Matrix - Vector] multiplication
Matrix - Matrix]

Let $A = \left[\begin{array}{c|c|c|c} a_1 & a_2 & \cdots & a_n \\ \downarrow & \downarrow & & \downarrow \end{array} \right]$ $\quad x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$

Then $Ax = x_1 a_1 + x_2 a_2 + \cdots + x_n a_n$
$= $ lc of Cols of A

A is matrix & x is a vector

Let $A = \begin{bmatrix} y_1 \rightarrow \\ \hline y_2 \rightarrow \\ \hline \vdots \\ \hline y_m \rightarrow \end{bmatrix}$ and $x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix}$

Then
$x^T A = x_1 y_1 + x_2 y_2 + \cdots + x_m y_m$
$=$ lc of rows of A

$AB = C$. Then

- $C(i,j) =$ dot product of i^{th} row of A and j^{th} col. of B
  - usual rule - by dot product - row and col. multiplication
- j^{th} col of C is A times j^{th} col of B
  Hence each col of C is lc of cols of A
- i^{th} row of C is i^{th} row of A times matrix B
  Hence each row of C is lc of rows of B
- ~~Remember~~ Also Recall! multiplication by - outer product - col and row rule.

Elementary Row operations in A

Type I	row exchange	$R_i \leftrightarrow R_j$
Type II		$R_i \rightarrow R_i + \alpha R_j$
Type III		$R_i \rightarrow \alpha R_i \quad (\alpha \neq 0)$

Row operations in A does not change solution of $Ax = b$ (provided we apply same operations in vector b also)

Gauss - Elimination

given $Ax = b$

Take $[A|b] \rightarrow$ augmented matrix

$\Big\downarrow$ By row operations

$[U|\tilde{b}] \longrightarrow$ solve by backward substitution
we say $U \rightarrow$ upper triangular matrix

Gauss - Jordan

$Ax = b$

Take $[A|b] \rightarrow$ augmented matrix

$\Big\downarrow$ By row operations

$[I|\tilde{b}] \rightarrow \tilde{b}$ is the solution

We use Gauss - Jordan to find A^{-1}.

Take $[A|I]$

$\Big\downarrow$ By row operations

$[I|B]$ Then $\underline{B = A^{-1}}$

Remember! Here we have assumed that A^{-1} exists
$\Updownarrow$
$Ax = b$ has a unique solution.

LU/PLU decomposition

Let $A = n \times n$ matrix. Then, by elementary row operations, A can be converted into following form, say

$$E_3 P_3 E_2 P_2 E_1 P_1 A = U$$

$$\Rightarrow \quad E_3' E_2' E_1' P_3 P_2 P_1 A = U$$

$\rightarrow$ $E_3' = E_3$
$E_2' = P_3 E_2 P_3^{-1}$
$E_1' = P_3 P_2 E_1 P_2^{-1} P_3^{-1}$
etc.

$$\Rightarrow \quad PA = LU \quad \text{where } P = P_3 P_2 P_1$$
$$\& \ L = E_1'^{-1} E_2'^{-1} E_3'^{-1}$$

A^{-1} exists, then always PLU decomposition exists

(Remember! PLU is not unique, depends on your choice of P_i's.)

When each $P_i = I$ (means row exchange, Type I row operation, is not needed, then $A = LU$ exists and it is unique)

Ques Solve

$$2\sin x - \cos y + 3\tan z = 3$$
$$4\sin x + 2\cos y - 2\tan z = 10$$
$$6\sin x - 3\cos y + \tan z = 9$$

Assume $\sin x = X$, $\cos y = Y$, $\tan z = Z$

Then given system is $Ax = b$

where $A = \begin{bmatrix} 2 & -1 & 3 \\ 4 & 2 & -2 \\ 6 & -3 & 1 \end{bmatrix}$ $x = \begin{bmatrix} X \\ Y \\ Z \end{bmatrix}$ $b = \begin{bmatrix} 3 \\ 10 \\ 9 \end{bmatrix}$

first solve for x

$$(A|b) = \left[\begin{array}{ccc|c} \boxed{2} & -1 & 3 & 3 \\ 4 & 2 & -2 & 10 \\ 6 & -3 & 1 & 9 \end{array}\right] \sim \left[\begin{array}{ccc|c} \boxed{2} & -1 & 3 & 3 \\ 0 & \boxed{4} & -8 & 4 \\ 0 & 0 & \boxed{-8} & 0 \end{array}\right]$$

$Z = 0$; $Y = 1$; and $X = 2$

See $\sin x = 2$ is not solvable in $\mathbb{R}$

Hence original system does not have any solution.

Ques Using Gauss-Jordan method first find A^{-1} then solution:

$Ax = b$ where $A = \begin{bmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{bmatrix}$ $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ $b = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

$$[A \mid I] = \left[\begin{array}{ccc|ccc} \boxed{2} & 1 & 1 & 1 & 0 & 0 \\ 4 & -6 & 0 & 0 & 1 & 0 \\ -2 & 7 & 2 & 0 & 0 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 2 & 1 & 1 & 1 & 0 & 0 \\ 0 & \boxed{-8} & -2 & -2 & 1 & 0 \\ 0 & 8 & 3 & 1 & 0 & 1 \end{array}\right] \begin{array}{l} r_1 \\ r_2 - 2r_1 \\ r_3 + r_1 \end{array} \sim \left[\begin{array}{ccc|ccc} 2 & 1 & 1 & 1 & 0 & 0 \\ 0 & -8 & -2 & -2 & 1 & 0 \\ 0 & 0 & 1 & -1 & 1 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} \boxed{1} & 1/2 & 1/2 & 1/2 & 0 & 0 \\ 0 & \boxed{1} & 1/4 & 1/4 & -1/8 & 0 \\ 0 & 0 & \boxed{1} & -1 & 1 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & \frac{1}{2} & 0 & 1 & -\frac{1}{2} & -\frac{1}{2} \\ 0 & \boxed{1} & 0 & 1/2 & -3/8 & -\frac{1}{4} \\ 0 & 0 & \boxed{1} & -1 & 1 & 1 \end{array}\right] \begin{array}{l} r_1 - \frac{1}{2} r_3 \\ r_2 - \frac{1}{4} r_3 \\ r_3 \end{array}$$

$-\frac{1}{2} + \frac{3}{16} = \frac{-8+3}{16} = -\frac{5}{16}$

$-\frac{1}{2} + \frac{1}{8}$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{3}{4} & -\frac{5}{16} & -\frac{3}{8} \\ 0 & 1 & 0 & \frac{1}{2} & -\frac{3}{8} & -\frac{1}{4} \\ 0 & 0 & 1 & -1 & 1 & 1 \end{array}\right]$$

Thus $A^{-1} = \begin{bmatrix} 3/4 & -5/16 & -3/8 \\ 1/2 & -3/8 & -1/4 \\ -1 & 1 & 1 \end{bmatrix}$

Solution $\begin{bmatrix} \frac{3}{4} - \frac{5}{16} - \frac{3}{8} \\ \frac{1}{2} - \frac{3}{8} - \frac{1}{4} \\ -1 + 1 + 1 \end{bmatrix} = \begin{bmatrix} \frac{12-5-6}{16} \\ \frac{4-3-2}{8} \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{16} \\ -\frac{1}{8} \\ 1 \end{bmatrix}$ ✓ **Answer**

Ques.

Find LU/PLU & then Solution for $Ax = b$

(i) $A = \begin{bmatrix} \boxed{1} & 0 & 1 \\ 2 & 2 & 2 \\ 3 & 4 & 5 \end{bmatrix} \xrightarrow{E_1} \begin{bmatrix} 1 & 0 & 1 \\ 0 & \boxed{2} & 0 \\ 0 & 4 & 2 \end{bmatrix} \xrightarrow{E_2} \begin{bmatrix} \boxed{1} & 0 & 1 \\ 0 & \boxed{2} & 0 \\ 0 & 0 & \boxed{2} \end{bmatrix} = U$

Thus $E_2 E_1 A = U$ where $E_1 = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix}$

$E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix}$

Hence $A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix} U$

Hence $A = LU$ where $L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix}$ and $U = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

Now solve $Ax = b$ where $b = \begin{bmatrix} 1 \\ 4 \\ 7 \end{bmatrix}$

$Ax = b$

$LUx = b \quad (\because A = LU)$

Take $Ux = c \Rightarrow Lc = b \rightarrow$ Solve first this by forward substitution

$\downarrow$

then $\boxed{Ux = c}$

$\hookrightarrow$ by backward subst.

Then

Hence $Lc = b \Rightarrow c = \begin{bmatrix} 1 \\ 2 \\ 7-3-4 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$

$Ux = c \Rightarrow x = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

(ii) $A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \xrightarrow{P_1} \begin{bmatrix} \boxed{1} & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \xrightarrow{E_1} \begin{bmatrix} \boxed{1} & 1 & 0 \\ 0 & \boxed{1} & 1 \\ 0 & 0 & \boxed{1} \end{bmatrix} = U$ (Unique solution exists)

Thus we have $E_1 P_1 A = U$

$\Rightarrow PA = LU$ where $P = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $L = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ +1 & 0 & 1 \end{bmatrix} = E_1^{-1}$

Solve $Ax = b$ where $b = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

$\Rightarrow PAx = Pb = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

put $PA = LU \rightarrow LUx = Pb$

$Lc = Pb \longrightarrow Ux = c$

$\Downarrow$ Forward Sub. $\qquad \Downarrow$ Backward Sub.

$c = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \qquad x = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$

(III) $A = \begin{bmatrix} \boxed{1} & 4 & 2 \\ -2 & -8 & 3 \\ 0 & 1 & 1 \end{bmatrix} \xrightarrow{E_1} \begin{pmatrix} 1 & 4 & 2 \\ 0 & 0 & 7 \\ 0 & 1 & 1 \end{pmatrix} \xrightarrow{P_1} \begin{bmatrix} \boxed{1} & 4 & 2 \\ 0 & \boxed{1} & 1 \\ 0 & 0 & \boxed{7} \end{bmatrix} = U$

Here $E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $P_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$

Thus we have

$$P_1 E_1 A = U$$

$\Rightarrow E_1' P_1 A = U$ where $E_1' = P_1 E_1 P_1^{-1}$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 2 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}$$

$\Rightarrow PA = LU$

where

$P = P_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$

$L = (E_1')^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$

$U = \begin{bmatrix} 1 & 4 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 7 \end{bmatrix}$

Solve $Ax = b$ where $b = \begin{bmatrix} -2 \\ 32 \\ 1 \end{bmatrix}$

$\Rightarrow PAx = Pb = \begin{bmatrix} -2 \\ 1 \\ 32 \end{bmatrix}$

$\Rightarrow LUx = Pb$

$Lc = Pb$

$\Downarrow$ By forward sub

$c = \begin{bmatrix} -2 \\ 1 \\ 28 \end{bmatrix}$

$Ux = c \xrightarrow{\text{By Backward Sub}} x = \begin{bmatrix} -2-4\times(-3)-2\times 4 \\ 1-1\times 4 = -3 \\ 4 \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \\ 4 \end{bmatrix}$