Vecter space

$(V, \oplus, \odot)$ is a vecter space over F if $\forall \alpha_1, \alpha_2 \in F$ and $x_1, x_2, x_3 \in V$ following properties hold (Always remember these 10 properties)

1. $x_1 \oplus x_2 \in V$
2. $x_1 \oplus x_2 = x_2 \oplus x_1$
3. $x_1 \oplus (x_2 \oplus x_3) = (x_1 \oplus x_2) \oplus x_3$
4. $\exists\ 0 \in V$ s.t. $x_1 \oplus 0 = x_1$
5. $\forall x \in V\ \exists\ y \in V$ s.t. $x \oplus y = 0$
6. $\alpha_1 \odot x_1 \in V$
7. $1 \odot x_1 = x_1$
8. $(\alpha_1 \alpha_2) \odot x_1 = \alpha_1 \odot (\alpha_2 \odot x_1)$
9. $\alpha_1 \odot (x_1 \oplus x_2) = \alpha_1 \odot x_1 + \alpha_1 \odot x_2$
10. $(\alpha_1 + \alpha_2) \odot x_1 = \alpha_1 \odot x_1 \oplus \alpha_2 \odot x_2$

Imp. facts

(i) $\oplus$ is called vecter addition

(ii) $\odot$ is called Scalar multiplication

Always remember ↳ Field operations which are different from vecter addition and Scalar mullip.

(iii) Other basic properties are consequence of above 10 properties :- E.g.

(a) $0 \odot x = 0$ (b) $\alpha \odot 0 = 0$

(c) $\alpha \odot x = 0 \Leftrightarrow$ either $\alpha = 0$ OR $x = 0$

(d) $-x$ (additive inverse of x) $= -1 \odot x$

Such rules can be proved by using 10 rules See Tutorial problem

Always remember following examples of vecter spaces with usual binary operations and field

$\mathbb{R}^n$, $\mathbb{C}^n$, F over F, $\mathbb{C}$ over $\mathbb{R}$, $\mathbb{R}^{m\times n}$, $\mathbb{C}^{m\times n}$,

P, P_n, $C[a,b]$, $C^{(K)}[a,b]$

Also remember → geometric connection of $\mathbb{R}^2$ and $\mathbb{R}^3$.

Ques Let $V = \mathbb{R}^2$. Define vector addition and scalar multiplication as follows

$$\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \oplus \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} a_1 + b_2 \\ a_2 + b_1 \end{bmatrix} \quad \text{and} \quad \alpha \odot \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} \alpha a_1 \\ \alpha a_2 \end{bmatrix}$$

Here $\mathbb{R}$ is the field of scalars. Does $(V, \oplus, \odot)$ form a real vector space.

Solution $\boxed{\text{See: does } \mathbb{R}^2 \text{ satisfy all 10 rules required for a VS wrt given operations}}$

① Since addition of two real numbers is a real number, $a_1 + b_2 \in \mathbb{R}$ and $a_2 + b_1 \in \mathbb{R}$ for any $a_1, a_2, b_1, b_2 \in \mathbb{R}$, Therefor

$$\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \oplus \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \in \mathbb{R}^2$$

② $\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \oplus \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} a_1 + b_2 \\ a_2 + b_1 \end{bmatrix}$ and $\begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \oplus \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} b_1 + a_2 \\ b_2 + a_1 \end{bmatrix}$

Hence $\oplus$ is not commutative.

Hence $(V, \oplus, \odot)$ is not a VS.

Question Elaborate: (i) $\alpha \odot 0 = 0$ (II) $0 \odot u = 0$

(III) $(-1) \odot u = -u$ (IV) $\alpha \odot u = 0 \Leftrightarrow$ either $\alpha = 0$ or $u = 0$

Solution $\boxed{\text{Remember: here } 0 \text{ vector represents additive identity. | and } -u \text{ is additive inverse of } u}$

(i) $\alpha \odot 0 = \alpha \odot (0 \oplus 0)$ [$\because x \oplus 0 = x \ \forall x$]

$= \alpha \odot 0 \oplus \alpha \odot 0$ [$\because$ distributive rule]

$\Rightarrow \alpha \odot 0 \oplus (-\alpha \odot 0) = \alpha \odot 0 \oplus \alpha \odot 0 + (-\alpha \odot 0)$ Here assume $-\alpha \odot 0$ is additive inverse of $\alpha \odot 0$

$\Rightarrow 0 = \alpha \odot 0 \oplus 0 = \alpha \odot 0$ $\square$

(II) similar to (i) part

(III) $(-1) \odot u \oplus u = (-1) \odot u \oplus 1 \odot u$

$= (-1+1) \odot u = 0 \odot u = 0$

Hence additive inverse of u that is represented by $-u$ here is $(-1) \odot u$, i.e. $-u = (-1) \odot u$

(IV) Let $\alpha \odot u = 0$ and $\alpha \neq 0$

Then

$u = 1 \odot u = (\alpha \frac{1}{\alpha}) \odot u$

$= \frac{1}{\alpha} \odot (\alpha \odot u)$

$= \frac{1}{\alpha} \odot 0$

$= 0$

Let $\alpha \odot u = 0$ and $u \neq 0$

Assume $\alpha \neq 0$. Then

$u = 1 \odot u = \frac{1}{\alpha} \odot (\alpha \odot u) = \frac{1}{\alpha} \odot 0 = 0$

which contradicts the fact that $u = 0$

Hence $\alpha = 0$.

Ques Suppose we define addition on $\mathbb{R}^2$ by the rule

$$\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} a_1+b_1 \\ 0 \end{bmatrix}.$$

Show that additive identity does not exist in $\mathbb{R}^2$ w.r.t. above rule.

Proof:- Let $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathbb{R}^2$ be an element s.t. $x_1 \neq 0 \neq x_2$.

Let $\begin{bmatrix} e_1 \\ e_2 \end{bmatrix}$ be additive identity w.r.t above rule. Then

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad \text{(by definition of additive identity)}$$

$$\Rightarrow \begin{bmatrix} x_1+e_1 \\ 0 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad \text{(by given rule of vector addition)}$$

$\Rightarrow x_2 = 0$

which contradicts the fact that $x_2 \neq 0$. Hence $\begin{bmatrix} e_1 \\ e_2 \end{bmatrix}$ is not the additive identive.

Ques Suppose we define addition on $\mathbb{R}^3$ by the rule

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} a_1 b_1 \\ a_2 b_2 \\ a_3 b_3 \end{bmatrix}.$$

Show that we have additive identity but additive inverse may not exist for some elements.

Solution Let $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \mathbb{R}^3$ be an element s.t. $x_1 \neq 0$, $x_2 \neq 0$, and $x_3 \neq 0$.

Let $\begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix}$ be additive identity w.r.t given rule. Then

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x_1 e_1 \\ x_2 e_2 \\ x_3 e_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \Rightarrow e_i = 1 \ \forall\ i = 1,2,3$$

Note that $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ works as additive identity also if any of x_i's is 0.

Hence $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ is additive identity.

Let $\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$ be additive inverse of $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

Then we must have

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

(NOTE that identity is here $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ not $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$)

$$\Rightarrow \begin{bmatrix} x_1 y_1 \\ x_2 y_2 \\ x_3 y_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 1/x_1 \\ 1/x_2 \\ 1/x_3 \end{bmatrix}$$

This inverse exists only when x_i's are not zero.

Therefore additive inverse does not exist for elements $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ when any of x_i's is zero.

Ques Let $\mathbb{R}^+$ be the set of all positive real numbers. Define vector addition $\oplus$ and scalar multiplication $\odot$ as follows:

$u \oplus v = uv \quad \forall\ u, v \in \mathbb{R}^+$

$\alpha \odot u = u^\alpha \quad \forall\ \alpha \in \mathbb{R}$ and $u \in \mathbb{R}^+$.

Prove $(\mathbb{R}^+, \oplus, \odot)$ is a VS over $\mathbb{R}$.

Solution We have to check all 10 properties of VS

1. Since multiplication of two +ive real numbers is a +ive real number, we obtain $u \oplus v = uv \in \mathbb{R}^+ \ \forall\ u, v \in \mathbb{R}^+$.

2. $u \oplus v = \underbrace{uv = vu}_{} = v \oplus u$

   → Recall! This is valid for real numbers by Field axioms.

3. $u \oplus (v \oplus z) = u \oplus (vz) = u(vz) = (uv)z = (u \oplus v) \oplus z$
   $\forall\ u, v, z \in \mathbb{R}^+$.

4. $u \oplus 1 = u \cdot 1 = u \quad \forall\ u \in \mathbb{R}^+$ (i.e. additive identity is 1)

   NOTE:- Here 0-vector is 1 in this space

   By definition 0-vector is always additive identity

5. $u \oplus \frac{1}{u} = 1 \quad \forall\ u \in \mathbb{R}^+$

   ↳ NOTE - in this space $-u = \frac{1}{u}$
   By our definition $-u$ is additive inverse of u

6. $\alpha \odot u = u^\alpha \in \mathbb{R}^+ \quad \forall\ \alpha \in \mathbb{R}$ and $u \in \mathbb{R}^+$

7. $1 \odot u = u^1 = u \quad \forall\ u \in \mathbb{R}^+$ [NOTE - here 1 is multiplicative identity of field $\mathbb{R}$]

8. $(\alpha_1 \alpha_2) \odot u = u^{\alpha_1 \alpha_2} = \alpha_1 \odot (\alpha_2 \odot u) = \alpha_1 \odot u^{\alpha_2} = (u^{\alpha_2})^{\alpha_1} = u^{\alpha_1 \alpha_2}$
   $\forall\ \alpha_1, \alpha_2 \in \mathbb{R}$ and $u \in \mathbb{R}^+$

9. $\alpha \odot (u \oplus v) = \alpha \odot (uv) = u^\alpha v^\alpha = (\alpha \odot u) \oplus (\alpha \odot v) \quad \forall\ \alpha \in \mathbb{R}$ and $u, v \in \mathbb{R}^+$.

10. $(\alpha_1 + \alpha_2) \odot u = u^{\alpha_1 + \alpha_2} = u^{\alpha_1} u^{\alpha_2} = (\alpha_1 \odot u) \oplus (\alpha_2 \odot u) \quad \forall\ \alpha_1, \alpha_2 \in \mathbb{R}$ and $u \in \mathbb{R}^+$.

Hence $(\mathbb{R}^+, \oplus, \odot)$ is a VS over $\mathbb{R}$.

NOTE:- Any set with some unusual rules may be a VS

Keyword to learn is $\boxed{\text{Subspace}}$

Definition ~~Let V be a vector~~
Let $(V, \oplus, \odot)$ be a vector space over F. Let S be a subset of V. Then S is called subspace of V if S itself is a vector space over F w.r.t. $\oplus$ and $\odot$.

Important Fact:

$\boxed{! \ 0 \text{ is additive identity}}$

$S \subset V$ is a subspace $\Leftrightarrow$ (i) $0 \in S$
(ii) $x \oplus y \in S \ \forall \ x, y \in S$
(iii) $\alpha \odot x \in S \ \forall \ \alpha \in F$ and $x \in S$

$\Leftrightarrow$ (i) $0 \in S$
(ii) $\alpha \odot x \oplus \beta \odot y \in S \ \forall \ \alpha, \beta \in F$, $x, y \in S$.

Always remember this condition $\longleftrightarrow$ $\boxed{\Leftrightarrow (\alpha \odot x) \oplus y \in S \ \forall \ \alpha \in F \text{ and } x, y \in S}$

↳ Think! To prove subspace we require to show only one condition NOT (10) conditions of Vector space.

Procedure to check any set a subspace or not

Step ① Check whether $0 \in S$. If NO then S cannot be a subspace.

Step ② Check $\alpha \odot x \oplus y \in S$. → First try to disprove this rule by examples. If not able to disprove them prove the rule for arbitrary $\alpha \in F$ and $x, y \in S$.

Benefit of step ① is ÷ you can conclude immediately if S is not a subspace.

Remember

Smallest subspace of V is zero - subspace that contains only 0-element & means 0 (additive identity) vector only.

Largest subspace of V is V itself

Sum of two subspaces is also a subspace.

Intersection of subspaces is always a subspace
Union of two subspaces need not be a subspace

Question Prove: Let $(V, \oplus, \odot)$ be a VS over F. Let S be any ~~subset (non-empty) of V~~ non empty set of V.

$S \subset V$ is a subspace $\Leftrightarrow (\alpha \odot x) \oplus y \in S \quad \forall \alpha \in F$ and $x, y \in S$.

Proof

($\Rightarrow$ part) obvious ($\because$ By definition S is a VS itself, therefore $\alpha \odot x \in S$ and $\alpha \odot x \oplus y \in S$ for any $\alpha \in F$ and $x, y \in S$)

($\Leftarrow$ part) given: $\alpha \odot x \oplus y \in S \quad \forall \alpha \in F$ and $x, y \in S$.

To prove: S is a subspace, i.e. S is a vectorspace in itself wrt $\oplus$ and $\odot$ over F.

Therefore, we have to show that all (10) properties of VS are satisfied if $(\alpha \odot x) \oplus y \in S \quad \forall \alpha \in F$ and $x, y \in S$.

① Take $\alpha = 1$ then given statement $(\alpha \odot x) \oplus y \in S$ gives $x \oplus y \in S \quad \forall x, y \in S$, i.e. S is closed wrt $\oplus$.

② $x \oplus y = y \oplus x$

③ $x \oplus (y \oplus z) = (x \oplus y) \oplus z$

All $x, y, z \in S$ satisfy these properties because $x, y, z \in S \Rightarrow x, y, z \in V$ and $(V, \oplus, \odot)$ is a Vector space.

④ Take $\alpha = 1$ and $y = -x$ then we have $1 \odot x \oplus (-x) = x \oplus (-x) = 0 \in S$.

And hence $x \oplus 0 = x \quad \forall x \in S$.

⑤ Take $\alpha = -1$ and $y = 0$, we obtain $-1 \odot x \oplus 0 \in S$
$\Rightarrow -x$ (additive inverse of each $x \in S$) $\in S$.

⑥ Take $y = 0$, we obtain $\alpha \odot x \in S \quad \forall \alpha \in F$ and $x \in S$
i.e. S is closed wrt scalar multiplication

⑦ Take $\alpha = 1$ and $y = 0$, we obtain $1 \odot x \oplus 0 = 1 \odot x \in S \quad \forall x \in S$.

⑧ $(\alpha_1 \alpha_2) \odot x = \alpha_1 \odot (\alpha_2 \odot x)$

⑨ $\alpha_1 \odot (x \oplus y) = (\alpha_1 \odot x) \oplus (\alpha_1 \odot y)$

⑩ $(\alpha_1 + \alpha_2) \odot x = (\alpha_1 \odot x) \oplus (\alpha_2 \odot x)$

all $x, y \in S$ and $\alpha_1, \alpha_2 \in F$ satisfy these rule as elements of S are elements of V also and $(V, \oplus, \odot)$ is a Vector space.

Question Prove! (i) Intersection of subspaces is a subspace.
(ii) Sum of two subspaces is a subspace.

Proof: (i) Let $S_1, S_2, \cdots$ be subspaces of $(V, \oplus, \odot)$.
Let $S = \bigcap_i S_i$

To prove - S is a subspace - show $(\alpha \odot x) \oplus y \in S \ \forall \alpha \in F$, $x, y \in S$.

Let $\alpha \in F$ and $x, y \in S$
$\Rightarrow \alpha \in F$ and $x, y \in S_i \ \forall i$
$\Rightarrow (\alpha \odot x) \oplus y \in S_i \ \forall i$ ($\because$ each S_i is a subspace)
$\Rightarrow (\alpha \odot x) \oplus y \in S$
Hence S is a subspace.

(ii) Let S_1 and S_2 be two subspaces of $(V, +, \cdot)$ over F.
Let $S = S_1 + S_2$.
To prove: S is a subspace.
Remember $S = S_1 + S_2 = \{x_1 + x_2 : x_1 \in S_1 \text{ and } x_2 \in S_2\}$
Let $\alpha \in F$ and $x, y \in S$.
Then $x = x_1 + x_2$ for some $x_1 \in S_1$ and $x_2 \in S_2$
$y = y_1 + y_2$ for some $y_1 \in S_1$ and $y_2 \in S_2$.

Now $\alpha x + y = \alpha x_1 + \alpha x_2 + y_1 + y_2$
$= \underbrace{(\alpha x_1 + y_1)}_{\in S_1} + \underbrace{(\alpha x_2 + y_2)}_{\in S_2} \in S$

$\boxed{\text{Remember } S \text{ is } L[S_1 \cup S_2]}$ $\rightarrow$ i.e. smallest subspace that contains $S_1 \cup S_2$

Important $S_1 \cup S_2$ need not be subspace for two subspaces $S_1 \& S_2$.
But $L[S_1 \cup S_2]$ is always a subspace.

! Remember: Union of two subspaces need not be a subspace.

Ques

Let W_1 and W_2 be subspaces of a vector space V such that $W_1 \cup W_2$ is also a subspace.

To prove $W_1 \subseteq W_2$ or $W_2 \subseteq W_1$

Solution Let $W_1 \cup W_2$ be a subspace

Assume $W_1 \nsubseteq W_2$ and $W_2 \nsubseteq W_1$

$\Rightarrow \exists\, x_1 \in W_1$ but $x_1 \notin W_2$ ——— (1)

and $\exists\, x_2 \in W_2$ but $x_2 \notin W_1$ ——— (2)

Since $x_1, x_2 \in W_1 \cup W_2 \Rightarrow x_1 + x_2 \in W_1 \cup W_2$ ($\because W_1 \cup W_2$ is a subspace)

$\Rightarrow x_1 + x_2 \in W_1$ or W_2

If $x_1 + x_2 \in W_1$

$\Rightarrow x_1 + x_2 + (-x_1) \in W_1$

$\Rightarrow x_2 \in W_1$

which contradicts – (2)

If $x_1 + x_2 \in W_2$

$\Rightarrow x_1 + x_2 + (-x_2) \in W_2$

$\Rightarrow x_1 \in W_2$

which contradicts – (1)

Hence either $W_1 \subseteq W_2$ OR $W_2 \subseteq W_1$.

Ques

Let W_1 and W_2 be subspaces of V s.t.

$$\left.\begin{array}{l} W_1 + W_2 = V \\ W_1 \cap W_2 = \{0\} \end{array}\right] \rightarrow \text{In this case } V \text{ is called direct sum of } W_1 \text{ and } W_2.$$

Then show that there are unique vectors $u_1 \in W_1$ and $u_2 \in W_2$ such that $u = u_1 + u_2$

Solution: By definition of sum of two subspaces, If $V = W_1 + W_2$, then for each $u \in V$, $\exists$ $u_1 \in W_1$ and $u_2 \in W_2$ s.t. $u = u_1 + u_2$.

Therefore, the only claim to prove is uniqueness of u_1 and u_2. Let

$u = u_1 + u_2$ and $u = u_3 + u_4$ also, where $u_1, u_3 \in W_1$, $u_2, u_4 \in W_2$ and $u_1 \neq u_3$, $u_2 \neq u_4$.

By construction, we obtain $u_1 - u_3 = u_4 - u_2$ — (1)

Since $u_1, u_3 \in W_1 \Rightarrow u_1 - u_3 \in W_1$ —— (2)

$\Rightarrow u_4 - u_2 \in W_1$ (by (1)) — (3)

Moreover $u_2, u_4 \in W_2 \Rightarrow u_4 - u_2 \in W_2$ —— (4)

$\Rightarrow u_1 - u_3 \in W_2$ (by (1)) — (5)

from (2) & (5), $u_1 - u_3 \in W_1 \cap W_2$

But it is given that $W_1 \cap W_2 = \{0\}$, i.e. $u_1 = u_3$

Similarly, from (3) and (4), we obtain $u_4 = u_2$

Hence the claim is proved.

Question

Determine which of the following subsets of $C[0,1]$ are subspaces.

Remember! (i) $0 \in S$ (ii) $\alpha x + y \in S \;\forall\; \alpha \in F,\ x, y \in S$

NOTE 0 vector of $C[0,1]$ is zero function.

(i) $\{f : f \in C[0,1] \text{ and } f(\tfrac{1}{2}) = 0\} = S$ (say)

clearly $0 \in S$. Let $f, g \in S$ then for any $\alpha \in \mathbb{R}$, we obtain

$(\alpha f + g)(\tfrac{1}{2}) = \alpha f(\tfrac{1}{2}) + g(\tfrac{1}{2}) = \alpha \cdot 0 + 0 = 0$

Hence $\alpha f + g \in S$. Yes S is a subspace.

(ii) $\{f \in C[0,1] : f(\tfrac{3}{4}) = 1\} \to$ NO 0-function is not here. It is not a subspace.

(iii) $\{f \in C[0,1] : f(0) = f(1)\} \to$ Yes.

(iv) $\{f \in C[0,1] : f(x) = 0$ only at a finite number of pts$\}$
↳ No. 0- does not belong to this set.

Question

which of the following subsets of $\mathbb{R}^{2\times 2}$ are subspaces

(i) All diagonal matrices —— Yes

(ii) All upper triangular matrices — Yes

(iii) All symmetric matrices — Yes

(iv) All invertible matrices — No

(v) All matrices which commute with fix B — Yes

(vi) All matrices with 0 determinant. — No

Hint: $0 \notin (iv)$; $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \in (vi)$ but sum $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \notin (vi)$.

<u>Ques</u> which of the following subsets of $\mathcal{P}$ are subspaces

(i) $\{p \in \mathcal{P}$ s.t. $\deg p \le 4\} \to$ Yes.

(ii) $\{p \in \mathcal{P}$ s.t. $\deg p = 4\} \to$ No

(iii) $\{p \in \mathcal{P}$ s.t. $\deg p \ge 4\} \to$ No

$\to 0 \notin$ (ii)/(iii)/(v)

(iv) $\{p \in \mathcal{P}$ s.t. $p(1) = 0\} \to$ Yes

(v) $\{p \in \mathcal{P}$ s.t. $p(2) = 1\} \longrightarrow$ No

(vi) $\{p \in \mathcal{P}$ s.t. $p'(1) = 0\} \longrightarrow$ Yes.

<u>Ques</u> which of the following subsets of $\mathbb{R}^4$ are subspaces

(i) $\left\{ \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} : \quad a - c - d = 0 \right.$

(ii) $: \quad b - c + d = 0$
$\quad a - c - d = 0$

(iii) $: \quad c - d = 0$

Yes

(iv) $: \quad a = 1 \longrightarrow$ No $\quad 0 \notin$ here

(v) $: \quad a \le b \longrightarrow$ No $\to$ <u>Think! why</u>

(vi) $: \quad a = b = c = d \to$ Yes

(vii) $: \quad a$ is an integer $\to$ $\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ is here but $\frac{1}{2}\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ is not $\to$ No
scalar multiplication is not closed

(viii) $: \quad a^2 - b^2 = 0 \to$ No. $\quad \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

(ix) $\left\{ \begin{pmatrix} a - c \\ b - a \\ c + b \\ a + b \end{pmatrix} : \quad a, b, c, d \in \mathbb{R} \right\} \to$ <u>Yes.</u>

Keyword to learn :- $\boxed{\text{(Linear) span}}$

Definition

Let $(V, \oplus, \odot)$ be a vector space.
Let S be a subset of V. Then linear span of S is denoted by $L[S]$ and is defined as

$L[S]$ = collection of all possible linear combinations of elements of S.

Important Facts

① S may be finite or infinite but $L[S]$ always contains infinite elements. Remember linear combination is always related with finitely many elements only.

② If S_1 and S_2 are different sets, $L[S_1]$ may be equal to $L[S_2]$.
E.g. In $\mathbb{R}^2$ Take

$S_1 = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$ $\quad S_2 = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \end{bmatrix}, \cdots \right\}$ i.e. $S_2 = \left\{ \begin{bmatrix} n \\ 0 \end{bmatrix} : n \in \mathbb{N} \right\}$

$S_3 = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$ $\quad S_4 = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$ Then $L[S_1] = L[S_2]$
$L[S_3] = L[S_4]$

③ $L[S]$ is always a subspace for any set S.
Even $L[S]$ is the smallest subspace of V that contains S.

④ Remember! what is $\boxed{\text{generator}}$

How you can decide a vector b lies in span of vectors $\{x_1, x_2, \cdots x_n\}$

By definition,
Vector b lies in span of vectors $\{x_1, x_2, \cdots, x_n\}$ If $\exists$ scalars $\alpha_1, \alpha_2, \cdots, \alpha_n$ s.t.

$$\alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 + \cdots + \alpha_n x_n = b$$

$$\Rightarrow \begin{bmatrix} x_1 \mid x_2 \mid \cdots \mid x_n \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{bmatrix} = b \rightarrow \boxed{\text{Recall matrix-vector multiplication}}$$

$\Rightarrow$ $\boxed{A\alpha = b}$ has a solution for α.

Here A is a matrix whose i^{th} column is vector x_i.

Ques Prove $L[S]$ is a subspace. Moreover $L[S]$ is the smallest subspace that contains S.

Solution We have to show that

$$\alpha x + y \in L[S] \quad \forall \quad x, y \in L[S] \text{ and } \alpha \in F.$$

Let $\alpha \in F$ and $x, y \in L[S]$. Therefore, there exist

$$x = \sum_{i=1}^{n} \alpha_i x_i \quad \text{and} \quad y = \sum_{j=1}^{m} \beta_j y_j \, . \quad \text{Hence}$$

$$\alpha x + y = \sum_{i=1}^{n} (\alpha \alpha_i) x_i + \sum_{j=1}^{m} \beta_j y_j \in L[S].$$

Hence $L[S]$ is a subspace.

Assume W is another subspace that contains S. Hence linear combination of any elements of S is in W, i.e. W contains $L[S]$. Hence $L[S]$ is the intersection of all subspaces that contain S. Hence $L[S]$ is the smallest subspace containing S.

Ques $M \subset N \Rightarrow L[M] \subset L[N]$.

Let $x \in L[M]$

$\Rightarrow \exists\, x_1, x_2, \cdots x_n \in M$ and $\alpha_i \in F$ $(i = 1, 2, \cdots n)$ s.t.

$$x = \alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n$$

Since x_i's $\in M$ and $M \subset N \Rightarrow x_i \in N \;\forall\; i = 1, 2, \cdots n$

$\Rightarrow x \in L[N]$.

Ques If M is a subspace then $L[M] = M$

Sol. Above, we have proved that $L[M]$ is the smallest subspace that containing M, since M is a subspace in itself, $L[M] = M$.

Ques $L[L(M)] = L[M]$ for any subset M.

Sol. Since $L(M)$ is a subspace, hence $L[L(M)] = L[M]$.

Keyword to learn LI/LD

Let $(V, +, \cdot)$ be a VS over F.
Let S be a subset of V.
Then set S is LI (linearly independent) if any arbitrary chosen n vectors, say $x_1, x_2, \ldots, x_n$, of S satisfy following properties

$$\alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n = 0 \Rightarrow \alpha_i's = 0 \ \forall \ i = 1, 2, \ldots, n$$

Otherwise S is called LD (linearly dependent)

Important points

① If $S = \{x_1, x_2, \ldots, x_n\} \subset V$. Then sometimes we say that vectors $x_1, x_2, \ldots, x_n$ are LI/LD instead of saying that set S is LI/LD.

② Remember! Vectors $x_1, x_2, \ldots, x_n$ are LD if $\exists$ scalars $\alpha_1, \alpha_2, \ldots, \alpha_n$ (not all zero) s.t.

$$\alpha_1 x_1 + \cdots + \alpha_n x_n = \boxed{\sum_{i=1}^{n} \alpha_i x_i} = 0 \longrightarrow \text{shorthand for sum.}$$

③ Let S_1, S_2 be two subsets of V s.t. $S_1 \subset S_2$. Then
(i) S_2 is LI $\Rightarrow$ S_1 is LI (ii) S_1 is LD $\Rightarrow$ S_2 is LD

④ S is LD $\Leftrightarrow$ $\exists$ at least one vector in S that can be written as lc of other vectors

⑤ ALWAYS REMEMBER $0 \in S \Rightarrow S$ is LD
($\Leftarrow$ need not be true)

Keywords to learn

Basis / dimension / co-ordinates

Let $(V, +, \cdot)$ be a VS over F.
Let S be a subset of V. Then S is called a basis of V if

(i) S is LI

(II) S spans/generates V.

The number of elements in a basis is called **dimension** of V.

If dimension of V is finite, then we say that V is a finite dimensional space. Otherwise V is called infinite dimensional space.

Example

space	standard basis	dimension
$\mathbb{R}^n$	$\{e_1, e_2, \cdots, e_n\}$ where e_j is the j^{th} column of $n\times n$ identity matrix	n
$\mathbb{R}^{m\times n}$	$\left\{\begin{bmatrix}1&0\\0&0\end{bmatrix}, \begin{bmatrix}0&1\\0&0\end{bmatrix}, \begin{bmatrix}0&0\\1&0\end{bmatrix}, \begin{bmatrix}0&0\\0&1\end{bmatrix}\right\}$ is standard basis of $\mathbb{R}^{2\times 2}$ ↳ Notice the pattern to write st. basis of $\mathbb{R}^{m\times n}$	mn
P_n	$\{1, x, x^2, \cdots, x^n\}$	$n+1$

particular case $\mathbb{R} \to$ is one dim. space st. basis is $\{1\}$.

dimension of $\mathbb{C}$ over $\mathbb{C}$ is one but dim of $\mathbb{C}$ over $\mathbb{R}$ is 2

P, $C^{(k)}[a,b]$, $C[a,b]$ are infinite dimensional vector spaces

Import facts

① there are many basis for a space but dim. is unique.

② Let dim $(V) = n$. Let S be any subset of V having m elements. Then

(i) S is always LD if $m > n$

(II) S cannot span V if $m < n$

} Basis is maximal LI set and minimal generator of space V.

See:- The meaning of statement one - If there are two basis of V - # elements are same - it can be proved by statement ②.

Remember:- ordered basis and coordinate

Ques Let S_1, S_2 be two subsets of VS V. Let $S_1 \subset S_2$. Then
(i) S_1 is LD $\Rightarrow$ S_2 is LD
(ii) S_2 is LI $\Rightarrow$ S_1 is LI

Solution

part (i)

Let S_1 be LD.

Let $S_2 = S_1 \cup \{v_1, v_2, v_3, \cdots\}$

Since S_1 is LD, for some vectors $s_i \in S_1$, we obtain

$$\sum_{i=1}^{n} \alpha_i s_i = 0 \Rightarrow \alpha_i \neq 0 \ \forall i \quad \text{— (1)}$$

$$\Rightarrow \sum_{i=1}^{n} \alpha_i s_i + \sum_{j=1}^{m} \beta_j v_j = 0$$ holds for some nonzero α_i's
(take all $\beta_j = 0$ & use (1))

$\Rightarrow$ S_2 is LD.

Part (ii) Let S_2 be LI.

Suppose S_1 is LD. Then by part (i) S_2 is also LD, which contradicts the fact that S_2 is LI.

Hence S_1 is LI.

Remember! Subset of LI set is LI.
Superset of LD set is LD.

But

Superset of LI may be LD
Subset of LD may be LI

Ques If u, v, and w are LI in a vector space V, then $u+v$, $v+w$, $w+u$ are also LI.

Sol given - u, v, w are LI, i.e.

$\alpha_1 u + \alpha_2 v + \alpha_3 w = 0 \Rightarrow \alpha_1 = \alpha_2 = \alpha_3 = 0$ — (1)

to prove: $u+v$, $v+w$, $w+u$ are LI.

Suppose $\beta_1(u+v) + \beta_2(v+w) + \beta_3(w+u) = 0$

$\Rightarrow (\beta_1 + \beta_3)u + (\beta_2 + \beta_1)v + (\beta_2 + \beta_3)w = 0$

$$\Rightarrow \left.\begin{cases} \beta_1 + \beta_3 = 0 \\ \beta_1 + \beta_2 = 0 \\ \beta_2 + \beta_3 = 0 \end{cases}\right] \text{ by (1)}$$

$\Rightarrow$ It is a hom. system

$$\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

this system has unique Solution $\begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$\Downarrow$

$\beta_1 = \beta_2 = \beta_3 = 0$

Hence $u+v$, $v+w$, $w+u$ are LI.

Ques Let S be LI in V. Let $v \in L(S)$. Show that $\{v\} \cup S$ is LD.

Sol: $v \in L(S)$. Therefore $\exists\, v_1, v_2, \cdots, v_n \in S$ s.t.

$v = \alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n$ for some α_i's $\in F$.

$\Rightarrow 1 \cdot v - \alpha_1 v_1 - \alpha_2 v_2 - \cdots - \alpha_n v_n = 0$

$\Rightarrow \{v, v_1, v_2, \cdots, v_n\}$ is LD in S

$\Rightarrow$ S is LD. $\because$ S is subset of $\{v, v_1, v_2, \cdots v_n\}$.

Ques Let S be LI in V. Let $v \notin L[S]$.
Then $\{v\} \cup S$ is also LI.

Solution Let $v_i \in S$ for $i = 1, 2, 3, \cdots n$
Let $v \notin L(S)$.

Consider

$$\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n + \alpha v = 0 \quad \text{— (1)}$$

Then (1) $\Rightarrow$ $\alpha = 0$ [Otherwise $v \in L(S)$] — (2)

Thus, (1) & (2) $\Rightarrow$ $\sum_{i=1}^{n} \alpha_i v_i = 0$, which implies

$\alpha_i = 0 \ \forall \ i = 1, 2, \cdots, n$ [$\because$ S is LI] — (3)

from (2) and (3), we obtain

(1) $\Rightarrow$ $\alpha = 0$ and $\alpha_i = 0 \ \forall \ i$

Hence $\{v\} \cup S$ is LI.

Ques If u, v, w, z are LI. Then $u+v, v+w, w, z$ are also LI.

Solution

$$\alpha_1 u + \alpha_2 v + \alpha_3 w + \alpha_4 z = 0 \Rightarrow \alpha_i = 0 \quad \text{— (1)}$$

Now obtain

$$\beta_1 (u+v) + \beta_2 (v+w) + \beta_3 w + \beta_4 z = 0$$

$$\Rightarrow \beta_1 u + (\beta_1+\beta_2) v + (\beta_2+\beta_3) w + \beta_4 z = 0$$

$$\Downarrow$$

$$\left.\begin{aligned} \beta_1 &= 0 \\ \beta_1+\beta_2 &= 0 \\ \beta_2+\beta_3 &= 0 \\ \beta_4 &= 0 \end{aligned}\right) \text{— by (1)} \quad \text{—— (2)}$$

check null space of above homog. system

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} \boxed{1} & 0 & 0 & 0 \\ 0 & \boxed{1} & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} \boxed{1} & 0 & 0 & 0 \\ 0 & \boxed{1} & 0 & 0 \\ 0 & 0 & \boxed{1} & 0 \\ 0 & 0 & 0 & \boxed{1} \end{bmatrix}$$

Thus, we have (1) $\Rightarrow$ (2) $\Rightarrow$ $\beta_1 = \beta_2 = \beta_3 = \beta_4 = 0$

Hence $u+v, v+w, w, z$ are LI.

Ques Write two different bases of $\mathbb{R}^4$ that have the vectors $\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$ in common.

Solution

$$B_1 = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\} \quad \text{Standard Basis}$$

Then by applying above problem get

$$B_2 = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$