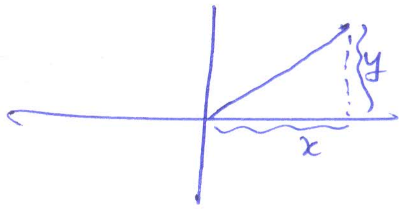
So, the argument of a complex no $z\ (\neq 0)$ $= x+iy$
is the set $\{ \theta_0 + 2n\pi : 0 \leq \theta_0 \leq 2\pi,\ n \in \mathbb{Z}$ &
$x = r\cos\theta,\ y = r\sin\theta \}$
$\parallel$
$\arg(z)$

: Many a times, we pick one value from $\arg(z)$,
say $0 \leq \theta_0 < 2\pi$: Principal argument of
$z\ (\neq 0)$.

$|z|$: <u>Modulus of z</u> : ($r = \sqrt{x^2+y^2}$: has geometric meaning

<u>Properties</u> : –

(i) $|z|$ is distance in the plane of the comp. no. z from 0.



(ii) ~~$|z_1|$~~ :

$|z_1 - z_2|$: is the distance in the plane of z_1 from z_2.

(iii) $|Re(z)| \leq |z|$ & $|Im(z)| \leq |z|$.

absolute value for a real no.

($z = x+iy$, then $|Re(z)| = |x|$
$|z| = \sqrt{x^2+y^2}$
Clearly $|x| \leq |z|$).

(iv) $z\bar{z} = |z|^2$ & $|z_1 z_2| = |z_1||z_2|$ & $|\bar{z}| = |z|$.

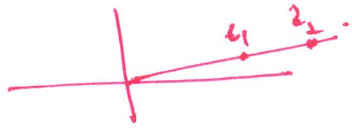
$\left(\begin{array}{l} z = x+iy \\ \bar{z} = x-iy \end{array} \right. \Rightarrow z\bar{z} = x^2 - y^2$
$= |z|^2$.)

(V) Triangle inequality

$$|z_1+z_2| \leq |z_1|+|z_2|$$

and the equality holds if and only if z_1 and z_2 lies on the same half ray through the origin in the complex plane.

underlined is same as saying that the argument of z_1 & z_2 differ by $2n\pi$.



Proof :- $|z_1+z_2|^2 = (z_1+z_2)\cdot\overline{(z_1+z_2)}$

$= (z_1+z_2)(\bar{z}_1+\bar{z}_2)$

$= z_1\bar{z}_1 + z_1\bar{z}_2 + z_2\bar{z}_1 + z_2\bar{z}_2$

$= |z_1|^2 + (z_1\bar{z}_2 + \overline{z_1\bar{z}_2}) + |z_2|^2$

Verify that $\overline{z_1\bar{z}_2} = \bar{z}_1 z_2$

$= |z_1|^2 + 2\cdot Re\underset{}{\overset{(\text{real no})}{(z_1\bar{z}_2)}} + |z_2|^2$ $\quad\left[\frac{z+\bar{z}}{2} = Re\, z\right]$

$\leq |z_1|^2 + |2\cdot Re(z_1\bar{z}_2)| + |z_2|^2$

$\leq |z_1|^2 + 2\cdot|z_1\bar{z}_2| + |z_2|^2$

$= |z_1|^2 + 2\cdot|z_1||\bar{z}_2| + |z_2|^2$

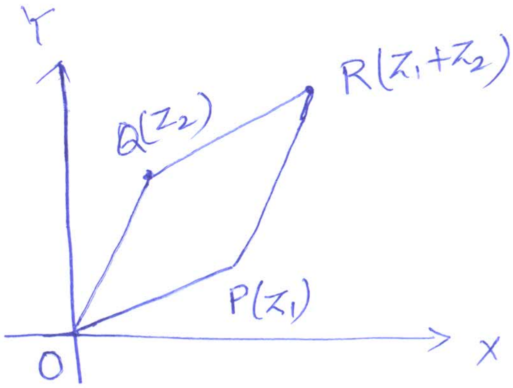
$= |z_1|^2 + 2\cdot|z_1||z_2| + |z_2|^2$ (all real no,)

$= (|z_1|+|z_2|)^2$

Take the positive sq. root both sides, then $|z_1+z_2| \leq |z_1|+|z_2|$.

The points on the Argand plane representing the sum, difference, product and division of two complex numbers:

Sum: Let the complex numbers z_1 and z_2 be represented by the points P and Q on the Argand plane. Complete the parallelogram OPRQ. Then the mid points of PQ and OR are the same. But mid-point of PQ is $\left(\dfrac{x_1+x_2}{2}, \dfrac{y_1+y_2}{2}\right)$ so that the co-ordinates of R are (x_1+x_2, y_1+y_2).

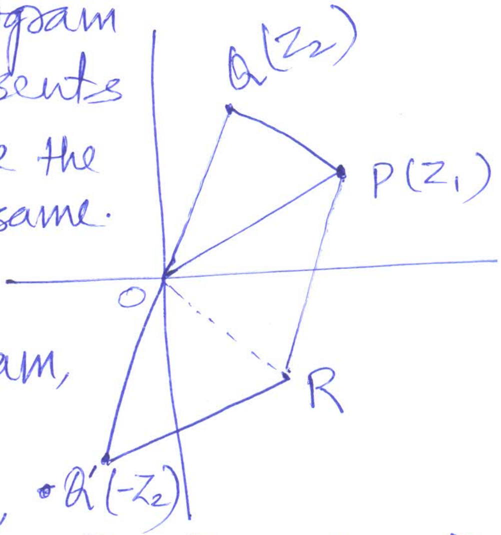


Thus the point R corresponds to the sum of the complex numbers z_1 and z_2.

In vector notation, we have

$$z_1+z_2 = \overrightarrow{OP} + \overrightarrow{OQ} = \overrightarrow{OP} + \overrightarrow{PR} = \overrightarrow{OR}. \longrightarrow (1)$$

Difference: We first represent $-z_2$ by Q' so that QQ' is bisected at O. Complete the parallelogram OPRQ'. Then the point R represents the complex number z_1-z_2, since the mid-point of PQ' and OR are the same.



s OQ is equal and parallel to RP, e see that ORPQ is a parallelogram, o that $\overrightarrow{OR} = \overrightarrow{QP}$.

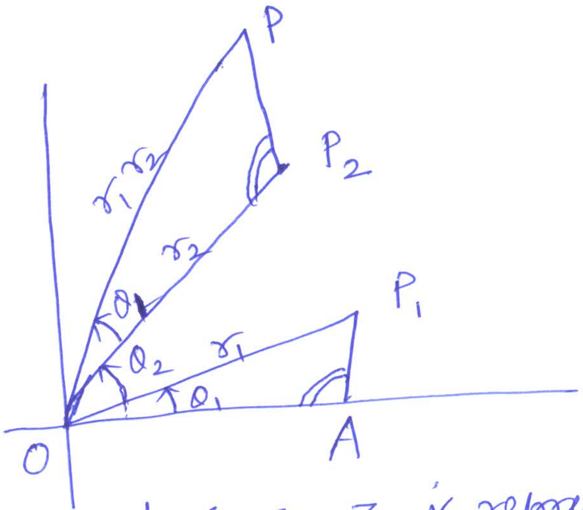
Thus we have in vectorial notation,

$$z_1-z_2 = \overrightarrow{OP} - \overrightarrow{OQ} = \overrightarrow{OP} + \overrightarrow{QO} = \overrightarrow{OP} + \overrightarrow{PR} = \overrightarrow{OR} = \overrightarrow{QP} \longrightarrow (2)$$

t follows that the complex number z_1-z_2 is represented y vector $\overrightarrow{QP}$, where the points P and Q represent e complex numbers z_1 and z_2, respectively.

Geometric representation of $z_1 z_2$:

Let P_1, P_2 represent the complex numbers
$$z_1 = x_1 + iy_1 = r_1(\cos\theta_1 + i\sin\theta_1)$$
$$z_2 = x_2 + iy_2 = r_2(\cos\theta_2 + i\sin\theta_2)$$
Measure off $OA = 1$ along OX. Construct ΔOP_2P on OP_2 directly similar to ΔOAP_1, so that
$$\frac{OP}{OP_1} = \frac{OP_2}{OA}$$



i.e. $OP = OP_1 \cdot OP_2 = r_1 r_2$

and $\angle AOP = \angle AOP_2 + \angle AOP_1$
$$= \theta_2 + \theta_1$$
$\therefore$ P represents the number
$$r_1 r_2 [\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)].$$
Hence the product of two complex numbers z_1, z_2 is represented by the point P such that

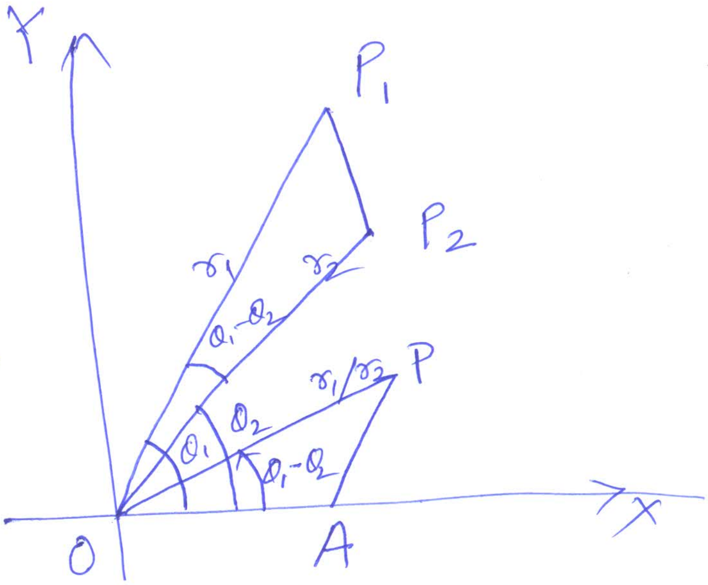
(i) $|z_1 z_2| = |z_1| \cdot |z_2|$

(ii) $\text{amp}(z_1 z_2) = \text{amp}(z_1) + \text{amp}(z_2)$.

Geometric representation of z_1/z_2:

Let P_1, P_2 represent the complex numbers $z_1 = x_1 + iy_1 = r_1(\cos\theta_1 + i\sin\theta_1)$
$z_2 = x_2 + iy_2 = r_2(\cos\theta_2 + i\sin\theta_2)$.

Measure off $OA = 1$, construct triangle OAP on OA directly ~~simi~~ similar to the triangle OP_2P_1 so that
$$\frac{OP}{OA} = \frac{OP_1}{OP_2}$$
$$\therefore OP = \frac{OP_1}{OP_2} = \frac{r_1}{r_2}$$



and $\angle XOP = \angle P_2OP_1 = \angle AOP_1 - \angle AOP_2$
$$= \theta_1 - \theta_2.$$
$\therefore$ P represents the number
$$(r_1/r_2)[\cos(\theta_1-\theta_2) + i\sin(\theta_1-\theta_2)].$$
Hence the complex number z_1/z_2 is represented by the point P such that (i) $|z_1/z_2| = \frac{|z_1|}{|z_2|}$ and (ii) $\text{amp}(z_1/z_2) = \text{amp}(z_1) - \text{amp}(z_2)$.

Remarks : 1.

$$x+iy = (x,0)+(0,1)(y,0)$$
$$= (x,0)+(0\cdot y-1\cdot 0,\ 0\cdot 0+1\cdot y)$$
$$= (x,0)+(0,y) = (x+0,\ 0+y)$$
$$= (x,y).$$

$\therefore$ $i=\sqrt{-1}$, $i^2=-1$, we have

$$(\sqrt{-1})^2 = \sqrt{-1}\cdot\sqrt{-1} = -1.$$

$$(\sqrt{-1})^2 = \sqrt{-1}\cdot\sqrt{-1} = -1.$$

Again, $(\sqrt{a}\cdot\sqrt{-1})^2 = (\sqrt{a}\cdot\sqrt{-1})(\sqrt{a}\cdot\sqrt{-1})$
$$= (\sqrt{a})^2(\sqrt{-1})^2 = -a.$$

Hence, $\sqrt{-a}$ means the product of $\sqrt{a}$ and $\sqrt{-1}$.

Therefore, $\sqrt{-a}\sqrt{-b} = \sqrt{a}\sqrt{-1}\sqrt{b}\sqrt{-1} = -\sqrt{ab}$ is correct but $\sqrt{-a}\sqrt{-b} = \sqrt{(-a)(-b)} = \sqrt{ab}$ is wrong.

2. For inequalities :

$$|a+b| \le |a|+|b|.$$

Triangle inequality.

For the same reason,

$|b|-|a| \le |a-b|$ and these inequalities can be combined to

$$|a-b| \ge \big||a|-|b|\big|.$$

A special case is $|\alpha+i\beta| \le |\alpha|+|\beta|$.

Foremost is Cauchy's inequality which states that

$$|a_1b_1 + \cdots + a_nb_n|^2 \leq (|a_1|^2 + \cdots + |a_n|^2)(|b_1|^2 + \cdots + |b_n|^2)$$

or $$\left|\sum_{i=1}^{n} a_ib_i\right|^2 \leq \sum_{i=1}^{n} |a_i|^2 \sum_{i=1}^{n} |b_i|^2 .$$

To prove it, let λ be an arbitrary complex number. We obtain

$$\sum_{i=1}^{n} |a_i - \lambda \bar{b}_i|^2 = \sum_{i=1}^{n} |a_i|^2 + |\lambda|^2 \sum_{i=1}^{n} |b_i|^2 - 2\,\mathrm{Re}\,\bar{\lambda} \sum_{i=1}^{n} a_ib_i .$$

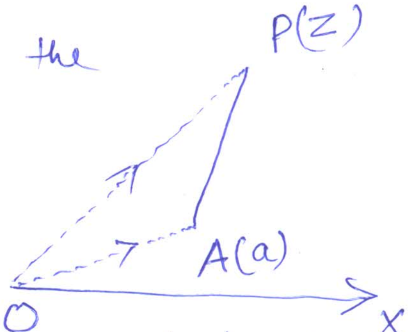
This expression is ≥ 0 for all λ. We can choose

$$\lambda = \frac{\sum_{i=1}^{n} a_ib_i}{\sum_{i=1}^{n} |b_i|^2} .$$

1. Q: The centre of a regular hexagon is at the origin and one vertex is given by $\sqrt{3} + i$ on the Argand diagram. Determine the other vertices.

Q: Find the locus of $P(z)$ when (i) $|z-a| = K$; (ii) $\mathrm{amp}(z-a) = \alpha$; where K and α are constants.

Hint: Let a and z be represented by A and P in the Argand plane, O being the origin.



Then $z - a = \vec{OP} - \vec{OA} = \vec{AP}$

(i) $|z-a| = K$ means that $AP = K$.

Thus the locus of $P(z)$ is a circle whose centre is $A(a)$ and radius K.

) $\mathrm{amp}(z-a) = \mathrm{amp}(\vec{AP}) = \alpha$, means AP always makes a constant angle with -axis. Thus locus of $P(z)$ is a straight line through $A(a)$ making angle α with OX.

Exercises: (Tutorial Sheet)

1. Find the values of

(i) $(1+2i)^3$, (ii) $\dfrac{5}{-3+4i}$, (iii) $\left(\dfrac{2+i}{3-2i}\right)^2$, (iv) $(1+i)^n+(1-i)^n$.

2. If $z = x+iy$ (x and y real), find the real and imaginary parts of

(i) z^4, (ii) $\dfrac{1}{z}$, (iii) $\dfrac{z-1}{z+1}$, (iv) $\dfrac{1}{z^2}$.

3. Show that

$$\left(\frac{-1 \pm i\sqrt{3}}{2}\right)^3 = 1 \quad \text{and} \quad \left(\frac{\pm 1 \pm i\sqrt{3}}{2}\right)^6 = 1$$

for all combinations of sign.

Square Roots:

If the given number is $\alpha + i\beta$, we are looking for a number $x+iy$ such that

$$(x+iy)^2 = \alpha + i\beta$$

This is equivalent to the system of equations

$$\left.\begin{aligned} x^2 - y^2 &= \alpha \\ 2xy &= \beta \end{aligned}\right\} \longrightarrow (1)$$

From these equations, we obtain

$$(x^2+y^2)^2 = (x^2-y^2)^2 + 4x^2y^2 = \alpha^2+\beta^2$$

Hence we must have $x^2+y^2 = \sqrt{\alpha^2+\beta^2}$, where the square root is positive or zero. Together with the first equation of (1) we find

$$\left.\begin{aligned} x^2 &= \tfrac{1}{2}\left\{\alpha + \sqrt{\alpha^2+\beta^2}\right\} \\ y^2 &= \tfrac{1}{2}\left\{-\alpha + \sqrt{\alpha^2+\beta^2}\right\} \end{aligned}\right\} \longrightarrow (2)$$

Observe that these quantities are positive or zero regardless of the sign of α.

In (2) we have two oppositive values for x and two for y. But these values cannot be combined arbitrarily, for the second equation of (1) is not a consequence of (2).

We must therefore be careful to select x and y so that their product has the sign of β. This leads to the general solution

$$\sqrt{\alpha+i\beta} = \pm\left(\sqrt{\frac{\alpha+\sqrt{\alpha^2+\beta^2}}{2}} + i\frac{\beta}{|\beta|}\sqrt{\frac{-\alpha+\sqrt{\alpha^2+\beta^2}}{2}}\right) \quad (3)$$

provided that $\beta \neq 0$. For $\beta = 0$ the values are

$$\pm\sqrt{\alpha}, \text{ if } \alpha \geq 0, \quad \pm i\sqrt{-\alpha}, \text{ if } \alpha < 0.$$

It is understood that all square roots of positive numbers are taken with the positive sign.

We found that the square root of any complex number exists and has two opposite values. They coincide only if $\alpha + i\beta = 0$. They are real if $\beta = 0$, $\alpha \geq 0$ and purely imaginary if $\beta = 0$, $\alpha \leq 0$. In other words, except for zero, only positive numbers have real square roots and only negative numbers have purely imaginary square roots.

Since both square roots are in general complex, it is not possible to distinguish between the positive and negative square root of a complex number. We could of course distinguish between upper and lower sign in (3), but this distinction is artificial and should be avoided. The correct way is to treat both square roots in a symmetric manner.

Ex: 1. Compute: $\sqrt{i}$, $\sqrt{-i}$, $\sqrt{1+i}$, $\sqrt{\frac{1-i\sqrt{3}}{2}}$.

2. Find the four values of $\sqrt[4]{-1}$.

3. Compute $\sqrt[4]{i}$ and $\sqrt[4]{-i}$

4. Solve the quadratic equation

$$z^2 + (\alpha+i\beta)z + \gamma + i\delta = 0.$$