Tutorial - 1

1. If possible let $\mathbb{C}$ be ordered.

Then, either $i > 0$ or $i < 0$ as $i \neq 0$.

If $i > 0 \Rightarrow i \cdot i > 0$

$\Rightarrow -1 > 0$ (contradiction)

Similarly for $i < 0$.

So, $\mathbb{C}$ can not be ordered.

2. i) let $z_1 = x_1 + iy_1$

$z_2 = x_2 + iy_2$

$$|z_1 + z_2| = \sqrt{(x_1 + x_2)^2 + (y_1 + y_2)^2}$$

$$|z_1| = \sqrt{x_1^2 + y_1^2}$$

$$|z_2| = \sqrt{x_2^2 + y_2^2}$$

Now, $(x_1 y_2 - y_1 x_2)^2 \geq 0$

$\Rightarrow 0 \leq (x_1 y_2 - y_1 x_2)^2$

[Adding $x_1^2 x_2^2 + y_1^2 y_2^2$ on both sides]

$\Rightarrow (x_1 x_2 + y_1 y_2)^2 \leq (x_1^2 + y_1^2)(x_2^2 + y_2^2)$

$\Rightarrow 2x_1 x_2 + y_1 y_2 \leq 2\sqrt{(x_1^2 + y_1^2)(x_2^2 + y_2^2)}$

[Adding $x_1^2 + x_2^2 + y_1^2 + y_2^2$ on both sides]

$$\Rightarrow (x_1+x_2)^2 + (y_1+y_2)^2 \leq (x_1^2+y_1^2) + (x_2^2+y_2^2) + 2\sqrt{(x_1^2+y_1^2)(x_2^2+y_2^2)}$$

[Square root both sides.]

$$\Rightarrow \sqrt{(x_1+x_2)^2+(y_1+y_2)^2} \leq \sqrt{x_1^2+y_1^2} + \sqrt{x_2^2+y_2^2}$$

$$\Rightarrow |z_1+z_2| \leq |z_1| + |z_2|$$

Similarly, $|z_1 - z_2| \leq |z_1| + |z_2|$

(Proved)

<u>or</u>

$$|z_1+z_2|^2 = (z_1+z_2)\overline{(z_1+z_2)}$$
$$= (z_1+z_2)(\bar{z}_1+\bar{z}_2)$$
$$= z_1\cdot\bar{z}_1 + z_2\cdot\bar{z}_1 + z_1\cdot\bar{z}_2 + z_2\cdot\bar{z}_2$$
$$= |z_1|^2 + |z_2|^2 + z_2\cdot\bar{z}_1 + \overline{z_1\cdot\bar{z}_2}$$
$$= |z_1|^2 + |z_2|^2 + 2\,\mathrm{Re}(z_1\bar{z}_2)$$
$$\leq |z_1|^2 + |z_2|^2 + 2|z_1|\cdot|z_2|$$
$$\leq (|z_1| + |z_2|)^2$$

$$\Rightarrow |z_1+z_2| \leq |z_1| + |z_2|$$

Similar for, $|z_1 - z_2| \leq |z_1| + |z_2|$

ii) Similar, iii) Similar.

iv) to show, $\sqrt{2}\,|z| \geq |Re(z)| + |Im(z)|$

Now, $|z|^2 = x^2 + y^2 = |Re(z)|^2 + |Im(z)|^2$

Also, $|x|^2 + |y|^2 \geq 2|x|\cdot|y|$

$\Rightarrow |z|^2 \geq 2|Re(z)|\cdot|Im(z)|$

Now, $2|z|^2 = |z|^2 + |z|^2$

$\geq |Re(z)|^2 + |Im(z)|^2 + 2|Re(z)|\cdot|Im(z)|$

$\geq (|Re(z)| + |Im(z)|)^2$

$\Rightarrow \sqrt{2}\,|z| \geq |Re(z)| + |Im(z)|$

3. To show that, $\left|\sum_{i=1}^{n} z_i\right| \leq \sum_{i=1}^{n} |z_i|$

Now this is true for $n=1$ and $n=2$

let us assume it is true for $n=k$

ie $\left|\sum_{i=1}^{k} z_i\right| \leq \sum_{i=1}^{k} |z_i|$

Now, $\left|\sum_{i=1}^{k+1} z_i\right| = \left|\sum_{i=1}^{k} z_i + z_{k+1}\right|$

$\leq \left|\sum_{i=1}^{k} z_i\right| + |z_{k+1}|$

$\leq \sum_{i=1}^{k} |z_i| + |z_{k+1}|$

$\leq \sum_{i=1}^{k+1} |z_i|$

By induction, it is true.

4. We know,

$$|z_1 \cdot \bar{z}_2| = |\bar{z}_1 \cdot z_2|$$

Now, $|z_1 \cdot \bar{z}_2| = \frac{1}{2}\left(|\bar{z}_1 \cdot z_2| + |\bar{z}_2 \cdot z_1| \right)$

$$\geq \frac{1}{2} |z_1 \cdot \bar{z}_2 + \bar{z}_1 \cdot z_2|$$

$$\geq \frac{1}{2} \left| z_1 \cdot \bar{z}_2 + \overline{(z_1 \cdot \bar{z}_2)} \right|$$

$$\geq \frac{1}{2} |Re(z_1 \cdot \bar{z}_2)|$$

$$\geq Re(z_1 \cdot \bar{z}_2).$$

2nd Part

When equality holds then let,

$z_1 = x_1 + iy_1$

$z_2 = x_2 + iy_2$

Then, $z_1 \cdot \bar{z}_2 = (x_1 + iy_1) \cdot (x_2 - iy_2)$

$= (x_1 x_2 + y_1 y_2) + i(x_2 y_1 - x_1 y_2)$

Now, $Re(z_1 \bar{z}_2) = |z_1 \cdot \bar{z}_2|$

$\Rightarrow (x_1 x_2 + y_1 y_2) = \sqrt{(x_1 x_2 + y_1 y_2)^2 + (x_2 y_1 - x_1 y_2)^2}$

$\Rightarrow \cancel{(x_1 x_2 + y_1 y_2)^2} = \cancel{(x_1 x_2 + y_1 y_2)^2} + (x_2 y_1 - x_1 y_2)^2$

$\Rightarrow x_2 y_1 - x_1 y_2 = 0$

$\Rightarrow x_2 y_1 = x_1 y_2$

$\cancel{\Rightarrow \frac{x_1}{y_1} = \frac{x_2}{y_2}}$

$\Rightarrow \frac{x_1}{x_2} = \frac{y_1}{y_2} = K$ (say) ie $x_1 = Kx_2$

$y_1 = Ky_2$

ie $z_1 = x_1 + iy_1 = Kx_2 + iKy_2$

$= K(x_2 + iy_2)$

$= K z_2$

ie $z_1 = K z_2$. where $K > 0$

Because, $K < 0$ have contradiction with the equation $\operatorname{Re}(z_1 \bar{z}_2) = |z_1 \cdot \bar{z}_2|$.

5. $p(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$

As z_1 is a root, we have

$p(z_1) = a_n z_1^n + a_{n-1} z_1^{n-1} + \cdots + a_1 z_1 + a_0 = 0$

$\Rightarrow \overline{a_n z_1^n + a_{n-1} z_1^{n-1} + \cdots + a_0} = 0$

$\Rightarrow a_n \overline{z_1^n} + a_{n-1} \cdot \overline{z_1^{n-1}} + \cdots + a_0 = 0$

$\Rightarrow a_n (\bar{z}_1)^n + a_{n-1} (\bar{z}_1)^{n-1} + \cdots + a_1 (\bar{z}_1) + a_0 = 0$

$\Rightarrow p(\bar{z}_1) = 0$

So, $\bar{z}_1$ is also a root.

6. i) To find the locus of $Re\left(\frac{1}{z}\right) = 2$

let, $z = x+iy$

$$Re\left(\frac{1}{z}\right) = 2$$

$$\Rightarrow Re\left(\frac{1}{x+iy}\right) = 2$$

$$\Rightarrow Re\left(\frac{x-iy}{x^2+y^2}\right) = 2$$

$$\Rightarrow \frac{x}{x^2+y^2} = 2$$

$$\Rightarrow 2x^2+2y^2-x = 0.$$

ii) $Re\left(z^2\right) \leq 1$

$$\Rightarrow Re\left((x+iy)^2\right) \leq 1$$

$$\Rightarrow Re\left(x^2+i2xy-y^2\right) \leq 1$$

$$\Rightarrow x^2-y^2 \leq 1.$$

7. $\sin z = \frac{1}{2i}\left(e^{iz}-e^{-iz}\right)$

Now, when $z = \pi + i\ln(2+\sqrt{5})$

then, $e^{iz} = e^{i(\pi+i\ln(2+\sqrt{5}))}$

$$= e^{i\pi} \cdot e^{-\ln(2+\sqrt{5})}$$

$$= e^{i\pi} \cdot e^{\ln(2+\sqrt{5})^{-1}}$$

$$= -1 \cdot \frac{1}{2+\sqrt{5}} = 2-\sqrt{5}$$

and, $e^{-iz} = e^{-i(\pi + i\ln(2+\sqrt{5}))}$

$= e^{-i\pi} \cdot e^{\ln(2+\sqrt{5})}$

$= (-1) \cdot (2+\sqrt{5})$

$= -(2+\sqrt{5})$

So, $e^{iz} - e^{-iz} = 2-\sqrt{5} + (2+\sqrt{5})$

$= 4$

So, $|\sin z| = \left| \frac{1}{2i} (e^{iz} - e^{-iz}) \right|$

$= \left| \frac{1}{2i} \times 4 \right|$

$= \left| \frac{2}{i} \right| = |-2i| = 2.$

8. For a complex number α, s.t. $|\alpha| < 1$

we need to show that, $\left| \frac{z-\alpha}{1-\bar{\alpha}z} \right| < 1$

if $|z| < 1$ and $\left| \frac{z-\alpha}{1-\bar{\alpha}z} \right| = 1$ if $|z| = 1$

Ans:- $|z-\alpha|^2 - |1-\bar{\alpha}z|^2$

$= (z-\alpha)(\bar{z}-\bar{\alpha}) - (1-\bar{\alpha}z)(1-\alpha\bar{z})$

$= (z\cdot\bar{z} - \alpha\bar{z} - \bar{\alpha}z + \alpha\cdot\bar{\alpha}) - (1 - \bar{\alpha}z - \alpha\bar{z} + \alpha\bar{\alpha}\cdot z\bar{z})$

$= |z|^2 - \cancel{\alpha\bar{z}} - \cancel{\bar{\alpha}z} + |\alpha|^2 - 1 + \cancel{\bar{\alpha}z} + \cancel{\alpha\bar{z}} - |\alpha|^2|z|^2$

$= (|z|^2 - 1)(1-|\alpha|^2) < 0$ if $|z| < 1$

ie if $|z| < 1$

$$|z-\alpha|^2 - |1-\bar{\alpha}z|^2 < 0$$

$$\Rightarrow |z-\alpha|^2 < |1-\bar{\alpha}z|^2$$

$$\Rightarrow \left|\frac{z-\alpha}{1-\bar{\alpha}z}\right| < 1.$$

if $|z| = 1$

Then, $|z-\alpha|^2 - |1-\bar{\alpha}z|^2 = 0$

$$\Rightarrow |z-\alpha|^2 = |1-\bar{\alpha}z|$$

$$\Rightarrow \left|\frac{z-\alpha}{1-\bar{\alpha}z}\right| = 1.$$

9. To show that, $z + \frac{1}{z}$ is real if $Im(z) = 0$ or $|z| = 1$

put $z = x+iy$

Then, $Im(z) = y$

and, $|z| = 1 \Rightarrow x^2+y^2 = 1$

Now, $z + \frac{1}{z}$

$$= x+iy + \frac{1}{x+iy}$$

$$= x+iy + \frac{x-iy}{x^2+y^2}$$

$$= \frac{x(x^2+y^2) + iy(x^2+y^2) + x - iy}{x^2+y^2}$$

So, $\operatorname{Im}\left(z+\frac{1}{z}\right) = \dfrac{y\left(x^2+y^2-1\right)}{x^2+y^2}$

if $y=0$ or $x^2+y^2=1$ then $\operatorname{Im}\left(z+\frac{1}{z}\right)=0$

<u>ie</u> $z+\frac{1}{z}$ is real.

10. If $|z|=1$, then to show, $|z^2-z+1| \leq 3$

Now, $|z^2-z+1| \leq |z^2|+|-z|+1$

$\leq |z|^2+|z|+1$

≤ 3.

<u>2nd Part</u>

to show, $|z^2-2| \geq 1$

$$||z^2|-|2|| \leq |z^2-2|$$

$$\Rightarrow ||z|^2-|2|| \leq |z^2-2|$$

$$\Rightarrow |1-2| \leq |z^2-2|$$

$$\Rightarrow |z^2-2| \geq 1.$$

11. i) We have $|z|=2$

$$|z^4-4z^2+3| = |(z^2-1)(z^2-3)|$$

$$\geq (|z|^2-1)(|z|^2-3) = 3$$

$$\Rightarrow \frac{1}{|z^4-4z^2+3|} \leq \frac{1}{3}.$$

12. $1+z+z^2+\cdots+z^n = \dfrac{1-z^{n+1}}{1-z}, \quad z \neq 1$

i) let us put, $z = e^{i\theta}$

then, $z^n = e^{in\theta}$

$= \cos n\theta + i \sin n\theta$

So, $1 + e^{i\theta} + e^{i2\theta} + \cdots + e^{in\theta} = \dfrac{1-e^{i(n+1)\theta}}{1-e^{i\theta}}$

$\Rightarrow (1+\cos\theta + \cos 2\theta + \cdots + \cos n\theta)$

$+ (\sin\theta + \cdots + \sin n\theta) = \dfrac{1-\cos(n+1)\theta - i\sin(n+1)\theta}{1-\cos\theta - i\sin\theta}$

$= \dfrac{\{1-\cos(n+1)\theta - i\sin(n+1)\theta\}\{1-\cos\theta + i\sin\theta\}}{(1-\cos\theta)^2+\sin^2\theta}$

$$= \frac{\{1-\cos(n+1)\theta\}(1-\cos\theta)+\sin(n+1)\theta\sin\theta + i\{1-\cos(n+1)\theta\}\sin\theta - i\sin(n+1)\theta(1-\cos\theta)}{(1-\cos\theta)^2+\sin^2\theta}$$

Now compare the real parts of the both sides

2) $1 + \cos\theta + \cos 2\theta + \cdots + \cos n\theta$

$$= \frac{\{1 - \cos(n+1)\theta\}(1-\cos\theta) + \sin(n+1)\theta \sin\theta}{(1-\cos\theta)^2 + \sin^2\theta}$$

$$= \frac{1 - \cos(n+1)\theta - \cos\theta + \cos(n+1)\theta\cos\theta + \sin(n+1)\theta\sin\theta}{1 - 2\cos\theta + \cos^2\theta + \sin^2\theta}$$

$$= \frac{1 - \cos(n+1)\theta - \cos\theta + \cos(n+\cancel{1}-\cancel{1})\theta}{2(1-\cos\theta)}$$

$$= \frac{1}{2} + \frac{\cos n\theta - \cos(n+1)\theta}{2(1-\cos\theta)}$$

$$= \frac{1}{2} + \frac{\cancel{2}\sin(n+\frac{1}{2})\theta \sin\frac{\theta}{2}}{\cancel{2}\cdot 2\sin^2\theta/2}$$

$$= \frac{1}{2} + \frac{\sin(n+\frac{1}{2})\theta}{2\sin\frac{\theta}{2}}$$ (Proved).

ii) Similar (compare imaginary part).

13. i) $x^8 - 16 = 0$

$\Rightarrow (x^4)^2 - (4)^2 = 0$

$\Rightarrow (x^4 - 4)(x^4 + 4) = 0$

$\Rightarrow (x^2 - 2)(x^2 + 2)(x^4 + 4) = 0$

$\Rightarrow (x - \sqrt{2})(x + \sqrt{2})(x^2 + 2)(x^4 + 4) = 0$

Now, $x - \sqrt{2} = 0 \Rightarrow x = \sqrt{2}$

$x + \sqrt{2} = 0 \Rightarrow x = -\sqrt{2}$

$x^2 + 2 = 0$

$\Rightarrow x^2 = -2 \Rightarrow x = \pm\sqrt{2}\, i$

$x^4 + 4 = 0$

$\Rightarrow x^4 = -4 \Rightarrow x^2 = \pm 2i$

Now, $x^2 = +2i = 2 \cdot e^{i\frac{\pi}{2}}$

$\Rightarrow x = \pm\sqrt{2}\, e^{i\frac{\pi}{2}\cdot\frac{1}{2}}$

$= \pm\sqrt{2}\, e^{i\frac{\pi}{4}} = \pm\sqrt{2}\left(\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right)$

$= \pm(1 + i)$

Again, $x^2 = -2i = -2 \cdot e^{i\pi/2}$

$\Rightarrow x = \pm\sqrt{2}\, i\, e^{i\pi/4}$

$= \pm\, i\,(1 + i)$

$= \pm(-1 + i)$.

So the roots are, $\pm\sqrt{2}$, $\pm\sqrt{2}\,i$, $\pm(1+i)$, $\pm(-1+i)$

iii) $z^4 - 4z^3 + 6z^2 - 4z + 5 = 0$

As i is a root, so conjugate $-i$ is also a root.

So, $(z-i)(z+i) = z^2+1$ is a factor.

ie $z^4 - 4z^3 + 6z^2 - 4z + 5 = 0$

$\Rightarrow (z^2+1)(z^2-4z+5) = 0$

Now, $z^2 - 4z + 5 = 0$

$\Rightarrow (z-5)(z+1) = 0$

$\Rightarrow z = 5, -1$

So, the roots are $z = \pm i, 5, -1$.

14. From previous exercise, the roots of z^4+4 are $\pm(1+i), \pm(-1+i)$

So, $z^4+4 = (z-1-i)(z+1+i)(z-1+i)(z+1-i)$

$= \{(z+1)+i\}\{(z+1)-i\}\{(z-1)+i\}\{(z-1)-i\}$

$= \{(z+1)^2+1\}.\{(z-1)^2+1\}$

$= (z^2+2z+1+1)(z^2-2z+1+1)$

$= (z^2+2z+2)(z^2-2z+2)$

15. i) $(-\sqrt{3}-i)^{-6}$

$= \left\{2\cdot\left(-\frac{\sqrt{3}}{2}-\frac{i}{2}\right)\right\}^{-6}$

$= 2^{-6}\cdot\left(\cos\frac{7\pi}{6}+i\sin\frac{7\pi}{6}\right)^{-6}$

$= 2^{-6}\cdot\left(e^{i\frac{7\pi}{6}}\right)^{-6}$

$= 2^{-6}\cdot e^{i7\pi}$

$= 2^{-6}\cdot\left(\cos 7\pi + i\sin 7\pi\right)$

$= \frac{-1}{26}.$

Rest is similar.