ICS141: Discrete Mathematics for Computer Science I

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based on slides by Dr. Baek and Dr. Still
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Provided by McGraw-Hill

Lecture 24

Chapter 5. Counting

5.1 The Basics of Counting

5.2 The Pigeonhole Principle

5.3 Permutations and Combinations

Review

- **Sum Rule:** If a task can be done in one of n_1 ways, or in one of n_2 ways, …, or in one of n_m ways, where none of the set of n_i ways of doing the task is the same as any of the set of n_j ways, for all pairs i and j with $1 \leq i < j \leq m$. Then the number of ways to do the task is $n_1 + n_2 + \cdots + n_m$.
- **Product Rule:** Suppose that a procedure can be broken down into a sequence of m successive tasks. If the task T_1 can be done in n_1 ways; the task T_2 can then be done in n_2 ways; ...; and the task T_m can be done in n_m ways, then there are $n_1 \cdot n_2 \cdots n_m$ ways to do the procedure.

The Product Rule: Example

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- Show that a set $\{x_1,\ldots, x_n\}$ containing n elements has 2^n subsets.
  - A subset can be constructed in n successive steps:
    - Pick or do not pick x_1, pick or do not pick x_2, …, pick or do not pick x_n .
  - Each step can be done in two ways.
  - Thus the number of possible subsets is $\underbrace{2\cdot 2\cdots 2}_{n \text{ factors}} = 2^n$.

The Product Rule: Example

- What is the value of *k* after the following code has been executed?

$k := 0$

for $i_1 := 1$ **to** n_1

  for $i_2 := 1$ **to** n_2

    …

    for $i_m := 1$ **to** n_m

      $k := k + 1$

$$n_1 \cdot n_2 \cdots n_m$$

The Product Rule: Example

- How many functions are there from a set with *m* elements to one with *n* elements?

$$n^m$$

- How many one-to-one functions are there from a set with *m* elements to one with *n* elements?

$$n \cdot (n-1)(n-2) \cdots (n-m+1)$$

- More examples in the textbook

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IP Address Example

- In version 4 of the Internet Protocol (IPv4)
  - Internet address is a string of 32 bits
  - Network number (*netid*) + host number (*hostid*)
  - Valid computer addresses are in one of 3 types:
    - A ***class A*** IP address consists of 0, followed by a 7-bit "netid" $\neq \mathbf{1}^7$, and a 24-bit "hostid"
    - A ***class B*** address has 10, followed by a 14-bit netid and a 16-bit hostid.
    - A ***class C*** address has 110, followed by a 21-bit netid and an 8-bit hostid.
  - The 3 classes have distinct headers (0, 10, 110)
  - Hostids that are all 0s or all 1s are not allowed.

e.g., hawaii.edu is 128.171.224.100

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IP Address Example

Bit Number	0	1	2	3	4	8	16	24	31
Class A	0	netid				hostid			
Class B	1	0	netid				hostid		
Class C	1	1	0	netid				hostid	
Class D	1	1	1	0	Multicast Address				
Class E	1	1	1	1	0	Address			

- How many valid computer addresses are there?

IP Address Solution

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- # of addresses

  = (# class A) + (# class B) + (# class C)

  (by sum rule)
- # class A = (# valid netids)×(# valid hostids)

  (by product rule)
- # valid class A netids = $2^7 - 1 = 127$.
- # valid class A hostids = $2^{24} - 2 = 16{,}777{,}214$.
- Continuing in this fashion we find the answer is:

  3,737,091,842 (3.7 billion IP addresses)

Set Theoretic Version

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- If A is the set of ways to do task 1, and B the set of ways to do task 2, and if A and B are disjoint, then:
  - The ways to do either task 1 or 2 are $A \cup B$, and $|A \cup B| = |A| + |B|$
  - The ways to do both task 1 and 2 can be represented as $A \times B$, and $|A \times B| = |A| \cdot |B|$

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Inclusion-Exclusion Principle

- Suppose that *k* out of *m* ways of doing task 1 <u>also</u> simultaneously accomplish task 2.
  - And there are also *n* ways of doing task 2.
- Then, the number of ways to accomplish "Do either task 1 or task 2" is $m + n - k$.
- Set theory: If *A* and *B* are not disjoint, then $|A \cup B| = |A| + |B| - |A \cap B|$.
  - If they are disjoint, this simplifies to $|A| + |B|$.

Inclusion-Exclusion Example

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- Some hypothetical rules for passwords:
  - Passwords must be 2 characters long
  - Each character must be a letter a ~ z, a digit 0 ~ 9, or one of the 10 punctuation characters ! @ # $ % ^ & * ( )
  - Each password must contain *at least one* digit or punctuation character

Setup of Problem

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- A legal password has a digit or punctuation character in position 1 **or** position 2.
  - These cases overlap, so the principle applies.
- # of passwords with OK symbol in position #1 $= (10 + 10) \times (10 + 10 + 26) = 20 \times 46 = 920$
- # with OK symbol in pos. #2 $= 46 \times 20 = 920$
- # with OK symbol both places $= 20 \times 20 = 400$
- Answer: $920 + 920 - 400 = 1{,}440$

Tree Diagrams

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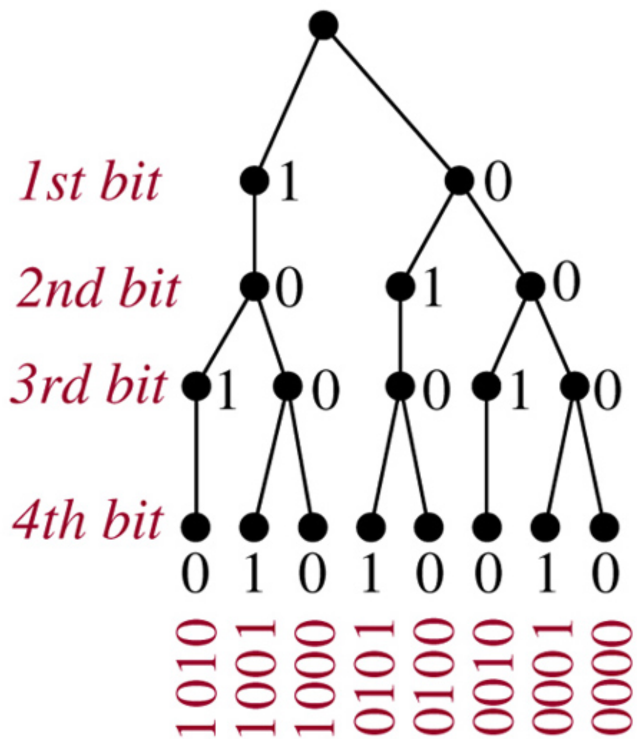
- A tree diagram can be used in many different counting problems.
- To use trees in counting, we use a branch to represent each possible choice.
- We represent the possible outcomes by the leaves, which are the endpoints of branches not having other branches starting at them.

Tree Diagrams: Example

- How many bit strings of length four do not have two consecutive 1s?



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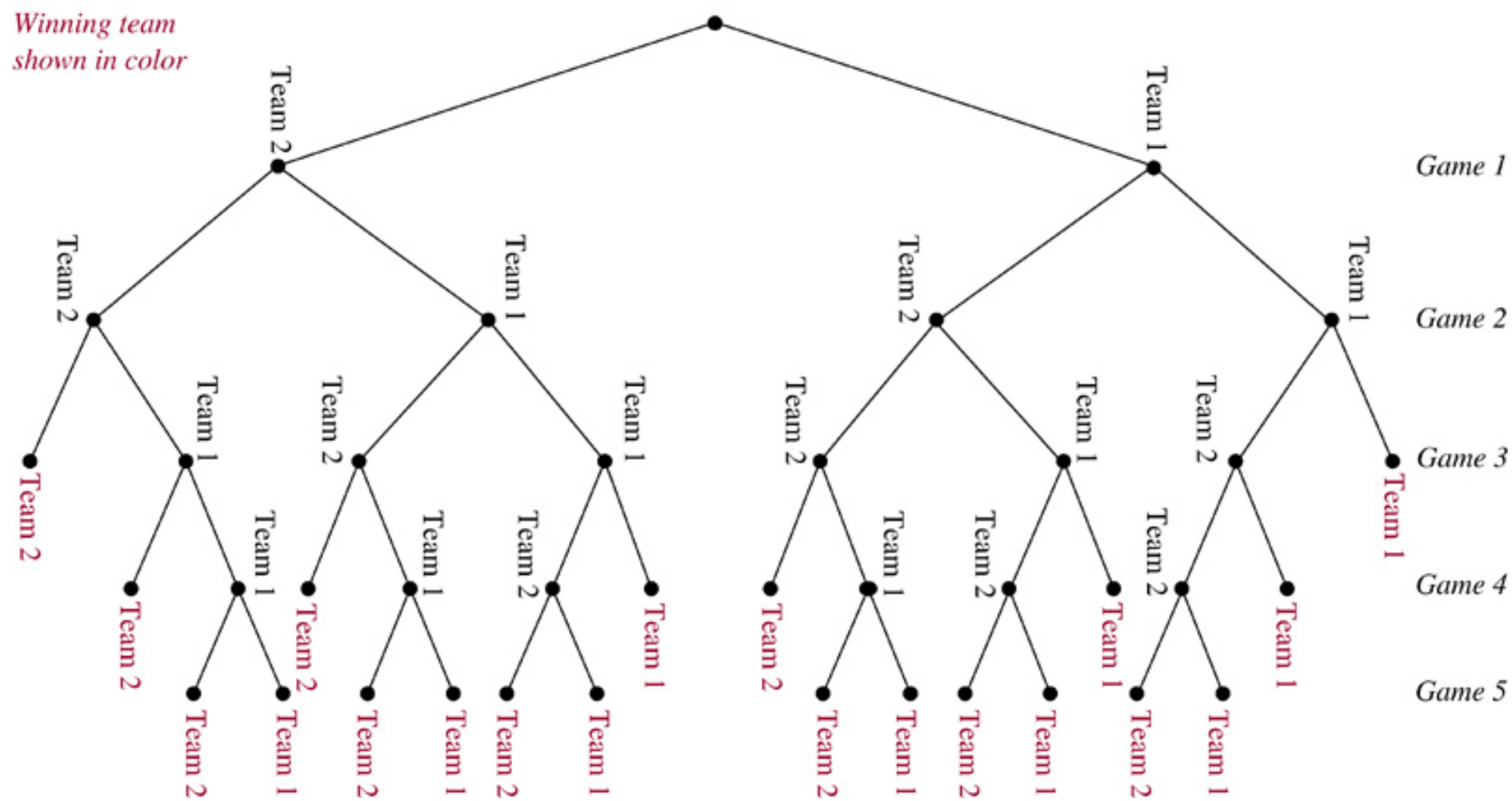
Tree Diagrams: Example

- A playoff between two teams consists of at most five games. The first team that wins three games wins the playoff. In how many different ways can the playoff occur?



The Pigeonhole Principle

- A.k.a. the "Dirichlet drawer principle" or the "Shoe Box Principle".
- If $k + 1$ or more objects are assigned to k places, then at least 1 place must be assigned 2 or more objects.
- In terms of the assignment function:
  - If $f: A \to B$ and $|A| \geq |B| + 1$, then some element of B has more than two preimages under f.
    - I.e., f is not one-to-one.

The Pigeonhole Principle

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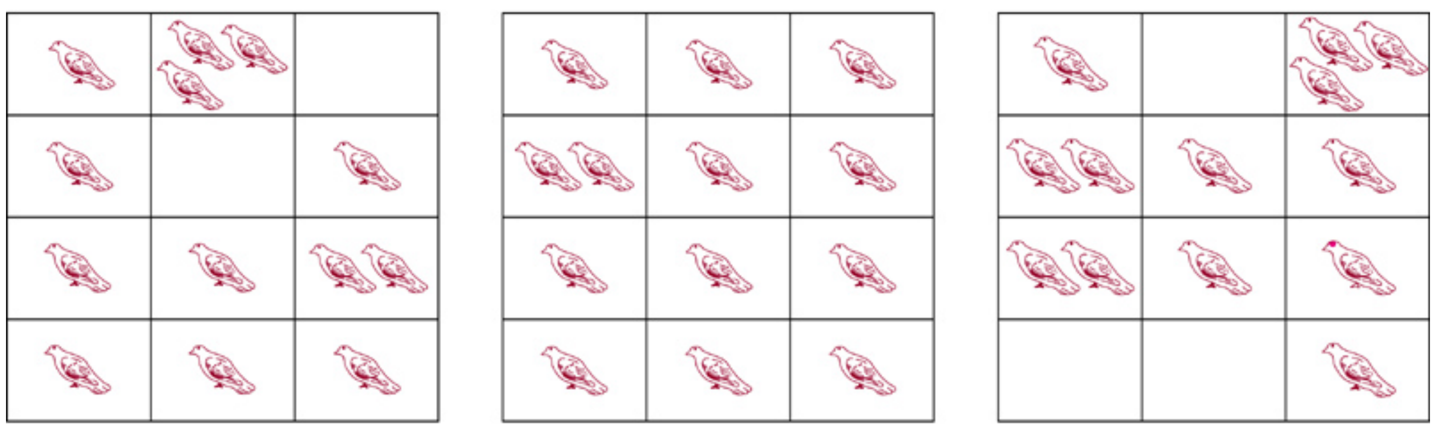
- Proof by contradiction:
  - If the conclusion is false, each pigeonhole contains at most one pigeon and in this time, we can account for at most *k* pigeons.
  - Since there are $k + 1$ pigeons, we have a contradiction.



(a) (b) (c)

Pigeonhole Principle: Example

- There are 101 possible numeric grades (0% ~ 100%) rounded to the nearest integer.
  - Also, there are >101 students in a class.
- Therefore, there must be at least one (rounded) grade that will be shared by at least 2 students at the end of the semester.
  - i.e., the function from students to rounded grades is *not* a one-to-one function.

Another Example of P.P.

- 10 persons have first names as Alice, Bernare, and Charles, and last names as Lee, McDuff, and Ng. Show that at least two persons have the same first and last names.
- Solution:
  - 9 possible names for the 10 persons → 10 pigeons and 9 pigeonholes.
  - Assignment of names to people = assignment of pigeonholes to the pigeons
  - By the Pigeonhole Principle, some name (pigeonhole) is assigned to at least two persons (pigeons).

Generalized Pigeonhole Principle

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- If N objects are assigned to k places, then at least one place must be assigned at least $\lceil N/k \rceil$ objects.

- *E.g.*, there are $N = 280$ students in a class. There are $k = 52$ weeks in the year.
  - Therefore, there must be at least 1 week during which at least $\lceil 280/52 \rceil = \lceil 5.38 \rceil = 6$ students in the class have a birthday.

Proof of G.P.P.

- By contradiction.

  Suppose every place has < $\lceil N/k \rceil$ objects, thus ≤ $\lceil N/k \rceil - 1$.

- Then the total number of objects is at most

$$k\left(\left\lceil \frac{N}{k} \right\rceil - 1\right) < k\left(\left(\frac{N}{k} + 1\right) - 1\right) = k\left(\frac{N}{k}\right) = N$$

- So, there are less than *N* objects, which contradicts our assumption of *N* objects! ■

G.P.P. Example I

- Given: There are 280 students in a class.
  - Without knowing anybody's birthday, what is the largest value of *n* for which we can prove using the G.P.P. that at least *n* students must have been born in the same month?
- Answer:

$$\lceil 280/12 \rceil = \lceil 23.3 \rceil = 24$$

G.P.P. Example II

- What is the least number of area codes needed to guarantee that the 25 million phones in a state can be assigned distinct 10-digit telephone numbers?
  - Phone #: NXX-NXX-XXXX
    - N: 2 ~ 9 and X: any digit
- Solution
  - NXX-XXXX: $(8\cdot 10\cdot 10)\cdot(10\cdot 10\cdot 10\cdot 10) = 8$ million
  - By G.P.P. at least $\lceil 25{,}000{,}000/8{,}000{,}000 \rceil = 4$ phones have the identical numbers
  - Hence, at least 4 area codes are required

Fun Pigeonhole Proof

- **Example 4:** $\forall n \in \mathbf{N}$, $\exists$ a multiple $m > 0$ of n such that m has only 0's and 1's in its decimal expansion!

- **Proof:** Consider the n+1 decimal integers 1, 11, 111, …, $\underbrace{1\cdots1}_{n+1}$. They have only n possible remainders mod n.

  So, take the difference of two that have the same remainder. The result is the answer! □

A Specific Case

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- Let n=3. Consider 1, 11, 111, 1111.
  - 1 mod 3 = 1
  - 11 mod 3 = 2
  - 111 mod 3 = 0 ← Lucky extra solution.
  - 1,111 mod 3 = 1

Note same residue

- 1,111 − 1 = 1,110 = 3·370.
  - It has only 0's and 1's in its expansion.
  - Its remainder mod 3 = 0, so it's a multiple of 3.

Baseball Example

- Suppose that next June, the Marlins baseball team plays at least 1 game a day, but ≤ 45 games total. Show there must be some sequence of consecutive days in June during which they play *exactly* 14 games.
  - **Proof:** Let a_j be the number of games played on or before day j. Then, $a_1,\ldots,a_{30} \in \mathbf{Z}^+$ is a sequence of 30 distinct integers with $1 \leq a_j \leq 45$.
    Therefore $a_1+14,\ldots,a_{30}+14$ is a sequence of 30 distinct integers with $15 \leq a_j+14 \leq 59$.
    Thus, $(a_1,\ldots,a_{30},a_1+14,\ldots,a_{30}+14)$ is a sequence of 60 integers from the set {1,..,59}.
    By the Pigeonhole Principle, two of them must be equal, but $a_i \neq a_j$ for $i \neq j$. So, $\exists ij$: $a_i = a_j+14$.
    Thus, 14 games were played on days $a_j+1, \ldots, a_i$.

Baseball Problem Illustrated

- Example of $\{a_i\}$: Note all elements are distinct.
  - 1, 2, 4, 5, 7, 8, 10, 11, 13, 14, 16, 17, 19, 21, 22, 23, 25, 27, 29, 30, 31, 33, 34, 36, 37, 39, 40, 41, 43, 45
  - Then $\{a_i+14\}$ is the following sequence: 15, 16, 18, 19, 21, 22, 24, 25, 27, 28, 30, 32, 33, 35, 36, 37, 39, 41, 43, 44, 45, 47, 48, 50, 51, 53, 54, 55, 57, 59
- In any 60 integers from 1-59 there must be some duplicates, indeed we find the following ones:
  - 16, 19, 21, 22, 25, 27, 30, 33, 36, 37, 39, 41, 43, 45

Thus, for example, exactly 14 games were played during days
3 to 11:
2+1+2+1+2+1+2+1+2