Let V and W be two vector spaces over F.
A vector function $f: V \to W$ is called
linear transformation from V into W iff
$\forall\, x, y \in V$ and $\forall\, \alpha \in F$

$$\begin{bmatrix} \text{(i)} & f(\alpha x) = \alpha f(x) \\ \text{(ii)} & f(x+y) = f(x) + f(y) \end{bmatrix} \equiv f(\alpha x + y) = \alpha f(x) + f(y)$$

$$|||$$

$\forall\, x, y \in V$ and $\alpha, \beta \in F$ $\quad f(\alpha x + \beta y) = \alpha f(x) + \beta f(y)$

$$|||$$

$\forall\, x_i \in V$ and $\alpha_i \in F$, $i = 1, 2, \dots, n$

$$f(\alpha_1 x_1 + \alpha_2 x_2 + \dots + \alpha_n x_n) = \alpha_1 f(x_1) + \alpha_2 f(x_2) + \dots \alpha_n f(x_n)$$

in short

$$f\left(\sum_{i=1}^{n} \alpha_i x_i\right) = \sum_{i=1}^{n} \alpha_i f(x_i)$$

Important facts

① $f(0) = 0$ (here $f: V \to W$ is a $\overset{\text{linear transformation}}{LT}$)
Here LHS 0 means 0 vector in V
& RHS 0 —— 0 —— W.

② We represent linear transformations by L/T (normaly) instead of function representation f/g etc.

③ Let V be finite dimensional with basis $\{v_1, v_2, \dots, v_n\}$.
Let $w_1, w_2, w_3, \dots, w_n \in W$. Then there exists a linear transformation $L: V \to W$ s.t.

$$L(v_i) = w_i \quad \forall\ i = 1, 2, 3, \dots, n.$$

Moreover, L is unique

Note:- difference b/t linear function and linear transformation; ① is important.

The meaning of 3rd statement on last page is:-

We can always find a LT from U into W if we know images of any basis of U.

Why?? Suppose $\{u_1, u_2, u_3, \dots, u_n\}$ be a basis of U. Then any arbitrary $x \in U$ can be written as

$$x = \alpha_1 u_1 + \alpha_2 u_2 + \dots + \alpha_n u_n$$

$$\Rightarrow T(x) = T(\alpha_1 u_1 + \alpha_2 u_2 + \dots + \alpha_n u_n)$$

$$= \alpha_1 T(u_1) + \alpha_2 T(u_2) + \dots + \alpha_n T(u_n) \quad (\because T \text{ is linear})$$

$$= \sum_{i=1}^{n} \alpha_i T(u_i) \rightarrow$$ These are known, put the values and find the mathematical expression for $T(x)$ when x is arb. in U.

Note that Moreover, such LT is unique.

Tutorial Sheet Problems

<u>Idea</u>:- We can find LT if image of any basis is given.
Also remember def. $L(\alpha X + Y) = \alpha L(X) + L(Y)$.

<u>Ques</u> Find a LT, if possible

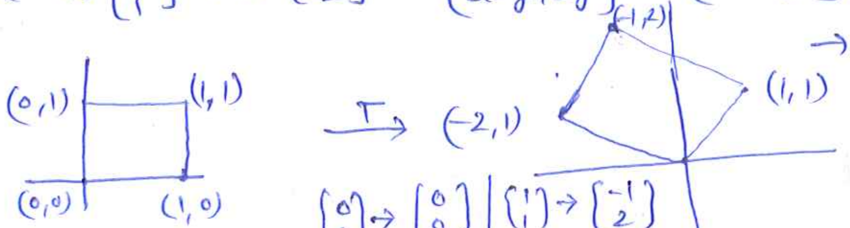
(i) $T: \mathbb{R}^2 \to \mathbb{R}^2$ s.t. $T\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}1\\1\end{bmatrix}$ and $T\begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix}-1\\2\end{bmatrix}$

$\left[\begin{array}{cc|c}1 & 1 & x\\ 0 & 1 & y\end{array}\right]$

<u>Sol</u>: $T\begin{bmatrix}x\\y\end{bmatrix} = T\left((x-y)\begin{bmatrix}1\\0\end{bmatrix} + y\begin{bmatrix}1\\1\end{bmatrix}\right)$

$= (x-y)\,T\begin{bmatrix}1\\0\end{bmatrix} + y\,T\begin{bmatrix}1\\1\end{bmatrix}$

$= (x-y)\begin{bmatrix}1\\1\end{bmatrix} + y\begin{bmatrix}-1\\2\end{bmatrix} = \begin{bmatrix}x-y-y\\x-y+2y\end{bmatrix} = \begin{bmatrix}x-2y\\x+y\end{bmatrix}$



→ Shows that it is ||gram

$\begin{bmatrix}0\\0\end{bmatrix} \to \begin{bmatrix}0\\0\end{bmatrix}$ | $\begin{bmatrix}1\\1\end{bmatrix} \to \begin{bmatrix}-1\\2\end{bmatrix}$

$\begin{bmatrix}1\\0\end{bmatrix} \to \begin{bmatrix}1\\1\end{bmatrix}$ | $\begin{bmatrix}0\\1\end{bmatrix} \to \begin{bmatrix}-2\\1\end{bmatrix}$

(II) $T: \mathbb{R}^2 \to \mathbb{R}^2$ s.t. $T\begin{bmatrix}2\\3\end{bmatrix} = \begin{bmatrix}4\\5\end{bmatrix}$, $T\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$

$\left[\begin{array}{cc|c}2 & 1 & x\\ 3 & 0 & y\end{array}\right] \sim \left[\begin{array}{cc|c}2 & 1 & x\\ 0 & -3/2 & y-\frac{3}{2}x\end{array}\right]$

$-\frac{2}{3}y + x$

$\frac{1}{2}\left[x + \frac{2}{3}y - x\right]$

$T\begin{bmatrix}x\\y\end{bmatrix} = T\left(\frac{1}{3}y\begin{bmatrix}2\\3\end{bmatrix} + \left(x - \frac{2}{3}y\right)\begin{bmatrix}1\\0\end{bmatrix}\right)$

$= \frac{1}{3}y\,T\begin{bmatrix}2\\3\end{bmatrix} + \left(x-\frac{2}{3}y\right)T\begin{bmatrix}1\\0\end{bmatrix}$

$= \frac{1}{3}y\begin{bmatrix}4\\5\end{bmatrix} + \left(x - \frac{2}{3}y\right)\begin{bmatrix}0\\0\end{bmatrix} = \begin{bmatrix}4/3\,y\\5/3\,y\end{bmatrix}$

(III) $T: \mathbb{R}^2 \to \mathbb{R}^3$ s.t. $T\begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix}1\\0\\1\end{bmatrix}$, $T\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}1\\0\\0\end{bmatrix}$, $T\begin{bmatrix}1\\2\end{bmatrix} = \begin{bmatrix}2\\1\\1\end{bmatrix}$

See: $\begin{bmatrix}1\\1\end{bmatrix} + \begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}1\\2\end{bmatrix} \Rightarrow T\left(\begin{bmatrix}1\\2\end{bmatrix}\right)$ must equal to $T\begin{bmatrix}1\\1\end{bmatrix} + T\begin{bmatrix}0\\1\end{bmatrix}$

But it does not hold here.

Ques Find a LT $T: \mathbb{R}^3 \to \mathbb{R}^3$ whose range is spanned by the vectors $\begin{bmatrix}1\\0\\-1\end{bmatrix}$ and $\begin{bmatrix}1\\2\\2\end{bmatrix}$.

Sol. Take $T\begin{bmatrix}1\\0\\0\end{bmatrix} = \begin{pmatrix}1\\0\\-1\end{pmatrix}$ $\quad T\begin{bmatrix}0\\1\\0\end{bmatrix} = \begin{pmatrix}1\\2\\2\end{pmatrix}$ & $T\begin{bmatrix}0\\0\\1\end{bmatrix} = ??$

~~write~~

Remember! To find LT | image of a basis is required. Since Range is spanned by some vectors so map a basis to given vectors.

In place of ?? write any ~~vector~~ lc of $\begin{bmatrix}1\\0\\-1\end{bmatrix}$ and $\begin{bmatrix}1\\2\\2\end{bmatrix}$ easy choice is $\begin{bmatrix}0\\0\\0\end{bmatrix}$.

Hence

$$T\begin{bmatrix}x\\y\\z\end{bmatrix} = x\begin{bmatrix}1\\0\\-1\end{bmatrix} + y\begin{pmatrix}1\\2\\2\end{pmatrix} + z\begin{bmatrix}0\\0\\0\end{bmatrix} = \begin{bmatrix}x+y\\2y\\-x+2y\end{bmatrix}$$

→ Answer is not unique depends on our choice of ??.

Ques Find a nonzero LT $T: \mathbb{R}^2 \to \mathbb{R}^2$ which maps all vectors on the line $y=x$ onto the origin.

Sol. given $T\begin{bmatrix}1\\1\end{bmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$

Take $T\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}1\\3\end{pmatrix}$ → This vector is any thing in $\mathbb{R}^2$ except $\begin{bmatrix}0\\0\end{bmatrix}$ otherwise we get zero LT.

→ This vector is a vector s.t. it makes basis of $\mathbb{R}^2$ with $\begin{bmatrix}1\\1\end{bmatrix}$

$$\begin{bmatrix}1 & 1\\1 & 0\end{bmatrix}\begin{bmatrix}a\\b\end{bmatrix} - \begin{bmatrix}x\\y\end{bmatrix} \quad \begin{bmatrix}1 & 1\\0 & -1\end{bmatrix}\begin{bmatrix}x\\y-x\end{bmatrix}$$

Under above choice

$$T\begin{bmatrix}x\\y\end{bmatrix} = T\left(y\begin{bmatrix}1\\1\end{bmatrix} + (x-y)\begin{bmatrix}1\\0\end{bmatrix}\right)$$
$$= y\begin{bmatrix}0\\0\end{bmatrix} + (x-y)\begin{pmatrix}1\\3\end{pmatrix} = \begin{pmatrix}x-y\\3x-3y\end{pmatrix}$$

→ This is unique under a choice. But If we change our choice here then we get a different LT.

Example of LT

(1) $L : \mathbb{R}^2 \to \mathbb{R}^3$ s.t. $L\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 + x_2 \\ 3x_1 + 2x_2 \\ x_2 \end{bmatrix}$

Show

$$L\left(\alpha\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = L\left(\begin{bmatrix} \alpha x_1 \\ \alpha x_2 \end{bmatrix}\right) = \begin{bmatrix} \alpha x_1 + \alpha x_2 \\ 3\alpha x_1 + 2\alpha x_2 \\ \alpha x_2 \end{bmatrix}$$

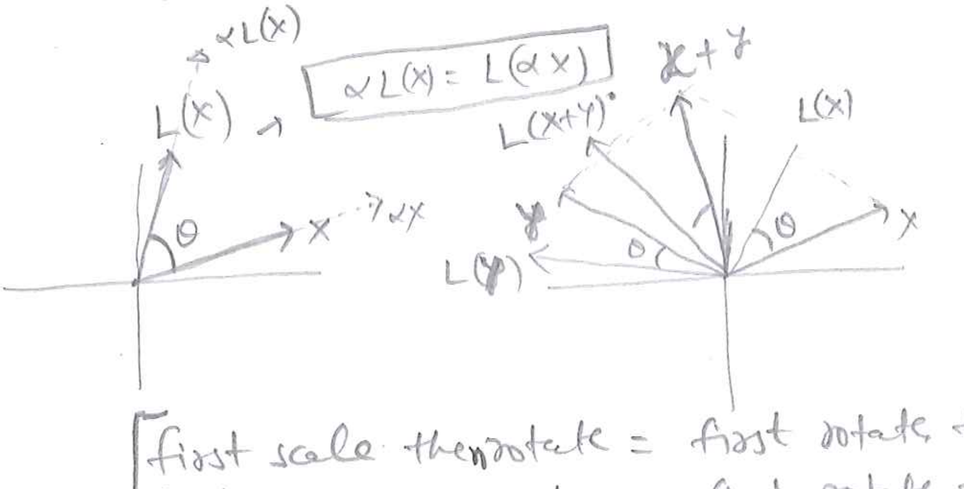
$$= \alpha \begin{bmatrix} x_1 + x_2 \\ 3x_1 + 2x_2 \\ x_2 \end{bmatrix} = \alpha L\left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\right)$$

$$L\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}\right) = L\left(\begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 + y_1 + x_2 + y_2 \\ 3(x_1 + y_1) + 2(x_2 + y_2) \\ x_2 + y_2 \end{bmatrix}$$

$$= \begin{bmatrix} x_1 + x_2 \\ 3x_1 + 2x_2 \\ x_2 \end{bmatrix} + \begin{bmatrix} y_1 + y_2 \\ 3y_1 + 2y_2 \\ y_2 \end{bmatrix} = L\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + L\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

(2) $L : \mathbb{R}^2 \to \mathbb{R}^2$ s.t. $L\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 + x_2 \\ x_2 \end{bmatrix} \;\Big|\; \begin{bmatrix} x_1 + x_2 \\ 0 \end{bmatrix} \;\Big|\; \begin{bmatrix} x_1 + x_2 \\ 1 \end{bmatrix}$

(3) Let $L : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation s.t. $L(X)$ rotates X (anticlockwise) by angle θ.



$L(X+Y) = L(X) + L(Y)$

first scale then rotate = first rotate then scale
first sum then rotate = first rotate then sum

You can verify it by simple geometry.

Hence rotational map is a linear transformation.

4. See any matrix $A \in \mathbb{R}^{m\times n}$

for each $x \in \mathbb{R}^n$, $Ax \in \mathbb{R}^m$

Hence A is a function from $\mathbb{R}^n$ to $\mathbb{R}^m$

Is it a linear transformation?

Answer is yes. $\boxed{\because A(\alpha x + y) = \alpha Ax + Ay}$

$\boxed{\text{Great Result}}$ Let V, W be two finite dim. VS over F.

Let $L : V \to W$ be a linear transformation.

Let $B_1 = \{v_1, v_2, \dots, v_n\}$

and $B_2 = \{w_1, w_2, \dots, w_m\}$

be bases of V and W resp. (i.e. $\dim V = n$, $\dim W = m$)

Then we can find a matrix A of order $m \times n$ such that

$$L(x) = Ax \quad \forall x \in V.$$

- Thus, any LT can be expressed by a matrix
- Remember! (i) $L : V \to W$ $\left.\begin{matrix}\dim(V) = n \\ \dim(W) = m\end{matrix}\right) \Leftrightarrow$ A is $m \times n$ matrix

  (ii) Representation of A depends on choice of B_1 and B_2.

Topic to learn

Matrix representation of T

Let $T: U \to W$ be a LT.

Let $B_1 = \{u_1, u_2, \cdots, u_n\}$ be a basis of U

$B_2 = \{w_1, w_2, \cdots, w_m\}$ be a basis of W.

Step 1 Find image of each element of Basis B_1 under T, i.e. Find $T(u_i)$ for each $i = 1, 2, 3, \cdots n$

Step 2 Find co-ordinate vector a_i for each $T(u_i)$ in terms of basis B_2.

Note that each $a_i \in \mathbb{R}^m$ for $i = 1, 2, \cdots n$,

because
$$T(u_i) = \sum_{j=1}^{m} (a_i)_j \, w_j \qquad i = 1, 2, \cdots n$$

Step 3 Matrix
$$T_{B_1, B_2} = \left[\begin{array}{c|c|c|c} a_1 & a_2 & \cdots & a_n \\ \downarrow & \downarrow & & \downarrow \end{array} \right]$$
$$= [a_{ij}]_{\substack{i=1,2,\cdots n \\ j=1,2,\cdots m}}$$
$$= A \text{ (say)}$$

Note (i) A is $m \times n$ matrix

(ii) If x is coordinate vector of any point $u \in U$ wrt basis B_1 then Ax is the co-ordinate vector of Tu wrt basis B_2 in space W

Hence matrix representation A of a linear transformation $T: U \to W$ depends on the choice of bases B_1 of U and B_2 of W. Therefore we write

$$T_{B_1, B_2} \equiv A$$

Remember

If $\quad T_{B_1, B_2}(u) = w \quad \forall\, u \in U$

then $\quad Ax = y \quad$ where x is coordinate vector of u in B_1.

and y is coordinate vector of w in B_2.

$$\boxed{(T\,u)\text{'s coordinate wrt } B_2 = A(u\text{'s coordinate wrt } B_1) \quad \forall\, u \in U}$$

Topic to learn: Relation between coordinate of a point wrt different bases

Let $x \in V$ (V is a vector space)

Let $[x]_{B_1}$ represents coordinate of x wrt basis B_1 of V.

Let $[x]_{B_2}$ ———————— B_2 of V.

Then

$$\boxed{[x]_{B_1} = \left[(id)_{B_2,B_1}\right][x]_{B_2}} \quad \text{OR} \quad \boxed{[x]_{B_2} = \left[(id)_{B_1,B_2}\right][x]_{B_1}}$$

$$\because \left[(id)_{B_2,B_1}\right]^{-1} = \left[(id)_{B_1,B_2}\right]$$

$$\boxed{[x]_{B_1} = \left[(id)_{B_1,B_2}\right]^{-1}[x]_{B_2}}$$

Proof:-

Let $[x]_{B_1} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$

$[x]_{B_2} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$

$B_1 = \{e_1, e_2, \dots, e_n\}$

$B_2 = \{\bar{e}_1, \bar{e}_2, \dots, \bar{e}_n\}$

Then

$$[x]_{B_1} = \sum_{i=1}^{n} x_i e_i \quad \text{— (1)}$$

$$[x]_{B_2} = \sum_{j=1}^{n} y_j \bar{e}_j \quad \text{— (2)}$$

Let $\bar{e}_j = \sum_{i=1}^{n} p_{ij} e_i \quad \forall j = 1, 2, \dots, n$ — (3)

Then

$$[x]_{B_2} = \sum_{j=1}^{n} y_j \sum_{i=1}^{n} p_{ij} e_i$$

$$= \sum_{j=1}^{n} \sum_{i=1}^{n} (p_{ij} y_j) e_i \quad \text{— (4)}$$

from (1) & (4), we get $x_i = \sum_{i=1}^{n} p_{ij} y_j$

$$[x]_{B_1} = \begin{bmatrix} x_1 \\ \vdots \\ x_2 \end{bmatrix} = P \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} \text{ where}$$

$$= P[x]_{B_2}$$

$P = [p_{ij}]_{i,j=1}^{n}$

$\Downarrow$

a matrix whose $(i,j)^{th}$ element is p_{ij}

From (3), it is clear that each col. of matrix P is the co-ordinate of $\bar{e}_j$ wrt basis $B = \{e_1, e_2, \dots, e_n\}$. Therefore

$$P = (id)_{B_2, B_1}$$

Question Find coordinate of vector $\begin{bmatrix}1\\5\end{bmatrix}$ wrt basis $B_1 = \left\{\begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}\right\}$ and then convert the coordinate vector wrt. basis $B_2 = \left\{\begin{bmatrix}-1\\4\end{bmatrix} \begin{bmatrix}7\\6\end{bmatrix}\right\}$

Solution Let $[x]_{B_1}$ is the coordinate vector of $\begin{bmatrix}1\\5\end{bmatrix}$ wrt B_1

Then

$$A[x]_{B_1} = \begin{bmatrix}1\\5\end{bmatrix} \Rightarrow [x]_{B_1} = \begin{bmatrix}1\\4\end{bmatrix}$$

$$\left[\begin{array}{cc|c}1 & 0 & 1\\1 & 1 & 5\end{array}\right] \sim \left[\begin{array}{cc|c}1 & 0 & 1\\0 & 1 & 4\end{array}\right]$$

Find $(id)_{B_1, B_2}$

$$id\begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix}1\\1\end{bmatrix} = \frac{1}{34}\begin{bmatrix}-1\\4\end{bmatrix} + \frac{5}{34}\begin{bmatrix}7\\6\end{bmatrix}$$

$$id\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}0\\1\end{bmatrix} = \frac{7}{34}\begin{bmatrix}-1\\4\end{bmatrix} + \frac{1}{34}\begin{bmatrix}7\\6\end{bmatrix}$$

$$\left[\begin{array}{cc|c|c}-1 & 7 & 1 & 0\\4 & 6 & 1 & 1\end{array}\right]$$

$$\sim \cancel{\left[\begin{array}{cc|c|c}-1 & 7 & 1 & 0\\0 & 22 & 3 & 1\end{array}\right]} \left[\begin{array}{cc|cc}-1 & 7 & 1 & 0\\0 & 34 & 5 & 1\end{array}\right]$$

$$\sim \cancel{\left[\begin{array}{cc|c|c}1 & -7 & -1 & 0\\0 & 1 & 3/22 & 1/22\end{array}\right]} \left[\begin{array}{cc|cc}1 & -7 & -1 & 0\\0 & 1 & \frac{5}{34} & \frac{1}{34}\end{array}\right]$$

$$-1 + \frac{35}{34} = \frac{1}{34}$$

Thus

$$(id)_{B_1, B_2} = \begin{bmatrix}\frac{1}{34} & \frac{7}{34}\\ \frac{5}{34} & \frac{1}{34}\end{bmatrix}$$

Therefore,

$$[x]_{B_2} = \begin{bmatrix}1/34 & 7/34\\5/34 & 1/34\end{bmatrix}\begin{bmatrix}1\\4\end{bmatrix} = \begin{bmatrix}29/34\\9/34\end{bmatrix} \checkmark$$

Verify this by finding coordinate vector of point $\begin{bmatrix}1\\5\end{bmatrix}$ directly wrt basis B_2.

Topic to learn: Relation b/t different matrices wrt different bases.

Let $T : U \to W$ be a LT.

Let $T_{B_1, B_2} = A$ and $T_{B_3, B_4} = B$

Then

$$\boxed{(id)_{B_2, B_4} A = B (id)_{B_1, B_3}}$$

where $id : W \to W$ is identity LT and $(id)_{B_2, B_4}$ is its matrix.

and $(id)_{B_1, B_3}$ is the matrix of identity linear transformation from U into U wrt Bases B_1 and B_3 respectively.

$\boxed{\text{Important case}}$

Linear transformation / operator

$\boxed{T : U \to U \qquad \dim U = n}$

Let $T_{B_1, B_1} \equiv A$

and $T_{B_2, B_2} \equiv B$

Then

$$(id)_{B_1, B_2} A = B \, (id)_{B_1, B_2}$$

$$\Rightarrow \quad A = \left[(id)_{B_1, B_2}\right]^{-1} B \left[(id)_{B_1, B_2}\right]$$

OR

$$A = \left[(id)_{B_2, B_1}\right] B \left[(id)_{B_2, B_1}\right]^{-1}$$

<u>Re-write</u>

$$T_{B_1, B_1} = \left[(id)_{B_2, B_1}\right] T_{B_2, B_2} \left[(id)_{B_2, B_1}\right]^{-1}$$

OR

$$T_{B_2, B_2} = \left[(id)_{B_1, B_2}\right] T_{B_1, B_1} \left[(id)_{B_2, B_1}\right]$$

$$= \left[(id)_{B_2, B_1}\right]^{-1} T_{B_1, B_1} \left[(id)_{B_2, B_1}\right]$$

Ques
Let a LT, $T : \mathbb{R}^2 \to \mathbb{R}^3$ be defined as $T\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_1 + x_2 \\ x_2 \end{bmatrix}$

Find matrix representation of $\left.\begin{matrix} T_{B_1, B_2} \\ T_{B_3, B_4} \end{matrix}\right)$ Also see relation between these matrices.

where $B_1 =$ standard basis of $\mathbb{R}^2$,
$B_2 =$ standard —— $\mathbb{R}^3$,
$B_3 = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$ $\quad B_4 = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix} \right\}$

Solution Let $T_{B_1, B_2} = A$. We obtain

$$T\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = 1\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 1\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + 0\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$T\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = 0\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 1\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + 1\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Thus $A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix}$

Let $T_{B_3, B_4} = B$. We obtain

$$T\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \frac{2}{3}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 0\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} + \frac{1}{3}\begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c|c} \boxed{1} & 1 & 1 & 1 & 1 \\ 0 & \boxed{1} & 3 & 1 & 2 \\ 0 & 2 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|cc} \boxed{1} & 1 & 1 & 1 & 1 \\ 0 & \boxed{1} & 3 & 1 & 2 \\ 0 & 0 & \boxed{-6} & -2 & -3 \end{array}\right] \sim \left[\begin{array}{ccc|c|c} \boxed{1} & 1 & 1 & 1 & 1 \\ 0 & \boxed{1} & 3 & 1 & 2 \\ 0 & 0 & \boxed{1} & 1/3 & 1/2 \end{array}\right] \quad \begin{bmatrix} 1 - 0 - 1/3 = 2/3 \\ 1 - 3 \cdot \frac{1}{3} = 0 \\ 1/3 \end{bmatrix} \quad \begin{bmatrix} 1 - 1 \cdot \frac{1}{2} - 1 \cdot \frac{1}{2} = 0 \\ 2 - 3 \cdot \frac{1}{2} = 1/2 \\ 1/2 \end{bmatrix}$$

$$T\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = 0\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}$$

Hence $\quad B = \begin{bmatrix} 2/3 & 0 \\ 0 & 1/2 \\ 1/3 & 1/2 \end{bmatrix}$

We know relation is $\quad (id)_{B_2, B_4} A = B (id)_{B_1, B_3}$

Verify this relation by finding

$(id)_{B_2, B_4} = \begin{bmatrix} 1 & -1/3 & -1/3 \\ 0 & 0 & 1/2 \\ 0 & 1/3 & -1/6 \end{bmatrix}$ and $(id)_{B_1, B_3} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$

To obtain $(id)_{B_2, B_4}$, we do the following calculation

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 3 & 0 & 1 & 0 \\ 0 & 2 & 0 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 3 & 0 & 1 & 0 \\ 0 & 0 & -6 & 0 & -2 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 3 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & +\frac{1}{3} & -\frac{1}{6} \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 0 & -1/3 & 1/6 \\ 0 & 1 & 0 & 0 & 0 & 1/2 \\ 0 & 0 & 1 & 0 & \frac{1}{3} & -1/6 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -1/3 & -1/3 \\ 0 & 1 & 0 & 0 & 0 & 1/2 \\ 0 & 0 & 1 & 0 & 1/3 & -1/6 \end{array}\right]$$

<u>To obtain</u> $(id)_{B_1, B_3}$

$$\left[\begin{array}{cc|cc} \boxed{1} & 1 & 1 & 0 \\ 0 & \boxed{1} & 0 & 1 \end{array}\right] \sim \left[\begin{array}{cc|cc} 1 & 0 & 1 & -1 \\ 0 & 1 & 0 & 1 \end{array}\right]$$

<u>Note</u>: If B_1, B_2 are standard bases it is always each to check $\boxed{A\,(id)_{B_3, B_1} = (id)_{B_4, B_2}\, B}$ $\because \left[(id)_{B_1, B_3}\right]^{-1} = \left[(id)_{B_3, B_1}\right]$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 0 & 2 & 0 \end{bmatrix} \begin{bmatrix} 2/3 & 0 \\ 0 & 1/2 \\ 1/3 & 1/2 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 0 & 2 & 0 \end{bmatrix} \begin{bmatrix} 2/3 & -2/3 \\ 0 & 1/2 \\ 1/3 & \frac{1}{2}-\frac{1}{3}=\frac{1}{6} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix}$$

<u>Recall</u> $T_{B_1, B_2} u = w \equiv Ax = y$

Here for any $u = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in U$; $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ $\because B_1$ is standard basis.

$w = y$ meet perfectly due to the fact that B_2 is also standard basis.

But see in case

$T_{B_3, B_4}(u) = w$, we have $Bx = y$

Let $u = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ then $x = \begin{bmatrix} x_1 - x_2 \\ x_2 \end{bmatrix}$ $\qquad \left[\begin{array}{cc|c} 1 & 1 & x_1 \\ 0 & 1 & x_2 \end{array}\right]$

Check $\quad Bx = \begin{bmatrix} 2/3 & 0 \\ 0 & 1/2 \\ 1/3 & 1/2 \end{bmatrix} \begin{pmatrix} x_1 - x_2 \\ x_2 \end{pmatrix} = \begin{bmatrix} 2/3\, x_1 - \frac{2}{3} x_2 \\ 1/2\, x_2 \\ 1/3\, x_1 + 1/6\, x_2 \end{bmatrix} = y$

$\|$

Co-ordinates of $\begin{bmatrix} x_1 \\ x_1 + x_2 \\ x_2 \end{bmatrix}$ wrt basis B_4

Check

$\left[\begin{array}{ccc|c} \boxed{1} & 1 & 1 & x_1 \\ 0 & \boxed{1} & 3 & x_1 + x_2 \\ 0 & 2 & 0 & x_2 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & x_1 \\ 0 & 1 & 3 & x_1 + x_2 \\ 0 & 0 & -6 & x_2 - 2x_1 - 2x_2 = -2x_1 - x_2 \end{array}\right]$

3^{rd} coordinate is $\frac{1}{3} x_1 + \frac{1}{6} x_2$ ✓

2^{nd} ——— is $x_1 + x_2 - 3\left(\frac{1}{3} x_1 + \frac{1}{6} x_2\right)$

$= x_1 + x_2 - x_1 - \frac{1}{2} x_2 = \frac{1}{2} x_2$ ✓

1^{st} ——— $x_1 - \frac{1}{2} x_2 - \frac{1}{3} x_1 - \frac{1}{6} x_2$

$= \frac{2}{3} x_1 - \frac{2}{3} x_2$ ✓

Ques
Let $T: \mathbb{R}^2 \to \mathbb{R}^2$ be a LT defined as $T\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 \\ 2x_2 \end{bmatrix}$. Then find (17)

(i) T_{B_1, B_2} where B_1 and B_2 are standard basis

(ii) T_{B_3, B_4} where $B_3 = B_4 = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}$.

Solution

(i)
$$\left.\begin{aligned} T\begin{pmatrix} 1 \\ 0 \end{pmatrix} &= \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 1\begin{bmatrix} 1 \\ 0 \end{bmatrix} + 0\begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ T\begin{pmatrix} 0 \\ 1 \end{pmatrix} &= \begin{bmatrix} 1 \\ 2 \end{bmatrix} = 1\begin{bmatrix} 1 \\ 0 \end{bmatrix} + 2\begin{bmatrix} 0 \\ 1 \end{bmatrix} \end{aligned}\right\} \Rightarrow T_{B_1, B_2} = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} (= A \text{ say})$$

(ii)
$$\left.\begin{aligned} T\begin{bmatrix} 1 \\ 0 \end{bmatrix} &= \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 1\begin{bmatrix} 1 \\ 0 \end{bmatrix} + 0\begin{pmatrix} 1 \\ 1 \end{pmatrix} \\ T\begin{bmatrix} 1 \\ 1 \end{bmatrix} &= \begin{bmatrix} 2 \\ 2 \end{bmatrix} = 0\begin{bmatrix} 1 \\ 0 \end{bmatrix} + 2\begin{bmatrix} 1 \\ 1 \end{bmatrix} \end{aligned}\right\} \Rightarrow T_{B_3, B_4} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} (= B \text{ say})$$

Check relation $\quad (id)_{B_2, B_4} A = B \, (id)_{B_1, B_3}$

$$\Rightarrow \quad A \, (id)_{B_3, B_1} = (id)_{B_4, B_2} B$$

See $\quad (id)_{B_3, B_1} = (id)_{B_4, B_2} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$.

Therefor we obtain

$$\boxed{\begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}}$$

Let U and V be two vector spaces over $\mathbb{R}$.
Let $L(U, V)$ be set all all possible linear transformations $T : U \to V$, Let $\dim U = n$ & $\dim V = m$.
We already know that $\mathbb{R}^{m \times n}$ is the set of all matrices of order $m \times n$.

By our great result: we know for each element of $L(U,V)$ we have one element in $\mathbb{R}^{m \times n}$ and visa versa.

Let T_{B_1, B_2} is represented by matrix A Then

These words are valid for T also

① $\underbrace{\text{Kernel/Null space of } T}$ = $\underbrace{\text{Null space of } A}$
Basis are same. (Remember → nullity ←)

② $\underbrace{\text{range of } T}$ = $\underbrace{\text{Column space of } A}$
basis are same. (Remember → rank ←)

③ T is 1-1 $\Leftrightarrow$ $N(A) = \{0\}$ only $\Leftrightarrow$ nullity of $A = 0$

④ T is onto $\Leftrightarrow$ rank $(A) = m = \#$of rows in A

⑤ T is bijective $\Leftrightarrow$ A^{-1} exists [here $m = n$]
(here $U \equiv V$) rank $(A) = m = n$

⑥ rank-nullity Theorem is valid for T and A both.
(e.g. If $n > m$ then T is not 1-1)

$$\boxed{\text{rank} + \text{nullity} = \dim . U}$$

→ Always Remember

$\mathbb{R}^{m \times n}$ is a VS. See $L(U,V)$ is also a Vector space over $\mathbb{R}$ [operations are $(T_1 + T_2)(x) = T_1(x) + T_2(x)$
$(\alpha T_1)(x) = \alpha\, T_1(x)$

Ques Find matrix representation of T (If T is LT) wrt given basis of domain & co domain. Take standard basis If particular choice of bases are not given. Also find range and null space, whenever applicable.

(i) $T: P_3 \to \mathbb{R}^3$ defined as $T(a_0 + a_1 x + a_2 x^2 + a_3 x^3) = \begin{bmatrix} a_0 + a_1 + 2a_3 \\ 2a_1 + a_2 \\ a_3 + a_1 \end{bmatrix}$

Sol: $T(1) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

$T(x) = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$

$T(x^2) = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

$T(x^3) = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$

matrix is $\begin{bmatrix} 1 & 1 & 0 & 2 \\ 0 & 2 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} = A$ (say)

Find N(A) and C(A).

$$\begin{bmatrix} \boxed{1} & 1 & 0 & 2 \\ 0 & \boxed{2} & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} \boxed{1} & 1 & 0 & 2 \\ 0 & \boxed{2} & 1 & 0 \\ 0 & 0 & \boxed{-1/2} & 1 \end{bmatrix} \sim \begin{bmatrix} \boxed{1} & 1 & 0 & 2 \\ 0 & \boxed{1} & 1/2 & 0 \\ 0 & 0 & \boxed{1} & -2 \end{bmatrix}$$

$$\sim \begin{bmatrix} \boxed{1} & 1 & 0 & 2 \\ 0 & \boxed{1} & 0 & 1 \\ 0 & 0 & \boxed{1} & -2 \end{bmatrix} \sim \begin{bmatrix} \boxed{1} & 0 & 0 & 1 \\ 0 & \boxed{1} & 0 & 1 \\ 0 & 0 & \boxed{1} & -2 \end{bmatrix}$$

Here # LI columns are 3 = rank(A) = 3 = # rows

and basis of null space is $\left\{ \begin{bmatrix} -1 \\ -1 \\ 2 \\ 1 \end{bmatrix} \right\}$

Hence the given transformation is not (1-1)

but onto.

Topic to learn
From matrix to LT

$\boxed{\text{Remember}}$ $[T_{B_1,B_2}]$ is given. To find T.

(Tu)'s coordinate in basis B_2

$= T_{B_1,B_2}$ (u's coordinate in basis B_1)

Hence to find the formula for Tu

i) Find u's coordinate in B_1 i.e. find $[u]_{B_1}$

ii) $Tu = [T_{B_1,B_2}][u]_{B_1}$

But Remember Tu is $[Tu]_{B_2}$

Question Find a LT $T: \mathbb{R}^2 \to \mathbb{R}^2$ whose matrix representation is $\begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$ (i) with respect to standard basis $B_1 = B_2$.

(ii) with respect to $B_3 = B_4 = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$

Solution (i) given $T_{B_1,B_2} = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$

Let $u = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ be arb. vector in $\mathbb{R}^2$.

$[u]_{B_1} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ Hence, $Tu = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 \\ 2x_2 \end{bmatrix}$

Hence formula for T is $T: \mathbb{R}^2 \to \mathbb{R}^2$ defined as $T\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 \\ 2x_2 \end{bmatrix}$.

(ii) given $T_{B_3,B_4} = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$ Let $u = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ Then

$[u]_{B_3} = \begin{bmatrix} x_1 - x_2 \\ x_2 \end{bmatrix}$ Hence formula for $T: \mathbb{R}^2 \to \mathbb{R}^2$ is

$Tu = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} x_1 - x_2 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ 2x_2 \end{bmatrix}$ This vector is same as $\begin{bmatrix} x_1 + x_2 \\ 2x_2 \end{bmatrix}$ in B_4.