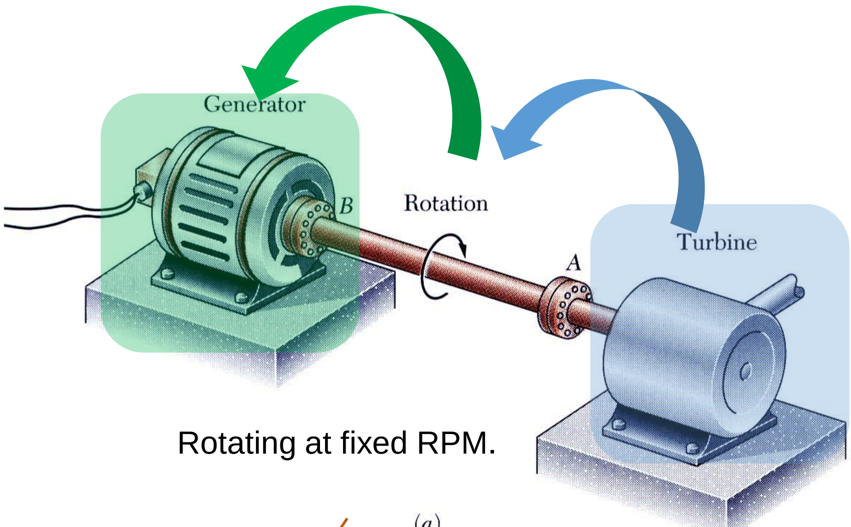
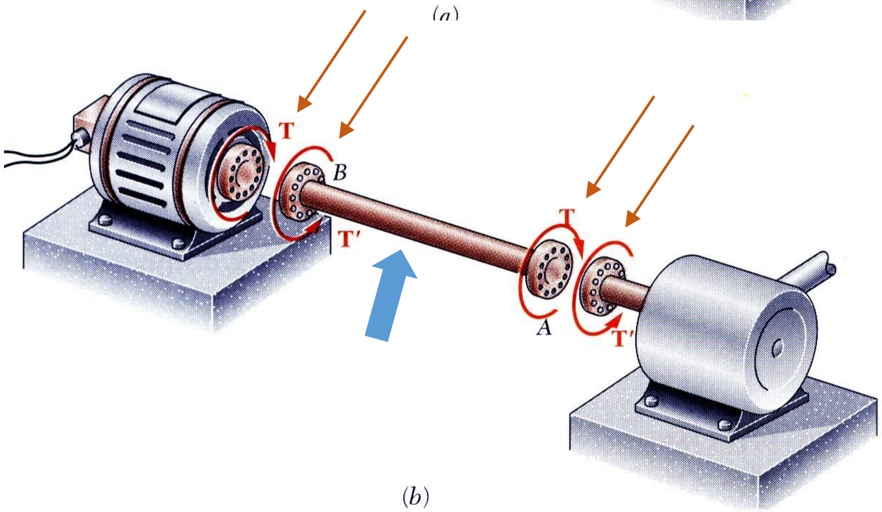
# Torsion
# Lecture 13

Engineering Mechanics - ME102

Rishi Raj

# Torsional Loads on Circular Shafts

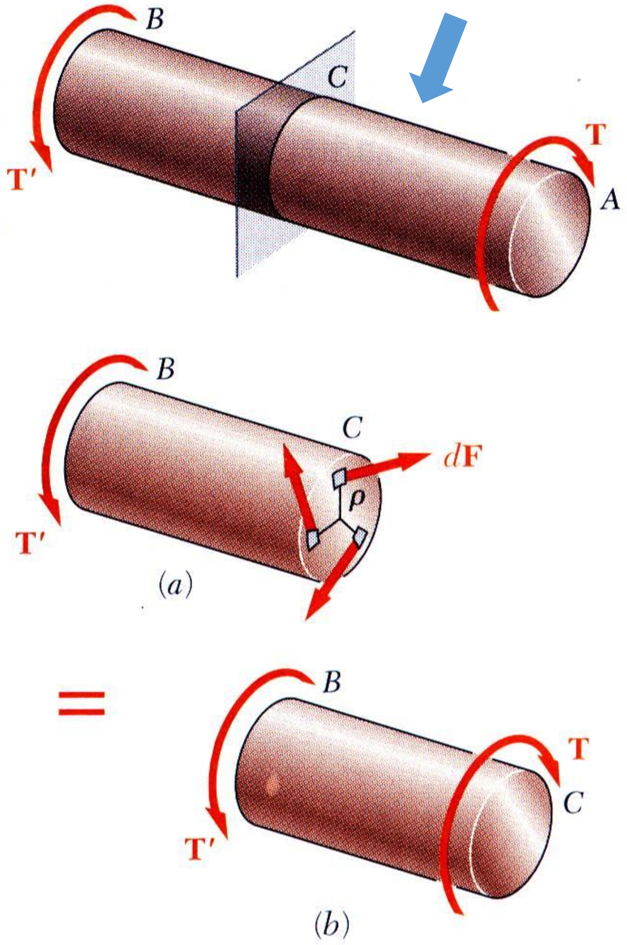



(b)

- Interested in stresses and strains of circular shafts subjected to **twisting couples** or *torques*
- Turbine exerts torque *T* on the shaft
- Shaft transmits the torque to the generator
- Generator creates an equal and opposite torque *T*'
- Let us consider the shaft in the next slide

# Net Torque Due to Internal Stresses




- Net of the internal shearing stresses is an internal torque, equal and opposite to the applied torque,

$$\boxed{T} = \int \rho \, dF = \int \rho(\tau \, dA)$$

- Although the net torque due to the shearing stresses is known, the distribution of the stresses is not

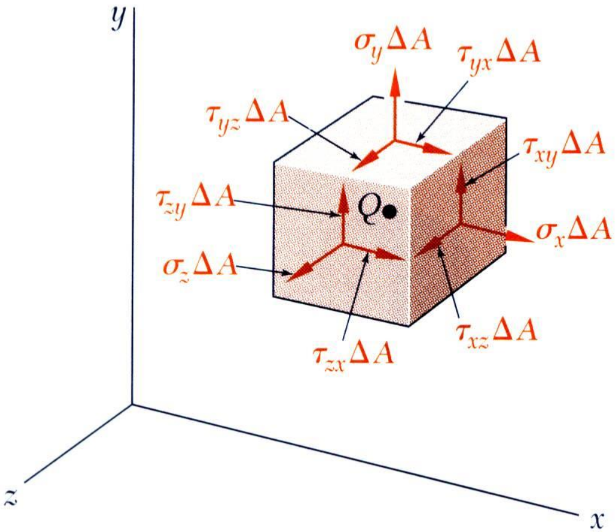
- Distribution of shearing stresses is ***statically indeterminate*** – must consider shaft deformations

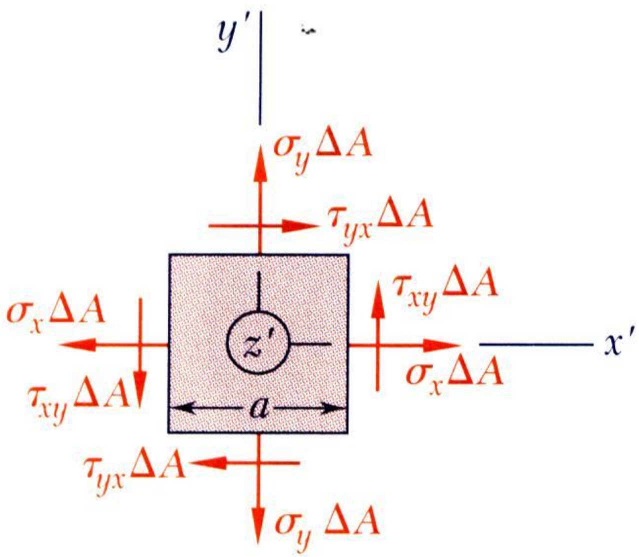
- Unlike the normal stress due to axial loads, the ***distribution of shearing stresses due to torsional loads can not be assumed uniform***.

# Do you remember?

- At a  point, shear stress cannot take place in one plane only, an equal shear stress must be exerted on another plane perpendicular to the first one.

# State of Stress




- Stress components are defined for the planes cut parallel to the *x*, *y* and *z* axes. For equilibrium, equal and opposite stresses are exerted on the hidden planes.

  - The combination of forces generated by the stresses must satisfy the conditions for equilibrium:

$$\sum F_x = \sum F_y = \sum F_z = 0$$

$$\sum M_x = \sum M_y = \sum M_z = 0$$

  - Consider the moments about the *z* axis:

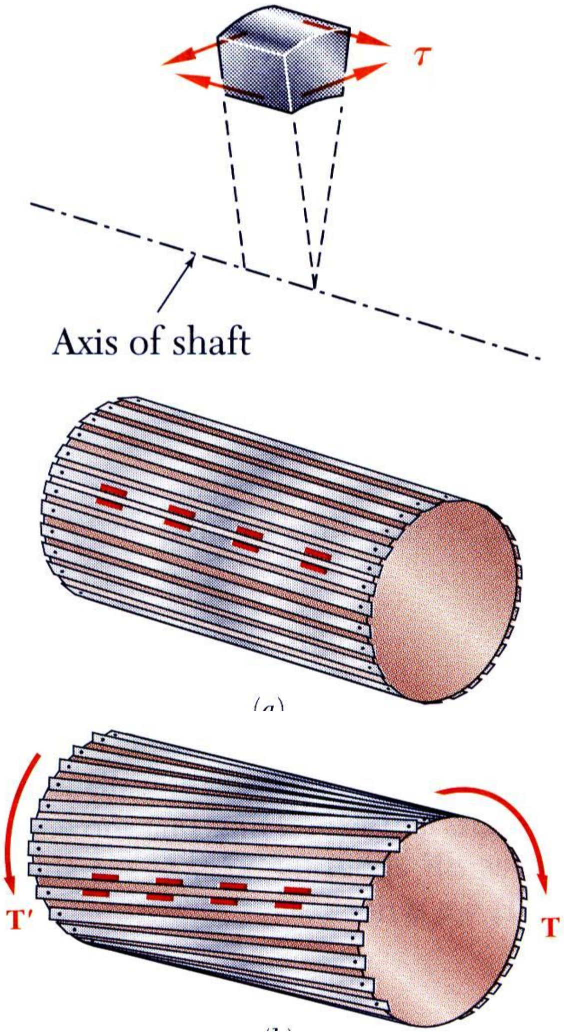
$$\sum M_z = 0 = (\tau_{xy}\Delta A)a - (\tau_{yx}\Delta A)a$$

$$\tau_{xy} = \tau_{yx}$$

similarly, $\tau_{yz} = \tau_{zy}$ and $\tau_{xz} = \tau_{zx}$

  - It follows that only 6 components of stress are required to define the complete state of stress

# Axial Shear Components

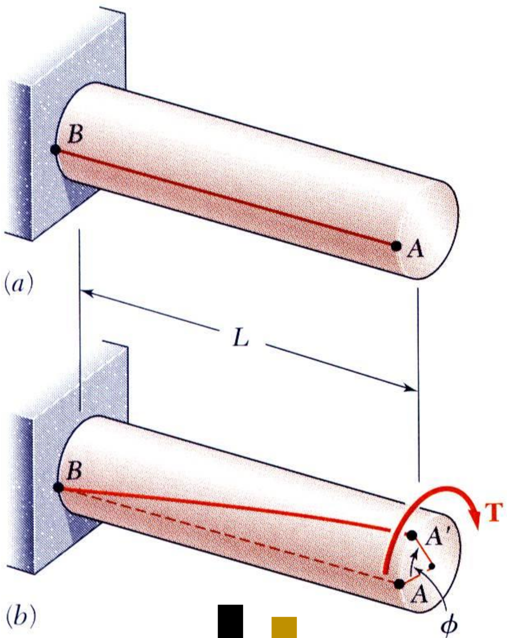


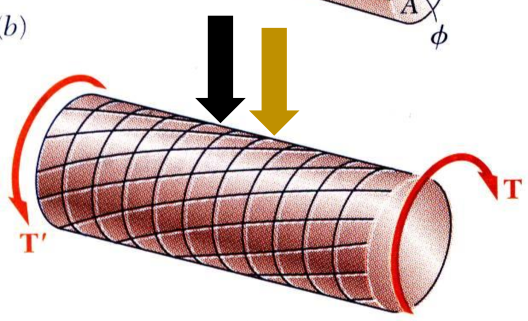

- Torque applied to shaft produces shearing stresses on the faces perpendicular to the axis.

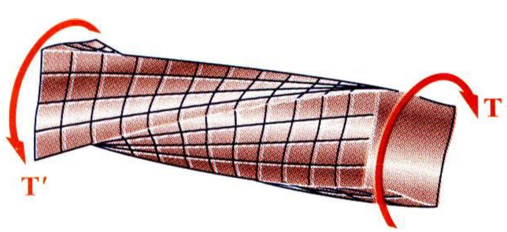
- ***Conditions of equilibrium require the existence of equal stresses on the faces of the two planes containing the axis of the shaft***
- The existence of the axial shear components is demonstrated by considering a shaft made up of axial slats.

  The slats slide with respect to each other when equal and opposite torques are applied to the ends of the shaft.

# Shaft Deformations



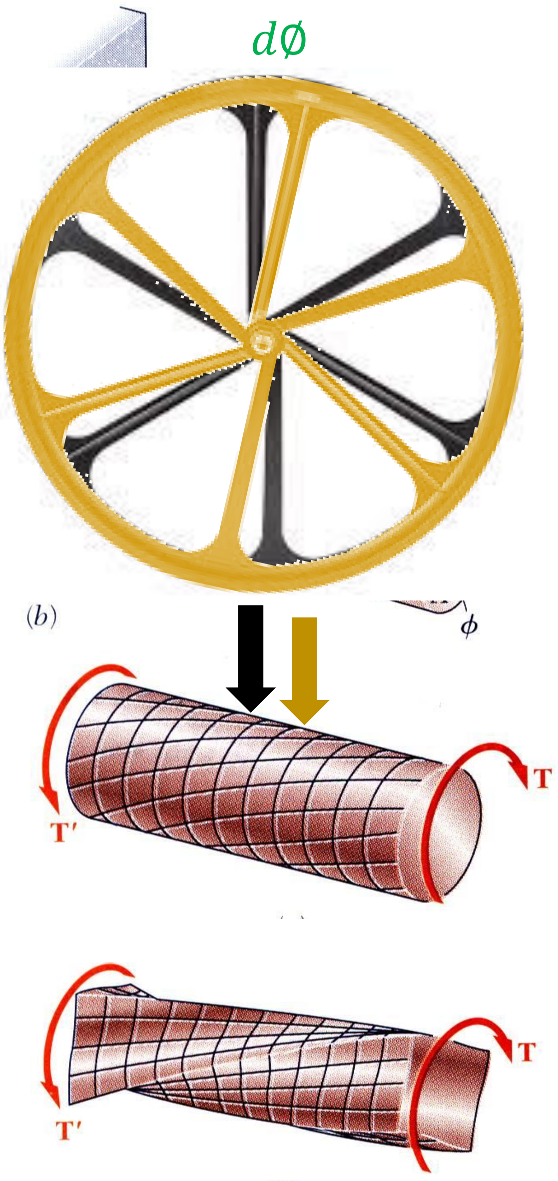

- From observation, the angle of twist of the shaft is proportional to the applied torque and to the shaft length.

$$\phi \propto T$$

$$\phi \propto L$$

- When subjected to torsion, every cross-section of a circular shaft remains plane and **undistorted**.
- However, two cross sections rotate by different amounts ($d\emptyset$)
- Cross-sections for hollow and solid circular shafts remain plain and undistorted because a circular shaft is axisymmetric.
- Cross-sections of noncircular (non-axisymmetric, w.r.t. to shaft axis) shafts are distorted when subjected to torsion.

# Shaft Deformations

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# **Recall:** Shearing Strain

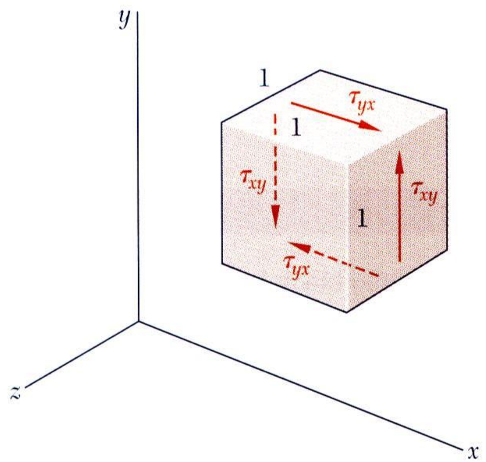



Fig. 2.46

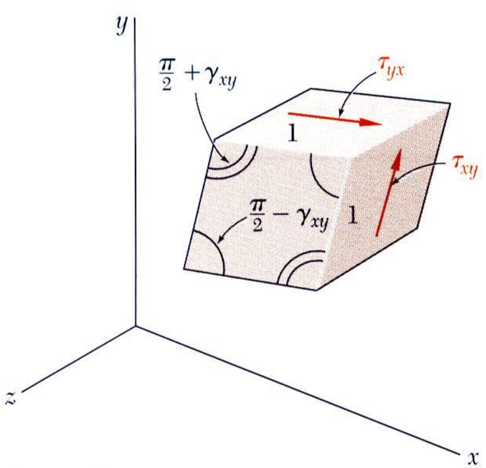



Fig. 2.47

- A cubic element subjected to a shear stress will deform into a rhomboid. The corresponding *shear* strain is quantified in terms of the change in angle between the sides,

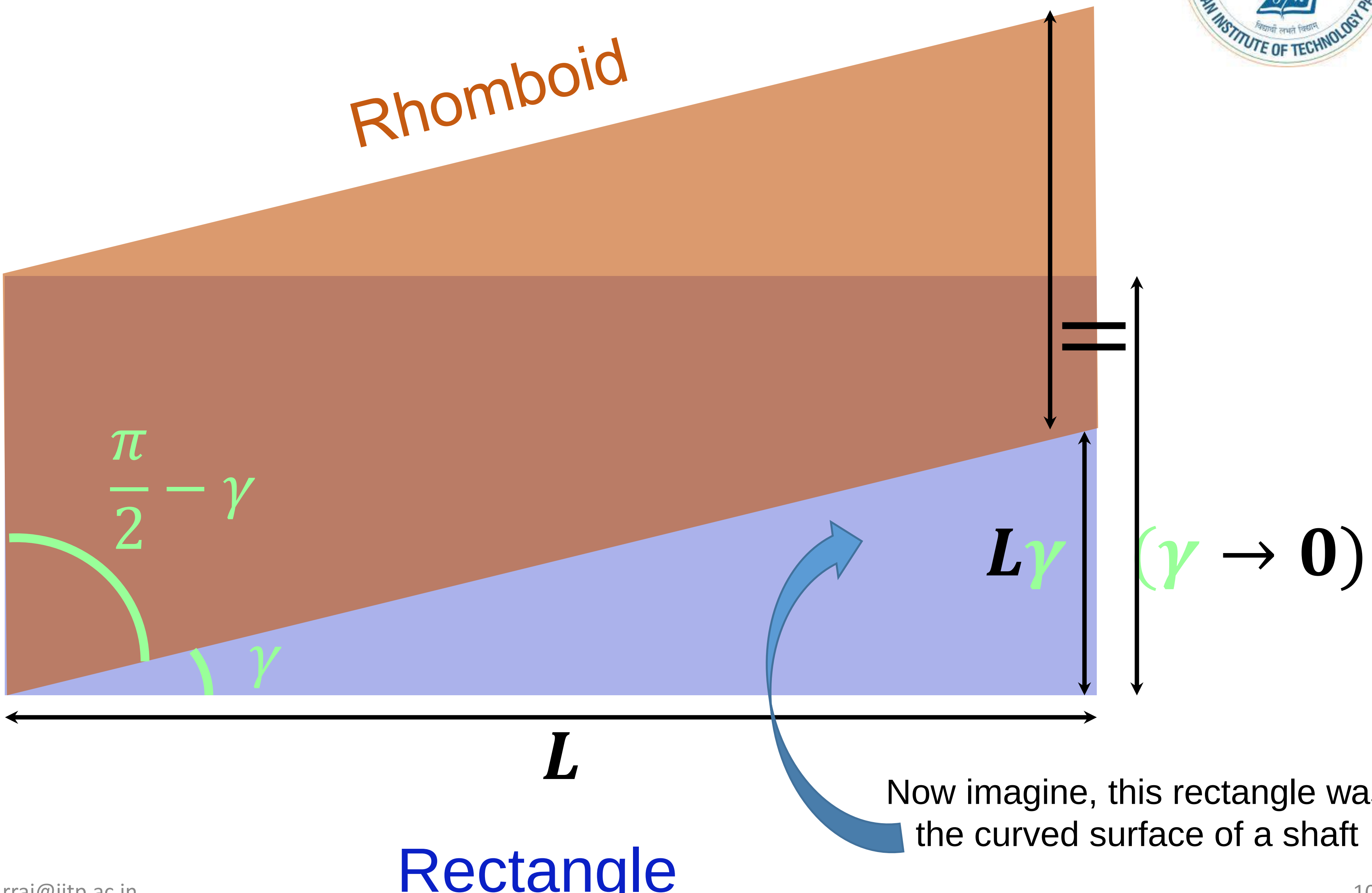
$$\tau_{xy} = f\left(\gamma_{xy}\right)$$

- A plot of shear stress vs. shear strain is similar the plots of normal stress vs. normal strain except that the **strength values are approximately half**. For small strains,

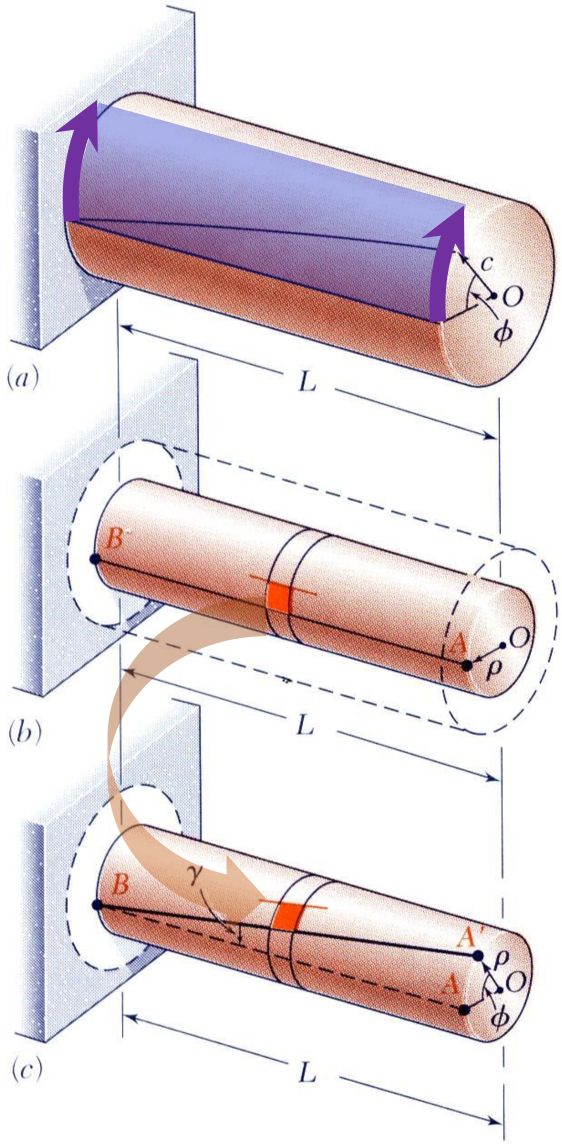
$$\tau_{xy} = G\gamma_{xy} \quad \tau_{yz} = G\gamma_{yz} \quad \tau_{zx} = G\gamma_{zx}$$

where $G$ is the modulus of rigidity or shear modulus.

# Shearing Strain



# Shearing Strain




- Consider an interior section of the shaft. As a torsional load is applied, an element on the interior **cylinder** deforms into a **rhombus**.

- Since the ends of the element remain planar, the **shear strain** $\boldsymbol{\gamma}$ is proportional to **angle of twist** Ø.
- It follows that

$$AA' = L\gamma = \rho\varphi \quad \text{or} \quad \gamma = \frac{\rho\varphi}{L}$$

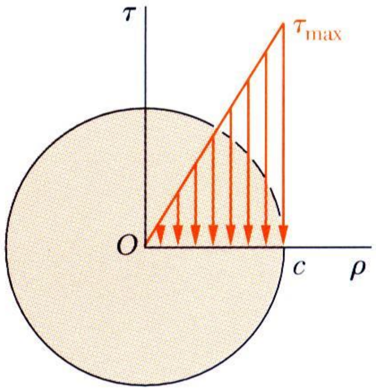
- Shear strain is proportional to twist and radius

$$\gamma_{\max} = \frac{c\phi}{L} \quad \text{and} \quad \boxed{\gamma = \frac{\rho}{c}\gamma_{\max}}$$

$c$ = surface radius

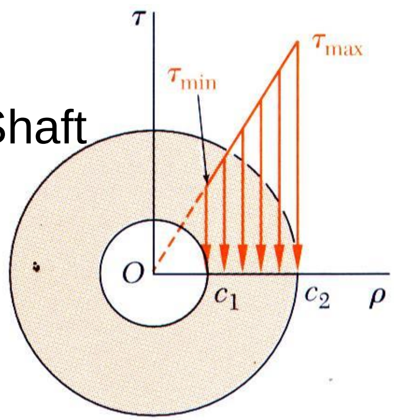
- Shear strain in a shaft varies linearly with the distance from the axis of the shaft

# Stresses in Elastic Range




$$J = \frac{1}{2}\pi c^4$$

Hollow Shaft




$$J = \frac{1}{2}\pi\left(c_2^4 - c_1^4\right)$$

- Multiplying the previous equation by the shear modulus,

$$G\gamma = \frac{\rho}{c} G\gamma_{max}$$

From Hooke's Law, $\tau = G\gamma$, so

$$\tau = \frac{\rho}{c}\tau_{max}$$

The shearing stress varies linearly with the radial position in the section.

- Recall that the sum of the moments from the internal stress distribution is equal to the torque on the shaft at the section,

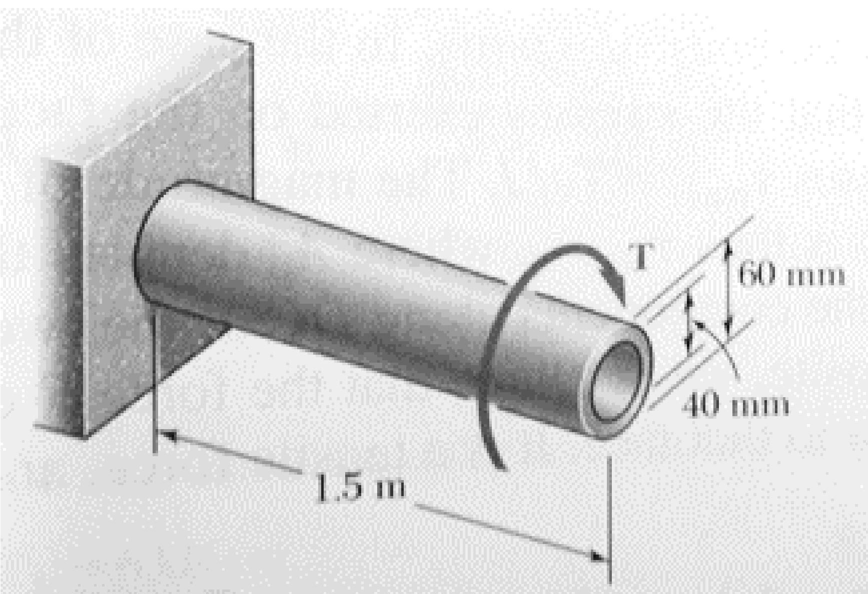
$$T = \int \rho\tau \, dA = \frac{\tau_{max}}{c}\int \rho^2 \, dA = \frac{\tau_{max}}{c} J$$

- The results are known as the elastic torsion formulas,

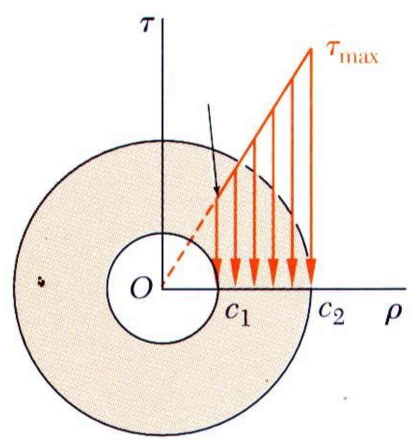
$$\tau_{max} = \frac{Tc}{J} \quad \text{and} \quad \tau = \frac{T\rho}{J}$$

# Problem 1

A hollow cylindrical shaft is 1.5 m long and has inner and outer diameters respectively equal to 40 and 60 mm. (a)What is the largest torque that can be applied to the shaft if the shearing stress is not to exceed 120 MPa? (b) What is the corresponding minimum value of shearing stress in the shaft?



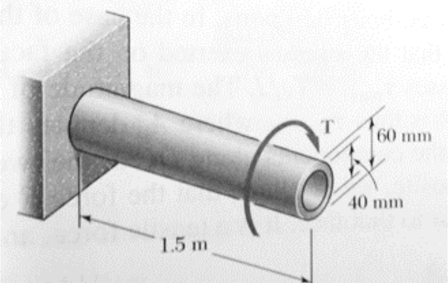

Where will the shaft experience largest shearing stress?




$$@\ \rho = c_2$$

$$\tau_{max} = 120 \text{ MPa}$$

# Solution




*(a) Largest Permissible Torque.* The largest torque **T** that can be applied to the shaft is the torque for which $\tau_{max} = 120$ MPa. Since this value is less than the yield strength for steel, we can use Eq. (3.9). Solving this equation for $T$, we have

$$T = \frac{J\tau_{max}}{c} \tag{3.12}$$

Recalling that the polar moment of inertia $J$ of the cross section is given by Eq. (3.11), where $c_1 = \frac{1}{2}(40 \text{ mm}) = 0.02$ m and $c_2 = \frac{1}{2}(60 \text{ mm}) = 0.03$ m, we write

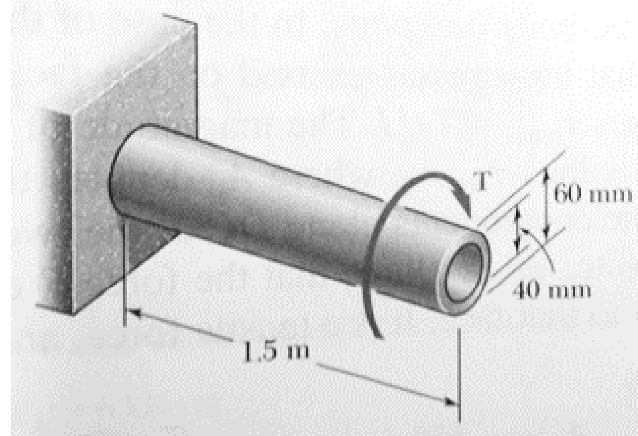
$$J = \tfrac{1}{2}\pi\,(c_2^4 - c_1^4) = \tfrac{1}{2}\pi(0.03^4 - 0.02^4) = 1.021 \times 10^{-6}\text{ m}^4$$

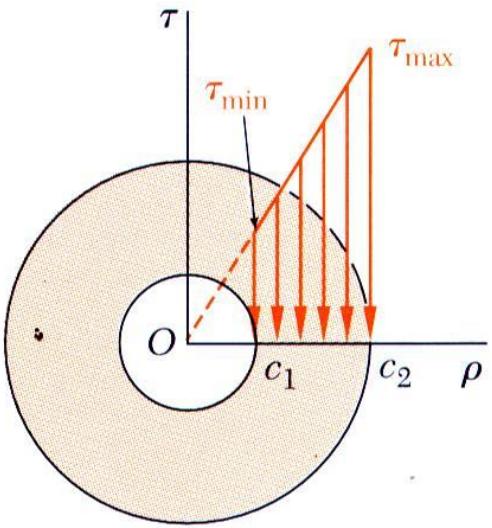
Substituting for $J$ and $\tau_{max}$ into (3.12), and letting $c = c_2 = 0.03$ m, we have

$$T = \frac{J\tau_{max}}{c} = \frac{(1.021 \times 10^{-6}\text{ m}^4)(120 \times 10^6\text{ Pa})}{0.03\text{ m}} = 4.08\text{ kN}\cdot\text{m}$$

# Solution

Where will the shaft experience minimum shearing stress?



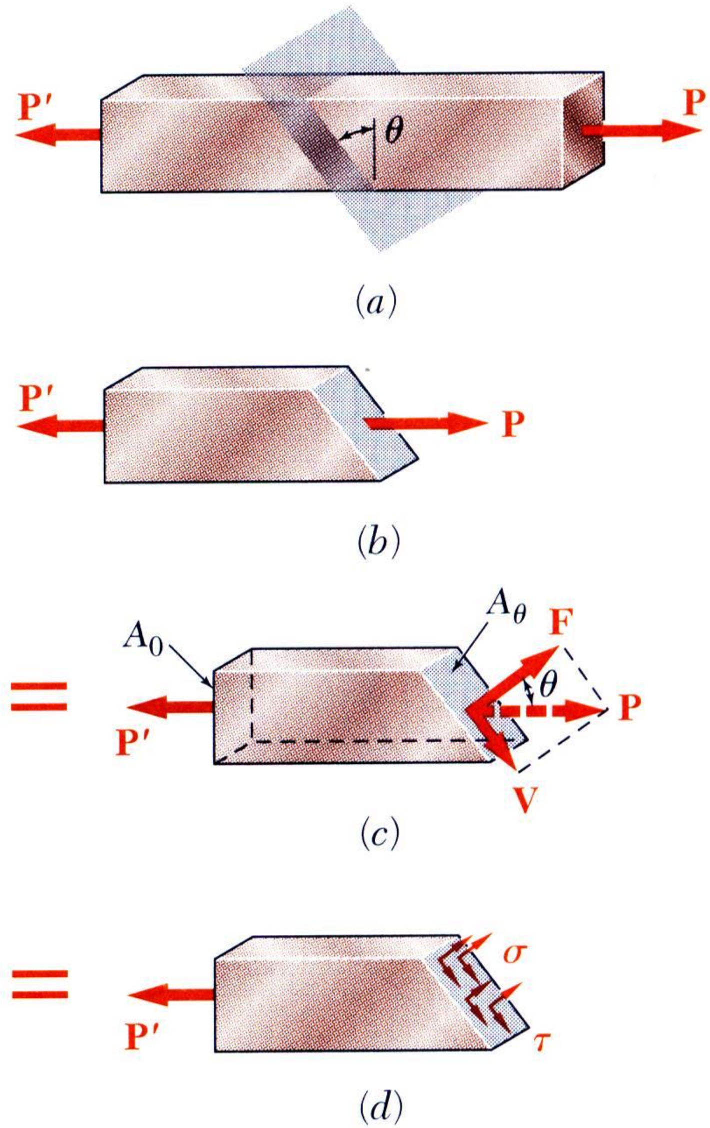




$$@\ \rho = c_1$$

*(b) Minimum Shearing Stress.* The minimum value of the shearing stress occurs on the inner surface of the shaft. It is obtained from Eq. (3.7), which expresses that $\tau_{min}$ and $\tau_{max}$ are respectively proportional to $c_1$ and $c_2$:

$$\tau_{min} = \frac{c_1}{c_2}\tau_{max} = \frac{0.02 \text{ m}}{0.03 \text{ m}}(120 \text{ MPa}) = 80 \text{ MPa}$$

# Recall: Axial Load → Shear Stress




- Pass a section through the member forming an angle $\theta$ with the normal plane.
- From equilibrium conditions, the distributed forces (stresses) on the plane must be equivalent to the force *P*.
- Resolve *P* into components normal and tangential to the oblique section,

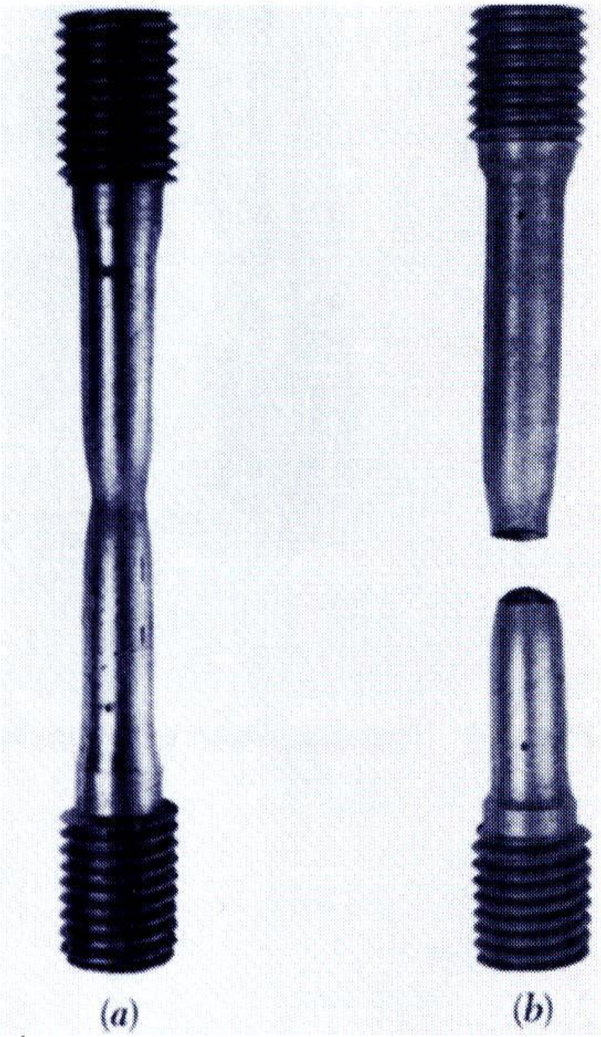
$$F = P\cos\theta \qquad V = P\sin\theta$$

- The average normal and shear stresses on the oblique plane are

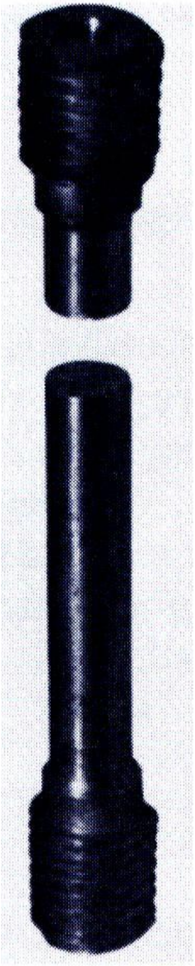
$$\sigma = \frac{F}{A_\theta} = \frac{P\cos\theta}{A_0 / \cos\theta} = \frac{P}{A_0}\cos^2\theta$$

$$\tau = \frac{V}{A_\theta} = \frac{P\sin\theta}{A_0 / \cos\theta} = \frac{P}{A_0}\sin\theta\cos\theta$$

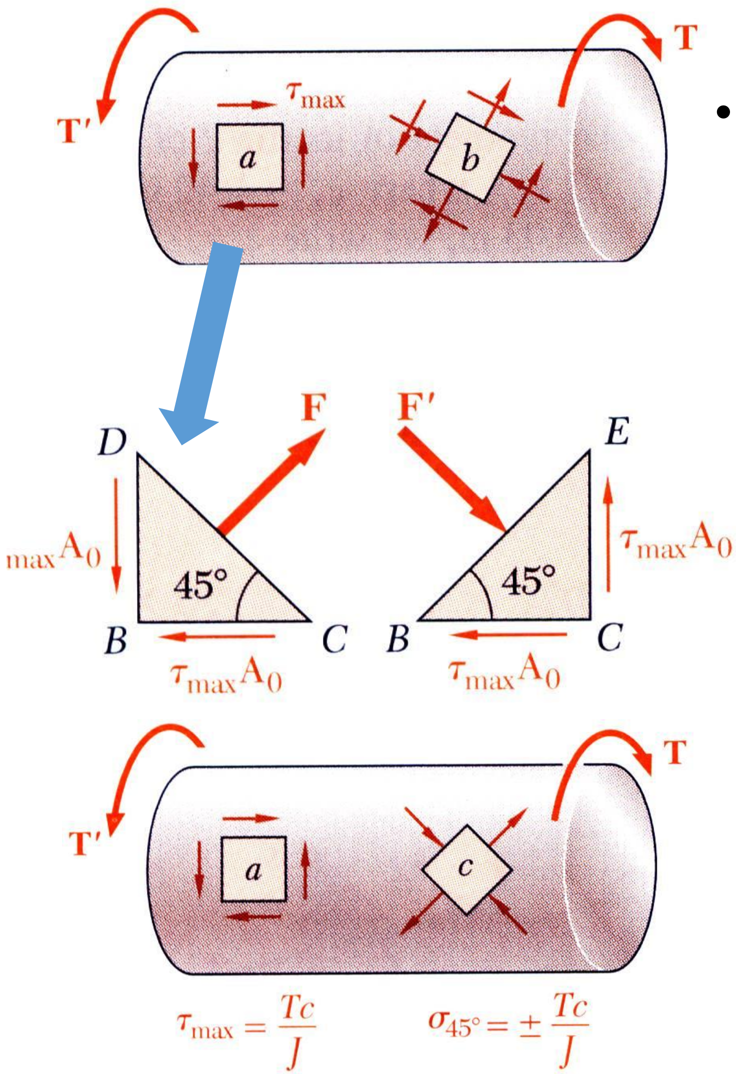
# Ductile




# Brittle



# Torsion → Normal Stresses



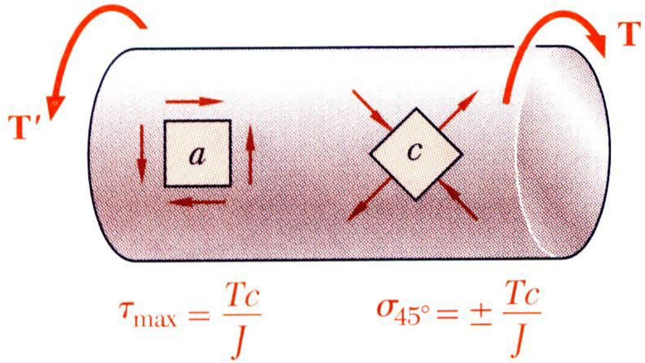

- Elements with faces **parallel** and **perpendicular** to the shaft axis are subjected to shear stresses only. **Normal stresses, shearing stresses** or a **combination of both** may be found for other orientations.
- Consider an element at 45° to the shaft axis,

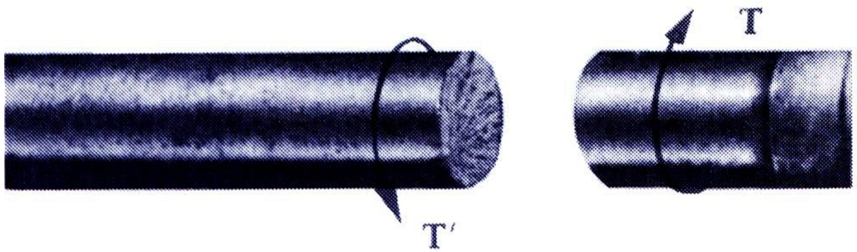
$$F = 2(\tau_{max} A_0)\cos 45 = \tau_{max} A_0 \sqrt{2}$$

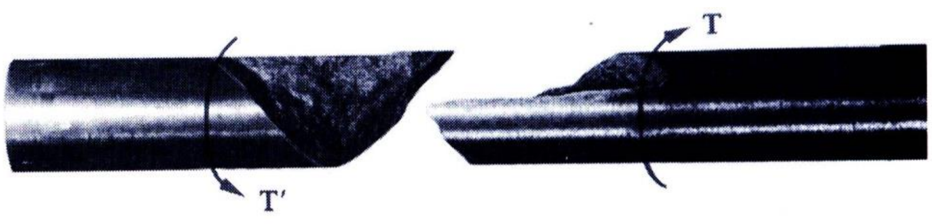
$$\sigma_{45^o} = \frac{F}{A} = \frac{\tau_{max} A_0 \sqrt{2}}{A_0 \sqrt{2}} = \tau_{max}$$

- Element *a* is in pure shear.
- Element *c* is subjected to a tensile stress on two faces and compressive stress on the other two.
- Note that all stresses for elements *a* and *c* have the same magnitude

# Torsional Failure Modes



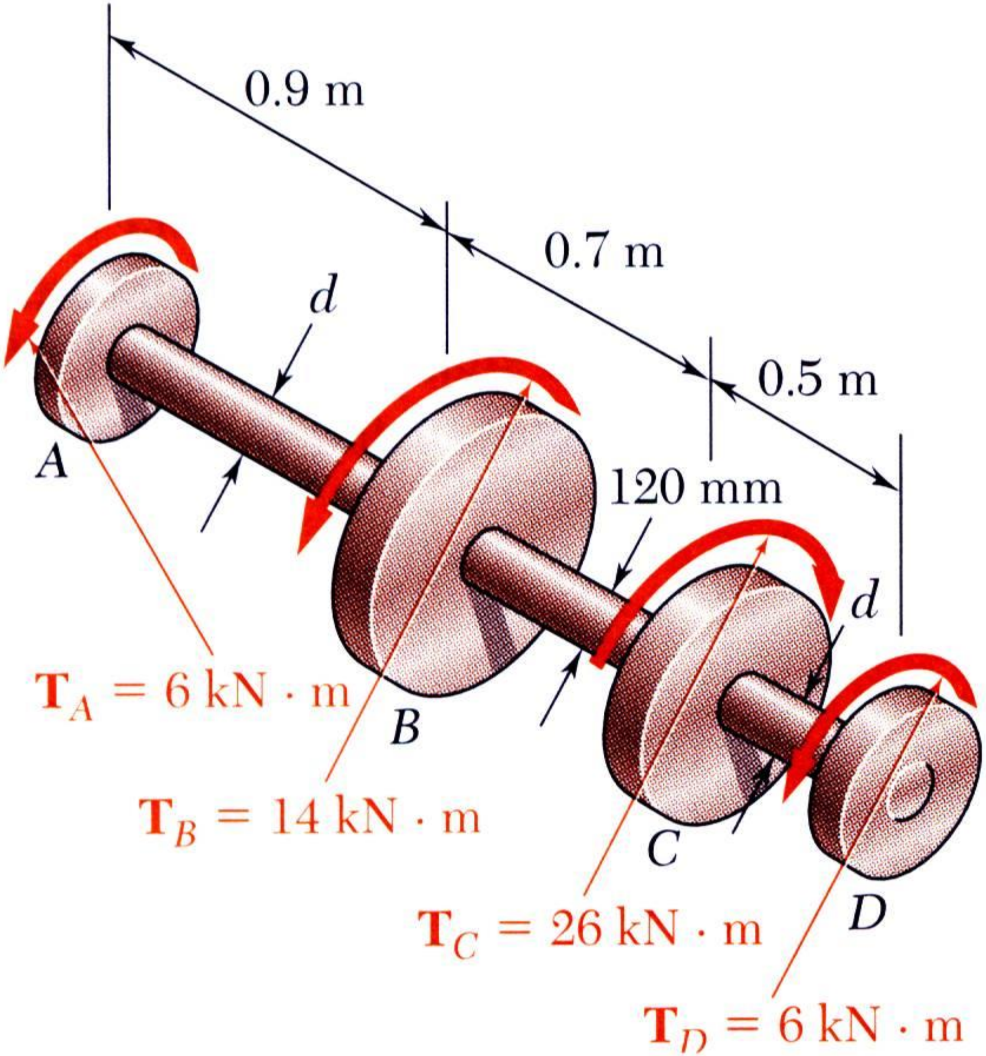




- Ductile materials generally fail in shear. Brittle materials are weaker in tension than shear.

- When subjected to torsion, a ductile specimen breaks along a plane of maximum shear, i.e., a plane perpendicular to the shaft axis.

- When subjected to torsion, a brittle specimen breaks along planes perpendicular to the direction in which tension is a maximum, i.e., along surfaces at 45$^o$ to the shaft axis.

# Problem 2



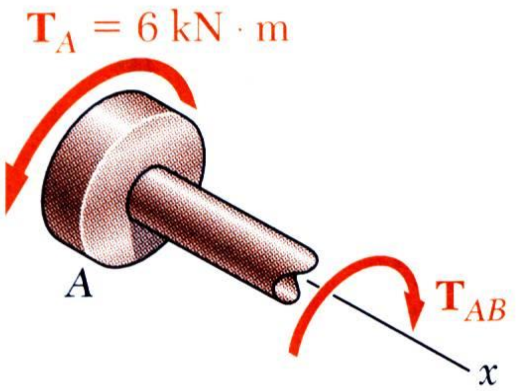

Shaft *BC* is hollow with inner and outer diameters of 90 mm and 120 mm, respectively. Shafts *AB* and *CD* are solid of diameter *d*. For the loading shown, determine

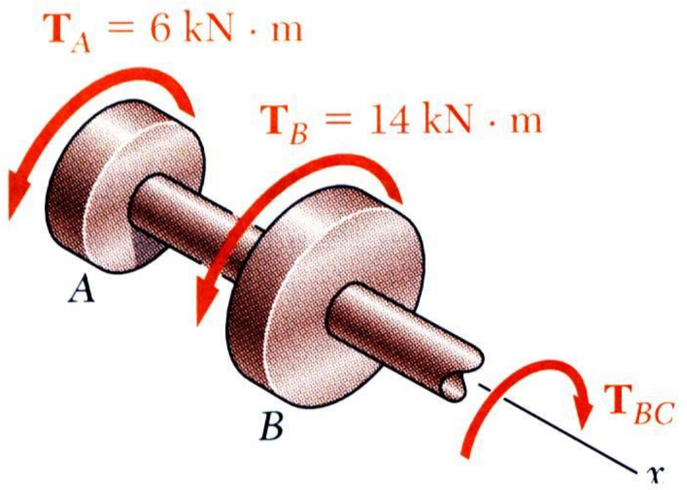
(a) the minimum and maximum shearing stress in shaft *BC*,

(b) the required diameter *d* of shafts *AB* and *CD* if the allowable shearing stress in these shafts is 65 MPa.

SOLUTION:

- Cut sections through shafts *AB* and *BC* and perform static equilibrium analysis to find torque loadings



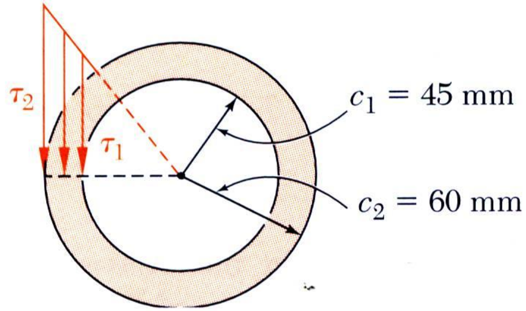

$$\sum M_x = 0 = (6\,\mathrm{kN \cdot m}) - T_{AB}$$

$$T_{AB} = 6\,\mathrm{kN \cdot m} = T_{CD}$$

$$\sum M_x = 0 = (6\,\mathrm{kN \cdot m}) + (14\,\mathrm{kN \cdot m}) - T_{BC}$$

$$T_{BC} = 20\,\mathrm{kN \cdot m}$$

- Apply elastic torsion formulas to find minimum and maximum stress on shaft *BC*




$$J = \frac{\pi}{2}\left(c_2^4 - c_1^4\right) = \frac{\pi}{2}\left[(0.060)^4 - (0.045)^4\right]$$

$$= 13.92 \times 10^{-6}\,\mathrm{m}^4$$

$$\tau_{max} = \tau_2 = \frac{T_{BC}c_2}{J} = \frac{(20\,\mathrm{kN \cdot m})(0.060\,\mathrm{m})}{13.92 \times 10^{-6}\,\mathrm{m}^4}$$

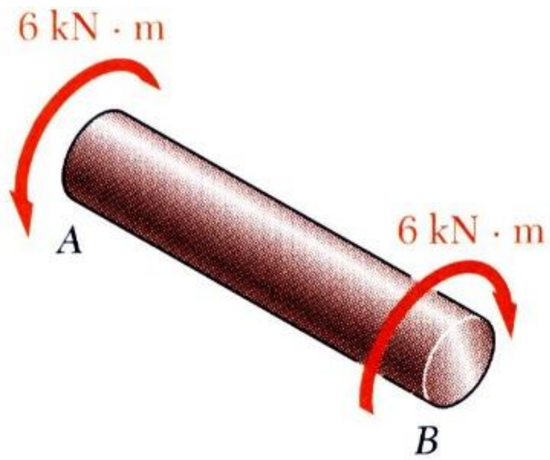
$$= 86.2\,\mathrm{MPa}$$

$$\frac{\tau_{min}}{\tau_{max}} = \frac{c_1}{c_2} \qquad \frac{\tau_{min}}{86.2\,\mathrm{MPa}} = \frac{45\,\mathrm{mm}}{60\,\mathrm{mm}}$$

$$\tau_{min} = 64.7\,\mathrm{MPa}$$

$$\boxed{\begin{aligned}\tau_{max} &= 86.2\,\mathrm{MPa}\\ \tau_{min} &= 64.7\,\mathrm{MPa}\end{aligned}}$$

- Given allowable shearing stress and applied torque, invert the elastic torsion formula to find the required diameter



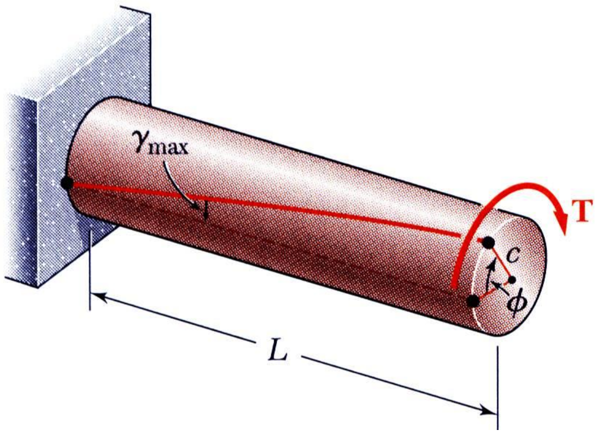

$$\tau_{max} = \frac{Tc}{J} = \frac{Tc}{\frac{\pi}{2}c^4} \qquad 65MPa = \frac{6\,\mathrm{kN \cdot m}}{\frac{\pi}{2}c^3}$$

$$c = 38.9 \times 10^{-3}\,\mathrm{m}$$

$$\boxed{d = 2c = 77.8\,\mathrm{mm}}$$

What about shaft CD?

# Angle of Twist in Elastic Range

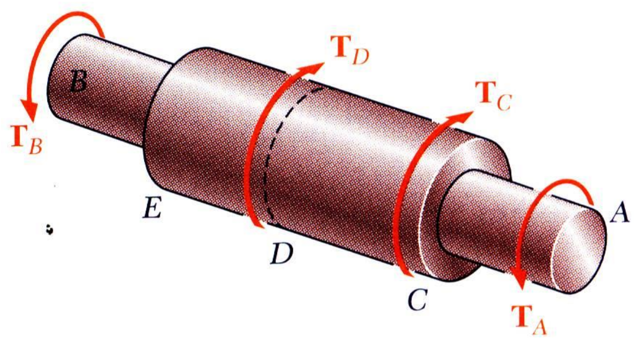



- Recall that the angle of twist and maximum shearing strain are related,

$$\gamma_{\max} = \frac{c\phi}{L}$$

- In the elastic range, the shearing strain and shear are related by Hooke's Law,

$$\gamma_{\max} = \frac{\tau_{\max}}{G} = \frac{Tc}{JG} \qquad T = \frac{J\tau_{\max}}{c}$$



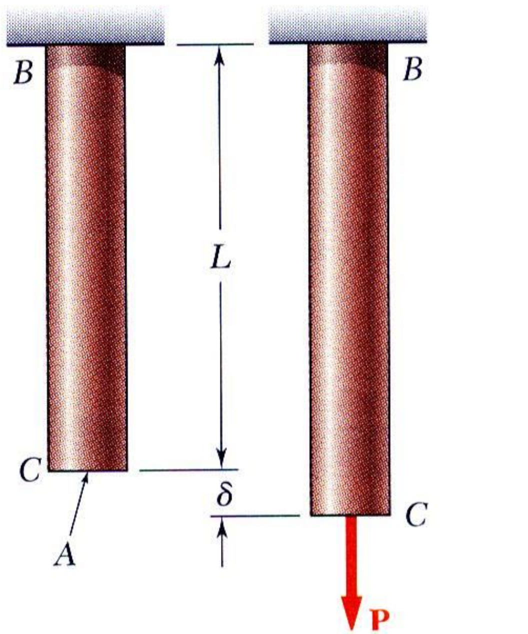

- Equating the expressions for shearing strain and solving for the angle of twist,

$$\phi = \frac{TL}{JG}$$

- If the torsional loading or shaft cross-section changes along the length, the angle of rotation is found as the sum of segment rotations

$$\phi = \sum_i \frac{T_i L_i}{J_i G_i}$$

# Comparison: Deformations Under Axial and Torsional Loadings




- From Hooke's Law:

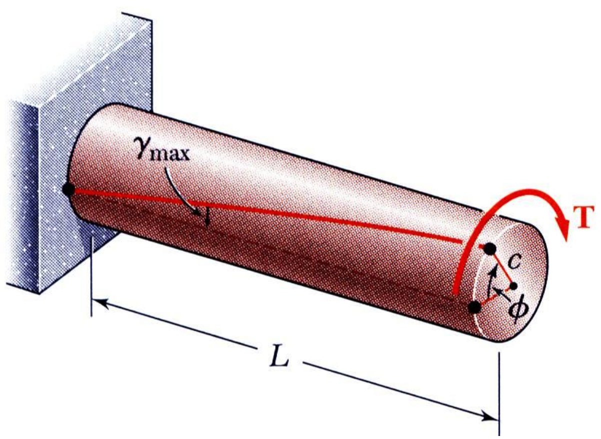
$$\sigma = E\varepsilon \qquad \tau = G\gamma$$

- Deformation

$$\delta = \frac{PL}{AE} \qquad \varphi = \frac{TL}{JG}$$

- With variations in loading, cross-section or material properties,

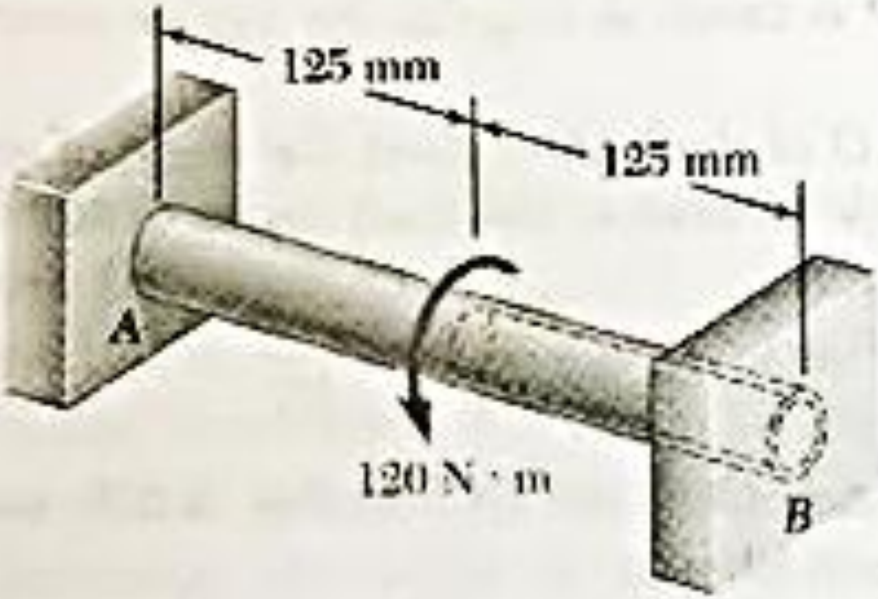
$$\delta = \sum_i \frac{P_i L_i}{A_i E_i} \qquad \varphi = \sum_i \frac{T_i L_i}{J_i G_i}$$



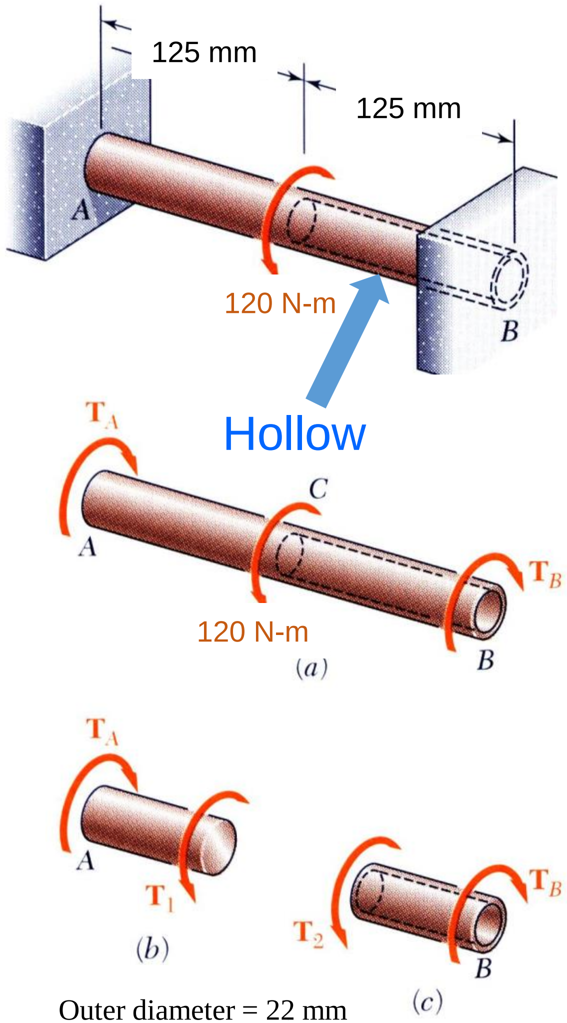

| **Axial** | | **Torsional** |
|---|---|---|
| $\boldsymbol{\delta\ (m)}$ | $\rightarrow$ | $\boldsymbol{\emptyset\ (-)}$ |
| $\boldsymbol{P\ (N)}$ | $\rightarrow$ | $\boldsymbol{T\ (N-m)}$ |
| $\boldsymbol{L\ (m)}$ | $\rightarrow$ | $\boldsymbol{L\ (m)}$ |
| $\boldsymbol{A\ (m^2)}$ | $\rightarrow$ | $\boldsymbol{J\ (m^4)}$ |
| $\boldsymbol{E\ (Pa)}$ | $\rightarrow$ | $\boldsymbol{G\ (Pa)}$ |

# Problem 3

**05** A circular shaft $AB$ consists of a 250-mm-long, 22-mm-diameter steel cylinder, in which a 125-mm-long, 16-mm-diameter cavity has been drilled from end $B$. The shaft is attached to fixed supports at both ends, and a 120-N · m torque is applied at its midsection (Fig. 3.25). Determine the torque exerted on the shaft by each of the supports.



# Statically Indeterminate Shafts




Outer diameter = 22 mm
Inner diameter = 16 mm

- Given the shaft dimensions and the applied torque, we would like to find the torque reactions at *A* and *B*.
- From a free-body analysis of the shaft,

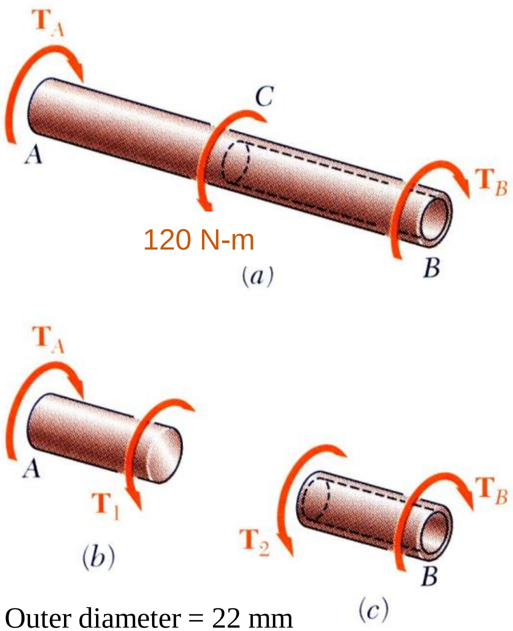
$$T_A + T_B = 120 \text{ N} - \text{m}$$

which is not sufficient to find the end torques. The problem is statically indeterminate.

- Divide the shaft into two components which must have compatible deformations,

$$\phi = \phi_1 + \phi_2 = \frac{T_A L_1}{J_1 G} - \frac{T_B L_2}{J_2 G} = 0 \qquad T_B = \frac{L_1 J_2}{L_2 J_1} T_A$$

# Statically Indeterminate Shafts




Outer diameter = 22 mm
Inner diameter = 16 mm

- Divide the shaft into two components which must have compatible deformations,

$$\phi = \phi_1 + \phi_2 = \frac{T_A L_1}{J_1 G} - \frac{T_B L_2}{J_2 G} = 0 \qquad T_B = \frac{L_1 J_2}{L_2 J_1} T_A$$

Substituting the numerical data

$$L_1 = L_2 = 125 \text{ mm}$$
$$J_1 = \tfrac{1}{2}\pi(0.011 \text{ m})^4 = 230 \times 10^{-6} \text{ m}^4$$
$$J_2 = \tfrac{1}{2}\pi[(0.011 \text{ m})^4 - (0.008 \text{ m})^4] = 165.6 \times 10^{-6} \text{ m}^4$$

we obtain

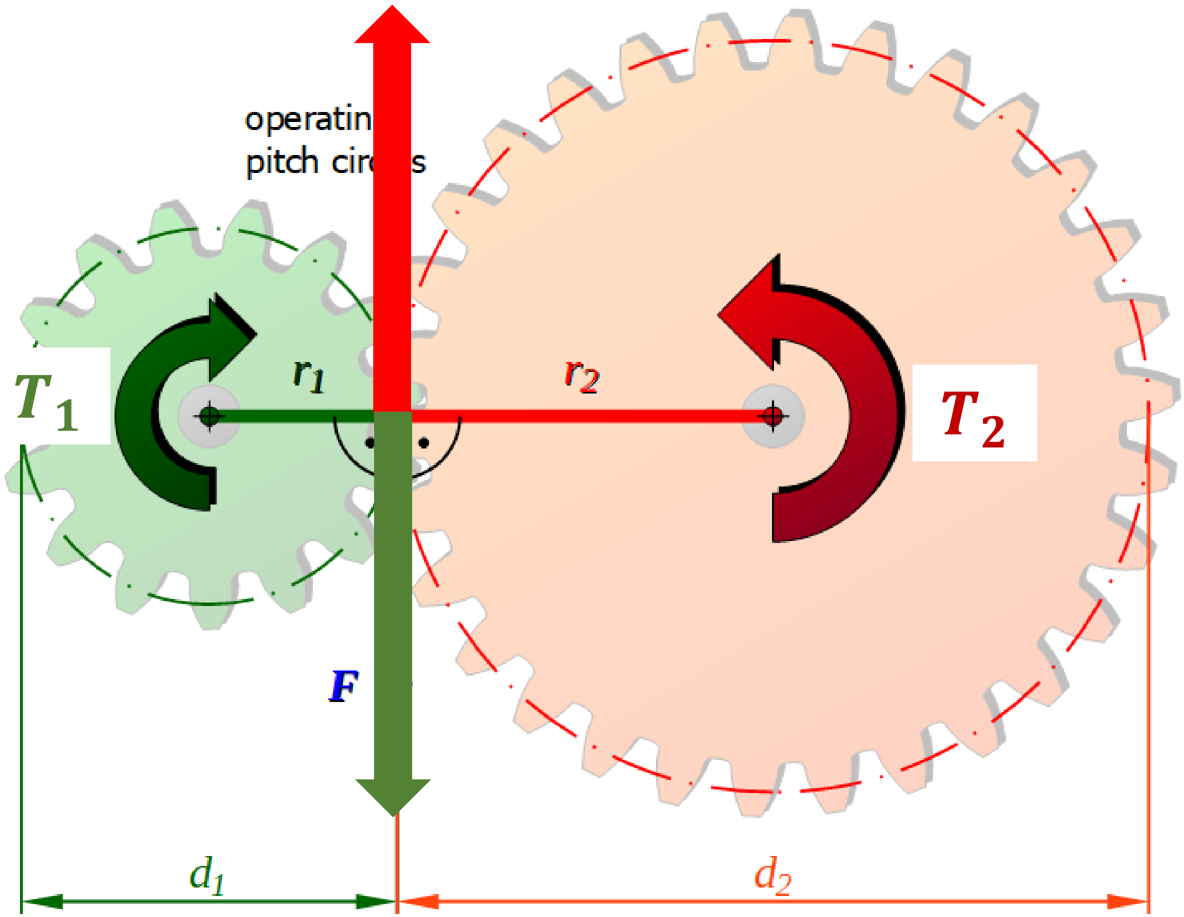
$$T_B = 0.72\, T_A$$

- Substitute into the original equilibrium equation,

$$T_A + \frac{L_1 J_2}{L_2 J_1} T_A = 120 \text{ N} - \text{m}$$

$$T_A = 69.8 \text{ N} - \text{m} \qquad T_B = 50.2 \text{ N} - \text{m}$$

# Torque Transmission: Newton's 3rd Law




$$F_1 = F_2 = \mathbf{F}$$

$$\frac{T_1}{r_1} = \frac{T_2}{r_2}$$

Peripheral distance covered should be same

$$r_1\theta_1 = r_2\theta_2$$

Peripheral velocity should be same

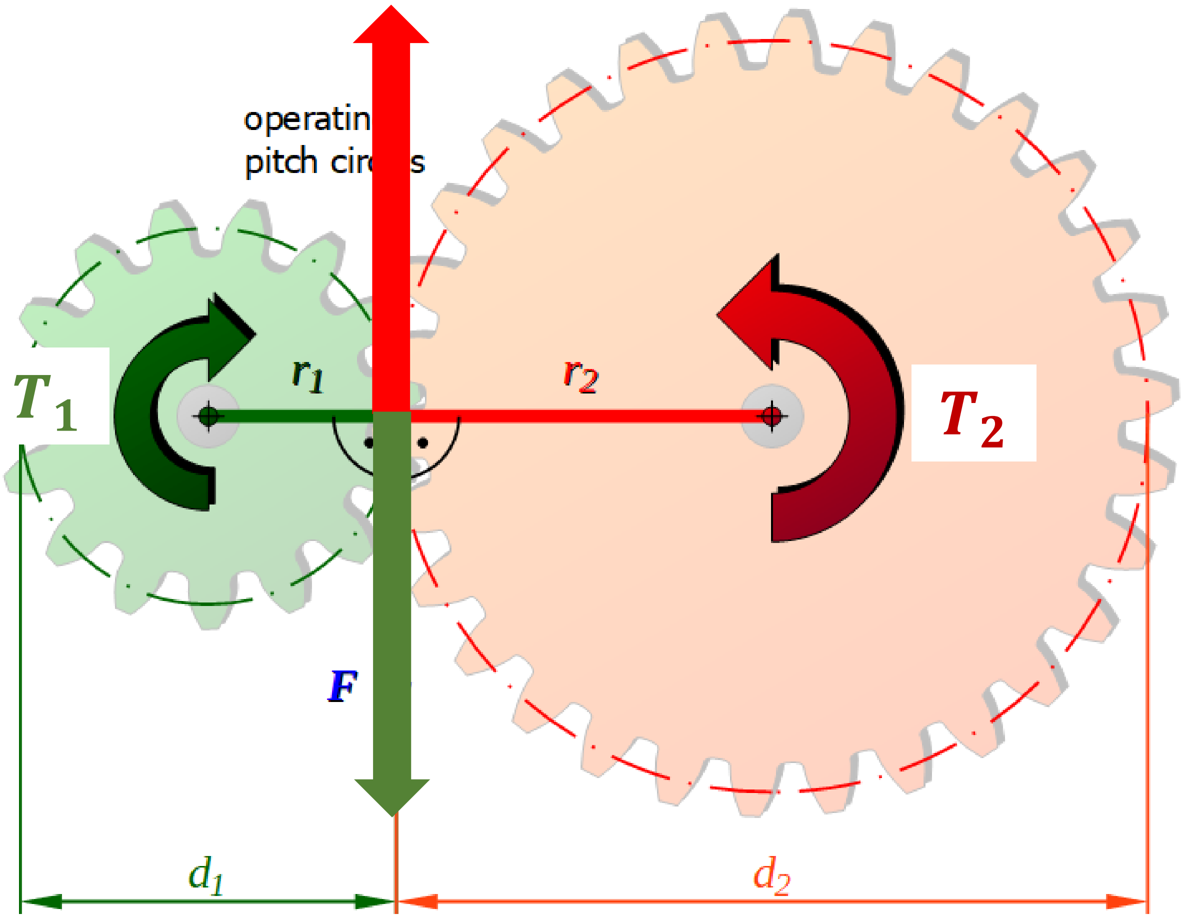
$$r_1\omega_1 = r_2\omega_2$$

$$\frac{\omega_1}{r_2} = \frac{\omega_2}{r_1}$$

Number of teeth interacted

$$r_1\theta_1 = r_2\theta_2 \rightarrow r_1\, 2\pi\frac{N}{N_1} = r_2 2\pi\frac{N}{N_2} \rightarrow \frac{r_1}{N_1} = \frac{r_2}{N_2}$$

# Torque Transmission: Newton's 3rd Law




$$\frac{T_1}{T_2} = \frac{r_1}{r_2} = \frac{\theta_2}{\theta_1} = \frac{\omega_2}{\omega_1} = \frac{N_1}{N_2}$$

# Problem 4

Two solid shafts are connected by gears as shown. Knowing that G = 77.2 GPa for each shaft, determine the angle through which end A rotates when $T_A$ = 1200N · m.

Given the direction of torque at A, B will rotate clockwise, C will rotate counter-clockwise

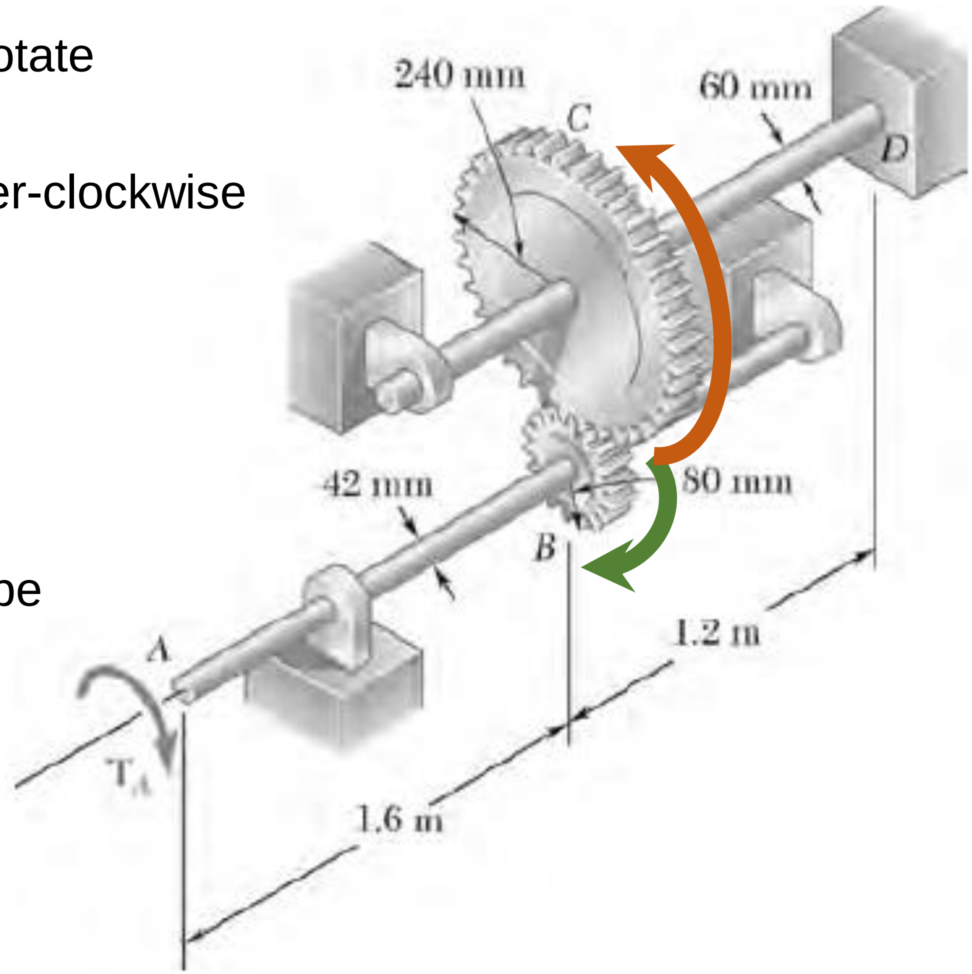
w.r.t. to D (fixed support), C **twists** counter-clockwise

if C twists counter-clockwise, B **rotates** opposite, *i.e.* clockwise w.r.t. D

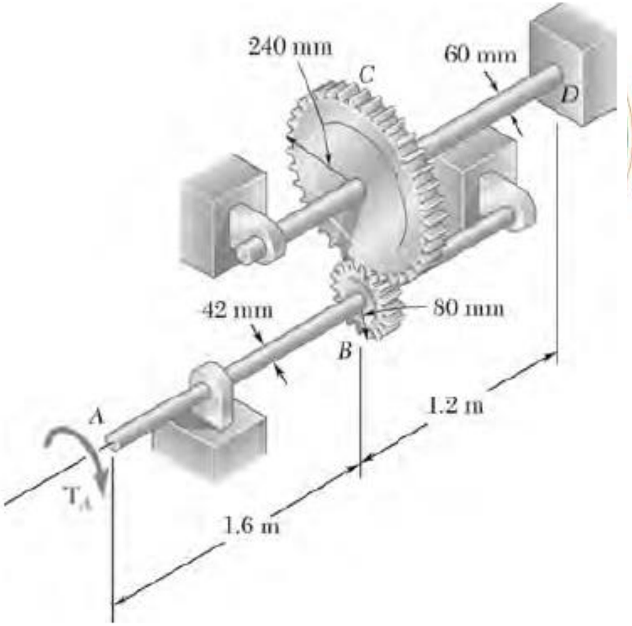
w.r.t. to B, A **twists** clockwise

Hence, the twists $\phi_{B/D}$ and $\phi_{A/B}$ should be added to get $\phi_{A/D}$

$$\phi_{A/D} = \phi_{A/B} + \phi_{B/D}$$



# Solution

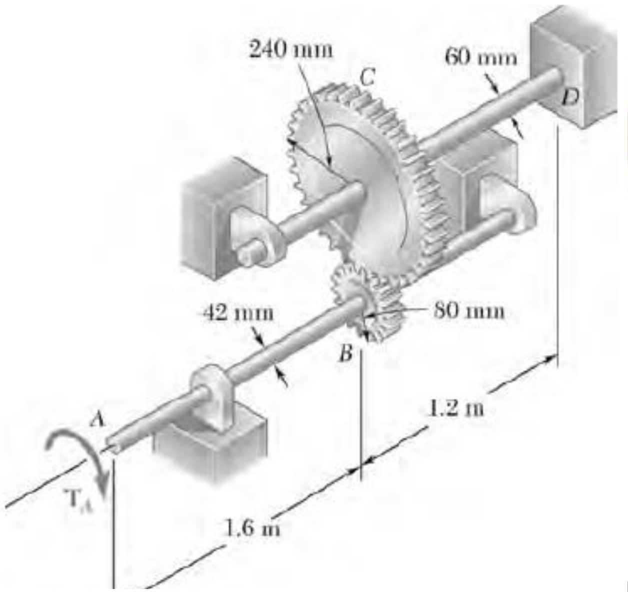



Given $T_{AB}$ = 1200 n-m, find $T_{CD}$

Knowing $T_{CD}$, find $\phi_{C/D}$

# Solution

Due to rotation of C, find rotation in B $\phi_{B/D}$



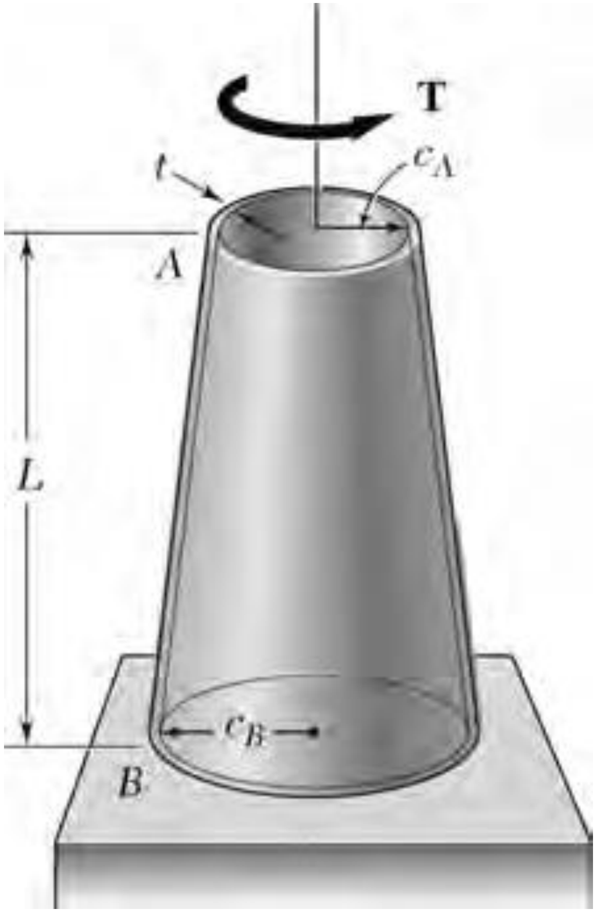

Knowing $T_{AB}$, find $\phi_{A/B}$

# Problem 5

The long, hollow, tapered shaft AB has a uniform thickness t. Denoting by G the modulus of rigidity, shown that the angle of twist at end A is

$$\phi_A = \frac{TL}{4\pi G t} \frac{c_A + c_B}{c_A^2 c_B^2}$$

$$\varphi = \sum_i \frac{T_i L_i}{J_i G_i}$$



# Design of Transmission Shafts

- Principal transmission shaft performance specifications are:
  - power
  - speed

- Designer must select shaft material and cross-section to meet performance specifications **without exceeding allowable shearing stress.**

- Determine torque applied to shaft at specified power and speed,

$$P = T\omega = 2\pi f T$$

$$T = \frac{P}{\omega} = \frac{P}{2\pi f}$$

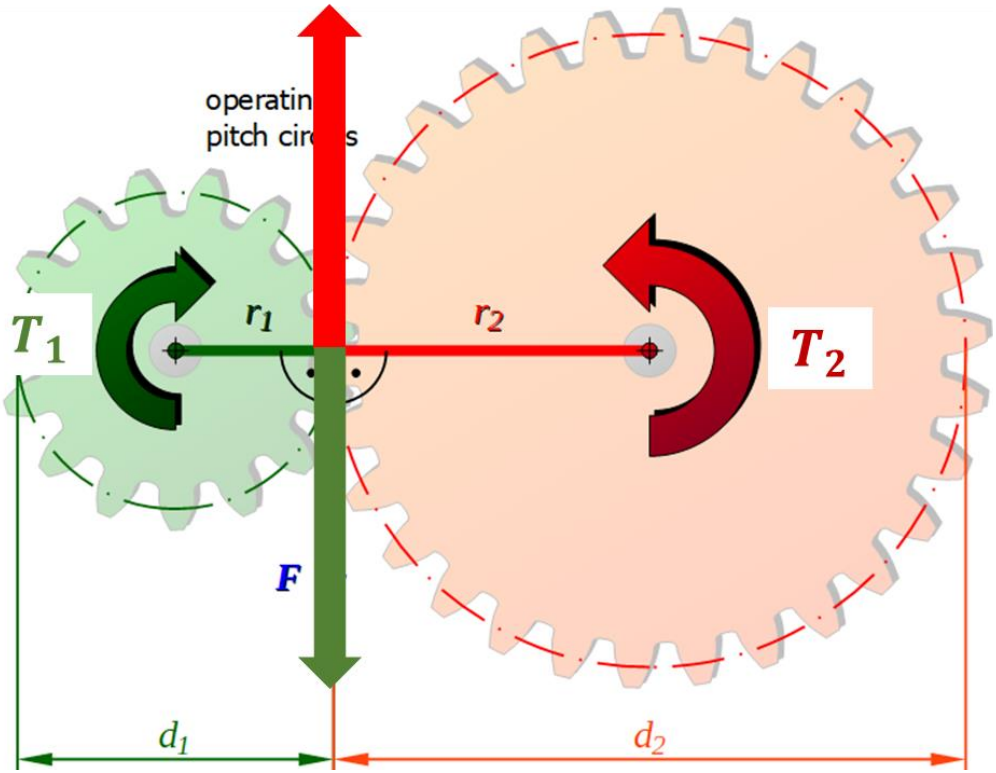
- Find shaft cross-section which will not exceed the maximum allowable shearing stress,

$$\tau_{\max} = \frac{Tc}{J}$$

$$\frac{J}{c} = \frac{\pi}{2} c^3 = \frac{T}{\tau_{\max}} \quad (\text{solid shafts})$$

$$\frac{J}{c_2} = \frac{\pi}{2c_2}\left(c_2^4 - c_1^4\right) = \frac{T}{\tau_{\max}} \quad (\text{hollow shafts})$$

# Power Transmission




- Derive the relation between $P_1$ and $P_2$

$$\frac{T_1}{T_2} = \frac{r_1}{r_2} = \frac{\theta_2}{\theta_1} = \frac{\omega_2}{\omega_1} = \frac{N_1}{N_2}$$

$$\frac{P_1}{P_2} = \frac{T_1}{T_2}\frac{\omega_1}{\omega_2} = 1$$