ODE

Differential Equation ÷ (D· Eq'n)

An equation involving derivatives of one or more dependent variables w r t one or more independent variables is called a D.Eq'n.

e.g.

$$\left.\begin{aligned} \frac{dy}{dx} &= ay \\ \frac{dy}{dx} &= k(a-y) \\ \frac{d^2y}{dt^2} &= \frac{F}{m} \\ \frac{d^4y}{dx^2} + b\frac{d^2y}{dx^2} + 5y &= \sin x \end{aligned}\right\} \begin{array}{l} \underline{\text{ODE}} \\ \text{Ordinary} \\ \text{Differential} \\ \text{Equations} \end{array}$$

$$\left.\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}\right\} \text{Partial Differential Equation.}$$

ODE ÷

$$f(x, y, y', y'', --) = 0$$

Order of a DE ÷ The order of the highest ordered derivative involved in a D. Eq'n is called the order of the D. Eq'n.

Degree of a DE ÷

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Firs

Why we read this subject:

D Eq'ns form a language in which the basic rules of science are expressed.

$$\begin{tikzcd}[column sep=large]
\text{D. Eq'n} \arrow[rr, "\text{solving}"] & & \text{Behavior of model} \arrow[dl, "\text{interpretation}"] \\
& \text{Physical } \cancel{\text{model}} \text{ problem} \arrow[ul, "\text{modeling}"] &
\end{tikzcd}$$

→ **First order ODE**

Solution:

(i) Geometrical methods (Quantitative methods).
(ii) Analytical methods (Exact OR Symbolic methods)
(iii) Numerical methods

First order ODE

Standard form of a I_{st} order ODE

$$\frac{dy}{dx} \left(\equiv y'\right) = f(x,y) \rightarrow \text{derivative form}$$

$$M(x,y)\,dx + N(x,y)\,dy = 0 \longrightarrow \text{differential form}$$

The equation in one of these forms may be written in another form

e.g.

$$\frac{dy}{dx} = \frac{x+y}{x-y} \quad \text{— (1)}$$

OR

$$(x+y)\,dx + (y-x)\,dy = 0 \quad \text{— (2)}$$

In ~~dif~~ derivative form (1) y is dependent var. and x is independent variable.

In differential form, we may treat either x as dependent or y as dependent variable or in ~~other~~ reverse order also.

But In expression $M(x,y)\,dx + N(x,y)\,dy = 0$,

In this course we will treat y as dependent and x as independent variables, unless otherwise specifically is stated.

First order Exact D Eq'n

Method 1.

Exact Ist order ODEs.

Any Ist order D Eq'n

$$M(x,y)\,dx + N(x,y)\,dy = 0 \quad \text{— (i)}$$

is said to be exact if $\exists$ $f(x,y)$ s.t.

write first it

$$df \overset{\text{Def.}}{=} \left(\frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy\right) = M(x,y)\,dx + N(x,y)\,dy \quad \text{— (ii)}$$

and then it

In this case solution ~~is~~ of (i) is

$$f(x,y) = c \quad \text{— (iii)}$$

[Integration of ii]

Discussion:

from definition (i) is exact then $\exists$ $f(x,y)$ s.t.

$$\frac{\partial f}{\partial x} = M \quad \& \quad \frac{\partial f}{\partial y} = N \quad \text{— (a)}$$

$$\Rightarrow \frac{\partial^2 f}{\partial y\,\partial x} = \frac{\partial M}{\partial y} \quad \& \quad \frac{\partial^2 f}{\partial x\,\partial y} = \frac{\partial N}{\partial x} \quad \text{— (b)}$$

If $\frac{\partial^2 f}{\partial y\,\partial x}$ & $\frac{\partial^2 f}{\partial x\,\partial y}$ are continuous then

$$\frac{\partial^2 f}{\partial y\,\partial x} = \frac{\partial^2 f}{\partial x\,\partial y}$$

i.e. $$\boxed{\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}}$$

So if M and N have continuous partial derivatives then (i) is called exact if

$$\boxed{\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}} \Longrightarrow$$ This is the required condition to check the exactness of (i).

So equation
$M\,dx + N\,dy = 0$ is exact
$\Updownarrow$
$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Procedure to find f :-

If D Eq'n

$M(x,y)\,dx + N(x,y)\,dy = 0$ — (i) is exact, then

by definition $\exists$ f(x, y) s.t.

$$\frac{\partial f}{\partial x} = M \text{ — (ii(a))} \quad \& \quad \frac{\partial f}{\partial y} = N \text{ — (ii(b))}$$

Step 3

from (ii(a)) :-

$$f = \int M(x,y)\,\partial x + \phi(y) \quad \text{— (iii)}$$

$$\Rightarrow \frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\left[\int M(x,y)\,\partial x + \phi(y)\right]$$

$$= \frac{\partial}{\partial y}\int M(x,y)\,\partial x + \frac{d\phi}{dy}$$

Now from (ii) (b)

$$N(x,y) = \frac{\partial}{\partial y}\int M(x,y)\,\partial x + \frac{d\phi}{dy} \quad \text{— (iv)}$$

find ϕ from (IV) and put that in (III).

D

Step 1 :- write given Eq'n in differential form ① i.e. $M\,dx + N\,dy = 0$

Step 2 :- check ; ~~whether~~ ① is exact OR not

by $\boxed{\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}}$

If exact → go on Step 3.

No exact → ??X **stop** you can't solve the given D Eq'n by this method.

$$\frac{dy}{dx} = -\frac{\text{[illegible]}}{2x^2+2y}$$

Example :-

Solve :-

$$(3x^2+4xy)\,dx + (2x^2+2y)\,dy = 0 \quad \text{— (i)}$$

$$M = 3x^2+4xy$$
$$N = 2x^2+2y$$
$$\frac{\partial M}{\partial y} = 4x \; ; \; \frac{\partial N}{\partial x} = 4x$$

Hence, eq'n is exact.

So $\exists\, f(x,y)$ s.t.

$$\frac{\partial f}{\partial x} = M = 3x^2+4xy \quad \text{— (ii)}$$

and $$\frac{\partial f}{\partial y} = N = 2x^2+2y \quad \text{— (iii)}$$

from (ii)

$$f = x^3 + 2x^2y + \phi(y) \quad \text{— (iv)}$$

$$\Rightarrow \frac{\partial f}{\partial y} = 2x^2 + \frac{d\phi}{dy} \quad \text{— (v)}$$

from (iii) & (v)

$$2x^2 + \frac{d\phi}{dy} = 2x^2+2y$$

$$\Rightarrow \frac{d\phi}{dy} = 2y \Rightarrow \phi = y^2 \quad \text{— (vi)}$$

from (vi) and (iv) :-

$$f = x^3 + 2x^2y + y^2$$

Hence solution of (1) is

$$\boxed{x^3+2x^2y+y^2 = c}$$

If Eq'n is exact, you can solve that using some basics

i.e. same Eq'n can be written as

$$(3x^2)\,dx + (4xy\,dx + 2x^2\,dy) + (2y)\,dy = 0$$

$$\Rightarrow d(x^3) + d(2x^2y) + d(y^2) = 0$$

$$\Rightarrow \boxed{x^3+2x^2y+y^2 = c}$$

(IVP)

Solve (1) with $y(0) = 1$

$\Rightarrow c = 1$

Thus, solution is

$$\boxed{x^3+2x^2y+y^2 = 1}$$

Method 2
separable
method

C

T

If an Eq'n is in the form

$$f(x)\,dx + g(y)\,dy = 0 \quad — (1)$$

it is called separable and its
then, Sol'n is

$$\int f(x)\,dx + \int g(y)\,dy = c \quad - (2)$$

(1) is exact and (2) is soln obtained as the previous procedure

General type of separable Equation

Further eq'n is also separable If that can be written in form

$$f(x)\,F(y)\,dx + G(x)\,g(y)\,dy = 0 \quad — (3)$$

$$(3) \times \frac{1}{F(y)\,G(x)} \Longrightarrow \frac{f(x)}{G(x)}\,dx + \frac{g(y)}{F(y)}\,dy = 0 \quad - (4)$$

so solution is

$$\int \frac{f(x)}{G(x)}\,dx + \int \frac{g(y)}{F(y)}\,dy = 0 \quad - (5)$$

Note :-
sol'n (5) may have one or some more or less sol'ns

Definition
Integrating Factor:

A function is called IF If that convert a non-exact Eq'n into an exact Eq'n after multiplication

Non-exact Eq. × $\boxed{\text{a func}^n\ u}$ → exact equation

In this process, resulting exact eq'n may have lost or gained some solutions.

→ Given non-exact ~~solution~~ D. Eq'n and resulting exact D. Eq'n are called essentially equivalent

→ means solutions are (considerable) same.

Example :

$$(x-4)y^4 dx - x^3(y^2-3)\,dy = 0 \quad \text{— ①}$$

$$\Rightarrow \frac{x-4}{x^3}dx - \frac{y^2-3}{y^4}dy = 0$$

$$\Rightarrow \left(\frac{1}{x^2} - \frac{4}{x^3}\right)dx - \left(\frac{1}{y^2} - \frac{3}{y^4}\right)dy = 0$$

$$\Rightarrow \left(-\frac{1}{x} + \frac{4}{2x^2}\right) - \left(-\frac{1}{y} + \frac{3}{3y^3}\right) = c$$

$$\Rightarrow -\frac{1}{x} + \frac{2}{x^2} + \frac{1}{y} - \frac{1}{y^3} = c \quad \text{— ②}$$

$\frac{dy}{dx} = \alpha y$ — See its solu^n by computer.

$$\frac{dy}{y} = \alpha\, dx$$

$$\ln y = \alpha x + c$$

$$\Rightarrow y = e^{(\alpha x + c)}$$

$$= e^{c} \cdot e^{\alpha x}$$

$$\boxed{y = c\,e^{\alpha x}}$$

$y=0$ is not included in family ②

but it is a solution of ① because it satisfies

$$\frac{dy}{dx} = \frac{(x-4)y^4}{x^3(y^2-3)}$$

obvious, while linear equations always have integrating factors of a certain special form. We shall return to the question raised above in Section 2.4. Our object here has been merely to introduce the concept of an integrating factor.

Exercises from SL Ross

In Exercises 1–10 determine whether or not each of the given equations is exact; solve those that are exact.

1. $(3x + 2y)\,dx + (2x + y)\,dy = 0.$ $\quad \frac{3x^2}{2} + 2yx + \frac{y^2}{2} = c$
2. $(y^2 + 3)\,dx + (2xy - 4)\,dy = 0.$
3. $(2xy + 1)\,dx + (x^2 + 4y)\,dy = 0.$ $\quad x^2y + x + 2y^2 = c$
4. $(3x^2y + 2)\,dx - (x^3 + y)\,dy = 0.$
5. $(6xy + 2y^2 - 5)\,dx + (3x^2 + 4xy - 6)\,dy = 0.$ $\rightarrow 3x^2y + 2xy^2 - 5x - 6y = c$
6. $(\theta^2 + 1)\cos r\,dr + 2\theta \sin r\,d\theta = 0.$
7. $(y \sec^2 x + \sec x \tan x)\,dx + (\tan x + 2y)\,dy = 0.$ $\rightarrow y\tan x + \sec x + y^2 = c$
8. $\left(\dfrac{x}{y^2} + x\right) dx + \left(\dfrac{x^2}{y^3} + y\right) dy = 0.$
9. $\left(\dfrac{2s - 1}{t}\right) ds + \left(\dfrac{s - s^2}{t^2}\right) dt = 0.$ $\quad \frac{s^2 - s}{t} = c$
10. $\dfrac{2y^{3/2} + 1}{x^{1/2}}\,dx + (3x^{1/2}y^{1/2} - 1)\,dy = 0.$

Solve the initial-value problems in Exercises 11–16.

11. $(2xy - 3)\,dx + (x^2 + 4y)\,dy = 0, \quad y(1) = 2.$
12. $(3x^2y^2 - y^3 + 2x)\,dx + (2x^3y - 3xy^2 + 1)\,dy = 0, \quad y(-2) = 1.$
13. $(2y \sin x \cos x + y^2 \sin x)\,dx + (\sin^2 x - 2y \cos x)\,dy = 0, \quad y(0) = 3.$
14. $(ye^x + 2e^x + y^2)\,dx + (e^x + 2xy)\,dy = 0, \quad y(0) = 6.$
15. $\left(\dfrac{3 - y}{x^2}\right) dx + \left(\dfrac{y^2 - 2x}{xy^2}\right) dy = 0, \quad y(-1) = 2.$
16. $\dfrac{1 + 8xy^{2/3}}{x^{2/3}y^{1/3}}\,dx + \dfrac{2x^{4/3}y^{2/3} - x^{1/3}}{y^{4/3}}\,dy = 0, \quad y(1) = 8.$
17. In each of the following equations determine the constant A such that the equation is exact, and solve the resulting exact equation:

    (a) $(x^2 + 3xy)\,dx + (Ax^2 + 4y)\,dy = 0.$

    (b) $\left(\dfrac{1}{x^2} + \dfrac{1}{y^2}\right) dx + \left(\dfrac{Ax + 1}{y^3}\right) dy = 0.$ $\quad M_y = N_x$

    $\Rightarrow -\frac{2}{y^3} = \frac{A}{y^3} \Rightarrow A = -2$