Sem - II :

1] Matrix :

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & & & a_{2n} \\ \vdots & & & \\ a_{m1} & \cdots & \cdots & a_{mn} \end{bmatrix}$$

→ Raw

colown

$m \times n$

Raw   colown -

$A = [a_{ij}]_{m \times n}$ $m = n$ then square matrix.

$[A+B]_{ij} = [a_{ij} + b_{ij}]$

size A = size B.

$(m \times n)$

1] $c \in R$, $A = [a_{ij}]_{m \times n}$

$\therefore [cA] = [ca_{ij}]_{m \times n}$, multiplies all entries

Zero matrix → all entries zero.

Identity matrix → If $a_{ij} = 1$ for $(i = j)$ & $a_{ij} = 0$ for $(i \neq j)$

It is always a square matrix.

Equal matrix : Two matrix A and B are equal if

(1) A and B have same size.

(2) $a_{ij} = b_{ij}$ $\forall$ i & j

Scalar : kI multiple of I

2] sum of Matrices :

let A and B be two matrices. $m \times n$

then sum of A & B..

(i) $A + 0 = A = 0 + A$

(ii) $A + B = B + A$ (iii) $A - B = A + (-B)$ (iv) $A + (B + C) = (A + B) + C$

* Transpose of $A_{m\times n}$

The transpose of a matrix A is the matrix B of size $(n\times m)$ in such that $B_{ij} = A_{ji}$

and is denoted by $\boxed{A^t = B}$

prop :
$$(A+B)^t = A^t + B^t$$
$$(cA)^t = c(A)^t$$
$$(A^t)^t = A.$$

* Linear Combination of Vectors.

let $v_1, v_2, v \ldots v_n \in S$ Then v is said to be in the Linear combination of $v_1, v_2, \ldots v_n$, if $\exists\ c_1, c_2, \ldots, c_n \in R$ s.t.

$$v = \sum_{i=1}^{n} c_i v_i$$

* Matrix Multiplication.

let $A_{m\times n}$ and $B_{n\times p}$ matrix then the product of A and B is the matrix C of order $m\times p$ s.t. $C = A.B$, where

$$C_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$$

: To prove $(AB)^t = B^t A^t$

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & \cdots & & a_{2n} \\ a_{m1} & \cdots & \cdots & a_{mn} \end{bmatrix}_{m\times n} \qquad X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}_{n\times 1}$$

$$AX = \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \cdots a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 \cdots a_{2n}x_n \\ a_{m1}x_1 \cdots \cdots a_{mn}x_n \end{bmatrix} = \text{P.T.O.}$$

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$$\Rightarrow \begin{bmatrix} a_{11}x_1 \\ a_{21}x_1 \\ \vdots \\ a_{n1}x_1 \end{bmatrix} + \begin{bmatrix} a_{12}x_2 \\ a_{22}x_2 \\ \vdots \\ a_{n2}x_2 \end{bmatrix} \cdots + \begin{bmatrix} a_{1n}x_n \\ \vdots \\ a_{mn}x_n \end{bmatrix}$$

$$= x_1 \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} + \cdots + \begin{bmatrix} a_{1n} \\ \vdots \\ a_{mn} \end{bmatrix} x_n$$

$$AX = A_1x_1 + \cdots A_nx_n$$

If $A_{m\times n}$, $B_{n\times p}$ Then $C_{mp} = A\cdot B$.

Then j^{th} column of $C = A.$ (j^{th} column of B).

Then j^{th} row of C = (j^{th} row of A) $\cdot B$.

* System of linear Equation.

By the system of linear eq^n's in 'n' variables $x_1, x_2, \ldots x_n$, we mean a collection of linear eq^n's in these variables. A system of m linear eq^n's in n variables can be written as.

(1) $$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots a_{2n}x_n = b_2 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{cases}$$

If all $b_i = 0$, $i = 1, 2, \ldots, m$
Then system of linear eq^{n}s is called homogenous.

~~If~~ system of linear eq^n (I) has sol^n then it is consistent.
If not then it is <u>inconsistent</u>.

Solution of a system of linear eq^n (I) is a tupple $(s_1, s_2, \ldots s_n)$ of numbers that make each eq^n true statement when $s_1, s_2, \ldots, s_n$ are satisfied for $x_1, x_2 \ldots x_n$ respectively.

★ <u>Classifications of linear systems:</u>

→ Given a system of linear eq^{n}s (I) in 'n' variables precisely one of the following three type is true.

(a) The system has exactly one sol^n.
(b) The system has infinite many sol^n.
(c) The system has no sol^n.

* Matrix formulation of system of linear equations.

→ Suppose we have :

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2$$
$$\vdots$$
$$a_{m1}x_1 + \cdots + a_{mn}x_n = b_m$$

Thus we write as :

$$\textcircled{O} \leftarrow \begin{bmatrix} a_{11} & a_{12} \cdots a_{1n} \\ \vdots & \\ a_{m1} & \cdots\cdots a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

$\therefore AX = B \rightarrow$ constant matrix

coefficient matrix, variable matrix

This has solⁿ if B can be written as linear combination of columns of A.

* For $AX = b$

$[A|b] \rightarrow$ Augmented matrix of the system of linear equation (I)

For [Ex] :

$$x_1 + x_2 - 2x_3 = 1$$
$$2x_1 - 3x_2 + x_3 = 8$$
$$3x_1 + x_2 + 4x_3 = 7.$$

$$\Rightarrow \begin{bmatrix} 1 & 2 & -2 \\ 2 & -3 & 1 \\ 3 & 1 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 8 \\ 7 \end{bmatrix}$$

Augmented matrix is

$$\left[\begin{array}{ccc|c} 1 & 2 & 2 & 1 \\ 2 & -3 & 1 & 8 \\ 3 & 1 & 4 & 7 \end{array}\right]$$

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d^n

for] $x_1 + x_2 - 2x_3 = 1 \rightarrow (1)$

$2x_1 - 3x_2 + x_3 = +8 \rightarrow (2)$

$3x_1 + x_2 + 4x_3 = 7 \rightarrow (3)$

Eqn (2) $\rightarrow$ Eq2 + (-2) Eq (1)

& Eqn (3) Eq 3 + (-3) Eq (1)

$\therefore$ ~~$5x_2 + 3x_1$~~ $= 7$

$x_1 + x_2 - 2x_3 = 1 \rightarrow$ (iv)

$-5x_2 + 5x_3 = 6 \rightarrow$ (v)

$-2x_2 + 10x_3 = 4$ — (vi)

In $\begin{bmatrix} 1 & 1 & -2 \\ 2 & -3 & 1 \\ 3 & 1 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 8 \\ 7 \end{bmatrix}$

$R_2 \rightarrow R_2 - 2(R_1)$

$R_3 \rightarrow R_3 - 3R_1$

$\therefore \begin{bmatrix} 1 & 1 & -2 \\ 0 & -5 & 5 \\ 0 & -2 & 10 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 6 \\ 4 \end{bmatrix}$

Instead if we write augmented matrix

-10
$+2$

$R_2 \rightarrow 2R_2 - R_3$

$\begin{bmatrix} 1 & 1 & -2 \\ 0 & -8 & 0 \\ 0 & -2 & 10 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 8 \\ 4 \end{bmatrix}$ no need to do separately on this

$R_1 \rightarrow 2R_1 + R_3$

$\begin{bmatrix} 2 & 0 & 6 \\ 0 & -8 & 0 \\ 0 & -2 & 10 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 6 \\ 8 \\ 4 \end{bmatrix}$

$\therefore 2x_1 + 6x_3 = 6 \Rightarrow x_1 + 3x_3 = 3$

$-8x_2 = 8 \Rightarrow x_2 = -1$

$-2x_2 + 10x_3 = 4 \Rightarrow x_2 - 5x_3 = -2$

$\therefore -1 - 5x_3 = -2$

$\therefore -5x_3 = -1$

$x_3 = \frac{1}{5}$

& $x_1 + \frac{3}{5} = 3$

$\therefore x_1 = 3 - \frac{3}{5} = \frac{12}{5}$

* Row operations :

→ There are three types of elementary row operations

(a) Interchanging the rows ; $R_i \leftrightarrow R_j$.

(b) Multiplying all the entries of one row by a non zero constant i.e. $R_2 \leftrightarrow CR_I$

(c) Adding the multiple of one row to another row.

$R_1 = R_i + CR_J$

#. Two linear systems in same variables are called equivalent if their solution's set are same.

* Row equivalent matrices

→ A matrix A and B is said to be row equivalent ($A \sim B$) if there is a sequence of elementary row operations →

* Upper triangular matrix:
A square matrix A is called upper triangular if the $a_{ij} = 0$ for all $i > j$

$$\begin{bmatrix} 2 & 3 & 1 \\ 0 & 4 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

* lower triangular is $a_{ij} = 0$ for all $i < j$

$$\begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 9 \\ 3 & 2 & 1 \end{bmatrix}$$

* Symmetric matrix:
A square matrix A is symmetric if $A^T = A$.

* Skew symmetric if $A^T = -A$.

*

$$A = \frac{(A + A^t) + (A - A^t)}{2}$$

$$= \frac{(A + A^T)}{2} + \frac{(A - A^t)}{2}$$

↓ symmetric ↓ Skew symmetric:

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Summary: Gauss Elimination Method.

$$AX = b \rightarrow (1)$$

Step 1: Write augmented matrix $[A \mid b]$.

Step 2] By applying elementary row operations. convert (1) in the following form.

$$[U \mid \tilde{b}]$$

$U \rightarrow$ Upper triangular matrix.

Step 3] Solve this by using back substitution.

Q] Solve the following system of linear eqns by using Gauss elimination -

$$x_1 + x_2 + 6x_3 = 7$$
$$3x_1 + x_3 = 6$$
$$x_1 + x_2 + 4x_3 = 7.$$

$\Rightarrow$

$$\left[\begin{array}{ccc|c} 1 & 1 & 6 & 7 \\ 3 & 0 & 1 & 6 \\ 1 & 1 & 4 & 7 \end{array}\right]$$

$\therefore R_3 \rightarrow R_1 - R_3$

$$\therefore \left[\begin{array}{ccc|c} 1 & 1 & 6 & 7 \\ 3 & 0 & 1 & 6 \\ 0 & 0 & 2 & 0 \end{array}\right]$$

$R_2 \rightarrow R_2 - 3R_1$

$$\left[\begin{array}{ccc|c} 1 & 1 & 6 & 7 \\ 0 & -3 & -17 & -15 \\ 0 & 0 & 2 & 0 \end{array}\right]$$

$\therefore \quad 2x_3 = 0$

$\therefore \quad x_3 = 0$

$-3x_2 - 17x_3 = -15$

$\therefore \quad x_2 = 5.$

& $\quad x_1 + x_2 + 6x_3 = 7$

$\therefore \quad x_1 + 5 + \cancel{0} = 7$

$\therefore \quad \cancel{x}$

$x_1 = 2.$

✱ Elementary matrix :

If we operate elementary row operation on the Identity matrix, the resultant is called a elementary matrix.

Ex : $I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$R_2 \to R_2 - 2R_1$

$E = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

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NSO
Day 2

Guass elimination method.

$[A \mid b]$
$\downarrow$ Row operations.
$[U \mid \tilde{b}]$
$\hookrightarrow$ Upper triangular.

Note : a) If we get some row of $[U \mid \tilde{b}]$ of the form $(0,0,0 \mid t)$ $t \neq 0 \in \tilde{b}$, then system of equations has no sol^n.

b) If number of non zero rows = number of unknows then system of equations has unique sol^n.

c) If number of non zero rows < number of unknows then system has infinitely many sol^{n}.

* Invertible matrix :

1. A square matrix is invertible if $\exists$ a matrix B s.t. $AB = I = BA$, Then $A^{-1} = B$

2. $(AB)C = A(BC)$ → associative law.
3. $AB = BA$ is not always true.
4. $A(B+C) = AB + BC$.
5. $(A+B)C = AC + BC$

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Book → ① Gilbert Strange 4^{th} ed^n
② Hoffman & kunje

Q. To prove : $(AB)^{-1} = B^{-1}A^{-1}$

let $AB = C$.

$\therefore (AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1}$

$= AIA^{-1} = AA^{-1} = I$

Thus, $(AB)^{-1} = (B^{-1}A^{-1})$

Q. To prove : $(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$

$(ABC)(C^{-1}B^{-1}A^{-1}) = AB(CC^{-1})(B^{-1}A^{-1})$

$= ABI(B^{-1}A^{-1})$

$= A(BB^{-1})A^{-1} = I$.

Ex: $x_1 + 2x_2 - x_3 = 1$

$2x_1 + x_2 + 4x_3 = 2$

$3x_1 + 3x_2 + 4x_3 = 1$

$\Rightarrow \left[\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 2 & 1 & 4 & 2 \\ 3 & 3 & 4 & 1 \end{array}\right]$

→ $R_3 \to R_2 - R_3$

① $\left[\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 2 & 1 & 4 & 2 \\ -1 & -2 & 0 & 1 \end{array}\right]$

② $R_3 \to R_1 + R_3$

$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 2 & 1 & 4 & 2 \\ 0 & 0 & -1 & 2 \end{array}\right]$

$R_2 \to 4R_3 + R_2$

$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 2 & 1 & 0 & 10 \\ 0 & 0 & -1 & 2 \end{array}\right]$

$R_1 \to R_1 - R_3$

$\left[\begin{array}{ccc|c} 1 & 2 & 0 & -1 \\ \cancel{2} & \cancel{1} & \cancel{0} & \cancel{10} \\ 0 & 0 & -1 & 2 \end{array}\right]$

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$R_2 \to 2R_1 - R_2$

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & 2 & 0 & -1 \\ 0 & 3 & 0 & -12 \\ 0 & 0 & -1 & 2 \end{array}\right]$$

$x - 8 = -1$
$x = 7$

Thus $x_1 + 2x_2 = -1$ $\quad x_1 = 7$

$3x_2 = -12$ $\quad x_2 = -4$

$-x_3 = 2 \Rightarrow x_3 = -2$

* LU decomposition :

Guass jordan method :

Suppose we have the following system of L.E.

$AX = b \to$ (I)

Step-I. Write (I) in the following form:

$[A|b] \to$ (II)

Step-II. Convert (II) by using elementary row operations.

$[I|\tilde{b}]$

* In $AX = b \to$ (I)

If A is invertible the eq^n (I) has unique solution.

6 Q] Find A^{-1}

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{bmatrix}$$

$AB = I$

$$\therefore \begin{bmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{bmatrix} B = I \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} \cancel{6} & \cancel{-5} & \cancel{0} \\ \cancel{4} & \cancel{-6} & \cancel{0} \\ -2 & \cancel{7} & 2 \end{bmatrix} B = \begin{bmatrix} 2 & \cancel{0} & \cancel{-1} \\ \cancel{0} & \cancel{1} & \cancel{0} \\ \cancel{0} & \cancel{0} & \cancel{1} \end{bmatrix}$$

12 0 -6

0 5 0

$R_1 \times \cancel{7}$

& $\cancel{R_3 \times 5}$

42 -35 0

-14 35 10

$$\begin{bmatrix} \cancel{28} & \cancel{0} & 10 \\ \cancel{4} & \cancel{-6} & \end{bmatrix}$$

36 -30 0

~~20 30 0~~

$$\begin{bmatrix} 56 & & 0 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{bmatrix} = \begin{bmatrix} 12 & 0 & -6 \\ 0 & 5 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{bmatrix} = \begin{bmatrix} \frac{12}{56} & 0 & \frac{-6}{56} \\ 0 & 5 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

[ut 1]

Q.1]
$$x + y + z = 4$$
$$2x + 5y - 2z = 3$$
$$x + 7y - 7z = 5$$

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 2 & 5 & -2 & 3 \\ 1 & 7 & -7 & 5 \end{array}\right]$$

$$= \begin{bmatrix} 1 & 1 & 1 \\ 2 & 5 & -2 \\ 0 & -6 & 8 \end{bmatrix} \begin{bmatrix} 4 \\ 3 \\ -1 \end{bmatrix}$$

-6

$$= \left[\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 0 & 3 & -4 & -5 \\ 0 & 6 & -8 & 1 \end{array}\right] = \left[\begin{array}{cccc} 1 & 1 & 1 & 4 \\ 0 & -3 & 4 & 5 \\ 0 & 0 & 0 & 11 \end{array}\right]$$

Q]
$$2\sin x - \cos y + 3\tan z = 3$$
$$4\sin x + 2\cos y - 2\tan z = 10$$
$$6\sin x - 3\cos y + \tan z = 9.$$

$$\Rightarrow \left[\begin{array}{ccc|c} 2 & -1 & 3 & 3 \\ 4 & 2 & -2 & 10 \\ 6 & -3 & 1 & 9 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 2 & -1 & 3 & 3 \\ 2 & 1 & -1 & 5 \\ 0 & 0 & 8 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 2 & -1 & 3 & 3 \\ 4 & 0 & 2 & 8 \\ 0 & 0 & 8 & 0 \end{array}\right]$$

$2x - y + z = 3$

$\therefore$ $z = 0$.

$z = 0$.

$4 - y = 3$

$\therefore$ $4x + 2z = 8$

$\underline{y = 1}$

$\therefore$ $\underline{x = 2}$

$Ax = b$.
$A \to LU$
$LUx = b$.
$Ux = c$
$Lc = b$.

Q4) $A = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 2 & 2 \\ 3 & 4 & 5 \end{bmatrix}$, $b = \begin{bmatrix} 1 \\ 4 \\ 7 \end{bmatrix}$

$Ax = b$.
$A = LU$.
$LUx = b$
$Ux = c$
$LC = b$.

$A = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 2 & 2 \\ 3 & 4 & 5 \end{bmatrix}$, $b = \begin{bmatrix} 1 \\ 4 \\ 7 \end{bmatrix}$

$R_2 \to R_2 - 2R_1$ $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 3 & 4 & 5 \end{bmatrix}$ $\xrightarrow{R_3 \to R_3 - 3R_1}$ $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 4 & 2 \end{bmatrix}$ $\xrightarrow{R_3 \to R_3 - 2R_2}$ $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

$E = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $F = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix}$ $G \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -5 & 0 & 1 \end{bmatrix}$

1 0 0
0 1 0
0 0 1

A'

0 0 1
0 2 0

$G = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 0 \end{bmatrix}$

everytime on I and not on previous.

$$\begin{matrix} 0 & 0 & 1 \\ 0 & -2 & 0 \\ -1 & 0 & 0 \end{matrix} \qquad \begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix} \qquad \begin{matrix} 0 & 1 & 0 \\ -1 & 0 & 0 \end{matrix}$$

$$\begin{matrix} 0 & -2 & 1 \\ -1 & 2 & 1 \end{matrix}$$

practice.

~~$R_2 - 5R_1$~~ $\rightarrow$
$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & -2 & 1 \end{bmatrix}$$
$R_3 \rightarrow R_3 - 2R_2 - R_1$

* How to get inverse of Elementary matrix → ?

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Inverse of E.

write diagonal entries as it is and.

change sign of off diagonal entries.

$$GFEA = U = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$A = (GFE)^{-1}U$$

$$= \underbrace{(E^{-1}F^{-1}G^{-1})}_{L}U$$

$$\therefore A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$\Downarrow \qquad \Downarrow$$

$$L \qquad U$$

$c_1 = 0$

$c_2 = 2$

$c_3 = 2$

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ 7 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$x_3 = 0$

$x_2 = 1$

$x_1 = 1$

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$$

Q. 5] Gauss Jordan

$2x + y + z = 1$

$4x - 6y = 1$

$-2x + 7y + 2z = 1$

Q] Find the inverse of A

$$A : \begin{bmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{bmatrix}$$

we write → $[A \mid I]$

$$\left[\begin{array}{ccc|ccc} 2 & 1 & 1 & 1 & 0 & 0 \\ 4 & -6 & 0 & 0 & 1 & 0 \\ -2 & 7 & 2 & 0 & 0 & 1 \end{array}\right]$$

→ $R_2 \to R_2 + (-2)R_1$

$R_3 \to R_3 + R_1$

$$\Rightarrow \left[\begin{array}{ccc|ccc} 2 & 1 & 1 & 1 & 0 & 0 \\ 0 & -8 & -2 & -2 & 1 & 0 \\ 0 & 8 & 3 & 1 & 0 & 1 \end{array}\right]$$

$R_3 \to R_3 + R_2$

$$\left[\begin{array}{ccc|ccc} 2 & 1 & 1 & 1 & 0 & 0 \\ 0 & -8 & -2 & -2 & 1 & 0 \\ 0 & 0 & 1 & -1 & 1 & 1 \end{array}\right]$$

$R_1 \to \frac{1}{2} R_1$

$$\left[\begin{array}{ccc|ccc} 1 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & -8 & -2 & -2 & 1 & 0 \\ 0 & 0 & 1 & -1 & 1 & 1 \end{array}\right]$$

$R_2 \to -\frac{1}{8} R_2$

$$\left(\begin{array}{ccc|ccc} 1 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 1 & \frac{1}{4} & \frac{1}{4} & -\frac{1}{8} & 0 \\ 0 & 0 & 1 & -1 & 1 & 1 \end{array}\right)$$

$-\frac{1}{8} \quad -\frac{2}{8}$

$\frac{3}{8}$

$R_2 \to R_2 - \frac{1}{4} R_3$

$$\begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{2} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & -\frac{3}{8} & \frac{1}{4} \\ -1 & 1 & 1 \end{bmatrix}$$

$R_1 \to R_1 - \frac{1}{2} R_3$

$$\begin{bmatrix} 1 & \frac{1}{2} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{3}{8} & \frac{1}{4} \\ -1 & 1 & 1 \end{bmatrix}$$

$R_1 \to R_1 - \frac{1}{2} R_2$

$\frac{1}{2} + \frac{3}{16}$

$\frac{8}{16} \quad \frac{11}{1}$

$\frac{1}{8} - \frac{7}{8}$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{3}{4} & \frac{11}{16} & \frac{3}{8} \\ \frac{1}{2} & -\frac{3}{8} & \frac{1}{4} \\ -1 & 1 & 1 \end{bmatrix}$$

$\frac{1}{8} - \frac{1}{8} \times \frac{1}{4}$

$\frac{3}{8}$

ans $$\begin{bmatrix} \frac{3}{4} & -\frac{5}{16} & -\frac{3}{8} \\ \frac{1}{2} & -\frac{3}{8} & -\frac{1}{4} \\ -1 & 1 & 1 \end{bmatrix}$$

Q] Apply elimination to produce the factors L and U for $A = \begin{bmatrix} 3 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 3 \end{bmatrix}$

Let A $n \times n$ matrix. Then by elementary row operation A can be converted into the following form.

$$E_3 P_3 E_2 P_2 \; E_1 P_1 A = U.$$

$$\Rightarrow E_3' E_2' E_1' P_3 P_2 P_1 A = U;$$

where $E_3' = E_3$, $E_2' = P_3 E_2 P_3^{-1}$
$E_1' = P_3 P_2 E_1 P_2^{-1} P_3^{-1}$

Note that if A^{-1} exist then PLU decomposition exists. PLU need not be unique, it depends on the choice of P_i's.

Q] $A = \begin{pmatrix} 1 & 2 & 1 \\ 1 & 2 & 2 \\ 2 & 1 & 1 \end{pmatrix}$ PLU

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 2 & 2 \\ 2 & 1 & 1 \end{bmatrix}$$

$R_2 \to R_2 + (-1) R_1$

$$\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 0 & 1 \\ 2 & 1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} EA = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 0 & 1 \\ 2 & 1 & 1 \end{bmatrix}$$

$R_2 \Leftrightarrow R_3$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} EA = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$PEA = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} PEA = \begin{bmatrix} \; \end{bmatrix}$$

$R_2 \to R_2 - 2R_1$

$$\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} PEA = \begin{bmatrix} 1 & 2 & 1 \\ 0 & -3 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$GPEA = U$$

$$PA = LU$$

$$P_1 A = LU$$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

$$U = \begin{bmatrix} 1 & 2 & 1 \\ 0 & -3 & -1 \\ 0 & 0 & 1 \end{bmatrix} \quad P_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

Cholesky decomposition:

* $A = LU$
$= L \cdot L^t$

A should be symmetric matrix and A is positive definite matrix.

$x^t A X > 0 \quad \forall X \neq 0$

Ex: $A = \begin{bmatrix} 25 & 15 & -5 \\ 15 & 18 & 0 \\ -5 & 0 & 11 \end{bmatrix}$

$IA = A$

E $\quad R_2 \rightarrow R_2 + \left(-\frac{1}{25} \times 15\right) R_1$

F $\quad R_3 \rightarrow R_3 + \left(\frac{1}{5}\right) R_1$

$FEA = \begin{bmatrix} 25 & 15 & -5 \\ 0 & 9 & 3 \\ 0 & 3 & 10 \end{bmatrix}$

$R_3 \rightarrow R_3 + \left(-\frac{1}{3}\right) R_2$

$GFEA = \begin{bmatrix} 25 & 15 & -5 \\ 0 & 9 & 3 \\ 0 & 0 & 9 \end{bmatrix} = U$

where $E = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{3}{5} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $F = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \frac{1}{5} & 0 & 1 \end{bmatrix}$

$G = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -\frac{1}{3} & 1 \end{bmatrix}$

$A = LU$

$L = E^{-1} F^{-1} G^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ \frac{3}{5} & 1 & 0 \\ -\frac{1}{5} & \frac{1}{3} & 1 \end{bmatrix}$

$$\cancel{A = LU} = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{3}{5} & 1 & 0 \\ -\frac{1}{5} & \frac{1}{3} & 1 \end{bmatrix} \begin{bmatrix} 25 & 15 & -5 \\ 0 & 9 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} \cancel{25} & \cancel{15} & -5 \\ \cancel{-15} & \cancel{0} & \\ & & \end{bmatrix}$$

$$A = L \times I \times U. \qquad A = \begin{bmatrix} 1 & 0 & 0 \\ \frac{3}{5} & 1 & 0 \\ -\frac{1}{5} & \frac{1}{3} & 1 \end{bmatrix} \begin{bmatrix} 25 & 15 & -5 \\ 0 & 9 & 3 \\ 0 & 0 & 9 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ \frac{3}{5} & 1 & 0 \\ -\frac{1}{5} & \frac{1}{3} & 1 \end{bmatrix} \underbrace{\begin{bmatrix} 25 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix}}_{D} \underbrace{\begin{bmatrix} \frac{1}{25} & 0 & 0 \\ 0 & \frac{1}{9} & 0 \\ 0 & 0 & \frac{1}{9} \end{bmatrix}}_{D^{-1}} \underbrace{\begin{bmatrix} 25 & 15 & -5 \\ 0 & 9 & 3 \\ 0 & 0 & 0 \end{bmatrix}}_{U}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ \frac{3}{5} & 1 & 0 \\ -\frac{1}{5} & \frac{1}{3} & 1 \end{bmatrix} \begin{bmatrix} 25 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix} \begin{bmatrix} 1 & \frac{3}{5} & -\frac{1}{5} \\ 0 & 1 & \frac{1}{3} \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ \frac{3}{5} & 1 & 0 \\ -\frac{1}{3} & \frac{1}{3} & 1 \end{bmatrix} \begin{bmatrix} 5 & 0 & 0 \\ 6 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 5 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & \frac{3}{5} & -\frac{1}{5} \\ 0 & 1 & \frac{1}{3} \\ 0 & 0 & 1 \end{bmatrix}$$

HW:

1) $A = \begin{bmatrix} 4 & 12 & -16 \\ 12 & 37 & -43 \\ -16 & -43 & 98 \end{bmatrix}$

2) $\begin{bmatrix} 1 & 2 & 2 \\ 2 & 8 & 6 \\ 2 & 0 & 24 \end{bmatrix}$

* Binary operator:

* $S \times S \to S$, where $S \neq \phi$

$(S, *) \to$ algebraic structure.

$(Z, +)$

$(Z, \cdot)$

* Field:

Let F be an non empty set and $(F, \oplus, \odot)$ algebraic structure

Then $(F, \oplus, \odot)$ is called field if it satisfies the following

I) for all $x, y, z \in F$

$x \oplus (y \oplus z) = (x \oplus y) \oplus z$

II) There exist an element $y \in F$ s.t.

$x \oplus y = x = y \oplus x \quad \forall x \in F$

then y is called the indentity under $\oplus$ and it is denoted by zero.

(iii) for each $x \in F$, $\exists\, y \in F$ s.t.

$$x \oplus y = 0 = y \oplus x$$

y is called the inverse of x and is denoted by '$-x$'.

(iv) Commutative law hold i.e.

$$x \oplus y = y \oplus x \quad \forall\, x, y \in F$$

(v) for all $x, y, z \in F$, $x \odot (y \odot z) = (x \odot y) \odot z$

(associative law holds under multiplication.

(vi) There is an element $y \in x$ such that

$$x \odot y = x = y \odot x \quad \forall\, x \in F.$$

and we denote y by '1'.

(vii) for each $x \neq 0$, $\exists\, y \in F$ s.t.

$$x \odot y = 1 = y \odot x.$$

y is called multiplicative inverse and it is denoted by x^{-1}.

(viii) Commutative law hold under $\odot$,

$$x \odot y = y \odot x \quad \forall\, x, y \in F$$

(x) For all $x, y, z \in F$.

$x \odot (y \oplus z) = x \odot y + x \odot z$

Ex $(\mathbb{R}, +, \cdot)$, $(\mathbb{C}, +, \cdot)$, $(\mathbb{Q}, +, \cdot)$

(1) $(\mathbb{Z}, +, \cdot)$ is not a field $2 \in \mathbb{Z}$, $\nexists y \in \mathbb{Z}$

s.t. $2y = 1$

$y = \frac{1}{2} \notin \mathbb{Z}$.

Ex: $S = \{0, 1, 2, 3, 4\}$

$\times : S \times S \to S$

$+ : S \times S \to S$

$+ : (a, b)$ = remainder when $a+b$ is divided by

$\times : (a, b)$ = remainder when $a \cdot b$ divided by 5.

Is $(S, +, \times)$ a field p?

yachi

Multiplicative identity

ज्यावेळी answer 1 yene.

ex:

2 chi multiplicative identity is 3

because $2 \times 3 = 6 \div 5$

the remainder (1)

Group : $(S, *)$ is called group if (i), (ii), (iii) are satisfied in the definition of field.

(ii) $(S, *)$ is called abelian grp if (i), (ii), (iii) & (iv) are satisfied.

$\Rightarrow$ $[F, \oplus] \Rightarrow$ abelian

$(F - \{0\}, \odot)$ abelian.

$a \odot (b \oplus c) = a \odot b + a \odot c$

* Vector space :-

$F \to$ field.

V — Non empty set.

$(V, \oplus) \to$

$\oplus : V \times V \to V$

$\cdot \quad F \times V \to V$.

Let V be a non empty set V is called vector space over the field F if it satisfies the following.

(i) $(\alpha \oplus \beta) \oplus \gamma = \alpha \oplus (\beta \oplus \gamma) \ \forall \alpha, \beta, \gamma \in V$

(ii) There is an element $0 \in V$ s.t.

$0 \oplus \alpha = \alpha \oplus 0 = \alpha \quad \forall \alpha \in V$,

0 is called the identity $\oplus$ in V.

(iii) for each $\alpha \in V$, $\exists$ an eliment $\beta \in V$ s.t.

$$\alpha \oplus \beta = 0 = \beta \oplus \alpha,$$

β is called additive inverse under $\oplus$ of α and is denoted by $-\alpha$.

(iv) $\alpha \oplus \beta = \beta \oplus \alpha \quad \forall\ \alpha, \beta \in V$ (commutative law).

For vector space, V should be ~~abelian group~~ and should follow (1), (2), (3) & (iv).

(v) $c \odot \alpha \in V$ (under scalar multiplication it is closed)

(vi) $1 \odot \alpha = \alpha \quad \forall\ \alpha \in V$

(vii) $(c_1 \cdot c_2) \odot \alpha = c_1 \odot c_2 \odot \alpha \quad \forall\ c_1, c_2 \in F, \alpha \in V$.

(viii) $(c_1 + c_2) \odot \alpha = c_1 \odot \alpha \oplus c_2 \odot \alpha$

$\forall\ c_1, c_2 \in F, \alpha \in V$

(ix) $c \odot (\alpha \oplus \beta) = c \odot \alpha \oplus c \odot \beta$

Note: (i) The element of the field F is called <u>Scalar</u>.

(ii) The element of V is called as <u>vector</u>

(iii) The operation $\oplus$ is called vector addition

(iv) The operation $\odot$ is called scalar multiplication

Prove that :

If $V(F)$ is a vector space. Then

(a) Prove that additive identity is unique.

(b) Prove that additive inverse is unique.

Proof : (a) let $0, 0_1$ be two additive identity under $\oplus$ on V.

since 0 is additive identity

$$0 \oplus \alpha = \alpha \oplus 0 = \alpha \quad \forall \alpha \in V.$$

so,

$$0 \oplus 0_1 = 0_1 \oplus 0 = 0_1 \rightarrow (1)$$

so, $\alpha \oplus 0_1 = \alpha = 0_1 + \alpha \quad \forall \alpha$

Thus $0 = 0_1$

(b) let $\alpha \in V$ and β, γ be two additive inverse of α.

$$\alpha \oplus \beta = 0 = \beta \oplus \alpha \rightarrow (1)$$

$$\alpha \oplus \gamma = 0 = \gamma \oplus \alpha \rightarrow (ii)$$

$$\beta = \beta \oplus 0 = \beta \oplus (\alpha + \gamma) \text{ by } (2)$$

$$= (\beta + \alpha) \oplus \gamma \quad \text{as associative law holds}$$

$$= 0 \oplus \gamma \quad (\text{By } (1))$$

$$= \gamma$$

Ex. (1) : The space of n - tuple $\mathbb{R}^n$.

$$V = \mathbb{R}^n = \{ (a_1, a_2, \ldots, a_n) \mid a_i \in \mathbb{R} \}$$

$$\oplus : V \times V \rightarrow V$$

let $\alpha, \beta \in V = \mathbb{R}^n$.

$\alpha = (a_1, a_2 \cdots, a_n)$ and $\beta = (b_1, b_2 \cdots, b_n)$

$\alpha \oplus \beta = (a_1+b_1, a_2+b_2, \cdots, a_n+b_n)$

$c \odot \alpha = (ca_1, ca_2 \cdots, ca_n)$

~~let $\alpha \in V$~~ and

Ex 2) $V = \{ A_{m \times n} \mid a_{ij} \in R \}$.

$\oplus : (A \oplus B)_{ij} = (a_{ij} + b_{ij})$

$(c \odot A)_{ij} = (c a_{ij})$.

Ex: 3

let S be a non empty set and F be a field.

$V = \{ f \mid f : S \to F \}$.

The vector addition on V and scalar multiplication defined as follows.

$(f \oplus g)(x) = f(x) + g(x) \quad \forall x \in S$

and $c \in F$, $f \in V$, $(c \odot f)(x) = c \cdot f(x)$.

Q* Prove that V is a vector space over the field F.

Proof: It is closed under $\oplus$, $(f \oplus g)(x)$
$= f(x) + g(x) = (f+g)x$.

(2) Associative law holds, we need to show that $(f \oplus g) \oplus h = f \oplus (g \oplus h)$

$$((f \oplus g) \oplus h)(s) = (f \oplus g)(s) + h(s)$$
$$= f(s) + g(s) + h(s) \quad \forall s \in S$$
$$= f(s) + (g(s) + h(s))$$

(since $f(s), g(s), h(s) \in F$, and in field associative law under addition holds).

$$(f \oplus g) \oplus h = f \oplus (g \oplus h)$$

$$(-f)(x) = -f(x)$$

Tuts

solve by cholesky decomposition.

Q]
$$A = \begin{bmatrix} 1 & -1 & 2 \\ -1 & 5 & -4 \\ 2 & -4 & 6 \end{bmatrix}$$

$R_2 \to R_2 + R_1$, $R_3 \to R_3 - 2R_1$

$$\begin{bmatrix} 1 & -1 & 2 \\ 0 & 4 & -2 \\ 0 & -2 & 2 \end{bmatrix}$$ $R_3 \to \frac{1}{2}R_2 + R_3$

Elementary

$$\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ -2 & 0 & 1 \end{pmatrix} \quad G = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \frac{1}{2} & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 2 \\ 0 & 4 & -2 \\ 0 & 0 & 1 \end{bmatrix} = U$$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 2 & -\frac{1}{2} & 1 \end{bmatrix}$$

$$D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \sqrt{D} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

vector space

Q] 1) $R^2(R)$

$(a_1, a_2) + (b_1, b_2) = (a_1 + b_1, 0)$

$(a_1, a_2) + (e_1, e_2) = (a_1, a_2)$

$(a_1 + e_1, 0) = (a_1, a_2)$

Q]
$R^3(R)$

$(a_1, a_2, a_3) + (b_1, b_2, b_3) = (a_1, b_1 + a_2 b_2 + a_3 b_3)$

Q.2] $R^+(R)$.

$u \oplus v = uv$

$\alpha \otimes u = u^\alpha$

$\Rightarrow$ sol^n.

(ii) $(u \oplus v) \oplus w = (uv) \oplus w$

$= uvw$

$= uvw$

$= u \oplus (v \oplus w)$

(iii) $v = 1$

$u \oplus v = u \cdot 1 = u$

(iv) $u \oplus v = uv = 1 = e$

$\boxed{v = u^{-1}}$

(v) $u \oplus v = uv = vu = v \oplus u$

(vi) $\alpha \otimes u = u^\alpha \in R^+$

(vii) $\alpha \otimes (\beta \otimes u) = \alpha \otimes (u^\beta)$

$= (u^\beta)^\alpha = u^{\beta\alpha}$

$= u^{\alpha\beta}$

$= (\alpha\beta) \otimes u$

(viii) $1 \otimes u = u$.

① $u \oplus v \in V$

2] $(u \oplus v) \oplus w$

$= u \oplus (v \oplus w)$

3] $\exists e$ s.t. $u \oplus e$

$= e \oplus u$

$= u \quad \forall u \in V$

iv] For each $u \in V$ $\exists u' \in V$

s.t. $u \oplus u' = u' \oplus u$

$= e$

v] $u \oplus v = v \oplus u$

vi] $\alpha \otimes u \in V$

vii] $\alpha \otimes (\beta \otimes u)$

$= (\alpha\beta) \otimes u$

viii] $1 \otimes u = u \quad \forall u \in V$.

ix] $\alpha \otimes (u \oplus v) = (\alpha \otimes u) \oplus (\alpha \otimes u)$

x] $(\alpha \oplus \beta) \otimes u$

$= (\alpha \otimes u) \oplus (\beta \otimes u)$

Shailesh kumar Tiwari.

Q] $R^2(R)$.

i) $(a_1, a_2) \oplus (b_1, b_2)$

$= (a_1 + b_2, a_2 + b_1)$.

$\Rightarrow (b_1 + b_2) \oplus (a_1, a_2)$

$= (b_1 + a_2, b_2 + a_1)$

Thus it doesn't for vector space.

Q.4] i) $\alpha \otimes 0 = 0 \quad \forall \alpha \in F \qquad 0 \in V$

supposing $V(+)$ is vector space.

$\alpha \otimes (0 \oplus 0) = (\alpha \otimes 0) \oplus (\alpha \otimes 0)$

$\alpha \otimes 0 = 0 = (\alpha \otimes 0) \oplus (\alpha \otimes 0)$

$\alpha \otimes 0 = 0$

*

$V \neq \phi$ set.

$\oplus : V \times V \to V$ ($\alpha \oplus \beta \in V$ $\forall \alpha, \beta \in V$)

(i) $(\alpha \oplus \beta) \oplus \gamma = \alpha \oplus (\beta \oplus \gamma)$
$\forall \alpha, \beta, \gamma \in V$

(ii) $\exists\, 0$, $0 \oplus \alpha = \alpha = 0 \oplus \alpha$ $\forall \alpha \in V$.

(iii) For each $\alpha \in V$, $\exists\, \beta \in V$
$\alpha \oplus \beta = 0 = \beta \oplus \alpha$.

(iv) $\alpha \oplus \beta = \beta \oplus \alpha$ $\forall \alpha, \beta \in V$.

$\odot : F \times V \to V$
i.e. $c \in F$, $\alpha \in V$
$c \odot \alpha \in V$.

(v) $1 \odot \alpha = \alpha$ $\forall \alpha \in V$
$\in F$

(vi) $(c_1 c_2) \odot \alpha = c_1 \odot (c_2 \odot \alpha)$
$\forall c_1, c_2 \in F$, $\alpha \in V$.

(vii) $(c_1 + c_2) \odot \alpha = c_1 \odot \alpha \oplus c_2 \odot \alpha$

(viii) $c \odot (\alpha \oplus \beta) = c \odot \alpha \oplus c \odot \beta$
$\forall c \in F$,
$\alpha, \beta \in V$.

Ex. (3) let F be a field.

$(f \oplus g)(s) = f(s) + g(s)$ $V = \{f ; f : F \to F\}$

$\oplus : V \times V \to V.$

$(f \oplus g)(x) = f(x) + g(x) \ \forall x \in F$

$(c \odot f)(x) = cf(x) \ \forall x \in F$

Ex let $V = R^2 = \{(x, y) \mid x, y \in R\}$

$\oplus$: $(x_1, x_2) \oplus (y_1, y_2)$

$= (x_1 + y_1 + 11, x_2 + y_2 - 3)$

$\odot$: $\alpha \odot (x_1, y_1) = (\alpha x_1 + \alpha - 1, \alpha y_1 - 3\alpha + 3)$

Q) Prove under $\oplus$, and scalar multiplicator $\odot$, V (is a vector space over the field R)

(1) Associative law holds in V under $\oplus$

let $(x_1, x_2), (y_1, y_2), (z_1, z_2) \in V$.

Now $(x_1, x_2) \oplus ((y_1, y_2) \oplus (z_1, z_2))$

$$= (x_1, x_2) \oplus (y_1 + z_1 + 1, y_2 + z_2 - 3)$$

$$= (x_1 + y_1 + z_1 + 1 + 1, x_2 + y_2 + z_2 - 3 - 3) \rightarrow (1)$$

Now,

we want additive identity

$$(x_1 + y_1 + 1, x_2 + y_2 - 3) = (x_1, x_2)$$

$$\Rightarrow y_1 = -1, y_2 = 3$$

$\Rightarrow (-1, 3)$ is the zero vector

* Existence of additive inverse

suppose (y_1, y_2) is the additive inverse of (x_1, x_2).

$\therefore (x_1, x_2) \oplus (y_1, y_2) = (-1, 3)$

$(x_1 + y_1 + 1), x_2 + y_2 - 3) = (-1, 3)$

$\Rightarrow x_1 + y_1 + 1 = -1$ & $x_2 + y_2 - 3 = 3$

$\Rightarrow y = -2 - x$, , $y_2 = 6 - x_2$

$(y_1, y_2) = (-2 - x_1, 6 - x_2)$

New polynomial

⇒ let F be a field and $f : F \to F$ which have a rule
$f(x) = a_0 + a_1 x + \cdots + a_n x^n$,
where $a_i \in F$
f, is called polynomial funct.

largest $n \in N$ s.t. $a_n \neq 0$ then $\deg f = n$.

let $V = \{ p(x),$ polynomial function over the field $F \}$

⊕ : $p(x) \oplus q(x) = (a_0 + b_0) + (a_1 + b_1)x + \cdots + (a_n + b_n)x^n$

where $p(x) = a_0 + a_1 x + \cdots + a_n x^n$
$q(x) = b_0 + b_1 x + \cdots b_n x^n$.

⊙ : $cp(x) = ca_0 + ca_1 x + \cdots + ca_n x^n$

For proving subspace :

We need to show that .

(1) $W \neq \phi \subseteq V$.

(2) $\forall \alpha, \beta \in W, \quad c\alpha + \beta \in W \quad \forall c \in F$.

Ex (1) let $V(F)$ and $W = \{0\} \subseteq V$. Then W is subspace of V.

(2) let V be a vector space over the field F. Then $W = V$ is a subspace of V.

Ex. (3) let $V = \{ [a_{ij}]_{n \times n} \mid a_{ij} \in R \}$ be a vector space over the field R. Suppose $W = \{ A \in V \mid A^t = A \}$ prove that W is a subspace of V.

$\Rightarrow$

Cond (1) To prove that $W \neq \phi \subseteq V$.

Since $O_{n \times n} = \begin{bmatrix} 0 & 0 & 0 \\ & \ddots & \\ 0 & & 0 \end{bmatrix}_{n \times n} \in V$

we know that

$0^t = 0$

$\Rightarrow 0 \in W$ i.e $W \neq \phi$.

let A and $B \in W$.

~~Now~~ $(cA + B)^t = (cA)^t + B^t$

$= cA^t + B^t$

$c \in R, A, B \in W$

$= cA + B$.

$\therefore (CA+B) \in W.$

$\therefore W$ is a subspace of V.

Ex. 3 Prove.

$W = \{A \in V \mid A^t = -A\}$ — same question

Sin $0 \in W \neq \phi$

let $A, B \in W$ i.e.

$A^t = -A$ & $B^t = -B$.

Now,

$$(CA+B)^t = CA^t + B^t$$
$$= -CA - B$$
$$= -(CA+B)$$

$\therefore W$ is a subspace of V.

Ex. let $V = \{f \mid f : R \to R\}$ be a vector space over the field R. Suppose that

$$V_e = \{f \in V \mid f(-x) = f(x) \ \forall x \in R\}$$
$$= \{\text{all even functions of } R\}$$

Prove that V_e is a subspace of V.

$\Rightarrow$ rro orm.

$0 \in V_e \; \therefore V_e \neq \phi$.

Now, we need to prove
$c\cancel{A} + B \in V_e$.

$0(-x) = 0 \quad \forall x \in R$.

$\Rightarrow 0 \in V_e \neq \phi$.

let f and g $\in V_e$.

$f(-x) = f(x)$
& $g(-x) = g(x)$.

$\therefore \cancel{cf(-x) + g(-x) =}$

$(cf+g)(-x) = cf(-x) + g(-x)$
$= cf(x) + g(x)$

$= (cf+g)x$

$(cf+g)(-x) = = (cf+g)x$

$\Rightarrow cf+g \in V_e$

$\therefore$ ~~W~~ is subspace over V

prove subspace or not

Ex) let $V = \{ [a_{ij}]_{n \times n} \mid a_{ij} \in R \}$ be a vector space over the field R.

$W = \{ A \in V \mid \text{trace } A = 0 \}$.

Ex 2. $W_2 = \{ A \in V \mid \text{trace } A = 1 \}$ X

Ex 3 $W_3 = \{ A \in V \mid \text{s.t. } a_{11} \geq 0 \}$ X.

Ex 4. $V = R^n (R)$

$W_1 = \{ (a_1, a_2, a_n) \in R^n \mid a_1 = a_n = a_2 \}$

* Let V be a vector space over the field F.

Suppose that W_1 and W_2 are two subspaces of V. Prove that $W_1 \cap W_2$ is a subspace of V.

proof: Since W_1 and W_2 are subspaces of V, hence $0 \in W_1$ and $0 \in W_2$

$\Rightarrow 0 \in W_1 \cap W_2 \neq 0.$

Let $\alpha \in W_1 \cap W_2$ & $\beta \in W_1 \cap W_2$

$\Rightarrow \alpha \in W_1$ & $\alpha \in W_2$ & $\beta \in W_1$ & $\beta \in W_2$.

$c\alpha + \beta \in W_1$

& $c\alpha + \beta \in W_2$

$\therefore \left[c\alpha + \beta \in W_1 \cap W_2 \right]$

$\therefore W_1 \cap W_2$ is a subspace over the vector space V.

Let V be a vector space over the field F. The intersection of any collection of subspaces of V is a subspace.

P.f. Let $\{W_t\}$ be a collection of subspace of V.

& $0 \in W_t \ \forall t \Rightarrow 0 \in \cap_t W_t$

$\cap W_t \neq \phi$

$\alpha, \beta \in W_t \ \forall t$

Th

Union of two subspaces need not to be subspace.

Ex. $V = \mathbb{R}^2 = \{(x, y) \mid x, y \in \mathbb{R}\}$

be a vector space over the field $\mathbb{R}$.

Let $W_1 = \{(x, 0) \mid x \in \mathbb{R}\}$ and $W_2 = \{(0, y) \mid y \in \mathbb{R}\}$

are two subspaces of V. But $W_1 \cup W_2$ is not a subspace.

$$W_1 \cup W_2 = \{(x, 0) \cup (0, y) \mid x, y \in \mathbb{R}\}$$

$$\alpha = (1, 0), \quad \beta = (0, 1) \in W_1 \cup W_2$$

$$\alpha + \beta = (1, 1) \notin W_1 \cup W_2$$

Thm ⇒ Let V be a vector space over the field F. Suppose W_1 and W_2 are two subspaces of V.

Then $W_1 \cup W_2$ is subspace of V iff one is contained in other.

either $W_1 \subseteq W_2$ or $W_2 \subseteq W_1$.

Proof : let $W_1 \cup W_2$ is a subspace of V then we have to prove that either

$$W_1 \subseteq W_2 \text{ or } W_2 \subseteq W_1$$

let $\alpha \in W_1$ & $\beta \in W_2$.

$$\alpha \text{ \& } \beta \in W_1 \cup W_2$$

$\therefore \alpha + \beta \in W_1 \cup W_2$ (As $W_1 \cup W_2$ is subspace.

$\Rightarrow$ either $\alpha + \beta \in W_1$ or $\alpha + \beta \in W_2$.

If $\alpha + \beta \in W_1$ and $\alpha \in W_1$,

$-\alpha \in W_1$.

(W_1 is subspace $\alpha \in W_1 \Rightarrow -\alpha \in W_1$)

$\Rightarrow (-\alpha) + (\alpha + \beta) \in W_1$

$\Rightarrow (-\alpha + \alpha) + \beta \in W_1$

$\Rightarrow 0 + \beta \in W_1$

$\Rightarrow \beta \in W_1 \Rightarrow W_2 \subseteq W_1$

($\because \beta \in W_2$ $\because \beta$ is arbitrary)

Some problems :

$V = \{f : R \to R\}$

$V_e = \{f \mid f \text{ is even}\}$

$V_o = \{f \mid f \text{ is odd function}\}$.

Is $V_e \cup V_o$ a subspace ?

$\Rightarrow$ Sol^n $\Rightarrow$ Here neither $V_e \subseteq V_o$, nor $V_o \subseteq V_e$,

Thus it is not a subspace.

☆ Sum of Subspaces.

Let $W_1, W_2 \dots, W_K$ be subspaces of a vector space V over the field F. The sum of $W_1, \dots W_K$ is defined as

$$W = W_1 + W_K + \dots + W_K = \{\alpha_1 + \alpha_2 + \dots + \alpha_K \mid \alpha_i \in W_i \ \forall\ i = 1, 2, \dots, K\}$$

Prove that W is a subspace of V.

Sol^n : Since $W_i \subseteq W \Rightarrow W \neq \phi$ $\because j = 1, \dots, k$,

suppose, $\alpha, \beta \in W$

since $\alpha \in W$ so $\exists$ $\alpha_1, \alpha_2, \dots, \alpha_K$ s.t.

$\alpha = \alpha_1 + \alpha_2 + \dots + \alpha_K$. $\alpha_i \in W_i = i = 1, 2, 3, \dots$

$\beta = \beta_1 + \beta_2 + \dots + \beta_K$, $\beta_i \in W_i$, $i = 1, 2, 3, \dots$

Now, $c\alpha + \beta = (c\alpha_1 + c\alpha_2 + \dots + c\alpha_K) + (\beta_1 + \beta_2 + \dots + \beta_K)$

where $\alpha_i, \beta_i \in W_i$, $i = 1, 2, \dots K$.

But ~~under vector~~ addition, it is commutative.

i.e. $\alpha + \beta = \beta + \alpha \ \forall\ \alpha, \beta \in V$.

$= (c\alpha_1 + \beta_1) + (c\alpha_2 + \beta_2) + \cdots + (c\alpha_k + \beta_k)$

$\quad \in W_1 \qquad \in W_2 \qquad \in W_k.$

$\Rightarrow c\alpha + \beta \in W.$

$\Rightarrow W$ is subspace of V.

Gn: $V = \{ A_{n\times n} \mid a_{ij} \in \mathbb{R} \}$ $\quad V(\mathbb{R})$.

$W_1 = \{ A \in V \mid A^t = A \}$

$W_2 = \{ A_1 \in V \mid A^t = -A \}$.

find $W_1 + W_2$.

Gn- $V = \{ f \mid f: \mathbb{R} \to \mathbb{R} \}$, $V(\mathbb{R})$

$V_e = \{ f \mid f$ is even $\}$

$V_o = \{ f \mid f$ is odd $\}$ find $V_e + V_o$.

Gn: let $V = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \,\middle|\, a,b,c,d \in \mathbb{R} \right\}$. $V(\mathbb{R})$ is vector space over the field $\mathbb{R}$ under the following operation

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \oplus \begin{bmatrix} x & y \\ z & w \end{bmatrix} = \begin{bmatrix} a+x & b+y \\ c+z & d+w \end{bmatrix}$$

$$t \odot \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} ta & tb \\ tc & td \end{bmatrix}$$

Now, $W_1 = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \middle| \ a,b,c \in \mathbb{R} \right\}$, $W_2 = \left\{ \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \middle| \ a,b,c, \in \mathbb{R} \right\}$.

Find sum $W_1 + W_2$

$$W_1 + W_2 = \left\{ \begin{bmatrix} a & b \\ c & 0 \end{bmatrix} + \begin{bmatrix} x & y \\ 0 & z \end{bmatrix} = \begin{bmatrix} a+x & b+y \\ c+0 & 0+z \end{bmatrix} \right\}$$

$a, x, b, y, c, z, \in \mathbb{R}$.

we want to show that $V = W_1 + W_2$

$V \subseteq W_1 + W_2$.

and $W_1 + W_2 \subseteq V$

suppose $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in V$ but.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ c & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & d \end{bmatrix}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in W_1 + W_2$$

$$\Rightarrow V \subseteq W_1 + W_2 \rightarrow (1)$$

since $W_1 + W_2$ is subspace of V,

$\Rightarrow W_1 + W_2 \subseteq V \rightarrow$ (2).

from (1) & (2), $W_1 + W_2 = V$.

★ Span (S) :

let V be the vector space over the field F and S be a subset V. The subspace spanned by S is defined to be the intersection $\cap$ of all subspaces of V which contains S.

denoted $L(S) = \bigcap_i W_i$

Remark $W = L(S) = \text{span}(S)$ is the smallest subspace of V containing S.

Linear combination of vectors

A vector $\beta \in V$ is said to be a LCOC $\alpha_1, \alpha_2, \ldots \alpha_n$.

$\exists$ scalars $c_1, c_2, c_3 \ldots c_n \in F$ s.t.

$$\beta = c_1 \alpha_1 + \ldots + c_n \alpha_n$$

$$\beta = \sum_{i=1}^{n} c_i \alpha_i$$

$V = R^3$,

$\alpha_1 = (1, 1, 1)$, $\alpha_2 (1, 1, 0)$, $(\alpha = 3, 0, 1)$

★ Missed after class - 2

Important facts :

① for any non-empty set except $\{0\}$. if field has infinite no. of elements, then $L(S)$ has infinite number of elements.

② $L(S)$ is the smallest subspace of V containing S.

③ $L(S) = \cap_i W_i$, where W_i is the subspace of V containing S.

④ If $L(S) = W$, then S is called generator of W.

⑤ If S_1 and S_2 are two different sets, $L(S_1)$ & $L(S_2)$ may be equal.

⑥ How you can decide a vector $\beta \in V$, belongs to the in the span of $(\alpha_1, \alpha_2, \dots, \alpha_n)$ ★

suppose $c_1, \dots, c_n \in F$ and

$$\beta = c_1\alpha_1 + c_2\alpha_2 + \dots + c_n\alpha_n$$

$$\Rightarrow [\alpha_1, \alpha_2, \dots, \alpha_n] \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = \beta$$

$$AX = \beta$$ ★

Ex : let V be a vector space of all polynomials over the field F.
let S be the subset of V consisting of the polynomials function $f_0, f_1, \dots, f_n, \dots$ defined by $f_n(x) = x^n$, $n = 0, 1, 2, \dots n$.
Then prove that $L(S) = V$

Soln

Sol^n: Clearly $L(S) \subseteq V \rightarrow ①$

let $\alpha \in V$

$\alpha = a_0 + a_1 x + \cdots + a_n x^n$

$= a_0 f_0(x) + a_1 f_1(x) + \cdots + a_n f_n(x)$

$= (a_0 f_0 + a_1 f_1 + \cdots + a_n f_n)(x)$

$\alpha = L(S)(x)$

$\therefore \alpha \in L(S)$

$\therefore V \subseteq L(S) \rightarrow ②$

$\therefore L(S) = V. \rightarrow$ from ① & ②.

☆ Row space of $A_{m \times n}$.

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ a_{m1} & \cdots & \cdots & a_{mn} \end{bmatrix} \Rightarrow R_1 = (a_{11}, a_{12}, \ldots, a_{1n}) \in F^n$$

$S = \{R_1, R_2, \ldots, R_m\} \subseteq F^n$

$L(S)$ is called row space of A.

☆ Linearly dependent

A set of vectors $(\alpha_1, \ldots, \alpha_n)$ is said to be L.D. if there are scalars $c_1, c_2, \ldots, c_n \in F$, not all zero s.t.

$c_1\alpha_1 + c_2\alpha_2 + c_3\alpha_3 \cdots + c_n\alpha_n = 0.$

Ex:
$V = \mathbb{R}^2$, $S = \{ \underset{\alpha_1}{(1,1)}, \underset{\alpha_2}{(2,2)} \}$.

$c_1\alpha_1 + c_2\alpha_2 = \underline{\underline{0}}$.

$(c_1 + 2c_2, c_1 + 2 \cdot c_2) = (0,0)$

$c_1 = -2, \quad c_2 = 1$

one sol^n

So, there are ∞ sol^n of the eq^n
Hence it is Linear dependent.

Ex: $V = \mathbb{R}^2$

$S = \{ (1,1), (1,0), (0,1) \}$

Is S an L.D. ?

$c_1\alpha_1 + c_2\alpha_2 + c_3\alpha_3 = 0 = (0,0)$

x : put 1, 1, 0
y : & then 1, 0, 1
for c_1, c_2, c_3
respectively
to get

$(c_1 + c_2, c_1 + c_3) = (0,0)$

$\Rightarrow c_1 + c_2 = 0, \quad c_1 + c_3 = 0$.

take $c_3 = 1, c_1 = -1$
& $c_2 = 1$.

$\cancel{\beta_2}$

* Linearly independent.

A set of vectors $\{\alpha_1, \dots, \alpha_n\}$ is said to be L.I. if,
$c_1\alpha_1 + \dots + c_n\alpha_n = 0$

for $\Rightarrow c_i = 0 \;\forall\; i = 1, 2, \dots, n$.

Ex → $V = \mathbb{R}$.

$S = \{1\}$

So here, S is L.I. Because $c_1 \alpha_1 = 0$ then $c = 0$ only sol^n.

Ex (2) $S_2 = \{1, 2\}$

here $c_1 + 2c_2 = 0$

thus ∞ sol^n thus L.D.

Note ✱ : (1) Any set containing $\underline{0}$ vector is L.D.

(2) If S is L.D. and $S \subseteq S_1$, then S_1 is also linearly Dependent.

(3) Any subset of L.I. set is L.I.

($S_1 \subseteq S$, S is L.I. then S_1 is L.I.)

(4) Suppose S is infinite set then, S is L.I. iff every subset of S is L.I.

✱ Basis :

let V be a vector space over the field F. A subset S of V is called basis of V If.

(1) S is L.I.

(2) $L(S) = V$

* Dimension of vector space.

⇒ let V be a vector space over the field F and S be a basis of V.

⇒ Cardinality of S is called dimension of V.

i.e. $|S| = \dim V$.

⇒ $\dim V$ = no. of elements in S.

Example: let $V = R$

Suppose $S_1 = \{5\}$, $S_2 = \{8\}$

$S_3 = \{5, 6\}$

$S_4 = \{-1\}$

prove that.

① S_1 is basis ②, S_2 is a basis

③ S_3 is not a basis but $L(S)$

Example

let $V = R^2$, $R^2(R)$

$S_1 = \{(1,0), (0,1)\}$, $S_2 = \{(1,1), (1,0)\}$,

$S_3 = \{(1,1), (0,1)\}$, $S_4 = \{(1,1), (1,2)\}$.

$S_5 = \{(1,2), (2,5)\}$, $S_6 = \{(1,1), (1,0), (3,4)\}$

$S_7 = \{(5,6)\}$

prove that ①: S_1, S_2, S_3, S_4, S_5 are basis of V and find dim V.

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(2) S_6, S_7 are not not basis of V.

Tutorial

Tut sheet 1-2.

* $V = W_1 \oplus W_2$ $\left\{ \begin{array}{l} W_1 + W_2 = V \\ W_1 \cap W_2 = \{0\} \end{array} \right\}$.

Q.1)

$u \in V$

$u = u_1 + u_2$ let $u = u_1' + u_2'$

$u_1 + u_2 = u_1' + u_2'$

$\underbrace{u_1 + (-u_1')}_{\in W_1} = \underbrace{u_2' + (-u_2)}_{\in W_2}$

$\therefore u_1 + (-u_1')$ & $u_2' + (-u_2) \in W_1 \cap W_2$

But $W_1 \cap W_2 = 0$

$\therefore u_1 = u_1'$

& $u_2 = u_2'$

$S = \{a_1, a_2 \dots a_n\} \subseteq V$

$L(S) = \{$ ~~set of~~ all possible l.c. of elements of $S \}$

$= \{\alpha_1 a_1 + \alpha_1 \dots + \alpha_s a_s \mid \alpha_i \in F\}$

$= \{ \sum_{i=1}^{s} \alpha_i a_i \mid \alpha_i \in F \}$

Q.1 2) $S = \{(1, 2, 3), (1, 1, -1), (3, 5, 5)\}$

whether
1) $(0, 0, 0)$
2) $(1, 1, 0)$
3) $(4, 5, 0)$
4) $(1, -3, 5)$

$\in L(S)$

$\notin L(S)$

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let $(0,0,0) \in L(S)$

$(0,0,0) = \alpha(1,2,3) + \beta(1,1,-1) + \gamma(3,5,5)$

$\therefore \alpha + \beta + 3\gamma = 0$

$2\alpha + \beta + 5\gamma = 0$

& $3\alpha - \beta + 5\gamma = 0$

$\therefore -\alpha + 2\beta = 0$

$\therefore \boxed{\beta = \frac{\alpha}{2}} \to ①$

$\therefore$ from ② & ③,

$\cancel{5\alpha + 5\beta + 15\gamma = 0}$

$\cancel{8\alpha + 3\beta + 15\gamma = 0}$

$-\cancel{\alpha}$ $\quad 9\alpha - 3\beta + 15\gamma = 0$

$5\alpha + 5\beta + 15\gamma = 0$

$\therefore \cancel{4\alpha - 8\beta = 0}$

$5\alpha + 10\gamma = 0$

$\therefore \alpha = -2\gamma \to ①$

$\therefore \alpha + \frac{\alpha}{2} + 3\left(\frac{\alpha}{-2}\right) = 0$

$\alpha = 0, \beta, \gamma = 0.$

2:) $(1,1,0)$

$\alpha + \beta + 3\gamma = 1$

$\therefore 2\alpha + \beta + 5\gamma = 1$

$3\alpha - \beta + 5\gamma = 0.$

$\therefore \alpha + 2\gamma = 0$

$\therefore \frac{-\alpha}{2} = \gamma$

$\therefore 5\alpha + 10\gamma = 1 \to ①$

$5\alpha + 10\left(\frac{-\alpha}{2}\right) = 1$ No soln.

$0 = 1$

Q. 13) $(i+i, 1-i) = \alpha(1+i, 1) + \beta(1, 1-i)$

$\therefore\ i+1 = \alpha(i+1) + \beta \quad = \alpha i + \alpha + \beta$

$1-i = \alpha + \beta(1-i) \quad = \alpha + \beta - \beta i$

$\therefore\ 2i = \alpha i + \beta i$

$\therefore\ \underline{\underline{2 = \alpha + \beta}}$

$1 - i = 2 - \beta i$

$\therefore\ 1 - 2 - i = -\beta i$

$\therefore\ -1 - i = -\beta i$

$\therefore\ \cancel{1+i = \beta i}$

$1 + i = \beta i$

$\therefore\ 1 - i = \beta$

$\alpha = 1 + i$

Q. 14]. ii) M is a subspace of $V \Leftrightarrow L[M] = M$.

$\Rightarrow$

$M, N \subseteq V$

i) $M \subset N$

$L(M) \subset L(N)$

$x \in L(M)$

$x = \alpha_1 a_1 + \alpha_2 \cdots + \alpha_s a_s$

$= \alpha_1 a_1 + \cdots + 0 \cdot (s+1) + 0 \cdot a_t$

$\in L(N)$

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2) M is a subspace,

$L[M] = M$

$x \in L(M)$

$x = \alpha_1 a_1 + \alpha_2 a_2 + \cdots + \alpha_s a_n \quad \in M.$

$\alpha_1 a_1 + \alpha_2 a_2 + \cdots \alpha_s a_s \in M$

$\alpha_1 a_1, \alpha_2 a_2, \alpha_3 a_3 \cdot \in M$

$a_1, a_2 \cdots a_s \in M.$

3] $L(L[M]) = L[M]$.

$\Downarrow$

$= \alpha \sum_{i=0}^{s} \alpha_i a_i \quad a, \alpha_i \in F.$

$= \sum_{i=0}^{s} \underbrace{\alpha \alpha_i}_{\beta} a_i, \quad \alpha, a_i \in F$

$x = \alpha \alpha_1 a_1 + \alpha \alpha_2 a_2 + \cdots \alpha \alpha_s a_s$

$= \beta_1 a_1 + \beta_2 a_2 \cdots + \beta_s a_s$

$= \in L[M].$

★ $R^n(R)$

$W = \{ \underset{\alpha_1}{(1,0,\dots,0)}, \underset{\alpha_2}{(0,1,0,\dots,0)}, \dots, \underset{\alpha_n}{(0,0,\dots,0,1)} \}$

Prove that W is a basis of R^n.

Sol^n : suppose we have scalars

$a_1, a_2, \dots, a_n \in R$ s.t.

$a_1\alpha_1 + a_2\alpha_2 + a \dots a_n\alpha_n = 0$

$\Rightarrow (a_1, a_2, \dots, a_n) = (0, 0, \dots 0)$.

$\Rightarrow a_i = 0 \ \forall\ i = 1, \dots, n$.

W is L.I.

2) $R^n = L(W)$

let $B \in R^n$

$B = (x_1, x_2 \dots x_n) \in R^n$.

suppose $\exists$ scalars $a_1\ a_2 \dots a_n$ s.t.

$\beta = \sum_{i=1}^{n} a_i\alpha_i$

$\Rightarrow (x_1, x_2 \dots x_n) = (a_1, \dots a_n)$

$\Rightarrow a_i = x_i \ \forall i = 1, 2, 3 \dots, n$

$B \in L'(W)$

$\Rightarrow R^n \subseteq L(W)$

& $L(W) \subseteq R^n$

$\therefore L(W) = R^n$.

let $V = \{ A_{m\times n} \mid a_{ij} \in \mathbb{R} \}$ be a vector space over the field $\mathbb{R}$.

$$W = \left\{ \begin{bmatrix} 1 & 0 & 0 & \cdots \\ 0 & 0 & 0 & \cdots 0 \\ 0 & \cdots & & 0 \end{bmatrix}_{m\times n}, \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \ldots, \begin{bmatrix} \cdots & 0 & 0 & 0 \\ & 0 & 0 & \\ 0 & 0 & 0 & 1 \end{bmatrix}_{m\times n} \right\}$$

$$\dim V = mn = |W|$$

$$\dim W = \frac{n(n-1)}{2} \quad \text{where } W = \{ A \in V \mid A^T = -A \}$$

$$\dim(W + W_1) = \dim(W) + \dim(W_1) - \dim(W \cap W_1)$$

Ex. let $V = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \,\middle|\, a, b, c, d \in \mathbb{R} \right\}$ be a vector space over $\mathbb{R}$

$$W_1 = \left\{ \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \,\middle|\, a, b, c \in \mathbb{R} \right\}$$

$$W_2 = \left\{ \begin{bmatrix} a & b \\ c & 0 \end{bmatrix} \,\middle|\, a, b, c \in \mathbb{R} \right\}$$

basis of W_1 $= \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$

basis of $W_2 = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \right\}$

Basis $w = \{\alpha_1, \alpha_2 \dots \alpha_n\}$.

let V be a vector space and β be a ordered basis of V.

$$\beta = \{\alpha_1, \alpha_2 \dots, \alpha_n\}$$

Q] for any $\alpha \in V$, $\exists$ scalars $a_1, a_2, \dots, a_n \in F$ s.t.

$$\alpha = a_1\alpha_1 + a_2\alpha_2 + \dots + a_n\alpha_n$$

$\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$ is called co-ordinate of α with respect to ordered basis β.

$$[\alpha]_\beta = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$$

Q] coordinate of α with respect to ordered basis $\beta = \{\alpha_1 \dots, \alpha_n\}$ is unique.

Sol^n $\Rightarrow$ suppose.

$$\alpha = \sum_{i=1}^{n} a_i\alpha_i \rightarrow (1)$$

$$\alpha = \sum_{i=1}^{n} c_i\alpha_i \rightarrow (2)$$

By (1) & (2) $\Rightarrow \sum_{i=1}^{n} (a_i - c_i)\alpha_i = 0$

$\Rightarrow a_i - c_i = 0 \quad \forall i = 1, 2, \dots, n$

$\Rightarrow a_i = c_i \quad \forall i$.

Q] Let S be a L.I. subset of a vector space V, suppose $v \in L(S)$. Then prove that $\{v\} \cup S$ is L.D.

Sol^n : given $v \in L(S)$; suppose $\exists$ $\alpha_1, \alpha_2, \dots \alpha_n \in S$ and $a_1, a_2, \dots a_n \in F$ s.t.

$V =$

$$v = a_1\alpha_1 + a_2\alpha_2 + \dots + a_n\alpha_n$$

$$\Rightarrow 1 \cdot v - a_1\alpha_1 - a_2\alpha_2 - \dots - a_n\alpha_n = 0.$$

$$\Rightarrow \{v\} \cup \{\alpha_1, \dots, \alpha_n\} \text{ L.D}$$

$$\Rightarrow \{v\}$$

Q] Let S be a L.I. subset of a vector space V and $v \notin L(S)$. prove that $\{v\} \cup S$ is L.I.

Sol^n $\Rightarrow$ let $\exists$ $c_1, c_2, \dots c_n \in F$ and $\alpha_1, \alpha_2, \dots, \alpha_n \in S$.

s.t. $c_1\alpha_1 + c_2\alpha_2 + \dots + c_n\alpha_n + cv = 0$, $c \in F$.

suppose $c \neq 0$ then $(v = -(c_1\alpha_1 + \dots + c_n\alpha_n)$

$$\Rightarrow v = -c^{-1}(c_1\alpha_1 + \dots + c_n\alpha_n)$$

$$v = -c^{-1}c_1\alpha_1 - c^{-1}c_2\alpha_2 - \dots - c^{-1}c_n\alpha_n$$

$v \in L(S)$

gives a contradiction

Thus $c = 0$. & $c_i = 0 \ \forall \ i = 1 \dots n$.

$v \cup S$ is L.I.

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If V is a vector space over the field F and B is a basis of V, If no. of element in B is finite then V is called finite dimensional vector space other wise infinite dimensional vector space.

$\Rightarrow$ $C(R) \longrightarrow$ finite dimension (dim = 2)

$R(R) \longrightarrow$ finite dimension (dim = 1)

$M_{m\times n}(R) \longrightarrow$ finite dimensions (dim mn)

$V = \{M_{m\times n} \mid a_{ij} \in C\}$, $V(C)$ is a vector space of dim mn.

$V = \{A_{m\times n} \mid a_{ij} \in C\}$. Then $V(R)$ is a vector space with dim '2mn'.

Theorem: Let V be a vector space over the field F and $B = \{\alpha_1, \alpha_2, \ldots, \alpha_n\} \subseteq V$ s.t. span (B) = V. Then any set containing more than 'n' vectors is L.D.

$\Rightarrow$ Sketch of proof:

Assume $B_j = \{B_1, B_2, \ldots, B_m\}$ $m > n$

$$B_j = \sum_{i=1}^{n} a_{ij}\alpha_i, \quad j = 1, 2, \ldots m.$$

$$x_i B_i + \cdots x_m B_m = \sum_{j=1}^{m} x_j B_j = \sum_{j=1}^{m}\sum_{i=1}^{n} a_{ij}\alpha_i$$

Ques.* let $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots \\ a_{m1} & a_{m2} & & a_{mn} \end{bmatrix}_{m\times n}$

$R_1 = (a_{11} \; a_{12} \; \ldots \; a_{1n}) \in F^n$

$\vdots$

$R_n = (a_{m1} \ldots \; a_{mn}) \in F^n$

Row space of A = span $\{R_1, R_2 \ldots R_m\}$ is a subspace of F^n

dim Row space of A = number of linearly independent vectors in S.

colown space of A : $S = \{c_1, c_2, \dots, c_n\}$ where c_j are columns of A.

Then column space of A = span (S), $c_i \in F^m$

Basis is a smallest spanning set.
Basis is a largest L.I. set.

Null space of $A_{m \times n}$: $AX = 0$

Null space of $A = \{x_{n\times 1} \in F^n \mid AX = 0\}$

Q.] If $W = \{x_{n\times 1} \in F^n \mid AX = 0\}$
Is W a subspace of F^n

$W \neq \phi$, $X, Y \in W$
$cX + Y \in W$ | $A(cX+Y) = cAX + AY = 0$.

★ Echelon form (Row Echelon Form (REF))
Suppose $A_{m\times n}$

→ Any matrix $A_{m\times n} \in F^{m\times n}$ can be reduced in the following form $F^{m\times n}$
(by using elementary row operations)

① All non zero rows are above any rows of all zeroes.

② The pivots are the first non zero entries in a non zero rows.

③ All entries in a column below a pivots entry are zero.

④ Each pivots of a row in a column to be the right of the pivot of the row above it.

* Row Reduced Echelon form (RREF):

(1) $A_{m\times n}$

Steps: (1) Reduced A in Echelon form.

(2) All pivots are only non zero entry in that column.

(3) All the pivots are 'one' (1).

⇒ Rank of Echelon matrix = no. of pivots in it.

Today 6/2/20

* $A_{m\times n}$.

Row space of A is a subspace of F^n.

Column space of A is a subspace of F^m.

$\dim C(A) \leq \min\{m,n\}$.

$\dim R(A) \leq \min\{m,n\}$

$N(A) = \{x \in F^n \mid AX = 0\}$

Null space of A or solution space of A is a subspace of F^n.

$N(A^T) = \{x \in F^m \mid A^T X = 0\}$.

Null space of A^T is a subspace of F^m

$A_{m\times n}$, $R_1, R_2 \ldots, R_m$ are rows of A and $C_1, C_2, \ldots, C_n$ are columns of A. Then,

(1) $R_i \cdot X = 0$, where $X \in N(A)$.

$R_i = (a_{i1}\ a_{i2} \ldots a_{in}) \begin{vmatrix} x_1 \\ x_2 \\ \vdots \end{vmatrix} = (a_{i1}x_1 + a_{i2}x_2 \cdots + a_{in}x_n)$

Row space

Rank of A = no. of pivots in RREF.

Row rank of A = dim R(A) = dim C(A) = column rank of (A)

rank (A) = dim R(A) = dim C(A).

* dim Null space of A is called nullity of A.

Example : Find a basis of column space, Row space and Null space of :

$$A = \begin{bmatrix} 1 & 3 & -1 & 2 & 3 \\ -2 & -1 & 2 & 1 & 1 \\ -1 & 2 & 1 & 3 & 4 \\ 0 & 5 & 0 & 5 & 7 \end{bmatrix}_{4\times 5}$$

$R_2 \to R_2 + 2R_1$

$R_3 \to R_3 + R_1$

$$\begin{bmatrix} 1 & 3 & -1 & 2 & 3 \\ 0 & 5 & 0 & 5 & 7 \\ 0 & 5 & 0 & 5 & 7 \\ 0 & 5 & 0 & 5 & 7 \end{bmatrix} \xrightarrow{R_2 \to R_2/5} \begin{bmatrix} 1 & 3 & -1 & 2 & 3 \\ 0 & 1 & 0 & 1 & 7/5 \\ 0 & 5 & 0 & 5 & 7 \\ 0 & 5 & 0 & 5 & 7 \end{bmatrix}$$

$R_3 \to R_3 - 5R_2$

$R_4 \to R_4 - 5R_2$ $\to$

$$\begin{bmatrix} 1 & 3 & -1 & 2 & 3 \\ 0 & 1 & 0 & 1 & 7/5 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$R_1 \to R_1 - 3R_2$

$$\begin{array}{c} \alpha_1 \leftarrow \\ \alpha_2 \leftarrow \\ \\ \\ \end{array} \begin{bmatrix} 1 & 0 & -1 & -1 & -\frac{4}{5} \\ 0 & 1 & 0 & 1 & 7/5 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Basis of row space is α_1 & α_2

Basis of row space $= \{(1, 0, -1, -1, -\frac{4}{5}), (0, 1, 0, 1, \frac{7}{5})\}$

dim $R(A) = 2$.

Basis of column space of $A = \left\{ \begin{pmatrix} 1 \\ 2 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} -3 \\ -1 \\ 2 \\ 5 \end{pmatrix} \right\}$

dim $C(A) = 2$

$\therefore$ rank $= 2$

$AX = 0 \iff A'X = 0$, where A' is RREF of A.

$$\begin{bmatrix} 1 & 0 & -1 & -1 & 6/5 \\ 0 & 1 & 0 & 1 & 7/5 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

~~$x_5 \in R$~~

~~$x_4 = 0$~~

$x_1 - x_3 - x_4 - \frac{4}{5} x_5 = 0 \Rightarrow x_1 = x_3 + x_4 + \frac{6}{5} x_5$

$x_2 + x_4 + \frac{7}{5} x_5 = 0 \qquad x_2 = -x_4 - \frac{7}{5} x_5$

$$N(A) = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} \middle| AX = 0 \right\}, \quad \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} x_3 + x_4 + \frac{6}{5} x_5 \\ -x_4 - \frac{7}{5} x_5 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}$$

$$= x_3 \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} \frac{6}{5} \\ -\frac{7}{5} \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Basis

#

#

Q

Basis of null space of A

$= \{ (1,0,1,0,0), (1,-1,0,1,0), (\frac{6}{5}, -\frac{7}{5}, 0, 0, 1) \}$

Dim null space (A) = 3 = nullity of A,

\# dim C(A) + dim $N(A^T) = m$.

\# If A m x n matrix then, rank A + nullity of A = n.

rank A + nullity of $A^T = m$.

\# If rank of $A_{m \times n}$ is r then, dim $N(A) = n - r$.

Q.] Find a basis spanned by a set

$S = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$

$\alpha_1 = (1, -2, 0, 3, -4) \in R^5$.

$\alpha_2 = (3, 2, 8, 1, 4) \in R^5$.

$\alpha_3 = (2, 3, 7, 2, 3) \in R^5$.

$\alpha_4 = (-1, 2, 0, 4, -3) \in R^5$.

$$A = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{bmatrix} \begin{array}{l} \rightarrow \text{1st row} \\ \rightarrow \text{2nd row} \\ \rightarrow \text{3rd row} \\ \rightarrow \text{4th row.} \end{array}$$

Basis of subspace spanned by $\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$ = Basis of row space of A.

Example : $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}_{3\times 2}$ → make this 0, 0

then it comes in RREF.

$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$\therefore A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$

Thus basis of ~~row~~ space = $\{(1,0), (0,1)\}$.

dim $R(A) = 2$.

dim $N(A) = 0$.

$N(A) = \{(0,0)\}$.

For basis of column space come from original matrix

$= \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$ dim $C(A) = 2$.

Basis of $N(A^T) = \{$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\therefore \; x_1 + x_3 = 0 \quad \& \quad x_2 + x_3 = 0$

$x_1 = -x_3$

$x_2 = -x_3$

$$\therefore \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -x_3 \\ -x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$

$$\text{Basis of } N(A^T) = \left\{ \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \right\}$$

$$\dim N(A^T) = 1$$

Tutorial :

Q.1) (i). $\{e^x, e^{3x}\}$ in $C^\infty(\mathbb{R})$.

$c_1 e^x + c_2 e^{3x} = 0$ $\quad e^x = 1 + x + \frac{x^2}{2!} + \cdots$

a) $(c_1+c_2) + x\,(c_1+3c_2) + \frac{x^2}{2!}(c_1+9c_2) + \cdots = 0$

$c_1 + c_2 = 0 \rightarrow (1)$

$c_1 + 3c_2 = 0 \rightarrow (2)$

$2c_2 = 0 \Rightarrow c_2 = 0.$

$c_1 = 0.$

b) $\{x, |x|\}$ in $C[-1,1]$.

$c_1 x + c_2|x| = 0$ $\quad |x| = \begin{cases} x, & x \geq 0 \\ -x, & x \leq 0 \end{cases}$

for $x \geq 0$

$c_1 x + c_2 x = 0$

for $x < 0$

$c_1 x - c_2 x = 0$

$\therefore c_1 = c_2$

$\therefore c_1 = c_2 = 0.$

$\therefore$ L.I.

Q.1) (ii) $\{\sin x, \sin 2x, \cdots \sin nx\}$ in $[-\pi, \pi]$

$c_1 \sin x + c_2 \sin 2x + \cdots + c_n \sin nx = 0$

$\int_{-\pi}^{\pi} c_1 \sin^2 x + \int_{-\pi}^{\pi} c_2 \sin x \sin 2x + \cdots + \int_{-\pi}^{\pi} c_n \sin x \sin nx = \int_{-\pi}^{\pi} 0.$

$\Rightarrow c_1 + 0 + \cdots + 0 = 0$

$\therefore c_1 = 0$

similarly, $c_2 = 0$, & $c_n = 0$.

Q.2) (i).

$\{u+v, v+w, w+u\}$

given $\{u, v, w\}$ is L.I. set.

$c_1(u+v) + c_2(v+w) + c_3(w+u) = 0.$

$(c_1+c_3)u + (c_1+c_2)v + (c_2+c_3)w = 0.$

$c_1 + c_3 = 0 \quad \Rightarrow c_2 = 0$

$c_1 + c_2 = 0 \quad\quad c_3 = 0$

$c_2 + c_3 = 0. \quad\quad c_1 = 0.$

Q.2) (ii). $S_1 \subseteq S_2$.

(1) S_1 L.D $\Rightarrow S_2$ L.D.

(2) S_2 L.I $\Rightarrow S_1$ L.I.

Q.3) $S = \{v_1, v_2, \dots, v_n\}$ is L.D. & $v_1 \neq 0$.

$\exists$ v_k s.t.

$$v_k = \sum_{i=1}^{k-1} c_i v_i.$$

$c_1 v_1 + c_2 v_2 + \dots + c_n v_n = 0$

$\exists i$ s.t. $c_i \neq 0$.

Take highest index k s.t. $c_k \neq 0$

$c_1 v_1 + \dots + c_k v_k = 0.$

$c_k v_k = -c_1 v_1 - c_2 v_2 - \dots - c_{k-1} v_{k-1}$

$v_k = -c_k^{-1} c_1 v_1 - c_k^{-1} c_2 v_2 \dots - c_k^{-1} c_{k-1} v_{k-1}$

$v_k = L[v_1, v_2, \dots, v_{k-1}]$

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Q.4] $V = R^3_{\mathbb{R}}$ $F = R$.

(IV) (IV) $\{1, x-2, (x-2)^2, (x-2)^3\}$.

$c_1 + c_2(x-2) + c_3(x-2)^2 + c_4(x-2)^3 = 0$.

$c_1 + c_2 x - 2c_2 + c_3\{x^2 + 4 - 4x\}$
$+ c_4[x^3 - 8 - 6x^2 + 12x]$.

$(c_1 - 2c_2 + 4c_3 - 8c_4) = 0$.
$+ x(c_2 - 4c_3 + 12c_4) + x^2(c_3 - 6c_4)$
$+ c_4 x = 0$.

Q.5 i ii)

B) $= \{(2,1,0), (2,1,1), (2,2,1)\}$

a) $(1, 2, -1)$

$(1, 2, -1) = c_1(2,1,0) + c_2(2,1,1) + c_3(2,2,1)$

$1 = 2c_1 + 2c_2 + 2c_3 \rightarrow (1)$.

$2 = c_1 + c_2 + 2c_3 \rightarrow (2)$

$-1 = c_2 + c_3 \rightarrow (3)$

$\Rightarrow c_3 = -1 - c_2$.

Substitute in (1) & (2)

$1 = 2c_1 + 2c_2 + 2(-1-c_2)$

$2 = c_1 + c_2 + 2(-1-c_2)$

$1 = 2c_1 + 2c_2 - 2 - 2c_2$.

$2 = c_1 + c_2 - 2 - 2c_2$

$2c_1 = 3 \Rightarrow c_1 = \frac{3}{2}$

$4 = c_1 - c_2$

$c_2 = c_1 - 4 = \frac{3}{2} - 4 = \frac{3-8}{2} = \frac{-5}{2}$

$c_3 = -1 + \frac{5}{2} = \frac{3}{2}$

Q. 8) (iii) $\{ (x_1, x_2, x_3, x_4) \in \mathbb{R}^4 ;$

$x_1 + x_2 + 2x_3 = 0$

$2x_2 + x_3 = 0 \quad \& \quad x_1 - x_2 + x_3 = 0$

$\Downarrow$

$x_3 = -2x_2 .$

i.e.

$x_1 + x_2 + 2x_3 = 0 \rightarrow (1)$

$2x_2 + x_3 = 0 \rightarrow (2)$

$x_1 - x_2 + x_3 = 0 \rightarrow (3)$

$x_1 - 3x_2 = 0 \rightarrow (4)$

Thus taking $x_1 = 1$,

$(1, \frac{1}{3}, \frac{-2}{3}, 0) \rightarrow$ by taking $x_1 = 1$,

$\& \ (0,0,0,1) \}$

$B = \{ (1, \frac{1}{3}, \frac{-2}{3}, 0), (0,0,0,1) \}$

Q. 10] Find a basis for $U, W, U \cap W, U + W$.

(i) $U = \{ (x_1, x_2, x_3) : x_1 + x_2 + x_3 = 0 \}$

$x_3 = -x_1 - x_2$

$B_U = \{ (1,0,-1), (0,1,-1) \}$

$W = \{ (x_1, x_2, x_3) : 2x_1 + x_2 = 0 \}$

$x_2 = -2x_1 \quad , B_W = \{ (1,-2,0), (0,0,1$

$-(-2x_1)$

$U \cap W \Rightarrow x_3 = -x_1 + 2x_1$

$x_3 = -x_1 + 2x_1 = +x_1$

$x_3 = x_1$

$B_{U \cap W} = \{(1, -2, 1)\}$

10/2/20) → written in hemant's copy

Q] Solve :

$5x_1 - 3x_2 + 2x_3 = 3$

$2x_1 + 4x_2 - x_3 = 7$

$x_1 - 11x_2 + 4x_3 = 3$

$$\Rightarrow \left[\begin{array}{ccc|c} 5 & -3 & 2 & 3 \\ 2 & 4 & -1 & 7 \\ 1 & -11 & 4 & 3 \end{array}\right]$$

$$\left(\begin{array}{ccc|c} 5 & -3 & 2 & 3 \\ 2 & 4 & -1 & 7 \\ 1 & -11 & 4 & 3 \end{array}\right)$$

$$\frac{5}{2} \quad \left(\begin{array}{ccc|c} 5 & -3 & 2 & 3 \\ 2 & 4 & -1 & 7 \\ 0 & 26 & -9 & 11 \end{array}\right) \quad 7 \times \frac{5}{2} \quad \frac{35}{2} - 7$$

$$\frac{5}{2} \times 4 / 2 \quad 10$$

$$-\frac{5}{2} \quad 2 + \frac{5}{2} \quad 9 \quad \left(\begin{array}{ccc|c} 5 & -3 & 2 & 3 \\ 0 & -13 & \frac{9}{2} & \frac{21}{2} \\ 0 & 26 & -9 & 11 \end{array}\right) \quad \frac{35 - 14}{2} \quad \frac{21}{2}$$

* linear transformation.

Suppose V and W are two vector spaces over the same field F. A linear transformation T is a map from V to W. (i.e. $T: V \to W$) such that (i) $T(\alpha+\beta) = T(\alpha) + T(\beta)$.

and (ii) $T(c\alpha) = cT(\alpha) \;\forall\; \alpha, \beta \in V, c \in F$

or

$$T(c\alpha+\beta) = cT(\alpha) + T(\beta) \;\forall\; \alpha, \beta \in V,\ c \in F$$

Ex (1) : let V be a vector space over the field F.

$T = I : V \to V$, $T(\alpha) = \alpha \;\forall\; \alpha \in V$ is a linear tran. (L.T).

(ii) $O: V \to V$ s.t. $O(\alpha) = 0 \;\forall\; \alpha \in V$.

Ex: let $V = R^2$, $T: V \to V$ such that

$$T(x, y) = (x+y, x-y)$$

Prove that T = L.T.

Soln :

let $\alpha = (x_1, x_2)$, $\beta = (y_1, y_2) \in R^2$ and $c \in F = R$.

$$c\alpha + \beta = (cx_1 + y_1, cx_2 + y_2)$$

$$T(c\alpha + \beta) = T(cx_1 + y_1, cx_2 + y_2)$$
$$= (cx_1 + y_1 + cx_2 + y_2,\ cx_1 + y_1 - cx_2 - y_2) \to (1)$$

$$cT(\alpha) + T(\beta) = c(x_1 + x_2, x_1 - x_2) + (y_1 + y_2, y_1 - y_2)$$
$$= (cx_1 + cx_2 + y_1 + y_2,\ cx_1 - cx_2 + y_1 - y_2) \to (2)$$

$\Rightarrow (1) = (2)$

$\therefore$ T is L.T.

prove that :

$T: V \to W$, L.T. then $T(0_v) = 0_w$

sol^n : $0_v \in V$, $0_v + 0_v = 0_v$

$T(0_v) = T(0_v + 0_v) = T(0_v) + T(0_v)$

$0_w + T(0_v) = T(0_v) + T(0_v)$

$\Rightarrow 0_w = T(0_v)$

by ~~adding~~ $-T(0_v)$ on both sides we get

2] let $A_{m \times n}$ be fixed matrix and $V = F^{n \times 1}$, $W = F^{m \times 1}$

Define a $T: V \to W$ s.t. $T(X) = AX$.
Prove that T is a L.T.

sol^n : let $\alpha = x$, $\beta = y \in F^n$ and $c \in F$.

we have to show that :

$T(c\alpha + \beta) = cT(\alpha) + T(\beta)$

$T(cx + y) = cT(x) + T(y)$

$\therefore T$ is L.T.

1st person → 103

Q] $T : R^2 \to R^3$ defined by
$T(x, y) = (x+2y, x-3y, x)$

prove that T is a L.T.

→

$T : V \to W$.
B is a basis of V such that $T(B)$ is known,
$B = \{\alpha_1, \alpha_2, \ldots, \alpha_n)$ and
$T(\alpha_1), T(\alpha_2) \ldots, T(\alpha_n)$ are given.

let $\alpha \in V$, $\exists\ c_1, c_2, \ldots, c_n \in F$ s.t.
$\alpha = c_1\alpha_1 + c_2\alpha_2 + \ldots + c_n\alpha_n$.
$T(\alpha) = T(c_1\alpha_1 + c_2\alpha_2 + c_n\alpha_n)$
$= c_1T(\alpha_1) + c_2T(\alpha_2) + \ldots + c_nT(\alpha_n)$

Ext suppose $T : R^2 \to R^2$ such that
$T(1,0) = (1,1)$ and $T(1,1) = (-1,2)$

Find the linear Transformation.

⇒ let $\alpha \in R^2$.
$\therefore\ \alpha = (x, y) \in R^2$, Aim is to find $T(x,y)$

sol^n: $B = \{(1,0), (1,1)\}$; a basis of R^2.
$(x,y) = c_1(1,0) + c_2(1,1)$
$= (c_1+c_2, c_2) \Rightarrow c_2 = y\ \ c_1 = x-y$

$(x,y) = (x-y)(1,0) + y(1,1)$
$T(x,y) = (x-y)\,T(1,0) + y\,T(1,1)$
$= (x-y)(1,1) + y(-1,2)$

• $T(x,y) = (x-2y, x+y)$

Q]

find L.T. $T: \mathbb{R}^2 \to \mathbb{R}^3$ s.t.

$T(1,2) = (5,0,0)$, $T(1,1) = (4,0,0)$

Sol^n: Since $B = \{(1,2),(1,1)\}$ is a basis of $\mathbb{R}^2$

For any $(x,y) \in \mathbb{R}^2$ $\exists$ $c_1, c_2 \in \mathbb{R}$ s.t.

$(x,y) = c_1(1,2) + c_2(1,1)$.

$c_1 = y - x$, $c_2 = 2x - y$)

$(x,y) = (y-x)(1,2) + (2x-y)(1,1)$.

$T(x,y) = (y-x)\,T(1,2) + (2x-y)\,T(1,1)$

$= (y-x)(5,0,0) + (2x-y)(4,0,0)$,

$= (3x+y, 0, 0)$

* Let T be a L.T. from V into W

$T: \mathbb{R}^2 \to \mathbb{R}^2$ s.t.

$T(x,y) = (x+y, x-y)$

$B_1 = \{(1,0),(0,1)\}$. find a matrix representation of T

w.r.t. B_1.

Sol^n:

$T(1,0) = (1,1)$ | Coordinate of $T(1,0) = c_1(1,0) + c_2(0,1)$

$T(0,1) = (1,-1)$ | $(1,1) = (c_1, c_2)$

$\Rightarrow c_1 = 1, c_2 = 1$

$A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = [T]_{B_1}$

$B_1 = \{(1,1), (1,2)\}$ is a basis of R^2

and $B' = \{(1,1,0), (0,0,1), (0,1,0)\}$

is a basis of R^3.

find a matrix representation of T w.r.t. basis B_1, B_1'.

Sol^n: find $T(1,1)$ and $T(1,2)$. then

$A = \Big[[T(1,1)]_{B_1'} \; [T(1,2)]_{B_1'} \Big]_{3\times 2}$