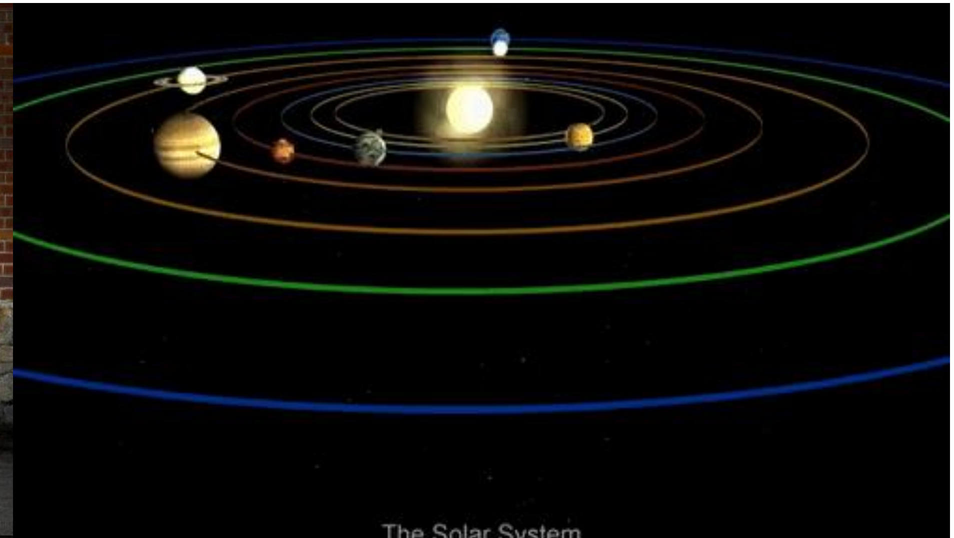
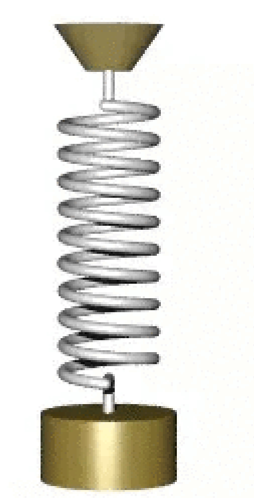
Work , Energy and Conservation laws





GIFs:GiPHY, Gfycat

Predicting the motion of a system under known constraints

$$m\frac{d^2x}{dt^2} = F(x)$$

Solve differential equation

Work, energy and conserved quantities

Work Energy Theorem from Force Equation

$$m\int_{x_a}^{x_b} \frac{dv}{dt}dx = \int_{x_a}^{x_b} F(x)dx$$

Change variable from *x* to *t*

$$m\int_{t_a}^{t_b} \frac{dv}{dt}vdt = \int_{x_a}^{x_b} F(x)dx$$

$$m\int_{t_a}^{t_b} \frac{d}{dt}\left(\frac{1}{2}v^2\right)dt = \int_{x_a}^{x_b} F(x)dx$$

Work Energy Theorem from Force Equation

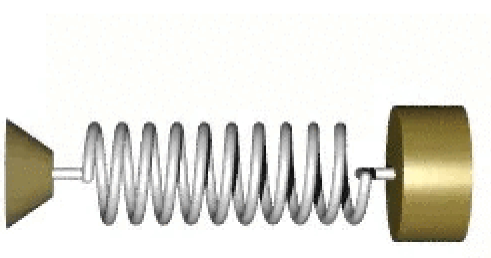
$$W_{ba} = K_b - K_a$$

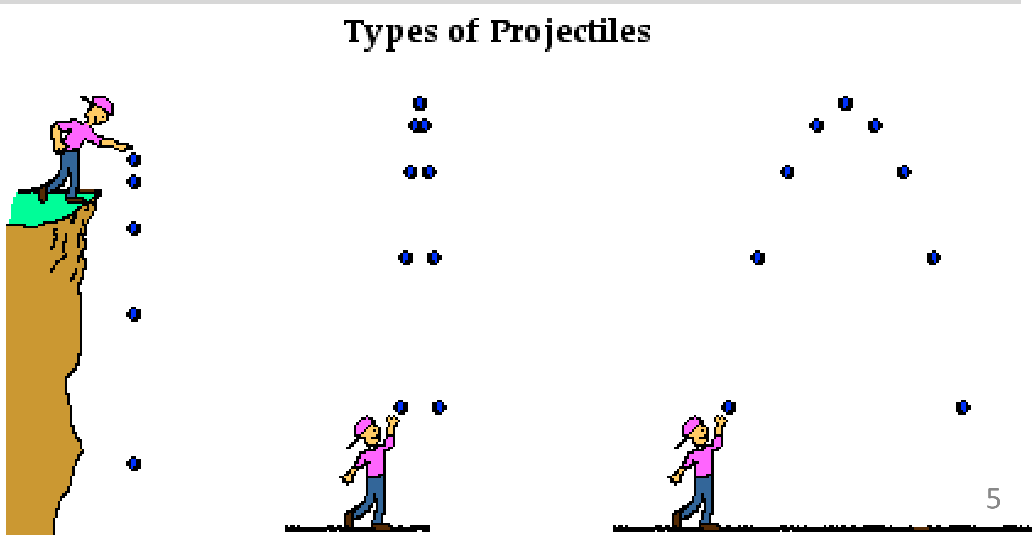
The change in kinetic energy equals work done on the particle

Some applications of Work-Energy relation in 1-D

Examples

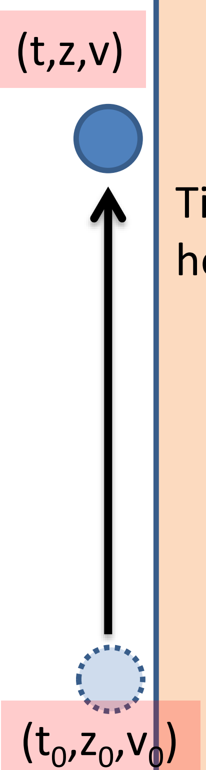
1. Mass thrown upward in a Uniform Gravitational field
2. Simple Harmonic Motion
3. Vertical Motion in an inverse square field





Maximum distance covered when Mass thrown upward in a Uniform Gravitational field

$$m\frac{d^2z}{dt^2} = -mg$$



1 Equation of motion

Integrate once with I.C.

$$v = v_0 - gt$$

Time at which ball reaches maximum height:

$$t_m = \frac{v_0}{g}$$

Integrate once again with I.C.

$$z = z_0 + v_0 t - g\frac{t^2}{2}$$

$$z_m = z_0 + v_0 t_m - g\frac{t_m^{\ 2}}{2}$$

Eliminate time using t_m.

$$z_m = z_0 + \frac{v_0^{\ 2}}{2g}$$

2 Work energy theorem

$$\frac{1}{2}\cancel{m v_m^{\ 2}} - \frac{1}{2}m v_0^{\ 2} = -mg\int_{z_0}^{z_m} dz$$

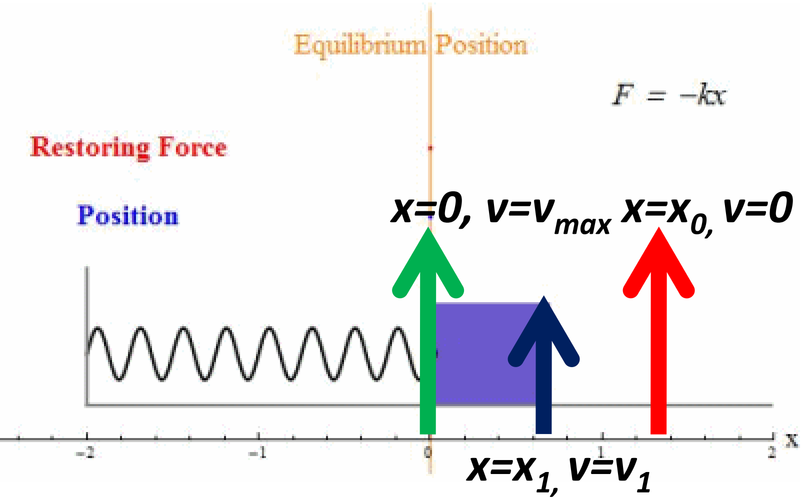
Integrate once with I.C.

$$-\frac{1}{2}m v_0^{\ 2} = -mg\left[z_m - z_0\right]$$

$$z_m = z_0 + \frac{v_0^{\ 2}}{2g}$$

The solution makes no reference to time!

Solving the equation of Simple Harmonic Motion



$$\frac{1}{2}mv^2 - \frac{1}{2}mv_0{}^2 = -k\int_{x_0}^{x} x\,dx$$

Case I

Choose Initial conditions: at $t=0$, $v=0$, $x=x_0$

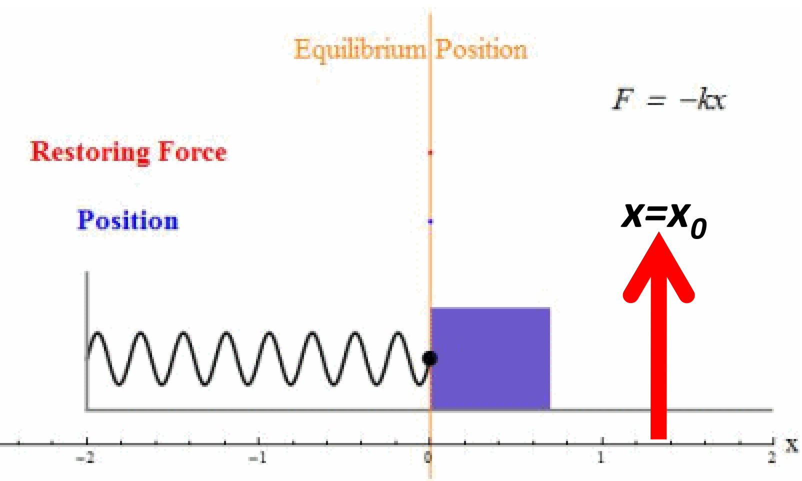
Case II

Choose Initial conditions: at $t=0$, $V=V_{max}$, $x=0$

Case III

Choose Initial conditions: at $t=0$, $V=V_1$, $x=x_1$

Solving the equation of Simple Harmonic Motion



$$\frac{1}{2}mv^2 - \cancel{\frac{1}{2}mv_0^{\ 2}} = -k\int_{x_0}^{x} x\,dx$$

Case I

Choose Initial conditions: at *t=0, v=0, x=x_0*

$$v^2 = \frac{k}{m}\left[x_0^{\ 2} - x^2\right]$$

$$v = \sqrt{\frac{k}{m}}\sqrt{\left[x_0^{\ 2} - x^2\right]}$$

Integrating again after separating variables

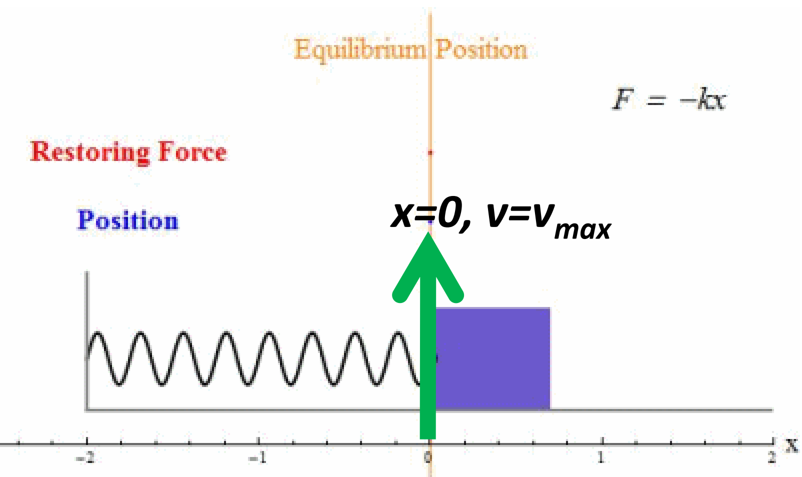
$$\int_{x_0}^{x}\int_{x_0}^{x}\frac{dx}{\sqrt{\left[x_0^{\ 2} - x^2\right]}} = \sqrt{\frac{k}{m}}\,t\,|t$$

$$\left[\sin^{-1}\left(\frac{x}{x_0}\right) - \sin^{-1}(1)\right] = \omega t$$

$$\text{Hint}: x = x_0 \sin(\theta)$$

$$x = x_0 \cos(\omega t)$$

Solving the equation of Simple Harmonic Motion



$$\frac{1}{2}mv^2 - \frac{1}{2}mv_0{}^2 = -k\int_{x_0}^{x} x\,dx$$

Case II

Choose Initial conditions: at *t=0, V=V$_{max}$, x=0*

$$v^2 - v_{\max}{}^2 = -\frac{k}{m}\left[x^2\right]$$

$$v = \sqrt{v_{\max}{}^2 - \frac{k}{m}x^2}$$

Integrating again after separating variables

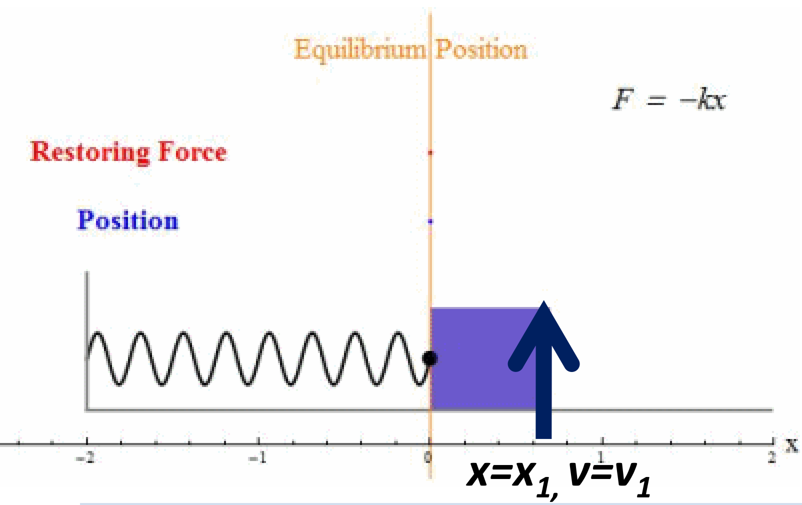
$$\int_0^x \cdot \int_0^x \frac{dx}{\sqrt{v_{\max}{}^2 - \frac{k}{m}x^2}} = t|t$$

$$\left[\sin^{-1}\left(\frac{\omega x}{v_{\max}}\right)\right] = \omega t$$

$$\text{Hint}: x = \frac{v_{\max}}{\omega}\sin(\theta)$$

$$x = \frac{v_{\max}}{\omega}\sin\left(\omega t\right)$$

Solving the equation of Simple Harmonic Motion



$$\frac{1}{2}mv^2 - \frac{1}{2}m{v_0}^2 = -k\int_{x_0}^{x} x\,dx$$

Case III

Choose Initial conditions: at *t=0, V=V*$_1$ *, x=x*$_1$

$$v^2 - {v_1}^2 = -\frac{k}{m}\left[x^2 - {x_1}^2\right]$$

$$v = \sqrt{{v_1}^2 - \frac{k}{m}\left[x^2 - {x_1}^2\right]}$$

Integrating again after separating variables

$$\int_{x_1}^{x} \int_{x_1}^{x} \frac{dx}{\sqrt{{v_1}^2 - \frac{k}{m}\left[x^2 - {x_1}^2\right]}} = t!t$$

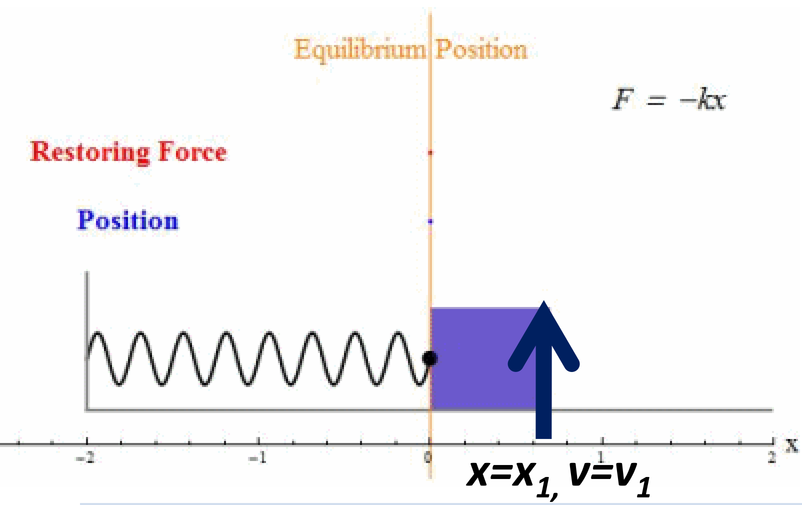
$$\sin^{-1}\left(\frac{\omega x}{A}\right) = \omega t + \sin^{-1}\left(\frac{\omega x_1}{A}\right)$$

$$\text{Hint} : x = \frac{A}{\omega}\sin(\theta)$$

$$A^2 = {v_1}^2 + \omega^2 {x_1}^2$$

$$x = \frac{A}{\omega}\sin\left(\omega t + \sin^{-1}\left(\frac{\omega x_1}{A}\right)\right)$$

Solving the equation of Simple Harmonic Motion



$$\frac{1}{2}mv^2 - \frac{1}{2}mv_0{}^2 = -k\int_{x_0}^{x} x\,dx$$

Case III

Choose Initial conditions: at *t=0, V=V₁ , x=x₁*

$$v^2 - v_1{}^2 = -\frac{k}{m}\left[x^2 - x_1{}^2\right]$$

$$v = \sqrt{v_1{}^2 - \frac{k}{m}\left[x^2 - x_1{}^2\right]}$$

Integrating again after separating variables

$$\int_{x_1}^{x} \cdot \int_{x_1}^{x} \frac{dx}{\sqrt{v_1{}^2 - \frac{k}{m}\left[x^2 - x_1{}^2\right]}} = t!t$$

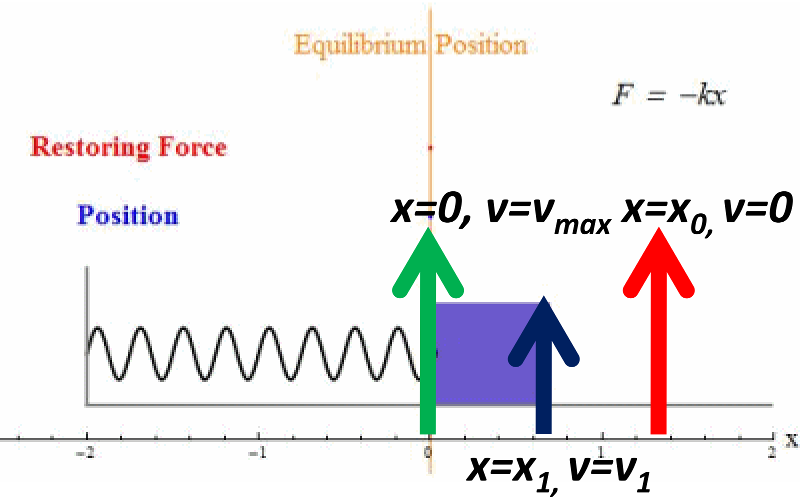
$$\sin^{-1}\left(\frac{\omega x}{A}\right) = \omega t + \sin^{-1}\left(\frac{\omega x_1}{A}\right)$$

$$\text{Hint}: x = \frac{A}{\omega}\sin(\theta)$$

$$A^2 = v_1{}^2 + \omega^2 x_1{}^2$$

$$x = B\sin\omega t + C\cos\omega t$$

Solving the equation of Simple Harmonic Motion



$$\frac{1}{2}mv^2 - \frac{1}{2}mv_0^{\ 2} = -k\int_{x_0}^{x} x\,dx$$

Case I

Choose Initial conditions: at $t=0, v=0, x=x_0$

$$x = x_0 \cos(\omega t)$$

Case II

Choose Initial conditions: at $t=0, V=V_{max}, x=0$

$$x = \frac{v_{max}}{\omega}\sin(\omega t)$$

Case III

Choose Initial conditions: at $t=0, V=V_1, x=x_1$

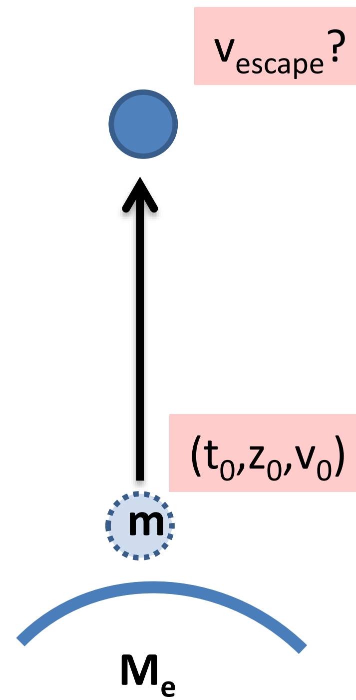
$$x = B\sin\omega t + C\cos\omega t$$

Vertical Motion in an inverse square field

What is the minimum value of velocity for escaping the earth and maximum altitude?

$$F = -\frac{GM_e m}{r^2}$$

Apply work energy theorem



$$\frac{1}{2}mv^2 - \frac{1}{2}mv_0{}^2 = -\int_{R_e}^{r} \frac{GM_e m}{r^2} dr$$

$$\frac{1}{2}mv^2 - \frac{1}{2}mv_0{}^2 = GM_e m\left(\frac{1}{r} - \frac{1}{R_e}\right)$$

At maximum height, v=0

$$v_0{}^2 = 2GM_e\left(\frac{1}{R_e} - \frac{1}{r_{max}}\right)$$

Vertical Motion in an inverse square field

$$v_0{}^2 = 2GM_e\left(\frac{1}{\mathrm{R_e}} - \frac{1}{r_{\max}}\right)$$

$$v_0{}^2 = 2g\,\mathrm{R_e}\left(1 - \frac{\mathrm{R_e}}{r_{\max}}\right)$$

Escape velocity is the initial velocity needed to move r_{max} to infinity

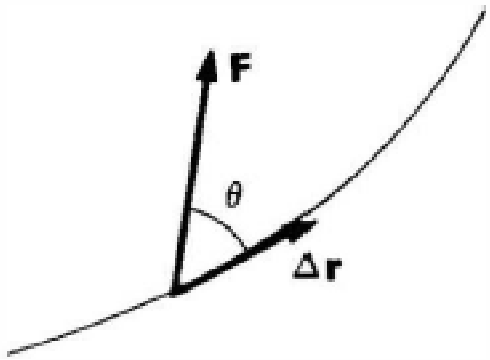
$$v_{esc} = \sqrt{2g\,\mathrm{R_e}}$$

Earth's radius R_e = 6.4x10^6 m $\Rightarrow$ v_{esc}=1.1x10^4 m/s

HW: Find the energy required to eject a 1000 kg spacecraft from the surface of the earth.

Work Energy Theorem in 3D

$$\vec{F} = m\frac{d\vec{v}}{dt}$$



$$\int_a^b \vec{F}\cdot d\vec{r} = \int_a^b m\frac{d\vec{v}}{dt}\cdot d\vec{r}$$

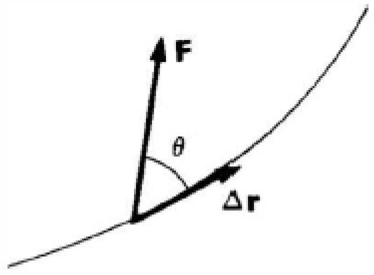
$$= \frac{1}{2}mv_b^2 - \frac{1}{2}mv_a^2$$

$$W_{ba} = K_b - K_a$$

W_{ba} : Work done on the particle by the total force

HW: $\frac{d\vec{v}}{dt}\cdot\vec{v} = \frac{1}{2}\frac{d}{dt}(v^2)$

Work Energy Theorem- Physical interpretation



$$\int_a^b \vec{F} \cdot d\vec{r} = \frac{1}{2} m\left[v_b^2 - v_a^2\right]$$

$$W_{ba} = K_b - K_a$$

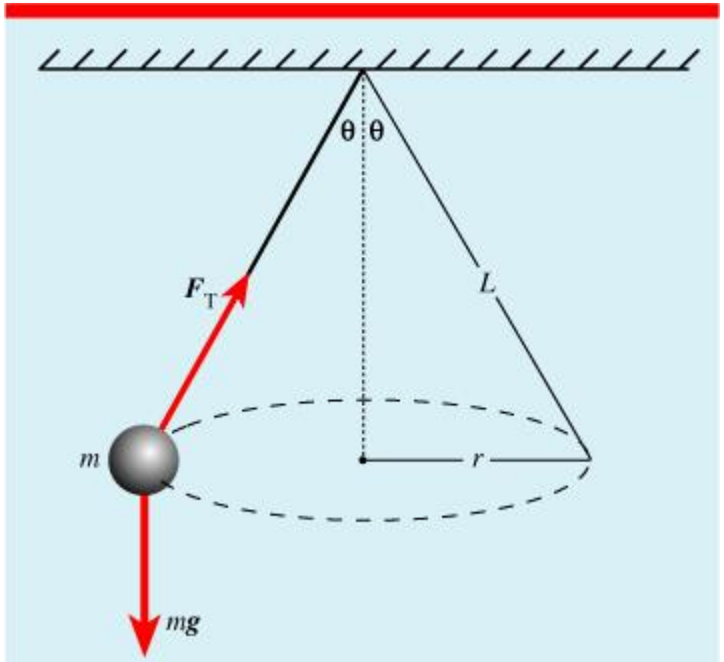
For infinitesimally small displacement $\overrightarrow{\Delta r}$,

$$\Delta W = \vec{F} \cdot \Delta\vec{r} = F cos\theta \Delta r = F_{\parallel} \Delta r$$

$F_{\perp}$ **does no work!!**

For a finite displacement, the work done on the particle is the sum of the contributions $\Delta W = F_{\parallel} \Delta r$ from each segment of the path, in the limit where the size of each segment approaches zero.

Conical Pendulum



Since the velocity is constant, the work energy theorem tells that no net work is being done on the mass.

String force and weight force separately do no work.

Work Energy Theorem- Physical interpretation

$$W_{ba} = K_b - K_a$$

$$\int_a^b \vec{F} \cdot d\vec{r} = \int_a^b m \frac{d\vec{v}}{dt} \cdot d\vec{r}$$

Work Energy Theorem- is a mathematical consequence of Newton's second law.

Evaluation of this integral depends on knowing what path the particle actually follows

Solution → Work-Energy Theorem

Constrained motion

Conservative forces

Conservative Forces

Definition:

The forces whose work integral does not depend on the particular path but only on the end points are called as Conservative Forces.

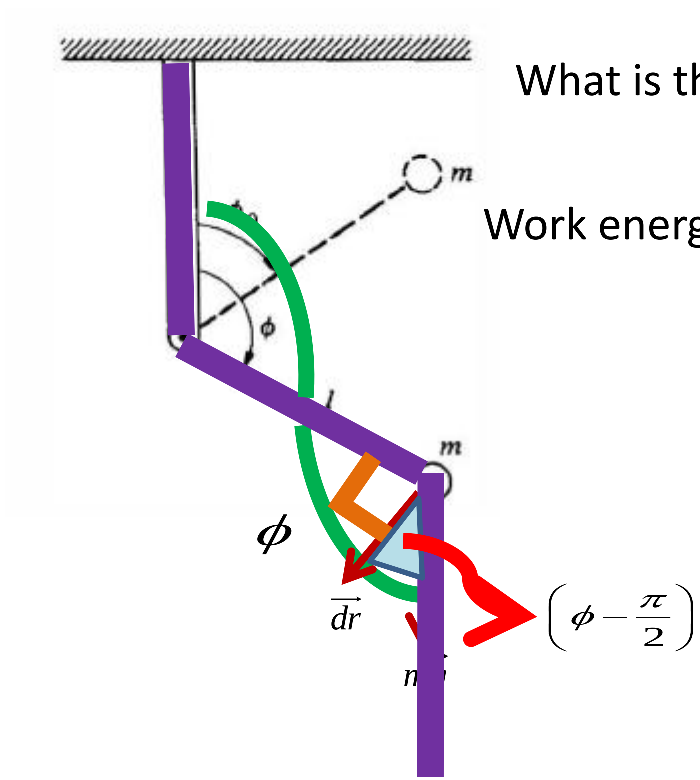
Examples are work done by a uniform force, central force etc

Constrained motion

The motion in which external constraints act to keep the particle on a predetermined trajectory. (The constraining force does no work)

Examples: Roller coasters, conical pendulums

Example: The inverted pendulum



What is the velocity of mass m when the rod is at angle ϕ?

Work energy theorem gives:

$$\int_a^b \vec{F}\cdot d\vec{r} = \frac{m}{2}\left[v_\varphi^2 - v_{\varphi_0}^2\right]$$

$$\int_a^b m\vec{g}\cdot d\vec{r} = \frac{m}{2}v_\varphi^2$$

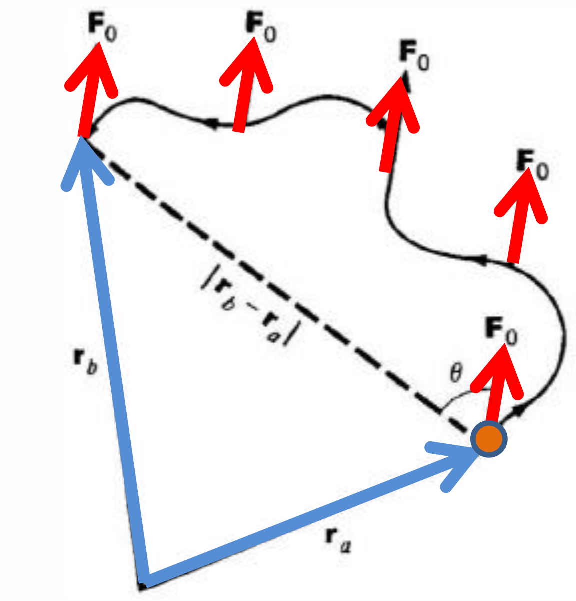
$$\int_a^b mgdr\cos\left(\phi-\frac{\pi}{2}\right) = \frac{m}{2}v_\phi{}^2$$

$$\int_{\phi_0}^{\phi} mgl\sin(\phi)d\phi = \frac{m}{2}v_\phi{}^2$$

$$v_\phi = \sqrt{2gl\left(\cos\phi_0 - \cos\phi\right)}$$

$$v_{\max} = 2\sqrt{gl}$$

Work done by a uniform Force



$$W_{ba} = \int_a^b \vec{F} \cdot d\vec{r}$$

$$W_{ba} = \int_a^b F_0 \hat{n} \cdot d\vec{r}$$

$$W_{ba} = F_0 \hat{n} \cdot \left[\hat{e}_x \int_{x_a}^{x_b} dx + \hat{e}_y \int_{y_a}^{y_b} dy + \hat{e}_z \int_{z_a}^{z_b} dz \right]$$

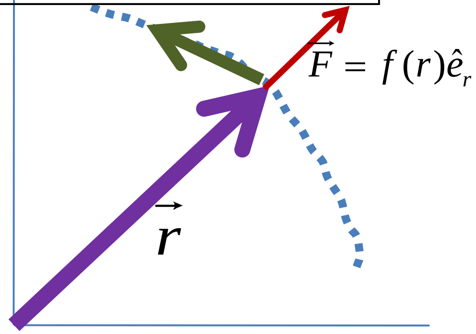
$$W_{ba} = F_0 \cos\theta \left| r_b - r_a \right|$$

For a constant force, work done only depends on the net displacement and not on the Path followed.

Work Done by Central Force $\vec{F} = f(r)\hat{e}_r$

Central force is a radial force which depends only on the distance from the origin

$$\vec{dr} = dr\hat{e}_r + rd\theta\hat{e}_\theta$$



Motion in a plane

$$W_{ba} = \int_a^b \vec{F} \cdot d\vec{r}$$

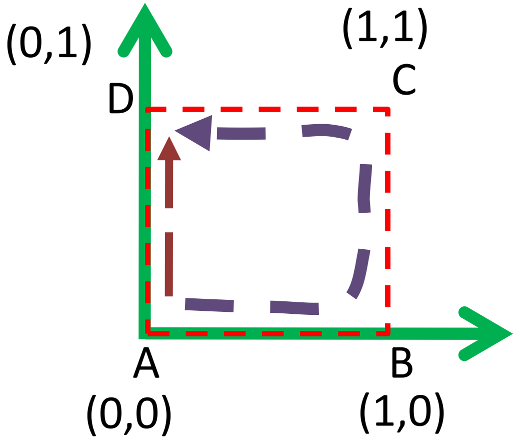
$$W_{ba} = \int_a^b f(r)\hat{e}_r \cdot [dr\hat{e}_r + rd\theta\hat{e}_\theta]$$

$$W_{ba} = \int_a^b f(r)dr$$

Note that work done only depends on initial and final radial distances, and not on the particular path.

Non-Conservative Force

- **The forces whose work is different for different paths between the initial and final points.**
- **Example is Friction force**
  - **Different paths will offer different friction**



HW: Evaluate path integral for

$$\vec{F} = A(xy\hat{i} + y^2\hat{j})$$

Path-1: ABCD
Path-2:AD

Concept of Potential Energy

For a conservative force

$$W_{ba} = \int_a^b \vec{F} \cdot d\vec{r} = fun(r_b) - fun(r_a) = \boxed{-V(r_b) + V(r_a)}$$

V(r) is known as potential energy function

$$W_{ba} = K_b - K_a$$

Position *'a'* and *'b'* are arbitrary, hence the relation is true at any point.

$$K_a + V_a = K_b + V_b = E$$

This proves that if Force is conservative, total energy E of the system is independent of position of particle.

Necessary conditions for Conservative Force

$$V(\vec{r}) - V(\vec{r}_0) = -\int_{r_0}^{r} \vec{F}(\vec{r}\,') \cdot d\vec{r}\,'$$

If V(r) is path independent,

$$dV(\vec{r}) = -\vec{F}(\vec{r}) \cdot d\vec{r}$$

$$dV(\vec{r}) = -\left[F_x dx + F_y dy + F_z dz\right]$$

Alternatively,

$$\frac{\partial V}{\partial x} dx + \frac{\partial V}{\partial y} dy + \frac{\partial V}{\partial z} dz = -\left[F_x dx + F_y dy + F_z dz\right]$$

$$\vec{F}\left(\vec{r}\right) = -\nabla V(\vec{r})$$

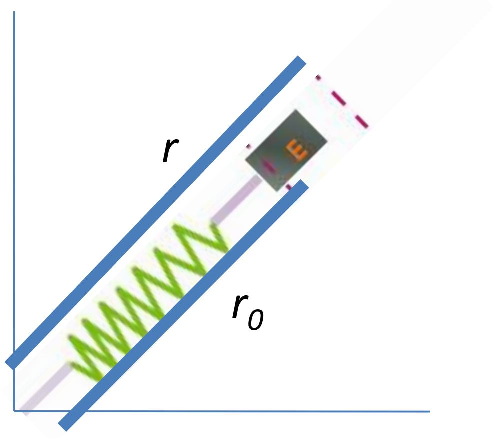
Curl of F is $\vec{\nabla} \times \vec{F}\left(\vec{r}\right) = -\left[\vec{\nabla} \times \nabla V(\vec{r})\right] = 0$

Curl of F is zero

Examples

- Find potential energy of Spring Force

$$\vec{F}\left(\vec{r}\right) = -k(r - r_0)\hat{e}_r$$



Since the central force is conservative,

$$V(\vec{r}) - V(\vec{r}_0) = -\int_{\vec{r}_0}^{\vec{r}} -k(r - r_0)dr$$

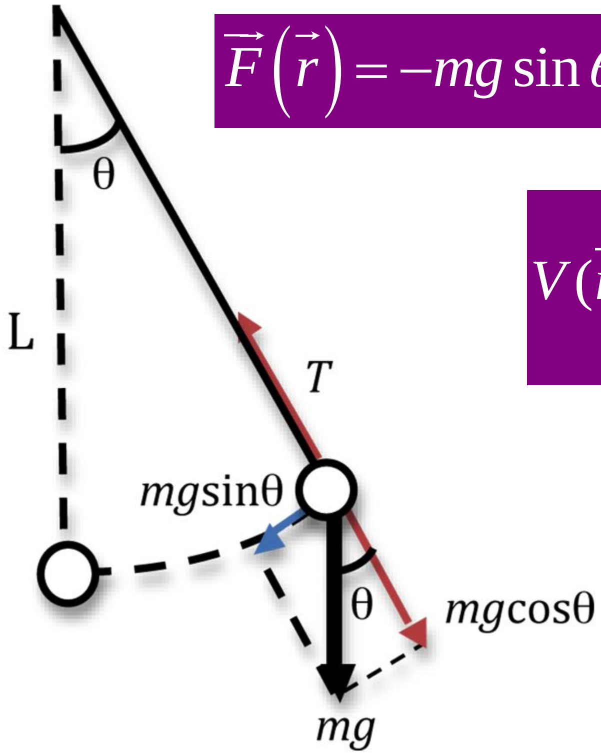
$$V(\vec{r}) - V(\vec{r}_0) = \frac{1}{2}k(r - r_0)^2$$

Choosing potential energy to be zero at equilibrium,

$$V(r) = \frac{1}{2}k(r - r_0)^2$$

Examples

- Find potential energy of Simple pendulum



$$\vec{F}\left(\vec{r}\right) = -mg\sin\theta\hat{e}_\theta$$

Since the central force is conservative,

$$V(\vec{r}) - V(\vec{r}_0) = -\int_{\vec{r}_0}^{\vec{r}} -mg\sin\theta\hat{e}_\theta.\left[dr\hat{e}_r + rd\theta\hat{e}_\theta\right]$$

$$V(\vec{r}) - V(\vec{r}_0) = \int_0^\theta mgl\sin\theta d\theta$$

$$V(\vec{r}) - V(\vec{r}_0) = mgl(1-\cos\theta)$$

Choosing potential energy to be zero at equilibrium,,

$$V(\vec{r}) = mgl(1-\cos\theta)$$

Examples

- Central force is conservative by showing that curl F=0

$$\vec{F} = f(r)\hat{e}_r$$

$$\vec{F} = f(r)\left[\frac{x}{r}\hat{e}_x + \frac{y}{r}\hat{e}_y + \frac{z}{r}\hat{e}_z\right]$$

$$\left[\vec{\nabla}\times\vec{F}\right]_z = \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} = \frac{\partial\left(\dfrac{yf(r)}{r}\right)}{\partial x} - \frac{\partial\left(\dfrac{xf(r)}{r}\right)}{\partial y} = 0$$

Likewise for the x and y components

Examples

- Central force is conservative by showing that curl F=0

$$\vec{F} = f(r)\hat{e}_r$$

$$\vec{F} = f(r)\left[\frac{x}{r}\hat{e}_x + \frac{y}{r}\hat{e}_y + \frac{z}{r}\hat{e}_z\right]$$

$$\left[\vec{\nabla}\times\vec{F}\right]_z = \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} = \frac{\partial\left(\frac{yf(r)}{r}\right)}{\partial x} - \frac{\partial\left(\frac{xf(r)}{r}\right)}{\partial y} = 0$$

$$\frac{\partial\left(\frac{yf(r)}{r}\right)}{\partial x} = \frac{y}{r}f'(r)\frac{\partial r}{\partial x} + yf(r)\frac{\partial\left(\frac{1}{r}\right)}{\partial x} = \frac{y}{r}f'(r)\frac{x}{r} - yf(r)\frac{1}{r^2}\frac{x}{r}$$