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ICS141: Discrete Mathematics for Computer Science I

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Lecture 25

Chapter 5. Counting

5.3 Permutations and Combinations

5.4 Binomial Coefficients

5.5 Generalized Permutations and Combinations

Permutations

- A ***permutation*** of a set S of distinct elements is an **ordered** sequence that contains each element in S exactly once.
  - E.g. $\{A, B, C\} \rightarrow$ six permutations:

    ABC, ACB, BAC, BCA, CAB, CBA
- An **ordered** arrangement of r distinct elements of S is called an ***r-permutation** of S*.
- The number of r-permutations of a set with $n = |S|$ elements is

$$\boxed{P(n,r) = n(n-1)(n-2)\cdots(n-r+1) = \frac{n!}{(n-r)!}\ ,\ \ 0 \le r \le n\ .}$$

- $P(n,n) = n!/(n-n)! = n!/0! = n!$ (Note: $0! = 1$)

Permutation Examples

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- **Example**: Let $S = \{1, 2, 3\}$.
  - The arrangement 3, 1, 2 is a permutation of S ($3! = 6$ ways)
  - The arrangement 3, 2 is a 2-permutation of S ($3 \cdot 2 = 3!/1! = 6$ ways)
- **Example**: There is an armed nuclear bomb planted in your city, and it is your job to disable it by cutting wires to the trigger device. There are 10 wires to the device. If you cut exactly the right three wires, in exactly the right order, you will disable the bomb, otherwise it will explode! If the wires all look the same, what are your chances of survival?

$P(10,3) = 10 \times 9 \times 8 = 720$,
so there is a 1 in 720 chance that you'll survive!

More Permutation Examples

- **Example 6**: Suppose that a sales woman has to visit eight different cities. She must begin her trip in a specified city, but she can visit the other seven cities in any order she wishes. How many possible orders can the saleswoman use when visiting these cities?

First city is determined, and the remaining seven can be ordered arbitrarily: $7! = 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 5040$

- **Example 7**: How many permutations of the letters ABCDEFGH contain the string ABC?

ABC must occur as a block, i.e. consider it as one object
Then, it'll be the number of permutations of six objects (ABC, D, E, F, G, H), which is $6! = 720$

Another Example

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- How many ways are there to pick a set of 3 people from a group of 6?
  - There are 6 choices for the first person, 5 for the second one, and 4 for the third one, so there are $6 \cdot 5 \cdot 4 = 120$ ways to do this.
  - This is not the correct result!
  - For example, picking person C, then person A, and then person E leads to the same group as first picking E, then C, and then A.
  - However, these cases are counted separately in the above equation.
- So how can we compute how many different subsets of people can be picked (that is, we want to disregard the order of picking)?

Combinations

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- An ***r-combination*** of elements of a set *S* is an **unordered** selection of *r* elements from the set. Thus, an *r*-combination is simply a subset $T \subseteq S$ with *r* members, $|T| = r$.
- **Example**: S = {1, 2, 3, 4}, then {1, 3, 4} is a 3-combination from S
- **Example**: How many distinct 7-card hands can be drawn from a standard 52-card deck?
  - The order of cards in a hand doesn't matter.
  - Notation: C(*n*, *r*) or $\binom{n}{r}$, where $n = 52$ and $r = 7$

Calculate $C(n, r)$

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- Consider that we can obtain the *r*-permutation of a set in the following way:
  - First, we form all the *r*-combinations of the set (there are $C(n, r)$ such *r*-combinations)
  - Then, we generate all possible orderings in each of these *r*-combinations (there are $P(r, r)$ such orderings in each case).
  - Therefore, we have:

$$P(n,r) = C(n,r) \cdot P(r,r)$$

$$C(n,r) = \frac{P(n,r)}{P(r,r)} = \frac{n(n-1)\cdots(n-r+1)}{r!}$$

$$= \frac{n!}{(n-r)!} \cdot \frac{1}{r!} = \frac{n!}{r!(n-r)!}$$

Combinations

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- The number of *r*-combinations of a set with $n = |S|$ elements is

$$C(n,r) = \binom{n}{r} = \frac{P(n,r)}{P(r,r)} = \frac{n!/(n-r)!}{r!} = \frac{n!}{r!(n-r)!}$$

- Note that $C(n, r) = C(n, n-r)$
  - Because choosing the *r* members of *T* is the same thing as choosing the $(n - r)$ non-members of *T*.

$$C(n,n-r) = \binom{n}{n-r} = \frac{n!}{(n-r)!\left[n-(n-r)\right]!} = \frac{n!}{(n-r)!r!}$$

Combination Example I

- How many distinct 7-card hands can be drawn from a standard 52-card deck?
  - The order of cards in a hand doesn't matter.

- Answer:

$C(52, 7) = P(52, 7) / P(7, 7) = 52! / (7!\cdot 45!)$

$= (52\cdot \cancel{51}\cdot \cancel{50}\cdot \cancel{49}\cdot \cancel{48}\cdot 47\cdot 46) / (\cancel{7}\cdot \cancel{6}\cdot \cancel{5}\cdot \cancel{4}\cdot \cancel{3}\cdot \cancel{2}\cdot 1)$

17 10 7 $\cancel{8}$ $\cancel{2}$

$52\cdot 17\cdot 10\cdot 7\cdot 47\cdot 46 = 133{,}784{,}560$

Combination Example II

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- $C(4, 3) = 4$, since, for example, the 3-combinations of a set {1, 2, 3, 4} are {1, 2, 3}, {1, 3, 4}, {2, 3, 4}, {1, 2, 4}.
  - $C(4, 3) = P(4, 3) / P(3, 3) = 4! / (3! \times 1!)$
    $= (4 \times 3 \times 2) / (3 \times 2 \times 1) = 4$
- How many ways are there to pick a set of 3 people from a group of 6 (disregarding the order of picking)?
  - $C(6, 3) = 6! / (3! \times 3!)$
    $= (6 \times 5 \times 4) / (3 \times 2 \times 1) = 20$
  - There are 20 different groups to be picked

Combination Example III

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- A soccer club has 8 female and 7 male members. For today's match, the coach wants to have 6 female and 5 male players on the grass. How many possible configurations are there?
  - $C(8, 6) \times C(7, 5)$

$$= \{P(8, 6) / P(6, 6)\} \times \{P(7, 5) / P(5, 5)\}$$

$$= \{8!/ (2! \times 6!)\} \times \{7!/ (2! \times 5!)\}$$

$$= \{(8 \times 7) / 2!\} \times \{(7 \times 6) / 2!\}$$

$$= 28 \times 21$$

$$= 588$$

Binomial Coefficients

- Expressions of the form $C(n, r)$ are also called ***binomial coefficients***
  - Coefficients of the expansion of powers of binomial expressions
  - Binomial expression is a simply the sum of two terms such as $x + y$
- Example:

$$
\begin{aligned}
(x+y)^3 &= (x+y)(x+y)(x+y) \\
&= (xx+xy+yx+yy)(x+y) \\
&= xxx+xxy+xyx+xyy+yxx+yxy+yyx+yyy \\
&= C(3,0)x^3 + C(3,1)x^2y + C(3,2)xy^2 + C(3,3)y^3 \\
&= x^3 + 3x^2y + 3xy^2 + y^3
\end{aligned}
$$

The Binomial Theorem

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- Let x and y be variables, and let n be a nonnegative integer. Then

$$(x+y)^n = \binom{n}{0}x^n + \binom{n}{1}x^{n-1}y + \binom{n}{2}x^{n-2}y^2 + \cdots + \binom{n}{n-1}xy^{n-1} + \binom{n}{n}y^n$$
$$= \sum_{j=0}^{n}\binom{n}{j}x^{n-j}y^j$$

- To obtain a term of the form $x^{n-j}y^j$, it is necessary to choose $(n-j)$ x's from the n terms, so that the other j terms in the product are y's. Therefore, the coefficient of $x^{n-j}y^j$ is $C(n, n-j) = C(n, j)$.
- The binomial theorem gives the coefficients of the expansion of powers of binomial expressions.

Examples

- $(a + b)^9 \rightarrow$ the coefficient of $a^5b^4 = C(9, 4)$
- The coefficient of $x^{12}y^{13}$ in the expansion of $(2x - 3y)^{25}$
  - By binomial theorem

$$(2x + (-3y))^{25} = \sum_{j=0}^{25} \binom{25}{j} (2x)^{25-j} (-3y)^j$$

  - The coefficient of $x^{12}y^{13}$ is obtained when $j = 13$

$$C(25,13) \cdot 2^{12} \cdot (-3)^{13} = -\frac{25!}{13! \cdot 12!} 2^{12} 3^{13}$$

- $(x + y + z)^9 \rightarrow$ the coefficient of $x^2y^3z^4 = C(9, 2) \cdot C(7, 3)$

Corollaries

- $\sum_{k=0}^{n} \binom{n}{k} = 2^n$, where n is a nonnegative integer.
  - Proof: $2^n = (1+1)^n = \sum_{k=0}^{n} \binom{n}{k} 1^{n-k} 1^k = \sum_{k=0}^{n} \binom{n}{k}$
- $\sum_{k=0}^{n} (-1)^k \binom{n}{k} = 0$, where n is a positive integer.
  - Proof: $0 = 0^n = (1+(-1))^n = \sum_{k=0}^{n} \binom{n}{k} 1^{n-k} (-1)^k = \sum_{k=0}^{n} (-1)^k \binom{n}{k}$
  - It implies that: $\binom{n}{0} + \binom{n}{2} + \binom{n}{4} + \cdots = \binom{n}{1} + \binom{n}{3} + \binom{n}{5} + \cdots$

Corollaries (cont.)

- $\sum_{k=0}^{n} 2^k \binom{n}{k} = 3^n$, where n is a nonnegative integer.

  - Proof:

$$3^n = (1+2)^n = \sum_{k=0}^{n} \binom{n}{k} 1^{n-k} 2^k$$

$$= \sum_{k=0}^{n} \binom{n}{k} 2^k$$

Generalized Permutations and Combinations

- Permutations and combinations allowing repetitions.
  - How many strings of length *r* can be formed from the English alphabet?
  - How many different ways are possible when we select a dozen donuts from a box that contains four different kinds of donuts?
- Permutations where not all objects are distinguishable.
  - The number of ways we can rearrange the letters of the word *MISSISSIPPI*

Permutations with Repetitions

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- **<u>Theorem 1</u>**: The number of *r*-permutations of n objects with repetition allowed is n^r.
  - Proof: There are *n* ways to select an element of the set for each of *r* positions with repetition allowed. By the product rule, the answer is given as *r* multiples of *n*.
- **Example**: How many strings of length *r* can be formed from the English alphabet?
  - Answer: 26^r

Combinations with Repetitions

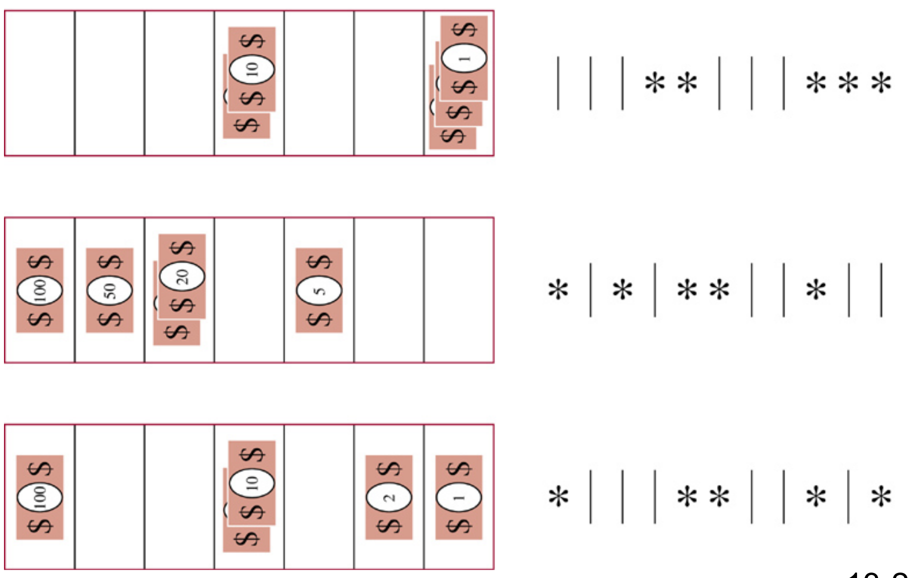
- An example
  - How many ways are there to select four pieces of fruit from a bowl containing apples, oranges, and pears if there are at least four pieces of each type of fruit in the bowl?
    - In this case, the order in which the pieces are selected does not matter, only the types of fruit, not the individual piece, matter.

Combinations with Repetitions

- Example Rephrased: The number of 4-combinations with repetition allowed from a 3-element set {apple, orange, pear}
  - All four in same type: 4 apples, 4 oranges, 4 pears [3 ways]
  - Three in same type: two cases for each of 3 apples, 3 oranges, 3 pears [2*3=6 ways]
  - Two diff. pairs with each pair in same type [3 ways]
  - Only one pair in same type [3 ways]
  - Total 15 ways
- Can be generalized:
  - The number of ways to fill 4 slots from 3 categories with repetition allowed

Example

- How many ways are there to select five bills from a cash box containing \$1 bills, \$2 bills, \$5 bills, \$10 bills, \$20 bills, \$50 bills, and \$100 bills?
  - The order in which the bills are chosen doesn't matter
  - The bills of each denomination are indistinguishable
  - At least five bills of each type



- $C(7-1+5, 5)$
  $= C(11, 5)$
  $= 11! / (5!\cdot 6!)$
  $= 462$

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Combinations with Repetitions

- **Theorem 2**: The number of *r*-combinations from a set with *n* elements with repetition allowed is:

$$\boxed{C(n+r-1,r) = C(n+r-1,n-1)}$$

- Other representations with the same meaning
  - # of ways to fill *r* slots from *n* categories with repetition allowed
  - # of ways to select *r* elements from *n* categories of elements with repetition allowed

Proof of Theorem 2

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- Represent each r-combinations from a set with n elements with repetition allowed by a list of n – 1 bars and r stars.
  - n – 1 bars: used to mark off n different cells (categories)
  - r stars: each star in *i-th* cell (if any) represents an element that is selected for the *i-th* category
- # of different lists that containing n – 1 bars and r stars
  - = # of ways to chose the r positions to place the r stars from $n + r - 1$ positions [**$C(n + r - 1, r)$**]
  - = # of ways to chose the n – 1 positions to place the n – 1 bars from $n + r - 1$ positions [**$C(n + r - 1, n - 1)$**]

More Examples

- How many ways can I fill a box holding 100 pieces of candy from 30 different types of candy?
  - Solution: Here #stars = 100, #bars = 30 – 1, so there are $C(100+29,100) = 129! / (100!\cdot29!)$ different ways to fill the box.
- How many ways if I must have at least 1 piece of each type?
  - Solution: Now, we are reducing the #stars to choose over to (100 – 30) stars, so there are $C(70+29, 70) = 99! / (70!29!)$

When to Use Generalized Combinations

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- Besides categorizing a problem based on its order and repetition requirements as a generalized combination, there are a couple of other characteristics which help us sort:
  - In generalized combinations, having all the slots filled in by only selections from one category is allowed;
  - It is possible to have more slots than categories.

More Integer Solutions & Restrictions

- How many integer solutions are there to:

  $$a + b + c + d = 15,$$

  when $a \geq -3$, $b \geq 0$, $c \geq -2$ and $d \geq -1$?
- In this case, we alter the restrictions and equation so that the restrictions "go away." To do this, we need each restriction ≥ 0 and balance the number of slots accordingly.
- Hence $a \geq -3+3$, $b \geq 0$, $c \geq -2+2$ and $d \geq -1+1$, yields $a + b + c + d = 15+3+2+1 = 21$
- So, there are C(21+4–1,21) = C(24,21) = C(24,3) = (24x23x22)/(3x2) = 2024 solutions.

Distributing Objects into Distinguishable Boxes

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- Distinguishable (or labeled) objects to distinguishable boxes
  - How many ways are there to distribute hands of 5 cards to each of four players from the standard deck of 52 cards?

    C(52,5)C(47,5)C(42,5)C(37,5)

- Indistinguishable (or unlabeled) objects to distinguishable boxes
  - How many ways are there to place 10 indistinguishable balls into 8 distinguishable bins?

    C(8+10-1, 10) = C(17,10) = 17! / (10!7!)

Distributing Distinguishable Objects into Indistinguishable Boxes

- How many ways are there to put <u>4 different employees</u> into <u>3 indistinguishable offices</u>, when each office can contain any number of employees?
  - all four in one office: C(4,4) = 1
  - three + one: C(4,3) = 4
  - two + two: C(4,2)/2 = 3
  - two + one + one: C(4,2) = 6

Distributing Indistinguishable Objects into Indistinguishable Boxes

- How many ways are there to pack 6 copies of same book into 4 identical boxes, where each box can contain as many as six books?
  - List # of books in each box with the largest # of books, followed by #s of books in each box containing at least 1 book, in order of decreasing # of books in a box.

6

5,1

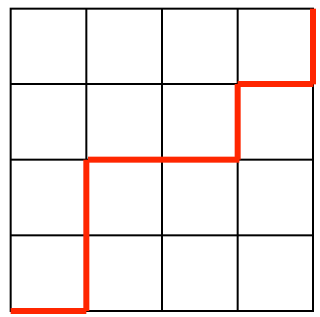
4,2 4,1,1

3,3 3,2,1 3,1,1,1

2,2,2 2,2,1,1

Another Combination Example

- How many routes are there from the lower left corner of an $n \times n$ square grid to the upper right corner if we are restricted to traveling only to the right or upward.
- Solution



R: right
U: up

  - route → *RUURRURU* : a string of n *R*'s and n *U*'s
  - Any such string can be obtained by selecting n positions for the *R*'s, without regard to the order of selection, from among the 2n available positions in the string and then filling the remaining position with *U*'s.
  - Thus there are C(2n,n) possible routes.

Pascal's Identity

- Let n and k be positive integers with $n \geq k$. Then

$$\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}$$

- Proof
  - Let X be a set with n elements. Let $a \notin X$.
  - Then $C(n+1, k)$ is the number of k-element subsets of $Y = X \cup \{a\}$ (Y contains $n+1$ elements).
  - Now the k-element subsets of Y can be divided into two disjoint classes:

    1. subsets of Y not containing $a \rightarrow k$-element subsets of X $\rightarrow C(n,k)$

    2. subsets of Y containing $a \rightarrow (k–1)$-element subsets of X together with $a \rightarrow C(n, k–1)$
  - $\therefore C(n+1, k) = C(n, k–1) + C(n, k)$

Pascal's Triangle

- We can write the binomial coefficient in triangular from

$$\binom{0}{0}$$
$$\binom{1}{0}\ \binom{1}{1}$$
$$\binom{2}{0}\ \binom{2}{1}\ \binom{2}{2}$$
$$\binom{3}{0}\ \binom{3}{1}\ \binom{3}{2}\ \binom{3}{3}$$
$$\binom{4}{0}\ \binom{4}{1}\ \binom{4}{2}\ \binom{4}{3}\ \binom{4}{4}$$
$$\binom{5}{0}\ \binom{5}{1}\ \binom{5}{2}\ \binom{5}{3}\ \binom{5}{4}\ \binom{5}{5}$$
$$\binom{6}{0}\ \binom{6}{1}\ \binom{6}{2}\ \binom{6}{3}\ \binom{6}{4}\ \binom{6}{5}\ \binom{6}{6}$$
$$\binom{7}{0}\ \binom{7}{1}\ \binom{7}{2}\ \binom{7}{3}\ \binom{7}{4}\ \binom{7}{5}\ \binom{7}{6}\ \binom{7}{7}$$
$$\binom{8}{0}\ \binom{8}{1}\ \binom{8}{2}\ \binom{8}{3}\ \binom{8}{4}\ \binom{8}{5}\ \binom{8}{6}\ \binom{8}{7}\ \binom{8}{8}$$

(a)

By Pascal's identity:

$$\binom{6}{4} + \binom{6}{5} = \binom{7}{5}$$

```
                1
              1   1
            1   2   1
          1   3   3   1
        1   4   6   4   1
      1   5  10  10   5   1
    1   6  15  20  15   6   1
  1   7  21  35  35  21   7   1
1   8  28  56  70  56  28   8   1
```

(b)

Vandermonde's Identity

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- Let *m*, *n*, and *r* be nonnegative integers with *r* not exceeding either *m* or *n*. Then

$$\binom{m+n}{r} = \sum_{k=0}^{r} \binom{m}{r-k}\binom{n}{k}$$

- Proof:

  Suppose that there are *m* items in one set and *n* items in a second set. Then the total number of ways to pick *r* elements from the union of these sets is $\binom{m+n}{r}$

Proof cont.

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Another way to pick *r* elements from the union is to pick *k* elements ($0 < k < r$) from the second set and then $r - k$ elements from the first set. By the product rule, this can be done in

$$\binom{m}{r-k}\binom{n}{k}$$

The total number of ways to pick *r* elements from the union also equals

$$\binom{m+n}{r} = \sum_{k=0}^{r}\binom{m}{r-k}\binom{n}{k}$$

More Theorems…

- Corollary

$$\binom{2n}{n} = \sum_{k=0}^{n} \binom{n}{k}^2, \text{ where } n \text{ is a nonnegative integer.}$$

- **Theorem 4**: Let *n* and *r* be nonnegative integers with *r* < *n*. Then

$$\binom{n+1}{r+1} = \sum_{j=r}^{n} \binom{j}{r}$$