HOME ASSIGNMENT

MA-201

Tarusi Mittal

Roll No. : 1901CS65
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Sign :
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Contact No : 9781337500

Question 1:-

$$a = \left|z + \frac{1}{z}\right| \longrightarrow ①$$

By triangle inequality

$$\left|z + \frac{1}{z}\right| \geq \left||z| - \left|\frac{1}{z}\right|\right| \text{ and } |z| + \left|\frac{1}{z}\right| \geq \left|z + \frac{1}{z}\right|$$

Putting $\left|z + \frac{1}{z}\right| = a$ from 1.

$$\Rightarrow z + \frac{1}{z} \geq a$$

$$\Rightarrow \frac{|z|^2 + 1}{|z|} \geq a$$

$$\Rightarrow \frac{|z|^2 + 1}{|z|} \geq a$$

$\Rightarrow$ since $|z| > 0$, we cross multiply

$$\underbrace{|z|^2 - |z| \times a + 1}_{\text{quadratic eq in } z} \geq 0 \begin{matrix} \nearrow r_1 \\ \searrow r_2 \end{matrix} \Big\} \text{roots}$$

$$r_1, r_2 = \frac{a \pm \sqrt{a^2 - 4}}{2}$$

allowed values of $|z| \Rightarrow$

$$\boxed{\begin{matrix} 0 \leq |z| \leq \dfrac{a - \sqrt{a^2-4}}{2} \\ \dfrac{a + \sqrt{a^2-4}}{2} \leq |z| \leq \infty \end{matrix}} \;\&$$

Putting $\left|z+\frac{1}{z}\right|=a$

$$a \geq \left||z|-\left|\frac{1}{|z|}\right|\right| \Rightarrow a \geq |z|-\frac{1}{|z|} \geq -a$$

$$\Rightarrow a \geq \frac{|z|^2-1}{|z|} \geq -a$$

$|z|^2 - a|z| - 1 \leq 0 \rightarrow x_1', x_2'$

$|z|^2 + a|z| - 1 \geq 0 \rightarrow x_1'', x_2''$

$|z| > 0$ & $|z| > x_1'$ & $|z| < x_2'$

$|z| > 0$ & $|z| > x_2^2$ & $|z| < x_1^2$

$$\boxed{0 \leq |z| \leq \frac{a+\sqrt{a^2+4}}{2}}$$

$$\boxed{\begin{array}{l} 0 < |z| \leq \frac{-a-\sqrt{a^2+4}}{2} \\ \& \ \frac{-a+\sqrt{a^2+4}}{2} \leq |z| \leq \infty \end{array}}$$

Taking intersection:

$$\boxed{\begin{array}{l} |z|_{min} = \frac{-a+\sqrt{a^2+4}}{2} \\ \\ |z|_{max} = \frac{a+\sqrt{a^2+4}}{2} \end{array}}$$

Question 2: $z = x + iy$; $f(z) = \sqrt{|xy|}$

$f(z) = u + iv$

$u(x, y) = \sqrt{|xy|}$ $\qquad v(x, y) = 0$

$\therefore \frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} = 0$ for all (x, y)

$$\frac{\partial u}{\partial x}(0,0) = \lim_{\substack{h \to 0 \\ h \in R}} \frac{u(h,0) - u(0,0)}{h} = 0$$

Similarly $\frac{\partial u}{\partial y} = 0$ at $(0, 0)$

Thus Cauchy Riemann equations are true at $(0, 0)$ on the other hand if $z = a(1+i)$

$\frac{f(z) - f(0)}{z} = \frac{|a|}{a(1+i)}$ does not have a limit as $a \to 0$ in R. Thus $f'(0)$ does not exist.

Question 3: Given $u = x^2 - y^2$ $\quad v = -y/(x^2+y^2)$

$$\Rightarrow \frac{\partial u}{\partial x} = 2x \; ; \; \frac{\partial v}{\partial x} = -2y$$

$$\frac{\partial^2 u}{\partial x^2} = 2 \qquad \frac{\partial^2 u}{\partial y^2} = -2$$

$\therefore \dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$; Thus u satisfies Laplace equation

$$\frac{\partial v}{\partial x} = \frac{0 + y(2x)}{(x^2+y^2)^2}$$

$$\frac{\partial^2 v}{\partial x^2} = \frac{2y(x^2+y^2)^2 - 2xy(2(x^2+y^2)(2x))}{(x^2+y^2)^4}$$

$$\frac{\partial v}{\partial y} = \frac{-(x^2+y^2) + y(2y)}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$$

$$\frac{\partial^2 v}{\partial y^2} = \frac{2y(x^2+y^2)^2 - (y^2-x^2)(2(x^2+y^2)(2y))}{(x^2+y^2)^4}$$

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = \frac{4y(x^2+y^2)^2 - 8x^2y(x^2+y^2) - 4(y(y^2-x^2)(x^2-y^2))}{(x^2+y^2)^4}$$

$$= 4y(x^2+y^2)^2 - 8x^4y - 8x^2y^3 - (4)(y^3-x^2y)(x^2+y^2)$$

$$= 4y(x^4+y^4+2x^2y^2) - 8x^4y - 8x^2y^3 - 4(y^3x^2 + y^5 - x^2y - x^2y^3) = 0$$

Thus both u and v satisfy Laplace equation

$\therefore$ If $f(z) = u + iv$ needs to be analytic if C-R equations are followed it is true.

Thus,

$\dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y}$	$\dfrac{\partial u}{\partial y} = \dfrac{-\partial v}{\partial x}$
$2x = \dfrac{y^2 - x^2}{(x^2+y^2)^2}$	$-2y = \dfrac{-2yx}{(x^2+y^2)^2}$

But this is not true.

Hence $f(z)$ is not analytic.

P.T.O.

Question 4:- Let f be an analytic function in a Rectangular region R.

Cauchy Goursat Theorem.

Suppose a function f is analytic with a continious all over in a simply connected domain D'.
Then for every closed contour C in D

$$\oint_C f(z)\cdot dz = 0$$

Rectangular domain proof:

Green theorem

$$\oint_C (L\,dx + M\,dy) = \iint_D \left(\frac{\partial M}{\partial x} - \frac{\partial L}{\partial y}\right) dx - dy \longrightarrow (1)$$

(Anti clockwise)

Now,

$$f(z) = u + iv \Rightarrow f'(z) = u_x + iv_x$$

$$= v_y - iv_y$$

By Cauchy Riemann Relation

$$[dz = dx + i\,dy]$$

$$\int_C f(z)\cdot dz = \oint_C (u+iv)(dx + i\,dy)$$

$$= \oint_c (u\,dx - v\,dy) + i \oint_c (v\,dx + u\,dy)$$

$$= -\iint \left(\frac{dv}{dx} + \frac{du}{dy} \right) dx \cdot dy$$

$$+ \; i \iint \left(\frac{du}{dx} - \frac{dv}{dy} \right) dx\,dy.$$

$$\left[\because \; \frac{dv}{dx} + \frac{du}{dy} = 0 \right.$$

$$\left. \& \; \frac{du}{dx} = \frac{dv}{dy} \right]$$

By Cauchy Riemann equation

$$\boxed{\int f(z)\,dz = 0}$$

<u>Hence Proved</u>

$$\int_{(0,3)}^{(2,4)} (2y + x^2)\,dx + (3x - y)\,dy.$$

$$x = 2t \quad , \quad y = t^3 + 3$$
$$dx = 2dt \qquad dy = 3t^2\,dt$$

$$\int_0^1 \left(2(t^3+3) + 4t^2\right) 2\,dt + (3\times 2t - t^3 - 3)\,3t^2\,dt$$

$$\int_0^1 (4t^3 + 12 + 8t^2)\,dt + (18t^3 - 3t^5 - 9t^2)\,dt$$

$$\int \frac{4t^4}{4} + 12t + \frac{8t^3}{3} + \frac{18t^4}{4} - \frac{3t^6}{6} - \frac{9t^2}{3}\Bigg|_0^1$$

$$\left(1 + 12 + \frac{8}{3} + \frac{\cancel{18}^{\,9}}{\cancel{4}_{\,2}} - \frac{1}{2} - 3\right) - (0)$$

$$1 + 12 + \frac{8}{3} + 4 - 3 = 14 + \frac{8}{3} = \boxed{\frac{50}{3}}$$

Question 5:- If a power series

$$\sum_{n=0}^{\infty} a_n (z-z_0)^n$$

z_0 = centre of disk of convergence

The radius of convergance r is a non negative real number or ∞ such that series converges if $|z-a| < r$ and diverges if $|z-a| > r$

Now,

$$\sum_{n=0}^{\infty} \frac{n\sqrt{2}+i}{1+2in} z^n$$

Using ratio test.

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n}$$

$$a_{n+1} = \frac{(n+1)\sqrt{2}+i}{1+2i(n+1)} z^{n+1}$$

$$a_n = \frac{n\sqrt{2}+i}{1+2in} z^n$$

$$\lim_{n\to\infty} \frac{a_{n+1}}{n} = \frac{1+\sqrt{2}+\sqrt{2}n}{1+2i+2in} \times \frac{1+2in}{n\sqrt{2}+i} \times z$$

$$= \frac{(1+\sqrt{2})/n + \sqrt{2}}{(1+2i)/n + 2i} \times \frac{1/n + 2i}{i/n + \sqrt{2}} \times z$$

$$= \frac{\sqrt{2}}{2i} \times \frac{2i}{\sqrt{2}} z = z$$

$\boxed{\text{Radius} = 1}$

$$\left(\text{if } \frac{a_{n+1}}{a_n} = |kz| \;;\; R = \frac{1}{|k|} \right)$$

Question 6:- Let $w = g(\xi)$ be an analytic function

ξ is an analytic function of z

Let $\xi = f(z)$ then $w = g(f(z))$

To prove : w is an analytic function of z

: $\frac{dw}{dz} = \frac{dw}{d\xi} \times \frac{d\xi}{dz}$

Now,

$$\frac{dw}{d\xi} = g'(\xi) = \lim_{h \to 0} \frac{g(\xi + h) - g(\xi)}{h}$$

$$\frac{d\xi}{dz} = f'(z) = \lim_{h \to 0} \frac{f(z+h) - f(z)}{h}$$

To find $\frac{dw}{dz}$:

$$w = g(f(z))$$

$$\frac{dw}{dz} = \lim_{h \to 0} \frac{g(f(z+h)) - g(f(z))}{h}$$

$$= \lim_{h \to 0} \frac{g(f(z+h)) - g(f(z))}{h} \times \frac{f(z+h) - f(z)}{f(z+h) - f(z)}$$

$$= \lim_{h \to 0} \frac{g(f(z+h)) - g(f(z))}{(f(z+h)) - f(z)} \times \frac{f(z+h) - f(z)}{h}$$

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$$\frac{dw}{dz} = \lim_{h \to 0} \frac{g(f(z+h)) - g(f(z))}{f(z+h) - f(z)} \cdot \frac{d\xi}{dz}$$

Let $k = f(z+h) - f(z)$

$k + f(z) = f(z+h)$

as $h \to 0$ $\quad k \to 0$

$$\therefore \frac{dw}{dz} = \lim_{k \to 0} \frac{g(k + f(z)) - g(f(z))}{k} \cdot \frac{d\xi}{dz}$$

$$\frac{dw}{dz} = \frac{dw}{d\xi} \cdot \frac{d\xi}{dz}$$

Hence Proved.