Linear First order ODEs

General form

$$a(x)\, y' + b(x)\, y = c(x) \quad -① \qquad \boxed{a(x) \neq 0}$$

It is linear because y and y' appear in linear form and there is not product of y and y' in the equation.

→ here equation looks like linear equation $a y_1 + b y_2 = c$

→ Remember y' is not a variable like in linear equation. It is rather than a function

Here we assume y is dependent variable and x is independent. It is obvious choice as y appears to be more restricted than variable x, because $a(x)$, $b(x)$ and $c(x)$ are arbitrary continuous functions. Why continuous - we will see in exist. and Uniq. theory.

Remember in practice, very often ind. var. is time 't'

$\boxed{\text{existence uniqueness}}$

Standard form

$$\boxed{y' + p(x)\, y = q(x)} \quad - ②$$

we can always convert ① int ② provided $a(x) \neq 0$ in the domain where solution is being searched, i.e. validity interval of ODE.

Recall in last lecture, we have used the following form of standard first order ODE

$$y' = f(x, y)$$

hence ② is in standard form as it can be written as

$$y' = -p(x)\, y + q(x). \quad - ③$$

But almost all authors write stand. form ② because here more emphasis is given to the word *linear*.

But you can write it in form ③ but then you must be cautious about the -ive sign of $p(x)$.

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Modeling

Population Growth Model :-

Let $x(t)$ be the population size of a city at time t.

Let b and d be birth and death rates. per individual per unit time.

Then

$$x(t+\Delta t) - x(t) = (b\,x(t) - d\,x(t))\,\Delta t$$

$$\Rightarrow \quad \Delta x(t) = (b-d)\,x(t)\,\Delta t$$

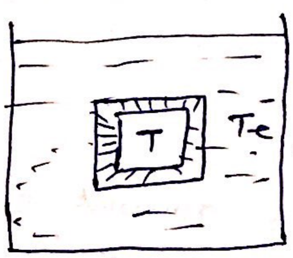
$$\Rightarrow \quad \frac{dx}{dt} = (b-d)\,x$$

$$= a x \quad (\text{say } b-d = a > 0)$$

↳ for increasing growth

$$\Rightarrow \quad \boxed{\frac{dx}{dt} - a x = 0}$$

Heat Conduction Model :-



T: temperature (inside)

T_e: temp. (outside)

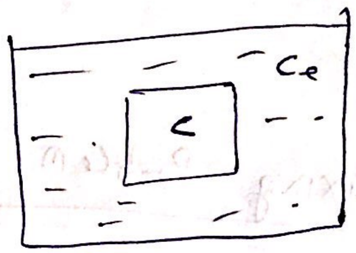
t: time

By Newton's cooling law :-

$$\frac{dT}{dt} = K(T_e - T)$$

$$\boxed{\frac{dT}{dt} + KT = KT_e}$$

Diffusion Model



C_e: concentration outside

C: concentration inside.

Fick's law of diffusion

$$\frac{dC}{dt} = K(C_e - C)$$

↳ rate of change of the concentration across a thin membrane

$$\boxed{\frac{dC}{dt} + KC = KC_e}$$

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Bank account model :-

$x(t)$: money in a A/c at time t.

$I(t)$: Interest rate (per money & year)

$q(t)$: deposit rate (per time)

so,

change of $x(t)$:-

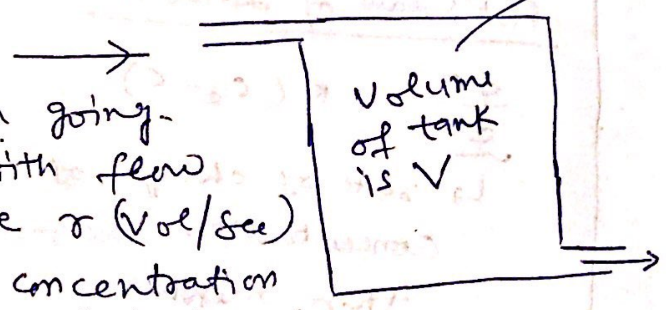
$$x(t+\Delta t) - x(t) = I(t)\, x(t)\, \Delta t + q(t)\, \Delta(t)$$

$$\Rightarrow \frac{\Delta x(t)}{\Delta t} = I(t)\, x(t) + q(t)$$

$$\Rightarrow \boxed{\frac{dx}{dt} - I(t)\, x(t) = q(t)}$$

Model for linear Equation.

Mixing Problem



$x(t) \equiv$ amount of salt in tank at time t.

Water going in with flow rate r (vol/sec) and concentration of salt in incoming water is c_e (gm/vol)

Water is coming out with flow rate r (same as incoming flow rate). Here concentration of salt in outflow water is same a concentration of fixed water inside the tank at time t.

Then

$$\frac{dx}{dt} = \boxed{\text{Inflow rate of Salt}} - \boxed{\text{outflow rate of Salt}}$$

$$= \underset{(\text{vol/sec})(\text{gm/vol})}{r c_e} - r\frac{x}{V} \Rightarrow \boxed{\frac{dx}{dt} + \frac{r}{V} x = r c_e}$$

How can we solve linear ODE of Ist order

Step ⓪ First write given ODE in standard form

Dot execute it in application

$$\boxed{y' + p(x)\, y = q(x)} \quad \text{--- } ①$$

This step is to justify step ① only

Step Ⓟ Find an IF for ①.

i.e. find a function, say $u(x)$, such that LHS of ① becomes complete derivate of a function after multiplying $u(x)$ to both sides of ①.

i.e.

$$u y' + u p y = u q$$

$$u y' + u' y = u q$$

$$\Rightarrow (uy)' = uq$$

If $up = u'$

i.e. $\frac{du}{dx} = up$

$\Rightarrow \frac{du}{u} = p\,dx$

$\Rightarrow u = e^{\int p\,dx}$

Think !! why !!

No need to use integration constant here

Step ① Take IF $u(x) = e^{\int p(x)dx}$

Step ② multiply both sides of ① by $u(x)$ and obtain

$$[u(x)\, y(x)]' = u(x)\, q(x)$$

$$\Rightarrow \boxed{u(x)\, y(x) = \int u(x)\, q(x) + C}$$

C is arbitrary constant

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In short

Given: $y' + p(x)y = q(x)$

→ Find IF $u(x) = e^{\int p(x)dx}$.

→ Write solution $y(x)$ as:

$$\boxed{y(x)\,u(x) = \int q(x)\,u(x)\,dx + C}$$

$$y(x) \times IF = \int (q(x) \times IF)\,dx + C$$

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Example :-

① $xy' - y = x^3$

Step ①

$$y' - \frac{1}{x}y = x^2 \quad —(i)$$

Step ②

$$u = e^{-\int \frac{1}{x}dx} = \frac{1}{x}$$

Step ③ multiply both sides of (i) by u.

$$\frac{1}{x}y' - \frac{1}{x^2}y = x$$

$$\Rightarrow \left(\frac{1}{x}y\right)' = x \quad —(ii)$$

Step ④ integrate (ii)

$$\frac{1}{x}y = \frac{x^2}{2} + c$$

$$\Rightarrow \boxed{y = \frac{x^3}{2} + cx} \text{ Ans.}$$

②

$$(1+\cos x)y' - (\sin x)y = 2x$$

$$y' - \frac{\sin x}{1+\cos x}y = \frac{2x}{(1+\cos x)} \quad —(i)$$

$$u = e^{-\int \frac{\sin x}{1+\cos x}dx} = (1+\cos x)$$

$$(1+\cos x)y' - (\sin x)y = 2x \quad —(ii)$$

[Note (i) & (ii) are same]

?? what is the meaning??

$\boxed{\text{(i) is already exact}}$

$$\Rightarrow ((1+\cos x)y)' = 2x$$

$$\xrightarrow{\int} \boxed{(1+\cos x)y = x^2 + C} \text{ Ans.}$$

$$y = \frac{x^2+c}{1+\cos x}$$

I/C $y(0) = 1$

$$1 = \frac{c}{2} \Rightarrow c = 2$$

Hence

$$\boxed{y = \frac{x^2+2}{1+\cos x}}$$

$\boxed{\text{Find its interval of validity??}}$

$\cancel{\frac{dy}{dx} - \frac{1}{x}y = 0}$

$\cancel{u = \frac{1}{x}}$

$\cancel{\left(\frac{1}{x}y\right)' = 0}$

$\cancel{\Rightarrow \boxed{\frac{y}{x} = c}}$

$\cancel{\frac{1}{x^2}\frac{dy}{dx} - \frac{1}{x^3}y = 0}$

$\cancel{\left(\frac{1}{x^2}y\right)' = 0}$

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