Improper Integrals of Rational Functions

A real integral $\int_a^b f(x)\,dx$ is called <u>improper integral</u> if,

(i) one or both limits are not finite, or

(ii) $f(x)$ has infinite discontinuity at 'a' or at 'b' or at some point c, $a < c < b$.

Defn: — The improper integral $\int_{-\infty}^{\infty} f(x)\,dx$ is defined as.

$$\int_{-\infty}^{\infty} f(x)\,dx = \lim_{a \to -\infty} \int_a^0 f(x)\,dx + \lim_{b \to \infty} \int_0^b f(x)\,dx \quad \text{——} (1)$$

(if both the limits exist)

If both the limits in (1) exist, then

$$\int_{-\infty}^{\infty} f(x)\,dx = \lim_{R \to \infty} \int_{-R}^{R} f(x)\,dx \quad \text{——} (2).$$

The expression on the right is called <u>Cauchy's principal value of the integral</u>.

<u>Note 1</u>: $P.V. \int_{-\infty}^{\infty} f(x)\,dx \left(\frac{\text{by}}{\text{defn in (2)}} \lim_{R \to \infty} \int_{-R}^{R} f(x)\,dx \right)$

may exist even though the limits in (1) do not exist.

for Eg: $\lim_{R \to \infty} \int_{-R}^{R} x\,dx = 0$,

But $\lim_{a \to -\infty} \int_a^0 x\,dx = -\infty$ and $\lim_{a \to \infty} \int_0^a x\,dx = \infty$

<u>Note 2</u>: If the limits in (1) exist (finite), then limit in (2) exist and $\int_{-\infty}^{\infty} f(x)\,dx = P.V. \int_{-\infty}^{\infty} f(x)\,dx$.

12.10.20

§ Real Integration I

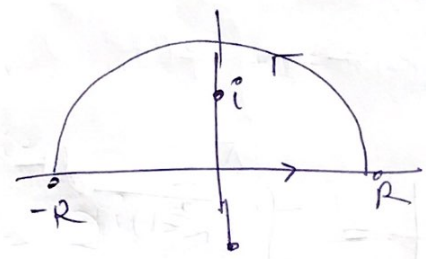
Examples : $I = \int_{-\infty}^{\infty} \frac{dx}{1+x^2}$ $\left(= \lim_{R\to\infty} \int_{-R}^{R} \frac{dx}{x^2+1} \right)$

(for this $f(x)$, limits in ① exist, and so we may proceed with finding ②)

Consider the function $f(z) = \frac{1}{z^2+1}$

it has singularities at $z = \pm i$: (both simple poles)

Choose C :



choose $R > 1$ (so that $z = i$ is inside the contour).

and $C := C_1 + C_2$ where C_1 : line segment joining $-R$ and R.

C_2 : semicircle $|z| = R$

(Note $z = -i$ is outside of C).

we take positive orientation on C.

By Cauchy's Residue Theorem

$$I' = \int_C \frac{1}{z^2+1}\,dz = 2\pi i \operatorname{Res}_{z=i} f(z) = 2\pi i \cdot \frac{1}{2i} = \pi \quad (1)$$

By contour integration using parametrization

$$\int_C \frac{dz}{z^2+1} = \int_{C_1} \frac{f(z)}{z^2+1}\,dz + \int_{C_2} \frac{f(z)\,dz}{z^2+1}$$

parametrization $C_1 : z(t) = t, \quad -R \leq t \leq R$

$C_2 : z(t) = Re^{it} \quad 0 \leq t \leq \pi$

So $$I' = \int_{-R}^{R} \frac{1}{(t^2+1)}\,dt + \int_0^{\pi} \frac{Rie^{it}}{R^2e^{i2t}+1}\,dt$$

$$\Rightarrow \int_{-R}^{R} \frac{1}{(t^2+1)} dt = I' - \int_0^{\pi} \frac{Rie^{it}}{R^2e^{i2t}+1} dt \quad \text{---} (2)$$

Now we take $\lim_{R\to\infty}$, Before that,

$$\left| \frac{1}{R^2e^{i2t}+1} \right| \le \left| \frac{1}{|R^2e^{i2t}|-1} \right| = \frac{1}{R^2-1}$$

and $|Rie^{it}| = R$

$$\Rightarrow \left| \int_0^{\pi} \frac{Rie^{it}}{R^2e^{i2t}+1} dt \right| \le \int_0^{\pi} \frac{R}{(R^2-1)} dt = \frac{R}{R^2-1} \cdot \pi.$$

This gives,

as $R \to \infty$, $\int_0^{\pi} \frac{iRe^{it}}{(R^2e^{2it}+1)} dt \to 0$. ——(3)

Now using (1), (2) & (3)

$$\lim_{R\to\infty} \int_{-R}^{R} \frac{dt}{(1+t^2)} = I = \pi.$$

So ~~the~~ $\cancel{\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \pi}$

$$\boxed{P.V. \int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \pi.}$$

Outline of the above method

- choose a suitable $f(z)$.
- choose a suitable contour
- evaluate the integration $\int_C f(z)\,dz$ using residues.
- evaluate __________ using parametrizatⁿ,

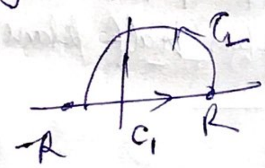
Real Integration II

Same method can be adopted for integration

$$I = \int_{-\infty}^{\infty} \frac{p(x)}{q(x)}\,dx \quad \text{where.}$$

- $p(x)$, $q(x)$ polynomials with real coeff.
- $q(x) \neq 0 \quad \forall x \in \mathbb{R}$
- $\deg q(x) \geq \deg p(x) + 2$.
- $p(x)$ and $q(x)$ have no factors.

For calculating I, choose $f(z) = \frac{p(z)}{q(z)}$

and C: 

with R large enough so that all the singularities of $f(z)$ lying in the upper half plane lie inside of C.

Then by C. Residue thⁿ.

$$I = \int_C f(z)\,dz = 2\pi i \sum_{\substack{z_k \text{ is} \\ \text{in upper half.}}} \operatorname{Res}_{z=z_k} f(z)$$

By contour integration (by parametrizatⁿ)

$$\int_C f(z)\,dz = \int_{C_1} f(z)\,dz + \int_{C_2} f(z)\,dz.$$

$$= \int_{-R}^{R} f(z)dz + \int_{0}^{\pi} f(Re^{it}) Ri e^{it} dt.$$

Then $\displaystyle\int_{-\infty}^{\infty} f(x)\,dx = \lim_{R\to\infty} \int_{-R}^{R} f(z)\,dz$

$$= \int_C f(z)dz + \int_0^{\pi} f(Re^{it})\, Ri e^{it} dt.$$

One can show that

$$\left|\int_0^{\pi} f(Re^{it}) \cdot Ri e^{it} dt\right| \longrightarrow 0 \text{ as } R \longrightarrow \infty.$$

Therefore, we get

$$\boxed{\int_{-\infty}^{\infty} f(x)dx = 2\pi i \sum_{z_k} \operatorname*{Res}_{z=z_k} f(z)}$$

z_k : sing. pt of $f(z)$ in upper half plane.

Eg: — $I = \int \frac{x^2}{(x^6+1)} dx$.

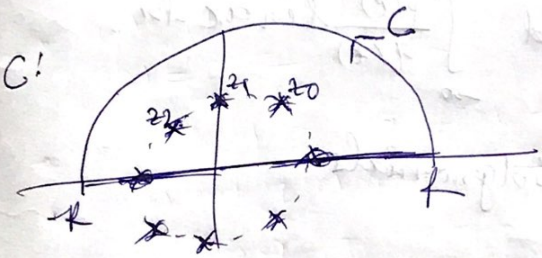
$f(z) = \frac{z^2}{z^6+1} = \frac{p(z)}{q(z)}$

singularities of $f(z)$ = 6^{th} root of (-1)

$= e^{i(\pi/6 + 2n\pi/6)}$ $n = 0, \dots, 5$

- $q(x) \neq 0 \ \forall x \in \mathbb{R}$
- $\deg q(z) = 6 \geq \deg p(z) + 2$.

C:



$z_0 = e^{i\pi/6}$, $z_1 = i$, $z_2 = e^{i5\pi/6}$. are inside C.

then $I = \oint_{-R} \ 2\pi i \sum_{k=0}^{2} \operatorname{Res}_{z=z_k} f(z) = 2\pi i (R_0 + R_1 + R_2)$

provided $\lim_{R\to\infty} \int_{C_2} f(z)dz = 0$.

To prove this, $\left|\frac{1}{z^6+1}\right| \leq \frac{1}{|z^6|-1} = \frac{1}{R^6-1}$ $\forall z \in C_2$

so $|f(z)| = \left|\frac{z^2}{z^6+1}\right| \leq \frac{|z^2|}{R^6-1} = \frac{R^2}{R^6-1}$

So, $\left| \int_{C_R} f(z)\,dz \right| \leq \frac{R^2}{R^6-1} \cdot \underbrace{\text{length of } C_R}_{\pi R}$

$$= \frac{\pi R^3}{R^6-1}$$

$\Rightarrow \lim_{R\to\infty} \int_{C_R} f(z)\,dz = 0.$

Real-Integration-III

Following a similar method, we can evaluate

$$\int_{-\infty}^{\infty} \frac{p(x)}{q(x)} \cos ax\,dx \quad \text{and} \quad \int_{-\infty}^{\infty} \frac{p(x)}{q(x)} \sin ax\,dx.$$

where, $p(x)$, $q(x)$: real polynomials.

- $q(x) \neq 0 \ \forall x \in \mathbb{R}$
- $\deg q(x) \geq \deg p(x) + 2$

(In fact $\deg q(x) \geq \deg p(x) + 1$ can also be solved)

Caution: - Choosing $f(z) = \frac{p(z)}{q(z)} \cdot \cos az$ or $\frac{p(z)}{q(z)} \sin az$

will not help.

→ For now, suppose $a > 0$.

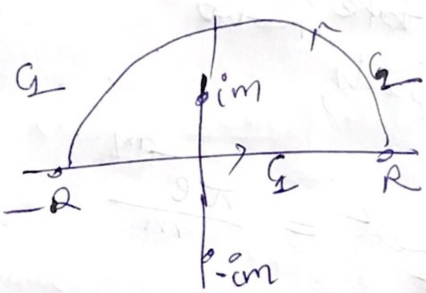
→ choose $f(z) = \frac{p(z)}{q(z)} \cdot e^{iaz}$

Eg:- $I = \int_{-\infty}^{\infty} \frac{\cos ax}{x^2 + m^2} dx$ $m > 0$, $a > 0$

Choose $f(z) = \frac{e^{iaz}}{z^2 + m^2}$

Singular pt. of $f(z) = \pm im$ (both poles of order 1.)

choose $C := C_1 + C_2$



By Residue thm

$$\frac{1}{2\pi i} \int_C f(z) dz = \operatorname{Res}_{z=im} f(z) = \lim_{z \to im} (z - im) f(z)$$

$$= \frac{e^{-am}}{2im} \quad \text{— (1)}$$

Parametrizatn

$$\int_C f(z) dz = \int f(z) dz + \int f(z) dz$$

$C_1 : z(t) = t$, $-R \le t \le R$ $C_2 : |z| = R$, $0 \le \theta \le \pi$

$$= \int_{-R}^{R} \frac{e^{iat} dt}{(t^2 + m^2)} + \int_{C_2} \frac{e^{iaz}}{z^2 + m^2} dz \quad \text{— (2)}$$

$$\Rightarrow \int_{-R}^{R} \frac{e^{iat} dt}{t^2 + m^2} = \frac{2\pi i e^{-am}}{2im} + \int_{C_2} \frac{e^{iaz} dz}{z^2 + m^2} \quad \text{— (3)}$$

[using (1) & (2)]

Now, $\left|\int_{C_2} \frac{e^{iaz}}{z^2+m^2}\,dz\right| \overset{(Ex *)}{\leq} \frac{1}{R^2-m^2}\int_{C_2} dz$

$$= \frac{\pi R}{R^2-m^2}.$$

$\Rightarrow \lim_{R\to\infty} \int_{C_2} \frac{e^{iaz}}{z^2+m^2} = 0.$

So, finally we take $\lim_{R\to\infty}$ and write the real and imaginary parts,

$$\int_{-\infty}^{\infty} \frac{\cos at}{t^2+m^2}\,dt = \frac{\pi e^{-am}}{m} \quad \text{and}$$

[Remember $a, m > 0$.]

$$\int_{-\infty}^{\infty} \frac{\sin at}{(a^2+m^2)}\,dt = 0$$

Solution for Ex * [This is where we use $a > 0$]

$|e^{iaz}| = |e^{ia(x+iy)}| = e^{-ay} = \frac{1}{e^{ay}} \leq 1$ if $ay > 0$

Now since $a > 0$ is given, we need $y > 0$ (that is why we choose contour in upper half plane)

and, $\frac{1}{|z^2+m^2|} \leq \frac{1}{||z|^2-m^2|} = \frac{1}{R^2-m^2}$

so $\left|\frac{e^{iaz}}{z^2+m^2}\right| \leq \frac{1}{R^2-m^2} \quad \forall z \in C_2$.

We can also write the value of $\int_{0}^{\infty} \frac{\cos at}{t^2+m^2}\,dt$.

Since $\int_{-\infty}^{\infty} f(t)\,dt = \int_{-\infty}^{0} f(t)\,dt + \int_{0}^{\infty} f(t)\,dt$,

we have that for even functions,

$$\int_{-\infty}^{\infty} f(t)\,dt = 2\int_{0}^{\infty} f(t)\,dt.$$

Using above, we get $\int_{0}^{\infty} \frac{\cos at}{t^2+m^2}\,dt = \frac{\pi e^{-am}}{2m}$

~~Example~~

<u>Caution</u> : $|e^{iaz}| \leq 1$ $\begin{cases} \text{when } y \geq 0, \text{ [given } a>0] \\ \text{when } y \leq 0 \text{ \& given } a<0 \end{cases}$

So, (1) when $a>0$ is given, we are forced to choose our contour in upper half plane i.e. $y \geq 0$

(2) when $a<0$, we are forced to choose our contour in lower half plane i.e. $y \leq 0$, in this method

~~Xam~~

Real- Integration IV

$$I = \int_0^{2\pi} g(\sin\theta, \cos\theta)\, d\theta \quad \text{for some function } g(\cdot).$$

Idea: Use $\sin\theta = \dfrac{e^{i\theta} - e^{-i\theta}}{2i}$; $\cos\theta = \dfrac{e^{i\theta} + e^{-i\theta}}{2}$

$$= \frac{z - z^{-1}}{2i} \qquad\qquad = \frac{z + z^{-1}}{2}$$

where $z = e^{i\theta}$.

and convert the integration to a contour integral over the unit circle $|z| = 1$. (or $z = e^{i\theta}$, $0 \le \theta \le 2\pi$)

Example
$$I = \int_0^{2\pi} \frac{d\theta}{a + b\cos\theta}, \quad a, b : \text{real}, \ |a| > |b|$$

If $b = 0$, $I = \int_0^{2\pi} \dfrac{d\theta}{a} = \dfrac{1}{a} 2\pi$.

We assume $b \neq 0$. Then

$$I = \frac{1}{b} \int_0^{2\pi} \frac{d\theta}{\frac{a}{b} + \cos\theta} = \frac{1}{b} \int_0^{2\pi} \frac{d\theta}{\alpha + \cos\theta}, \quad \alpha = \frac{a}{b} \ \& \ |\alpha| = \left|\frac{a}{b}\right| > 1$$

Now, take $C : z = e^{i\theta}$, $dz = ie^{i\theta} d\theta$,

then $$b \cdot I = \int_{C : |z| = 1} \frac{dz}{iz \cdot \left(\alpha + \left(\frac{z + z^{-1}}{2}\right)\right)} = \int_C \frac{dz}{iz\left(\frac{z^2 + 2\alpha z + 1}{2z}\right)}$$

$$= \frac{2}{i} \int_C \frac{dz}{\underbrace{(z^2 + 2\alpha z + 1)}_{f(z)}}$$

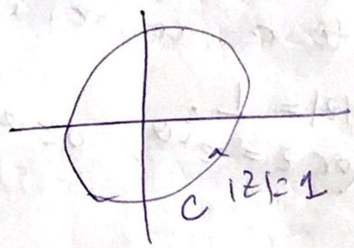
Sahu

we can apply Cauchy's - Residue theorem;

singularities of $f(z)$: $z^2 + 2\alpha z + 1 = 0$.

iff $z = -\alpha \pm \sqrt{\alpha^2 - 1}$

so, two simple poles of $f(z)$: $z_1 = -\alpha + \sqrt{\alpha^2 - 1}$
$z_2 = -\alpha - \sqrt{\alpha^2 - 1}$ $\Big\}$ both are real.



Are z_1 & z_2 inside of C ? Need to check $-1 \le z_i \le 1$?
(depends on α) (it is given that $|\alpha| > 1$).

$|\alpha| > 1$ & α is real

$\alpha > 1$, then z_1 is inside of C
z_2 is outside of C

$\alpha < -1$:
then z_1 is outside
z_2 is inside

(Need to check: $-1 \le z_i \le 1$

i.e $-1 \le -\alpha \pm \sqrt{\alpha^2 - 1} \le 1$

i.e $\alpha - 1 \le \pm\sqrt{(\alpha-1)(\alpha+1)} \le 1 + \alpha$.)

Suppose $\alpha > 1$, then $(\alpha - 1) < (\alpha + 1)$

$\Rightarrow (\alpha-1)(\alpha-1) < (\alpha-1)(\alpha+1) < (\alpha+1)(\alpha+1)$

$\Rightarrow (\alpha - 1) < \sqrt{(\alpha^2 - 1)} < (\alpha + 1)$.

$\Rightarrow (\alpha-1) < z_1 < (\alpha+1)$.

so z_1 is inside of C.

<u>For z_2:</u>

$-\sqrt{\alpha^2-1} < -\alpha+1$

$\Rightarrow z_2 = -\alpha-\sqrt{\alpha^2-1} < -2\alpha+1$
< -1. (as $\alpha>1$).

so z_2 is outside of C.

<u>Ex:</u> Verify that z_1 is outside and z_2 is inside when $\alpha < -1$.

<u>Now:</u> $|\alpha|>1$, α is real

$\alpha > 1$: z_1 is inside of C, z_2 is outside.

$\alpha < -1$: z_2 is inside, z_1 is outside

$$\int_C f(z)dz = 2\pi i \operatorname{Res}_{z=z_1} f(z) = \cancel{2}\pi i \cdot \frac{1}{\cancel{2}\sqrt{\alpha^2-1}}$$

$$\int_C f(z)dz = 2\pi i \operatorname{Res}_{z=z_2} f(z) = 2\pi i \frac{1}{-2\sqrt{\alpha^2-1}}$$

So finally.

$$I = \frac{2}{ib}\int_C f(z)dz = \begin{cases} \dfrac{2\pi i}{bi\sqrt{\alpha^2-1}} & \text{if } \alpha>1 \text{ i.e. } \frac{a}{b}>1 \\ \dfrac{-2\pi}{b\sqrt{\alpha^2-1}} & \text{if } \alpha<-1 \text{ i.e. } \frac{a}{b}<-1 \end{cases}$$

$\int_0^{2\pi} \frac{d\theta}{a+b\cos\theta}$

$$= \begin{cases} \dfrac{2\pi}{b\sqrt{\frac{(a^2-b^2)}{b^2}}} & \text{if } \frac{a}{b}>1 \\ \dfrac{-2\pi}{b\sqrt{(a^2-b^2)/b^2}} & \text{if } \frac{a}{b}<1. \end{cases}$$

Given that $|a| > |b|$

(i) $\frac{a}{b} > 1$ iff $a>0, b>0, \; a>b$, or

$a<0, b<0, \; a<b$.

(ii) $\frac{a}{b} < -1$ iff $a>0, \; b<0, \; a>-b$ or

$a<0, \; b>0, \; a<-b$.

Remember
$\sqrt{b^2} = \begin{cases} b \text{ if } b>0 \\ -b \text{ if } b<0. \end{cases}$, so we get

$$I = \begin{cases} \dfrac{2\pi}{b \cdot \sqrt{\frac{a^2-b^2}{b^2}}} & , \; a>0 \; b>0 \; a>b \\ \\ \dfrac{-2\pi}{\sqrt{a^2-b^2}} & a<0, \; b<0, \; a<b. \\ \\ \dfrac{2\pi}{\sqrt{a^2-b^2}} & a>0, b<0, a>-b \\ \\ \dfrac{-2\pi}{\sqrt{a^2-b^2}} & a<0, \; b>0, a<-b. \end{cases}$$

$$= \begin{cases} \dfrac{2\pi}{\sqrt{a^2-b^2}} & \text{if } a>0 \\ \\ \dfrac{-2\pi}{\sqrt{a^2-b^2}} & \text{if } a<0. \end{cases}$$

<u>Ex</u>: find $I = \int_0^{2\pi} \frac{d\theta}{1+b\sin\theta} d\theta \qquad |b|<1$