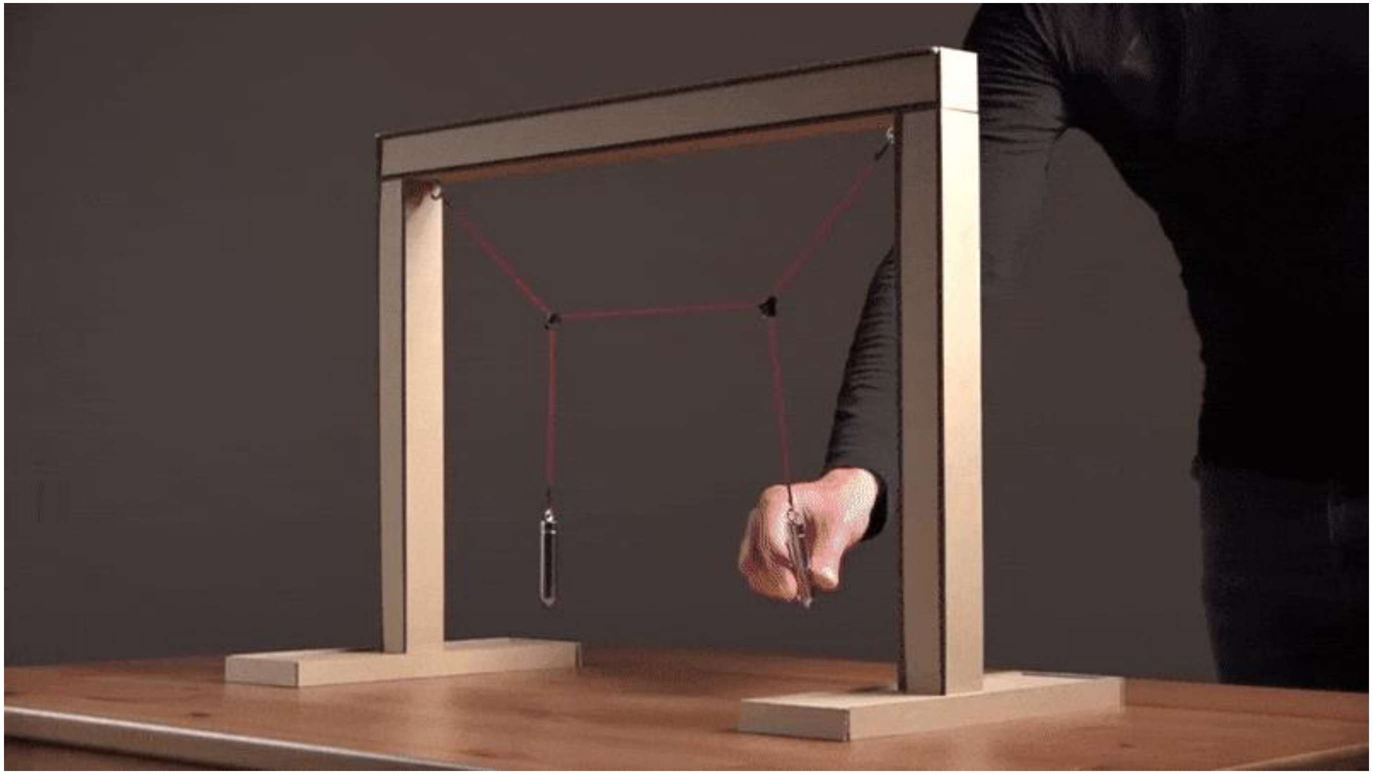
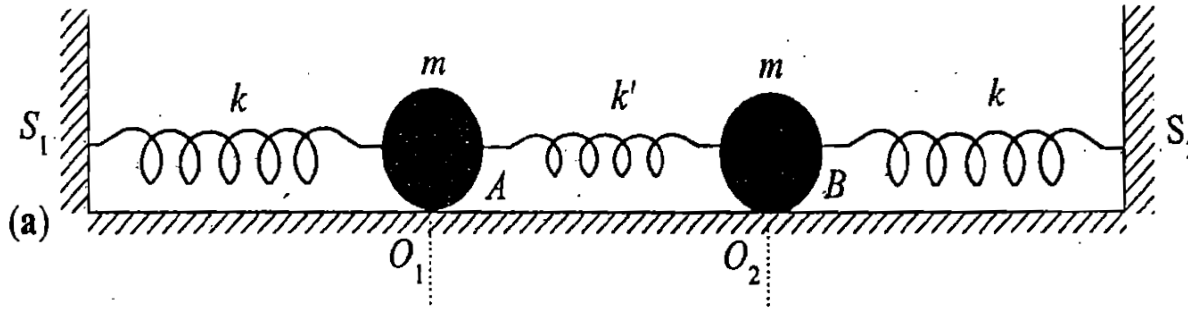
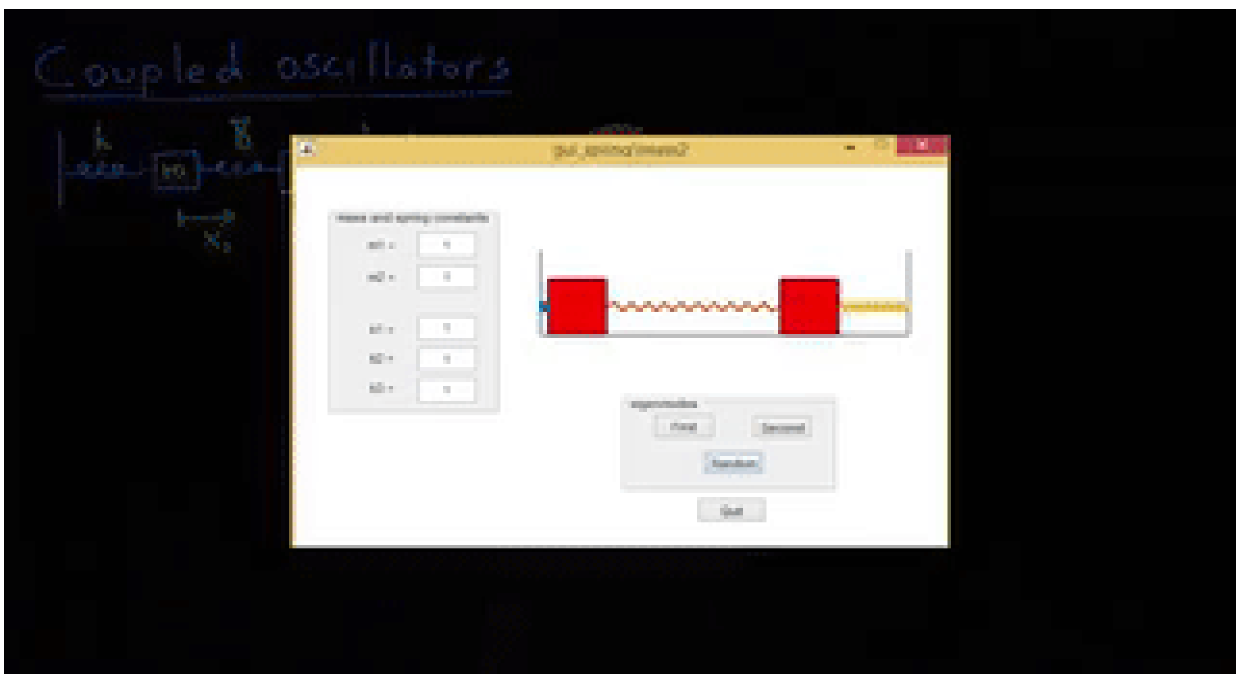
Coupled Oscillations

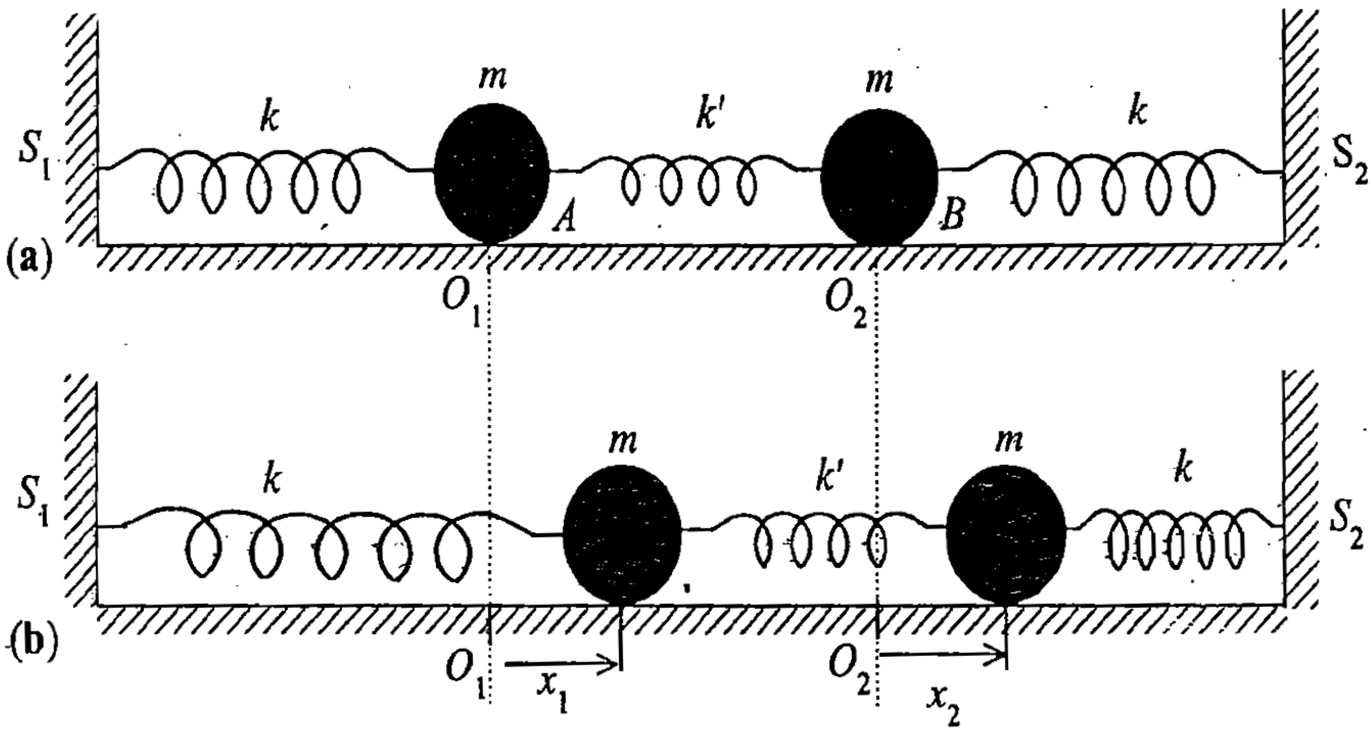


Coupled Oscillators





Coupled Oscillators



Equations of motion are

$$m\ddot{x}_1 = -kx_1 + k'(x_2 - x_1)$$

$$m\ddot{x}_2 = -kx_2 - k'(x_2 - x_1)$$

Simultaneous coupled differential equation

Matrix equation of coupled oscillator

$$\left.\begin{aligned} m\ddot{x}_1 &= -kx_1 + k'(x_2 - x_1) \\ m\ddot{x}_2 &= -kx_2 - k'(x_2 - x_1) \end{aligned}\right\} \quad \ldots (1)$$

$$m\ddot{x}_1 = (-k - k')x_1 + k'x_2$$

$$m\ddot{x}_2 = k'x_1 + (-k - k')x_2$$

$$m\frac{d^2}{dt^2}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{bmatrix} -k-k' & k' \\ k' & -k-k' \end{bmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$m\frac{d^2[X]}{dt^2} = K[X]$$

$$m\frac{d^2}{dt^2}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{bmatrix} -k-k' & k' \\ k' & -k-k' \end{bmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \quad \text{... (2)}$$

Dividing Eq. (2) by m on both sides and taking $\omega_0{}^2 = \dfrac{k}{m},\ \omega_c{}^2 = \dfrac{k'}{m}$

$$\frac{d^2[X]}{dt^2} = \begin{bmatrix} -\left(\omega_0{}^2+\omega_c{}^2\right) & \omega_c{}^2 \\ \omega_c{}^2 & -\left(\omega_0{}^2+\omega_c{}^2\right) \end{bmatrix}[X]$$

Let trail solution be $[X] = [V]e^{\alpha t}$

$$\frac{d^2[X]}{dt^2} = \begin{bmatrix} -\left(\omega_0{}^2 + \omega_c{}^2\right) & \omega_c{}^2 \\ \omega_c{}^2 & -\left(\omega_0{}^2 + \omega_c{}^2\right) \end{bmatrix} [X] \leftarrow [X] = [V] e^{\alpha t}$$

$$[V]\alpha^2 = \begin{bmatrix} -\left(\omega_0{}^2 + \omega_c{}^2\right) & \omega_c{}^2 \\ \omega_c{}^2 & -\left(\omega_0{}^2 + \omega_c{}^2\right) \end{bmatrix} [V]$$

$$\begin{bmatrix} -\left(\omega_0{}^2 + \omega_c{}^2\right) & \omega_c{}^2 \\ \omega_c{}^2 & -\left(\omega_0{}^2 + \omega_c{}^2\right) \end{bmatrix} [V] - \alpha^2 [I][V] = 0$$

Above equation is similar to the eigen value equation

$$\{[A] - \lambda [I]\}[V] = 0$$

Finding the eigen values λ of [A]

$$\left\{[A]-\lambda[I]\right\}[V]=0$$

$$\text{determinant: } \left|A-\lambda I\right|=0$$

Note: Please refer to classnotes (slide 49 onwards) from January 29, 2020 for more details on this method. Same method must be also discussed in MA102: matrices, Eigen values and Eigen vectors.

$$\left\|[A]-\lambda[I]\right\|=0$$

$$\begin{vmatrix} -\left(\omega_0{}^2+\omega_c{}^2\right)-\alpha^2 & \omega_c{}^2 \\ \omega_c{}^2 & -\left(\omega_0{}^2+\omega_c{}^2\right)-\alpha^2 \end{vmatrix}=0$$

$$\left[-(\omega_0^2+\omega_c^2)-\alpha^2\right]^2-\omega_c^4=0$$

Characteristic equation:

$$\left(\alpha^2\right)^2+2\left(\omega_0^2+\omega_c^2\right)\alpha^2+\omega_0^4+2\left(\omega_0^2\times\omega_c^2\right)=0$$

Eigen values of [A]:

$$\alpha_1=i\omega_0,\alpha_2=-i\omega_0,\alpha_3=i\sqrt{\omega_0^2+2\omega_c^2},\alpha_4=-i\sqrt{\omega_0^2+2\omega_c^2}$$

Find the eigen vectors of [A]

Finding Eigen vectors of [A]

Substitute each values of α^2 in the Eigen value equation

$$\left\{[A]-\lambda[I]\right\}[V]=0$$

Putting α_1^2 in the Eigen value equation to get first set of Eigen vectors

$$\begin{bmatrix} -\left(\omega_0{}^2+\omega_c{}^2\right)-\alpha_1^2 & \omega_c{}^2 \\ \omega_c{}^2 & -\left(\omega_0{}^2+\omega_c{}^2\right)-\alpha_1^2 \end{bmatrix}\begin{bmatrix} V_1^{\alpha 1} \\ V_2^{\alpha 1} \end{bmatrix}=0$$

Substituting $\alpha_1{}^2 = -\omega_0$

$$\begin{bmatrix} -\left(\omega_0{}^2 + \omega_c{}^2\right) + \omega_0{}^2 & \omega_c{}^2 \\ \omega_c{}^2 & -\left(\omega_0{}^2 + \omega_c{}^2\right) + \omega_0{}^2 \end{bmatrix} \begin{bmatrix} V_1^{\alpha 1} \\ V_2^{\alpha 1} \end{bmatrix} = 0$$

$$V_1^{\alpha 1} = V_2^{\alpha 1}$$

Eigen vector corresponding the eigen value $\lambda = \alpha_1^2$ is $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Similarly substituting $\alpha_2{}^2 = -\omega_0$

Eigen vector corresponding the eigen value $\lambda = \alpha_2^2$ is $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Putting $\alpha_3^2 = -(\omega_0^2 + 2\omega_c^2)$ in the Eigen value equation

$$\begin{bmatrix} -\left(\omega_0{}^2 + \omega_c{}^2\right) + \left(\omega_0^2 + 2\omega_c^2\right) & \omega_c{}^2 \\ \omega_c{}^2 & -\left(\omega_0{}^2 + \omega_c{}^2\right) + \left(\omega_0^2 + 2\omega_c^2\right) \end{bmatrix} \begin{bmatrix} V_1^{\alpha_3} \\ V_2^{\alpha_3} \end{bmatrix} = 0$$

$$V_1^{\alpha_3} = -V_2^{\alpha_3}$$

Eigen vector corresponding the eigen value $\lambda = \alpha_3^2$ is $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

Substitution of $\alpha_4{}^2 = -\left(\omega_0^2 + 2\omega_c^2\right)$

Eigen vector corresponding the eigen value $\lambda = \alpha_4^2$ is $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$$\frac{d^2}{dt^2}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -\left(\omega_0{}^2 + \omega_c{}^2\right) & \omega_c{}^2 \\ \omega_c{}^2 & -\left(\omega_0{}^2 + \omega_c{}^2\right) \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} e^{\alpha t}$$

$$\alpha_1 = i\omega_0, \alpha_2 = -i\omega_0, \alpha_3 = i\sqrt{\omega_0^2 + 2\omega_c^2}, \alpha_4 = -i\sqrt{\omega_0^2 + 2\omega_c^2}$$

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \begin{bmatrix} 1 \\ -1 \end{bmatrix} \quad \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

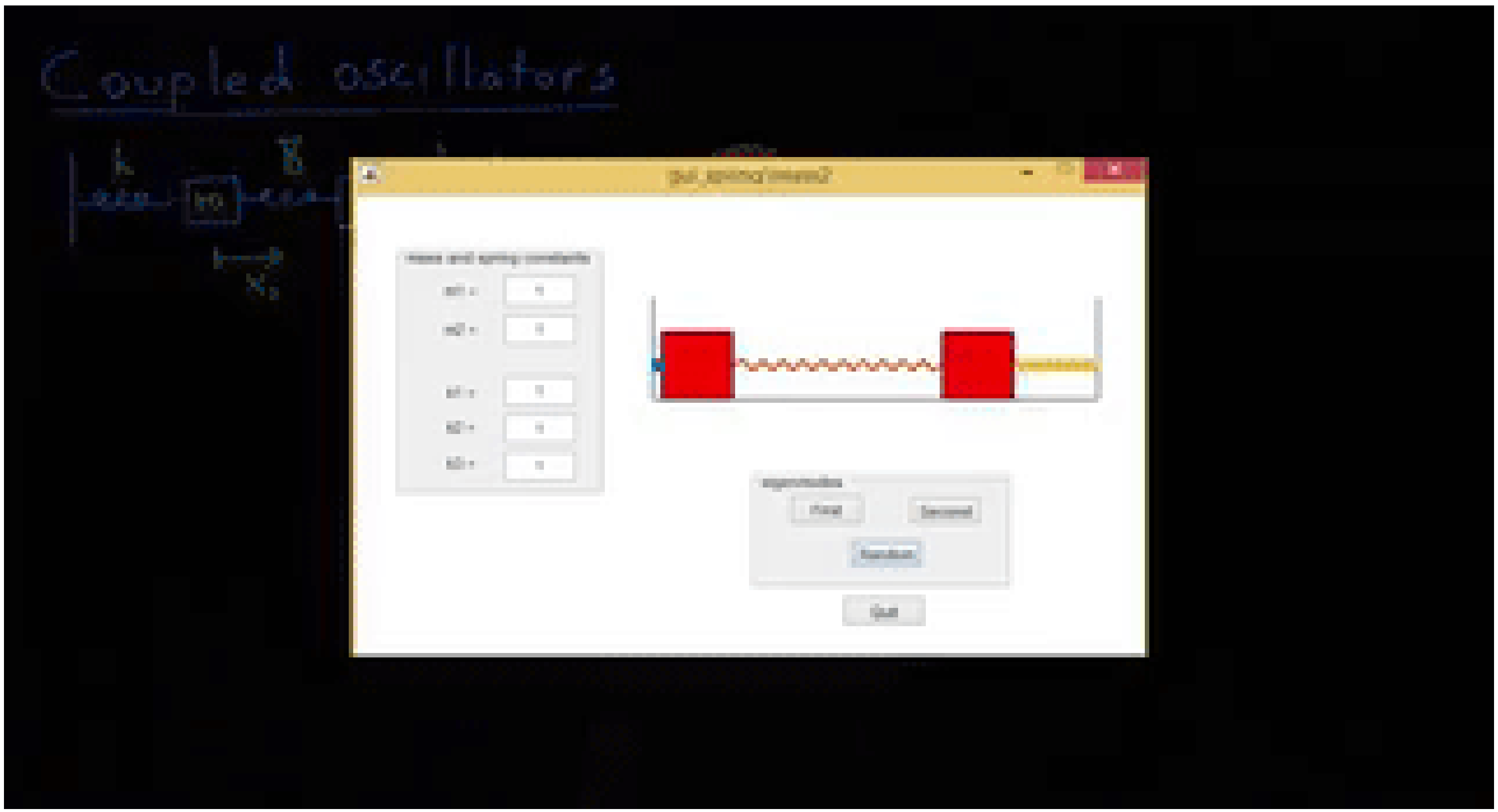
$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = a_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{i\omega_0 t} + a_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-i\omega_0 t} + a_3 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{i\sqrt{\omega_0^2 + 2\omega_c^2}\, t} + a_4 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-i\sqrt{\omega_0^2 + 2\omega_c^2}\, t}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = a_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{i\omega_0 t} + a_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-i\omega_0 t} + a_3 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{i\sqrt{\omega_0^2 + 2\omega_c^2}\, t} + a_4 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-i\sqrt{\omega_0^2 + 2\omega_c^2}\, t}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \boxed{a_1 e^{i\omega_0 t} + a_2 e^{-i\omega_0 t}} + \boxed{a_3 e^{i\sqrt{\omega_0^2 + 2\omega_c^2}\, t} + a_4 e^{-i\sqrt{\omega_0^2 + 2\omega_c^2}\, t}} \\ \boxed{a_1 e^{i\omega_0 t} + a_2 e^{-i\omega_0 t}} - \boxed{a_3 e^{i\sqrt{\omega_0^2 + 2\omega_c^2}\, t} - a_4 e^{-i\sqrt{\omega_0^2 + 2\omega_c^2}\, t}} \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} A_1 \cos(\omega_0 t + \phi_1) + A_2 \cos(\sqrt{\omega_0^2 + 2\omega_c^2}\, t + \phi_2) \\ A_1 \cos(\omega_0 t + \phi_1) - A_2 \cos(\sqrt{\omega_0^2 + 2\omega_c^2}\, t + \phi_2) \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} A_1 \cos(\omega_0 t + \phi_1) + A_2 \cos(\sqrt{\omega_0^2 + 2\omega_c^2}\, t + \phi_2) \\ A_1 \cos(\omega_0 t + \phi_1) - A_2 \cos(\sqrt{\omega_0^2 + 2\omega_c^2}\, t + \phi_2) \end{bmatrix}$$



Our dynamical system oscillates with frequencies ω_0 *and* $\sqrt{\omega_0^2 - 2\omega_c^2}$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \cancel{A_1\cos(\omega_0 t+\phi_1)} + \cancel{A_2\cos(\sqrt{\omega_0^2+2\omega_c^2}\,t+\phi_2)} \\ \cancel{A_1\cos(\omega_0 t+\phi_1)} - \cancel{A_2\cos(\sqrt{\omega_0^2+2\omega_c^2}\,t+\phi_2)} \end{bmatrix}$$

Spring in the middle does not stretch

Spring in the middle does stretch

Frequencies ω_0 *and* $\sqrt{\omega_0^2 - 2\omega_c^2}$: called the normal frequencies of the system

Normal Coordinates

Can we get some co-ordinates which will oscillate with one pure frequency??

Try by just adding x_1 and x_2 and subtracting x_1 and x_2

Normal Coordinates

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} A_1\cos(\omega_0 t + \phi_1) + A_2\cos(\cancel{\sqrt{\omega_0^2 + 2\omega_c^2}}\, t + \phi_2) \\ A_1\cos(\omega_0 t + \phi_1) - A_2\cos(\cancel{\sqrt{\omega_0^2 + 2\omega_c^2}}\, t + \phi_2) \end{bmatrix}$$

$$\begin{aligned} X_1 &= \frac{x_1 + x_2}{2} \\ &= \frac{1}{2}\left(2A_1\cos(\omega_0 t + \phi_1)\right) \\ &= A_1\cos(\omega_0 t + \phi_1) \end{aligned}$$

Normal Coordinates

$$\begin{bmatrix} x_1 \\ -x_2 \end{bmatrix} = \begin{bmatrix} \cancel{A_1 \cos(\omega_0 t + \phi_1)} + A_2 \cos(\sqrt{\omega_0^2 + 2\omega_c^2}\, t + \phi_2) \\ \cancel{-A_1 \cos(\omega_0 t + \phi_1)} + A_2 \cos(\sqrt{\omega_0^2 + 2\omega_c^2}\, t + \phi_2) \end{bmatrix}$$

$$\begin{aligned} X_2 &= \frac{x_1 - x_2}{2} \\ &= \frac{1}{2}\left(2A_2 \cos\left(\sqrt{\omega_0^2 + 2\omega_c^2 t} + \phi_2\right)\right) \\ &= A_2 \cos\left(\sqrt{\omega_0^2 + 2\omega_c^2 t} + \phi_2\right) \end{aligned}$$

Normal Coordinates

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} A_1 \cos(\omega_0 t + \phi_1) + A_2 \cos(\sqrt{\omega_0^2 + 2\omega_c^2}\, t + \phi_2) \\ A_1 \cos(\omega_0 t + \phi_1) - A_2 \cos(\sqrt{\omega_0^2 + 2\omega_c^2}\, t + \phi_2) \end{bmatrix}$$

$$X_1 = \frac{x_1 + x_2}{2}$$

$$X_2 = \frac{x_1 - x_2}{2}$$

$$X_1 = A_1 \cos(\omega_0 t + \phi_1)$$

$$X_2 = A_2 \cos(\sqrt{\omega_0^2 + 2\omega_c^2}\, t + \phi_2)$$

Original co-ordinates can be obtained by the inverse transformation

$$x_1 = X_1 + X_2$$

$$x_2 = X_1 - X_2$$

Normal Coordinates

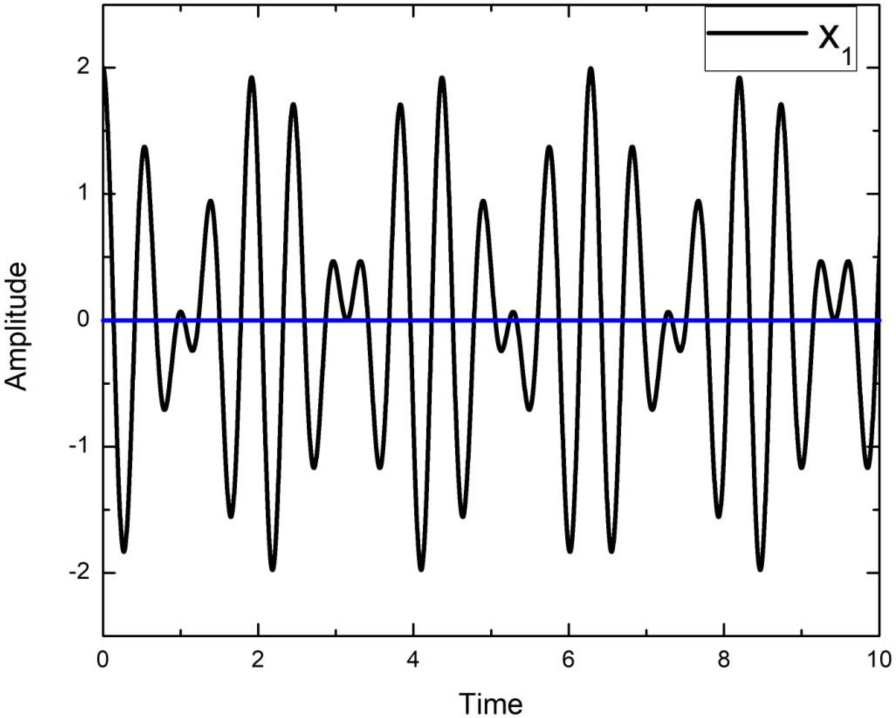
$$X_1 = A_1 \cos(\omega_0 t + \phi_1)$$

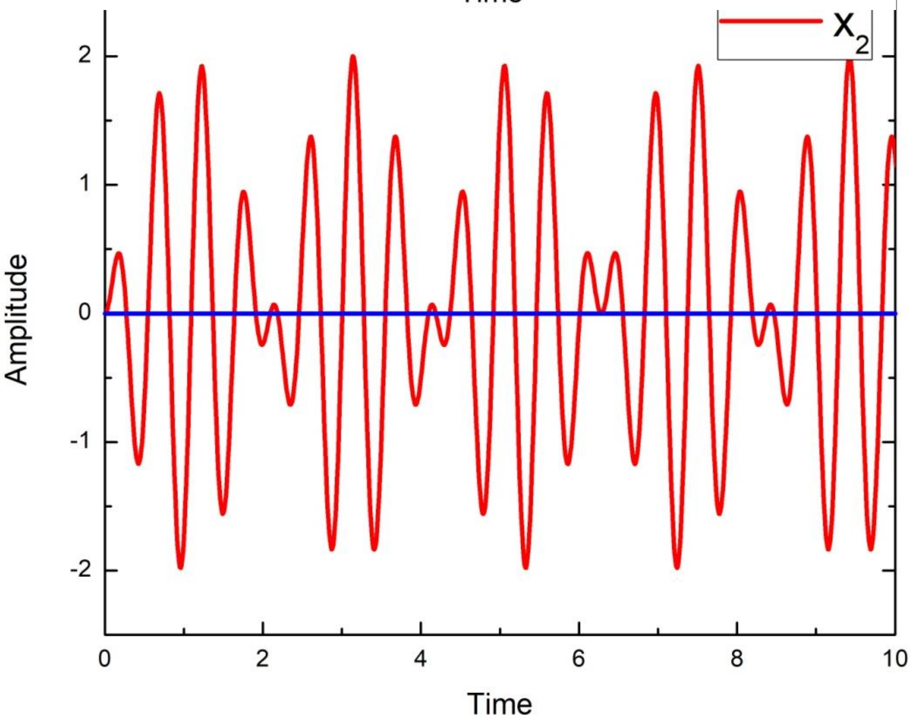
$$X_2 = A_2 \cos(\sqrt{\omega_0^2 + 2\omega_c^2}\, t + \phi_2)$$

- ✓ **Our dynamical system oscillates with frequencies $\boldsymbol{\omega_0}$ *and* $\boldsymbol{\sqrt{\omega_0^2 - 2\omega_c^2}}$: called the normal frequencies of the system**

- ✓ Since two independent coordinates are needed to specify system's instantaneous configuration (two degrees of freedom), it possesses *two* normal frequencies.

- ✓ It is a general rule that a dynamical system with N degrees of freedom possesses N normal frequencies.

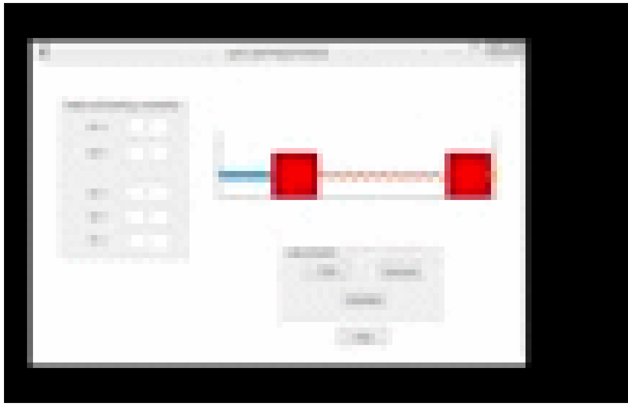




Plots: Amplitude v/s time

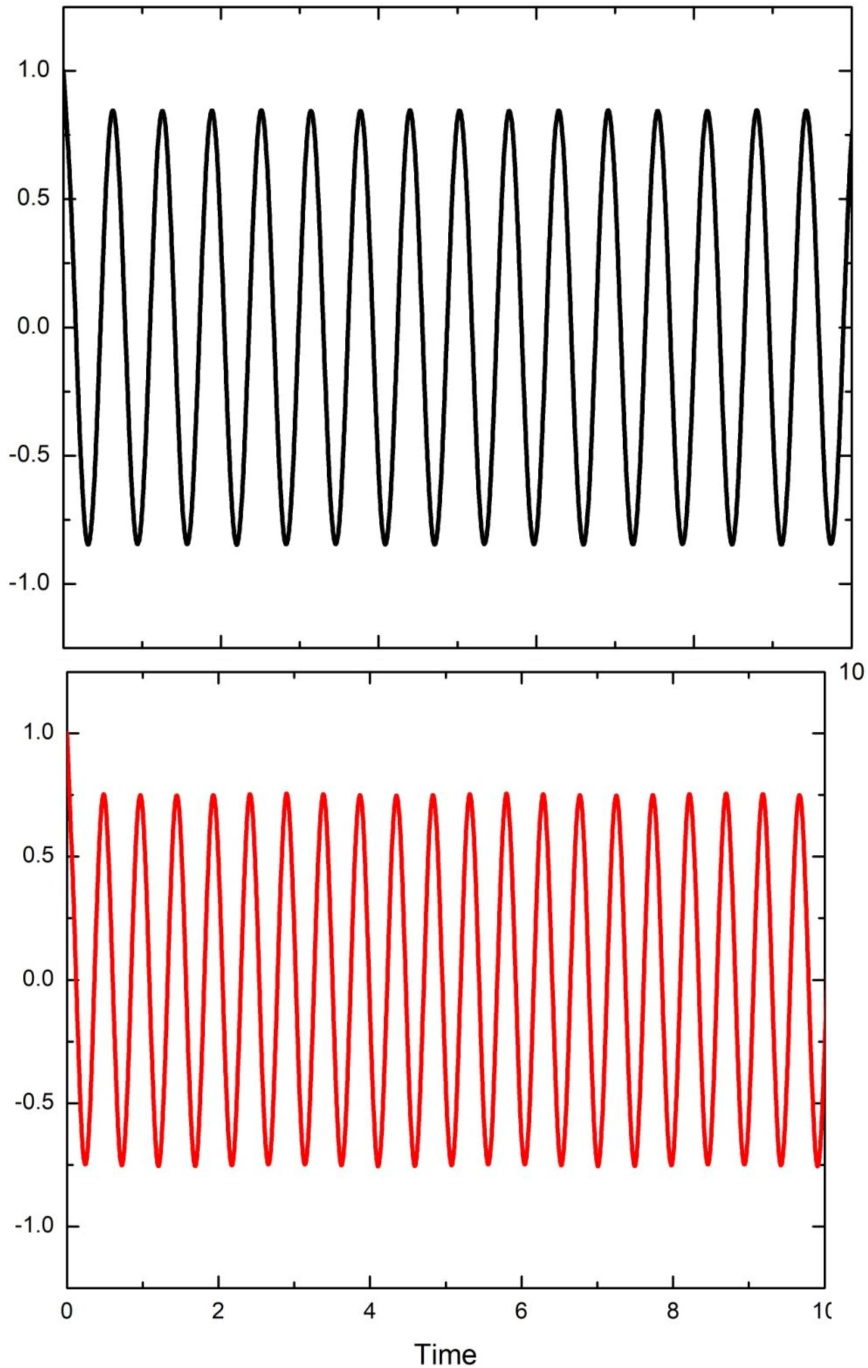
General Co-ordinates

$$x_1 = A_1 \cos(\omega_0 t + \phi_1) + A_2 \cos(\sqrt{\omega_0^2 + 2\omega_c^2}\, t + \phi_2)$$



$$x_2 = A_1 \cos(\omega_0 t + \phi_1) - A_2 \cos(\sqrt{\omega_0^2 + 2\omega_c^2}\, t + \phi_2)$$

Plots: Amplitude v/s time



Normal Co-ordinates

$$X_1 = A_1 \cos(\omega_0 t + \phi_1)$$

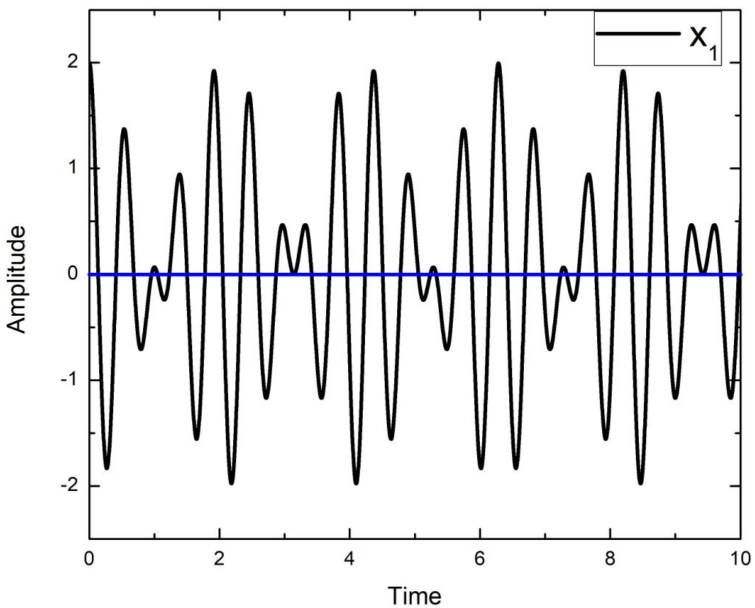
This pattern of motion corresponds to the two masses executing simple harmonic oscillation with the *same amplitude and phase*. Note that such an oscillation does not stretch the middle spring.

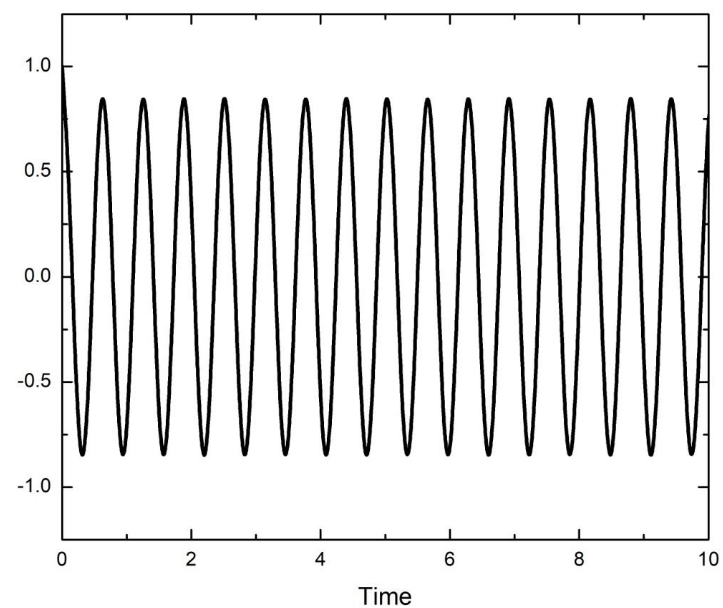
$$X_2 = A_2 \cos(\sqrt{\omega_0^2 + 2\omega_c^2}\, t + \phi_2)$$

This second pattern of motion corresponds to the two masses executing simple harmonic oscillation with the *same amplitude but in anti-phase.* Note that such an oscillation do stretch the middle spring.

Why Normal Co-ordinates?

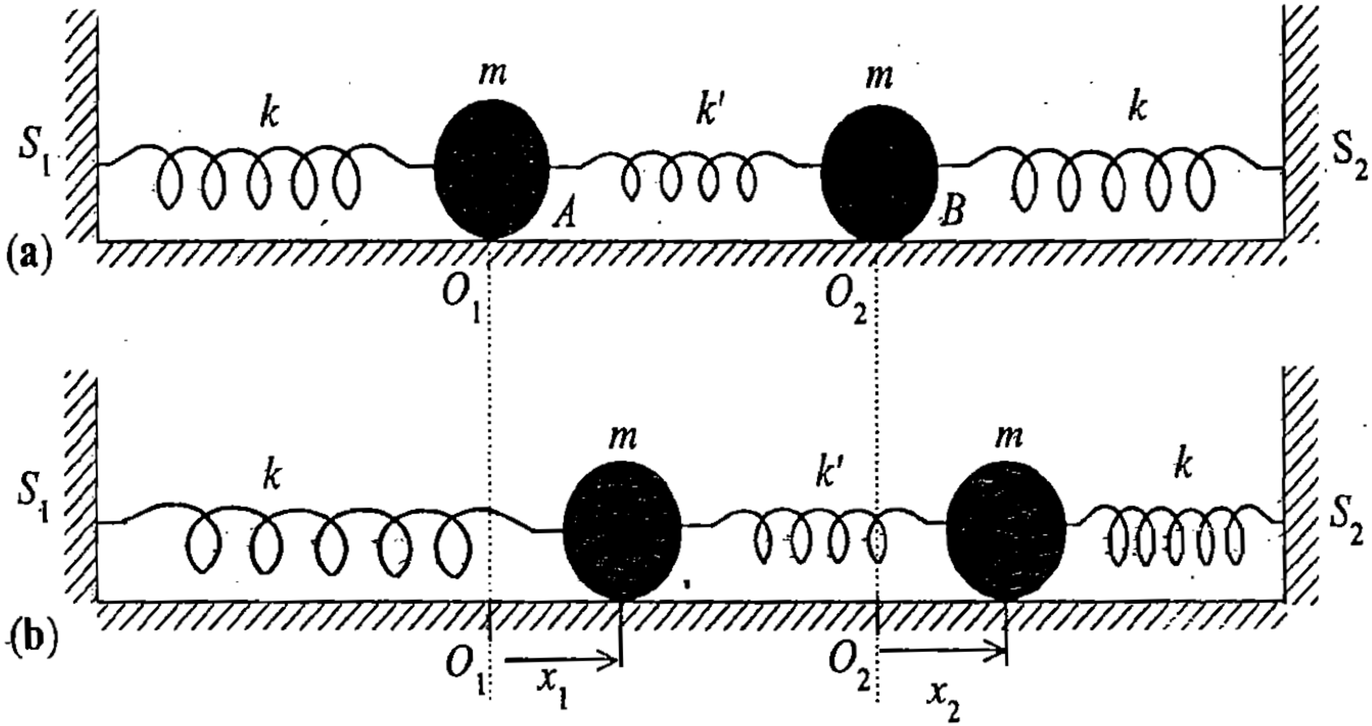
- This will help in knowing 2 different frequencies independently
- If you plot $X_1(t)$ and $X_2(t)$ v/s time you get nice cosine graph
- $x_1(t)$ and $x_2(t)$ v/s time was erratic in nature
- Out of phase and in-phase motion cannot get visualized though!





Is there any alternative way of solving the coupled differential equation?

Coupled Oscillators



Equations of motion are

$$\boldsymbol{m\ddot{x}_1 = -kx_1 + k'(x_2 - x_1)}$$

$$\boldsymbol{m\ddot{x}_2 = -kx_2 - k'(x_2 - x_1)}$$

Simultaneous coupled differential equation

Equations of motion are

$$m\ddot{x}_1 = -kx_1 + k'(x_2 - x_1) \quad \ldots (1)$$

$$m\ddot{x}_2 = -kx_2 - k'(x_2 - x_1) \quad \ldots (2)$$

(1)+(2) $$m\frac{d^2}{dt^2}\left(x_1 + x_2\right) = -k\left(x_1 + x_2\right) \quad \ldots (3)$$

(1)-(2) $$m\frac{d^2}{dt^2}\left(x_1 - x_2\right) = -k\left(x_1 - x_2\right) - 2k'\left(x_1 - x_2\right) \quad \ldots (4)$$

Let us consider

$$X_1 = x_1 + x_2$$
$$X_2 = x_1 - x_2$$

Substituting $$X_1 = x_1 + x_2$$ $$X_2 = x_1 - x_2$$ In (3) and (4)

$$m\frac{d^2}{dt^2}(x_1 + x_2) = -k(x_1 + x_2)$$

$$m\frac{d^2 X_1}{dt^2} = -kX_1 \quad \text{... (5)}$$

$$m\frac{d^2}{dt^2}(x_1 - x_2) = -k(x_1 - x_2) - 2k'(x_1 - x_2)$$

$$m\frac{d^2 X_2}{dt^2} = -[k + 2k']X_2 \quad \text{... (6)}$$

Eq. (5) and (6) are two decoupled equations. We can find their solutions by dividing both sides by m. Assuming $\omega_0 = \sqrt{\frac{k}{m}}$ and $\omega_c = \sqrt{\frac{k'}{m}}$.

$$\frac{d^2 X_1}{dt^2} = -\omega_0^2 X_1$$
$$\frac{d^2 X_2}{dt^2} = -(\omega_0^2 + 2\omega_c^2)X_2$$

$$\Rightarrow$$

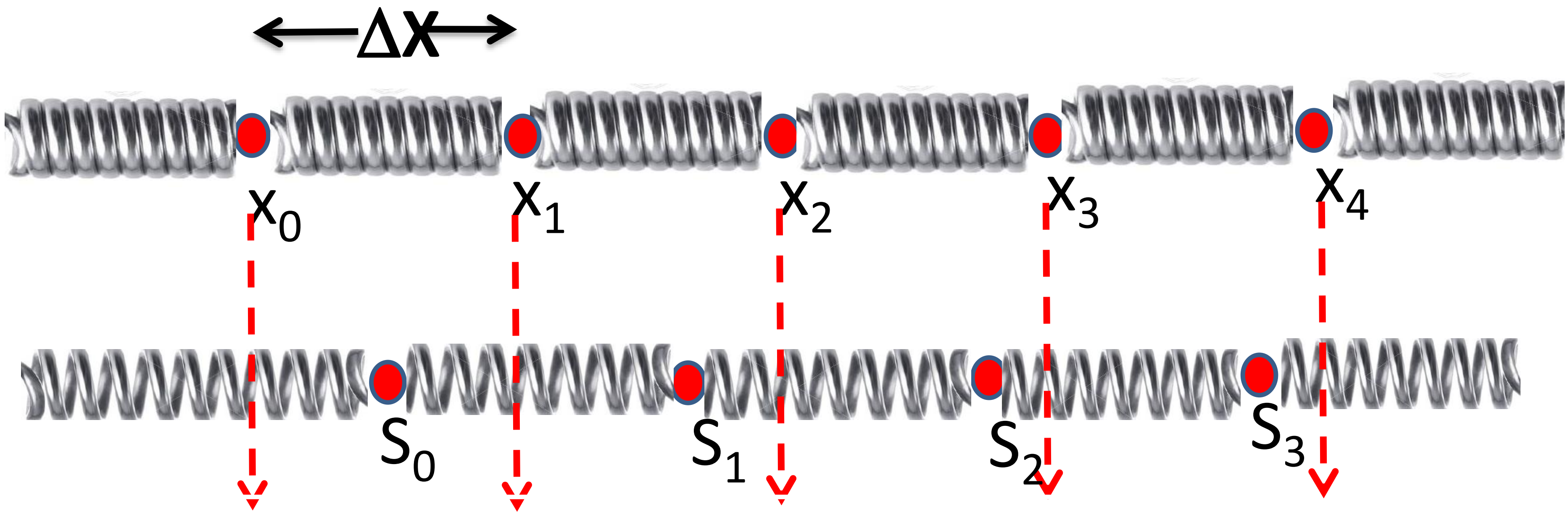
$$X_1 = A_1 \cos(\omega_0 t + \phi_1)$$
$$X_2 = A_2 \cos(\sqrt{\omega_0^2 + 2\omega_c^2}\, t + \phi_2)$$

Original co-ordinates can be obtained by inverse transform

$$x_1 = X_1 + X_2$$
$$x_2 = X_1 - X_2$$

Solving eigen values and eigen vectors for n-Coupled mass system

N-masses coupled oscillator



What will be the equation of motion for i^th^ mass?

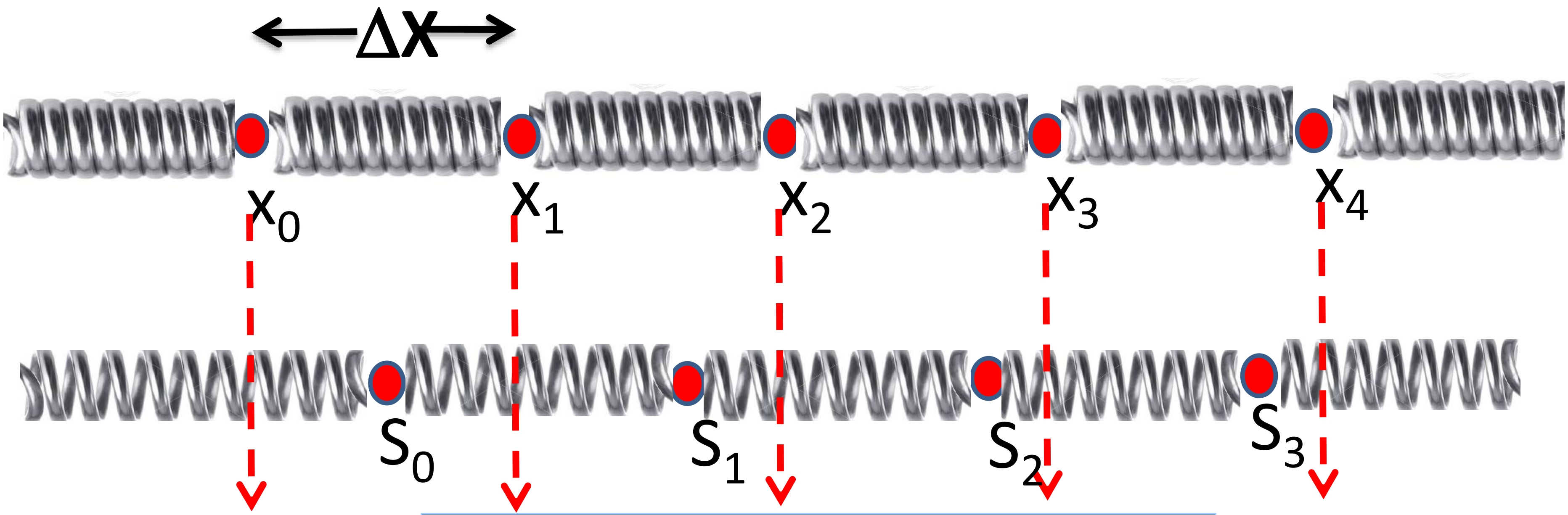
Equation of motion for central mass in the three mass system is

$$m\frac{d^2x_2}{dt^2} = -k\left(x_2 - x_1\right) + k\left(x_3 - x_2\right)$$

(Exercise problem from last class. Please see end of notes for the solution.)

$$m\frac{d^2S_i}{dt^2} = -k\left(S_i - S_{i-1}\right) + k\left(S_{i+1} - S_i\right)$$

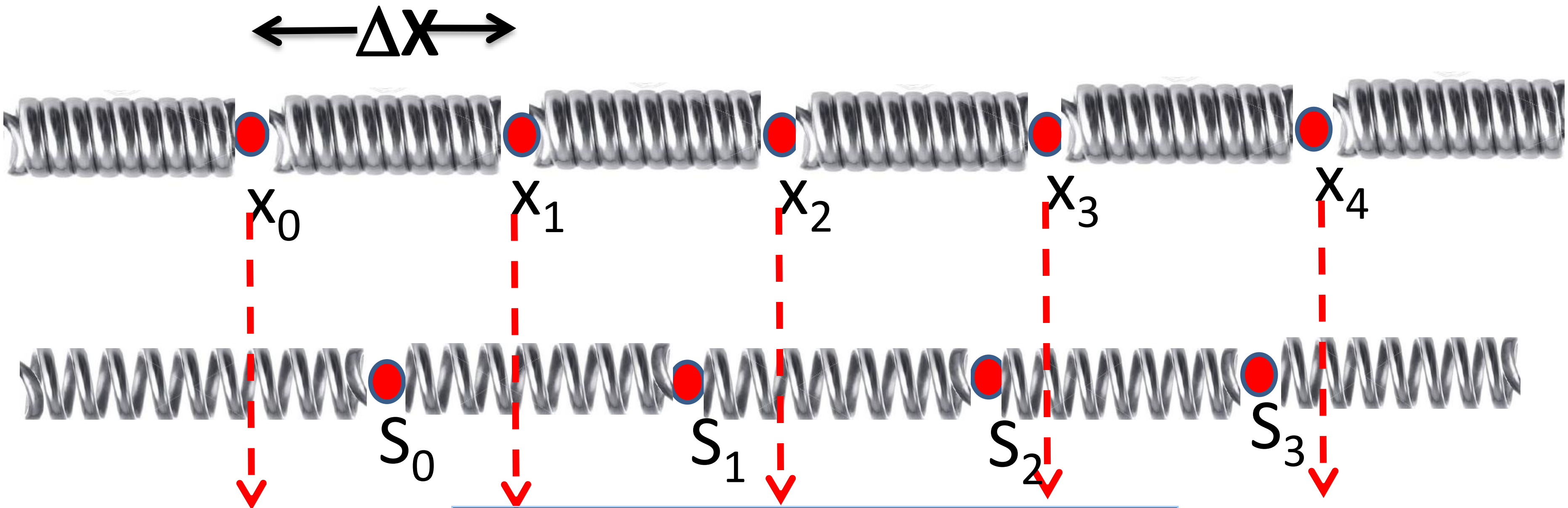
N-masses coupled oscillator



$$m\frac{d^2x_2}{dt^2} = -k\left(x_2 - x_1\right) + k\left(x_3 - x_2\right)$$

$$m\frac{d^2S_i}{dt^2} = -k\left(S_i - S_{i-1}\right) + k\left(S_{i+1} - S_i\right)$$

$$m\frac{d^2S_i}{dt^2} = kS_{i-1} - 2kS_i + kS_{i+1}$$



$$m\frac{d^2 S_i}{dt^2} = kS_{i-1} - 2kS_i + kS_{i+1}$$

Writing equations for all masses will give us following coupled differential equations

$$m\frac{d^2}{dt^2}\begin{pmatrix} \vdots \\ x_{n-1} \\ x_n \\ x_{n+1} \\ \vdots \end{pmatrix} = \begin{pmatrix} & & & \vdots & & & & \\ \cdots & k & -2k & k & & & & \\ & & k & -2k & k & & & \\ & & & k & -2k & k & \cdots & \\ & & & & \vdots & & & \end{pmatrix}\begin{pmatrix} \vdots \\ x_{n-1} \\ x_n \\ x_{n+1} \\ \vdots \end{pmatrix}$$

Solving n coupled masses equation

$$m\frac{d^2}{dt^2}\begin{pmatrix} . \\ . \\ S_{i-1} \\ S_i \\ S_{i+1} \\ . \\ . \end{pmatrix} = \begin{pmatrix} & & & \vdots & & & \\ \cdots & k & -2k & k & & & \\ & & k & -2k & k & & \\ & & & k & -2k & k & \cdots \\ & & & & \vdots & & \end{pmatrix}\begin{pmatrix} . \\ . \\ S_{i-1} \\ S_i \\ S_{i+1} \\ . \\ . \end{pmatrix}$$

Trail solution of the form:

$$\begin{bmatrix} . \\ . \\ S_{i-1} \\ S_i \\ S_{i+1} \\ . \\ . \end{bmatrix} = \begin{bmatrix} \\ \\ V_{i-1} \\ V_i \\ V_{i+1} \\ \\ \\ \end{bmatrix} e^{\alpha t}$$

Solve the eigen value equation, as discussed before and find the eigen values and eigen vectors for the above coupled differential equations.

α will have n values

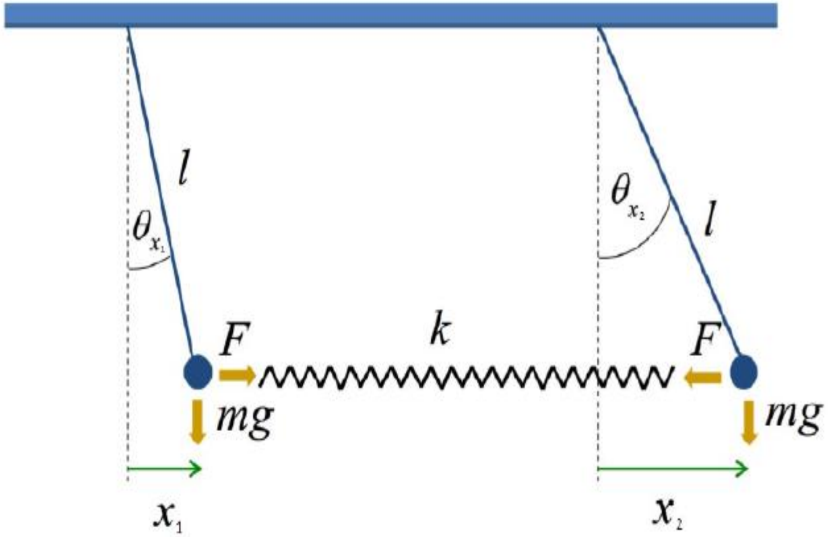
n normal modes will be present

Exercise1: Find general solution of following coupled equations:

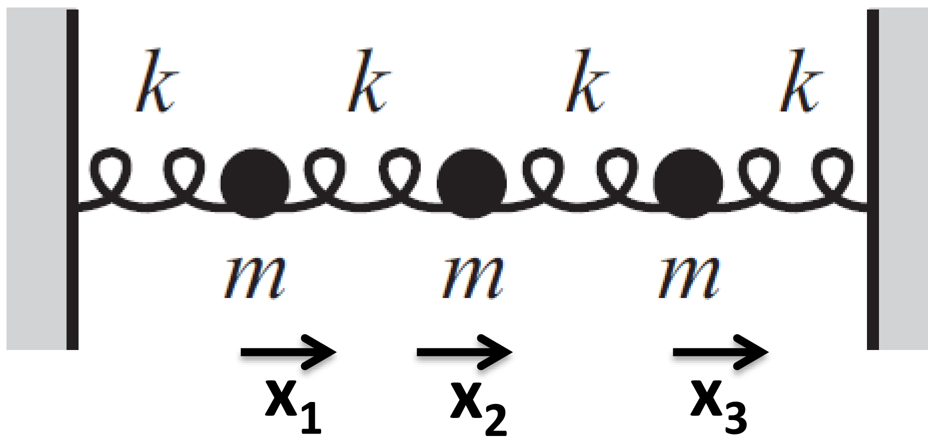
$$2\ddot{x} + \omega^2(5x - 3y) = 0$$
$$2\ddot{y} + \omega^2(5y - 3x) = 0$$

Find normal mode co-ordinates and normal mode frequencies for these equations.

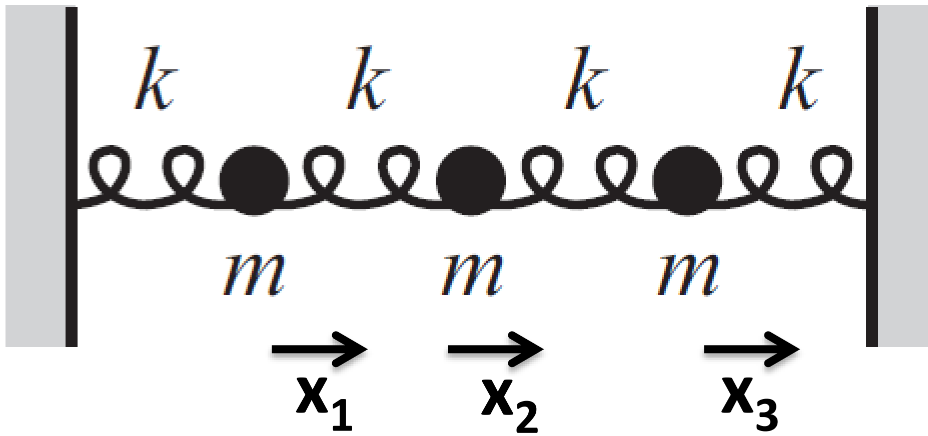
Exercise 2: Consider two pendula which are coupled together with a spring (having constant k) as shown in figure below. Assume that the displacement from the equilibrium positions are small enough that small angle approximation can be used and motion is approximately only in x-direction. Find the normal modes and normal co-ordinates for this system.



Solution: Write coupled equations for 3-masses coupled oscillator shown in figure below. Find the eigen values and eigen vectors.



3-masses coupled oscillator

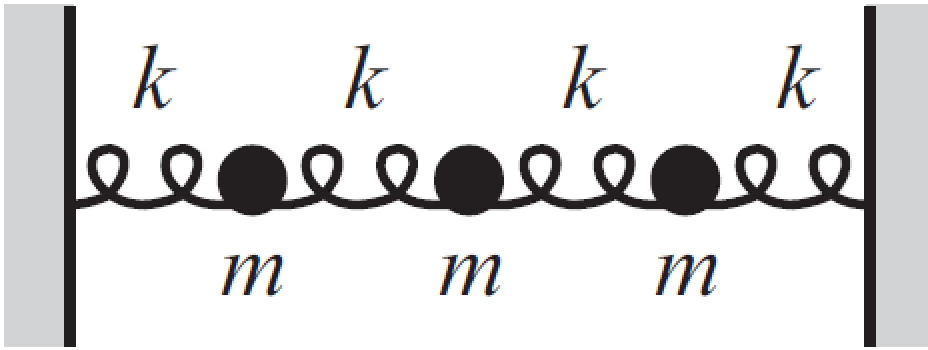


$$m\frac{d^2x_1}{dt^2} = -kx_1 + k\left(x_2 - x_1\right)$$

$$m\frac{d^2x_2}{dt^2} = -k\left(x_2 - x_1\right) + k\left(x_3 - x_2\right)$$

$$m\frac{d^2x_3}{dt^2} = -kx_3 - k\left(x_3 - x_2\right)$$

3-masses coupled oscillator



$$m\frac{d^2}{dt^2}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{bmatrix} -2k & k & 0 \\ k & -2k & k \\ 0 & k & -2k \end{bmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$m\frac{d^2}{dt^2}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{bmatrix} -2k & k & 0 \\ k & -2k & k \\ 0 & k & -2k \end{bmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} e^{\alpha t}$$

Characteristic equation

Eigen values and Eigen vectors

Normal modes and Normal co-ordinates

$$\frac{d^2}{dt^2}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{bmatrix} -2\omega_0^2 & \omega_0^2 & 0 \\ \omega_0^2 & -2\omega_0^2 & \omega_0^2 \\ 0 & \omega_0^2 & -2\omega_0^2 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} A_1 \\ A_2 \\ A_3 \end{bmatrix} e^{\alpha t}$$

$$\begin{bmatrix} -2\omega_0^2 & \omega_0^2 & 0 \\ \omega_0^2 & -2\omega_0^2 & \omega_0^2 \\ 0 & \omega_0^2 & -2\omega_0^2 \end{bmatrix} [A] - \alpha^2 [I][A] = 0$$

$$\{[K] - \lambda [I]\}[A] = 0$$

Finding Eigen values

$$\begin{vmatrix} -2\omega_0^2-\alpha^2 & \omega_0^2 & 0 \\ \omega_0^2 & -2\omega_0^2-\alpha^2 & \omega_0^2 \\ 0 & \omega_0^2 & -2\omega_0^2-\alpha^2 \end{vmatrix} = 0$$

$$\boxed{(2\omega_0^2+\alpha^2)}[(2\omega_0^2+\alpha^2)^2-2\omega_0^4]=0$$

$$\alpha_1 = i\sqrt{2}\omega_0,\ \ \alpha_2 = -i\sqrt{2}\omega_0$$

$$\alpha^4+4\omega_0^2\alpha^2+2\omega_0^4=0$$

$$\alpha^2 = (-2\pm\sqrt{2})\omega_0{}^2$$

$$\alpha_3 = i\sqrt{\left(2+\sqrt{2}\right)}\omega_0,\ \ \alpha_4 = -i\sqrt{\left(2+\sqrt{2}\right)}\omega_0$$

$$\alpha_5 = i\sqrt{\left(2-\sqrt{2}\right)}\omega_0,\ \ \alpha_6 = -i\sqrt{\left(2-\sqrt{2}\right)}\omega_0$$

$$\omega = \pm\sqrt{2}\,\omega_0 \quad \Longrightarrow \quad \begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix} \propto \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix},$$

$$\omega = \pm\sqrt{2+\sqrt{2}}\,\omega_0 \quad \Longrightarrow \quad \begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix} \propto \begin{pmatrix} 1 \\ -\sqrt{2} \\ 1 \end{pmatrix},$$

$$\omega = \pm\sqrt{2-\sqrt{2}}\,\omega_0 \quad \Longrightarrow \quad \begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix} \propto \begin{pmatrix} 1 \\ \sqrt{2} \\ 1 \end{pmatrix}.$$