Online Bilevel Optimization: Regret Analysis of Online Alternating Gradient Methods

Davoud Ataee Tarzanagh[1] Parvin Nazari[2] Bojian Hou[1] Li Shen [1] Laura Balzano [3]

University of Pennsylvania[1], Amirkabir University of Technology[2], University of Michigan[3]

Online Bilevel Optimization (OBO)

OBO is a Stackelberg leader-follower game with

- **Leader's** decision $\mathbf{x}_t \in \mathcal{X} \subset \mathbb{R}^{d_1}$ and cost $f_t : \mathcal{X} \times \mathbb{R}^{d_2} \to \mathbb{R}$; and
- **Follower's** decision $\mathbf{y}_t \in \mathbb{R}^{d_2}$ and cost $g_t : \mathcal{X} \times \mathbb{R}^{d_2} \to \mathbb{R}$.

Goal: Select $\mathbf{x}_t \in \mathcal{X}$ to minimize *bilevel regret*.

- Bilevel dynamic regret:

$$\text{BD-Reg}_T := \sum_{t=1}^{T} \left(f_t(\mathbf{x}_t, \mathbf{y}_t^*(\mathbf{x}_t)) - f_t(\mathbf{x}_t^*, \mathbf{y}_t^*(\mathbf{x}_t^*)) \right), \quad \text{where}$$

$\mathbf{y}_t^*(\mathbf{x}) \in \arg\min_{\mathbf{y} \in \mathbb{R}^{d_2}} g_t(\mathbf{x}, \mathbf{y})$ and $\mathbf{x}_t^* \in \arg\min_{\mathbf{x} \in \mathcal{X}} f_t(\mathbf{x}, \mathbf{y}_t^*(\mathbf{x}))$.

- Bilevel local regret: For $F_{t,\mathbf{u}}(\mathbf{x}, \mathbf{y})$ defined in (TAF):

$$\text{BL-Reg}_T := \sum_{t=1}^{T} \|\nabla F_{t,\mathbf{u}}(\mathbf{x}_t, \mathbf{y}_t^*(\mathbf{x}_t))\|^2.$$

Time-Averaged Function and its Hypergradient

$$F_{t,\mathbf{u}}(\mathbf{x}, \mathbf{y}) := (1/W) \sum_{i=0}^{w-1} u_i f_{t-i}(\mathbf{x}, \mathbf{y}). \qquad \text{(TAF)}$$

Here, $W = \sum_{i=0}^{w-1} u_i$, $f_t \equiv 0$ for $t \le 0$, $\{u_i\}_{i=0}^{w-1}$ is a positive decreasing sequence with $u_0 = 1$ and $\mathbf{M}_t(\mathbf{x}, \mathbf{y})$ satisfies

$$\nabla^2_{\mathbf{xy}} g_t(\mathbf{x}, \mathbf{y}) + \mathbf{M}_t(\mathbf{x}, \mathbf{y}) \nabla^2_{\mathbf{y}} g_t(\mathbf{x}, \mathbf{y}) = 0.$$

Time-Averaged Hypergradient:

$$\tilde{\nabla} F_{t,\mathbf{u}}(\mathbf{x}, \mathbf{y}) := (1/W) \sum_{i=0}^{w-1} u_i \tilde{\nabla} f_{t-i}(\mathbf{x}, \mathbf{y}),$$

where $\tilde{\nabla} f_t(\mathbf{x}, \mathbf{y}) := \nabla_{\mathbf{x}} f_t(\mathbf{x}, \mathbf{y}) + \mathbf{M}_t(\mathbf{x}, \mathbf{y}) \nabla_{\mathbf{y}} f_t(\mathbf{x}, \mathbf{y})$.

OAGD Algorithm for Bilevel Regret Minimization

- **REQUIRE:** Initial values $(\mathbf{x}_1, \mathbf{y}_1) \in \mathcal{X} \times \mathbb{R}^{d_2}$; parameters $w, T, K_1, K_2, \ldots, K_T \in \mathbb{N}$; stepsizes $\{(\alpha_t, \beta_t) \in \mathbb{R}^2_{++}\}_{t=1}^{T}$; and weights $\{u_i\}_{i=0}^{w-1}$ with $1 = u_0 \ge u_1 \ge \ldots \ge u_{w-1} > 0$.

FOR $t = 1$ **to** T
  Set $\mathbf{z}_t^1 \leftarrow \mathbf{y}_t$
  FOR $k = 1$ **to** K_t
    Update $\mathbf{z}_t^{k+1} \leftarrow \mathbf{z}_t^k - \beta_t \nabla_{\mathbf{z}} g_t(\mathbf{x}_t, \mathbf{z}_t^k)$
  ENDFOR
  Update $\mathbf{y}_{t+1} \leftarrow \mathbf{z}_t^{K_t+1}$
  Update $\mathbf{x}_{t+1} \leftarrow \Pi_{\mathcal{X}} \left[\mathbf{x}_t - \alpha_t \tilde{\nabla} F_{t,\mathbf{u}}(\mathbf{x}_t, \mathbf{y}_{t+1}) \right]$
ENDFOR

Theory

Assumptions:

A1) f_t is $\ell_{f,0}$-Lipschitz continuous; $g_t(\mathbf{x}, \mathbf{y})$ is μ_g-strongly convex in $\mathbf{y}$ for any $\mathbf{x} \in \mathcal{X}$; and $\nabla f_t, \nabla g_t$, and $\nabla^2 g_t$ are respectively $\ell_{f,1}, \ell_{g,1}$, and $\ell_{g,2}$-Lipschitz continuous.

A2) For any $\mathbf{x}, \mathbf{x}' \in \mathcal{X}$, $\|\mathbf{x} - \mathbf{x}'\| \le D$, for some $D > 0$, and $\|\mathbf{y}_1 - \mathbf{y}_1^*(\mathbf{x}_1)\| \le D'$, for some $D' > 0$.

A3) For all $t \in [T]$, $|f_t(\mathbf{x}, \mathbf{y})| \le M$ for some finite constant $M > 0$.

Regret Bounds

- (**Strongly-Convex**) Under Assumptions **A1** and **A2**, and $\{f_t\}_{t=1}^{T}$ being strongly convex:

$$\text{BD-Reg}_T \le \mathcal{O}\Big(1 + \min\left\{S_{1,T}, S_{2,T}\right\}\Big),$$

where $S_{p,T} := P_{p,T} + Y_{p,T}$ with

$$P_{p,T} := \sum_{t=2}^{T} \|\mathbf{x}_{t-1}^* - \mathbf{x}_t^*\|^p, \quad \text{and} \quad Y_{p,T} := \sum_{t=2}^{T} \|\mathbf{y}_{t-1}^*(\mathbf{x}_{t-1}^*) - \mathbf{y}_t^*(\mathbf{x}_t^*)\|^p.$$

- (**Convex**) Under Assumptions **A1** and **A2**, and the convexity of $\{f_t\}_{t=1}^{T}$:

$$\text{BD-Reg}_T \le \mathcal{O}\Big(1 + S_{1,T} + Y_{2,T}\Big).$$

- (**Nonconvex**) Under Assumptions **A1** and **A3**, and $\mathcal{X} \equiv \mathbb{R}^{d_1}$:

$$\text{BL-Reg}_T \le \mathcal{O}\Big(\frac{T}{W} + H_{1,T} + H_{2,T}\Big),$$

where $H_{p,T} := \sum_{t=2}^{T} \sup_{\mathbf{x} \in \mathbb{R}^{d_1}} \|\mathbf{y}_{t-1}^*(\mathbf{x}) - \mathbf{y}_t^*(\mathbf{x})\|^p$.

- **OAGD achieves a problem-dependent dynamic regret bound.**
- **OAGD can achieve a sublinear local regret bound given $H_{p,T} = o(T)$ and $W = o(T)$.**

Experiments

- **Online Parametric Loss Tuning for Imbalanced Data**: OAGD trains on MNIST with class sampling at 0.6^i ($i = 0$ to 9). Uses 128-sample batches; for $w > 1$, merges with $w - 1$ previous batches. Define training and validation losses, respectively, as

$$g(\mathbf{x}_{t-1}, \mathbf{y}_t; \mathcal{D}_t^{\text{tr}}) = -\log \frac{e^{\gamma_{b_t}[\mathbf{y}_t(\mathbf{a}_t)]_{b_t} + \Delta_{b_t}}}{\sum_{j=1}^{J} e^{\gamma_j[\mathbf{y}_t(\mathbf{a}_t)]_j + \Delta_j}}, \qquad f(\mathbf{y}_t(\mathbf{x}_t); \mathcal{D}_t^{\text{val}}) = -u_{b_t} \log \frac{e^{[\mathbf{y}_t(\mathbf{a}_t)]_{b_t}}}{\sum_{j=1}^{J} e^{[\mathbf{y}_t(\mathbf{a}_t)]_j}}.$$

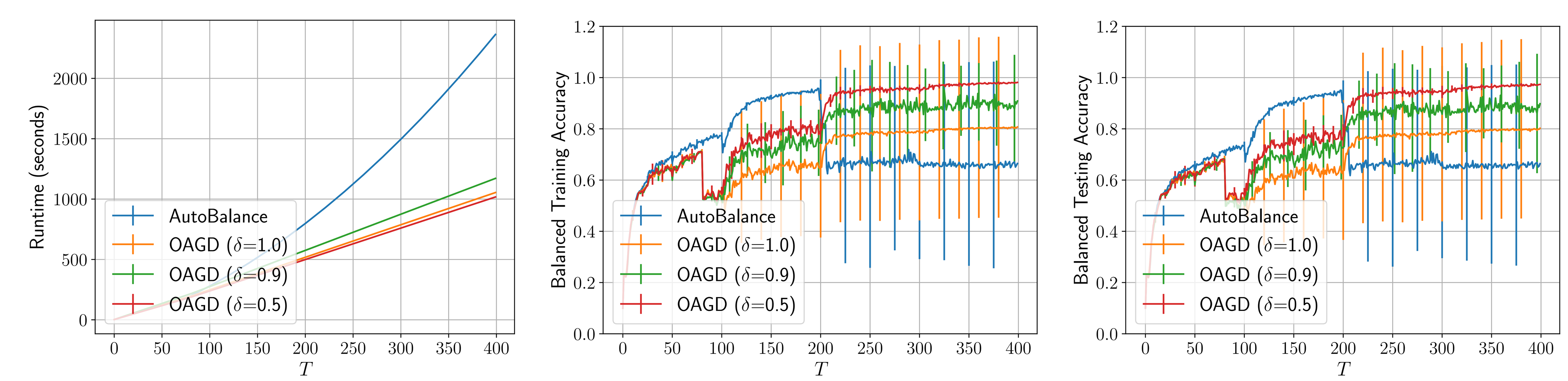

Figure: Performance comparison (mean±std) on loss tuning for imbalanced MNIST data across five runs.

- **Online Meta Learning**: Uses FC100 for a 5-way 5-shot classification. Each timestep features 25 train/test samples. If $w > 1$, OAGD includes data from $w - 1$ previous tasks.

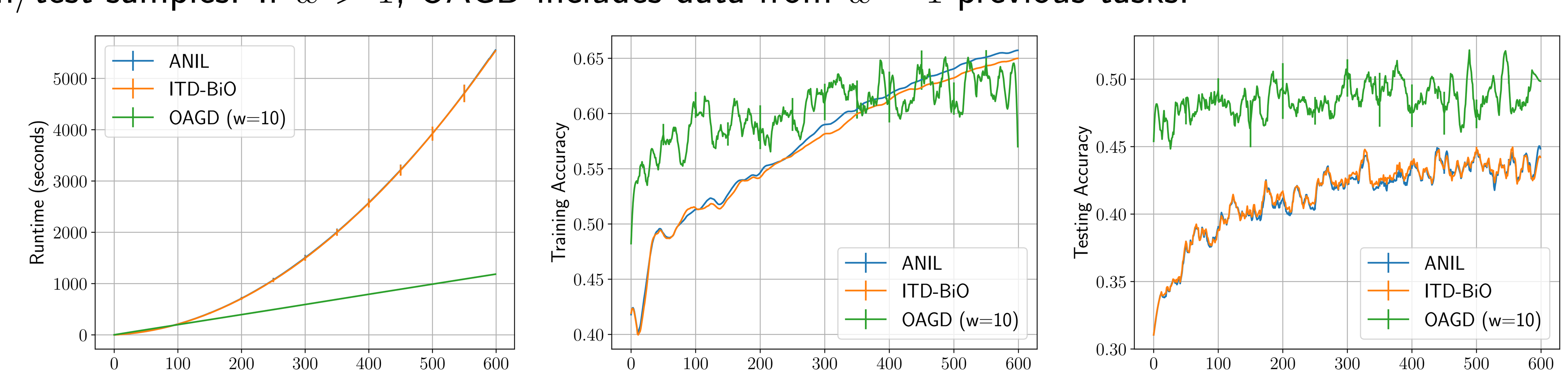

Figure: Performance comparison (mean±std) on meta-learning for FC100 data across five runs.

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