# Lecture-8

6/10/2020

## Back to integration :

Recall : How do we find $I = \int_C f(z)\,dz$ ?

I : parametrizing C and convert it to real integration

Eg. if C : circle of radius r, center at z=0

for $\int_C \frac{1}{z}\,dz$ we write $z(t) = re^{it}$, $0 \le t \le 2\pi$

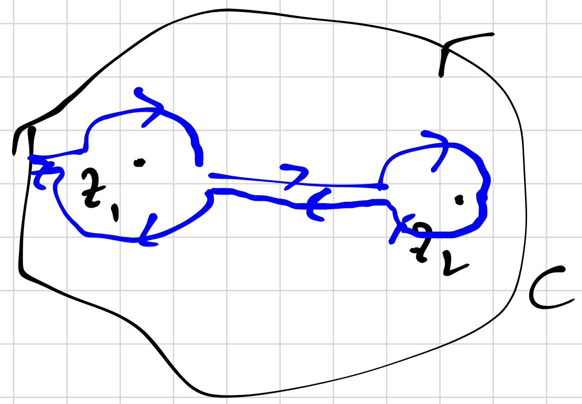
$dz = r\,i\,e^{it}\,dt$

so $I = \int_0^{2\pi} \frac{1}{re^{it}} \cdot r\,i\,e^{it}\,dt = i \cdot \int_0^{2\pi} dt = 2\pi i$

II If C : simple closed contour and f(z) is analytic inside and on C, then apply Cauchy's-Goursat th$^m$ to say

$$I = \int_C f(z)\,dz = 0.$$

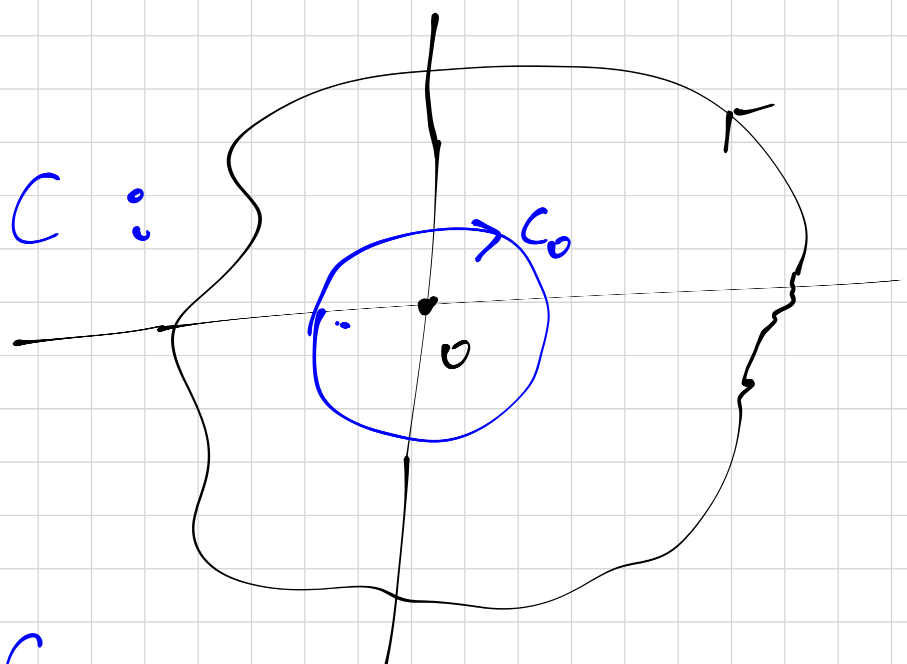
III $C$ : as above and
$f(z)$ is analytic inside and on $C$
except some finite no. of points $z_1, \ldots z$




then we can apply Cauchy Goursat th^m for multi domain to say

$$\int_C f(z)\,dz = -\sum_{k=1}^{n} \int_{C_k} f(z)\,dz$$

Eg. $f(z) = \frac{1}{z}$, $C$ :




then $\int_C f(z)\,dz = -\int_{C_0} \frac{1}{z}\,dz$.

So still, we need to calculate $\int_{C_k} f(z)\,dz$

- either use parametrizat^n.
- NOTE : $f(z)$ is analytic inside and on each $C_k$ except the point $z_k$.

So one can write Laurent series of $f(z)$

(inside $C_k$), precisely $0 < |z - z_k| <$ radius of $C_k$

That is $$f(z) = \sum_{n\geq 0}^{\infty} a_n^k (z-z_k)^n + \sum_{n=1}^{\infty} \frac{b_n^k}{(z-z_k)^n}$$

where $a_n^k = \int \cdots$

$$b_n^k = \frac{1}{2\pi i}\int_{C'} \frac{f(z)}{(z-z_0)^{-n+1}}\, dz \qquad n = 1, 2, \cdots$$

Particularly, $$b_1^k = \frac{1}{2\pi i}\int_{C'} f(z)\, dz \qquad \text{———} (*)$$

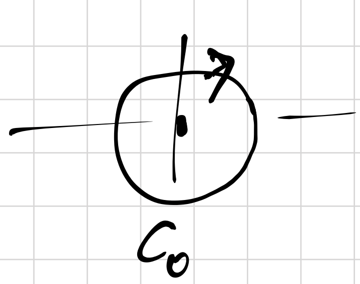
So for our calculation, we can take $C' = -C_k$ and then use $(*)$

Then we get $$I = \int_C f(z)\,dz = -\sum_{k=1}^{n} \int_{C_k} f(z)\,dz$$

$$= \sum_{k=1}^{n} \cdot\, 2\pi i\ b_1^k$$

for eg. $$\int_C \frac{1}{z}\,dz = \int_{-C_0} \frac{1}{z}\,dz = 2\pi i \cdot b_1$$

$$= 2\pi i \cdot 1 = 2\pi i$$




[Since $f(z) = \frac{1}{z}$ is itself the Laurent series near $0$]

We develop the above ideas more formally below..

## Definition

I. A point $z_0$ is called a singular point of a function $f(z)$ if (i) $f(z)$ is not analytic at $z_0$.

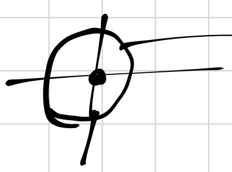
(ii) $f(z)$ is analytic at some point in every nbd of $z_0$.

II. A singular point is called an isolated singular point if (iii) $f(z)$ is analytic in a deleted neighbourhood $0 < |z - z_0| < R$ of $z_0$ for some $R > 0$

[So $f(z)$ has Laurent series expn in $0<|z-z_0|<R$]

Eg: ① $f(z) = \dfrac{z+1}{z^3(z^2+1)}$ not analytic at $z = 0, i, -i$

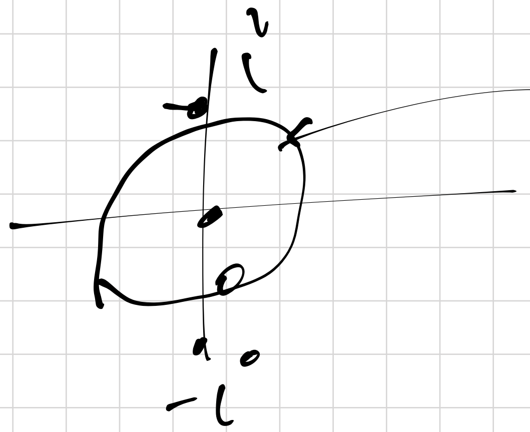
(i) ✓

(ii)  → has points where $f(z)$ is analytic. (take any pt other than $0, i, -i$)

[Similarly for $i$ & $-i$]

So all three are singular pt.

(iii)



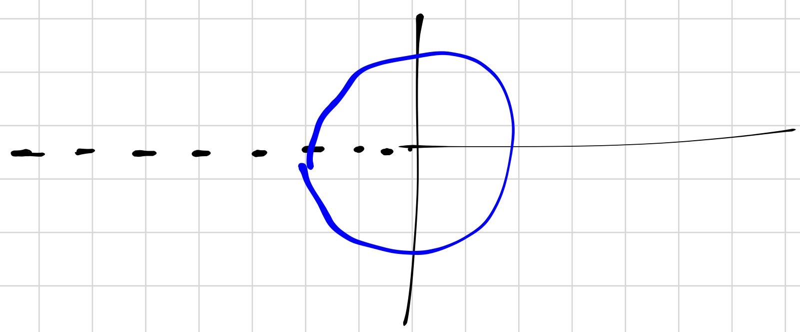

In this deleted neighbourhood of $z=0$, (keeping radius $<1$) $f(z)$ is analytic.

So $z=0$ is an isolated singular pt.

Similarly $z=\pm i$ are isolated singular pts

(2) Take the principal branch of $\log z$

$$\operatorname{Log} z = \ln|z| + i\theta \qquad -\pi<\theta<\pi,\ z\neq 0$$



At $z=0$ : (i) not analytic at $z=0$

(ii) Take any point other than pts on the negative real axis.

(iii) No,

So $z=0$ is a singular point not an isolated singular point.

(Laurent series can not be written in nbd of $z=0$)

Def$^n$ :- Suppose $z_0$ is an isolated singular point of $f(z)$. Let

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n}, \qquad \text{---} (\#)$$

in $0 < |z-z_0| < R$ for some $R > 0$.

The coefficient of $\frac{1}{(z-z_0)}$ in $(\#)$

$= b_1$

is called the residue of $f(z)$ at $z_0$.

written as.

$$b_1 := \operatorname*{Res}_{z=z_0} f(z)$$

## Cauchy - Residue Theorem.

Suppose $C$ is a simple closed contour and positively oriented.

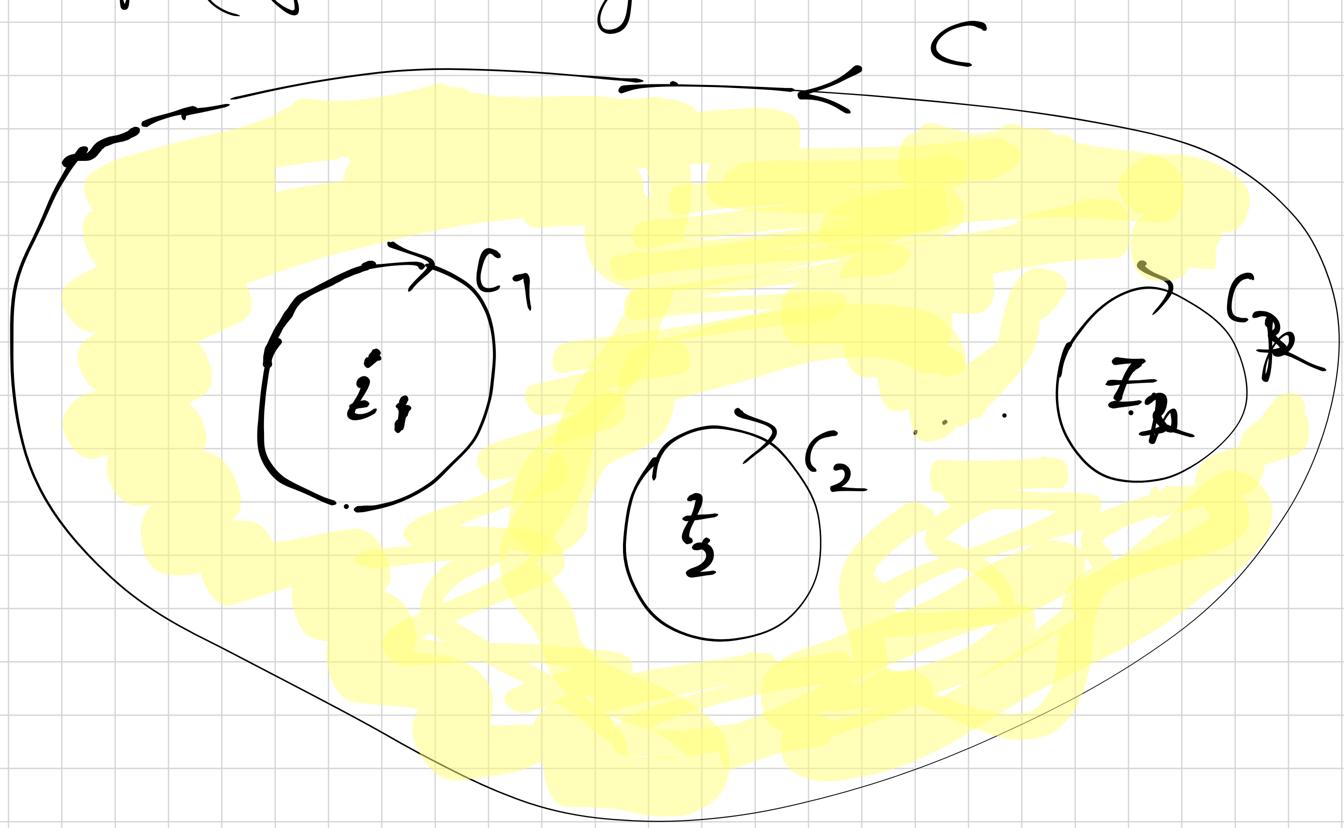
Supp. $f(z)$ is analytic inside and on $C$ except at a finite number of singular points $z_1, \ldots, z_n$ inside of $C$.

Then

$$\int_C f(z)\, dz = 2\pi i \sum_{k=1}^{n} \operatorname*{Res}_{z=z_k} f(z)$$

**Ex:** If $f(z)$ has finite no. of singular points, then they are all isolated.

Proof (of Cauchy's Residue theorem).




$f(z)$ is analytic inside and on $C$.

- Let $C_1, \ldots, C_k$ be circles with orientation as above around $z_1, \ldots z_k$ respectively lying inside of $C$, and have no common points.

- Now, $f(z)$ is analytic in the shaded multiply connected domain,

So, by Cauchy's Goursat theorem,

$$\int_C f(z)\,dz = \sum_{k=1}^{n} \int_{C_k} f(z)\,dz$$

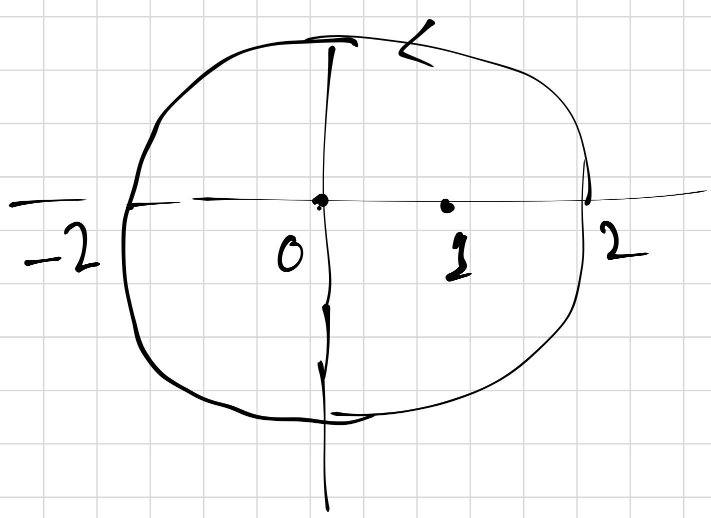
By earlier discussion,

$$\int_{C_k} f(z)\,dz = \operatorname{Res}_{z=z_k} f(z) \quad , \ k=1,2,\ldots$$

$\boxtimes$

**Eg:** Evaluate $I = \int_C \frac{5z-2}{z(z-1)}\,dz$ where

$C: \quad |z| = 2$ , counterclockwise




→ $z = 0, 1$ are singular pts of $f(z)$.

→ By Cauchy-Residue thm,

$$I = \operatorname{Res}_{z=0} f(z) + \operatorname{Res}_{z=1} f(z).$$

→ Write Laurent series in a deleted nbd. of $z=0$ & $z=1$

→ for $0 < |z| < 1/2$, we have

$\Rightarrow$ $\dfrac{5z-2}{z(z-1)} = \dfrac{5z-2}{z} \cdot \dfrac{-1}{1-z}$

$= \left(5 - \dfrac{2}{z}\right)\left(-1 - z - z^2 - \cdots\right)$

$\operatorname{Res}_{z=0} f(z) = b_1 = \text{coeff of } \dfrac{1}{z} = 2$

$\Rightarrow$ for $0 < |z-1| < 1/2$, we can write

$\dfrac{5z-2}{z(z-1)} = \dfrac{5z-5+3}{z(z-1)} = \dfrac{5}{z} + \dfrac{3}{z(z-1)}$

$= \left(5 + \dfrac{3}{(z-1)}\right)\left[\dfrac{1}{1+(z-1)}\right]$

$= \left(5 + \dfrac{3}{z-1}\right)\left(1 - (z-1) + (z-1)^2 \cdots\right)$

$\operatorname{Res}_{z=1} f(z) = \text{coeff of } \dfrac{1}{(z-1)} = 3.$

so $I = 2\pi i(2+3) = 10\pi i$

# Type of isolated Singular points

Let $z_0$ be an isolated sing. pt. of $f(z)$ and $\forall z$ with $0 < |z - z_0| < R$, we have the Laurent series,

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \underbrace{\frac{b_1}{(z-z_0)} + \frac{b_2}{(z-z_0)^2} + \cdots}_{\text{principal part of } f(z) \text{ at } z_0}$$

principal part of $f(z)$ at $z_0$.

**Case. I**

principal part is nonzero & finite

i.e. $\exists\ m \geq 1$ s.t

$b_m \neq 0$ & $b_{m+1} = b_{m+2} = \cdots = 0$

i.e. $f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \frac{b_1}{(z-z_0)} + \cdots + \frac{b_m}{(z-z_0)^m}$

$\forall\ 0 < |z - z_0| < R$.

Such $z_0$ is called pole of order $m$.

**Case. II**

principal part is zero

i.e. $b_1 = b_2 = \cdots = 0$

Such $z_0$ is called removable singularity

**Case. III**

principal part is infinite

i.e. $\exists$ a sequence $(n_k)$ s.t. $b_{n_k} \neq 0$.

$z_0$ is called essential singularity

→ How do we identify the type of singularity?

<u>Method-I</u> : By writing out the Laurent series in a deleted nbd of that pt.

<u>Eg.</u> ① $f(z) = \dfrac{1-\cos z}{z^2}$

$z=0$ : a singular point. and

$f(z) = \dfrac{1}{2!} - \dfrac{z^2}{4!} + \dfrac{z^4}{6!} \cdots \quad , \; 0<|z|<\infty$

<u>so</u> $z=0$ is a removable singularity.

② $f(z) = e^{1/z}$, $z=0$ is a singular point.

$= 1 + \dfrac{1}{z} + \dfrac{1}{2!\,z^2} + \dfrac{1}{3!\,z^3} + \cdots$

So $z=0$ is an essential singular point.

Eg. $f(z) = \dfrac{5z-2}{z(z-1)}$

$z=0$, we have. $f(z) = \left(5 - \dfrac{2}{z}\right)\left(-1 - z - z^2 - \dots\right)$

$0 < |z| < 1/2$

$= \sum a_n z^n + \dfrac{2}{z}$

So $z=0$ is a pole of order 1.

also called simple pole.

$z=1$. we have, for $0 < |z-1| < 1/2$

$f(z) = \left(5 + \dfrac{3}{z-1}\right)\left(1 - (z-1) + (z-1)^2 - \dots\right)$

So, $z=1$ is a pole of order 1 or a sim p

Method-II : Use the next theorems.

Theorem I Suppose $z_0$ is an isolated sing. pt of $f(z)$. Then TFAE (the following are equivalent)

① $f(z)$ has a removable singularity at $z_0$.

② $\lim_{z \to z_0} f(z)$ exists (and is finite)

③ $f(z)$ is analytic and bounded in a deleted nbd of $z_0$.

Apply Thm I to $f(z) = \frac{\sin z}{z}$.

①

$$\lim_{z \to 0} \frac{\sin z}{z} = 1 \quad , \quad \text{so } z=0 \text{ is a removable sing. pt.}$$

② $f(z) = \frac{1}{z}$,

$$\lim_{z \to 0} f(z) = \infty \quad \text{so } z=0 \text{ is not a remov. sing pt.}$$

Theorem: II: Suppose $z_0$ is an isolated sing. of $f(z)$. Then $z_0$ is a pole of order $m$ if and only if

$$f(z) = \frac{\phi(z)}{(z-z_0)^m}, \quad z \neq z_0$$

where $\phi(z)$ is analytic at $z_0$
$\phi(z_0) \neq 0$.

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Moreover, (i) $\operatorname{Res}_{z=z_0} f(z) = \phi(z_0)$ if $m = 1$.

(ii) $\operatorname{Res}_{z=z_0} f(z) = \dfrac{\phi^{(m-1)}(z_0)}{(m-1)!}$ if $m \geq 2$

Eg:

$$f(z) = \frac{5z-2}{z(z-1)} = \frac{\phi(z)}{z}$$

where $\phi(z) = \dfrac{5z-2}{z}$, is analytic at $z=1$

$$\phi(z=1) = 3 \neq 0.$$

so $z=1$ is a pole of order 1.

& $\operatorname{Res}_{z=1} f(z) = \phi(1) = 3.$

Do same for $z=0$

**Theorem III** : Suppose $z_0$ is an isolated singular pt of $f(z)$. Then $z_0$ is a pole of order $m$ if and only if $\lim_{z \to z_0} (z-z_0)^m f(z) = a \neq 0, \infty$.

**Theorem IV** Suppose $z_0$ is an isolated singular pt of $f(z)$. Then $z_0$ is a pole if and only if $\lim_{z \to z_0} f(z) = \infty$.

**Eg**:- $f(z) = \dfrac{1}{z^2(e^z - 1)}$

$z = 0$ and $2n\pi i$, $n \in \mathbb{Z}$ are **isolated** singular points.

$e^z - 1 = 0$
iff $e^x e^{iy} = 1$
iff $e^x = 1$ and $y = 2n\pi$
i.e. $z = 0 + 2n\pi i$

Now

$$\lim_{z \to 0} (z-0)^3 f(z) = \lim_{z \to 0} \frac{z}{e^z - 1} = 1$$

So, $z = 0$ is a pole of order 3.

for $z = 2n\pi i$,

Ex:- find $\lim_{z \to 2n\pi i} (z - 2n\pi i) f(z) = \frac{1}{(2n\pi i)^2}$

So, $z = 2n\pi i$ is a simple pole.
$n \neq 0$

——— x ——— x ——— x ———