Properties of Probability Laws

Tuesday, January 12, 2021 8:45 AM

Discrete Sample Space

$$\Omega = \{ s_1, s_2 \cdots s_n \}$$

$$P(\underbrace{\{ s_{i_1}, s_{i_2}, s_{i_t} \}}_{\text{event}}) = P(\{ \{s_{i_1}\}, \{s_{i_2}\}, \cdots \{s_{i_t}\} \})$$

$$= P(\{s_{i_1}\}) + P(\{s_{i_2}\}) + \cdots + P(\{s_{i_t}\})$$

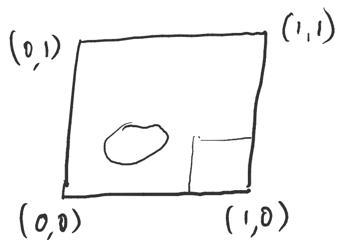
Discrete Prob law

If the sample space consists of a finite no of possible outcomes then the prob law is specified by the probabilities of the events that consist of a single outcome.

If the sample space consists of n possible outcomes which are "equally likely", the the prob of any event A is given by

$$P(A) = \frac{\text{Number of elements in } A}{n}$$

Continuous Case

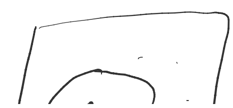


$$\Omega = \bigcup_{0 \le x, y \le 1} \{(x,y)\}$$

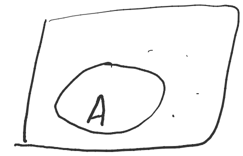
$$1 = P(\Omega) = P\left(\bigcup_{0 \le x, y \le 1} \{(x,y)\}\right) \overset{???}{=} \sum_{0 \le x, y \le 1} P(\{x,y\}) = \sum 0 = 0$$

Properties of Prob laws

1. $P(\emptyset) = 0$ $\qquad P(\Omega) = P(\Omega) + P(\emptyset)$



1. $P(\emptyset) = 0$ $\qquad P(\Omega) = P(\Omega) + P(\emptyset)$



2. $P(A^c) = 1 - P(A)$ $\qquad \Omega = A \cup A^c$

3. If $A \subseteq B$, then $P(A) \leq P(B)$

$$B = \underline{\underline{A}} \cup (A^c \cap B)$$

$$P(B) = P(A) + \underbrace{P(A^c \cap B)}_{\geq 0}$$

4. $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$$A \cup B = A \cup (A^c \cap B) \Rightarrow P(A \cup B) = P(A) + \underline{P(A^c \cap B)}$$

&

$$\Rightarrow B = (A \cap B) \cup (A^c \cap B) \Rightarrow P(B) = P(A \cap B) + \underline{P(A^c \cap B)}$$

5. For any events $A_1, A_2 \cdots A_n$

$$P\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} P(A_i) - \sum_{i<j} P(A_i \cap A_j) + \sum_{i<j<k} P(A_i \cap A_j \cap A_k) - \cdots$$

$$+ (-1)^{n-1} P\left(\bigcap_{i=1}^{n} A_i\right)$$

By using induction; (Ex)

6. Boole's Inequality (Union Bound)

For any seq of events $A_1\ A_2\ A_3 \cdots$

$$P\left(\bigcup_{i=1}^{n} A_i\right) \leq \sum_{i=1}^{n} P(A_i)$$

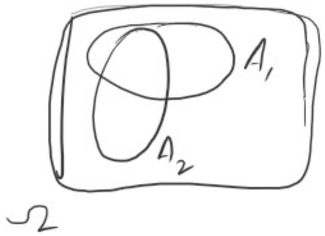
[Special case when events are disjt $=$]

$$P\left(\bigcup_{i=1}^{n} A_i\right) \le \sum_{i=1} P(A_i)$$

special case when events are disj

$=$

$$\underline{\underline{P(A_1 \cup A_2)}} \le \underline{\underline{P(A_1)}} + \underline{\underline{P(A_2)}}$$



Pf: $B_1 = A_1$, $B_2 = A_2 \setminus A_1$, $B_3 = A_3 \setminus \{A_1 \cup A_2\}$ ... $B_i = A_i \setminus \left\{\bigcup_{t=1}^{i-1} A_t\right\}$

$\bigcup A_i = \bigcup B_i$ and B_is are disjt.

and $B_i \subseteq A_i$

$$P\left(\bigcup_{i=1}^{\infty} A_i\right) = P\left(\bigcup_{i=1}^{\infty} B_i\right) = \sum_{i=1}^{\infty} P(B_i) \le \sum_{i=1}^{\infty} P(A_i)$$

7. <u>Bonferroni's Inequality:</u>

For any events $A_1, A_2 \cdots A_n$ and for any r $1 \le r \le n$

$$P\left(\bigcup_{i=1}^{n} A_i\right) \overset{\le}{\underset{\ge}{\text{or}}} \sum_{i=1}^{n} P(A_i) - \sum_{i<j} P(A_i \cap A_j) + \cdots\cdots + (-1)^{r-1} \sum_{i_1 < i_2 \cdots < i_r} P\left(A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_r}\right)$$

as r is odd or even

Pf: Using induction: $n = 2$ $P(A_1 \cup A_2) \le P(A_1) + P(A_2)$

$$P\left(\underline{A_1} \cup \underline{\bigcup_{i=2}^{n} A_i}\right) = P(A) + P\left(\bigcup_{i=2}^{n} A_i\right) - P\left(\underbrace{A_1 \cap \bigcup_{i=2}^{n} A_i}_{\bigcup^{n} (A_i \cap A_1)}\right)$$

$$\bigcup_{i=2}^{n}(A_i \cap A_1)$$

$$= P(A) + \underbrace{P\left(\bigcup_{i=2}^{n} A_i\right)}_{\le r} - \underbrace{P\left(\bigcup_{i=2}^{n} (A_i \cap A_1)\right)}_{(r-1)\le}$$

(Ex)