Q: Prove that the function $u(x,y) = x^3 - 3xy^2$ is harmonic and obtain its conjugate.

Sol^n: We have

$$u(x,y) = x^3 - 3xy^2 \longrightarrow (1)$$

$$\therefore \frac{\partial u}{\partial x} = 3x^2 - 3y^2 \; ; \quad \frac{\partial u}{\partial y} = -6xy$$

$$\frac{\partial^2 u}{\partial x^2} = 6x \; ; \quad \frac{\partial^2 u}{\partial y^2} = -6x .$$

Hence, $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$. Thus, $u(x,y)$ is a harmonic function.

$$\therefore \frac{\partial v}{\partial y} = \frac{\partial u}{\partial x} = 3x^2 - 3y^2 .$$

Integrating this differential equation with respect to y, taking x as fixed, we have

$$v = 3x^2 y - y^3 + g(x)$$

where $g(x)$ is some real function of x.

Now, using the other one of the Cauchy - Riemann equations $\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$, we obtain the identity

$$6xy + g'(x) = 6xy \qquad \text{or } g'(x) = 0 .$$

so that $g(x) = c$, where c is an arbitrary real constant. Hence the conjugate of $u(x,y)$ is

$$v(x,y) = 3x^2 y - y^3 + c .$$

Remarks: It may be observed that the analytic function $f(z) = u(x,y) + i v(x,y)$ is given by

$$f(z) = x^3 - 3xy^2 + i(3x^2 y - y^3 + c)$$

$$= x^3 + i^2 3xy^2 + 3x^2(iy) + (iy)^3 + ic$$

$$= (x+iy)^3 + ic = z^3 + ic$$

This implies,

$f'(z) = 3z^2$; which exists for all finite values of z. Therefore, $f(z)$ is analytic in the finite complex plane.

Second Method:

$$f'(z) = u_x - i u_y$$

$$= (3x^2 - 3y^2) - i(-6xy)$$

$$= (3z^2 - 0) - i(-6z \cdot 0)$$

$$= 3z^2$$

$$\therefore f(z) = \cancel{3z^3 + c}\ z^3 + c.$$

Complex Integration:

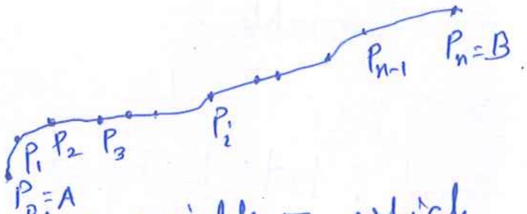
If $f(z)$ be a continuous function of the complex variable $z = x+iy$ defined at all points of a curve C having end points A and B. Divide C into n parts at the points

$$A = P_0(z_0),\ P_1(z_1), \ldots, P_i(z_i), \ldots, P_n(z_n) = B.$$

Let $\delta z_i = z_i - z_{i-1}$ and ξ_i be any point on the arc $P_{i-1}P_i$. Then the limit of the sum

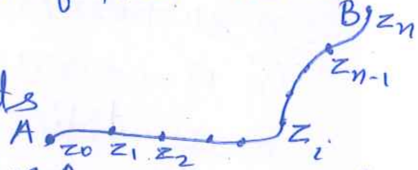
$$\sum_{i=1}^{n} f(\xi_i)\,\delta z_i$$

as $n \to \infty$ in such a way that the length of the chord δz_i approaches zero, is called the line integral of $f(z)$ taken along the path C i.e. $\int_C f(z)\,dz$.



Equivalent Definition:

Let $f(z)$ be a continuous function of the complex variable z, which is not necessarily analytic but has a definite value at each point of a rectifiable arc C with equation of this arc be $z = x(t) + i y(t)$, where $\alpha \le t \le \beta$.



Divide the arc into n portions by the points

$\{a = z_0, z_1, z_2, \ldots, z_n = b\}$, z_0 being A and z_n being B; let these points correspond to the values

$$\alpha = t_0 < t_1 < t_2 < \cdots < t_{n-1} < t_n = \beta.$$

Consider the sum

$$\sum_{r=1}^{n} f(\xi_r)(z_r - z_{r-1})$$

where ξ_r is any point in the arc $z_{k-1}z_k$.

If for every choice of point ξ_r and for every partition P this sum tends to a unique limit l as $n \to \infty$ and $\max\{|t_1 - t_0|, |t_2 - t_1|, \ldots, |t_n - t_{n-1}|\} \to 0$ i.e. $\|P\| \to 0$, then we write

$$I = \int_C f(z)\,dz \longrightarrow (1)$$

It is also pointed out that continuity of $f(z)$ on C is a sufficient condition for the existence the the integral (1).

$$\therefore \int_C f(z)\,dz = \lim_{\substack{n \to \infty \\ \|P\| \to 0}} \sum_{r=1}^{n} f(\xi_r)(z_r - z_{r-1})$$

where $z_{r-1} < \xi_r < z_r \longrightarrow (2)$

Complex Integration

In the case of a real variable, a distinction is made between definite and indefinite integrals — the formal being regarded as the limit of a sum and the latter as a process inverse to differentiation. We make a similar distinction between definite and indefinite integrals of a complex variable. Definite integrals of complex variable are usually known as line integrals. As in the case of real variable, an indefinite integral of a complex variable function whose derivative equals a given analytic function in a region. However, the theory of definite integrals of a real variable does not extend straight way to the domain of complex variable.

Rectifiable curve:

Def^n: Let L be a (continuous) curve with equation

$$z = x(t) + iy(t) \quad \{\alpha \leq t \leq \beta\}$$

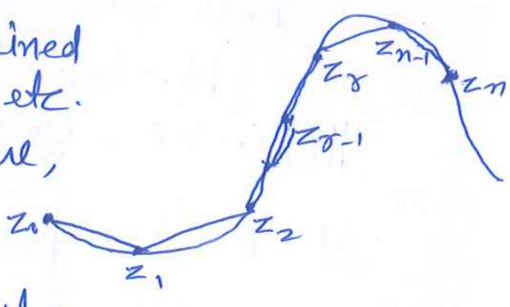
and suppose we ~~dev~~ divide the interval $[\alpha, \beta]$ into n subintervals $[t_{K-1}, t_K]$ $(K = 1, 2, \dots, n)$ by introducing the $(n-1)$ intermediate points $t_1, \dots, t_{n-1}$ satisfying the inequalities $\alpha = t_0 < t_1 < \dots < t_{n-1} < t_n = \beta$.

The set $P = \{t_0, t_1, \dots, t_n\} \longrightarrow (1)$

is called a ~~path~~ partition of the interval $[\alpha, \beta]$ and the largest of the numbers $t_1 - t_0, t_2 - t_1, \dots, t_n - t_{n-1}$ i.e. the maximum length of the subintervals is called the norm of the partition P, denoted by $\|P\|$.

Let $z_0, z_1, \dots, z_n$ be the points on the curve corresponding to the ~~the~~ values of $t_0, t_1, \dots, t_n$ i.e. $z(t_K) = z_K$. Evidently the length of the polygon curve inscribed in L obtained by joining successively z_0, z_1 and z_2 etc. by straight line segments as in figure, is given by

$$\sum_{K=1}^{n} |z_K - z_{K-1}| \longrightarrow (2)$$



The curve L is said to be rectifiable if

$$\sup \sum |z_K - z_{K-1}| = l < \infty$$

where the least upper bound or supremum is taken over all possible partitions given by (1).

Remaining of (1) page

Contours :

Let $z = z(t) = x(t) + iy(t) \longrightarrow (1)$
where t runs through the interval $\alpha \le t \le \beta$
and $x(t)$, $y(t)$ are continuous functions of t represents a continuous arc L in the complex plane.

If the equation (1) satisfied by more than one values of t in the given range, then the point z or say the point (x, y) is a multiple point of the arc.
A continuous arc without multiple point is called Jordan arc.

If for a point z on a Jordan arc, z as expressed in equation (1) is one valued and $x(t)$, $y(t)$ are continuous and if $x'(t)$ and $y'(t)$ are continuous in the range $\alpha \le t \le \beta$, then the arc is called a regular arc of a Jordan curve.
[Regular arc is always rectifiable.]
A Jordan curve consisting of continuous chain of a finite number of regular arcs is called a contour.

Remaining of (2) :

Also, $\int_C f(z)\,dz$ is called the complex line integral or simply the line integral of $f(z)$ along C or the definite integral of $f(z)$ from a to b along C.

Real line Integral :

Let $P(x,y)$ and $Q(x,y)$ be real valued functions of x and y continuous at all points of curve C. Then the real line integral of $P\,dx + Q\,dy$ along the curve C is denoted by

$$\int_C \{P(x,y)\,dx + Q(x,y)\,dy\} \text{ or simply by } \int_C P\,dx + Q\,dy .$$

Note : If $f(z) = u(x,y) + iv(x,y)$ and noting that $dz = dx + i\,dy$.

$$\therefore \int_C f(z)\,dz = \int_C \big(u(x,y)\,dx - v(x,y)\,dy\big) + i\int_C \big(v(x,y)\,dx + u(x,y)\,dy\big)$$

which shows that the evaluation of the line integral of a complex function can be reduced to the evaluation of two line integrals of real functions.

Evaluation of Integrals by definition :

The evaluation of integral (complex integral) by the direct application of the definition is quite difficult. However, we give below integrals of some simple functions.

Ex: Using the definition of an integral as the limit of sum evaluate the following integrals :

(i) $\int_C dz$ (ii) $\int_C |dz|$ (iii) $\int_C z\,dz$

where C is any rectifiable arc joining the points $z=a$ to $z=b$.

Solⁿ :- We first note that the integral exists since the integrand is continuous on C in each case.

(i) Using the definition of complex integral, we have

$$\int_C dz = \lim_{n\to\infty} \sum 1\cdot(z_r - z_{r-1}) \quad [\because f(z)=f(\xi_r)=1]$$

$$= \lim_{n\to\infty} \left[(z_1-z_0)+(z_2-z_1)+\cdots+(z_n-z_{n-1})\right]$$

$$= \lim_{n\to\infty} [z_n - z_0] = b-a.$$

(ii) $$\int_C |dz| = \lim_{n\to\infty} \sum_{r=1}^{n} |z_r - z_{r-1}| = \lim_{n\to\infty}\left[|z_1-z_0|+|z_2-z_1|+\cdots+|z_n-z_{n-1}|\right]$$

$$= \lim_{n\to\infty}\left[\text{chord } z_1z_0 + \text{chord } z_2z_1 + \cdots + \text{chord } z_nz_{n-1}\right]$$

$$= \left[\text{arc } z_1z_0 + \text{arc } z_2z_1 + \cdots + \text{arc } z_nz_{n-1}\right]$$

$$= \text{Arc length of } C.$$

(iii) $$\int_C z\,dz = \lim_{n\to\infty} \sum_{r=1}^{n} \xi_r(z_r - z_{r-1}) \longrightarrow (1)$$

Since ξ_r is arbitrary. Thus taking $\xi_r \equiv z_r$ and z_{r-1}, successively in (1), we get

$$\int_C z\,dz = \lim_{n\to\infty} \sum_{r=1}^{n} z_r(z_r - z_{r-1}) \longrightarrow (2)$$

$$\therefore \int_C z\,dz = \lim_{n\to\infty} \sum_{r=1}^{n} z_{r-1}(z_r - z_{r-1}) \longrightarrow (3)$$

Adding (2) and (3), we get

$$2\int_C z\,dz = \lim_{n\to\infty} \sum (z_r^2 - z_{r-1}^2) = \lim_{n\to\infty}(z_n^2 - z_0^2)$$

$$\therefore \int_C z\,dz = \frac{b^2-a^2}{2} \quad \text{Ans.}$$

Evaluation of Integrals:

Note: In case the curve C is a closed, then the end points a and b coincide and as such

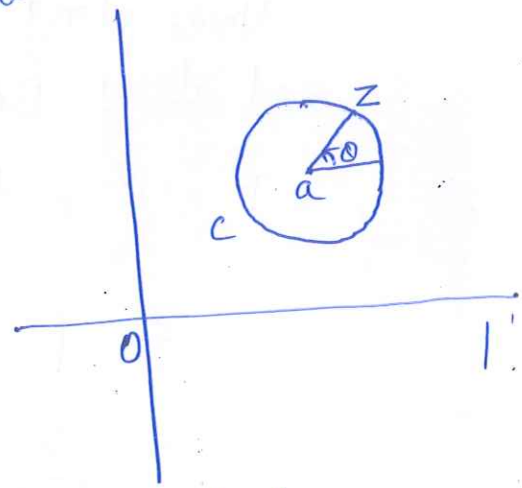
$$\int_C dz = 0 \quad \text{and} \quad \int_C z\,dz = 0.$$

Example: Prove that

(i) $\int_C \frac{dz}{z-a} = 2\pi i$ (ii) $\int_C (z-a)^n dz = 0$ [n, any integer $\neq -1$],

where C is the circle $|z-a| = r$.

Sol^n: The parametric equation of C is $z-a = re^{i\theta}$ where θ varies from 0 to 2π as z describes C once in the positive (anti-clockwise) sense.



(i) $$\int_C \frac{dz}{z-a} = \int_0^{2\pi} \frac{1}{re^{i\theta}} \cdot ire^{i\theta}\, d\theta$$

$$= i\int_0^{2\pi} d\theta = 2\pi i.$$

(ii) $$\int_C (z-a)^n dz = \int_0^{2\pi} r^n e^{ni\theta} \cdot ire^{i\theta}\, d\theta$$

$$= ir^{n+1}\int_0^{2\pi} e^{(n+1)\theta i}\, d\theta$$

$$= \frac{r^{n+1}}{n+1}\left[e^{(n+1)\theta i}\right]_0^{2\pi}, \quad \text{provided } n \neq -1.$$

$$= \frac{r^{n+1}}{n+1}\left[e^{2(n+1)\pi i} - 1\right] = 0 \quad \left[\because e^{2(n+1)\pi i} = 1\right].$$

(3) Ex: Evaluate $\int_0^{2+i} (\bar{z})^2 dz$ along

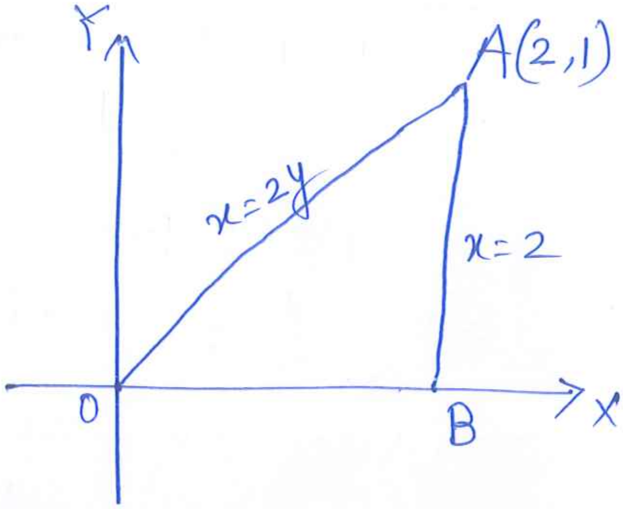
(i) the line $y = \frac{x}{2}$

(ii) the real axis to 2 and then vertically to $2+i$.

Sol^n:- (i) Along the line OA

$x = 2y,\quad z = (2+i)y,\quad \bar{z} = (2-i)y$

and $dz = (2+i)\,dy$



$$\therefore I = \int_0^{2+i} (\bar{z})^2 dz$$

$$= \int_0^1 (2-i)^2 y^2 (2+i)\, dy$$

$$= 5(2-i)\left[\frac{y^3}{3}\right]_0^1$$

$$= \frac{5}{3}(2-i) \quad \underline{\text{Ans}}$$

(ii) $$I = \int_{OB} (\bar{z})^2 dz + \int_{BA} (\bar{z})^2 dz$$

Now, along OB, $z = x$, $\bar{z} = x$, $dz = dx$;

and along BA, $z = 2+iy$, $\bar{z} = 2-iy$, $dz = i\,dy$

$$\therefore I = \int_0^2 x^2 dx + \int_0^1 (2-iy)^2 \cdot i\,dy$$

$$= \left[\frac{x^3}{3}\right]_0^2 + \int_0^1 \left[4y + (4-y^2)i\right] dy$$

$$= \frac{8}{3} + \left[4\frac{y^2}{2} + i\left\{4y - \frac{y^3}{3}\right\}\right]_0^1$$

$$= \frac{8}{3} + 4\cdot\frac{1}{2} + i\left\{4\cdot 1 - \frac{1}{3}\right) = \frac{14}{3} + \frac{11i}{3}$$

$$= \frac{1}{3}(14 + 11i) \quad \underline{\text{Ans}}.$$

Cauchy's Integral Theorem :

Green's Theorem :

Let C be a positively oriented, piecewise smooth, simple closed curve in a plane and let D be the region bounded by C. If L and M are functions of x and y defined on an open region containing D and have continuous partial derivatives there, then

$$\oint_C (L\,dx + M\,dy) = \iint_D \left(\frac{\partial M}{\partial x} - \frac{\partial L}{\partial y}\right) dx\,dy$$

where path of integration along C is anti-clockwise.

There are several forms of Cauchy's theorem, but they differ in their topological rather than in their analytical content.
It is natural to begin with a case in which the topological considerations are trivial.

Cauchy's Fundamental Theorem :

St : If $f(z)$ is analytic, with a continuous derivative, in a simply connected domain G and C is a closed contour lying in G, then

$$\int_C f(z)\,dz = 0 .$$

Proof: Let $f(z) = u(x,y) + i v(x,y)$ $\longrightarrow$ (1)
be an analytic function with a continuous derivative in a simply connected domain G and C is a closed contour lying in G.
Then by Cauchy-Riemann equations for analytic functions

$$f'(z) = u_x + i v_x = v_y - i u_y \longrightarrow (2)$$

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for all points in the domain. Since $f'(z)$ is continuous, the four partial derivatives u_x, u_y, v_x and v_y must also be continuous in G.

Also, $dz = dx + i\,dy$, then

$$\int_C f(z)\,dz = \int_C (u+iv)(dx+i\,dy)$$

$$= \int_C (u\,dx - v\,dy) + i\int (v\,dx + u\,dy) \longrightarrow (3)$$

Therefore, by Green's theorem which states that

$$\int_C (P\,dx + Q\,dy) = \iint_G \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy \longrightarrow (4)$$

Then from (3), we have

$$\int_C f(z)\,dz = -\iint_G \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) dx\,dy + i\iint_G \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) dx\,dy$$

$$= -\iint_G \left(\frac{\partial v}{\partial x} - \frac{\partial v}{\partial x}\right) dx\,dy + i\iint_G \left(\frac{\partial v}{\partial y} - \frac{\partial v}{\partial y}\right) dx\,dy$$

(By using C-R equations)

$$= 0$$

$$\therefore \int_C f(z)\,dz = 0.$$ Proved.

Note: This form of Cauchy's theorem is of much practical utility in applied mathematics in as much as the continuity of the four partial derivatives u_x, u_y, v_x, v_y is generally assumes on physical grounds......

The absolute value of a complex integral :

Th: If $f(z)$ is continuous on a closed contour C of length l and $|f(z)| \le M$ for every point z on C, then

$$\left| \int_C f(z)\,dz \right| \le Ml$$

$$\text{or} \quad \left| \int_C f(z)\,dz \right| \le \int_C |f(z)|\,|dz|$$

Proof : By the definition of complex line integral, we have

$$\int_C f(z)\,dz = \lim_{n\to\infty} \sum f(\xi_K)(z_K - z_{K-1})$$

where ξ_K is any point on the arc $z_K z_K'$

Now, $$\left| \sum_{K=1}^{n} f(\xi_K)(z_K - z_{K-1}) \right| \le \sum_{K=1}^{n} |f(\xi_K)|\,|z_K - z_{K-1}|$$

$$\le M \sum_{K=1}^{n} |z_K - z_{K-1}| \quad (\because |f(z)| \le M,\ \forall z \text{ on } C)$$

$$\therefore \left| \int_C f(z)\,dz \right| \le M \lim_{n\to\infty} \sum_{K=1}^{n} |z_K - z_{K-1}| \longrightarrow (1)$$

But $$\lim_{n\to\infty} \sum_{K=1}^{n} |z_K - z_{K-1}| = l \ (\text{length of the contour } C) \longrightarrow (2)$$

Hence from (1) and (2), we get

$$\left| \int_C f(z)\,dz \right| \le \int_C |f(z)|\,|dz| .$$

Some properties of complex integral :

(I) $\int_L (f(z) + \phi(z))\,dz = \int_L f(z)\,dz + \int_L \phi(z)\,dz$

(II) $\int_L f(z)\,dz = -\int_{-L} f(z)\,dz$

(III) $\int_{L_1+L_2} f(z)\,dz = \int_{L_1} f(z)\,dz + \int_{L_2} f(z)\,dz$

where the final point of L_1 coincides with the initial point of L_2.

In general, if $L = L_1 + L_2 + \cdots + L_n$ where the final point of L_K coincides with the initial point of L_{K+1} $(K = 1, 2, \ldots, n-1)$, then

$$\int_L f(z)\,dz = \int_{L_1} f(z)\,dz + \int_{L_2} f(z)\,dz + \cdots + \int_{L_n} f(z)\,dz .$$

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(IV) $\int_L c\, f(z)\,dz = c\int_L f(z)\,dz$, where c is any complex constant.

(V) $\int_L [c_1 f_1(z) + c_2 f_2(z) + \cdots + c_n f_n(z)]\,dz$
$= c_1 \int_L f_1(z)\,dz + c_2 \int_L f_2(z)\,dz + \cdots + c_n \int_L f_n(z)\,dz.$

This follows from the properties III and IV.

(VI) $\left|\int_L f(z)\,dz\right| \le \int_L |f(z)|\,|dz|$.

The Circle

The equation

$$az\bar{z} + \alpha z + \bar{\alpha}\bar{z} + c = 0 \longrightarrow (1)$$

represents a real circle or a straight line provided $\alpha\bar{\alpha} > ac \longrightarrow (2)$

where a, c are real constants, α is a complex constant and z a complex variable. For if we write $\alpha = a_1 + ia_2$ and $\bar{\alpha} = a_1 - ia_2$, $z = x+iy$ and $\bar{z} = x-iy$, the equation (1) become

$$a(x^2+y^2) + (a_1+ia_2)(x+iy) + (a_1-ia_2)(x-iy) + c = 0.$$

$$\text{or, } ax^2 + ay^2 + 2a_1x - 2a_2y + c = 0 \longrightarrow (3)$$

which is a real ~~real~~ circle if its radius is positive.

The centre and radius of (3) are

$$\left(-\frac{a_1}{a}, \frac{a_2}{a}\right) \text{ and } \sqrt{\left\{\frac{a_1^2}{a^2} + \frac{a_2^2}{a^2} - \frac{c}{a}\right\}} = \sqrt{\left\{\frac{a_1^2+a_2^2-ac}{a^2}\right\}}$$

[Note that centre and radius of the circle $x^2+y^2+2gx+2fy+c=0$ are $(-g,-f)$ and $\sqrt{g^2+f^2-c}$].

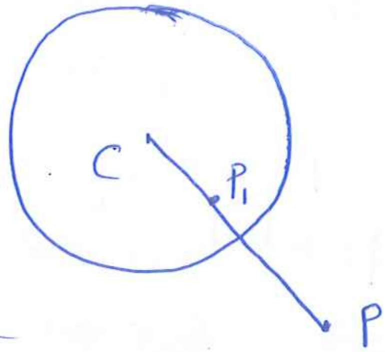
Hence the centre and the radius of the circle in the form (1) are $-\frac{a_1-ia_2}{a} = -\frac{\bar{\alpha}}{a}$ and

$$\sqrt{\frac{\alpha\bar{\alpha}-ac}{a^2}}.$$

It follows that (1) represents a real circle or a straight line if $\alpha\bar{\alpha} - ac > 0$ i.e. $\alpha\bar{\alpha} > ac$ which is the condition (2).

Inverse Points with respect to a circle:

Let S denote a circle of radius r and centre C. Then the two points P and P′ collinear with C are said to be inverse points with respect to the circle provided



$$CP' \cdot CP = r^2 \longrightarrow (1)$$

We now find the relation between the inverse points with respect to the circle

$$\alpha z\bar{z} + \alpha z + \bar{\alpha}\bar{z} + c = 0 \longrightarrow (2)$$

where a, c are real and α is a complex constant.

We have already shown that the centre C of this circle is $-\frac{\bar{\alpha}}{\alpha}$ and the radius r is given by $r^2 = \frac{\alpha\bar{\alpha} - ac}{a^2} \rightarrow (3)$

Let P_1, P be the inverse points with respect to the circle with affixes z' and z respectively. The condition (1) in this case becomes

$$\left|z' + \frac{\bar{\alpha}}{a}\right|\left|z + \frac{\bar{\alpha}}{a}\right| = r^2 \longrightarrow (4)$$

Also, since C, P_1, P are collinear, we have

$$\arg\left(z' + \frac{\bar{\alpha}}{a}\right) = \arg\left(z + \frac{\bar{\alpha}}{a}\right)$$

which may be written as

$$\arg\left(z' + \frac{\bar{\alpha}}{a}\right) = -\arg\left(\bar{z} + \frac{\alpha}{a}\right)$$

[$\because \arg z = -\arg\bar{z}$ for any complex number z.]

or, $$\arg\left(z' + \frac{\bar{\alpha}}{a}\right)\left(\bar{z} + \frac{\alpha}{a}\right) = 0 \longrightarrow (5)$$

The equation (5) shows that

$\left(\bar{z} + \frac{\bar{\alpha}}{a}\right)\left(\bar{z} + \frac{\alpha}{a}\right)$ is a positive real number.

Also, (4) may be written as

$$\left|z' + \frac{\bar{\alpha}}{a}\right|\left|\bar{z} + \frac{\alpha}{a}\right| = r^2 \longrightarrow (6)$$

[$\because |z| = |\bar{z}|$ for any complex number z].

Hence the conditions (5) and (6) are equivalent to a single condition

$$\left(z' + \frac{\bar{\alpha}}{a}\right)\left(\bar{z} + \frac{\alpha}{a}\right) = r^2 \quad \text{or} \quad z'z + \frac{z'\alpha}{a} + \frac{\bar{z}\bar{\alpha}}{a^2} + \frac{\alpha\bar{\alpha}}{a^2} = \frac{\alpha\bar{\alpha} - ac}{a^2} \text{ by (3).}$$

or, $$a z'\bar{z} + \alpha z' + \bar{\alpha}\bar{z} + c = 0 \longrightarrow (7)$$

The equation (7) gives a relation between z and its inverse z'. Thus to obtain a relation between z and its inverse z', we replace z by z' and leave z' unchanged in (2).

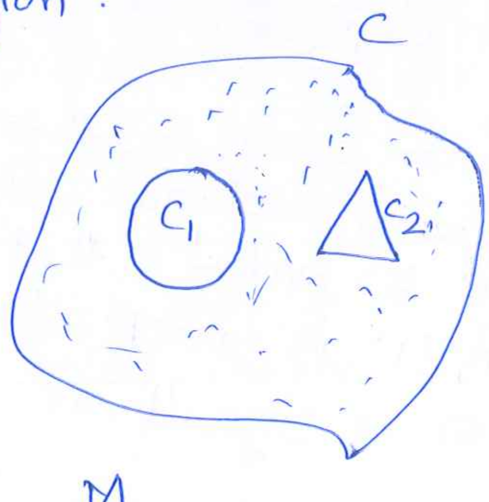
Simply and Multiply connected Regions

Def^n: A region is said to be connected region if any two points of the region G can be connected by a curve which lies entirely within the region.

A connected region is simply connected region if every closed curve in the region can be shrunk to a point without passing out of the region otherwise it is said to be multiply connected. In other words, if all the points of the area bounded by any single closed curve C drawn in the region G are the points of the region G, then it is called the simply connected region, otherwise it is called the multiply connected region.

For example let there be a number of ~~closed~~ closed curve C, C_1, C_2, ... all drawn in a certain region G. If all the points of the area lying between the closed curves C, C_1, C_2, ... the area which is interior to C and exterior to the other curves C_1, C_2, ... are the points of the region G, then the region G is multi-connected region.

Simply connected region



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Cauchy's Integral Formula:

Statement: Let $f(z)$ be an analytic function in a simply connected domain Q bounded by a rectifiable Jordan curve C and is ~~continuous~~ continuous on C. Then

$$f(z_0) = \frac{1}{2\pi i}\int_C \frac{f(z)\,dz}{z-z_0}$$, where z_0 is within C.

OR,

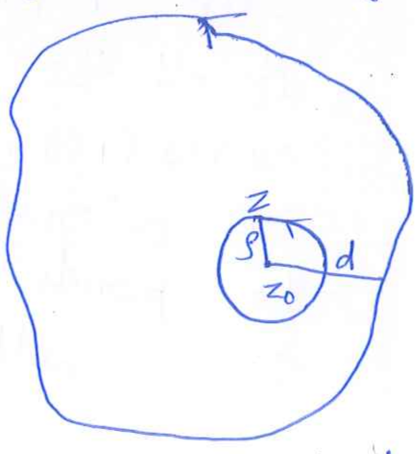
If $f(z)$ is analytic within and on a closed contour C and z_0 is any point within C, then

$$f(z_0) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z-z_0}dz.$$

Proof: Let $f(z)$ be an analytic function within and on a closed contour C and z_0 is any point within C.

We describe a circle C_1 defined by the equation $|z-z_0| = \rho$, where $\rho < d$ (the distance of z_0 from C).

Then the function $\phi(z) = \frac{f(z)}{z-z_0}$ is analytic in the doubly connected region bounded by C and C_1.



Hence, we have $\int_C \phi(z)\,dz = \int_{C_1} \phi(z)\,dz$.

$$\text{or, } \frac{1}{2\pi i}\int_C \frac{f(z)}{z-z_0}dz = \frac{1}{2\pi i}\int_{C_1}\frac{f(z)}{z-z_0}dz \longrightarrow (1)$$

where C and C_1 are both traveled in the counter-clockwise direction.

It is evident that the integral on the right-hand side of (1) is independent of ρ and so we may choose ρ as small as we please.

Now, $\frac{1}{2\pi i}\int_{C_1}\frac{f(z)}{z-z_0}dz = \frac{1}{2\pi i}\int_{C_1}\frac{f(z)-f(z_0)}{z-z_0}dz + \frac{1}{2\pi i}\int_{C_1}\frac{f(z_0)}{z-z_0}dz \longrightarrow (2)$

Writing $z-z_0=\rho e^{i\theta}$, $dz=\rho i e^{i\theta}d\theta$, we have

$$\int_{C_1}\frac{f(z_0)}{z-z_0}dz = f(z_0)\int_0^{2\pi}\frac{\rho i e^{i\theta}}{\rho e^{i\theta}}d\theta = f(z_0)\int_0^{2\pi} i\,d\theta = 2\pi i f(z_0).$$

$\therefore \frac{1}{2\pi i}\int_{C_1}\frac{f(z_0)}{z-z_0}dz = f(z_0).$

Hence, from (2), we can write

$$\frac{1}{2\pi i}\int_{C_1}\frac{f(z)}{z-z_0}dz = \frac{1}{2\pi i}\int_{C_1}\frac{f(z)-f(z_0)}{z-z_0}dz + f(z_0).$$

or, $\frac{1}{2\pi i}\int_{C_1}\frac{f(z)}{z-z_0}dz - f(z_0) = \frac{1}{2\pi i}\int_{C_1}\frac{f(z)-f(z_0)}{z-z_0}dz \longrightarrow (3)$

Since $f(z)$ is continuous at z_0, for a given $\epsilon>0$, there exists a $\delta>0$ such that

$$|f(z)-f(z_0)|<\epsilon \longrightarrow (4)$$

for all z satisfying the inequality $|z-z_0|<\delta$.

Since ρ is at our choice, we can take $\rho<\delta$ so that the inequality (4) is satisfied for all points on C_1.

Hence

$$\left|\frac{1}{2\pi i}\int_{C_1}\frac{f(z)-f(z_0)}{z-z_0}dz\right| = \left|\frac{1}{2\pi i}\int_0^{2\pi}\frac{f(z)-f(z_0)}{\rho e^{i\theta}}\cdot\rho i e^{i\theta}d\theta\right|$$

$$\le \frac{1}{2\pi}\int_0^{2\pi}|f(z)-f(z_0)|\,d\theta < \frac{1}{2\pi}\int_0^{2\pi}\epsilon\,d\theta. \quad (\text{by } (4)).$$

$$= \frac{1}{2\pi}\cdot 2\pi\epsilon = \epsilon.$$

Thus, $\left|\frac{1}{2\pi i}\int_{C_1}\frac{f(z)}{z-z_0}dz - f(z_0)\right| < \epsilon \longrightarrow (5)$

It then follows from (3) and (5) that $\left|\frac{1}{2\pi i}\int_{C_1}\frac{f(z)}{z-z_0}dz - f(z_0)\right|<\epsilon \longrightarrow (6)$

Since ϵ was arbitrary and the left hand side of (6) does not depend upon ρ, we conclude that $\frac{1}{2\pi i}\int_{C_1}\frac{f(z)}{z-z_0}dz - f(z_0)=0$ or $\frac{1}{2\pi i}\int_{C_1}\frac{f(z)}{z-z_0}dz = f(z_0)$.

Finally from (1) & (7), we have $f(z_0) = \frac{1}{2\pi i}\int_C\frac{f(z)}{z-z_0}dz$. Proved. $\longrightarrow (7)$

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The Derivatives of an analytic function

① Th: If a function $f(z)$ is analytic within on a simple closed contour C, then its derivative at any point z_0 inside C is given by

$$f'(z_0) = \frac{1}{2\pi i}\int_C \frac{f(z)}{(z-z_0)^2}\,dz.$$

② Th: The derivative of an analytic function is itself an analytic function.

(3) Th: If $f(z)$ is analytic in a simply connected domain G containing a simple closed contour C, then $f(z)$ has derivatives of all orders at each point z_0 inside C with

$$f^n(z_0) = \frac{\lfloor n}{2\pi i}\int_C \frac{f(z)\,dz}{(z-z_0)^{n+1}}.$$

Proof :- ① Let z_0+h be any point in the neighbourhood of the point z_0 so that h is at our choice. Then by Cauchy's integral formula, we have

$$f(z_0+h) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z-(z_0+h)}\,dz$$

Hence,

$$\frac{f(z_0+h)-f(z_0)}{h} = \frac{1}{2\pi i h}\int_C \left(\frac{1}{z-z_0-h} - \frac{1}{z-z_0}\right) f(z)\,dz$$

$$= \frac{1}{2\pi i}\int_C \frac{f(z)}{(z-z_0-h)(z-z_0)}\,dz$$

$$= \frac{1}{2\pi i}\int_C \frac{(z-z_0)f(z)}{(z-z_0-h)(z-z_0)^2}\,dz$$

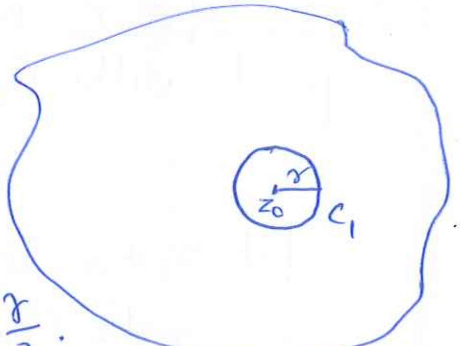
$$= \frac{1}{2\pi i}\int_C \frac{(z-z_0-h+h)f(z)}{(z-z_0-h)(z-z_0)^2}\,dz$$

$$= \frac{1}{2\pi i}\int_C \frac{f(z)\,dz}{(z-z_0)^2} + \frac{h}{2\pi i}\int_C \frac{f(z)\,dz}{(z-z_0-h)(z-z_0)^2} \quad \rightarrow ①$$

The result follows on taking the limits as $h \to 0$ if we can show that the last term of ① approaches to zero. For this describe a circle $C_1 : |z-z_0| = r$ such that C_1 lies entirely within C. Then by Cauchy's theorem for multiply connected regions, we have

$$\frac{h}{2\pi i}\int_C \frac{f(z)\,dz}{(z-z_0-h)(z-z_0)^2} = \frac{h}{2\pi i}\int_{C_1} \frac{f(z)\,dz}{(z-z_0-h)(z-z_0)^2} \longrightarrow ②$$

Choose h small enough so that $|h| \le \frac{r}{2}$, therefore z_0+h lies within C_1. Since $f(z)$ is analytic on C_1, it is bounded, therefore, for some positive real constant M, we have $|f(z)| \le M$.



Also, $|z-z_0-h| \geq |z-z_0| - |h| \geq r - \frac{r}{2} = \frac{r}{2}$.

Now, $$\left|\frac{h}{2\pi i}\int_{C_1} \frac{f(z)dz}{(z-z_0-h)(z-z_0)^2}\right| \le \frac{|h|}{2\pi}\int_{C_1} \frac{|f(z)|\,|dz|}{|z-z_0-h|\,|z-z_0|^2}$$

$$\le \frac{M|h|}{2\pi r^2\left(\frac{r}{2}\right)}\int_{C_1}|dz|$$

$$\le \frac{M|h|}{\pi r^3}\cdot 2\pi r$$

Thus $$\left|\frac{h}{2\pi i}\int_{C_1} \frac{f(z)\,dz}{(z-z_0-h)(z-z_0)^2}\right| \le \left(\frac{2M}{r^2}\right)|h| \longrightarrow ③$$

From ② and ③, we find that

$$\left|\frac{h}{2\pi i}\int_C \frac{f(z)\,dz}{(z-z_0-h)(z-z_0)^2}\right| \le \left(\frac{2M}{r^2}\right)|h|$$

and it follows that the left-hand side approaches to zero as $h \to 0$.

Hence $\lim_{h\to 0} \frac{f(z_0+h)-f(z_0)}{h} = \frac{1}{2\pi i}\int_C \frac{f(z)\,dz}{(z-z_0)^2}$.

Thus $f(z)$ is differentiable at z_0 and $f'(z_0) = \frac{1}{2\pi i}\int_C \frac{f(z)\,dz}{(z-z_0)^2} \longrightarrow ④$

Note: The formula (4) for the derivative $f'(z_0)$ can be written formally by differentiating the integral in Cauchy's integral formula

$$f(z_0) = \frac{1}{2\pi i}\int_C \frac{f(z)}{(z-z_0)}\,dz$$

with respect to z_0 under the integral sign.

$$f'(z_0) = \frac{1}{2\pi i}\int_C \frac{d}{dz_0}\left(\frac{f(z)}{z-z_0}\right)dz$$

$$= \frac{1}{2\pi i}\int_C \frac{f(z)}{(z-z_0)^2}\,dz.$$

Proof (2): Let $f(z)$ be an analytic function in the domain G. If C is any closed contour in G and $z=z_0$ be any point within C, then by theorem, we have

$$f'(z_0) = \frac{1}{2\pi i}\int_C \frac{f(z)}{(z-z_0)^2}\,dz \longrightarrow (1)$$

Let z_0+h be a point in the neighbourhood of the point z_0 inside C, then

$$\frac{f'(z_0+h)-f'(z_0)}{h} = \frac{1}{2\pi i h}\int_C \left[\frac{1}{(z-z_0-h)^2} - \frac{1}{(z-z_0)^2}\right] f(z)\,dz$$

$$= \frac{1}{2\pi i h}\int_C \left[\frac{(z-z_0)^2-(z-z_0-h)^2}{(z-z_0-h)^2(z-z_0)^2}\right] f(z)\,dz$$

$$= \frac{2}{2\pi i}\int_C \frac{(z-z_0)-h/2}{(z-z_0)^2(z-z_0-h)^2}\, f(z)\,dz$$

$$= \frac{\lfloor 2}{2\pi i}\int_C \frac{\{(z-z_0-h)+h\}\{(z-z_0-h)+\frac{h}{2}\}}{(z-z_0-h)^2(z-z_0)^3}\, f(z)\,dz$$

$$= \frac{\lfloor 2}{2\pi i}\int_C \frac{(z-z_0-h)^2+\frac{3h}{2}(z-z_0-h)+\frac{h^2}{2}}{(z-z_0-h)^2(z-z_0)^3}\, f(z)\,dz$$

$$= \frac{\lfloor 2}{2\pi i}\int_C \frac{f(z)\,dz}{(z-z_0)^3} + \frac{\lfloor 2\, h}{2\pi i}\int_C \frac{\frac{3}{2}(z-z_0)-h}{(z-z_0-h)^2(z-z_0)^3}\, f(z)\,dz$$

$$\therefore \frac{f'(z_0+h)-f'(z_0)}{h} - \frac{\lfloor 2}{2\pi i}\int_C \frac{f(z)}{(z-z_0)^3}\,dz$$

$$= \frac{\lfloor 2\, h}{2\pi i}\int_C \frac{\frac{3}{2}(z-z_0)-h}{(z-z_0-h)^2(z-z_0)^3} f(z)\,dz$$

$$= \frac{\lfloor 2}{2\pi i}\int_C f(z)\left[\frac{\frac{3}{2}h(z-z_0)-h^2}{(z-z_0)^3(z-z_0-h)^2}\right]dz$$

$$= \frac{\lfloor 2}{2\pi i}\int_{C_1} \frac{h\left\{\frac{3}{2}(z-z_0)-h\right\}}{(z-z_0)^3(z-z_0-h)^2} f(z)\,dz$$

where C_1 is the circle $|z-z_0|=\rho$ lying entirely within C.

Hence, $$\left|\frac{f'(z_0+h)-f'(z_0)}{h} - \frac{\lfloor 2}{2\pi i}\int_C \frac{f(z)}{(z-z_0)^3}\,dz\right|$$

$$\leq \frac{\lfloor 2}{2\pi}|h|\int_{C_1} \frac{\frac{3}{2}|z-z_0|+|-h|}{|(z-z_0)|^3\,|z-z_0-h|^2}|f(z)|\,|dz|$$

$$\leq \frac{\lfloor 2}{2\pi}|h|\,\frac{\frac{3}{2}\rho+|h|}{\rho^3\cdot\left(\frac{1}{2}\rho\right)^2}\cdot M\cdot 2\pi\rho \longrightarrow (2)$$

where M is the upper bound of $f(z)$ in C. [If we choose h such that $|h|<\frac{1}{2}\rho$ and so $|z-z_0-h|\geq\frac{\rho}{2}$ etc.]

Hence, when $h\to 0$, the right hand side of (2) also tends to zero and we have

$$\lim_{h \to 0} \frac{f'(z_0+h) - f'(z_0)}{h} = \frac{\lfloor 2}{2\pi i} \int_C \frac{f(z)}{(z-z_0)^3} dz.$$

Thus, $f'(z)$ is differentiable at z_0 and

$$f''(z_0) = \frac{\lfloor 2}{2\pi i} \int_C \frac{f(z)}{(z-z_0)^3} dz \longrightarrow (3)$$

Therefore, the formula holds for $n=2$.

Now, suppose that the formula is true for $n-1$, so that we assume that

$$f^{n-1}(z_0) = \frac{\lfloor n-1}{2\pi i} \int_C \frac{f(z)}{(z-z_0)^n} dz ;$$

and $f^{n-1}(z_0+h) = \dfrac{\lfloor n-1}{2\pi i} \displaystyle\int_C \frac{f(z)}{(z-z_0-h)^n} dz$

$$\therefore f^{n-1}(z_0+h) - f^{n-1}(z_0) = \frac{\lfloor n-1}{2\pi i} \int_C f(z) \left[\frac{1}{(z-z_0-h)^n} - \frac{1}{(z-z_0)^n} \right] dz$$

$$= \frac{\lfloor n-1}{2\pi i} \int_C f(z) \left[\frac{(z-z_0)^n - (z-z_0-h)^n}{(z-z_0)^n (z-z_0-h)^n} \right] dz.$$

Also, $(z-z_0)^n - (z-z_0-h)^n = [z-z_0-(z-z_0-h)]$

$$[(z-z_0)^{n-1} + (z-z_0)^{n-2}(z-z_0-h) + \cdots + ((z-z_0)-h)^{n-1}]$$

$$= h \sum_{r=1}^{n} (z-z_0)^{n-r} (z-z_0-h)^{r-1}.$$

Hence, we get

$$\frac{f^{n-1}(z_0+h) - f^{n-1}(z_0)}{h} - \frac{\lfloor n}{2\pi i} \int_C \frac{f(z)}{(z-z_0)^{n+1}} dz =$$

$$= \frac{\lfloor n-1}{2\pi i}\int_C f(z)\,\frac{\sum_{r=1}^{n}(z-z_0)^{n-r+1}(z-z_0-h)^{r-1}-n(z-z_0-h)^n}{(z-z_0)^{n+1}(z-z_0-h)^n}\,dz$$

$$= \frac{\lfloor n-1}{2\pi i}\int_{C_1} f(z)\,\frac{\sum_{r=1}^{n}\left[(z-z_0)^{n-r+1}(z-z_0-h)^{r-1}-n(z-z_0-h)^n\right]}{(z-z_0)^{n+1}(z-z_0-h)^n}\,dz$$

where C_1 is the circle $|z-z_0|=\rho$ lying entirely within C.

$$= \frac{\lfloor n-1}{2\pi i}\sum_{r=1}^{n}\int_{C_1} f(z)\,\frac{h\sum_{s=0}^{n-r}(z-z_0)^{n-r-s}(z-z_0-h)^s}{(z-z_0)^{n+1}(z-z_0-h)^{n-r+1}}\,dz$$

$$= \frac{h\lfloor n-1}{2\pi i}\sum_{r=1}^{n}\int_{C_1}\sum_{r=0}^{n}\frac{f(z)\,dz}{(z-z_0)^{r+s+1}(z-z_0)^{n-r+1}}$$

As before, $|z-z_0-h| \geq \frac{\rho}{2}$ and so

$$\left|\frac{f^{(n-1)}(z_0+h)-f^{(n-1)}(z_0)}{h}-\frac{\lfloor n}{2\pi i}\int_C\frac{f(z)}{(z-z_0)^{n+1}}dz\right|$$

$$\leq \frac{|h|\lfloor n-1}{2\pi}\sum_{r=1}^{n}\sum_{s=0}^{n}\frac{M\cdot 2\pi\rho}{\rho^{r+s+1}\left(\frac{1}{2}\rho\right)^{n-r-s+1}}$$

$\longrightarrow 0$ as $h\longrightarrow 0$.

$$\therefore \lim_{h\to 0}\frac{f^{n-1}(z_0+h)-f^{n-1}(z_0)}{h}=\frac{\lfloor n}{2\pi i}\int_C\frac{f(z)}{(z-z_0)^{n+1}}dz$$

or, $$f^n(z_0)=\frac{\lfloor n}{2\pi i}\int_C\frac{f(z)}{(z-z_0)^{n+1}}dz.$$

Hence the formula holds for all values of n. Thus $f(z)$ has derivatives of all orders and these are all analytic at z_0. Thus, the theorem is completely established.