Lecture - 1.

Complex numbers :

rectangular coordinate

Def^n :- A complex number is an ordered pair (x, y), $x, y \in \mathbb{R}$, and written as $z := x + iy$ (just a symbol)
where i stands for $\sqrt{-1}$. (i.e. $i \cdot i = -1$)

Not^n $\mathbb{C}$: set of all complex nos.

(So, note that complex no. system provides an entity which is $\sqrt{-1}$ and which is not there in real no. system)

Eg : $3 + 2i$, $(1 \Rightarrow) \; 1 + 0i$, $0 + i$, $\pi + i$

Define :- ~~the~~ For a complex no $z = x + iy$,
define real part of $z := x =: Re(z)$
imaginary part of $z := y =: Im(z)$

Quest : Is 0 a complex no.? Is 1 a complex no.?

Basic Arithmetic :

(1) Two complex nos z_1 and z_2 are equal if and only if

$$Re(z_1) = Re(z_2) \quad \text{and} \quad Im(z_1) = Im(z_2)$$

(2) complex addition/~~subtraction~~ subtraction

$$\underset{x_1 + iy_1}{z_1} \pm \underset{x_2 + iy_2}{z_2} := \big(x_1 \underset{\text{real operat}^n}{\pm} x_2\big) \pm i\big(y_1 \underset{\text{real operat}^n}{\pm} y_2\big)$$

where the operations on R.H.S. are the operations of real nos.

(3) $$\underset{\text{complex mult}^n}{z_1 \cdot z_2} = (x_1 + iy_1)(x_2 + iy_2) = x_1x_2 + x_1(iy_2) + (iy_1)x_2 + (iy_1)(iy_2)$$

(treat them as binomial)

$$= (x_1x_2 - y_1y_2) + i(x_1y_2 + y_1x_2)$$

Use $i^2 = -1$

So, $\text{Re}(z_1 z_2) = x_1 x_2 - y_1 y_2$ and $\text{Im}(z_1 z_2) = x_1 y_2 + x_2 y_1$

(4) **Division** : Let $z_2 \neq 0 + i.0$

$$\frac{z_1}{z_2} = \frac{x_1 + i y_1}{x_2 + i y_2} = \frac{(x_1 + i y_1)(x_2 - i y_2)}{(x_2 + i y_2)(x_2 - i y_2)}$$

$$\xlongequal[\text{both the multiplication}]{\text{perform}} \frac{(x_1 x_2 - y_1 y_2) + i(x_2 y_1 - x_1 y_2)}{(x_2^2 + y_2^2)}$$

$$= \underbrace{\frac{x_1 x_2 - y_1 y_2}{(x_2^2 + y_2^2)}}_{\text{Re}\left(\frac{z_1}{z_2}\right)} + i \underbrace{\left(\frac{x_2 y_1 - x_1 y_2}{x_2^2 + y_2^2}\right)}_{\text{Im}\left(\frac{z_1}{z_2}\right)}$$

(5) **Properties :**

i) $z_1 + z_2 = z_2 + z_1$ [follows from the similar properties of real no.s.].

Ex. Verify.

ii) $z_1 z_2 = z_2 \cdot z_1$

iii) $z_1 + (z_2 + z_3) = (z_1 + z_2) + z_3$.

(iv) $z_1 \cdot (z_2 z_3) = (z_1 \cdot z_2) z_3$

(v) $z_1 \cdot (z_2 + z_3) = z_1 z_2 + z_1 z_3$

(vi) For $\underset{x+iy}{\overset{\shortparallel}{z}} \in \mathbb{C}$, $-z := -x + i(-y)$, then $z + (-z) = 0$

(vii) $z \in \mathbb{C}$, $z \neq 0$, $\frac{1}{z} = \frac{1 + 0i}{z} \xlongequal[\text{defined in (4)}]{\text{as}} \frac{x}{x^2+y^2} + i\left(\frac{-y}{x^2+y^2}\right)$

then $z \cdot \frac{1}{z} = 1$.

The complex plane:

$\mathbb{C} = \{a+ib,\ a,b \in \mathbb{R}\}$: the set of complex no.

→ ~~~~ For $x \in \mathbb{R}$, $x = x + i\cdot 0$ is a complex no.

$\boxed{\text{In other words, } \mathbb{R} \subseteq \mathbb{C}}$

(Particularly, $1 = \underbrace{1}_{Re(1)} + i\cdot\underbrace{0}_{Im(1)}$, and $0 = \underbrace{0}_{Re(0)} + i\cdot\underbrace{0}_{Im(0)}$)

→ There are ① arithmetic operations (addition, multi;) for real nos.

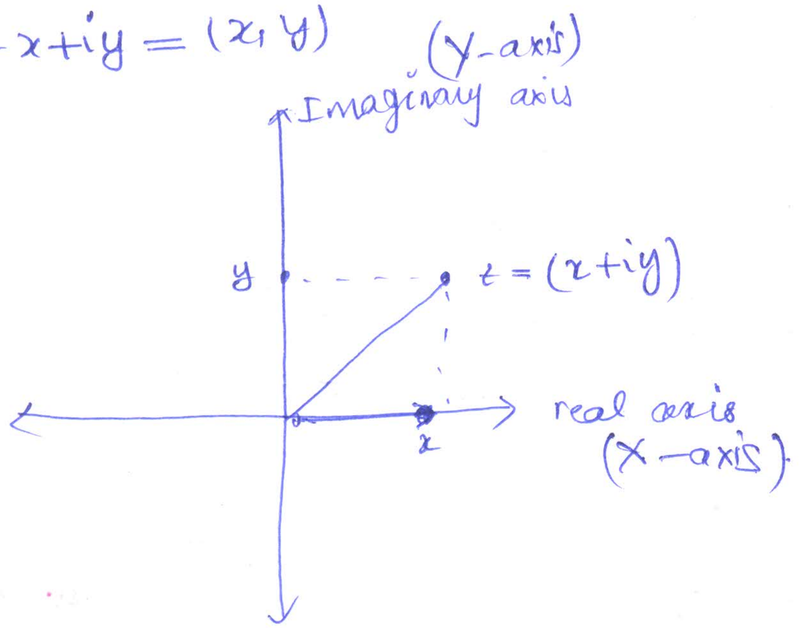
② " for complex nos. (as we defined)

Ex:- $\boxed{\text{Notice that}}$: operations of $\mathbb{C}$, when restricted to $\mathbb{R}$, are same as the operations in $\mathbb{R}$.

→ Inside $\mathbb{C}$, we have $\mathbb{R}$: $x + i\cdot 0$

we have : $0 + ib$: purely imaginary nos.

④ $z = x+iy = (x, y)$



X-Y-plane.

(6) **Conjugation** : For a complex no $z = x + iy$,
define conjugate of z as $x - iy$ $(= x + i(-y))$
written as $\bar{z} := x - iy$.

Properties :-

(i) $\bar{\bar{z}} = z$.

(ii) $\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$

(iii) $\overline{z_1 \cdot z_2} = \bar{z}_1 \bar{z}_2$

Verify :- $z_1 = x_1 + iy_1$, $z_2 = x_2 + iy_2$, then $z_1 + z_2 = (x_1 + x_2) + i(y_1 + y_2)$

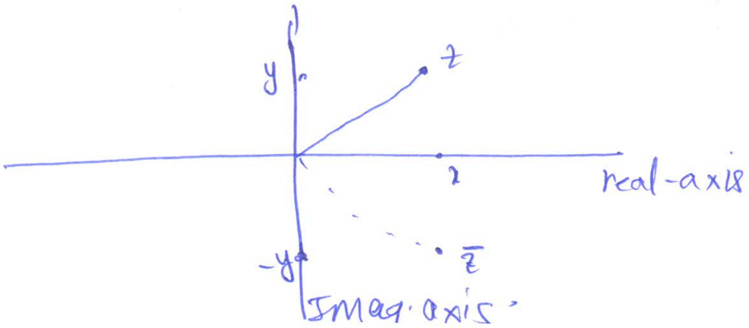
$\Rightarrow \overline{z_1 + z_2} = (x_1 + x_2) - i(y_1 + y_2)$

$\bar{z}_1 = x_1 - iy_1$, $\bar{z}_2 = x_2 - iy_2$, then $\bar{z}_1 + \bar{z}_2 = (x_1 + x_2) - i(y_1 + y_2)$

Ex :- Verify the other equalities by comparing L.H.S. and R.H.S.

(iv) $Re(z) = \dfrac{z + \bar{z}}{2}$; $Im(z) = \dfrac{z - \bar{z}}{2}$

(v) If $z = x + i \cdot 0$, then $\bar{z} = 0$
If $z = 0 + i \cdot y$ then $\bar{z} = -z$



: $\bar{z}$: reflection of z ~~w.r.t.~~ through X-axis.

Fact I So, the conjugate of complex no. is itself ~~iff~~ if and only if it is a real no.

Proof:- let $z = x + i.0$

then $\bar{z} = x - i.0 = z$.

Conversely let $z = \bar{z}$

$\Rightarrow x+iy = x-iy$

$\Rightarrow x = x, \quad \underbrace{y = -y}$.

$\Updownarrow$

thus $z = x + i.0$ is a real no. $\quad y = 0$.

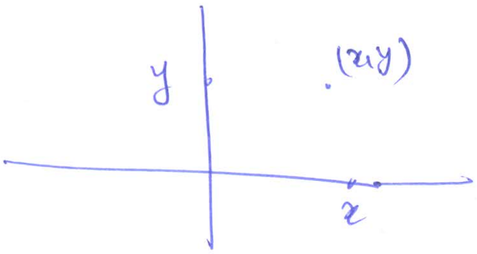
Ex II ~~the conjugate of a complex~~

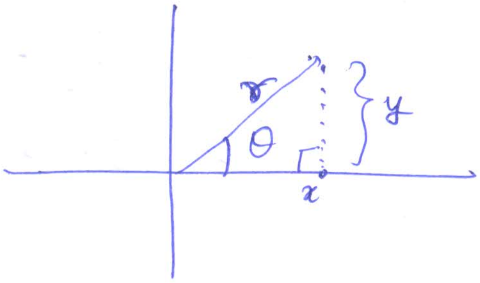
$\bar{z} = -z$ if and only if $z = 0 + iy$ is purely imaginary

§ Polar representation

Recall that in coordinate geometry, ~~we~~ points in plane can be represented using polar coordinates.

$z = x + iy$





(x, y) : polar representation : (r, θ)

where $x = r\cos\theta$
$y = r\sin\theta$

OR $r = \sqrt{x^2+y^2}$

$\theta := \tan^{-1} \frac{y}{x}$

(θ is not defined for 0).

- So, we have a polar representation of complex no. (as above)

$(0 \neq) z = x + iy$, (r, θ) where $r = \sqrt{x^2+y^2}$
$\theta = \tan^{-1} y/x$.

r : modulus of the complex no $= |z|$ Notation

θ : argument of the comp. no.

- (argument of z is not unique) : If (r, θ) is polar representⁿ then $(r, 2n\pi + \theta)$ is also a polar representⁿ

~~So The argument of $z\,(\neq 0)$ is the set $\{\theta_0 + 2n\pi : n \in \mathbb{Z}$ $0 \le \theta_0 < 2\pi$~~