Linear ODEs of Second and Higher order

- How can we solve 2^{nd} order linear ODEs.
- Can we extend such techniques to Higher order linear ODEs.
- Can we solve 2^{nd} and higher order non-linear ODEs.

Notation:

$$y' \equiv \frac{dy}{dx} \; ; \quad y'' \equiv \frac{d^2y}{dx^2}$$

$$y^{(n)} \equiv \frac{d^n y}{dx^n}$$

Definition of Linear ODEs

$$\boxed{a(x)\, y' + b(x)\, y = c(x)} \rightarrow I^{st} \text{ order}$$

$a(x) \neq 0$

$\rightarrow$ To solve it:- we write

$$y' + p(x)\, y = q(x)$$

- IF $u(x) = e^{\int p(x)dx}$
- $y(x)\,u(x) = \int q(x)\,u(x)\,dx + C$

$$a_0(x)\, y'' + a_1(x)\, y' + a_2(x)\, y = F(x) \text{ where } a_0(x) \neq 0 \rightarrow 2^{nd} \text{ order linear ODE}$$

$\rightarrow$ standard form:

$$\boxed{y'' + p(x)\, y' + q(x)\, y = r(x)}$$

Imp.

n^{th} order ODEs:- $\quad f(x, y, y', y'', y^{(3)}, \cdots y^{(n)}) = 0$

n^{th} order linear ODE

$$\boxed{a_0(x)\, y^{(n)} + a_1(x)\, y^{(n-1)} + a_2(x)\, y^{(n-2)} + \cdots + a_{n-1}(x)\, y' + a_n(x)\, y = F(x)}$$

where $a_0(x) \neq 0$ on I

n^{th} order linear ODE is called homogeneous if RHS function $F(x) \equiv 0$ on I.

$\rightarrow$ input term / forcing term / control / driving term

Sol. y is called response of given $F(x)$

Some other ways for writting Linear ODEs

Consider n^{th} order linear ODE

$$\boxed{a_0 y^{(n)} + a_1 y^{(n-1)} + a_2 y^{(n-2)} + \cdots + a_{n-1} y' + a_n y = F(x)}$$

where a_i's are functions of x;

$x \in I = [a, b]$ and $i = 0, 1, 2, \ldots n$

→ at some places, we write

$$a_0 D^n y + a_1 D^{n-1} y + a_2 D^{n-2} y + \cdots + a_{n-1} D y + a_n y = F(x)$$

$$\boxed{\text{i.e. } D^n y = \frac{d^n y}{dx^n} = y^{(n)}}$$

OR, we write

$$(a_0 D^n + a_1 D^{n-1} + a_2 D^{n-2} + \cdots + a_{n-1} D + a_n) y = F(x)$$

OR in short we write

$$\boxed{L y = F}$$

where

$$L \equiv a_0 D^n + a_1 D^{n-1} + \cdots + a_{n-1} D + a_n$$

is an operator that involves derivatives.

e.g.

$L = x D^2 + 2D + x^3$

and $y = x^4$

then

$Ly = x \cdot 4 \cdot 3 \cdot x^2 + 2 \cdot 4x^3 + x^7$

$= 12x^3 + 8x^3 + x^7$

Remember our favorate equation of linear Algebra $Ax = b$;

In $Ax = b$; for a given b, we find x.

In ODE also; for a given F, we find y s.t. $Ly = F$

The only difference is,

in $Ax = b$; x and b are vectors in finite dimensional spaces

and in $Ly = F$; y and F are functions in some function space.

Otherwise both equations are same as far as linearity is concerned.

$Ax = b$	$Ly = F$
① A is a linear operator i.e. $A(\alpha_1 x_1 + \alpha_2 x_2) = \alpha_1 A x_1 + \alpha_2 A x_2$ $\forall\ \alpha_1, \alpha_2$ (scalars) $\in \mathbb{R}$, x_1, x_2 vectors.	① L is a linear operator $L(\alpha_1 y_1 + \alpha_2 y_2) = \alpha_1 L y_1 + \alpha_2 L y_2$ $\forall\ \alpha_1, \alpha_2$ scalars, y_1, y_2 vectors (here functions).
e.g. $L = D+2$ then $(D+2)(\alpha_1 y_1 + \alpha_2 y_2) = D(\alpha_1 y_1 + \alpha_2 y_2) + 2(\alpha_1 y_1 + \alpha_2 y_2)$ $= \alpha_1 (D+2) y_1 + \alpha_2 (D+2) y_2 = \alpha_1 L y_1 + \alpha_2 L y_2$	due to the fact that differentiation is a linear operator
② General solution of $Ax = b$ is $\boxed{x = x_n + x_p}$ where $A x_n = 0$ and x_p is any (particular) solution of $Ax = b$ i.e. $A x_p = b$	② General solution of $Ly = F$ is $y = y_c + y_p$ where $L y_c = 0$ and $L y_p = F$ $y_c \to$ Complementary function $y_p \to$ particular solution. y_c is solution of corresponding homogeneous equation and y_p is a particular solution of $Ly = F$.

Superposition principle of Linear ODEs.

Consider n^{th} order linear equation

$\boxed{Ly = F}$ where $L = a_0 D^n + a_1 D^{n-1} + \cdots + a_{n-1} D + a_n$

If y_1 is a solution of $Ly = f_1$
and y_2 is a solution of $Ly = f_2$
then $c_1 y_1 + c_2 y_2$ solves $Ly = c_1 f_1 + c_2 f_2$
where c_1, c_2 are scalars

Pf
Given
$Ly_1 = f_1$ — ①
$Ly_2 = f_2$ — ②
Now
$L(c_1 y_1 + c_2 y_2)$
$= c_1 L y_1 + c_2 L y_2$
$= c_1 f_1 + c_2 f_2$

OR we can write

If y_1 is response of $Ly = F$ for input f_1
and y_2 is response of $Ly = F$ for input f_2
then $c_1 y_1 + c_2 y_2$ is response of $Ly = F$ for input $c_1 f_1 + c_2 f_2$
where c_1, c_2 are constants

Engineers like the keyword superposition principle
Mathematicians love to say that L is a linear operator

Both are same things as clear from proof.

Now, let us continue to read the table of similarities b/t $Ax=b$ and $Ly=F$

$Ax=b$; $A\equiv m\times n$; $\text{rank}(A)=r$	$Ly=F$; $L=a_0D^n+a_1D^{n-1}+\dots+a_{n-1}D+a_nD$
(1) A is a linear operator	(1) L is a linear operator.
(2) General sol. of $Ax=b$ $X=X_n+X_p$ Here, $AX_n=0$ i.e. $X_n=\alpha_1X_1+\alpha_2X_2+\dots\alpha X_p$ where $p+r=n$ and $\{X_1,X_2,\dots X_p\}$ is a basis of null space of A.	(2) general sol. of $Ly=F$ is $y=y_c+y_p$ $\left.\begin{array}{l} Ly_c=0 \\ Ly_p=F \end{array}\right\}$ $y_c\to$ Complementary func; $y_p\to$ particular solution and $y_c=c_1y_1+c_2y_2+\dots+c_ny_n$ where $y_1,y_2\dots y_n$ are n LI solutions of homogeneous equation $Ly=0$ and $c_1,c_2\dots c_n$ are arbitrary real constants.

<u>Note:-</u> $Ly=0$ is an ODE of order n to solve it, somehow we have to do n integrations, so general solution of $Ly=0$ (OR $Ly=F$) is an n-parameter family. All n-parameters are part of y_c. y_p does not contain any parameter.

So why name is particular sol. for y_p

See I^{st} order linear ODE
$y'=py+q(x)$; in case p is constant, remember solution formula

$$y(t)=\underbrace{e^{pt}y_0}_{y_c}+\underbrace{\int_{t_0}^{t}e^{p(t-s)}q(s)\,ds}_{y_p} \quad \text{where } y(t_0)=y_0$$

check y_c solve hom.-eqn $y'=py$ &	y_p is a particular solution.

Some Interesting results for

$$\boxed{y'' + p(x)y' + q(x)y = 0} \quad — \quad (*)$$

<u>Th</u> Let y_1 and y_2 be two solutions of $(*)$. Then $y = c_1 y_1 + c_2 y_2$; c_1, c_2 are arbitrary constants, is also a solution of $(*)$.

<u>Proof.</u> Given: $y_1'' + p(x)y_1' + q(x)y_1 = 0$ — (1)

$y_2'' + p(x)y_2' + q(x)y_2 = 0$ — (2)

Now $(c_1 y_1 + c_2 y_2)'' + p(x)(c_1 y_1 + c_2 y_2)' + q(x)(c_1 y_1 + c_2 y_2)$

$= c_1(y_1'' + p(x)y_1' + q(x)y_1) + c_2(y_2'' + p(x)y_2' + q(x)y_2)$

$= c_1 \cdot 0 + c_2 \cdot 0 = 0$.

Thus $y = c_1 y_1 + c_2 y_2$ is a solution of $(*)$.

Similar result you can easily prove for n^{th} order linear homogeneous ODEs

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Before next result, we recall one keyword from Linear algebra: <u>Wronskian</u>

If $y_1(x)$ and $y_2(x)$ are two function then their Wronskian is denoted by $W(y_1, y_2)$ and is defined as

$$W(y_1, y_2) = \begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix}$$

$$= y_1 y_2' - y_2 y_1'$$

$\boxed{\text{See } W(y_1, y_2) \text{ is a function of } x}$

Th. 2 Let y_1 and y_2 be two solutions of

$$y'' + p(x)y' + q(x)y = 0 \quad \text{—— } (*)$$

on $[a,b]$. Then $W(y_1,y_2)$ is either identically zero on $[a,b]$ or never zero on $[a,b]$. Further:

$\boxed{y_1 \text{ and } y_2 \text{ are LD (linearly dependent) on } [a,b]} \Leftrightarrow \boxed{W(y_1,y_2) \equiv 0 \text{ on } [a,b]}$

Pf $W(y_1,y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1y_2' - y_2y_1'$ —— (1)

$\Rightarrow \frac{dW}{dx} = y_1y_2'' + y_1'y_2' - y_2'y_1' - y_2y_1'' = y_1y_2'' - y_2y_1''$ —— (2)

Since y_1 and y_2 are solution of $(*)$, we obtain

$$y_1'' + py_1' + qy_1 = 0 \quad \text{— (3)}$$

and $$y_2'' + py_2' + qy_2 = 0 \quad \text{— (4)}$$

$$y_2 \times (2) - y_1 \times (3) \Longrightarrow (y_2y_1'' - y_1y_2'') + p(y_2y_1' - y_1y_2') = 0 \quad \text{— (4)}$$

Using values of W and $\frac{dW}{dx}$ from (1) and (2), respectively, in equation (4), we obtain

$$\frac{dW}{dx} + pW = 0$$

$$\Rightarrow \boxed{W = c\,e^{-\int p(x)\,dx}} \quad c \text{ is arbitrary constant}$$

Thus $W \equiv 0$ if $c = 0$
and W is never zero if $c \neq 0$ $\Big]$ $\because e^{-\int p(x)dx}$ is never zero for any $x \in [a,b]$

↳ Using this formula, we can find Wronskian of two solutions to any ODE upto multiplication of a constant.

$x^4y'' - 2x^3y' - x^8y = 0$; $W(y_1,y_2) = c\,e^{-\int -\frac{2}{x}dx} = \boxed{cx^2}$ ✓

Now we prove second part of theorem

($\Rightarrow$) Let y_1 and y_2 be LD.

Assume $y_1 = \alpha y_2$ (α is an scalar)

Then $W(y_1, y_2) = y_1 y_2' - y_2 y_1'$

$= \alpha y_2 y_2' - \alpha y_2 y_2' = 0 \quad \forall x \in [a, b]$

i.e. $W \equiv 0$ on $[a, b]$

($\Leftarrow$) Let $W(y_1, y_2) \equiv 0$

$\Rightarrow y_1 y_2' - y_2 y_1' = 0$

$\Rightarrow \dfrac{y_1 y_2' - y_2 y_1'}{(y_1)^2} = 0$ [Assume y_1 is never zero on $[a, b]$

$\Rightarrow \left(\dfrac{y_2}{y_1}\right)' = 0 \Rightarrow \dfrac{y_2}{y_1} = \alpha$ (α is constant)

$\Rightarrow \boxed{y_2 = \alpha y_1}$

Imp. Remark

Let $f_1(x)$ and $f_2(x)$ be two differentiable function on I.

Then $\boxed{f_1(x) \text{ and } f_2(x) \text{ are LD on } I \Rightarrow W(f_1, f_2) \equiv 0 \text{ on } I}$

i.e. $W(f_1, f_2) \not\equiv 0$ on $I \Rightarrow f_1(x)$ and $f_2(x)$ are LI.

$f_1(x) = 9\cos 2x$ $f_2(x) = 2\cos^2 x - 2\sin^2 x$ $W(f_1, f_2) = 0$ (check!) So f_1 and f_2 are LD	$f_1(x) = x^2$ on $[-1, 1] = I$ $f_2(x) = x^4$ $W(f_1, f_2) = \begin{vmatrix} x^2 & x^4 \\ 2x & 4x^3 \end{vmatrix} = 3x^5 \not\equiv 0$ on I So f_1 and f_2 are LI on I

Can x^2 and x^4 be two solutions to $\circledast$ on $I = [-1, 1]$? $\boxed{\text{NO}}$ why??

Th ③ (w/p) Existence and uniqueness thm of linear ODEs

Consider the 2^{nd} order linear ODE

$$y'' + p(x)y' + q(x)y = r(x) \quad \text{—— } \#$$

If p, q and r are continuous functions on $[a,b]$ and $x_0 \in [a,b]$. Then # has a unique solution on I s.t.

$$y(x_0) = c_0$$
$$y'(x_0) = C_1$$

⑪ If p, q and r are continuous on I. If $x_0 \in I$, They the following IVP has a unique solution on I

$$\begin{cases} y'' + p(x)y' + q(x)y = r(x) \\ y(x_0) = c_0 \\ y'(x_0) = C_1 \end{cases}$$

Similar result holds for n^{th} order linear ODE : Let $f(x)$ and $a_i(x)$'s be continuous functions on $[a,b]$ $(i = 0,1,2,\ldots n)$. Let $x_0 \in [a,b]$. Then the following IVP has a unique solution on I

$$\begin{cases} a_0 y^{(n)} + a_1 y^{(n-1)} + a_2 y^{(n-2)} + \cdots + a_{n-1} y' + a_n y = \\ y(x_0) = c_0 \\ y'(x_0) = c_1 \\ y''(x_0) = c_2 \\ \vdots \\ y^{(n-1)}(x_0) = c_{n-1} \end{cases}$$

☞ General solution of n^{th} order ODE has n-parameters. To find particular solution for general sol., we need n-conditions to find value of n parameters.

1st order case : Validate thm from Picards

$$y' + p(x)y = q(x) \quad \text{so } f(x,y) = -p(x)y + q(x)$$

$$\Rightarrow y' = -p(x)y + q(x)$$

If p and q are continuous on I then f and $\frac{\partial f}{\partial y}$ are bdd in nbhd of (x_0, y_0)

Initial Condition $y(x_0) = y_0$

Th (4)

Let y_1 and y_2 be two LI solutions of

$$y'' + p(x)y' + q(x)y = 0 \quad \text{——} \quad (*)$$

on I. Then

(a) $y = c_1 y_1 + c_2 y_2$ is a solution of $(*)$, where $c_1, c_2 \in \mathbb{R}$.

(b) If y is a solution of $(*)$ on I, then $\exists\ c_1, c_2 \in \mathbb{R}$ such that $y = c_1 y_1 + c_2 y_2$.

Hence, general solution of $(*)$ is

$$\boxed{y = c_1 y_1 + c_2 y_2}$$

Th (1) is true for any two solutions y_1 and y_2; need not be LI.

Pf: (part (a)) already proved in Th (1)

Just verify $c_1 y_1 + c_2 y_2$, where c_1 and c_2 are arbitrary constants, satisfies $(*)$ if y_1 and y_2 solve $(*)$. Part (a) is true for any two solutions y_1 and y_2 of $(*)$.

(part (b)) Let y be a solution of $(*)$ such that

$$y(x_0) = \alpha$$

and $y'(x_0) = \beta$

for some $x_0 \in I$.

First we show that $\exists\ c_1$ and c_2 s.t.

$$\left.\begin{aligned} c_1 y_1(x_0) + c_2 y_2(x_0) &= \alpha \\ c_1 y_1'(x_0) + c_2 y_2'(x_0) &= \beta \end{aligned}\right\} \quad \text{——} \quad (1)$$

i.e. system $A \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$ has unique solution, where $A = \begin{bmatrix} y_1(x_0) & y_2(x_0) \\ y_1'(x_0) & y_2'(x_0) \end{bmatrix}$

Since y_1 and y_2 are LI so determinant of A is non-zero and hence (1) has unique solution.

Thus, we have c_1 and c_2 such that solutions $c_1 y_1 + c_2 y_2$ satisfies conditions (1).

Therefore by Existence and uniqueness thm

$$\boxed{y = c_1 y_1 + c_2 y_2}.$$

Algorithm to find general solution to homogeneous ODE

$$\boxed{y'' + p(x) y' + q(x) y = 0} \quad -(1)$$

Step 1 Find two LI solutions $y_1(x)$ and $y_2(x)$ of (1).

Step 2 general solution to (1)

$$\boxed{y(x) = c_1 y_1(x) + c_2 y_2(x)} \quad -(*)$$

where c_1 and c_2 are two arbitrary constants.

→ Thus general solution to (1) is a two parameter family of curves.

To find particular solution to IVP

$$y'' + p(x) y' + q(x) y = 0$$
$$\left.\begin{array}{l} y(x_0) = \alpha \\ y'(x_0) = \beta \end{array}\right] \rightarrow \text{Initial Conditions}$$

we put initial values in (*) and find values of c_1 and c_2.

Now, question is: how to find LI solutions y_1 and y_2 of ODE (*)

To set the method, first we prove following two theorems.

Theorem (A):–

Let $w(x) = u(x) + i\,v(x)$ be a complex solution of $y'' + p(x) y' + q(x) y = 0$, —— (*)

then $u(x)$ and $v(x)$ are two real solutions of (*).

Pf. Given: $w = u + iv$ is a solution of (*).

Therefore $(u+iv)'' + p(u+iv)' + q(u+iv) = 0$

$$\Rightarrow (u'' + pu' + qu) + i(v'' + pv' + qv) = 0$$

Thus, we obtain

$$u'' + pu' + qu = 0$$

and $v'' + pv' + qv = 0$

Hence u and v are two real solutions of (*).

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Theorem (B)

If $y_1(x)$ is a solution of

$$y'' + p(x)y' + q(x)y = 0, \quad — (*)$$

then there exists a function $u(x)$ such that $y_2(x) = y_1(x) \cdot u(x)$ is also a solution of (*).

Pf. Given: y_1 is a solution of (*), i.e.

$$y_1'' + p y_1' + q y_1 = 0 \quad — (1)$$

Now, If $y_2 = y_1 \cdot u$ is a solution of (*), then

$$(y_1 u)'' + p(y_1 u)' + q(y_1 u) = 0$$

$$\Rightarrow (y_1'' u + 2y_1' u' + y_1 u'') + p(y_1' u + y_1 u') + q y_1 u = 0$$

$$\Rightarrow \underbrace{(y_1'' + p y_1' + q y_1)}_{0 \text{ by } (1)} u + y_1 u'' + (2y_1' u' + p y_1 u') = 0$$

$$\Rightarrow y_1 u'' + (2y_1' + p y_1) u' = 0 \quad — (2)$$

Thus, If we are able to find a u s.t. (2) holds then $y_2 = y_1 u$ is also a solution of (*).

Take $u' = F$ — (3)

Then (2) reduces to

$$y_1 F' + (2y_1' + p y_1) F = 0$$

$$\Rightarrow \quad \frac{F'}{F} = -\frac{2y_1'}{y_1} - p$$

By integration, we obtain

$$\ln F = -2\ln y_1 - \int p(x)\,dx$$

$$\Rightarrow \quad F \cdot y_1^2 = e^{-\int p(x)\,dx}$$

$$\Rightarrow \quad F = \frac{1}{y_1^2} e^{-\int p(x)\,dx}$$

By using (3), we obtain

$$u'(x) = \frac{1}{y_1^2} e^{-\int p(x)\,dx}$$

$$\Rightarrow \quad \boxed{u(x) = \int \frac{1}{y_1^2} e^{-\int p(x)\,dx}\,dx}$$

□.

Relook

If we know that $y_1(x)$ is a solution of

$$y'' + p(x)y' + q(x)y = 0 \quad \text{—— } (*)$$

Then $y_2(x) = y_1(x)\,u(x)$ is also a solution of $(*)$

where $u(x) = \int \frac{1}{y_1^2} e^{-\int p(x)\,dx}\,dx$

→ This is called <u>method of order reduction</u>

why?

<u>Remember</u> If y_1 is a solution of $(*)$, then αy_1 (α is constant) is always a solution of $(*)$. But in Thm. (B) u is a function of x. In Thm (B), y_1 and y_2 are LI.