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ICS141: Discrete Mathematics for Computer Science I

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Lecture 7

Chapter 1. The Foundations

1.6 Introduction to Proofs

Chapter 2. Basic Structures

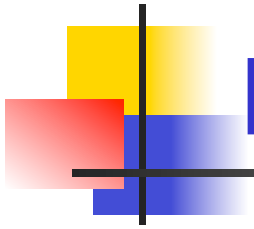
2.1 Sets

Proof Terminology

- A ***proof*** is a valid argument that establishes the truth of a mathematical statement
- ***Axiom*** (or ***postulate***): a statement that is assumed to be true
- ***Theorem***
  - A statement that has been proven to be true
- ***Hypothesis***, ***premise***
  - An assumption (often unproven) defining the structures about which we are reasoning

More Proof Terminology

- ***Lemma***
  - A minor theorem used as a stepping-stone to proving a major theorem.
- ***Corollary***
  - A minor theorem proved as an easy consequence of a major theorem.
- ***Conjecture***
  - A statement whose truth value has not been proven. (A conjecture may be widely believed to be true, regardless.)



Proof Methods

- For proving a statement p alone
  - ***Proof by Contradiction*** (indirect proof): Assume $\neg p$, and prove $\neg p \rightarrow \mathbf{F}$.

Proof Methods

- For proving implications $p \rightarrow q$, we have:
  - ***Trivial* proof:** Prove q by itself.
  - ***Direct* proof:** Assume p is true, and prove q.
  - ***Indirect* proof:**
    - ***Proof by Contraposition*** ($\neg q \rightarrow \neg p$): Assume $\neg q$, and prove $\neg p$.
    - ***Proof by Contradiction***: Assume $p \wedge \neg q$, and show this leads to a contradiction. (i.e. prove $(p \wedge \neg q) \rightarrow \mathbf{F}$)
  - ***Vacuous* proof:** Prove $\neg p$ by itself.

Direct Proof Example

- **Definition:** An integer n is called *odd* iff $n=2k+1$ for some integer k; n is *even* iff $n=2k$ for some k.
- **Theorem:** Every integer is either odd or even, but not both.
  - This can be proven from even simpler axioms.
- **Theorem:**
  (For all integers n) If n is odd, then n^2 is odd.

  Proof:

  If n is odd, then $n = 2k + 1$ for some integer k.

  Thus, $n^2 = (2k + 1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1$.

  Therefore n^2 is of the form $2j + 1$ (with j the integer $2k^2 + 2k$), thus n^2 is odd. ■

Indirect Proof Example: Proof by Contraposition

- **Theorem:** (For all integers n)
  If $3n + 2$ is odd, then n is odd.
- **Proof:**

  (Contrapositive: If n is even, then $3n + 2$ is even)

  Suppose that the conclusion is false, *i.e.*, that n is even.

  Then $n = 2k$ for some integer k.

  Then $3n + 2 = 3(2k) + 2 = 6k + 2 = 2(3k + 1)$.

  Thus $3n + 2$ is even, because it equals $2j$ for an integer $j = 3k + 1$. So $3n + 2$ is not odd.

  We have shown that $\neg(n \text{ is odd}) \rightarrow \neg(3n + 2 \text{ is odd})$, thus its contrapositive $(3n + 2 \text{ is odd}) \rightarrow (n \text{ is odd})$ is also true. ■

Vacuous Proof Example

- Show $\neg p$ (i.e. p is false) to prove $p \rightarrow q$ is true.

- **Theorem:** (For all n) If n is both odd and even, then $n^2 = n + n$.
- **Proof:**

  The statement "n is both odd and even" is necessarily false, since no number can be both odd and even. So, the theorem is vacuously true. ■

Trivial Proof Example

- Show q (i.e. q is true) to prove $p \rightarrow q$ is true.

- **Theorem:** (For integers n) If n is the sum of two prime numbers, then either n is odd or n is even.
- **Proof:**

  Any integer n is either odd or even. So the conclusion of the implication is true regardless of the truth of the hypothesis. Thus the implication is true trivially. ■

Proof by Contradiction

- A method for proving p.
  - Assume $\neg p$, and prove both q and $\neg q$ for some proposition q. (Can be anything!)
  - Thus $\neg p \rightarrow (q \wedge \neg q)$
  - $(q \wedge \neg q)$ is a trivial contradiction, equal to **F**
  - Thus $\neg p \rightarrow$ **F**, which is only true if $\neg p =$ **F**
  - Thus p is true

Rational Number

- Definition:

  The real number r is *rational* if there exist integers p and q with $q \neq 0$ such that $r = p/q$. A real number that is not rational is called *irrational*.

Proof by Contradiction Example

- **Theorem:** $\sqrt{2}$ is irrational.
  - **Proof:**
  - Assume that $\sqrt{2}$ is rational. This means there are integers x and y ($y \neq 0$) with no common divisors such that $\sqrt{2} = x/y$.

    Squaring both sides, $2 = x^2/y^2$, so $2y^2 = x^2$. So x^2 is even; thus x is even (see earlier).

    Let $x = 2k$. So $2y^2 = (2k)^2 = 4k^2$. Dividing both sides by 2, $y^2 = 2k^2$. Thus y^2 is even, so y is even.

    But then x and y have a common divisor, namely 2, so we have a contradiction.

    Therefore, $\sqrt{2}$ is irrational. ■

Proof by Contradiction

- Proving implication $p \rightarrow q$ by contradiction
  - Assume $\neg q$, and use the premise p to arrive at a contradiction, i.e. $(\neg q \wedge p) \rightarrow \mathbf{F}$

    $(p \rightarrow q \equiv (\neg q \wedge p) \rightarrow \mathbf{F})$
  - How does this relate to the proof by contraposition?
  - ***Proof by Contraposition*** $(\neg q \rightarrow \neg p)$: Assume $\neg q$, and prove $\neg p$.

Proof by Contradiction Example: Implication

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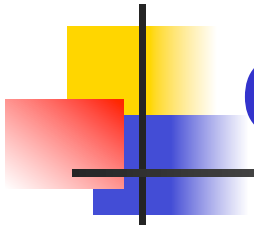
- **Theorem:** (For all integers n)
  If $3n + 2$ is odd, then n is odd.
- **Proof:**

  Assume that the conclusion is false, *i.e.*, that n is even, and that $3n + 2$ is odd.

  Then $n = 2k$ for some integer k and $3n + 2 = 3(2k) + 2 = 6k + 2 = 2(3k + 1)$. Thus $3n + 2$ is even, because it equals $2j$ for an integer $j = 3k + 1$.

  This contradicts the assumption "$3n + 2$ is odd".

  This completes the proof by contradiction, proving that if $3n + 2$ is odd, then n is odd. ■



Circular Reasoning

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- The fallacy of (explicitly or implicitly) assuming the very statement you are trying to prove in the course of its proof. Example:
- Prove that an integer n is even, if n^2 is even.
- **Attempted proof:**

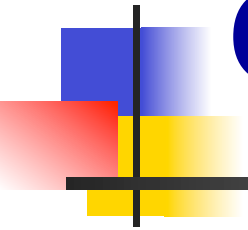
  Assume n^2 is even. Then $n^2 = 2k$ for some integer k.

  Dividing both sides by n gives $n = (2k)/n = 2(k/n)$.

  So there is an integer j (namely k/n) such that $n = 2j$. Therefore n is even.

  - Circular reasoning is used in this proof. Where?

*Begs the question: How do you show that $j = k/n = n/2$ is an integer, without **first** assuming that n is even?*



Chapter 2

Basic Structures:
Sets, Functions, Sequences, and Sums

2.1 Sets

- A ***set*** is a new type of structure, representing an ***unordered*** collection (group) of zero or more ***distinct*** (different) objects. The objects are called ***elements*** or ***members*** of the set.
  - Notation: $x \in S$
- Set theory deals with operations between, relations among, and statements about sets.
- Sets are ubiquitous in computer software systems.
  - (E.g. data types `Set`, `HashSet` in `java.util`)

Basic Notations for Sets

- For sets, we'll use variables S, T, U,…
- We can denote a set S in writing by listing all of its elements in curly braces:
  - $\{a,b,c\}$ is the set whose elements are a, b, and c
- ***Set builder notation***:
  - For any statement $P(x)$ over any domain,
    $\{x \mid P(x)\}$ is *the set of all x such that* $P(x)$ is true
  - Example: {1, 2, 3, 4}
    $= \{x \mid x \text{ is an integer where } x > 0 \text{ and } x < 5\}$
    $= \{x \in \mathbf{Z} \mid x > 0 \text{ and } x < 5\}$

Basic Properties of Sets

- Sets are inherently ***unordered***:
  - No matter what objects a, b, and c denote, $\{a, b, c\} = \{a, c, b\} = \{b, a, c\} = \{b, c, a\} = \{c, a, b\} = \{c, b, a\}$.
- All elements are ***distinct*** (unequal); multiple listings make no difference!
  - If $a = b$, then $\{a, b, c\} = \{a, c\} = \{b, c\} = \{a, a, b, a, b, c, c, c, c\}$.
  - This set contains (at most) 2 elements!

Definition of Set Equality

- Two sets are declared to be equal *if and only if* they contain exactly the same elements.
- In particular, it does not matter *how the set is defined or denoted.*
- Example:

  The set {1, 2, 3, 4}

  $= \{x \mid x \text{ is an integer where } x > 0 \text{ and } x < 5\}$

  $= \{x \mid x \text{ is a positive integer where } x^2 < 20\}$

Infinite Sets

- Conceptually, sets may be *infinite* (*i.e.,* not *finite*, without end, unending).
- Symbols for some special infinite sets:

  N = {0, 1, 2,…} the set of **N**atural numbers.

  Z = {…, –2, –1, 0, 1, 2,…} the set of **Z**ntegers.

  $\mathbf{Z}^+$ = {1, 2, 3,…} the set of positive integers.

  Q = $\{p/q \mid p,q \in \mathbf{Z}, \text{ and } q \neq 0\}$

  the set of Rational numbers.

  R = the set of "**R**eal" numbers.
- "Blackboard Bold" or double-struck font is also often used for these special number sets.