How to describe a rigid body?

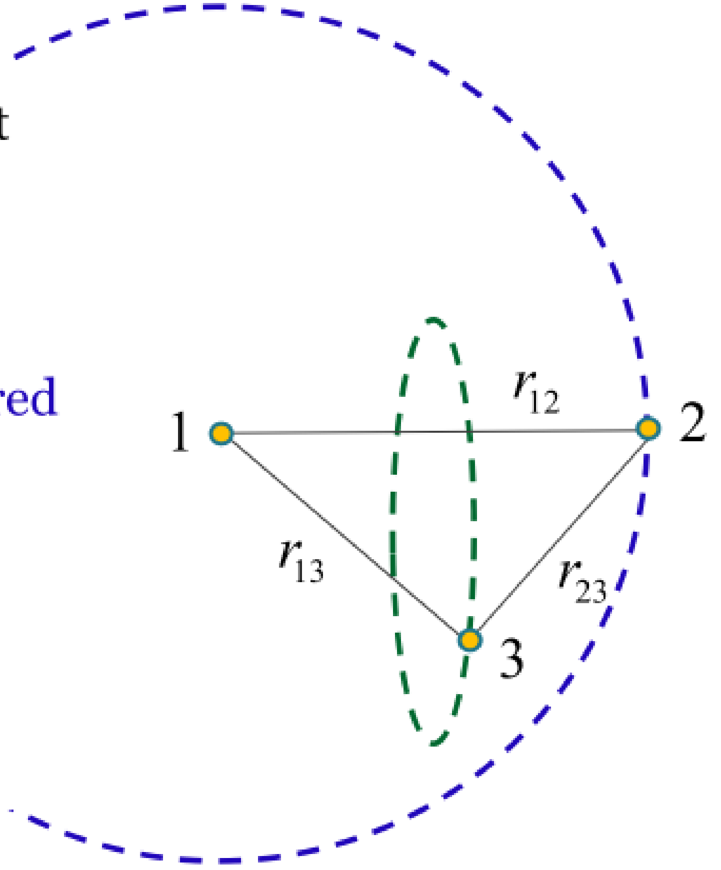
Again, NOT ALL 9 coordinates for the 3 ref pts are independent !

⟹ Let count…

- 3 coordinates needed to specify the 1st point
- 2 more coordinates to fix the 2nd point

  (Since point #2 can be anywhere on the surface of a *sphere* (fixed distance) centered on the 1st point.)
- 1 more coordinate to fix the 3rd point

  (Since point #3 must lie on a *circle* with fixed distances to the first two points.)

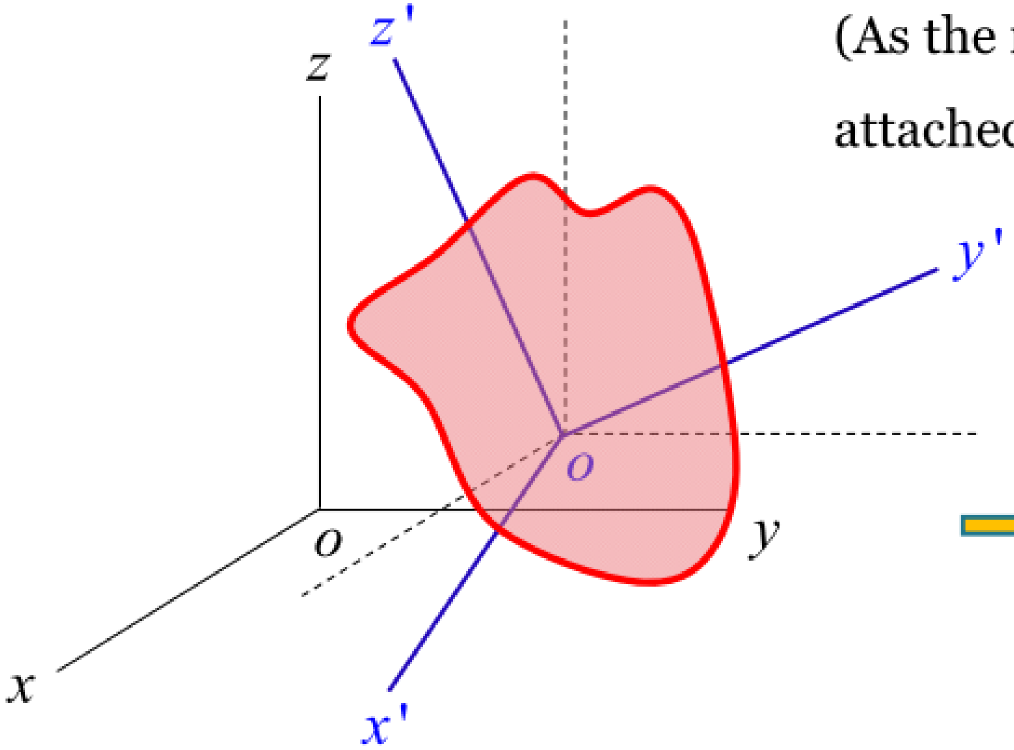


⟹ **Total 6 degrees of freedom**

Fixed and Body Axes for a Rigid Body

We will use two sets of coordinates:

- 1 set of external "fixed" coordinates (x, y, z) (unprimed)
- 1 set of internal "body" coordinates (x', y', z') (primed)



(As the name implied, the "body" axes are attached to the rigid body.)

⇒ 3 coords to specify the origin o of the "body" axes wrt the fixed system

⇒ 3 coords to specify the *orientation* of the "body" axes wrt to the translated "fixed" system (dotted axes).

Euler's and Chasles's Theorems

Useful general principle for the analysis of rigid bodies...

Euler's Theorem:

⇨ A general displacement of a rigid body with 1 pt fixed in space is equivalent to a rotation about some axis.

Chasles's Theorem:

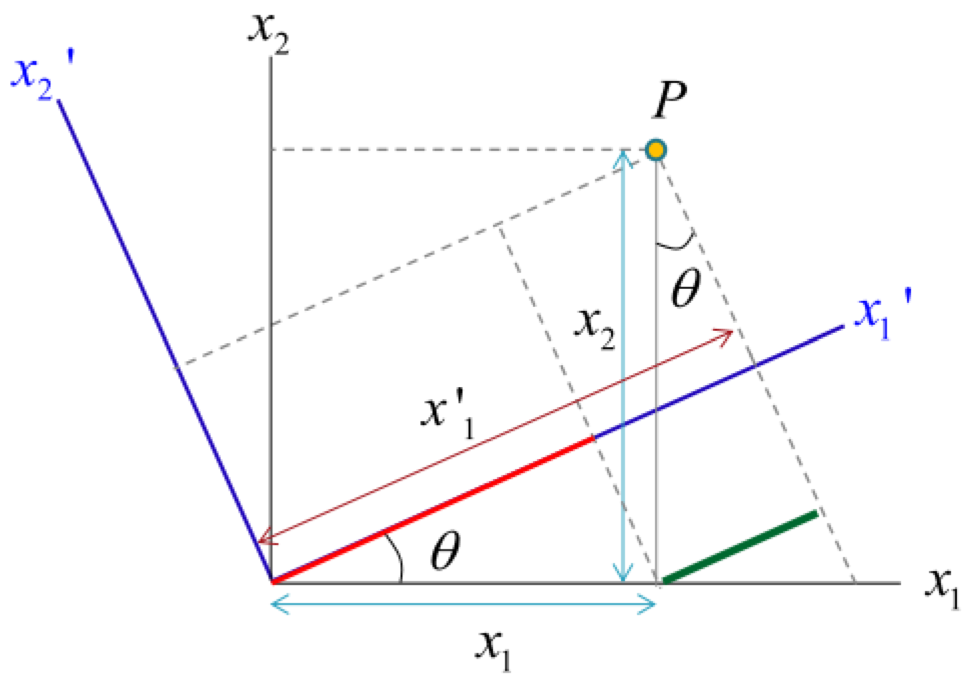
⇨ A general displacement = a translation + a rotation

Rotation

Relating x'_1 from (x_1, x_2) : green line $= x_2 \sin\theta$

red line $= x_1 \cos\theta$

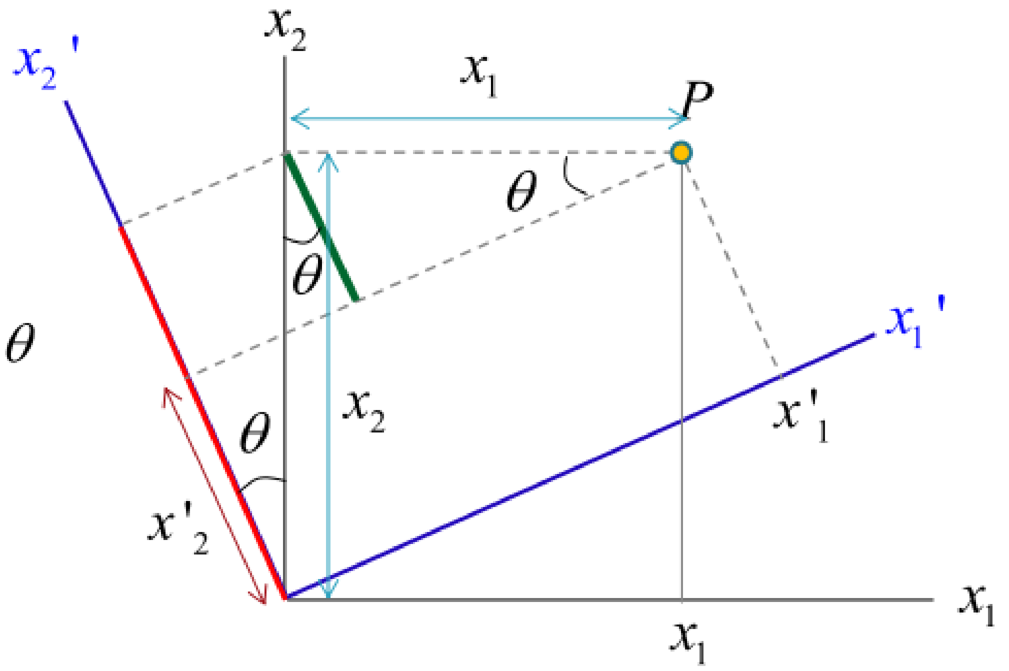
$$\Rightarrow \quad x'_1 = \text{red} + \text{green} = x_1 \cos\theta + x_2 \sin\theta$$



Similarly, relating x'_2 from (x_1, x_2) : green line $= x_1 \sin\theta$

red line $= x_2 \cos\theta$

$$\Rightarrow \quad x'_2 = \text{red} - \text{green} = x_2 \cos\theta - x_1 \sin\theta = -x_1 \sin\theta + x_2 \cos\theta$$

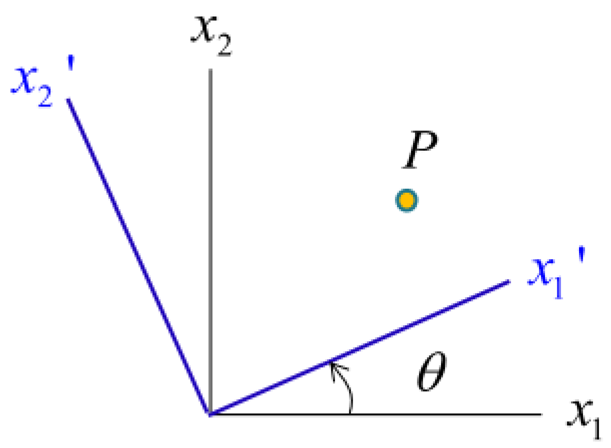


Rotation Matrix

These two transformation equations can be put in a compact matrix form:

$$\begin{pmatrix} x'_1 \\ x'_2 \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \mathbf{R} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

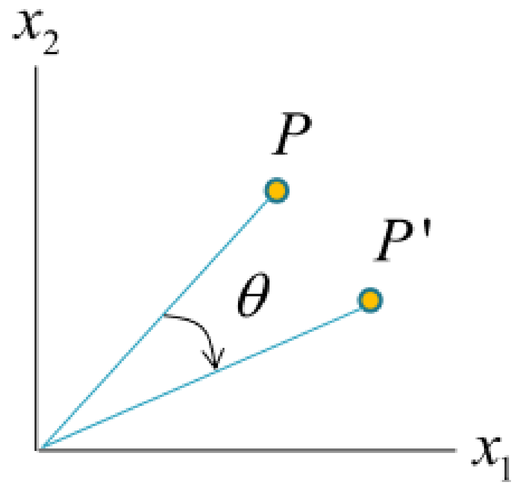
R is called a rotation matrix and it takes P from the unprimed frame to the prime frame.



P is "fixed" and the prime frame rotates counter-clockwise

→ This is the "passive" point of view.
(convention: + → counterclockwise)

point P (or vector) and rotating it by θ in the clockwise direction in the same frame.



→ This is the "active" point of view.
(convention: + → clockwise)

Rotation matrix for 3D

- In 3 dimensions, it is particular simple to get the matrix for a rotation about one of the coordinate axes:

→ Just put a “1” in the diagonal corresponding to that axis and “squeeze” the 2D rotation matrix into the rest of the entries .

i.e., to rotate about the x_3 axis:

$$\begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

i.e., to rotate about the x_2 axis:

$$\begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix}$$

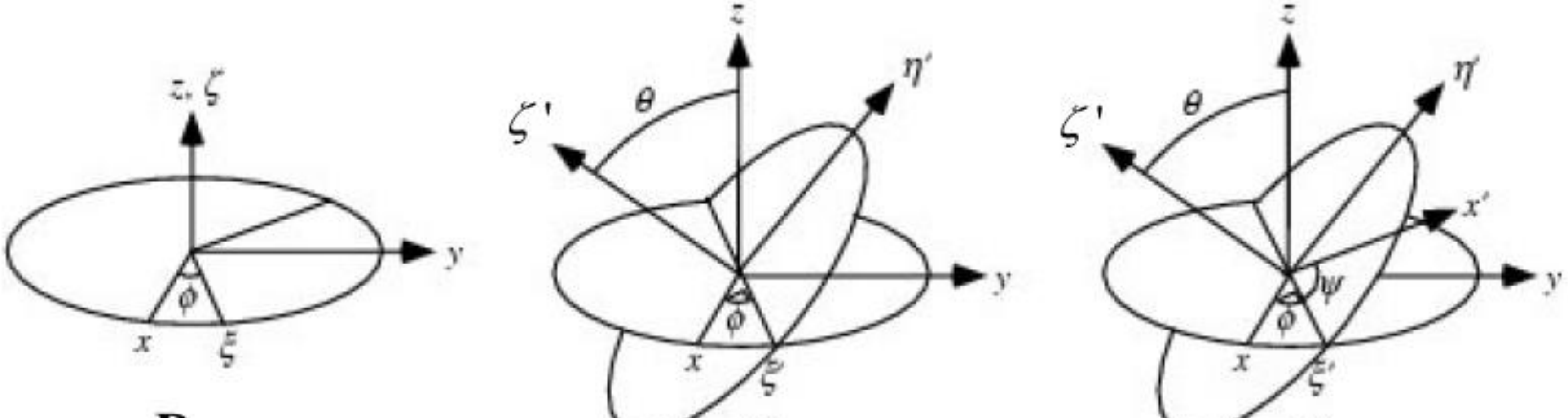
Euler's Angle

- However, a general rotation about an *arbitrary* axis is not quite so simple. In general all entries will be nonzero !

- But, one can build up a general rotation as separate successive rotations.

- There are many conventions... We will choose one called the

  Euler's Angles ϕ, θ, ψ consisting of a particular sequence of 3 rotations (**D**, **C**, **B**) along three principle axes:



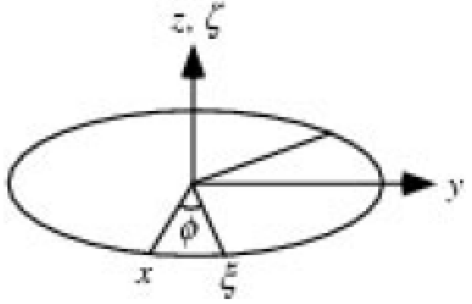
D **C** **B**

More on Euler's Angle

Start with the "fixed" axes (x, y, z)

→ The first rotation **D** is about z by an angle ϕ

$(space\ axes)(x, y, z) \rightarrow (\xi, \eta, \zeta)$

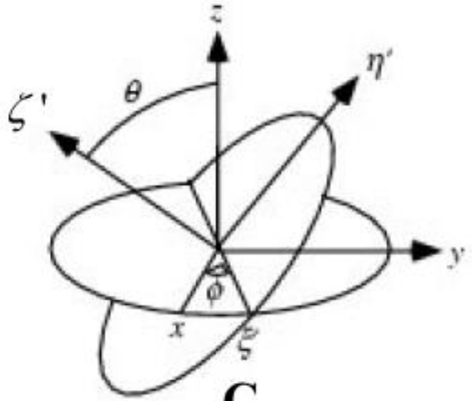


D

$$\mathbf{D} = \begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

→ The second rotation **C** is about the new ξ (new x-axis) by an angle θ :

$$(\xi, \eta, \zeta) \rightarrow (\xi', \eta', \zeta')$$

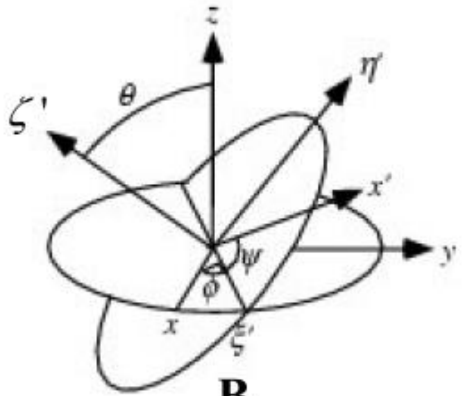


C

$$\mathbf{C} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{pmatrix}$$

→ The third rotation **B** is about the new ζ' (new z-axis) by an angle ψ :

$$(\xi', \eta', \zeta') \rightarrow (x', y', z')\ (body\ axes)$$



B

$$\mathbf{B} = \begin{pmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

More on Euler's Angle

- Then the general rotation from the "fixed" axes (x, y, z) to the "body" axes (x', y', z') is given by the product (note the specific *order*):

$$\mathbf{A}(\phi,\theta,\psi) = \mathbf{B}(\psi)\mathbf{C}(\theta)\mathbf{D}(\phi) \qquad \text{(note the order of operations !)}$$

(The sequence of rotated angles (ϕ,θ,ψ) are called the Euler's angles.)

$$\mathbf{A} = \begin{pmatrix} \cos\psi\cos\phi - \cos\theta\sin\phi\sin\psi & \cos\psi\sin\phi + \cos\theta\cos\phi\sin\psi & \sin\theta\sin\psi \\ -\sin\psi\cos\phi - \cos\theta\sin\phi\cos\psi & -\sin\psi\sin\phi + \cos\theta\cos\phi\cos\psi & \sin\theta\cos\psi \\ \sin\theta\sin\phi & -\sin\theta\cos\phi & \cos\theta \end{pmatrix}$$

$$\mathbf{x}'(body) = \mathbf{A}\mathbf{x}(fixed)$$

Infinitesimal Rotations

In matrix notations, we have, $\mathbf{x}' = (\mathbf{I} + \boldsymbol{\varepsilon})\mathbf{x}$

$\boldsymbol{\varepsilon}$ is assumed to be small

- Now, let consider an *inverse* of an infinitesimal transformation, $\mathbf{A} = \mathbf{I} + \boldsymbol{\varepsilon}$

It is given simply by: $\mathbf{A}^{-1} = \mathbf{I} - \boldsymbol{\varepsilon}$

CHECK:

$$\mathbf{A}\mathbf{A}^{-1} = (\mathbf{I} + \boldsymbol{\varepsilon})(\mathbf{I} - \boldsymbol{\varepsilon}) = \mathbf{I} \quad \text{(to first order)}$$

- Then, let see what property will $\boldsymbol{\varepsilon}$ have if we want this infinitesimal transformation to be *orthogonal*.

- For *orthogonal* transformations, we need to have $\mathbf{A}^T = \mathbf{A}^{-1}$

- This then gives, $\mathbf{A}^T = \mathbf{I} + \boldsymbol{\varepsilon}^T = \mathbf{A}^{-1} = \mathbf{I} - \boldsymbol{\varepsilon} \Rightarrow \boldsymbol{\varepsilon}^T = -\boldsymbol{\varepsilon}$

$\Rightarrow$ $\boldsymbol{\varepsilon}$ is an anti-symmetric matrix

More on Infinitesimal Rotations

- Further note that,

$$\det \mathbf{A} = \det\left(\mathbf{I} + \boldsymbol{\varepsilon}\right) = +1 \quad \text{(to first order)}$$

An infinitesimal *orthogonal* transformation corresponds to a proper rotation.

- Since $\boldsymbol{\varepsilon}$ is anti-symmetric, we can write it generally in component form as,

$$\boldsymbol{\varepsilon} = \begin{bmatrix} 0 & d\Omega_3 & -d\Omega_2 \\ -d\Omega_3 & 0 & d\Omega_1 \\ d\Omega_2 & -d\Omega_1 & 0 \end{bmatrix}$$

(Clearly, the three quantities $\left(d\Omega_1, d\Omega_2, d\Omega_3\right)$ can be identified as three independent parameters specifying the infinitesimal rotation.)

Euler's Equations

We have seen how to describe the kinematic properties of a rigid body. Now, we would like to get equations of motion for it.

1. We will follow the Lagrangian Formalism that we have developed.

2. For generalized coordinates, we will use the Euler's angles with one point of the rigid body being **fixed** (no translation, just rotation)

3. As we have seen previously, the rotational kinetic energy is given by

$$T = \frac{1}{2}\boldsymbol{\omega}\cdot\mathbf{I}\cdot\boldsymbol{\omega} = \frac{1}{2}\omega_i I_{ij}\omega_j$$

4. Choose the body axes to coincide with the Principal axes, then

$$T = \frac{1}{2} I_{ii}\omega_i^2 \qquad (\text{ no sum; } \; I_{ij} \text{ is diagonized! })$$

Euler's Equations

Note:

-We still have the freedom to align $\hat{x}_3\,(\hat{\mathbf{z}})$(from the body axes) to any one of the 3 Principal axes.

- The three Euler's angles (ϕ, θ, ψ) give the orientation of the Principal axes of the body relative to the fixed axes.

5. A general rotation (an inf. one here) $d\boldsymbol{\Omega}$ along a given axis in the body frame can be decomposed into three rotations along the Euler's angles. Then, the time rate of change of this rotation $\boldsymbol{\omega} = d\boldsymbol{\Omega}/dt$ can also be written as,

$$\boldsymbol{\omega} = \boldsymbol{\omega}_\phi + \boldsymbol{\omega}_\theta + \boldsymbol{\omega}_\psi$$

(we write this as a sum since the angular changes are infinitesimal)

- These three different pieces correspond to time rate of change of the individual rotations along each of the three Euler's angles.

Euler's Equations

$$\mathbf{D} = \begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\mathbf{C} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{pmatrix}$$

$$\mathbf{B} = \begin{pmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

-Now, our task is to express each of these in terms of the body coordinate (x_1, x_2, x_3) We will go through the three individual Euler steps now:

a) $\boldsymbol{\omega}_\phi$: We are in the fixed axes and we do a rotation along the $x_3(\hat{\mathbf{z}})$

→ In the fixed axes, we have $(\boldsymbol{\omega}_\phi)_{fixed} = \begin{pmatrix} 0 \\ 0 \\ \dot\phi \end{pmatrix}$

→ To express it in the body axes, we apply the Euler rotations **BCD**

$$(\boldsymbol{\omega}_\phi)_{body} = \mathbf{BCD}\begin{pmatrix} 0 \\ 0 \\ \dot\phi \end{pmatrix} = \begin{pmatrix} \dot\phi\sin\psi\sin\theta \\ \dot\phi\cos\psi\sin\theta \\ \dot\phi\cos\theta \end{pmatrix}$$

Note: Since $(0, 0, \dot\phi)^T$ is already in the $\hat{\mathbf{z}}$ direction, $\mathbf{D}\begin{pmatrix} 0 \\ 0 \\ \dot\phi \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \dot\phi \end{pmatrix}$

Euler's Equations

$$\mathbf{C} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{pmatrix}$$

b) $\boldsymbol{\omega}_\theta$: This is the (2nd) rotation along the "line of nodes" (x_1 $(\hat{\mathbf{x}})$ in the intermediate (ξ, η, ζ) coordinate system)

$$\mathbf{B} = \begin{pmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

→ In the intermediate axes, we have $(\boldsymbol{\omega}_\theta)_\xi = \begin{pmatrix} \dot{\theta} \\ 0 \\ 0 \end{pmatrix}$

→ To express it in the body axes, we apply the Euler rotations **BC**

$$(\boldsymbol{\omega}_\theta)_{body} = \mathbf{BC}\begin{pmatrix} \dot{\theta} \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \dot{\theta}\cos\psi \\ -\dot{\theta}\sin\psi \\ 0 \end{pmatrix}$$

Note: Since $(\dot{\theta}, 0, 0)^T$ is already in the $\hat{\mathbf{x}}$ direction, $\mathbf{C}\begin{pmatrix} \dot{\theta} \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \dot{\theta} \\ 0 \\ 0 \end{pmatrix}$

Euler's Equations

$$\mathbf{B} = \begin{pmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

c) $\boldsymbol{\omega}_\psi$: Finally, the last rotation is along the $x_3\left(\hat{\mathbf{z}}\right)$ of the $\left(\xi', \eta', \zeta'\right)$

→ In the $\left(\xi', \eta', \zeta'\right)$ axes, we have $\left(\boldsymbol{\omega}_\psi\right)_{\zeta'} = \begin{pmatrix} 0 \\ 0 \\ \dot{\psi} \end{pmatrix}$

→ To express it in the body axes, we apply the Euler rotations **B**

$$\left(\boldsymbol{\omega}_\psi\right)_{body} = \mathbf{B}\begin{pmatrix} 0 \\ 0 \\ \dot{\psi} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \dot{\psi} \end{pmatrix}$$

Note: Since $\left(0, 0, \dot{\psi}\right)^T$ is already in the $\hat{\mathbf{z}}$ direction, $\mathbf{B}\begin{pmatrix} 0 \\ 0 \\ \dot{\psi} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \dot{\psi} \end{pmatrix}$

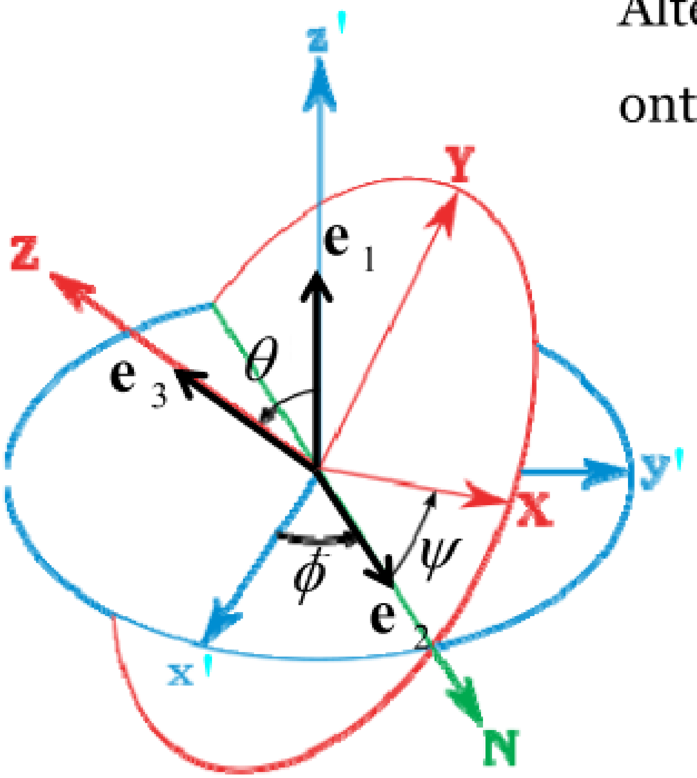
Euler's Equations

→ Putting all three pieces together, we have

$$\boldsymbol{\omega} = \boldsymbol{\omega}_\phi + \boldsymbol{\omega}_\theta + \boldsymbol{\omega}_\psi = \begin{pmatrix} \dot\phi \sin\psi \sin\theta + \dot\theta \cos\psi \\ \dot\phi \cos\psi \sin\theta - \dot\theta \sin\psi \\ \dot\phi \cos\theta + \dot\psi \end{pmatrix}$$

These are the correct components of $\boldsymbol{\omega}$ expressed in the "body" frame using the Euler's angles.

Euler's Equations



Alternatively, one can geometrically project $\boldsymbol{\omega}$ onto the "body" axes (red frame)

$$\boldsymbol{\omega} = \dot{\phi}\,\mathbf{e}_1 + \dot{\theta}\,\mathbf{e}_2 + \dot{\psi}\,\mathbf{e}_3$$

$$\Downarrow$$

$$\boldsymbol{\omega} = \omega_x \hat{\mathbf{x}} + \omega_y \hat{\mathbf{y}} + \omega_z \hat{\mathbf{z}}$$

Note: the basis set $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ defining an infinitesimal rotation along the Euler angles is NOT an orthogonal set of vectors.

(Note: Here and onward, space frame is primed and body frame is unprimed.)

Derivation of Euler's Equations

Now, we will continue with our equation of motion for a rotating rigid body.

$$T = \frac{1}{2} I_i \omega_i^2$$

I is diagonalized since we've chosen the body axes to lay along the principal axes and we will call the nonzero diagonal elements, $I_{ii} = I_i$

Without further assuming the nature of the applied forces acting on this system, we will use the following general form of the E-L equation:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} = Q_i$$

Q_i is the generalized force (including forces derivable from conservative and non-conservative sources)

Derivation of Euler's Equations

Let calculate the equation of motion explicitly for ψ :

$$\frac{\partial T}{\partial \dot{\psi}} = \frac{\partial T}{\partial \omega_i}\frac{\partial \omega_i}{\partial \dot{\psi}} = \left(I_i\omega_i\right)\frac{\partial \omega_i}{\partial \dot{\psi}} \quad \text{(E's sum)}$$

$$= \left(I_1\omega_1\right)\frac{\partial \omega_1}{\partial \dot{\psi}} + \left(I_2\omega_2\right)\frac{\partial \omega_2}{\partial \dot{\psi}} + \left(I_3\omega_3\right)\frac{\partial \omega_3}{\partial \dot{\psi}}$$

$$= \left(I_1\omega_1\right)(0) + \left(I_2\omega_2\right)(0) + \left(I_3\omega_3\right)(1)$$

$$= I_3\omega_3$$

$$T = \frac{1}{2}I_i\omega_i^2$$

$$\boldsymbol{\omega} = \begin{pmatrix} \dot{\phi}\sin\psi\sin\theta + \dot{\theta}\cos\psi \\ \dot{\phi}\cos\psi\sin\theta - \dot{\theta}\sin\psi \\ \dot{\phi}\cos\theta + \dot{\psi} \end{pmatrix}$$

$$\Rightarrow \quad \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\psi}}\right) = I_3\dot{\omega}_3$$

Derivation of Euler's Equations

Now, we need

$$\frac{\partial T}{\partial \psi} = \frac{\partial T}{\partial \omega_i}\frac{\partial \omega_i}{\partial \psi} = I_1\omega_1\frac{\partial \omega_1}{\partial \psi} + I_2\omega_2\frac{\partial \omega_2}{\partial \psi} + I_3\omega_3\frac{\partial \omega_3}{\partial \psi}$$

Note that:

$$\frac{\partial \omega_1}{\partial \psi} = \dot{\phi}\cos\psi\sin\theta - \dot{\theta}\sin\psi = \omega_2 \qquad \frac{\partial \omega_3}{\partial \psi} = 0$$

$$\frac{\partial \omega_2}{\partial \psi} = -\dot{\phi}\sin\psi\sin\theta - \dot{\theta}\cos\psi = -\omega_1$$

Thus, we have,

$$\frac{\partial T}{\partial \psi} = I_1\omega_1\left(\omega_2\right) + I_2\omega_2\left(-\omega_1\right) + 0$$

Derivation of Euler's Equations

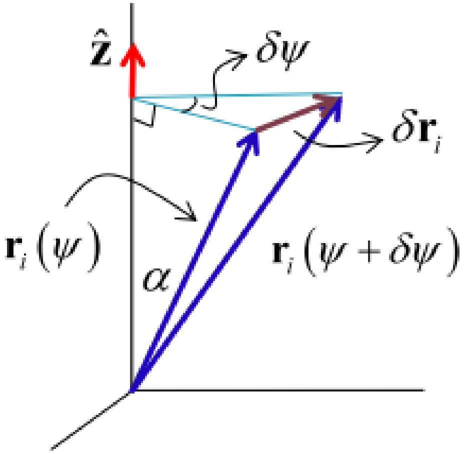
Now, we need to calculate the generalized force with respect to ψ :

Since the Euler angle ψ is associated with a rotation about the $\hat{\mathbf{z}}$ axis in the "body" frame, we have,

$$Q_\psi \equiv \sum_i \mathbf{F}_i \cdot \frac{\partial \mathbf{r}_i}{\partial \psi} = \sum_i \mathbf{F}_i \cdot \left(\hat{\mathbf{z}} \times \mathbf{r}_i\right)$$

$$= \hat{\mathbf{z}} \cdot \mathbf{N} = N_3$$

used
$\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = \mathbf{B} \cdot (\mathbf{C} \times \mathbf{A})$



$$|\delta \mathbf{r}_i| = |\mathbf{r}_i| \sin\alpha \, \delta\psi$$

$$\frac{\partial \mathbf{r}_i}{\partial \psi} = \hat{\mathbf{z}} \times \mathbf{r}_i$$

Derivation of Euler's Equations

Finally, putting everything together, the E-L equation gives,

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\psi}}\right) - \frac{\partial T}{\partial \psi} = Q_\psi$$

$$\Rightarrow \quad I_3\dot{\omega}_3 - I_1\omega_1\omega_2 + I_2\omega_1\omega_2 = N_3 \quad \Rightarrow \quad \boxed{I_3\dot{\omega}_3 - \left(I_1 - I_2\right)\omega_1\omega_2 = N_3}$$

- One can calculate the E-L equation for θ, ϕ but (they are ugly) we are not doing them here !

- There is a smarter way to get EOM for the other two dofs...

→ Since nothing required our choice of ω_3 to lay along $\hat{\mathbf{z}}(x_3)$. Then, by a symmetry argument, the other components of $\dot{\boldsymbol{\omega}}$ should have a SIMILAR form.

Derivation of Euler's Equations

This then gives,

$$\boxed{\begin{aligned} I_1\dot{\omega}_1 - (I_2 - I_3)\,\omega_2\omega_3 &= N_1 \\ I_2\dot{\omega}_2 - (I_3 - I_1)\,\omega_3\omega_1 &= N_2 \\ I_3\dot{\omega}_3 - (I_1 - I_2)\,\omega_1\omega_2 &= N_3 \end{aligned}}$$

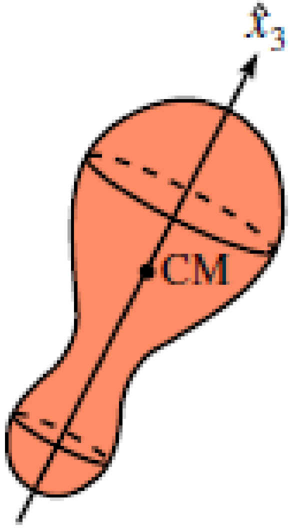
(same cyclic symmetry as the equation for ω_3)

In principle, one can get out the $\dot{\omega}_2$ *and* $\dot{\omega}_3$ equation by solving for $\dot{\omega}_2$ *and* $\dot{\omega}_3$ simultaneously from the θ, ϕ Euler-Lagrange equations.

These are called the Euler's Equations and the motion is described in terms of the Principal Moments !

Torque Free Motion of a Symmetric Top

A symmetric top means that: $I_1 = I_2 \neq I_3$



For concreteness, let $I_1 = I_2 > I_3$ (example will be a long cigar-like objects such as a juggling pin)

Euler equations (torque free) are:

$$I_1\dot{\omega}_1 = (I_2 - I_3)\omega_2\omega_3$$
$$I_2\dot{\omega}_2 = (I_3 - I_1)\omega_3\omega_1$$
$$I_3\dot{\omega}_3 = (I_1 - I_2)\omega_1\omega_2 = 0$$

$$\boxed{\begin{aligned} I_1\dot{\omega}_1 - (I_2 - I_3)\omega_2\omega_3 &= N_1 \\ I_2\dot{\omega}_2 - (I_3 - I_1)\omega_3\omega_1 &= N_2 \\ I_3\dot{\omega}_3 - (I_1 - I_2)\omega_1\omega_2 &= N_3 \end{aligned}}$$

Trivial case ($\boldsymbol{\omega}$ is along one of the principal axes):

$\Rightarrow$ $\boldsymbol{\omega}$ is along one of the eigendirection of $\mathbf{I}$ and $\mathbf{L} \parallel \boldsymbol{\omega}$

Then, from Euler equation, $\dot{\mathbf{L}} = \mathbf{I}\dot{\boldsymbol{\omega}} = -\boldsymbol{\omega}\times\mathbf{L} = 0,$ $\Rightarrow$ $\dot{\boldsymbol{\omega}} = 0$

Torque Free Motion of a Symmetric Top

Interesting case ($\boldsymbol{\omega}$ is NOT along one of the principal axes):

We still have, $\dot{\omega}_3 = 0$ since $I_1 = I_2$

$$\boxed{\omega_3 = const}$$

Note: $\hat{x}_3$ is along the body's symmetry axis.

And, the rest of the Euler equations give,

$$\dot{\omega}_1 = \left(\frac{I_2 - I_3}{I_1}\right)\omega_2\omega_3 = \left(\frac{I_1 - I_3}{I_1}\right)\omega_2\omega_3$$

$$\dot{\omega}_2 = \left(\frac{I_3 - I_1}{I_2}\right)\omega_3\omega_1 = \left(\frac{I_3 - I_1}{I_1}\right)\omega_3\omega_1$$

Note: $\boxed{I_1 = I_2}$

Torque Free Motion of a Symmetric Top

Let $\Omega = \left(\frac{I_3 - I_1}{I_1}\right)\omega_3 = const$

$$\dot{\omega}_1 = \left(\frac{I_2 - I_3}{I_1}\right)\omega_2\omega_3 = \left(\frac{I_1 - I_3}{I_1}\right)\omega_2\omega_3$$

$$\dot{\omega}_2 = \left(\frac{I_3 - I_1}{I_2}\right)\omega_3\omega_1 = \left(\frac{I_3 - I_1}{I_1}\right)\omega_3\omega_1$$

Then, the remaining two Euler's equations reduce simply to,

$$\dot{\omega}_1 = -\Omega\omega_2$$
$$\dot{\omega}_2 = \Omega\omega_1$$

Taking the derivative of the top equation and substitute the bottom on the right, we have,

$$\ddot{\omega}_1 = -\Omega\dot{\omega}_2 = -\Omega\left(\Omega\omega_1\right) = -\Omega^2\omega_1$$

Since, $\Omega^2 \geq 0$ we have the solution:

A, φ_0 will be determined by ICs

$$\boxed{\omega_1(t) = A\cos\left(\Omega t + \varphi_0\right) \quad \text{and} \quad \omega_2(t) = A\sin\left(\Omega t + \varphi_0\right)}$$

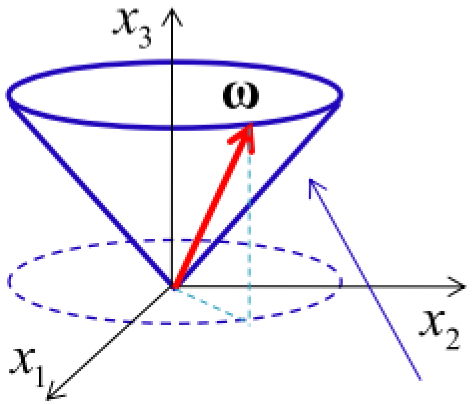
Torque Free Motion of a Symmetric Top

Looking at this deeper… First in the "body" frame,

- We know that ω_3 is a constant and ω_1 & ω_2 oscillates harmonically in a circle.

So, $\omega_1{}^2 + \omega_2{}^2 + \omega_3{}^2 = const$ ⇒ $|\boldsymbol{\omega}| = const$

⇒ In the "body" axes, this description for $\boldsymbol{\omega}$ can be visualized as $\boldsymbol{\omega}$ precessing about $\hat{x}_3$.



-The projection of $\boldsymbol{\omega}$ onto the x_3 axis is fixed.

-The projection of $\boldsymbol{\omega}$ onto the $x_1 - x_2$ plane rotates as a parametric circle with a rate of

$$\Omega = \left(\frac{I_3 - I_1}{I_1}\right)\omega_3 = const$$

(This is called the "body" cone)

Torque Free Motion of a Symmetric Top

Observations (in the fixed axes) cont:

3. The three vectors $\boldsymbol{\omega}, \mathbf{L}, \hat{\mathbf{x}}_3(body)$ always lie on a plane.

Consider the following product:

$$\mathbf{L}\cdot\left(\boldsymbol{\omega}\times\hat{\mathbf{x}}_3\right)$$ where $\hat{\mathbf{x}}_3$ is in the $\hat{\mathbf{z}}$ direction in the body axes

$$=\mathbf{L}\cdot\left(\omega_2\hat{\mathbf{x}}_1-\omega_1\hat{\mathbf{x}}_2\right)=\omega_2\left(\mathbf{L}\cdot\hat{\mathbf{x}}_1\right)-\omega_1\left(\mathbf{L}\cdot\hat{\mathbf{x}}_2\right)$$

Since the body axes are chosen to lie along the principal axes, we have

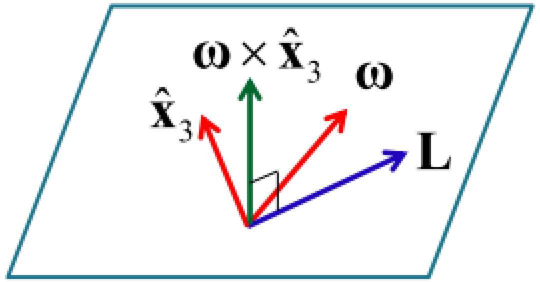
$$L_i = I_i\omega_i \ (no\ sum)$$

$$\Rightarrow \mathbf{L}\cdot\left(\boldsymbol{\omega}\times\hat{\mathbf{x}}_3\right)=\omega_2\left(I_1\omega_1\right)-\omega_1\left(I_2\omega_2\right)=0$$ (for a symmetric top) $I_1 = I_2$

Torque Free Motion of a Symmetric Top

Observations (in the fixed axes) cont:

This means that all three vectors $\boldsymbol{\omega}, \mathbf{L}, \hat{\mathbf{x}}_3$ always lie on a plane.



$$\mathbf{L} \cdot \left(\boldsymbol{\omega} \times \hat{\mathbf{x}}_3 \right) = 0$$

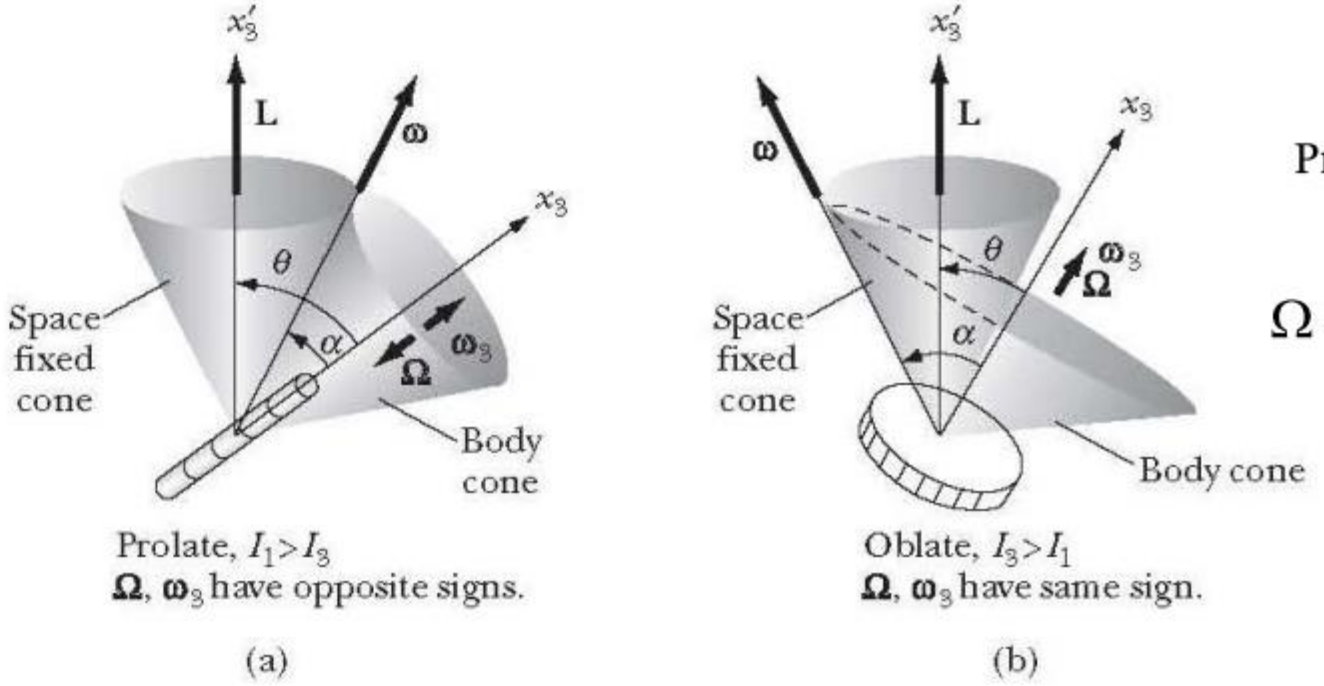
(for a torque free symmetric top)

Summary:

- $\boldsymbol{\omega}$ precesses around the "body" cone
- $\boldsymbol{\omega}$ also precesses around the "space" cone
- All three vectors $\boldsymbol{\omega}, \mathbf{L}, \hat{\mathbf{x}}_3$ always lie on a plane
- $\mathbf{L}$ is chosen to align with $\hat{\mathbf{x}}_3\,'$ in the space axes

Torque Free Motion of a Symmetric Top

This can be visualized as the body cone rolling either inside or outside of the space cone !



Precession Rate

$$\Omega = \left(\frac{I_3 - I_1}{I_1} \right) \omega_3$$

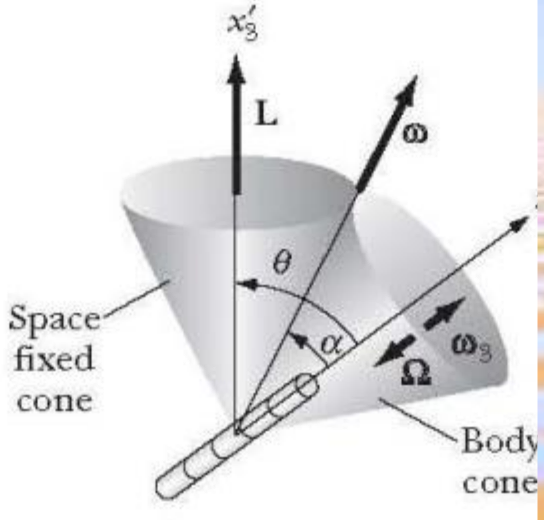
Case 1: $I_1 > I_3$

Case 2: $I_1 < I_3$

Torque Free Motion of a Symmetric Top

This can be visualized as the

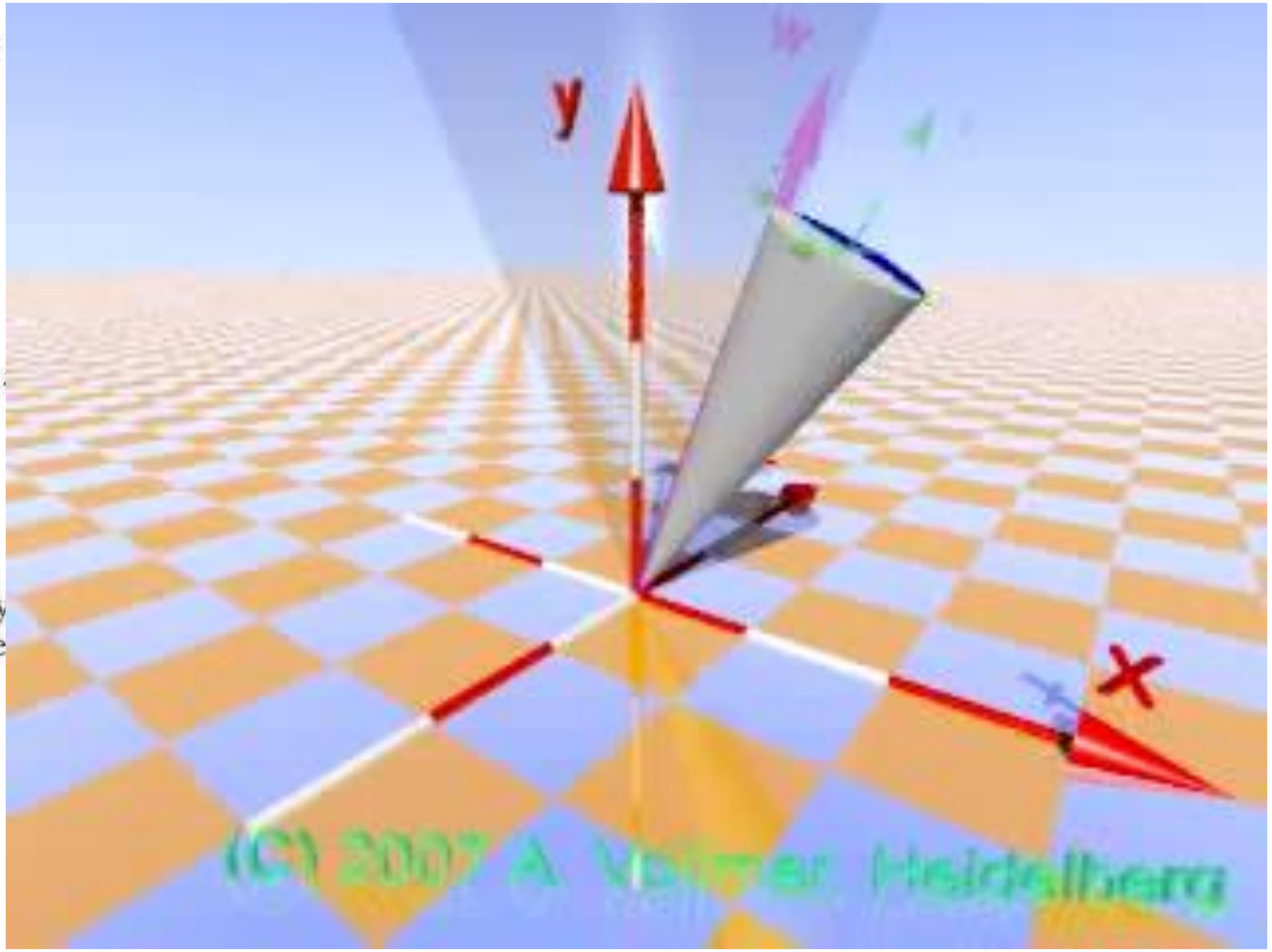
the space cone !



Prolate, $I_1 > I_3$
$\boldsymbol{\Omega}$, $\boldsymbol{\omega}_3$ have opposite signs.

(a)

Case 1: $I_1 > I_3$



Example of Symmetric Top: Earth's precession

The bulge at the equator makes I_3 slightly larger than I.

$$\mathbf{(I_3 - I)/I \approx 1/320}$$

$$\Omega = \left(\frac{1}{320}\right) 2\pi/(1day)$$

$\omega_3 = 2\pi/\text{day}$

For a person standing on the earth, the ω vector should precess around in its cone once every 320 days.

True value is ~ 430 days.

Chandler Wobble

Cause for the difference: Nonrigidity of the earth

Chandler Wobble: How to determine direction of ω?

Chandler Wobble: How big is the ω cone?

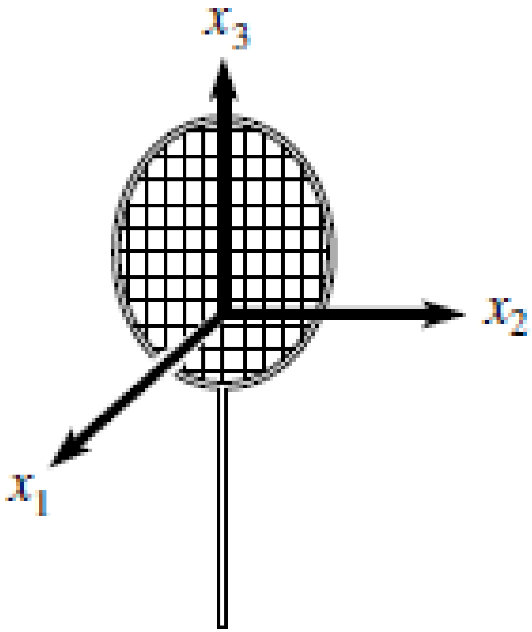
The ω Vector pierces the earth at a point on the order of 10m from the north pole.

$A/\omega_3 \approx (10\text{m})/R_E$

Half-angle of the cone ~ 10^{-4} degrees

Tennis Racket Theorem

If you try to spin a tennis racket around any of its three principal axes, you will notice that different things happen with different axes. Assuming that the principal moments (relative to the CM) are labeled according to $I_1>I_2>I_3$, you will find that the racket will spin nicely around the x_1 and x_3 axes, but it will wobble in a rather messy manner if you try to spin it around the x_2 axis.



Let us show that motion around the x_1 and x_3 axes is stable, whereas the motion around x_2 axis is unstable.

Tennis Racket Theorem

Case-1: Rotation around x_1–axis

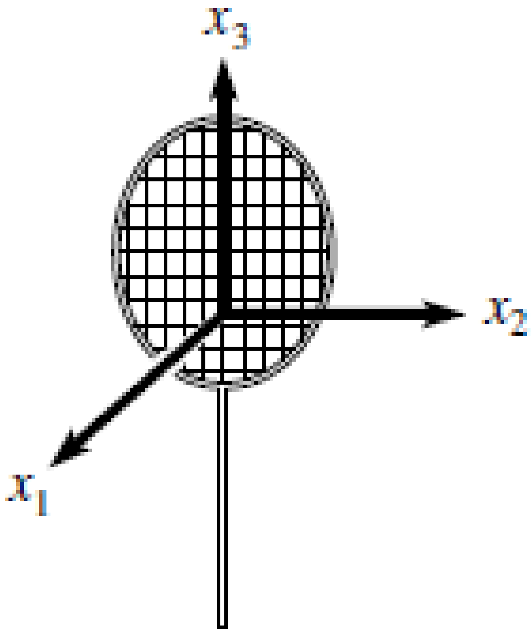
$$I_1\dot{\omega}_1 - (I_2 - I_3)\omega_2\omega_3 = N_1$$
$$I_2\dot{\omega}_2 - (I_3 - I_1)\omega_3\omega_1 = N_2$$
$$I_3\dot{\omega}_3 - (I_1 - I_2)\omega_1\omega_2 = N_3$$

$\Rightarrow \omega_2$ and ω_3 are much smaller than ω_1

Let $\omega_2 \longrightarrow \epsilon_2$ and $\omega_3 \longrightarrow \epsilon_3$

As gravity provides no torque around the CM, N=0

$I_1>I_2>I_3$



$$\dot{\omega}_1 - A\epsilon_2\epsilon_3 = 0$$

$$\dot{\epsilon}_2 + B\omega_1\epsilon_3 = 0$$

$$\dot{\epsilon}_3 - C\omega_1\epsilon_2 = 0$$

$$A \equiv \frac{I_2 - I_3}{I_1} > 0$$

$$B \equiv \frac{I_1 - I_3}{I_2} > 0$$

$$C \equiv \frac{I_1 - I_2}{I_3} > 0$$

Tennis Racket Theorem

Case-1: Rotation around x_1–axis

$\Rightarrow \omega_2$ and ω_3 are much smaller than ω_1

Let $\omega_2 \longrightarrow \epsilon_2$ and $\omega_3 \longrightarrow \epsilon_3$

$$\dot{\omega}_1 \approx 0 \;\rightarrow\; \omega_1 \sim constant$$

$$\boxed{\ddot{\epsilon}_2 = -B\omega_1\dot{\epsilon}_3 = -(BC\omega_1^2)\epsilon_2}$$

$$\dot{\epsilon}_3 - C\omega_1\epsilon_2 = 0$$

$$A \equiv \frac{I_2 - I_3}{I_1} > 0 \qquad B \equiv \frac{I_1 - I_3}{I_2} > 0 \qquad C \equiv \frac{I_1 - I_2}{I_3} > 0$$

$\vec{\omega} = (\omega_1, 0, 0)$ at all times
$\vec{L} \approx (I_1\omega_1, 0, 0)$ at all times

As there is no torque, direction of **L** is fixed and hence that of x_1
Racket does not wobble

Simple harmonic motion

ϵ_2 oscillates sinosoidally around zero

If ϵ_2 starts small, it remain small
Similarly ϵ_3 remains small

Tennis Racket Theorem

Case-2: Rotation around x_3–axis

The calculation is exactly same as previous case.

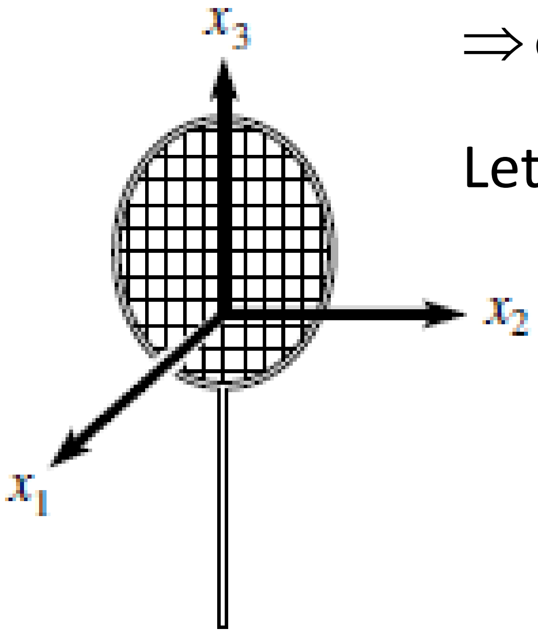
If ϵ_2 starts small, it remain small

Similarly ϵ_1 remains small

Racket does not wobble

$$I_1\dot{\omega}_1 - (I_2 - I_3)\omega_2\omega_3 = N_1$$
$$I_2\dot{\omega}_2 - (I_3 - I_1)\omega_3\omega_1 = N_2$$
$$I_3\dot{\omega}_3 - (I_1 - I_2)\omega_1\omega_2 = N_3$$

Case-3: Rotation around x_2–axis



$\Rightarrow \omega_1$ and ω_3 are much smaller than ω_2

$I_1 > I_2 > I_3$

Let $\omega_1 \longrightarrow \epsilon_1$ and $\omega_3 \longrightarrow \epsilon_3$ and torque N=0

$$\dot{\epsilon}_1 - A\omega_2\epsilon_3 = 0 \qquad A \equiv \frac{I_2 - I_3}{I_1} > 0$$

$$\dot{\omega}_2 + B\epsilon_1\epsilon_3 = 0 \qquad B \equiv \frac{I_1 - I_3}{I_2} > 0$$

$$\dot{\epsilon}_3 - C\omega_2\epsilon_1 = 0 \qquad C \equiv \frac{I_1 - I_2}{I_3} > 0$$

Tennis Racket Theorem

Case-2: Rotation around x_2–axis

$\Rightarrow \omega_1$ and ω_3 are much smaller than ω_1

$\ddot{\epsilon}_2 = -B\omega_1\dot{\epsilon}_3 = -(BC\omega_1^2)\epsilon_2$

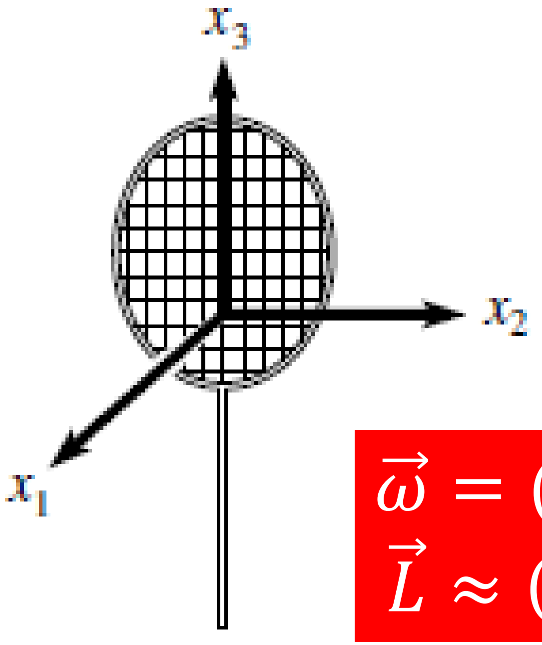
Let $\omega_1 \longrightarrow \epsilon_1$ and $\omega_3 \longrightarrow \epsilon_3$ and torque N=0

$$\dot{\epsilon}_1 - A\omega_2\epsilon_3 = 0$$

$$\dot{\omega}_2 + B\epsilon_1\epsilon_3 = 0$$

$$\dot{\epsilon}_3 - C\omega_2\epsilon_1 = 0$$

$\ddot{\epsilon}_1 = (AC\omega_2^2)\,\epsilon_1$



Positive coefficient $\Rightarrow$ exponentially growing motion, instead of an oscillatory one.

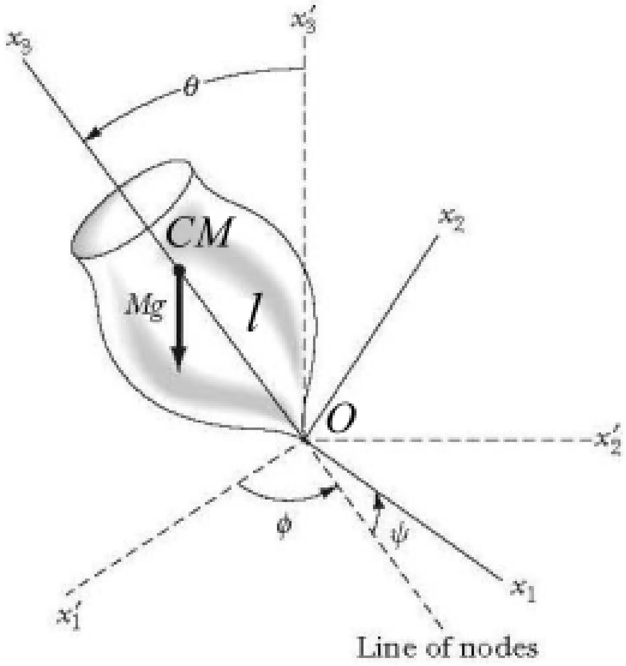
$\vec{\omega} = (0, \omega_2, 0)$ not valid
$\vec{L} \approx (0, I_2\omega_2, 0)$ not valid

ϵ_1 grows very quickly from its initial small value

Symmetric Top in an Uniform Gravity Field

So, we have $T = \frac{I_1}{2}\left(\dot{\phi}^2 \sin^2\theta + \dot{\theta}^2\right) + \frac{I_3}{2}\left(\dot{\phi}\cos\theta + \dot{\psi}\right)^2$

$$\boxed{L = T - U = \frac{I_1}{2}\left(\dot{\phi}^2 \sin^2\theta + \dot{\theta}^2\right) + \frac{I_3}{2}\left(\dot{\phi}\cos\theta + \dot{\psi}\right)^2 - Mgl\cos\theta}$$



Symmetric Top in an Uniform Gravity Field