Notes by
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Lecture-3-4 Notes

21/09/20

We discuss the Maximum Modulus principle and its applications

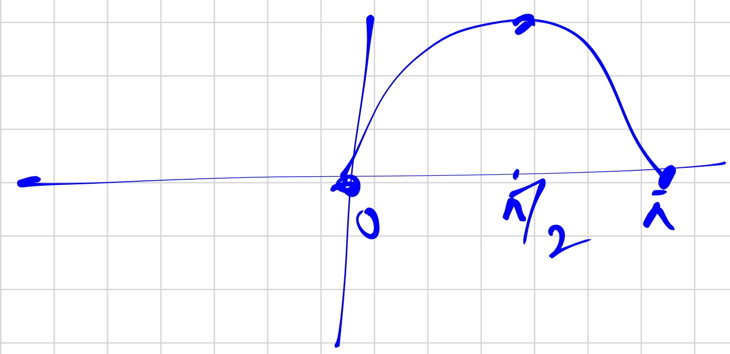
Ref: § 50 of Brown & Churchill

■ Consider $f(x) = \sin x$, $x \in [0, \pi]$ closed & bounded

: real valued function of real variable.

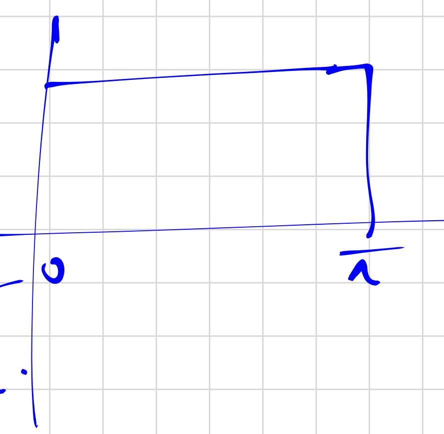
Then $\max_{x\in[0,\pi]} f(x) = f(\pi/2) = 1$

Note that $x = \pi/2$ is an interior pt. of domain



■ Consider $f(z) = \sin z$, $z \in R$: closed & bounded

Then $|f(z)| = \sqrt{\sin^2 x + \sinh^2 y}$



→ $\sin x$ achieves its max value at $x = \pi/2$

→ $\sinh^2 y$ achieves its max. value at $y = 1$.

Thus $\max_{z\in R} |f(z)| = f(\pi/2, 1)$

Note that $(\pi/2, 1)$ is not an interior pt. of R

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The above observation is not <u>a coincidence</u> as we discuss below.

Lemma (1) Suppose $|f(z)| \leq |f(z_0)|$ for all z in $B_\varepsilon(z_0)$ for some $\varepsilon > 0$ and $f(z)$ is analytic in $B_\varepsilon(z_0)$. Then

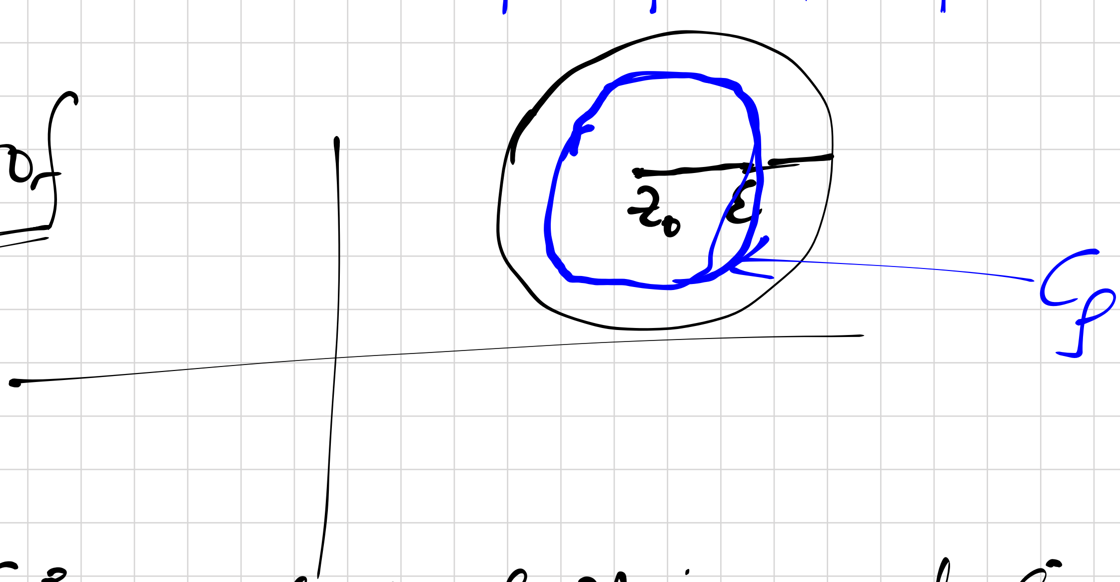
$$f(z) = f(z_0) \quad \forall \; z \in B_\varepsilon(z_0)$$

(2) Suppose $|f(z)| \leq |f(z_0)|$ for all z in some domain D with $z_0 \in D$ and $f(z)$ is analytic in D. Then

$$f(z) = f(z_0) \quad \forall \; z \in D$$

(We prove the 1st part of Lemma. We skip the proof of part 2.)

Proof



Given that $f(z)$ is analytic in the inside of $B_\varepsilon(z_0)$. Take a circle around z_0,

$$C_\rho : |z - z_0| = \rho \quad , \quad \text{with } 0 < \rho < \varepsilon$$

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Then C_ρ is inside of $B_\varepsilon(z_0)$.

So, $f(z)$ is analytic inside and on C_ρ.

Thus, by Cauchy's integral formula,

$$f(z_0) = \frac{1}{2\pi i}\int_{C_\rho} \frac{f(z)\,dz}{(z-z_0)}$$

Parametrization of C_ρ: $\quad z = z_0 + \rho e^{i\theta}$

$$0 \le \theta \le 2\pi$$

$$dz = \rho i e^{i\theta} d\theta.$$

So.
$$f(z_0) = \frac{1}{2\pi i}\int_0^{2\pi} \frac{f(z_0+\rho e^{i\theta})}{\rho e^{i\theta}} \rho i e^{i\theta} d\theta \qquad \text{—— (1)}$$

It is given that $|f(z)| \le |f(z_0)| \;\forall z \in B_\varepsilon(z_0)$

In particular $|f(z_0+\rho e^{i\theta})| \le |f(z_0)| \;\forall\, \theta$ —— (2)

Using (1) and (2),

$$|f(z_0)| \le \frac{1}{2\pi}\int_0^{2\pi} |f(z_0+\rho e^{i\theta})|\, d\theta$$

$$\le \frac{1}{2\pi}\int_0^{2\pi} |f(z_0)|\, d\theta = \frac{|f(z_0)|}{\cancel{2\pi}}\cdot\cancel{2\pi}$$

This is possible only if equality holds everywhere

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Solution (since the first and last numbers are same)

So, we get $|f(z_0)| = \frac{1}{2\pi}\int_0^{2\pi} |f(z_0+\rho e^{i\theta})|\,d\theta$

Now $L.H.S. = \frac{1}{2\pi}\int_0^{2\pi} |f(z_0)|\,d\theta = \text{''}$

$$\Rightarrow \frac{1}{2\pi}\int_0^{2\pi}\left(|f(z_0)| - |f(z_0+\rho e^{i\theta})|\right)d\theta = 0$$

Say $g(\theta) := |f(z_0)| - |f(z_0+\rho e^{i\theta})|$

Observe that (a) $g(\theta) \geq 0$ as $|f(z_0)| \geq |f(z)|$

(b) $g(\theta)$ is continuous on $[0,2\pi]$

Thus $\int_0^{2\pi} g(\theta) = 0$ iff $g(\theta) = 0 \ \forall \theta \in [0,2\pi]$

iff $|f(z_0)| = |f(z_0+\rho e^{i\theta})|$ $\forall \theta \in [0,2\pi]$

In other words,

$$|f(z_0)| = |f(z)| \quad \forall z \text{ lying on } C_\rho.$$

Note that in above discussion, ρ can be chosen arbitrarily with $0<\rho<\varepsilon$.
By choosing all the values of ρ, with $0<\rho<\varepsilon$, the circle C_ρ fill up the domain

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$B_\varepsilon(z_0)$.

So, we can conclude that $|f(z)| = |f(z_0)|$ $\forall z \in B_\varepsilon(z_0)$

Now it follows from the following exercise that $f(z) = f(z_0)$ $\forall z \in B_\varepsilon(z_0)$.

This completes the proof. ∎

Ex: If $f(z)$ is an analytic function in a domain D and $|f(z)|$ is constant $\forall z \in D$ Then $f(z)$ is constant $\forall z \in D$.

The next result is known as Maximum modulus principle.

Theorem: A nonconstant function $f(z)$ which is analytic in a domain D has no maximum absolute value in D

i.e. there is no point $z_0 \in D$ s.t.

$$|f(z)| \leq |f(z_0)| \quad \forall z$$

Proof: - Suppose the result is not true

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which means there is $z_0 \in D$ such that

$$|f(z)| \leq |f(z_0)| \quad \forall z \in D. \quad \text{Then}$$

using lemm, part 2, $f(z)$ is constant $\forall z \in D$.

which is a contradiction.

Thus we can conclude that the theorem is true. $\blacksquare$

The following corollary is immediate.

Corollary: Suppose that a function $f(z)$ is

1. continuous on a closed bounded region R
2. analytic in the interior of R (which means excluding boundary)
3. not constant in the interior of R.

and $M :=$ the maximum value of $|f(z)|$ is attained at a point z_0 i.e. $M = |f(z_0)|$

Then z_0 must lie on the boundary of R and not in the interior of R.

Ex: Under conditions (1) & (2), $|f(z)|$ has a maximum value, say M, which is acheived at some point $z_0 \in R$ i.e. $|f(z_0)| = M$

Same is true for minimum value of $|f(z)|$.

i.e.

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Remark :- (1) Maximum modulus principle does not hold for real valued funct

Eg : $f(x) = \sin x$ (already discussed)

(2) An analogue statement for minimum value of $f(z)$ can be also proved. It can be called "Minimum modulus principle" and will be discussed in Tutorial - 5.

Examples (1) Let $f(z) = 3z - 2i$

Find the maximum absolute value of $f(z)$ on $|z| \leq 3$.

Soln : (first note that the maximum value is acheived since the region $|z| \leq 3$ is closed and bounded).

By Maximum modulus principle,

$\max |f(z)|$ is acheived at a boundary point, z_0 with $|z_0| = 3$.

Now $f(z) = 3z - 2i = 3x + i(3y - 2)$

Soln

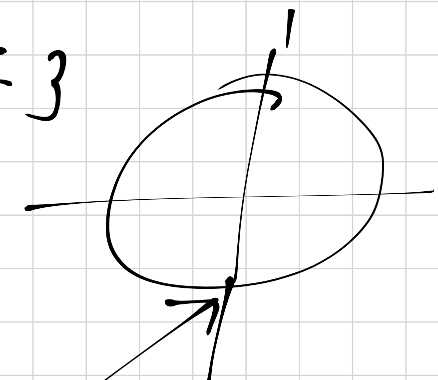
So, $|f(z)|^2 = 9x^2 + 9y^2 - 12y + 4$

$= 9|z|^2 - 12 \cdot \operatorname{Im} z + 4$.

We can put $|z| = 3$.

$|f(z)| = \sqrt{9 \cdot 9 - 12 \cdot \operatorname{Im} z + 4}$ is maximum when $\operatorname{Im} z$ is minimum.

minimum of $\operatorname{Im} z$ in $|z| \leq 3$ is at $z = -3i$



Thus $\max\limits_{z : |z| \leq 3} |f(z)| = |f(z_0 = -3i)|$

$= \sqrt{9 \cdot 9 - 12 \cdot (-3) + 4} = 11$

Eg: 2 $f(z) = \dfrac{z^2}{(z^3 - 10)}$

Find the absolute maximum attained in $\{ z \mid |z| \leq 2 \}$.

Soln: By Max'm modulus principle, $\max\limits_{|z| \leq 2} f(z)$ is attained at $|z| = 2$.

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Now $|f(z)| = \dfrac{|z|^2}{|z^3-10|}$ -

When $|z| = 2$, we can write $z = 2e^{i\theta}$. $0 \le \theta \le 2\pi$

so,

$|f(z)| = \dfrac{4}{|(2e^{i\theta})^3 - 10|}$ on $|z| = 2$

$$= \frac{4}{\sqrt{(8\cos 3\theta - 10)^2 + 8^2 \sin^2 3\theta}}.$$

$$= \frac{4}{\sqrt{64 + 100 - 160\cos 3\theta}}$$

is max^n when $\cos 3\theta$ is max'm which means $\cos 3\theta = 1$, $0 \le \theta \le 2\pi$

so $\max\limits_{|z| \le 2} |f(z)| = \dfrac{4}{\sqrt{64+100-160}} = \dfrac{4}{2} = 2$

at $z = 2e^{i\theta}$, with $\theta = 0, 2\pi/3, \ldots$

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