Polar form of Cauchy - Riemann equations

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}, \quad \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}.$$

Derivation: We have

$x = r\cos\theta$, $y = r\sin\theta$, so that $r^2 = x^2+y^2$ and $\theta = \tan^{-1}\frac{y}{x}$.

Now, $$\frac{\partial r}{\partial x} = \frac{x}{r} = \cos\theta$$

$$\frac{\partial r}{\partial y} = \sin\theta.$$

$$\frac{\partial \theta}{\partial x} = \frac{1}{1+\left(\frac{y}{x}\right)^2}\left(-\frac{y}{x^2}\right) = -\frac{\sin\theta}{r}$$

and $$\frac{\partial \theta}{\partial y} = \frac{1}{1+\left(\frac{y}{x}\right)^2}\left(\frac{1}{x}\right) = \frac{\cos\theta}{r}.$$

Now, $$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial x} + \frac{\partial u}{\partial \theta}\cdot\frac{\partial \theta}{\partial x}$$

$$= \frac{\partial u}{\partial r}\cos\theta - \frac{\partial u}{\partial \theta}\frac{\sin\theta}{r} \longrightarrow (1)$$

Similarly, $$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial r}\sin\theta + \frac{\partial u}{\partial \theta}\frac{\cos\theta}{r} \longrightarrow (2)$$

$$\frac{\partial v}{\partial x} = \frac{\partial v}{\partial r}\cos\theta - \frac{\partial v}{\partial \theta}\frac{\sin\theta}{r} \longrightarrow (3)$$

$$\frac{\partial v}{\partial y} = \frac{\partial v}{\partial r}\sin\theta + \frac{\partial v}{\partial \theta}\frac{\cos\theta}{r} \longrightarrow (4)$$

Since Cauchy - Riemann equations in cartesian form are

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}; \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \longrightarrow (5)$$

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Substituting in these equations from above, we have

$$\frac{\partial u}{\partial r}\cos\theta - \frac{\partial u}{\partial \theta}\frac{\sin\theta}{r} = \frac{\partial v}{\partial r}\sin\theta + \frac{\partial v}{\partial \theta}\frac{\cos\theta}{r} \longrightarrow (6)$$

and $$\frac{\partial u}{\partial r}\sin\theta + \frac{\partial u}{\partial \theta}\frac{\cos\theta}{r} = -\frac{\partial v}{\partial r}\cos\theta + \frac{\partial v}{\partial \theta}\frac{\sin\theta}{r} \longrightarrow (7)$$

Multiplying (6) by $\cos\theta$ and (7) by $\sin\theta$ and adding, we get

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \longrightarrow (8)$$

Again multiplying (6) by $\sin\theta$ and (7) by $\cos\theta$ and subtracting, we get

$$\frac{\partial u}{\partial \theta} = -r\frac{\partial v}{\partial r} \longrightarrow (9)$$

Thus, the C-R equations in polar form are

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \text{ and } \frac{\partial u}{\partial \theta} = -r\frac{\partial v}{\partial r}.$$

Note: Differentiating (8) partially w.r.t. r, we have

$$\frac{\partial^2 u}{\partial r^2} = -\frac{1}{r^2}\frac{\partial v}{\partial \theta} + \frac{1}{r}\frac{\partial^2 v}{\partial \theta\,\partial r} \longrightarrow (10)$$

Differentiating (9) partially w.r.t. θ, we get

$$\frac{\partial^2 u}{\partial \theta^2} = -r\frac{\partial^2 v}{\partial r\,\partial \theta} \longrightarrow (11)$$

Hence using (8), (10) and (11), we have

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = \frac{1}{r^2}\frac{\partial v}{\partial \theta} + \frac{1}{r}\frac{\partial^2 v}{\partial \theta\,\partial r} + \frac{1}{r}\left(\frac{1}{r}\frac{\partial v}{\partial \theta}\right)$$
$$+ \frac{1}{r^2}\left(-r\frac{\partial^2 v}{\partial r\,\partial \theta}\right) = 0 \quad \left(\because \frac{\partial^2 v}{\partial \theta\,\partial r} = \frac{\partial^2 v}{\partial r\,\partial \theta}\right)$$

Harmonic function :

Any function of x, y which has continuous partial derivatives of the first and second orders and satisfies Laplace's equation is called a Harmonic function.

If $f(z) = u + iv$ be analytic, then u, v both are harmonic functions since they satisfy Laplace's equation. In such a case, u and v are called conjugate harmonic function or simply conjugate functions.

5. If harmonic functions u and v satisfy Cauchy-Riemann equations, then $u + iv$ is an analytic function.

6. Orthogonal System :

Two family of curves $u(x,y) = c_1$ and $v(x,y) = c_2$ are said to form an orthogonal system if they intersect at right angles at each of their point of intersection.

Diff. $u(x,y) = c_1$, we get

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dx} = 0$$

$$\therefore \frac{dy}{dx} = -\frac{\partial u}{\partial x} \Big/ \frac{\partial u}{\partial y} = m_1 \text{ (say)}.$$

Similarly from $v(x,y) = c_2$, we get

$$\frac{dy}{dx} = -\frac{\partial v}{\partial x} \Big/ \frac{\partial v}{\partial y} = m_2 \text{, say}.$$

Now, the two families of curves will intersect orthogonally if $m_1 m_2 = -1$

$$\therefore \frac{\partial u}{\partial x} \cdot \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \cdot \frac{\partial v}{\partial y} = 0.$$

Example: If $w = f(z) = u + iv$ be an analytic function of $z = x + iy$, show that the curves u = const., v = const. represented on the z-plane intersect at right angles.

Soln:- $f(z) = u + iv$ is regular function of z, then functions u and v satisfy Cauchy-Riemann equations $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$.

Multiplying these, we get

$$\frac{\partial u}{\partial x} \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \cdot \frac{\partial v}{\partial y} = 0$$

which is the condition that the curve u = const. and v = const. intersect at right angles as shown above.

Note:- Hence if $f(z)$ is regular function of z, then the curves

$$u = R[f(z)] = \text{const.}$$

$$\text{and } v = I[f(z)] = \text{const.}$$

form an orthogonal system i.e. they intersect at right angles.

Ex!: Show that a harmonic function satisfies the formal differential equation $\dfrac{\partial^2 u}{\partial z\, \partial \bar{z}} = 0$.

Soln:- If u is a harmonic function, then

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

Now, $x = \frac{1}{2}(z+\bar{z})$, $y = \frac{1}{2i}(z-\bar{z})$

$$\therefore \frac{\partial u}{\partial \bar{z}} = \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial \bar{z}} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial \bar{z}} = \frac{1}{2}\frac{\partial u}{\partial x} - \frac{1}{2i}\frac{\partial u}{\partial y}$$

and
$$\frac{\partial^2 u}{\partial z\,\partial \bar{z}} = \frac{1}{2}\frac{\partial^2 u}{\partial x^2}\cdot\frac{\partial x}{\partial z} + \frac{1}{2}\frac{\partial^2 u}{\partial y\,\partial y}\cdot\frac{\partial y}{\partial z} - \frac{1}{2i}\frac{\partial^2 u}{\partial y^2}\frac{\partial y}{\partial z} - \frac{1}{2i}\frac{\partial^2 u}{\partial x\,\partial y}\cdot\frac{\partial x}{\partial z}$$

$$= \frac{1}{2}\cdot\frac{\partial^2 u}{\partial x^2}\cdot\frac{1}{2} + \frac{1}{2}\cdot\frac{\partial^2 u}{\partial y\,\partial x}\cdot\frac{1}{2i} - \frac{1}{2i}\frac{\partial^2 u}{\partial y^2}\cdot\frac{1}{2i} - \frac{1}{2i}\frac{\partial^2 u}{\partial x\,\partial y}\cdot\frac{1}{2}$$

$$= \frac{1}{4}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) = 0 \quad (\because u \text{ is harmonic}).$$

Proved.

Method of constructing a regular function:
(Milne-Thomsen's Method):

Since $f(z) = u(x,y) + iv(x,y)$ and $x = \frac{1}{2}(z+\bar{z})$, $y = \frac{1}{2i}(z-\bar{z})$, we may write

$$f(z) = u\left[\tfrac{1}{2}(z+\bar{z}), \tfrac{1}{2i}(z-\bar{z})\right] + iv\left[\tfrac{1}{2}(z+\bar{z}), \tfrac{1}{2i}(z-\bar{z})\right] \quad (1)$$

We may regard (1) as a formal identity in two independent variables $z, \bar{z}$. On putting $\bar{z} = z$, we get $f(z) = u(z,0) + iv(z,0)$.

Now, $f'(z) = \dfrac{\partial w}{\partial x} = u_x - iv_x = u_x - iu_y$ (by C-R equations).

Let $u_x = \phi_1(x,y)$, $u_y = \phi_2(x,y)$; then

$$f'(z) = \phi_1(x,y) - i\phi_2(x,y) = \phi_1(z,0) - i\phi_2(z,0).$$

Integrating, we get $f(z) = \int \phi_1(z,0)\,dz - i\int \phi_2(z,0)\,dz + c$ where c is arbitrary constant.

Similarly, if $v(x,y)$ be given, we have

$$f(z) = \int[\psi_1(z,0) + i\,\psi_2(z,0)]\,dz + c'$$ where $v_y = \psi_1(x,y)$ and $v_x = \psi_2(x,y)$.

Application of Cauchy - Riemann equations to find harmonic conjugate:

Theorem: If $f(z) = u + iv$ is an analytic function, where both $u(x,y)$ and $v(x,y)$ are conjugate functions, given one of these, say $u(x,y)$ to find the other $v(x,y)$.

Proof: Since v is a function of two real variables x and y, therefore,

$$dv = \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy$$

$$= -\frac{\partial u}{\partial y}dx + \frac{\partial u}{\partial x}dy \text{ (By C-R equations)}$$

The R.H.S. of this equation is of the form $Mdx + Ndy$ where $M = -\frac{\partial u}{\partial y}$ and $N = \frac{\partial u}{\partial x}$.

Therefore $\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}\left(-\frac{\partial u}{\partial y}\right) = -\frac{\partial^2 u}{\partial y^2}$

and $\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial^2 u}{\partial x^2}$

Now, since u is harmonic function, therefore it satisfies Laplace's equation, i.e.

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

$$\Rightarrow \frac{\partial^2 u}{\partial x^2} = -\frac{\partial^2 u}{\partial y^2}$$

$$\Rightarrow \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.$$

Thus $-\frac{\partial u}{\partial y}dx + \frac{\partial u}{\partial x}dy$ satisfies exact differential equation. Therefore

$$dv = -\frac{\partial u}{\partial y}dx + \frac{\partial u}{\partial x}dy$$

can be integrated and we can get $v(x,y)$.

If $f(z) = u(x,y) + iv(x,y)$ is analytic, then v is called a harmonic conjugate of u. Since ~~if~~ $if = i(u+iv) = -v+iu$ is analytic whenever f is analytic, we have the anti-symmetric property that v is a harmonic conjugate of u if and only if u is a harmonic conjugate of $-v$.

An Important Observation:

Since $x = \frac{1}{2}(z+\bar{z})$, $y = \frac{1}{2i}(z-\bar{z})$,

u and v can be regarded as functions of two independent variables z and $\bar{z}$. If u and v have first order continuous derivatives, the condition that w shall be independent of $\bar{z}$ is

$$\frac{\partial w}{\partial \bar{z}} = 0, \text{ or } \frac{\partial}{\partial \bar{z}}(u+iv) = 0.$$

or, $$\left(\frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial \bar{z}} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial \bar{z}}\right) + i\left(\frac{\partial v}{\partial x}\cdot\frac{\partial x}{\partial \bar{z}} + \frac{\partial v}{\partial y}\frac{\partial y}{\partial \bar{z}}\right) = 0.$$

or, $$\frac{1}{2}\frac{\partial u}{\partial x} - \frac{1}{2i}\frac{\partial u}{\partial y} + \frac{i}{2}\frac{\partial v}{\partial x} - \frac{1}{2}\frac{\partial v}{\partial y} = 0$$

or, $$\frac{\partial u}{\partial x} + i\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial x} - \frac{\partial v}{\partial y} = 0.$$

Hence, by equating real and imaginary parts to zero, we get

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

which are the Cauchy-Riemann equations.

It follows that if $f(z)$ is an analytic function of z, then x and y can occur in $f(z)$ only in the combination of $x+iy$.

1. **Example**: ~~If~~ $f(z) = z^2$. Find $\frac{d\,f(z)}{dz}$.

Sol^n:: Since $f(z) = z^2 = x^2 - y^2 + 2ixy$

$$\therefore\ f'(z) = \frac{d\,f(z)}{dz} = \frac{\partial f(z)}{\partial x} = 2x + i(2y)$$

$$= 2(x+iy) = 2z.$$

2. If $f(z) = \operatorname{Im}(z) = iy$. Determine the analyticity?

Sol^n:- We have

$f(z) = \operatorname{Im}(z) = iy = 0 + iy$. Hence ~~if~~ $u = 0,\ v = y$.

Then $\frac{\partial u}{\partial x} = 0,\ \frac{\partial u}{\partial y} = 0,\ \frac{\partial v}{\partial x} = 0,\ \frac{\partial v}{\partial y} = 1.$

Here all partial derivatives exist but Cauchy-Riemann equations are not satisfied for any value of z. Therefore f(z) is non-analytic.

3. Let $f(z) = |z|^4$. Check its analyticity.

Sol^n:- Let $f(z) = |z|^4 = (x^2+y^2)^2$

$$= x^4 + 2x^2y^2 + y^4.$$

Then $\frac{\partial u}{\partial x} = 4x^3 + 4xy^2,\ \frac{\partial u}{\partial y} = 4y^3 + 4x^2y.$

$$\frac{\partial v}{\partial x} = 0,\ \frac{\partial v}{\partial y} = 0.$$

Here, all partial derivatives exist but Cauchy-Riemann equations are not satisfied for the function f(z) so it cannot be analytic at any point.

4. The function $f(z) = e^z$ is analytic everywhere.

Sol^n:- Here $f(z) = e^z = e^{x+iy} = e^x e^{iy}$

$$= e^x(\cos y + i\sin y).$$

and if $f(z) = u + iv$, then

$$u(x,y) = e^x \cos y \quad \text{and} \quad v(x,y) = e^x \sin y.$$

The four partial derivatives

$$u_x = e^x \cos y, \quad u_y = -e^x \sin y$$

$$v_x = e^x \sin y, \quad v_y = e^x \cos y.$$

are continuous and satisfy Cauchy - Riemann equations. Moreover,

$$\frac{d\,e^z}{dz} = u_x + i v_x = e^x \cos y + i e^x \sin y$$

$$= e^x(\cos y + i\sin y) = e^x e^{iy}$$

$$= e^z.$$

Without further calculations, we conclude that $\sin z$ and $\cos z$ are functions analytic everywhere.

Moreover, $$\frac{d}{dz}(\sin z) = \frac{d}{dz}\left(\frac{e^{iz} - e^{-iz}}{2i}\right)$$

$$= \frac{e^{iz} + e^{-iz}}{2} = \cos z.$$

Similarly, $$\frac{d}{dz}(\cos z) = -\sin z.$$

Thus, all the formulas for differentiation of trigonometric functions are valid.

Th: If $f'(z) = 0$ in a domain D, then $f(z)$ is constant in D.

Proof: Since $f'(z) = 0$ in a domain D. So,

$$f'(z) = \frac{\partial f}{\partial x} = -i \frac{\partial f}{\partial y} = 0 \text{, for all points in } D.$$

i.e. $\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = 0$ for all points in D.

Now, $\frac{\partial u}{\partial x} = \frac{\partial u}{\partial y} = 0$ in D implies that $u(x,y)$ is constant along every horizontal and vertical line segments in D. Similarly, $\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} = 0$ in D implies that $v(x,y)$ is constant along every horizontal and vertical line segments in D.

Thus, $f(z) = u(z) + iv(z)$ is constant along every polygonal line in D whose sides are parallel to coordinate axes. Since any two points in D can be joined by such a line, therefore, $f(z_1) = f(z_2)$ for any pair of points $z_1, z_2 \in D$, so that $f(z)$ must be constant in D.

Derivative of $w = f(z)$ in polar form:

Here, we suppose $w = f(z) = u + iv$ be the given function. Then

$$\frac{dw}{dz} = \frac{\partial w}{\partial x}.$$

[same as $\frac{df(z)}{dz} = \frac{\partial f(z)}{\partial x}$

We use $\frac{dw}{dz}$.

Since in case of polar form w is a function of r and θ; and r, θ are functions of x and y. So

$$\frac{dw}{dz} = \frac{\partial w}{\partial x} = \frac{\partial w}{\partial r}\cdot\frac{\partial r}{\partial x} + \frac{\partial w}{\partial \theta}\cdot\frac{\partial \theta}{\partial x}$$

$$= \frac{\partial w}{\partial r}\cos\theta - \left(\frac{\partial u}{\partial \theta} + i\frac{\partial v}{\partial \theta}\right)\frac{\sin\theta}{r}$$

$(\because w = u + iv)$

$$= \frac{\partial w}{\partial r}\cos\theta - \left(-r\frac{\partial v}{\partial r} + ir\frac{\partial u}{\partial r}\right)\frac{\sin\theta}{r}$$

$$= \frac{\partial w}{\partial r}\cos\theta - i\left(\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}\right)r\cdot\frac{\sin\theta}{r}$$

$$= \frac{\partial w}{\partial r}\cos\theta - i\frac{\partial w}{\partial r}\sin\theta$$

$$= (\cos\theta - i\sin\theta)\frac{\partial w}{\partial r} = e^{-i\theta}\frac{\partial w}{\partial r}.$$

Similarly, $$\frac{dw}{dz} = \frac{\partial w}{\partial r}\cdot\frac{\partial r}{\partial x} + \frac{\partial w}{\partial \theta}\cdot\frac{\partial \theta}{\partial x}$$

$$= \left(\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}\right)\cos\theta - \frac{\partial w}{\partial \theta}\cdot\frac{\sin\theta}{r}$$

$$= \left(\frac{1}{r}\frac{\partial v}{\partial \theta} - \frac{i}{r}\frac{\partial u}{\partial \theta}\right)\cos\theta - \frac{\partial w}{\partial \theta}\cdot\frac{\sin\theta}{r}$$

$$= -\frac{i}{r}\left(\frac{\partial u}{\partial \theta} + i\frac{\partial v}{\partial \theta}\right)\cos\theta - \frac{\partial w}{\partial \theta}\cdot\frac{\sin\theta}{r}$$

$$= -\frac{i}{r}\frac{\partial w}{\partial \theta}\cos\theta - \frac{\partial w}{\partial \theta}\frac{\sin\theta}{r}$$

$$= -\frac{i}{r}(\cos\theta - i\sin\theta)\frac{\partial w}{\partial \theta} = -\frac{i}{r}e^{-i\theta}\frac{\partial w}{\partial \theta}.$$

Thus in polar form, $\frac{dw}{dz} = (\cos\theta - i\sin\theta)\frac{\partial w}{\partial r} = -\frac{i}{r}(\cos\theta - i\sin\theta)\frac{\partial w}{\partial \theta}.$

Note: $f'(z) = u_x + iv_x = v_y - iu_y$.

$\therefore |f'(z)|^2 = |u_x|^2 + |v_x|^2 = |v_y|^2 + |u_y|^2 = u_x v_y - u_y v_x$.

The last expression shows that $|f'(z)|^2$ is the Jacobian of u and v with resp. to x and y.

Example: Prove that the function

$$u = x^3 - 3xy^2 + 3x^2 - 3y^2 + 1$$

satisfies Laplace's equation and determine the corresponding analytic function $u + iv$.

Solⁿ: Here, $u = x^3 - 3xy^2 + 3x^2 - 3y^2 + 1 \longrightarrow (1)$

$\therefore \dfrac{\partial u}{\partial x} = 3x^2 - 3y^2 + 6x = \phi(x,y)$ (say)

$\dfrac{\partial u}{\partial y} = -6xy - 6y = \psi(x,y)$ (say)

Also, $\dfrac{\partial^2 u}{\partial x^2} = 6x + 6$, $\dfrac{\partial^2 u}{\partial y^2} = -6x - 6$.

$\therefore \dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 6x + 6 - 6x - 6 = 0$.

$\Rightarrow u$ satisfies Laplace's equation. Hence u is a harmonic function.

$$f'(z) = \phi(z,0) - \psi(z,0)$$
$$= 3z^2 + 6z.$$

Now, integrating it, we get

$$f(z) = \int (3z^2 + 6z)\, dz + c = z^3 + 3z^2 + c.$$

(2) Theorem: Let $|f(z)|$ be constant in a region where $f(z)$ is analytic. Then $f(z)$ is constant.

Proof: If $|f(z)| = |u + iv| = c$, then $u^2 + v^2 = c^2$.

Differentiating, we get

$$u\frac{\partial u}{\partial x} + v\frac{\partial v}{\partial x} = 0 \quad \& \quad u\frac{\partial u}{\partial y} + v\frac{\partial v}{\partial y} = 0.$$

Using Cauchy-Riemann equations, the above equation reduce to

$$u\frac{\partial u}{\partial x} - v\frac{\partial u}{\partial y} = 0 \ \& \ u\frac{\partial u}{\partial y} + v\frac{\partial u}{\partial x} = 0.$$

Eliminating $\frac{\partial u}{\partial y}$ from above equations, we get

$$(u^2+v^2)\frac{\partial u}{\partial x} = 0 \Rightarrow \frac{\partial u}{\partial x} = 0,$$

In like manner, we can show that

$$\frac{\partial u}{\partial y} = \frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} = 0.$$

Hence u, v are constants implies the result.

Q: Prove that $\lim_{z\to 0} \frac{\bar{z}}{z}$ does not exist.

Sol^n: We know that limit exists only when it is independent of the path along which z approaches zero.

First suppose that $z\to 0$ along x-axis. Then $y=0$ and $z = x+iy = x$. Also, $\bar{z} = x-iy = x$.

Therefore, $\lim_{z\to 0}\frac{\bar{z}}{z} = \lim_{x\to 0}\frac{x}{x} = 1.$

Again, suppose that $z\to 0$ along y-axis, then $x=0$, $z = iy$ and $\bar{z} = -iy$. Therefore, in this case

$$\lim_{z\to 0}\frac{\bar{z}}{z} = \lim_{y\to 0}\frac{-iy}{iy} = -1.$$

Since two values of limits are different, therefore limit does not exist.

Ex:- Show that $u(x,y) = e^{-x}[x\sin y - y\cos y]$ is harmonic and find its harmonic conjugate $v(x,y)$ such that $f(z) = u+iv$ is analytic.

Solⁿ:- We have $u(x,y) = e^{-x}[x\sin y - y\cos y] \longrightarrow (1)$

$$\frac{\partial u}{\partial x} = e^{-x}\sin y - xe^{-x}\sin y + ye^{-x}\cos y$$

$$\therefore \frac{\partial^2 u}{\partial x^2} = -2e^{-x}\sin y + xe^{-x}\sin y - ye^{-x}\cos y$$

$$\frac{\partial u}{\partial y} = xe^{-x}\cos y - e^{-x}\cos y + ye^{-x}\sin y$$

$$\frac{\partial^2 u}{\partial y^2} = -xe^{-x}\sin y + 2e^{-x}\sin y + ye^{-x}\cos y$$

Thus we get $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$. So that $u(x,y)$ is harmonic.

From Cauchy - Reimann equations

$$\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x} = e^{-x}\sin y - xe^{-x}\sin y + ye^{-x}\cos y \longrightarrow (2)$$

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} = -xe^{-x}\cos y + e^{-x}\cos y - ye^{-x}\sin y \longrightarrow (3)$$

Integrating (2) w.r.t. y, treating x as constant, we get

$$v(x,y) = ye^{-x}\sin y + xe^{-x}\cos y + g(x) \longrightarrow (4)$$

where $g(x)$ is an arbitrary real function of x. From (3) and (4), we obtain

$$\cancel{-ye^{-x}\sin y} + \cancel{e^{-x}\cos y} - \cancel{xe^{-x}\cos y} + g'(x) = \cancel{e^{-x}\cos y} - \cancel{xe^{-x}\cos y} - \cancel{ye^{-x}\sin y}$$

$$\therefore g'(x) = 0$$

So that $g(x) = c$ (an arbitrary real constant)

Hence from (4), we have

$$v(x,y) = e^{-x}(y\sin y + x\cos y) + c.$$

This is the required harmonic conjugate of $u(x,y)$.

Example: If $u(x,y) = e^x(x\cos y - y\sin y)$, find the analytic function $u + iv$.

Sol^n:- We have

$$u(x,y) = e^x(x\cos y - y\sin y) \longrightarrow (1)$$

$$\therefore \frac{\partial u}{\partial x} = e^x(x\cos y - y\sin y) + e^x(\cos y)$$
$$= xe^x\cos y - ye^x\sin y + e^x\cos y$$

$$\therefore \frac{\partial^2 u}{\partial x^2} = e^x\cos y + xe^x\cos y - ye^x\sin y + e^x\cos y$$

$$\frac{\partial u}{\partial y} = e^x(-x\sin y - \sin y - y\cos y)$$
$$= -xe^x\sin y - e^x\sin y - ye^x\cos y$$

$$\therefore \frac{\partial^2 u}{\partial y^2} = -xe^x\cos y - e^x\cos y + ye^x\sin y - e^x\cos y$$

$$\therefore \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \cancel{e^x\cos y} + \cancel{xe^x\cos y} - \cancel{ye^x\sin y} + \cancel{e^x\cos y}$$
$$- \cancel{xe^x\cos y} - \cancel{e^x\cos y} + \cancel{ye^x\sin y} - \cancel{e^x\cos y} = 0$$

Therefore, $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$. So that $u(x,y)$ is harmonic.

Now, $dv = \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy = -\frac{\partial u}{\partial y}dx + \frac{\partial u}{\partial x}dy$

$$= e^x(x\sin y + \sin y + y\cos y)dx + e^x(x\cos y - y\sin y + \cos y)dy$$

$\therefore v = \int_{y\text{ const.}} e^x(x\sin y + \sin y + y\cos y)dx + \int(\text{those terms which do not contain } x)dy + c$

$$= (x\sin y + \sin y + y\cos y)e^x - e^x\sin y + c$$
$$= e^x(x\sin y + y\cos y) + c, \text{ where } c \text{ is a constant.}$$

$$\therefore f(z) = u + iv = e^x[x\cos y - y\sin y + ix\sin y - iy\cos y] + ci$$
$$= e^x(x+iy)(\cos y + i\sin y) + ci$$
$$= e^{x+iy}.(x+iy) + ci = ze^z + ci. \text{ Ans.}$$

Second method: $v = xe^x\sin y + ye^x\cos y - e^x\cancel{\cos y}\sin y + e^x\sin y + C$

$$= xe^x\sin y + ye^x\cos y + C \text{ Ans} \quad (\text{by Diff. } \tfrac{\partial u}{\partial x} \text{ w.r.t. } y)$$

Ex: If $w = \log z$, find $\frac{dw}{dz}$ and determine where w is non-analytic.

Sol^n:- We have $w = f(z) = u + iv = \log(x+iy)$

$$= \frac{1}{2}\log(x^2+y^2) + i\tan^{-1}\frac{y}{x}$$

so that $u = \frac{1}{2}\log(x^2+y^2)$, $v = \tan^{-1}\frac{y}{x}$.

$$\therefore \frac{\partial u}{\partial x} = \frac{x}{x^2+y^2} = \frac{\partial v}{\partial y}$$

$$\frac{\partial u}{\partial y} = \frac{y}{x^2+y^2} = -\frac{\partial v}{\partial x}.$$

Since the C-R equations are satisfied and the partial derivatives are continuous except at $(0,0)$. Hence w is analytic everywhere except at $z=0$.

$$\therefore \frac{dw}{dz} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{x}{x^2+y^2} + i\frac{-y}{x^2+y^2} = \frac{x-iy}{(x+iy)(x-iy)}$$

$$= \frac{1}{x+iy} = \frac{1}{z} \quad (z \neq 0).$$

Ex: Prove that the function $f(z)$ defined by

$$f(z) = \frac{x^3(1+i) - y^3(1-i)}{x^2+y^2} \quad (z \neq 0), \quad f(0) = 0$$

is continuous and the C-R equations are satisfied at the origin, yet $f'(0)$ does not exist.

Sol^n:- We have $f(z) = \frac{x^3(1+i) - y^3(1-i)}{x^2+y^2}$, $(z \neq 0)$

$$= 0, \qquad (z = 0).$$

$$\therefore \lim_{z \to 0} f(z) = \lim_{\substack{x \to 0 \\ y \to 0}} \frac{x^3(1+i) - y^3(1-i)}{x^2+y^2} \quad \left[\text{Repeated limits}\right]$$

$$= \lim_{y \to 0} \frac{-y^3(1-i)}{y^2} = \lim_{y \to 0}[-y(1-i)] = 0.$$

$$\lim_{z \to 0} f(z) = \lim_{x \to 0}\lim_{y \to 0} \frac{x^3(1+i) - y^3(1-i)}{x^2+y^2} = \lim_{x \to 0}\frac{x^3(1+i)}{x^2}$$

$$= \lim_{x \to 0}[x(1+i)] = 0.$$

Also, $f(0) = 0$ (given).

Thus $\lim_{z \to 0} f(z) = f(0)$, when $x \to 0$ first and then $y \to 0$ and also vice-versa. Now, let both x and y tend to zero simultaneously along the path $y = mx$. Then

$$\lim_{z \to 0} f(z) = \lim_{\substack{y \to mx \\ x \to 0}} \frac{x^3(1+i) - y^3(1-i)}{x^2+y^2}$$

$$= \lim_{x \to 0} \frac{x^3(1+i) - m^3x^3(1-i)}{(1+m^2)\,x^2}$$

$$= \lim_{x \to 0} \frac{x\left[1+i - m^3(1-i)\right]}{1+m^2} = 0.$$

Hence, $\lim_{z \to 0} f(z) = f(0)$, in whatever manner $z \to 0$.

$\therefore$ $f(z)$ is continuous at the origin.

Now, $f(z) = \dfrac{x^3-y^3}{x^2+y^2} + i\,\dfrac{x^3+y^3}{x^2+y^2} = u(x,y) + i\,v(x,y)$

Also, $u(0,0) = 0$ and $v(0,0) = 0$ $\quad [\because f(0) = 0]$.

$$\therefore \left(\frac{\partial u}{\partial x}\right)_{(0,0)} = \lim_{x \to 0} \frac{u(x,0) - u(0,0)}{x} = \lim_{x \to 0} \frac{x}{x} = 1$$

$$\left(\frac{\partial u}{\partial y}\right)_{(0,0)} = \lim_{y \to 0} \frac{u(0,y) - u(0,0)}{y} = \lim_{y \to 0} \frac{-y}{y} = -1.$$

$$\left(\frac{\partial v}{\partial x}\right)_{(0,0)} = \lim_{x \to 0} \frac{v(x,0) - v(0,0)}{x} = \lim_{x \to 0} \frac{x}{x} = 1$$

and $$\left(\frac{\partial v}{\partial y}\right)_{(0,0)} = \lim_{y \to 0} \frac{v(0,y) - v(0,0)}{y} = \lim_{y \to 0} \frac{y}{y} = 1.$$

Hence, at $(0,0)$, $\dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y}$ and $\dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x}$.

Thus the C-R equations are satisfied at the origin.

But $$f'(0) = \lim_{z \to 0} \frac{f(z) - f(0)}{z} = \lim_{z \to 0} \frac{f(z)}{z} = \lim_{z \to 0} \frac{x^3 - y^3 + i(x^3+y^3)}{(x^2+y^2)(x+iy)}$$

If $z \to 0$ along the path $y = mx$, then

$f'(0) = \dfrac{1-m^3 + i(1+m^3)}{(1+m^2)(1+im)}$, which assumes different values as m varies. So, $f'(z)$ is not unique at $(0,0)$. Thus $f(z)$ is not analytic at the origin even though it is continuous and satisfies the C-R equations thereat.

Q: Show that the function $f(z) = u + iv$, where
$$f(z) = \frac{x^3(1+i) - y^3(1-i)}{x^2+y^2}, \quad (z \neq 0), \quad f(0) = 0$$
is continuous and that the Cauchy-Riemann equations are satisfied at the origin, yet $f'(0)$ does not exist.

Sol^n:- Here $f(z) = \dfrac{x^3(1+i) - y^3(1-i)}{x^2+y^2}$, $(z \neq 0)$ and $f(0) = 0$.

$$\therefore \quad u = \frac{x^3 - y^3}{x^2+y^2} \quad \text{and} \quad v = \frac{x^3+y^3}{x^2+y^2}$$

when $z \neq 0$, u and v are rational functions of x and y with non-zero denominators. It follows that they are continuous when $z \neq 0$. To test them for continuity at $z = 0$, we get on changing, to polars, $u = r(\cos^3\theta - \sin^3\theta)$ and $v = r(\cos^3\theta + \sin^3\theta)$, each of which tends to zero as $r \to 0$ whatever value θ may have.

Now, the actual values of u and v at the origin are zero since $f(0) = 0$. Since the actual and limiting values of u and v are equal at the origin, they are continuous there. Hence $f(z)$ is a continuous function for all values of z. Now at the origin

$$\frac{\partial u}{\partial x} = \lim_{x \to 0} \frac{u(x,0) - u(0,0)}{x} = \lim_{x \to 0} \frac{x - 0}{x} = 1.$$

Similarly $\dfrac{\partial u}{\partial y} = \lim_{y \to 0} \dfrac{-y - 0}{y} = -1$.

$$\frac{\partial v}{\partial x} = \lim_{x \to 0} \frac{x - 0}{x} = 1 \quad \text{and} \quad \frac{\partial v}{\partial y} = \lim_{y \to 0} \frac{y - 0}{y} = 1.$$

Hence $\dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y}$ and $\dfrac{\partial v}{\partial x} = -\dfrac{\partial u}{\partial y}$.

The Cauchy-Riemann equations are therefore satisfied,

Again, $$f'(0) = \lim_{z \to 0} \frac{f(z) - f(0)}{z} = \lim_{z \to 0} \frac{(x^3 - y^3) + i(x^3+y^3)}{(x^2+y^2)(x+iy)}$$

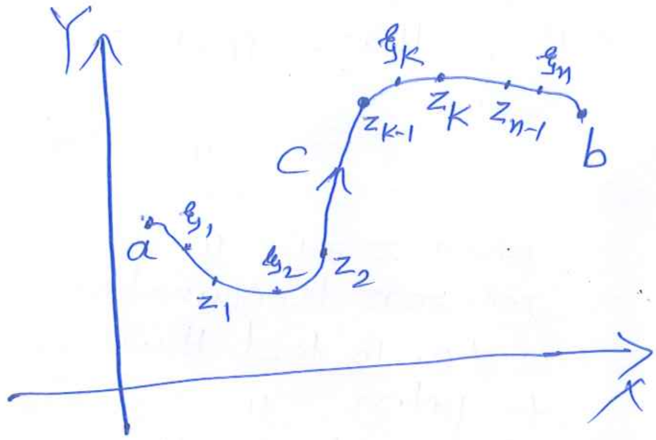
Now, let $z \to 0$ along $y = x$, then
$$f'(x) = \lim_{x \to 0} \frac{2i}{2(1+i)} = \frac{1}{2}(1+i)$$

Again let $z \to 0$ along x-axis, then
$$f'(0) = \lim_{x \to 0} \frac{x^2 + ix^3}{x^3} = (1+i) \quad [\because y = 0]$$

Since the two limits obtained are different, the function $f(z)$ is not differentiable at $z = 0$.

Q: If $f(z) = \dfrac{x^3 y (y - ix)}{x^6 + y^2}$, $(z \neq 0)$, $f(0) = 0$, prove that $\dfrac{f(z) - f(0)}{z} \to 0$ as $z \to 0$ along any radius vector but not as $z \to 0$ in any manner.

Let $f(z)$ be continuous at all points of a curve C, which we shall assume has a finite length i.e. C is a rectifiable curve.



Sol^n:- Let $z \to 0$ along $y = mx$. Then

$$\lim_{z \to 0} \frac{f(z)-f(0)}{z} = \lim_{z \to 0} \frac{x^3 y(y-ix)}{(x^6+y^2)(x+iy)}$$

$$= \lim_{x \to 0} \frac{x^3 mx(mx-ix)}{(x^6+m^2x^2)(x+imx)}$$

$$= \lim_{x \to 0} \frac{m(m-i)\cdot x^2}{(m^2+x^4)(1+im)} = 0.$$

Now, let $z \to 0$ along $y = x^3$.

Then $$\lim_{z \to 0} \frac{f(z)-f(0)}{z} = \lim_{x \to 0} \frac{x^6(x^3-ix)}{(x^6+x^6)(x+ix^3)}$$

$$= \lim_{x \to 0} \frac{(x^2-i)}{2(1+ix^2)} = -\frac{1}{2}i$$

Hence the result.

Q: Examine the nature of the function $f(z) = \dfrac{x^2y^5(x+iy)}{x^4+y^{10}}$, $z \neq 0$, $f(0)=0$ in a region including the origin.

Sol^n:- We have $$\frac{f(z)-f(0)}{z} = \left[\frac{x^2y^5(x+iy)}{x^4+y^{10}} - 0\right] \Big/ x+iy$$

$$= \frac{x^2y^5}{x^4+y^{10}}.$$

First let $z \to 0$ along $y = x$. Then $\lim_{z \to 0} \frac{f(z)-f(0)}{z} = 0$

Now, let $z \to 0$ along $y^5 = x^2$. Then $\lim_{z \to 0} \frac{f(z)-f(0)}{z} = \lim_{x \to 0} \frac{x^2 \cdot x^2}{x^4+x^4} = \frac{1}{2}$

Hence $f(z)$ is not analytic at $z=0$. The C-R equations are however satisfied.