.

we can define $(f_1+f_2)x = f_1(x) + f_2(x)$; $f_1f_2(x) = f_1(x).f_2(x)$

Ex. Let f_1 & f_2 be fns. from R to R s.t. $f_1(x) = x^2$ & $f_2(x) = x - x^2$

$(f_1+f_2)(x) = f_1(x) + f_2(x) = x^2 + (x - x^2) = x$

$f_1f_2(x) = f_1(x).f_2(x) = x^2.(x - x^2) = x^3 - x^4$.

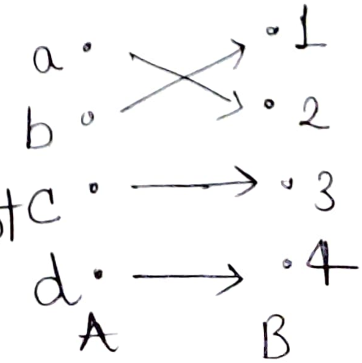


Types of functions

ⓐ One-to-One / Injective :-

fns. never assign the same value to two different elements

if $f(a) = f(b) \Rightarrow a = b$.



ⓑ Onto / Surjective :-

for every $b \in B$, there is an element $a \in A$ s.t. $f(a) = b$.

Co-domain = Range.

© **One-to-one Correspondence / Bijection :-**

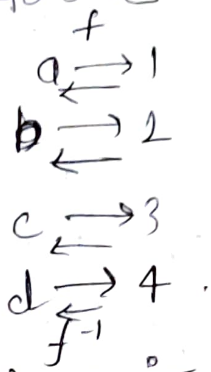
If a fn. is both one-to-one and onto then it will be a bijection.

Inverse Functions :-

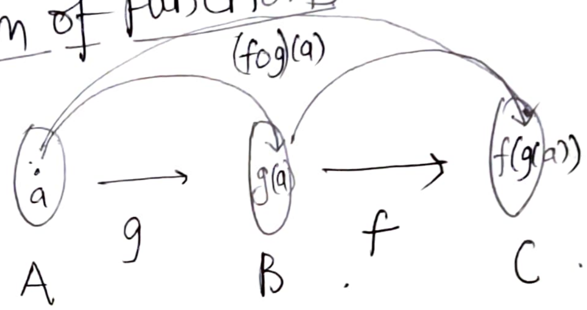
$f: A \to B$ is an one-to-one correspondence.

$f(a) = b$ $\quad f^{-1} \neq \frac{1}{f}$.

$f^{-1}: B \to A$ $\quad \boxed{f^{-1}(b) = a}$.



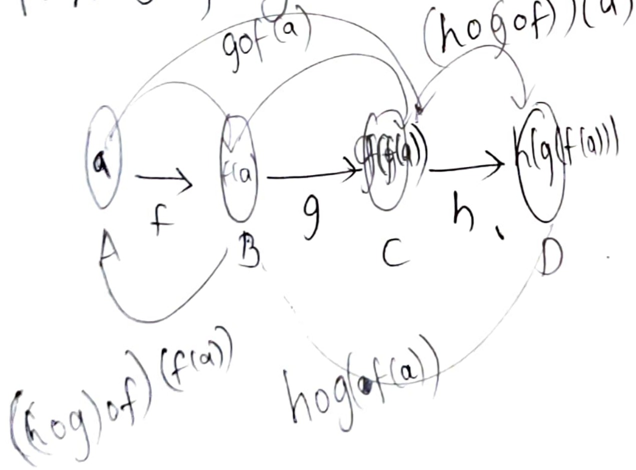
* **Composition of Functions :-**



Composition of functions is associative.

$f: A \to B$; $g: B \to C$; $h: C \to D$. $(ho(gof))(a)$ & $((hog)of)(a)$ are both fn. from A to D.

$f: A \to D$

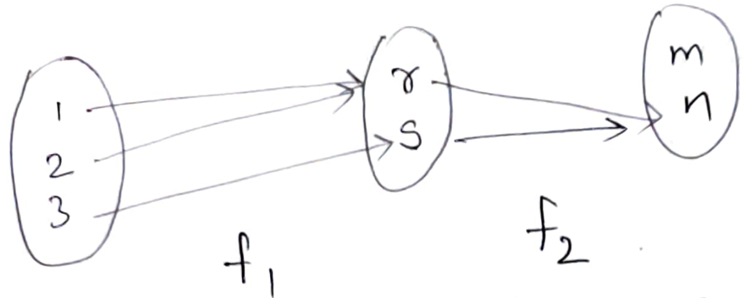


Ex. let $A = \{1,2,3\}$, $B = \{r,s\}$, and $C = \{m,n\}$.

$f_1: A \to B$ where $f_1 = \{(1,r), (2,r), (3,s)\}$

$f_2: B \to C$ where $f_2 = \{(r,n) (s,n)\}$.

$f_2 \circ f_1 = \{(1,n), (2,n), (3,n)\}$.



Composition of functions is associative.

$f : A \to B$; $g : B \to C$; $h : C \to D$.

$h \circ (g \circ f)$ & $(h \circ g) \circ f$ are both fn. from A to D.

$f(a) = b$, $g(b) = c$, $h(c) = d$.

$(a,b) \in f$ $\quad (b,c) \in g \quad (c,d) \in h$.

$(a,c) \in g \circ f$ | $(b,d) \in h \circ g$

$(a,d) \in h \circ (g \circ f)$ | $(a,d) \in (h \circ g) \circ f$.

$(h \circ g) \circ f = h \circ (g \circ f)$.

ALGEBRAIC STRUCTURES

Coins deposited	Merchandise
dime, dime	gum
dime, quarter	candy
quarter, dime	candy
quarter, quarter	chocolate

A = { dime, quarter }.

B = { gum, candy, choclate }

$f : A \times A \to B$. Binary operation on A

$f : A \times A \to A$. Closed binary operation on A.

ALGEBRAIC STRUCTURE

coins deposited	merchandise
dime, dime	gum
dime, quarter	candy
quarter, dime	candy
quarter, quarter	chocolate

$A = \{dime, quarter\}$
& $B = \{gum, candy, chocolate\}$

- $f: A \times A \longrightarrow B$ → binary operation on A (may be represented as □, *, ✡ etc)

  $f: A \times A \rightarrow A$ → closed binary operation

- (A, □, *, ✡) set A along with the list of ~~the~~ binary operations is called algebraic structure.

✡	dime	quarter
dime	biscuit	gum
quarter	gum	candy

*	dime	quarter
dime	chips	biscuit
quarter	biscuit	sandwich

$(\{dime, quarter\}, ✡, *) \longrightarrow AS$

→ SEMI GROUP

- Let $(A, \square)$ be an algebraic structure if

i) $\square$ is a closed operation w.r.t. A
ii) $\square$ is associative.

ie if $a, b, c \in A$ then $a \square (b \square c) = (a \square b) \square c$

eg- when A = set of all even integers $\{2, 4, 6, \dots\}$

$\square = +$ operation
$(A, +)$ is a semigroup.

→ MONOID

i) $\square$ is a closed operation w.r.t A
ii) $\square$ is associative
iii) There exists an identity element ~~in~~ 'e' in A s.t. $\forall x \in A$, $e \square x = x \square e = x$

eg- I: set of all whole nos. ~~[illegible]~~

+ : ordinary addition

'0' will be e

$(I, +)$

classmate

Date
Page

→ GROUP

i) $\Box$ is closed operation wrt A
ii) $\Box$ is associative
iii) There exists an identity element 'e' in A such that $\forall x \in A$, $e \Box x = x \Box e = x$

iv) For every element $x \in A$, there exists an inverse element $x^{-1} \in A$ st $x \Box x^{-1} = x^{-1} \Box x = e$

eg- $(I, +)$ where $e = 0$

for every $x \in I$, $-x \in I$

→ SUBGROUP

- $(A, \Box)$ be an algebric system and B a subset of A then $(B, \Box)$ will be called a subgroup if it is a group by itself

i) $\Box$ is a closed operation w.r.t. B

ii) $\Box$ is associative

iii) 'e' must exist in B as identity of $(B, \Box)$

iv) For every element $b \in B$, its inverse must be in B.

NOTE: For B to be a subgroup, A must be a group.

eg- I: set of integers

+: addition

E: set of even integers

→ ISOMORPHISM

-

□	a	b	c	d
a	a	b	c	d
b	b	a	a	c
c	b	d	d	c
d	a	b	c	d

A

$\star$	α	β	γ	δ
α	α	β	γ	δ
β	β	α	α	γ
γ	β	δ	δ	γ
δ	α	β	γ	δ

B

a, b, c, d to $\alpha, \beta, \gamma, \delta$
□ to $\star$
$(B, \star)$ from $(A, \square)$

f: one to one & onto
f^n

$f(a_1 \square a_2) = f(a_1) \star f(a_2)$

$f(a) = \alpha$
$f(b) = \beta$

NOTE: $f: \underset{a}{A} \longrightarrow \underset{b}{B}$ then no. of possible f^n's $= a^b$

TUTORIAL

1 Que: Let X, Y, Z be sets of sizes x, y, z and $W = X \times Y$ and E is set of all subsets of W. no. of f^n's from $Z \to E$ is ___.

Ans: $size(W) = xy$

$size(E) = 2^{xy}$

$f: Z \to E$

no. of f^n's possible $= (2^{xy})^z$
$= 2^{xyz}$

2 Que: Total no. of one to one f^n's from $A \to B$ where $|A| = m$ and $|B| = n$

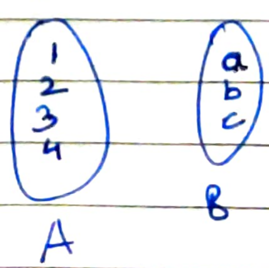
Ans:

$${}^nP_m = m! \times {}^nC_m$$

3 Que: Find total no. of non-onto f^n's from $A \to B$ where $|A| = 4$ & $|B| = 3$

Ans:

~~${}^3C_1 \times (2^4-1)(2^4-1) + {}^3C_2 \times (2^4-1) + {}^3C_3$~~



~~$= 3 \times 18 \times 18 + (3 \times 18 + 4$~~

~~$= 45(16)(+4$~~ ${}^3C_1 \times 2^4 - {}^3C_2 \times 1^4$

$= 3 \times 16 - 3 \times 1 = 45$

Let us consider

S_x = no. of f^ns in which 'x' is not mapped

Inclusion exclusion principle

$$n(S_a \cup S_b \cup S_c) = n(S_a) + n(S_b) + n(S_c) - n(S_a \cap S_b) - n(S_b \cap S_c) - n(S_a \cap S_c) + n(S_a \cap S_b \cap S_c)$$

$$S_b = S_c = S_a = 2^4$$

$$n(S_a \cap S_b) = 1 = n(S_b \cap S_c) = n(S_c \cap S_a)$$

$$n(S_a \cap S_b \cap S_c) = 0$$

$$\therefore n(S_a \cup S_b \cup S_c) = 16 + 16 + 16 - 1 - 1 - 1 + 0$$
$$= 45$$

4Que: given $f: A \rightarrow B$

given sizes $|A|$ & $|B|$

Find relation b/w them in case of:

i) One-one function : $|A| \leq |B|$

ii) Onto function : $|A| \geq |B|$

iii) Bijective function : $|A| = |B|$

5Que: Assuming $g \circ f$ is not empty, show that $g \circ f: A \rightarrow C$ where $f: A \rightarrow B$, $g: B \rightarrow C$

Ans: prove by contradiction

6 Que: $f(x) = x^2$

$g(x) = 2x+3$ Find fog & gof

Ans: $fog = f[g(x)] = f(2x+3) = (2x+3)^2$

$gof = g[f(x)] = g(x^2) = 2x^2+3$

	AS	S.G.	Monoid	G	A.G
N, +	✓	✓	×	×	×
N, −	×	×	×	×	×
N, ×	✓	✓	✓	×	×
N, /	×	×	×	×	×
Z, +	✓	✓	✓	✓	✓
Z, −	✓	×	×	×	×
Z, ×	✓	✓	✓	×	×
Z, /	×	×	×	×	×
R, +	✓	✓	✓	✓	✓
R, −	✓	×	×	×	×
R, ×	✓	✓	✓	×	×
R, /	~~✓~~	×	×	×	×
e, +	✓	✓	✓	✓	✓
e, ×	✓	✓	×	×	×
o, +	×	×	×	×	×
o, ×	✓	✓	✓	×	×

Q. $(\{1, \omega, \omega^2\}, \times)$; $\omega^3 = 1$

$\times$	1	ω	ω^2
1	1	ω	ω^2
ω	ω	ω^2	$\omega^3 = 1$
ω^2	ω^2	ω^3 $=1$	ω^4 $=\omega$

A.S. → ✓

S.G. → ✓

Monoid. → ✓

G → ✓

Q. $\{(1,-1,i,-i), \times\}$; $i=\sqrt{-1}$.

A.S. → ✓

S.G. → ✓

Monoid →

Group →

×	1	-1	i	-i
1	1	-1	i	-i
-1	-1	1	-i	i
i	i	-i	-1	1
-i	-i	i	1	-1

Q. $(N,*)$, $a*b = a^b$.

(a) Semi group

(b) Not semigroup ✓

(c) Monoid but not G

(d) Group.

$(a*b)*c = (a^b)*c = (a^b)^c = a^{bc}$ — ①

$a*(b*c) = a*(b^c) = a^{(b^c)}$ — ②

$1 \neq 2$

Q. $(Z,*)$, $a*b = \max(a,b)$

Semigroup

$(a*b)*c = \max(a,b)*c = \max(\max(a,b),c)$

$a*(b*c) = a*\max(b,c)$

$= \max(\cancel{\max}\ a, \max(b,c))$

$a*e = \max(a,e)$

$= a$

$e = -\infty$

but $-\infty$ does not exist in ~~real~~ Z ∴ not a monoid

★ Addition modulo. $(+_n)$

Q. $\{1,2,3,4\}$; $+_5$

$+_5$	1	2	3	4
1	2	3	4	0
2	3	4	0	1
3	4	0	1	2
4	0	1	2	3

(if sum ≥ n ; sum − n $+_n$)

$\{0,1,2,3\}$; $+_4$

$+_4$	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

Group

Identity = 0

Inverse : $0 \to 0$, $1 \to 3$, $2 \to 2$, $3 \to 1$

★ Multiplication Modulo ($\times_n$).

$\{0,1,2,3\}, \times_4$

$\times_4$	0	1	2	3
0	0	0	0	0
1	0	1	2	3
2	0	2	0	2
3	0	3	2	1

~~Semi Group~~

Monoid

homomorphism

$$(\mathbb{Z},+) \xrightarrow{f} (\mathbb{Z}_{2n},+)$$

$$f(a*b) = f(a) * f(b)$$

one-one

onto

↓

Isomorphism

★ Homomorphism

$(A,\square)$ and $(B,*)$ are A.S. ; f is a function from A to B $\forall\, a_1, a_2 \in A$; $f(a_1 \square a_2) = f(a_1) * f(a_2)$.

f: Homomorphic from $(A,\square)$ to $(B,*)$.

$(B,*)$ = Homomorphic image of $(A,\square)$.

$\square$	α	β	γ	δ	ϵ	ρ
α	α	β	α	α	γ	δ
β	β	α	γ	β	γ	ϵ
γ	α	γ	α	β	γ	ϵ
δ	α	β	β	δ	ϵ	ρ
ϵ	γ	γ	γ	ϵ	ϵ	ρ
ρ	δ	ϵ	ϵ	ρ	ρ	ρ

$*$	1	0	-1
1	1	1	0
0	1	0	-1
-1	0	-1	-1

$f(\alpha)=1$; $f(\beta)=1$; $f(\delta)=0$; $f(\epsilon)=0$, $f(\rho)=-1$, $f(\gamma)=1$.

f is homomorphic from $(\{\alpha,\beta,\gamma,\delta,\epsilon,\rho\}, \square)$ to $(\{-1,0,1\}, *)$

Congruence Relation :-

$(A, \square)$ is an A.S.
R be an equivalence relation.
R is called congruence relation on A w.r.t. $\square$
if (a_1, a_2) & (b_1, b_2) in R implies $(a_1 \square b_1, a_2 \square b_2)$
is also in R.

$\square$	a	b	c	d
a	a	a	d	c
b	b	a	d	a
c	c	b	a	b
d	c	d	b	a

	a	b	c	d
a	✓	✓		
b	✓	✓		
c			✓	✓
d			✓	✓

$\simeq (A, \square)$ Not a congruence relation.

★ Ring, Integral Domain, field

$(A, \square, *)$ is called a ring

① $(A, \square)$ is an Abelian Group.
② $(A, *)$ is semigroup
③ The operation '$*$' is distributive over $\square$

$$a * (b \square c) = (a * b) \square (a * c)$$
$$(b \square c) * a = (b * a) \square (c * a).$$

eg. $(Z, +, \cdot) \Rightarrow$ Ring $(Z, +)$ is A.G.
$(Z, \cdot)$ is S.G.

$$a \cdot (b + c) = a.b + a.c$$
$$(b + c) \cdot a = b.a + c.a.$$

$R = (A, \square, *)$.

Commutative ring

for ring R; $\forall x, y \in R$ $x * y = y * x$.

Ring with unity

for ring R $\exists e \in R$ s.t. $x * e = e * x = x$ $\forall x \in R$

Zero divisor of a ring

for ring R, $\exists$ $a, b \in R$

if $a \neq 0, b \neq 0$, then $a * b = 0$ } zero divisor of a ring.

eg $\left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix}, +, \cdot \right\}$; $(a, b, c, d) \in \mathbb{Z}$..

Ring without zero divisor $\Rightarrow$ $(\mathbb{Z}, +, \cdot)$

Integral Domain :- A ring R is called Integral Domain if
1. It is commutative.
2. It has unity.
3. It should be without zero divisor

eg $(\mathbb{Z}, +, \cdot)$; $(R, +, \cdot)$.

Field :- A ring R is called field if

1. It is commutative.
2. It has unity.
3. For every non-zero element it has invere w.r.t $*$ in R i.e. for Ring $(A, \square, *)$

$\forall\, a \in A,\ \exists\, b \in A$ s.t. $a * b = e = b * a$

e.g. $(R, +, \cdot)$ ✓ field ; $(Z, +, \cdot)$ × field

★ **COUNTING :-**

Pigeonhole Principle :- Suppose a flock of 20 pigeons comes home to roost into 19 pigeonholes. So, atleast 1 pigeonhole will have at leat 2 pigeons

If $k \in \mathbb{Z}^+$ and $k+1$ or more objects are placed into 'k' boxes, then there is at least 1 box containing 2 or more objects. (Generalization).

Corollary :- A fn. 'f' from a set with '$k+1$' or more elements to a set with 'k' elements is not one-to-one.
(Assignment).

Step 1 :- Suppose none of the 'k' boxes contain more than one object.

Recurrance Relation :-

$\{a_n\}$: seq. of powers of 2 $a_n = 2^n$ where $n = 0, 1, 2, \ldots$

$$\begin{cases} a_0 = 1 \\ a_{n+1} = 2a_n \end{cases} \Rightarrow \text{Recurrance Relation.}$$

Recursively defined ~~relati~~ fn. :-

Basic step : Specify the value of f at 0.

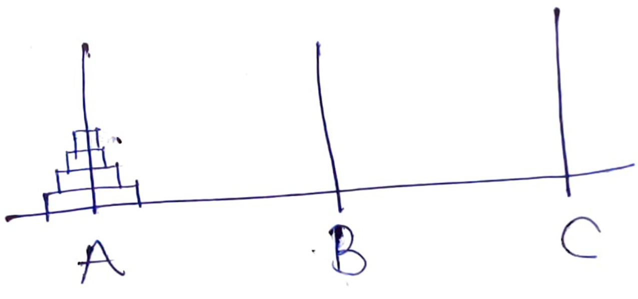
Recursive step : Rule for finding f at a particular integer from the value of f at lesser integers ex. $f(0) = 3$; $f(n+1) = 2f(n) + 3$.

ex. Give the recursive definition of a^x, where 'a' is a non-zero real no. & 'x' is non-negative integer.

$\Rightarrow$ $f(x) = a.f(x-1)$. $\quad a^0 = 1$

Recursively defined function. RR: $a^{n+1} = a.a^n$. for $n = 0, 1, 2, 3,$

Any sequence whose terms satisfy the RR is known as the solution to the RR.



Let $\{a_n\}$ be the no. of steps to move 'n' disks from A to B.

$f(n) = 2f(n-1) + 1$. $\quad a_1 = 1$

$\{a_{n-1}\} \rightarrow$ no. of steps to move $(n-1)$ disks from A to C. $\quad a_n = (a_{n-1}) + 1 + a_{n-1} = 2a_{n-1} + 1$.

Soln. $a_n = 2a_{n-1} + 1$

$= 2(2a_{n-2} + 1) + 1$

$= 2^2 a_{n-2} + 2 + 1$

$= 2^2(2a_{n-3} + 1) + 2 + 1$

$= 2^3 a_{n-3} + 2^2 + 2 + 1$

$=$ $\vdots$

$= 2^{n-1}.a_1 + 2^{n-2} + 2^{n-3} + \dots + 1$

Ex. Suppose that the no. of bacteria in a colony doubles every hour. If a colony begins with 5 bacteria, how many will be present in 'n' hours?

Ans :-

$$b(0) = 5$$

$$b(t) = 2.b(t-1)$$

$$b(n) = 2.b(n-1)$$
$$= 2(2\,b(n-2))$$
$$= 2^2.b(n-2)$$
$$= 2^3.b(n-3)$$

$$b(n) = 2^n.b(0) = 5.2^n \Rightarrow \boxed{b(n) = 5.2^n}$$

Generating functions

$a_0, a_1, a_2, \dots, a_k, \dots$ an infinite series of real nos.

$$G(x) = a_0 + a_1x + a_2x^2 + \dots + a_k.x^k + \dots$$
$$= \sum_{k=0}^{\infty} a_k.x^k$$

$\{a_k\}$ where $a_k = 2$; $\quad G(x) = \sum_{k=0}^{\infty} 2.x^k$

$a_k = k+1$; $\quad G(x) = \sum_{k=0}^{\infty} (k+1)x^k$.

→ GENERATING FUNCTIONS

- given sequence

$a_0, a_1, a_2, \ldots\ldots, a_k, \ldots\ldots$ an infinite series of real nos.,

then,

$$G(x) = a_0 + a_1 x + a_2 x^2 + \ldots\ldots + a_k x^k + \ldots\ldots$$

$$G(x) = \sum_{k=0}^{\infty} a_k x^k$$

- $\{a_k\}$ where $a_k = 2$ $\quad G(x) = \sum_{k=0}^{\infty} 2x^k$

$a_k = k+1$ $\quad G(x) = \sum_{k=0}^{\infty} (k+1)x^k$

NOTE: A generating func. need not have ∞ terms depending on the sequence

Que: What will be the generating function for the sequence 1, 1, 1, 1, 1?

Ans: $G(x) = 1 + x + x^2 + x^3 + x^4$

$$\Rightarrow G(x) = \frac{x^5 - 1}{x - 1}$$

→ THEOREM

- $f(x) = \sum_{k=0}^{\infty} a_k x^k$ & $g(x) = \sum_{k=0}^{\infty} b_k x^k$

$$f(x) + g(x) = \sum_{k=0}^{\infty} (a_k + b_k)\, x^k$$

$$f(x)\, g(x) = \sum_{k=0}^{\infty} \left\{ \sum_{j=0}^{k} a_j\, b_{k-j} \right\} x^k$$

~~eg~~

Que: $f(x) = \dfrac{1}{(1-x)^2}$ Find the ~~difference~~ coefficien ts $a_0, a_1, a_2, \ldots\ldots, a_k, \ldots$ in expansion

$$\sum_{k=0}^{\infty} a_k\, x^k$$

Ans:

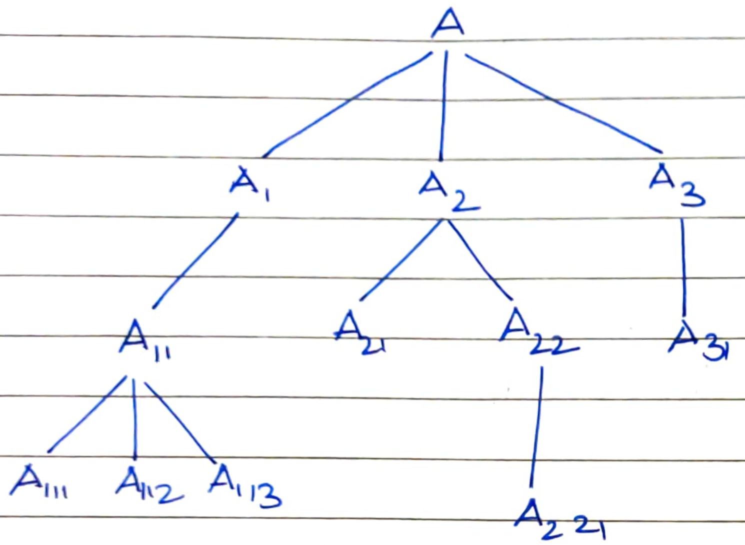
$$f(x) = \frac{1}{(1-x)} \times \frac{1}{(1-x)}$$

Using theorem,

$$f(x) = \sum_{k=0}^{\infty} \left\{ \sum_{j=0}^{k} (1) \right\} x^k$$

$$= \sum_{k=0}^{\infty} (k+1)\, x^k$$

→ TREE

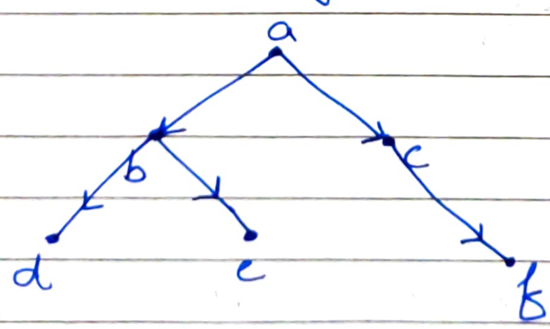


- Def: A connected undirected graph with no simple circuits.

NOTE:
- A tree has no self loops / parallel edges.
- A tree is a simple graph

- ROOTED TREE

  Root is a vertex in a tree. And all edges are directed away from the root



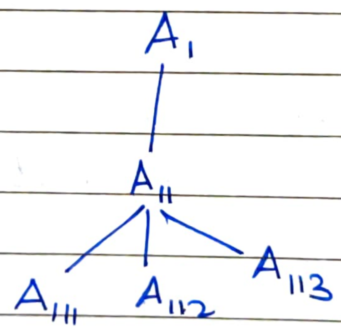
- **Terminologies**

In first tree drawn,

Parent: A_1 for A_{11}
Child: A_{11} for A_1
Siblings: A_{22} for A_{21}
Ancestors: A_{11} & A_1 & A for A_{111}
Descendants: A_{11}, A_{111}, A_{112}, A_{113} for A_1

Leaf: A vertex with no children is called a leaf. A_{111}, A_{112}, A_{113}, A_{21}, A_{221}, A_{31}
Internal vertices: Vertices with children.
Subtree: A tree from a given root

eg- subtree with root A_1



- **Theorem 1**

An undirected graph is a tree if and only if there is a unique simple path before any 2 of its vertices.

HW
PROOF:
Assignment

- Theorem 2

  A tree with 'n' vertices has 'n-1' edges.

PROOF: using induction

- Basic step ($n=1$)

  For 1 vertex, we have 0 edges.

  $\therefore$ Statement holds true.

- For $n=k$ assume statement to be true

  $\therefore$ For k vertices, $k-1$ edges.

- For $n=k+1$

Generalized Pigeonhole Principle (3 Mark)

$kn+1$ or more $\rightarrow$ Pigeons.

$n \rightarrow$ Pigeonhole.

$k+1$ or more pigeons in at least 1 pigeonhole.

Q. Find minimum number of students in a class to be sure that four of them were born in same month.

Ans :- n = no. of months $= 12$.

$k+1 = 4 \Rightarrow k = 4-1 = 3$

$kn+1 = 3(12)+1 = 37$.

minimum number of students are 37.

Note :- (Year has 365 days if not specified in question)

★ Homogeneous Recurrence Relation :-

$a_n = 3 \cdot a_{n-1}$; $a_0 = 5$

$x^n = 3 \cdot x^{n-1}$

$x = 3$

$a_n = A \cdot x^n$

$a_n = A 3^n$

$\Rightarrow \boxed{a_n = 5 \cdot 3^n}$

$\Rightarrow a_0 = 5$

$\Rightarrow 5 = A \cdot 3^0$

$\Rightarrow A = 5$

Q. $a_n = 2a_{n-1} + 2n$; $a_1 = 6$

(Non-homogeneous recurrence relation of first order). (5 Marks)

$\Rightarrow$ $$a_n = \underbrace{a_n^{(h)}}_{\text{homogenous solution}} + \underbrace{a_n^{(p)}}_{\text{particular solution}}.$$

For Homogenous part we can proceed like this,

$a_n = 2 \cdot a_{n-1}$ $\Rightarrow a_n^{(h)} = A \cdot x^n$

$x^n = 2 \cdot x^{n-1}$ $\Rightarrow \boxed{a_n^{(h)} = A \cdot 2^n}$

$\Rightarrow \boxed{x = 2}$

(On dividing both sides by x^{n-1}).

For Particular part we can proceed like this,

Let $(a_n)^p = cn + d$; be a particular sol^n.

$\Rightarrow$ $(cn+d) = 2[c(n-1)+d] + 2n$

$cn + d = 2cn - 2c + 2d + 2n$

$\Rightarrow$ $cn + d = 2(cn+d) - 2(c-n)$.

$\Rightarrow$ $0 = cn - 2c + d + 2n$.

$\Rightarrow$ $0 = n(c+2) + d - 2c$ — ①

$\Rightarrow$ $\cancel{n = \frac{2c-d}{c+2}}$

$c + 2 = 0$; $d - 2c = 0$.

$\boxed{c = -2}$; $d = 2c$

$\boxed{d = -4}$

(As n is variable for ① to be valid both $(c+2)$ & $(d-2c)$ must be equal to 0)

$$a_n^{(P)} = cn + d = (-2)n + (-4).$$

$$\Rightarrow \boxed{a_n^{(P)} = -2n - 4}$$

So, the final solution is

$$a_n = a_n^{(h)} + a_n^{(P)}$$

$$a_n = A.2^n - 2n - 4 \quad \text{———— ②}$$

we know ; $a_1 = 6$.

Put $n = 1$; in ②

$$a_1 = A.2^{(1)} - 2(1) - 4.$$

$$\Rightarrow 6 = 2A - 6$$

$$\Rightarrow 2A = 12 \Rightarrow \boxed{A = 6}$$

$$\Rightarrow \boxed{a_n = 6.2^n - 2n - 4}$$

is the required solution.

Non-homogeneous Recurrence Relation of second order (5 Marks).

$F_n = AF_{n-1} + BF_{n-2} + f(n)$.

(I) $f(n) = c \cdot x^n$.

$x^2 = Ax + B$ is characteristic eq^n of corresponding homogeneous recurrence relation.

$x_1, x_2 \Rightarrow$ roots of above quad. eq^n.

- If $x \neq x_1$ and $x \neq x_2$ then $(a_n)^{(P)} = Ax^n$.
- If $x = x_1$ or $x = x_2$ then $a_n^{(P)} = An \cdot x^n$.
- If $x = x_1 = x_2$ then $a_n^{(P)} = An^2 \cdot x^n$.

Q. $F_n = AF_{n-1} + B \cdot F_{n-2} + f(n)$.

if $x_1 = 2$; $x_2 = 5$.

$f(n)$	$a_n^{(P)}$
4 $\rightarrow 4 \cdot 1^n$ ($c \cdot x^n$)	A
$5 \cdot 2^n$	$An \cdot 2^n$
$8 \cdot 5^n$	$An \cdot 5^n$
4^n	$A \cdot 4^n$
$2n^2 + 3n + 1$	$An^2 + Bn + C$
cn	$cn + d$

e.g. $F_n = 3F_{n-1} + 10F_{n-2} + 7.5^n$; $F_0 = 4$, $F_1 = 3$.

For homogenous sol^n ;

$x^n = 3x^{n-1} + 10 . x^{n-2}$

$\Rightarrow x^2 = 3x + 10$

$\Rightarrow \boxed{x_1 = -2 ; x_2 = 5}$

for Homogeneous part ;

$a_n^{(h)} = A.(x_1)^n + B.(x_2)^n$

$$\boxed{a_n^{(h)} = A(-2)^n + B(5)^n} \quad \text{—— (1)}$$

for Particular part ;

$f(n) = 7.5^n = c.x^n \Rightarrow \underline{x = 5}$

$x = x_2$ & $x \neq x_1 \Rightarrow a_n^{(P)} = c.n(5)^n$

$\Rightarrow cn(5)^n = 3[c(n-1).5^{n-1}] + 10[c(n-2)5^{n-2}] + 7.5^n$

$\Rightarrow 25cn = 3[5c(n-1)] + 10[c(n-2)] + 7(25$

$\Rightarrow 25\cancel{cn} = 15\cancel{cn} - 15c + 10\cancel{cn} - 20c + 175$

$\Rightarrow 35c = 175 \Rightarrow c = \frac{175}{35} = 5$

$\Rightarrow \boxed{c = 5}$

$$\boxed{a_n^{(P)} = 5n(5)^n} \quad \text{—— (2)}$$

So, the final solution is

$$a_n = a_n^{(h)} + a_n^{(p)}$$

$$a_n = A(-2)^n + B(5)^n + 5n.(5)^n.$$

Using IC. $f_0 = 4$; $f_1 = 3$.

$$4 = A(-2)^0 + B(5)^0 + 5(0)(5)^0$$

$$\Rightarrow \boxed{4 = A+B} \text{ ——— (I)}$$

$$3 = A(-2)^1 + B(5)^1 + 5(1)(5)^1$$

$$\Rightarrow 3 = -2A + 5B + 25$$

$$\Rightarrow \boxed{2A - 5B = 22} \text{ ——— (II)}$$

(I) × 5 :- $20 = 5A + \cancel{5B}$.

(II) $22 = 2A - \cancel{5B}$

$$42 = 7A \qquad \boxed{A = 6}$$

$$\Rightarrow \boxed{B = -2}$$

$$\boxed{a_n = 6(-2)^n - 2(5)^n + 5n.(5)^n.}$$

NOTE: $\sum_{n=0}^{\infty} (n+1)\, a^n x^n = \frac{1}{(1-ax)^2}$

TUTORIAL

1 Que: Find a_n when $a_0 = 1$, $a_1 = 9$ & $n \geq 2$

$$a_n - 6a_{n-1} + 9a_{n-2} = 0$$

Ans: multiplying both sides by x^n

$$a_n x^n - 6a_{n-1} x^n + 9a_{n-2} x^n$$

Let $g(x) = \sum_{n=0}^{\infty} a_n x^n$

Taking Σ both sides

$$\sum_{n=2}^{\infty} a_n x_n - 6 \sum_{n=2}^{\infty} a_{n-1} x^n + 9 \sum_{n=2}^{\infty} a_{n-2} x^n = 0$$

$$\Rightarrow \sum_{n=0}^{\infty} a_n x^n - a_0 - a_1 x - 6x \left[\sum_{n=1}^{\infty} a_{n-1} x^{n-1} - a_0 \right] + 9x^2 \sum_{n=2}^{\infty} a_{n-2} x^{n-2}$$

$$\Rightarrow g(x) - 1 - 9x - 6x\,g(x) + 6x + 9x^2 g(x)$$

$$\Rightarrow g(x) \left[9x^2 - 6x + 1 \right] = 3x + 1$$

$$\Rightarrow g(x) = \frac{3x+1}{9x^2 - 6x + 1}$$

$$\Rightarrow g(x) = \frac{1}{(1-3x)^2} + 3x \times \frac{1}{(1-3x)^2}$$

$$= \sum_{n=0}^{\infty} (n+1) 3^n x^n + 3x \sum_{n=0}^{\infty} (n+1) 3^n x^n$$

$$= \sum_{n=0}^{\infty} (n+1)\, 3^n x^n + \sum_{n=0}^{\infty} (n+1)\, 3^{n+1} x^{n+1}$$

$$= 1 + \sum_{n=\cancel{0}}^{\infty} (n+1)\, 3^n x^n + \sum_{n=1}^{\infty} n\, 3^n x^n$$

$$= 1 + \sum_{n=1}^{\infty} (2n+1)\, 3^n x^n$$

$$\Rightarrow g(x) = \sum_{n=0}^{\infty} (2n+1)\, 3^n x^n \quad \Rightarrow a_n = (2n+1)\, 3^n$$

NOTE:

$$\sum_{n=0}^{\infty} (ax)^n = \frac{1}{1-ax} \qquad \Bigg| \qquad \sum_{n=1}^{\infty} n x^n = \frac{x}{(x-1)^2}$$

$$\sum_{n=0}^{\infty} (-1)^n x^n = \frac{1}{1+x}$$

Que: $a_n - 2a_{n-1} - 3a_{n-2} = 0$ $\quad n \geq 2$, $a_0 = 3$, $a_1 = 1$

Ans: multiplying with x^n

$$a_n x^n - 2a_{n-1} x^n - 3a_{n-2} x^n = 0$$

$$\Rightarrow \sum_{n=2}^{\infty} a_n x^n - 2\sum_{n=2}^{\infty} a_{n-1} x^n - 3\cancel{a_{n-2}} \sum_{n=2}^{\infty} a_{n-2} x^n = 0$$

$$\Rightarrow \sum_{n=0}^{\infty} a_n x^n - 3 - x - 2x\left[\sum_{n=\cancel{0}}^{\infty} a_{n-1} x^{n-1} - a_0\right] - 3x^2 \underbrace{\left[\sum_{n=2}^{\infty} a_{n-2} x^{n-2}\right]}_{=0}$$

$$\Rightarrow g(x) - 3 - x - 2x\, g(x) + 6x - 3x^2 g(x) = 0$$

$$g(x)\left[1-2x-3x^2\right] = 3-5x$$

$$\Rightarrow g(x) = \frac{3-5x}{(1-3x)(1+x)}$$

$$\Rightarrow g(x) = \frac{1}{1-3x} + \frac{2}{1+x}$$

$$= \sum_{n=0}^{\infty} (3x)^n + 2\sum_{n=0}^{\infty} (-1)^n x^n$$

$$= \sum_{n=0}^{\infty} 3^n x^n + 2(-1)^n x^n$$

$$\Rightarrow g(x) = \sum_{n=0}^{\infty} x^n\left(3^n + 2(-1)^n\right)$$

$$\therefore\ a_n = 3^n + (-1)^n 2$$

3Que: $\left.\begin{array}{l} a_n - 3a_{n-1} = n \\ \end{array}\right\}\ n \geq 1,\ a_0 = 1$

Ans: Multiplying by x^n both sides

$$a_n x^n - 3a_{n-1}x^n - n x^n = 0$$

$$\Rightarrow \sum_{n=1}^{\infty} a_n x^n - 3\sum_{n=1}^{\infty} a_{n-1} x^n = \sum_{n=1}^{\infty} n x^n$$

$$\Rightarrow \sum_{n=1}^{\infty} a_n x^n - 3\sum_{n=0}^{\infty} a_n x^{n+1} = \sum_{n=1}^{\infty} n x^n$$

$$\Rightarrow \sum_{n=0}^{\infty} a_n x^n - a_0 - 3x\sum_{n=0}^{\infty} a_n x^n = \frac{x}{(x-1)^2}$$

$\Rightarrow$

$$\Rightarrow g(x) - 1 - 3x\,g(x) = \frac{x}{(x-1)^2}$$

$$\Rightarrow g(x)\,[-3x+1] = \frac{x}{(x-1)^2} + 1$$