Lecture - 1
15/09/2020

§ Integration

Quick Recall

I. Cauchy (Fundamental th^{m}) theorem.

C : given simple closed contour
f(z) : analytic inside and on C
f'(z) : cont^s ″

Then

$$\int_C f(z)\,dz = 0$$

[Proof : Green's theorem]

II. Cauchy - Goursat Theorem.

C : given simple closed contour
f(z) : analytic$_s$ inside and on C

Then

$$\int_C f(z)\,dz = 0$$

III. II can be extended to simply connected domain D.

D : simply connected domain
f(z) : analytic throughout D,

Then $\int_C f(z)\, dz = 0$

where C : closed contour lying inside of D.

IV further extension to multiply connected domain

[Proof : uses Cauchy Goursat theorem]

V. Corollary of II

Suppose $f(z)$ is analytic throughout a simply connected domain D. Then $f(z)$ must have an antiderivative everywhere in D.

Proof :-

Lemma :- Suppose $f(z)$ is continuous on a domain D. Then the following are equivalent ;

(i) $f(z)$ has an antiderivative in D.

(ii) Integrals of $f(z)$ along contours lying entirely in D and extending from fixed points z_1 to z_2 all have the same value.

(iii) $\int_C f(z)\,dz = 0$ for any closed contour C lying inside of D.

Idea:

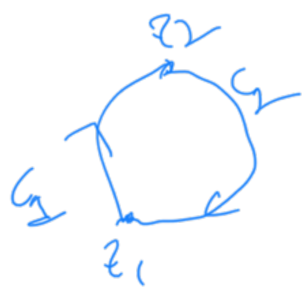
(ii) $\Longleftrightarrow$ (iii).

$$\int_{C_1} f(z)\,dz = \int_{C_2} f(z)\,dz \Longleftrightarrow \int_{C_1} f(z)\,dz - \int_{C_2} f(z)\,dz = 0$$

$$\left(-\int_{C_2} f(z)\,dz = \int_{-C_2} f(z)\,dz \right)$$

$$\Longleftrightarrow \int_{C_1} f(z)\,dz + \int_{-C_2} f(z)\,dz = 0$$

$$\Longleftrightarrow \int_{C_1 + (-C_2)} f(z)\,dz = 0.$$



~~Proof~~ $\longrightarrow$??

Proof of corollary:

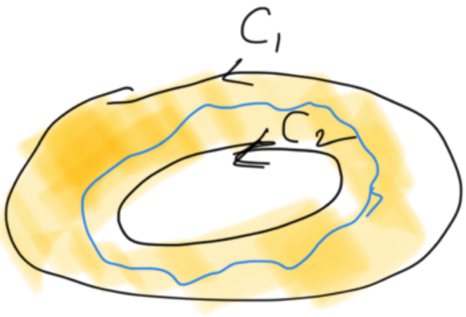
(iii) follows from Cauchy-Goursat Th.

So we get (i) [& (ii)]

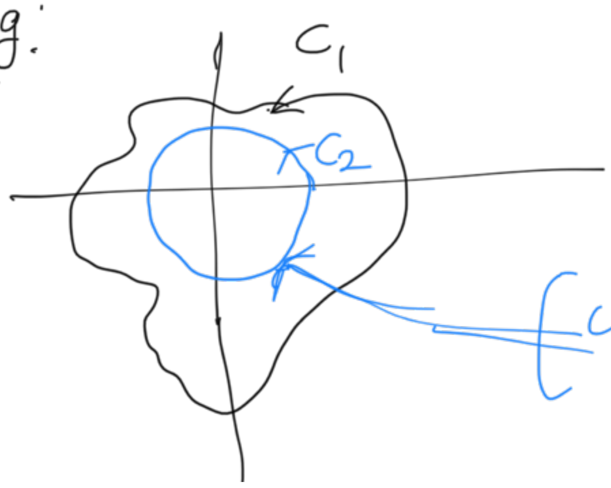
VI Corollary (B) of IV Let C_1 and C_2 be positively oriented simple closed contours.

Let C_2 is in the interior of C_1. Let $f(z)$ be analytic in the closed region which consists of C_1, C_2 and all points between them, Then

$$\int_{C_1} f(z)\,dz = \int_{C_2} f(z)\,dz$$



Eg:



$$\int_{C_1} f(z)\,dz = \int_{C_2} f(z)\,dz$$

[circle : parametrizat' is easy]

VII : Cauchy Integral formula

$f(z)$: analytic inside and on a simple closed contour C (positive sense)

z_0 : a point interior to C.

$$f(z_0) = \frac{1}{2\pi i}\int_C \frac{f(z)\,dz}{(z-z_0)}$$

[Proof : Use Corr B]

and so $\int_C \frac{f(z)\,dz}{z-z_0} = \int_{C_\rho} \frac{f(z)}{(z-z_0)}\,dz$

C_ρ : $z - z_0 = \rho e^{i\theta}$

$0 \le \theta \le 2\pi$

(manageable)

[Value of $f(z)$ inside of C are completely determined by its values on C]

analytic $\Rightarrow f'(z)$ exists

VIII Derivatives of Analytic function

$f(z)$: analytic inside and on a simple closed contour C (positive sense)

z_0 : a point interior to C.

Then

$$f'(z_0) = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z-z_0)^2} \, dz$$

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \int_C \frac{f(z)\,dz}{(z-z_0)^{n+1}}, \quad n = 1, 2, 3, \ldots$$

Proof: Cauchy integral formula.

Hidden information

$g(z) = f^{(n)}(z)$

$g'(z) = f^{(n+1)}(z)$ exists.

$\Rightarrow g(z)$ is cont.

(IX) Theorem : If a function $f(z)$ is analytic at a point, then its derivatives of all orders exist at that point.

Moreover, derivatives of all orders, are continuous at that point.

(X) Corollary C : If a $f(z) = u(x,y) + iv(x,y)$ is analytic at a point $z_0 = (x_0, y_0)$. Then $u(x,y)$ and $v(x,y)$ have continuous partial derivatives of all orders at that point derivatives of all orders at that point.

<u>Question</u> :- Go back to the statement of Cauchy fundamental th^m.

Using IX above, can we now drop the hypothesis "$f'(z)$ continuous" ??

and obtain a proof of Cauchy Goursat theorem ??

<u>Given</u> :- $f(z)$ analytic inside & on C

C : simple closed contour.

<u>Want</u> :- $\int_C f(z)\,dz = 0$.

<u>Proof</u> :- $f(z)$ is analytic inside & on C

$\Rightarrow f'(z)$ is cont^s "

By Cauchy's fundamental th^m,

$\int_C f(z)\,dz = 0$.

Is this correct?

XI (Morera's Theorem) ~~No~~

$f(z)$: cont^s on a domain D.

$\int_C f(z)\,dz = 0$ for any closed contour C

Then $f(z)$ is analytic throughout D.

[Proof: use the lemma, to get an antiderivative $F(z)$. i.e.

$$\begin{cases} F(z) \text{ is analytic in } D \text{ and} \\ F'(z) = f(z) \quad \forall z \in D \end{cases}$$

Now, By (IX), $F'(z)$ is analytic at all $z \in D$

$F'(z) = f(z)$.

XII (Cauchy Goursat theorem & Morera's theorem)

Let $f(z)$ be continuous in D. Then $f(z)$ is analytic in D if and only if $\int_C f(z)\,dz = 0$ for any closed contour C lying inside of D.

XIII (Cauchy's Inequality for analytic f^n)

Lemma:- Suppose that a function $f(z)$ is analytic inside and on a positively oriented circle $C_R : |z-z_0| = R$ and M_R be the maximum value of $|f(z)|$ on C_R, then

$$|f^{(n)}(z)| \le \frac{n!\, M_R}{R^n} \qquad n = 1, 2, \ldots \quad n=0$$

Proof:- (discussed already)

XIV **Theorem** (Liouville's theorem).

If $f(z)$ is an entire and bounded function in $\mathbb{C}$, then $f(z)$ is constant.

(Def^n: $f(z)$ is called entire if it is analytic in $\mathbb{C}$)

Proof:- Suppose $|f(z)| \le M \quad \forall z \in \mathbb{C}$

$$|f'(z)| \le \frac{M}{R} \quad \text{for every positive } R.$$

by lemma.

Letting $R \to \infty$, we get $f'(z) = 0 \quad \forall z \in \mathbb{C}$

$\Rightarrow$ $f(z)$ is constant by an earlier theorem

$\blacksquare$

Remark :-

Remark: $f(x) = \sin x$.

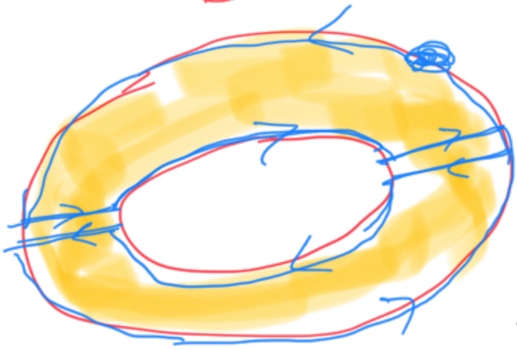
$|f(x)| \leq 1 \quad \forall \; x \in \mathbb{R}$.

diff'ble everywhere but not a constant funct^n.

Terminology

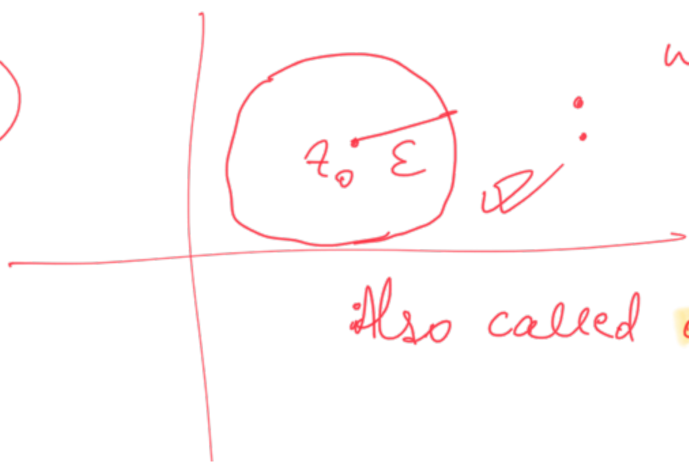
1. $\Rightarrow$ positively oriented
2. $\to$ nbd
3. $\to$ analytic at point
4. $\to$ bounded
5. $\to$ Domain (open & connected set)

(1)



a direction on the curve s.t. when you trace the curve along that direction then the domain must be on LHS.

(2)



We write

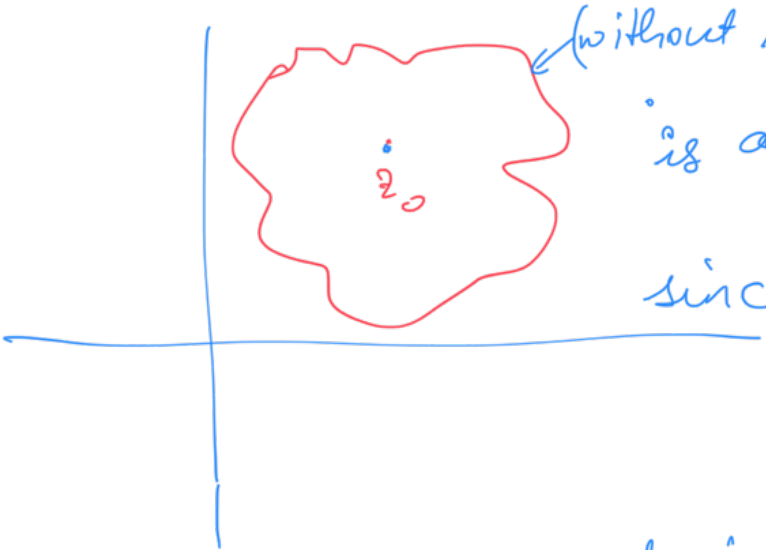
$$B_\varepsilon(z_0) := \{ z \in \mathbb{C} \mid |z - z_0| < \varepsilon \}$$

Also called open ball of radius ε

Definition ① : $S \subseteq \mathbb{C}$ is called an open set if for each $z_0 \in S$, there exists $B_\varepsilon(z_0) \subseteq S$ for some $\varepsilon > 0$.

② A neighbourhood of a point z_0 is an open set containing z_0.

Eg: $B_\varepsilon(z_0)$ is a nbd of z_0 (for $\varepsilon > 0$)

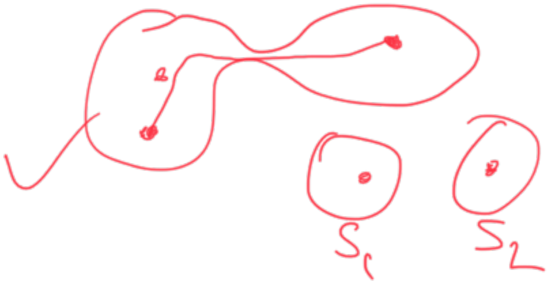


(without boundary) is also nbd of z_0, since it is an open set.

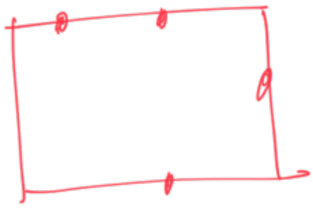
Eg: 

with boundaries $= \{ z \in \mathbb{C} \mid |z - z_0| \leq \varepsilon \}$ is not open.

(5) Open + Connected = domain



$D = S_1 \cup S_2$
is not connected.



without boundaries — domain

with boundaries — connected but not open

(4) We say $f(z)$ is analytic at a point z_0.
if $\exists$ a nbd of z_0 where $f'(z)$ exist.

$f(z)$ is analytic on a set S
if $f(z)$ is analytic at each
point of S.

⑤ $f(z)$ is bounded on $S \subseteq \mathbb{C}$

if $\exists\, M > 0$ s.t.

$$|f(z)| \leq M.$$

entire + bounded
$\Downarrow$
constant.

As a consequence of Liouville's theorem, we prove fundamental theorem of Algebra.

<u>Theorem</u>:- (Fundamental theorem of Algebra)

Let $P(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n$; $a_n \neq 0$ be a non constant polynomial of degree n, $(n \geq 1)$, Then $P(z)$ has atleast one root (in $\mathbb{C}$) i.e. there exists a $z_0 \in \mathbb{C}$ s.t.

$$P(z_0) = 0.$$

<u>Proof</u>:- (By way of contradiction).

Suppose $P(z) \neq 0 \quad \forall\, z \in \mathbb{C}$.

Define $f(z) := \dfrac{1}{P(z)}$ (well defined)

Ex. $P(z)$ is an entire function and $P(z) \neq 0 \; \forall z$

$\Rightarrow$ $f(z)$ is an entire function.

Claim: $f(z)$ is bounded $\forall z \in \mathbb{C}$.

Then by Liouville's theorem, $f(z)$ is constant

$\Rightarrow$ $P(z)$ is constant, which is a contradiction.

Proof of the claim:

On any closed region $|z| \leq R$ (arbitrary), $f(z)$ is continuous and so bounded. Ex

We have to prove the boundedness for $|z| > R$.

(Idea: $|f(z)| \leq M$ iff $|P(z)| \geq \frac{1}{M}$. ← we try to show this)

We have $P(z) = a_0 + a_1 z + \cdots + a_n z^n$

$\boxed{|z| > R}$

$$= z^n \left(\frac{a_0}{z^n} + \frac{a_1}{z^{n-1}} + \cdots + \frac{a_{n-1}}{z} + a_n \right)$$

$$= z^n (w + a_n)$$

where $w = \frac{a_0}{z^n} + \frac{a_1}{z^{n-1}} + \cdots + \frac{a_{n-1}}{z}$ —— (1)

$$|P(z)| = |z|^n \, |w + a_n|$$

$$\geq |z|^n \cdot \big| |w| - |a_n| \big| \quad \text{—— (2)}$$

$|z| > R$ h_w

for this factor, note that $|w| \leq \left|\frac{a_0}{z^n}\right| + \left|\frac{a_1}{z^{n-1}}\right| + \cdots + \left|\frac{a_{n-1}}{z}\right|$ ——— (2)

We can choose R_0 (>1) large enough such that $R_0 > 2n\left|\frac{a_i}{a_n}\right|$ for $i = 0 \ldots n-1$

Then, whenever $|z| \geq R_0$

$\Rightarrow |z|^{n-i} \geq R_0^{n-i} \underset{(R \geq 1)}{\geq} R_0 \overset{\text{by choice}}{>} 2n\left|\frac{a_i}{a_n}\right|$

$\forall\ i = 0, \ldots, n-1$

$\Rightarrow \left|\frac{2n\, a_i}{a_n}\right| < |z|^{n-i}$

$\Rightarrow \left|\frac{a_i}{z^{n-i}}\right| < \left|\frac{a_n}{2n}\right|$ for $i = 0, \ldots, n-1$

Then by (2),

$\rightarrow |w| < n \cdot \left|\frac{a_n}{2n}\right|$ whenever $|z| \geq R$

$\Rightarrow \big||w| - |a_n|\big| \geq \left|\frac{a_n}{2}\right|.$

By (1)

$|P(z)| \geq \underbrace{R_0^n \cdot \left|\frac{a_n}{2}\right|}_{\text{constant}}$ for all $|z| \geq R$

Thus

Thus $\quad |f(z)| \leq \dfrac{1}{|P(z)|} \leq \dfrac{2}{|a_n| R_0^n}$. for all $|z| \geq R_0$

For $|z| \leq R_0$, $f(z)$ is bounded.

Therefore $f(z)$ is bounded for all $z \in \mathbb{C}$.

Remark : (1) Fundamental theorem of Algebra does not hold true real nor.

Take $f(x) = x^2 + 1$: a nonconstant real poly
: have no root in $\mathbb{R}$.

(2) Liouville's theorem does not hold for real functions :

Take $f(x) = \sin x$: differentiable on $\mathbb{R}$
bounded ($|\sin x| \leq 1 \; \forall x \in \mathbb{R}$)

but $f(x)$ is not constant.

(3) Consider $f(z) = \sin z$, $z \in \mathbb{C}$.

Check the Hypothesis & conclusion of Liouville's Th^m.
