We already know two Analytic methods to find exact solution for I^{st} order ODEs

① For ODE $M\,dx + N\,dy = 0$ — (*) → In (*) $M \equiv M(x,y)$, $N \equiv N(x,y)$

when it is exact

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Its general solution is $\boxed{f(x,y) = C}$

and we know the algorithm to find f.

Separable and generalized separable cases are included in this case. See bottom of this page

② For ODE $y' = f(x,y)$ — (**)

when it is linear by writting this in standard form $y' + p(x)y = q(x)$

its solution algorithm contains two steps

step1 find I.F. $u(x) = e^{\int p(x)dx}$, Then find sol. $y(x)$ by

step2 $\boxed{y(x)\,u(x) = \int q(x)\,u(x)\,dx + C}$

Other methods can be divided in two main categories

③ → Solution of (*) by finding IF ← ~~the~~ is the topic of this lecture.

④ → Solution of (*)/(**) by substitutions that make the ODE in some solvable form.

Separable - $f(x)dx + g(y)dy = 0$

Generalized separable - $f(x)\,G(y)\,dx + g(y)\,F(x)\,dy = 0$

$$\text{I.F.} \equiv \frac{1}{F(x)\,G(y)}$$

Before actual start of the lecture, learn one good and faster trick to solve exact ODEs

$$M\,dx + N\,dy = 0$$

Its solution is

$$\boxed{\int M\,\partial x + \int N\,[\partial y] = C}$$

← This notation is not standard.

Meaning of notations

$\int M\,\partial x$ → integrate M partially w.r.t. x i.e. by treating y as constant

$\int N\,[\partial y]$ → integrate N partially w.r.t. y after ignoring the terms containing the variable x in N.

i.e. If $N = a(x) + b(y) + c(x,y) + d$

Then $\int N\,[\partial y] = \int (b(y) + d)\,\partial y$.

For example: Solve the following ODEs [All are exact]

ODE	solution
$(2x\cos y + 3x^2 y)\,dx + (x^3 - x^2\sin y - y)\,dy = 0$	$2\frac{x^2}{2}\cos y + 3\frac{x^3}{3}y - \frac{y^2}{2} = c$ $\Rightarrow \boxed{x^2\cos y + x^3 y - \frac{y^2}{2} = c}$
$(3x^2 + 4xy)\,dx + (2x^2 + 2y)\,dy = 0$	$x^3 + 2x^2 y + y^2 = c$
$y^2\,dx + 2xy\,dy = 0$	$xy^2 = c$

$\boxed{\text{Method \#3}}$ → Solution by finding IF

Suppose

$$M\,dx + N\,dy = 0 \quad \text{--- } (1)$$

is not exact, i.e. $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then

We try to find an IF for (1) ~~to~~ make (1) exact.

Suppose $u(x, y)$ is an IF for (1). Then

$$uM\,dx + uN\,dy = 0$$

is exact, i.e.

$$\frac{\partial}{\partial y}(uM) = \frac{\partial}{\partial x}(uN)$$

$$\Rightarrow \quad \frac{\partial u}{\partial y} M + u\frac{\partial M}{\partial y} = \frac{\partial u}{\partial x} N + u\frac{\partial N}{\partial x}$$

$$\Rightarrow \quad \boxed{M\frac{\partial u}{\partial y} - N\frac{\partial u}{\partial x} = \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)u} \quad \text{--- } (\#)$$

↳ See, solution of this PDE gives u. But it is not solvable in general.

The PDE is solvable in some cases

Case 1

Suppose u is a function of x alone, i.e. $u \equiv u(x)$

Then (#) becomes

$$-N\frac{du}{dx} = \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)u$$

$$\Rightarrow \quad \frac{du}{u} = \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)dx \quad \text{--- } (2)$$

The above ODE is in separable form if RHS

$\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)$ is a function of x alone. And if

$\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = f(x)$, then from (2), we obtain $\boxed{u(x) = e^{\int f(x)dx}}$.

Therefore, we have the following result:

$$\boxed{\text{Suppose } \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \text{ is a function of } x \text{ alone and if it is denoted by } f(x). \text{ Then } u(x) = e^{\int f(x)dx} \text{ is an IF for non-exact ODE } M\,dx + N\,dy = 0}$$

Similarly, we can obtain the following result by (#) If we assume u as a function of y alone.

$$\boxed{\text{Suppose } \frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) \text{ is a function of } y \text{ alone and If it is denoted by } f(y). \text{ Then } u(y) = e^{\int f(y)dy} \text{ is an IF for non-exact ODE } M\,dx + N\,dy = 0}$$

Pf: Suppose u is a function of y alone, i.e. $u = u(y)$

Then (#) becomes

$$M\frac{du}{dy} = \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)u \Rightarrow \frac{du}{u} = \frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) \quad (3)$$

The ODE (3) is in separable form If RHS is a function of y alone.

Hence the result is proved □.

Now, we prove the following result

$\boxed{\begin{array}{l}\text{Suppose } \dfrac{1}{Ny - Mx}\left[\dfrac{\partial M}{\partial y} - \dfrac{\partial N}{\partial x}\right] \text{ is a function} \\ \text{of } z = xy \text{ (product of } x \text{ and } y) \text{ and if it is} \\ \text{denoted by } f(z). \text{ Then } u(x,y) = u(z) = e^{\int f(z)dz} \\ \text{is an IF for non-exact ODE } Mdx + Ndy = 0\end{array}}$

<u>Pf:-</u> Suppose $u(x,y) = u(z)$; $z = xy$. Then

$$\frac{\partial u}{\partial y} = \frac{du}{dz} \cdot \frac{\partial z}{\partial y} = \frac{du}{dz} x$$

$$\frac{\partial u}{\partial x} = \frac{du}{dz} \cdot \frac{\partial z}{\partial x} = \frac{du}{dz} y.$$

By putting above values in #, we obtain

$$M \cdot x \frac{du}{dz} - Ny \frac{du}{dz} = \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) u$$

$$\Rightarrow \quad \frac{du}{u} = \frac{1}{Ny - Mx}\left[\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right] dz$$

Thus if $\dfrac{1}{Ny - Mx}\left(\dfrac{\partial M}{\partial y} - \dfrac{\partial N}{\partial x}\right) = f(z)$ then

$u = u(z) = e^{\int f(z)dz}$. Hence the result is proved.

Remember # is

$$M \frac{\partial u}{\partial y} - N \frac{\partial u}{\partial x} = \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) u \quad \text{— \#}$$

Similarly, we can prove the following result

If $\frac{1}{N-M}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)$ is a function of $z = x+y$ and if it is denoted by $f(z)$. Then $u(x,y) = u(z) = e^{\int f(z)dz}$ is an IF for non-exact ODE $Mdx + Ndy = 0$

Pf: $\boxed{\text{easy}}$ → Do yourself.

Summary of the lecture with more results

① If given ODE $M\,dx + N\,dy = 0$ is exact then its solution is

$$\int M\,\partial x + \int N\{\partial y\} = c$$

② If $\boxed{M\,dx + N\,dy = 0}$ is not exact, then try to find an IF as per the following results/cases

S.No.	Condition	IF $u \equiv u(x,y)$
①	$\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = f(x)$	$u = e^{\int f(x)\,dx}$
②	$\frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) = f(y)$	$u = e^{\int f(y)\,dy}$
③	$\frac{1}{N\cdot y - M\cdot x}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = f(z);\ z = x\cdot y$	$u = e^{\int f(z)\,dz}$
④	$\frac{1}{N - M}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = f(z);\ z = x + y$	$u = e^{\int f(z)\,dz}$
⑤	If M and N are polynomials in x and y.	Take $u = x^{\alpha} y^{\beta}$ and find suitable values of α and β.
⑥	$M_1(z)\,y\,dx + N_1(z)\,x\,dy = 0$ — (*); $z = xy$ If given ODE is of the form of (*) and $M\cdot x - N\cdot y \neq 0$ e.g. $(xy\sin(xy) + \cos(xy))\,y\,dx + (xy\sin(xy) - \cos(xy))\,x\,dy = 0$	$u = \frac{1}{Mx - Ny}$
⑦	If M and N are <u>homogeneous functions of same degree</u> and $Mx + N\cdot y \neq 0$	$u = \frac{1}{Mx + Ny}$
⑧	By Inspection	$M(tx, ty) = t^{\alpha} M(x,y)$ $N(tx, ty) = t^{\alpha} N(x,y)$ $M = x^2 + 3y^2$ $N = 2xy$

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Homogeneous functions of same degree

$$\boxed{\begin{aligned} M(tx, ty) &= t^{\alpha} M(x,y) \\ N(tx, ty) &= t^{\alpha} M(x,y) \end{aligned}}$$

put $t = \frac{1}{x}$

$$\Downarrow$$

$$M\left(1, \frac{y}{x}\right) = \frac{1}{x^{\alpha}} M(x,y)$$

$$N\left(1, \frac{y}{x}\right) = \frac{1}{x^{\alpha}} N(x,y)$$

$$\Leftrightarrow \quad \boxed{\begin{aligned} M(x,y) &\equiv x^{\alpha} M\left(1, \frac{y}{x}\right) \\ N(x,y) &\equiv x^{\alpha} N\left(1, \frac{y}{x}\right) \end{aligned}}$$

e.g. $\left.\begin{aligned} M(x,y) &= 3x^2 + 9xy + 5y^2 \\ N(x,y) &= 6x^2 + 4xy \end{aligned}\right) \Rightarrow \begin{aligned} M(tx,ty) &= t^2 M(x,y) \\ N(tx,ty) &= t^2 N(x,y) \end{aligned}$

$$\Downarrow$$

$$x^2 M\left(1, \frac{y}{x}\right) = x^2\left(3 + 9\frac{y}{x} + 5\frac{y^2}{x^2}\right) = \underbrace{3x^2 + 9xy + 5y^2}_{= M(x,y)}$$

$$x^2 N\left(1, \frac{y}{x}\right) = x^2\left(6 + 4\frac{y}{x}\right) = 6x^2 + 4xy = N(x,y)$$

Thus M and N are homogeneous functions of degree 2.

Check $\left.\begin{aligned} M(x,y) &= y + \sqrt{x^2 + y^2} \\ N(x,y) &= x \end{aligned}\right)$ are homogeneous of degree 1.

Examples

① Solve :- $(4x+3y^2)dx + 2xy\,dy = 0$ — (Ex1)

$$\left.\begin{array}{l} M \equiv 4x+3y^2 \\ N \equiv 2xy \end{array}\right) \Rightarrow \frac{\partial M}{\partial y} = 6y \; ; \; \frac{\partial N}{\partial x} = 2y \quad \text{(NOT exact ODE)}$$

$$\text{But } \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \frac{4y}{2xy} = \frac{2}{x} = f(x)$$

↳ (case ① holds)

Take IF $u = e^{\int \frac{2}{x}dx} = x^2$

By $u \times$ (Ex1), we obtain an exact ODE

$$(4x^3 + 3x^2y^2)dx + 2x^3y\,dy = 0$$

Its solution is $4\frac{x^4}{4} + 3\frac{x^3}{3}y^2 = C$

$$\Rightarrow \boxed{x^4 + x^3y^2 = c}$$

② Solve :- $\boxed{(3x^2-y^2)\,dy - 2xy\,dx = 0}$ — (Ex2)

$$\left.\begin{array}{l} M \equiv -2xy \\ N \equiv 3x^2-y^2 \end{array}\right) \Rightarrow \frac{\partial M}{\partial y} = -2x \quad \frac{\partial N}{\partial x} = 6x \quad \text{(Not exact ODE)}$$

$$\text{But } \frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) = \frac{1}{-2xy}8x = -\frac{4}{y} = f(y)$$

↳ (Case ② holds)

Take IF $u = e^{-\int \frac{4}{y}dy} = \frac{1}{y^4}$

By $u \times$ (Ex2), we obtain an exact ODE

$$\boxed{-\frac{2x}{y^3}dx + \left(\frac{3x^2}{y^4} - \frac{1}{y^2}\right)dy = 0}$$

solution is $-\frac{2}{y^3}\frac{x^2}{2} + \frac{1}{y} = c$

$$\Rightarrow \boxed{-\frac{x^2}{y^3} + \frac{1}{y} = c} \Rightarrow \boxed{\frac{y^2-x^2}{y^3} = c}$$

Note :- $y \equiv 0$ is a solution of red equation but it is not in the solution family for green.

Notice that the given ODE satisfies the condition of case (7) also.

ODE:- $(3x^2-y^2)\,dy - 2xy\,dx = 0$

$M \equiv -2xy$
$N \equiv 3x^2-y^2$

The degree of each term in M and N is 2.

$M = x^2 M(1, \frac{y}{x})$
$N = x^2 N(1, \frac{y}{x})$

and $Mx + Ny$
$= -2x^2y + 3x^2y - y^3 = x^2y - y^3 \neq 0$

Thus $u = \dfrac{1}{x^2y-y^3}$

Therefore, by u × ODE, we obtain an exact ODE

$$-\frac{2xy}{x^2y-y^3}\,dx + \frac{3x^2-y^2}{x^2y-y^3}\,dy = 0$$

$$\Rightarrow \boxed{\frac{2x}{y^2-x^2}\,dx + \frac{3x^2-y^2}{(x^2-y^2)y}\,dy = 0}$$

$$\Rightarrow \frac{2x}{y^2-x^2}\,dx + \frac{3x^2-3y^2+2y^2}{(x^2-y^2)y}\,dy = 0$$

$$\Rightarrow \frac{2x}{y^2-x^2}\,dx + \left(\frac{3}{y} + \frac{2y}{(x^2-y^2)}\right)dy = 0$$

$$\Rightarrow \frac{2x}{y^2-x^2}\,dx + \left(\frac{3}{y} + \frac{1}{x-y} - \frac{1}{x+y}\right)dy = 0$$

<u>Solution</u> $-\ln(y^2-x^2) + \ln y^3 = c$

$$\Rightarrow \boxed{\frac{y^3}{y^2-x^2} = c}$$

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Example

(3) Solve $\quad y\,dx + (x + 3x^3y^4)\,dy = 0$ —— (ODE 3)

$$\left.\begin{aligned} M &\equiv y \\ N &\equiv x + 3x^3y^4 \end{aligned}\right) \ni \quad \frac{\partial M}{\partial y} = 1 \; ; \quad \frac{\partial N}{\partial x} = 1 + 9x^2y^4 \quad \text{(ODE is not exact)}$$

But $\dfrac{1}{Ny - Mx}\left[\dfrac{\partial M}{\partial y} - \dfrac{\partial N}{\partial x}\right]$

$$= \frac{1}{\left(xy + 3x^3y^5 - xy\right)}\left[1 - 1 - 9x^2y^4\right] = -\frac{3}{xy}$$

$$= -\frac{3}{z} \; ; \quad z = xy$$

(case 3 holds)

Then IF: $u = e^{-\int \frac{3}{z} dz} = \dfrac{1}{z^3} = \dfrac{1}{x^3y^3}$

and by $\frac{1}{x^3y^3}$ ⊗ ODE, we obtain exact ODE

$$\frac{1}{x^3y^2}\,dx + \left(\frac{1}{x^2y^3} + 3y\right)dy = 0$$

and solution: $\boxed{-\dfrac{1}{2x^2y^2} + \dfrac{3y^2}{2} = C}$

Now, find an IF for (ODE 3) as suggested in case (5)

M & N are polynomials type in x & y.

Assume: $\quad u(x, y) = x^\alpha y^\beta$ then

$$x^\alpha y^{\beta+1}\,dx + \left(x^{\alpha+1}y^\beta + 3x^{\alpha+3}y^{\beta+4}\right)dy = 0 \quad \text{is exact}$$

i.e. $\quad \dfrac{\partial}{\partial y}\left(x^\alpha y^{\beta+1}\right) = \dfrac{\partial}{\partial x}\left(x^{\alpha+1}y^\beta + 3x^{\alpha+3}y^{\beta+4}\right)$

$$\Rightarrow \quad (\beta+1)\,x^\alpha y^\beta = (\alpha+1)\,x^\alpha y^\beta + 3(\alpha+3)\,x^{\alpha+2}y^{\beta+4}$$

Above equation is satisfied when $\left.\begin{aligned} \alpha + 3 &= 0 \\ \beta + 1 &= \alpha + 1 \end{aligned}\right) \Rightarrow \begin{aligned} \alpha &= -3 \\ \beta &= -3 \end{aligned}$

Thus $\quad \boxed{u = \dfrac{1}{x^3y^3}}$

Some time we get same IF by two different tricks. But this is not the situation always.

Example (4) To understand that all methods do not always give same IFs. Hence IF is <u>NOT</u> unique for an ODE.

$$\frac{dy}{dx} = \frac{y}{x} \quad \text{OR} \quad \boxed{y\,dx - x\,dy = 0} \text{ — (Ex 4)}$$

$$\left.\begin{array}{l} M \equiv y \\ N \equiv -x \end{array}\right) \Rightarrow \frac{\partial M}{\partial y} = 1 \, ; \quad \frac{\partial N}{\partial x} = -1 \quad \text{so equation is not exact}$$

Now let us find different-different IFs.

Case-1

$$\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)$$
$$= -\frac{1}{x}(1+2)$$
$$= -\frac{2}{x} = f(x)$$

So $u = e^{-\int \frac{2}{x} dx}$

$$\Rightarrow \boxed{u = \frac{1}{x^2}}$$

$\therefore$ $u \times$ ODE gives

$$\frac{y}{x^2}\,dx - \frac{1}{x}\,dy = 0$$

an exact ODE

Solution: $-\frac{y}{x} = c$

$$\Rightarrow \boxed{\frac{y}{x} = c}$$

Case-2

$$\frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)$$
$$= \frac{1}{y}(-1-1)$$
$$= -\frac{2}{y} = f(y)$$

So $u = e^{-\int \frac{2}{y} dy}$

$$\Rightarrow \boxed{u = \frac{1}{y^2}}$$

$\therefore$ $u \times$ ODE gives

$$\frac{1}{y}\,dx - \frac{x}{y^2}\,dy = 0$$

an exact ODE

Solution: $\boxed{\frac{x}{y} = c}$

Using generalized separable form

$$\boxed{u = \frac{1}{xy}}$$

$\therefore$ $u \times$ ODE

$$\Rightarrow \frac{1}{x}\,dx - \frac{1}{y}\,dy = 0$$

is separable and so exact.

Thus solution is

$$\ln x - \ln y = c$$
$$\Rightarrow \boxed{\frac{x}{y} = c}$$

By Inspection

$$\boxed{u = \frac{1}{x^2+y^2}}$$

As after multiplying this u, ODE becomes

$$\frac{y\,dx - x\,dy}{x^2+y^2} = 0$$

$$\Rightarrow d\left(\tan^{-1}\frac{y}{x}\right) = 0$$

Thus solution is

$$\tan^{-1}\frac{y}{x} = c$$
$$\Rightarrow \boxed{\frac{y}{x} = c}$$

Check

$$d\left(\tan^{-1}\frac{y}{x}\right)$$
$$\parallel$$
$$\frac{-y/x^2}{1+y^2/x^2}\,dx + \frac{1/x}{1+y^2/x^2}\,dy$$
$$= \frac{-y\,dx + x\,dy}{x^2+y^2}$$

We know that constant multiple of an IF is again an IF. But, in above example, we see that IFs may be totally different: $\frac{1}{x^2}, \frac{1}{y^2}, \frac{1}{xy}, \frac{1}{x^2+y^2}$