ICS141: Discrete Mathematics for Computer Science I

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based on slides by Dr. Baek and Dr. Still

Originals by Dr. M. P. Frank and Dr. J.L. Gross

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Lecture 12

Chapter 2. Basic Structures

2.4 Sequences and Summations

Summation Notation

- Given a sequence *{a_n}*, an integer *lower bound (or limit)* $j \geq 0$, and an integer *upper bound* $k \geq j$, then the *summation of {a_n} from a_j to a_k* is written and defined as follows:

$$\boxed{\sum_{i=j}^{k} a_i = a_j + a_{j+1} + \ldots + a_k}$$

- Here, i is called the *index of summation*.

$$\sum_{i=j}^{k} a_i = \sum_{m=j}^{k} a_m = \sum_{l=j}^{k} a_l$$

Generalized Summations

- For an infinite sequence, we write:

$$\sum_{i=j}^{\infty} a_i = a_j + a_{j+1} + \cdots$$

- To sum a function over all members of a set $X = \{x_1, x_2, \ldots\}$:

$$\sum_{x \in X} f(x) = f(x_1) + f(x_2) + \cdots$$

- Or, if $X = \{x \mid P(x)\}$, we may just write:

$$\sum_{P(x)} f(x) = f(x_1) + f(x_2) + \cdots$$

Simple Summation Example

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- $\sum_{i=2}^{4}\left(i^2+1\right)=$

- $1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}\cdots+\frac{1}{100}=\sum_{i=1}^{100}\frac{1}{i}$

More Summation Examples

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- An infinite sequence with a finite sum:

$$\sum_{i=0}^{\infty} 2^{-i} = 2^0 + 2^{-1} + \cdots = 1 + \frac{1}{2} + \frac{1}{4} + \cdots = 2$$

- Using a predicate to define a set of elements to sum over:

$$\sum_{\substack{(x \text{ is prime}) \wedge \\ x<10}} x^2 = 2^2 + 3^2 + 5^2 + 7^2$$

$$= 4 + 9 + 25 + 49 = 87$$

Summation Manipulations

- Some handy identities for summations:
  - Summing constant value

$$\sum_{n=i}^{j} c = (j-i+1)\cdot c$$

Number of terms in the summation

$$\sum_{n=1}^{3} 2 =$$

$$\sum_{n=-1}^{2} 2i$$

Summation Manipulations

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- Distributive law

$$\sum_{n=i}^{j} cf(n) = c\sum_{n=i}^{j} f(n)$$

$$\sum_{n=1}^{3}\left(4\cdot n^2\right) = 4\cdot 1^2 + 4\cdot 2^2 + 4\cdot 3^2$$

$$= 4\cdot\left(1^2 + 2^2 + 3^2\right)$$

$$= 4\sum_{n=1}^{3} n^2$$

Summation Manipulations

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- An application of commutativity

$$\sum_{n=i}^{j}\left(f(n)+g(n)\right)=\sum_{n=i}^{j}f(n)+\sum_{n=i}^{j}g(n)$$

$$\sum_{n=2}^{4}\left(n+2n\right)=(2+2\cdot 2)+(3+2\cdot 3)+(4+2\cdot 4)$$

$$=(2+3+4)+(2\cdot 2+2\cdot 3+2\cdot 4)$$

$$=\sum_{n=2}^{4}n+\sum_{n=2}^{4}2n$$

Index Shifting

$$\sum_{i=j}^{m} f(i) = \sum_{k=j+n}^{m+n} f(k-n)$$

$$\sum_{i=1}^{4} i^2 = 1^2 + 2^2 + 3^2 + 4^2$$

- Let $k = i + 2$, then $i = k - 2$

$$\sum_{k=1+2}^{4+2} (k-2)^2 = \sum_{k=3}^{6} (k-2)^2$$

$$= (3-2)^2 + (4-2)^2 + (5-2)^2 + (6-2)^2$$

More Summation Manipulations

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- Sequence splitting

$$\sum_{i=j}^{k} f(i) = \sum_{i=j}^{m} f(i) + \sum_{i=m+1}^{k} f(i) \quad \text{if } j \le m < k$$

$$\sum_{i=0}^{4} i^3 = 0^3 + 1^3 + 2^3 + 3^3 + 4^3$$

$$= (0^3 + 1^3 + 2^3) + (3^3 + 4^3)$$

$$= \sum_{i=0}^{2} i^3 + \sum_{i=3}^{4} i^3$$

More Summation Manipulations

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- Order reversal

$$\sum_{i=0}^{k} f(i) = \sum_{i=0}^{k} f(k-i)$$

$$\sum_{i=0}^{3} i^3 = 0^3 + 1^3 + 2^3 + 3^3$$

$$= (3-0)^3 + (3-1)^3 + (3-2)^3 + (3-3)^3$$

$$= \sum_{i=0}^{3} (3-i)^3$$

Example: Geometric Progression

- A *geometric progression* is a sequence of the form $a, ar, ar^2, ar^3, \ldots, ar^n, \ldots$ where $a,r \in \mathbf{R}$.
- The sum of such a sequence is given by:

$$\boxed{S = \sum_{i=0}^{n} ar^i}$$

- We can reduce this to *closed form* via clever manipulation of summations...

Geometric Sum Derivation

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- Here we go...

$$S = \sum_{i=0}^{n} ar^i$$

$$rS = r\sum_{i=0}^{n} ar^i = \sum_{i=0}^{n} rar^i = \sum_{i=0}^{n} arr^i = \sum_{i=0}^{n} ar^1 r^i$$

$$= \sum_{i=0}^{n} ar^{1+i} = \sum_{j=1}^{n+1} ar^{1+(j-1)} = \sum_{j=1}^{n+1} ar^j$$

$$= \left(\sum_{j=1}^{n} ar^j\right) + \sum_{j=n+1}^{n+1} ar^j = \left(\sum_{j=1}^{n} ar^j\right) + ar^{n+1} = \ldots$$

Derivation Example Cont...

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$$rS = \left(\sum_{i=1}^{n} ar^i\right) + ar^{n+1} = (ar^0 - ar^0) + \left(\sum_{i=1}^{n} ar^i\right) + ar^{n+1}$$

$$= ar^0 + \left(\sum_{i=1}^{n} ar^i\right) + ar^{n+1} - ar^0$$

$$= \left(\sum_{i=0}^{0} ar^i\right) + \left(\sum_{i=1}^{n} ar^i\right) + ar^{n+1} - a$$

$$= \left(\sum_{i=0}^{n} ar^i\right) + a(r^{n+1} - 1) = S + a(r^{n+1} - 1)$$

Concluding Long Derivation...

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$$rS = S + a(r^{n+1} - 1)$$

$$rS - S = a(r^{n+1} - 1)$$

$$S(r - 1) = a(r^{n+1} - 1)$$

$$S = \frac{a\left(r^{n+1} - 1\right)}{r - 1} \qquad \text{when } r \neq 1$$

When $r = 1$, $S = \sum_{i=0}^{n} ar^i = \sum_{i=0}^{n} a1^i = \sum_{i=0}^{n} a \cdot 1 = (n+1)a$

Example: Impress Your Friends

- Boast, "I'm so smart; give me any 2-digit number *n*, and I'll add all the numbers from 1 to *n* in my head in just a few seconds."
- *I.e.*, Evaluate the summation:

$$\sum_{i=1}^{n} i$$

- There is a simple closed-form formula for the result, discovered by Gauss at age 10!
  - And frequently rediscovered by many…

Gauss' Trick, Illustrated

- Consider the sum:

$$1+2+\ldots+(n/2)+((n/2)+1)+\ldots+(n-1)+n$$

$n+1$

$\vdots$

$n+1$

$n+1$

- We have *n*/2 pairs of elements, each pair summing to *n*+1, for a total of (*n*/2)(n+1).

Symbolic Derivation of Trick

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$k = n/2$

For case where n is even…

$$\sum_{i=1}^{n} i = \sum_{i=1}^{2k} i = \left(\sum_{i=1}^{k} i\right) + \sum_{i=k+1}^{n} i = \left(\sum_{i=1}^{k} i\right) + \sum_{j=0}^{n-(k+1)} (j + (k+1))$$

$$= \left(\sum_{i=1}^{k} i\right) + \sum_{j=0}^{n-(k+1)} ((n-(k+1)) - j) + (k+1))$$

$$= \left(\sum_{i=1}^{k} i\right) + \sum_{j=0}^{n-(k+1)} (n - j) = \left(\sum_{i=1}^{k} i\right) + \sum_{l=1}^{n-k} (n - (l-1))$$

$$= \left(\sum_{i=1}^{k} i\right) + \sum_{l=1}^{n-k} (n+1-l) = \left(\sum_{i=1}^{k} i\right) + \sum_{l=1}^{k} (n+1-l) = \ldots$$

Concluding Gauss' Derivation

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$$\sum_{i=1}^{n} i = \left(\sum_{i=1}^{k} i\right) + \sum_{i=1}^{k} (n+1-i) = \sum_{i=1}^{k} (i+n+1-i)$$

$$= \sum_{i=1}^{k} (n+1) = k(n+1) = \frac{n}{2}(n+1)$$

$$= \frac{n(n+1)}{2}$$

- So, you only have to do 1 easy multiplication in your head, then cut in half.
- Also works for odd *n* (prove this at home).

Some Shortcut Expressions

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TABLE 2 Some Useful Summation Formulae.

Sum	*Closed Form*	
$\sum_{k=0}^{n} ar^k \ (r \neq 0)$	$\frac{ar^{n+1} - a}{r - 1}, r \neq 1$	Geometric sequence
$\sum_{k=1}^{n} k$	$\frac{n(n+1)}{2}$	Gauss' trick
$\sum_{k=1}^{n} k^2$	$\frac{n(n+1)(2n+1)}{6}$	Quadratic series
$\sum_{k=1}^{n} k^3$	$\frac{n^2(n+1)^2}{4}$	Cubic series
$\sum_{k=0}^{\infty} x^k, \|x\| < 1$	$\frac{1}{1-x}$	
$\sum_{k=1}^{\infty}, kx^{k-1}, \|x\| < 1$	$\frac{1}{(1-x)^2}$	

Using the Shortcuts

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- Example: Evaluate $\sum_{k=50}^{100} k^2$
- Use series splitting. $\sum_{k=1}^{100} k^2 = \left(\sum_{k=1}^{49} k^2\right) + \sum_{k=50}^{100} k^2$
- Solve for desired summation. $\sum_{k=50}^{100} k^2 = \left(\sum_{k=1}^{100} k^2\right) - \sum_{k=1}^{49} k^2$
- Apply quadratic series rule. $= \frac{100 \cdot 101 \cdot 201}{6} - \frac{49 \cdot 50 \cdot 99}{6}$
- Evaluate. $= 338{,}350 - 40{,}425$

$= 297{,}925.$

Cardinality

- The sets *A* and *B* have the same ***cardinality*** if and only if there is a one-to-one correspondence from *A* to *B*.
- A set that is either finite or has the same cardinality as the set of positive integers is called **countable**.
- A set that is not countable is called **uncountable**.
- Example: Show that the set of odd positive integers is a countable set.

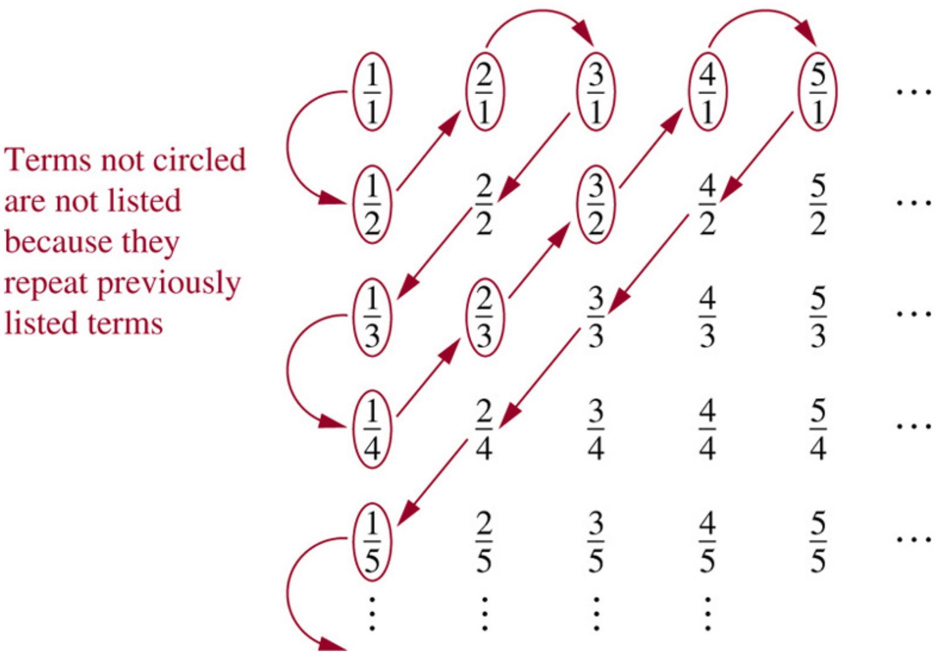
Consider the function $f(n) = 2n - 1$ from $\mathbf{Z}^+$ to the set of odd positive integers

$$
\begin{array}{ccccccccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & \ldots \\
\updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \\
1 & 3 & 5 & 7 & 9 & 11 & 13 & 15 & 17 & 19 & 21 & 23 & \ldots
\end{array}
$$

A one-to-one correspondence between $\mathbf{Z}^+$ and the set of odd positive integers.

Cardinality (cont.)

- An infinite set S is countable iff it is possible to list the elements of the set in a sequence (indexed by the positive integers)
  - $a_1, a_2, \ldots, a_n, \ldots$ is one-to-one mapping f: $\mathbf{Z}^+ \to S$ where $a_1 = f(1)$, $a_2 = f(2), \ldots, a_n = f(n), \ldots$
- Example: Show that the set of positive rational numbers is countable (see figure)



Summation Manipulations

- Useful identities:

$$\sum_{i=j}^{k} f(i) = \sum_{i=j}^{m} f(i) + \sum_{i=m+1}^{k} f(i) \quad \text{if } j \le m < k$$

(Sequence splitting.)

$$\sum_{i=0}^{k} f(i) = \sum_{i=0}^{k} f(k-i)$$

(Order reversal.)

$$\sum_{i=1}^{2k} f(i) = \sum_{i=1}^{k} \big(f(2i-1) + f(2i)\big)$$

(Grouping.)