



ICS141: Discrete Mathematics for Computer Science I

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Lecture 3

Chapter 1. The Foundations

1.2 Propositional Equivalences

1.3 Predicates and Quantifiers

1.2 Propositional Equivalence

- A **tautology** is a compound proposition that is **true** *no matter what* the truth values of its atomic propositions are!
 - e.g. $p \vee \neg p$ (“Today the sun will shine or today the sun will not shine.”) [What is its truth table?]
- A **contradiction** is a compound proposition that is **false** no matter what!
 - e.g. $p \wedge \neg p$ (“Today is Wednesday and today is not Wednesday.”) [Truth table?]
- A **contingency** is a compound proposition that is neither a tautology nor a contradiction.
 - e.g. $(p \vee q) \rightarrow \neg r$



Logical Equivalence

- Compound proposition p is **logically equivalent** to compound proposition q , written $p \equiv q$ or $p \Leftrightarrow q$, **iff** the compound proposition $p \leftrightarrow q$ is a tautology.
- Compound propositions p and q are logically equivalent to each other **iff** p and q contain the same truth values as each other in all corresponding rows of their truth tables.

Proving Equivalence via Truth Tables



- Prove that $\neg(p \wedge q) \equiv \neg p \vee \neg q$. (De Morgan's law)

p	q	$p \wedge q$	$\neg p$	$\neg q$	$\neg p \vee \neg q$	$\neg(p \wedge q)$
T	T	T	F	F	F	F
T	F	F	F	T	T	T
F	T	F	T	F	T	T
F	F	F	T	T	T	T

- Show that Check out the solution in the textbook!
 - $\neg(p \vee q) \equiv \neg p \wedge \neg q$ (De Morgan's law)
 - $p \rightarrow q \equiv \neg p \vee q$
 - $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$ (distributive law)



Equivalence Laws

- These are similar to the arithmetic identities you may have learned in algebra, but for propositional equivalences instead.
- They provide a pattern or template that can be used to match part of a much more complicated proposition and to find an equivalence for it and possibly simplify it.

Equivalence Laws

- *Identity:* $p \wedge \mathbf{T} \equiv p$ $p \vee \mathbf{F} \equiv p$
- *Domination:* $p \vee \mathbf{T} \equiv \mathbf{T}$ $p \wedge \mathbf{F} \equiv \mathbf{F}$
- *Idempotent:* $p \vee p \equiv p$ $p \wedge p \equiv p$
- *Double negation:* $\neg\neg p \equiv p$
- *Commutative:* $p \vee q \equiv q \vee p$ $p \wedge q \equiv q \wedge p$
- *Associative:* $(p \vee q) \vee r \equiv p \vee (q \vee r)$
 $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$

More Equivalence Laws

- *Distributive:*
$$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$$
$$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$$
- *De Morgan's:*
$$\neg(p \wedge q) \equiv \neg p \vee \neg q$$
$$\neg(p \vee q) \equiv \neg p \wedge \neg q$$
- *Absorption*
$$p \vee (p \wedge q) \equiv p \qquad p \wedge (p \vee q) \equiv p$$
- *Trivial tautology/contradiction:*
$$p \vee \neg p \equiv \mathbf{T} \qquad p \wedge \neg p \equiv \mathbf{F}$$

See Table 6, 7, and 8 of Section 1.2

Defining Operators via Equivalences



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Using equivalences, we can *define* operators in terms of other operators.

- Exclusive or: $p \oplus q \equiv (p \wedge \neg q) \vee (\neg p \wedge q)$
 $p \oplus q \equiv (p \vee q) \wedge \neg(p \wedge q)$
- Implies: $p \rightarrow q \equiv \neg p \vee q$
- Biconditional: $p \leftrightarrow q \equiv (p \rightarrow q) \wedge (q \rightarrow p)$
 $p \leftrightarrow q \equiv \neg(p \oplus q)$

This way we can “normalize” propositions

An Example Problem

- Show that $\neg(p \rightarrow q)$ and $p \wedge \neg q$ are logically equivalent.

$$\neg(p \rightarrow q) \quad [\text{Expand definition of } \rightarrow]$$

$$\equiv \neg(\neg p \vee q) \quad [\text{DeMorgan's Law}]$$

$$\equiv \neg(\neg p) \wedge \neg q \quad [\text{Double Negation}]$$

$$\equiv p \wedge \neg q$$

Another Example Problem

- Check using a symbolic derivation whether

$$(p \wedge \neg q) \rightarrow (p \oplus r) \equiv \neg p \vee q \vee \neg r$$

$$(p \wedge \neg q) \rightarrow (p \oplus r) \quad [\text{Expand definition of } \rightarrow]$$

$$\equiv \neg(p \wedge \neg q) \vee (p \oplus r) \quad [\text{Expand definition of } \oplus]$$

$$\equiv \neg(p \wedge \neg q) \vee ((p \vee r) \wedge \neg(p \wedge r))$$

[DeMorgan's Law]

$$\equiv (\neg p \vee q) \vee ((p \vee r) \wedge \neg(p \wedge r))$$

cont.

Example Continued...

$$(p \wedge \neg q) \rightarrow (p \oplus r) \equiv \neg p \vee q \vee \neg r$$

$$\begin{aligned} & (\neg p \vee q) \vee ((p \vee r) \wedge \neg(p \wedge r)) \quad [\vee \text{ Commutative}] \\ & \equiv (q \vee \neg p) \vee ((p \vee r) \wedge \neg(p \wedge r)) \quad [\vee \text{ Associative}] \\ & \equiv q \vee (\neg p \vee ((p \vee r) \wedge \neg(p \wedge r))) \quad [\text{Distribute } \vee \text{ over } \wedge] \\ & \equiv q \vee ((\neg p \vee (p \vee r)) \wedge (\neg p \vee \neg(p \wedge r))) \quad [\vee \text{ Assoc.}] \\ & \equiv q \vee ((\neg p \vee p) \vee r) \wedge (\neg p \vee \neg(p \wedge r)) \quad [\text{Trivial taut.}] \\ & \equiv q \vee (\mathbf{T} \vee r) \wedge (\neg p \vee \neg(p \wedge r)) \quad [\text{Domination}] \\ & \equiv q \vee (\mathbf{T} \wedge (\neg p \vee \neg(p \wedge r))) \quad [\text{Identity}] \\ & \equiv q \vee (\neg p \vee \neg(p \wedge r)) \end{aligned}$$

cont.



End of Long Example

$$(p \wedge \neg q) \rightarrow (p \oplus r) \equiv \neg p \vee q \vee \neg r$$

$$q \vee (\neg p \vee \neg(p \wedge r)) \quad [\text{DeMorgan's Law}]$$

$$\equiv q \vee (\neg p \vee (\neg p \vee \neg r)) \quad [\vee \text{ Associative}]$$

$$\equiv q \vee ((\neg p \vee \neg p) \vee \neg r) \quad [\text{Idempotent}]$$

$$\equiv q \vee (\neg p \vee \neg r) \quad [\text{Associative}]$$

$$\equiv (q \vee \neg p) \vee \neg r \quad [\vee \text{ Commutative}]$$

$$\equiv \neg p \vee q \vee \neg r \quad \blacksquare$$

Review: Propositional Logic (1.1-1.2)

- Atomic propositions: p, q, r, \dots
- Boolean operators: $\neg \wedge \vee \oplus \rightarrow \leftrightarrow$
- Compound propositions: $(p \wedge \neg q) \vee r$
- Equivalences: $p \wedge \neg q \leftrightarrow \equiv \neg(p \rightarrow q)$
- Proving equivalences using:
 - Truth tables
 - Symbolic derivations (series of logical equivalences) $p \equiv q \equiv r \equiv \dots$



1.3 Predicate Logic

- Consider the sentence

“For every x , $x > 0$ ”

If this were a true statement about the positive integers, it could not be adequately symbolized using only statement letters, parentheses and logical connectives.

*The sentence contains two new features: a **predicate** and a **quantifier***

Subjects and Predicates

- In the sentence “The dog is sleeping”:
 - The phrase “the dog” denotes the **subject** – the *object* or *entity* that the sentence is about.
 - The phrase “is sleeping” denotes the **predicate** – a property that the subject of the statement can have.
- In predicate logic, a **predicate** is modeled as a **propositional function $P(-)$** from subjects to propositions.
 - $P(x)$ = “ x is sleeping” (where x is any subject).
 - $P(\text{The cat})$ = “*The cat* is sleeping” (proposition!)

More About Predicates

- Convention: Lowercase variables $x, y, z...$ denote subjects; uppercase variables $P, Q, R...$ denote propositional functions (or predicates).
- Keep in mind that *the result of applying a predicate P to a value of subject x is the proposition*. But the predicate P , or the statement $P(x)$ **itself** (e.g. $P = \text{"is sleeping"}$ or $P(x) = \text{"x is sleeping"}$) is **not** a proposition.
 - e.g. if $P(x) = \text{"x is a prime number"}$,
 $P(3)$ is the *proposition* **"3 is a prime number."**

Propositional Functions

- Predicate logic *generalizes* the grammatical notion of a predicate to also include propositional functions of **any** number of arguments, each of which may take **any** grammatical role that a noun can take.

- *e.g.:*

let $P(x,y,z) = \text{“}x \text{ gave } y \text{ the grade } z\text{”}$

then if

$x = \text{“Mike”}, y = \text{“Mary”}, z = \text{“A”},$

then

$P(x,y,z) = \text{“Mike gave Mary the grade A.”}$

Examples

- Let $P(x): x > 3$. Then
 - $P(4)$ is ~~TRUE~~/FALSE $4 > 3$
 - $P(2)$ is TRUE/~~FALSE~~ $2 > 3$
- Let $Q(x, y): x$ is the capital of y . Then
 - $Q(\text{Washington D.C., U.S.A.})$ is TRUE
 - $Q(\text{Hilo, Hawaii})$ is FALSE
 - $Q(\text{Massachusetts, Boston})$ is FALSE
 - $Q(\text{Denver, Colorado})$ is TRUE
 - $Q(\text{New York, New York})$ is FALSE
- Read EXAMPLE 6 (pp.33)
 - If $x > 0$ then $x := x + 1$ (in a computer program)