ICS141: Discrete Mathematics for Computer Science I

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Lecture 5

Chapter 1. The Foundations

1.4 Nested Quantifiers

1.5 Rules of Inference

Previously…

TABLE 1 **Quantifiers.**

Statement	*When True?*	*When False?*
$\forall x P(x)$ $\exists x P(x)$	$P(x)$ is true for every x. There is an x for which $P(x)$ is true.	There is an x for which $P(x)$ is false. $P(x)$ is false for every x.

TABLE 2 **De Morgan's Laws for Quantifiers.**

Negation	*Equivalent Statement*	*When Is Negation True?*	*When False?*
$\neg\exists x P(x)$	$\forall x \neg P(x)$	For every x, $P(x)$ is false.	There is an x for which $P(x)$ is true.
$\neg\forall x P(x)$	$\exists x \neg P(x)$	There is an x for which $P(x)$ is false.	$P(x)$ is true for every x.

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Nesting of Quantifiers

- Example:

  Let the domain of x and y be people.

  Let $L(x,y)$ = "x likes y" (A statement with 2 free variables – not a proposition)

  - Then $\exists y\, L(x,y)$ = "There is someone whom x likes." (A statement with 1 free variable x – not a proposition)
  - Then $\forall x\, (\exists y\, L(x,y))$ =

    "Everyone has someone whom they like."

    (A __Proposition__ with __0__ free variables.)

Nested Quantifiers

- Nested quantifiers are quantifiers that occur within the scope of other quantifiers.
- The order of the quantifiers is important, unless all the quantifiers are universal quantifiers or all are existential quantifiers.

TABLE 1 Quantifications of Two Variables.

Statement	*When True?*	*When False?*
$\forall x \forall y P(x, y)$ $\forall y \forall x P(x, y)$	$P(x, y)$ is true for every pair x, y.	There is a pair x, y for which $P(x, y)$ is false.
$\forall x \exists y P(x, y)$	For every x there is a y for which $P(x, y)$ is true.	There is an x such that $P(x, y)$ is false for every y.
$\exists x \forall y P(x, y)$	There is an x for which $P(x, y)$ is true for every y.	For every x there is a y for which $P(x, y)$ is false.
$\exists x \exists y P(x, y)$ $\exists y \exists x P(x, y)$	There is a pair x, y for which $P(x, y)$ is true.	$P(x, y)$ is false for every pair x, y.

Nested Quantifiers

- Let the domain of x and y is **R**, and $P(x,y)$: $xy = 0$. Find the truth value of the following propositions.
  - $\forall x \, \forall y \, P(x, y)$ (F)
  - $\forall x \, \exists y \, P(x, y)$ (T)
  - $\exists x \, \forall y \, P(x, y)$ (T)
  - $\exists x \, \exists y \, P(x, y)$ (T)

R: set of real numbers

- $\forall x \, \exists y \, P(x,y) \not\equiv \exists y \, \forall x \, P(x,y)$
  - For every x, there exists y such that $x + y = 0$. (T)
  - There exists y such that, for every x, $x + y = 0$. (F)

Nested Quantifiers: Example

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- Let the domain = {1, 2, 3}. Find an expression equivalent to $\forall x\, \exists y\, P(x,y)$ where the variables are bound by substitution instead:
  - Expand from inside out or outside in.
  - Outside in:

$$\forall x\, \exists y\, P(x,y)$$

$$\equiv \exists y\, P(1,y) \wedge \exists y\, P(2,y) \wedge \exists y\, P(3,y)$$

$$\equiv [P(1,1) \vee P(1,2) \vee P(1,3)] \wedge$$
$$[P(2,1) \vee P(2,2) \vee P(2,3)] \wedge$$
$$[P(3,1) \vee P(3,2) \vee P(3,3)]$$

Quantifier Exercise

- If $R(x,y)$="x relies upon y," express the following in unambiguous English when the domain is all people

$\forall x(\exists y\, R(x,y))$ = Everyone has *someone* to rely on.

$\exists y(\forall x\, R(x,y))$ = There's a poor overburdened soul whom *everyone* relies upon (including himself)!

$\exists x(\forall y\, R(x,y))$ = There's some needy person who relies upon *everybody* (including himself).

$\forall y(\exists x\, R(x,y))$ = Everyone has *someone* who relies upon them.

$\forall x(\forall y\, R(x,y))$ = *Everyone* relies upon *everybody*, (including themselves)!

Negating Nested Quantifiers

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- Successively apply the rules for negating statements involving a single quantifier
- <u>Example</u>: Express the negation of the statement $\forall x\, \exists y\, (P(x,y) \wedge \exists z\, R(x,y,z))$ so that all negation symbols immediately precede predicates.
  - $\neg \forall x\, \boxed{\exists y\, (P(x,y) \wedge \exists z\, R(x,y,z))}$
    $\equiv \exists x\, \neg \boxed{\exists y}\, \boxed{(P(x,y) \wedge \exists z\, R(x,y,z))}$
    $\equiv \exists x\, \forall y\, \neg \boxed{(P(x,y) \wedge \exists z\, R(x,y,z))}$
    $\equiv \exists x\, \forall y\, (\neg P(x,y) \vee \boxed{\neg \exists z\, R(x,y,z)})$
    $\equiv \exists x\, \forall y\, (\neg P(x,y) \vee \forall z\, \neg R(x,y,z))$

Equivalence Laws

- $\forall x\, \forall y\, P(x,y) \equiv \forall y\, \forall x\, P(x,y)$

  $\exists x\, \exists y\, P(x,y) \equiv \exists y\, \exists x\, P(x,y)$

- $\forall x\, \mathbf{(}P(x) \land Q(x)\mathbf{)} \equiv \mathbf{(}\forall x\, P(x)\mathbf{)} \land \mathbf{(}\forall x\, Q(x)\mathbf{)}$

  $\exists x\, \mathbf{(}P(x) \lor Q(x)\mathbf{)} \equiv \mathbf{(}\exists x\, P(x)\mathbf{)} \lor \mathbf{(}\exists x\, Q(x)\mathbf{)}$

- Exercise:

  See if you can prove these yourself.

Notational Conventions

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- Quantifiers have higher precedence than all logical operators from propositional logic:

  $$(\forall x\, P(x)) \land Q(x)$$

- Consecutive quantifiers of the same type can be combined:

  $\forall x\, \forall y\, \forall z\, P(x,y,z) \equiv \forall x,y,z\, P(x,y,z)$

  or even $\forall xyz\, P(x,y,z)$

1.5 Rules of Inference

- ***An argument***: a sequence of statements that end with a conclusion
- Some forms of argument ("valid") never lead from correct statements to an incorrect conclusion. Some other forms of argument ("fallacies") can lead from true statements to an incorrect conclusion.
- ***A logical argument*** consists of a list of (possibly compound) propositions called premises/hypotheses and a single proposition called the conclusion.
- ***Logical rules of inference***: methods that depend on logic alone for deriving a new statement from a set of other statements. (Templates for constructing valid arguments.)

Valid Arguments (I)

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- Example: A logical argument

  If I dance all night, then I get tired.

  I danced all night.

  Therefore I got tired.

- Logical representation of underlying variables:

  p: I dance all night. q: I get tired.

- Logical analysis of argument:

$$
\begin{array}{ll}
p \rightarrow q & \text{premise 1} \\
\underline{p} & \underline{\text{premise 2}} \\
\therefore q & \text{conclusion}
\end{array}
$$

Valid Arguments (II)

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- A form of logical argument is ***valid*** if whenever every premise is true, the conclusion is also true. A form of argument that is not valid is called a ***fallacy***.

Inference Rules: General Form

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- An *Inference Rule* is
  - A pattern establishing that if we know that a set of *premise* statements of certain forms are all true, then we can validly deduce that a certain related *conclusion* statement is true.

$$\begin{array}{l} \quad \textit{premise}\ 1 \\ \quad \textit{premise}\ 2 \\ \quad \ldots \\ \hline \therefore\ \textit{conclusion} \end{array}$$

"$\therefore$" means "therefore"

Inference Rules & Implications

- Each valid logical inference rule corresponds to an implication that is a tautology.

$$\begin{array}{l} \quad \textit{premise 1} \\ \quad \textit{premise 2} \\ \quad \ldots \\ \hline \therefore \ \textit{conclusion} \end{array}$$

Inference rule

- Corresponding tautology:

$((\textit{premise 1}) \wedge (\textit{premise 2}) \wedge \cdots) \rightarrow \textit{conclusion}$

Modus Ponens

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- $$\begin{array}{l} \quad p \\ \quad p \to q \\ \hline \therefore q \end{array}$$
  Rule of ***Modus ponens*** (a.k.a. *law of detachment*)

  "the mode of affirming"

- $(p \land (p \to q)) \to q$ is a tautology

p	q	$p \to q$	$p \land (p \to q)$	$(p \land (p \to q)) \to q$
T	T	T	T	T
T	F	F	F	T
F	T	T	F	T
F	F	T	F	T

- Notice that the first row is the only one where premises are all true

Modus Ponens: Example

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If $\left\{\begin{array}{ll} p \rightarrow q & : \text{“If it snows today} \\ & \quad \text{then we will go skiing”} \\ p & : \text{“It is snowing today”} \end{array}\right\}$ assumed TRUE

Then $\therefore q$: “We will go skiing” is TRUE

If $\left\{\begin{array}{ll} p \rightarrow q & : \text{“If } n \text{ is divisible by 3} \\ & \quad \text{then } n^2 \text{ is divisible by 3”} \\ p & : \text{“} n \text{ is divisible by 3”} \end{array}\right\}$ assumed TRUE

Then $\therefore q$: “n^2 is divisible by 3” is TRUE

Modus Tollens

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- $$\begin{array}{l} \neg q \\ p \rightarrow q \\ \hline \therefore \neg p \end{array}$$ Rule of ***Modus tollens***

"the mode of denying"

- $(\neg q \wedge (p \rightarrow q)) \rightarrow \neg p$ is a tautology
- Example

If $p \rightarrow q$: "If this jewel is really a diamond then it will scratch glass"

$\neg q$: "The jewel doesn't scratch glass"

assumed TRUE

Then $\therefore \neg p$: "The jewel is not a diamond" is TRUE

More Inference Rules

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- $$\frac{p}{\therefore\ p \vee q}$$

  Rule of ***Addition***

  Tautology: $p \rightarrow (p \vee q)$

- $$\frac{p \wedge q}{\therefore\ p}$$

  Rule of ***Simplification***

  Tautology: $(p \wedge q) \rightarrow p$

- $$\frac{\begin{array}{c} p \\ q \end{array}}{\therefore\ p \wedge q}$$

  Rule of ***Conjunction***

  Tautology: $[(p) \wedge (q)] \rightarrow p \wedge q$

Examples

- State which rule of inference is the basis of the following arguments:
  - It is below freezing now. Therefore, it is either below freezing or raining now.
  - It is below freezing and raining now. Therefore, it is below freezing now.
- p: It is below freezing now.
  q: It is raining now.
  - $p \rightarrow (p \vee q)$ (rule of addition)
  - $(p \wedge q) \rightarrow p$ (rule of simplification)

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Hypothetical Syllogism

- $$\begin{array}{l} \quad p \rightarrow q \\ \quad q \rightarrow r \\ \hline \therefore p \rightarrow r \end{array}$$

  Rule of ***Hypothetical syllogism***

  Tautology:

  $[(p \rightarrow q) \wedge (q \rightarrow r)] \rightarrow (p \rightarrow r)$

- Example: State the rule of inference used in the argument:

  "If $\underbrace{\text{it rains today}}_{p}$, then $\underbrace{\text{we will not have a barbecue today}}_{q}$. If $\underbrace{\text{we do not have a barbecue today}}_{q}$, then $\underbrace{\text{we will have a barbecue tomorrow}}_{r}$. Therefore, if $\underbrace{\text{it rains today}}_{p}$, then $\underbrace{\text{we will have a barbecue tomorrow}}_{r}$."

Disjunctive Syllogism

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- $$\begin{array}{l} \quad p \vee q \\ \quad \neg p \\ \hline \therefore q \end{array}$$

  Rule of ***Disjunctive syllogism***

  Tautology: $[(p \vee q) \wedge (\neg p)] \rightarrow q$

- Example
  - $\underbrace{\text{Ed's wallet is in his back pocket}}_{p}$ or $\underbrace{\text{it is on his desk}}_{q}$. ($p \vee q$)
  - Ed's wallet is not in his back pocket. ($\neg p$)
  - Therefore, Ed's wallet is on his desk. (q)