

Online Bilevel Optimization: Regret Analysis of Online Alternating Gradient Methods

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Online Bilevel Optimization (OBO)

OBO is a Stackelberg leader-follower game with

- **Leader's** decision $\mathbf{x}_t \in \mathcal{X} \subset \mathbb{R}^{d_1}$ and cost $f_t: \mathcal{X} \times \mathbb{R}^{d_2} \rightarrow \mathbb{R}$; and
- **Follower's** decision $\mathbf{y}_t \in \mathbb{R}^{d_2}$ and cost $g_t: \mathcal{X} \times \mathbb{R}^{d_2} \rightarrow \mathbb{R}$.

Goal of the Leader: Select $\mathbf{x}_t \in \mathcal{X}$ to minimize *bilevel regret*.

- **Bilevel dynamic regret:**

$$\text{BD-Reg}_T := \sum_{t=1}^T (f_t(\mathbf{x}_t, \mathbf{y}_t^*(\mathbf{x}_t)) - f_t(\mathbf{x}_t^*, \mathbf{y}_t^*(\mathbf{x}_t^*))), \quad \text{where}$$

$\mathbf{y}_t^*(\mathbf{x}) \in \arg \min_{\mathbf{y} \in \mathbb{R}^{d_2}} g_t(\mathbf{x}, \mathbf{y})$ and $\mathbf{x}_t^* \in \arg \min_{\mathbf{x} \in \mathcal{X}} f_t(\mathbf{x}, \mathbf{y}_t^*(\mathbf{x}))$.

- **Bilevel local regret:** For $F_{t,u}(\mathbf{x}, \mathbf{y})$ defined in (TAF):

$$\text{BL-Reg}_T := \sum_{t=1}^T \|\nabla F_{t,u}(\mathbf{x}_t, \mathbf{y}_t^*(\mathbf{x}_t))\|^2.$$

Theory

Assumptions:

A1) f_t is $\ell_{f,0}$ -Lipschitz continuous; $g_t(\mathbf{x}, \mathbf{y})$ is μ_g -strongly convex in \mathbf{y} for any $\mathbf{x} \in \mathcal{X}$; and $\nabla f_t, \nabla g_t$, and $\nabla^2 g_t$ are respectively $\ell_{f,1}, \ell_{g,1}$, and $\ell_{g,2}$ -Lipschitz continuous.

A2) For any $\mathbf{x}, \mathbf{x}' \in \mathcal{X}$, $\|\mathbf{x} - \mathbf{x}'\| \leq D$, for some $D > 0$, and $\|\mathbf{y}_1 - \mathbf{y}_1^*(\mathbf{x}_1)\| \leq D'$, for some $D' > 0$.

A3) For all $t \in [T]$, $|f_t(\mathbf{x}, \mathbf{y})| \leq M$ for some finite constant $M > 0$.

Regret Bounds

- **(Strongly-Convex)** Under Assumptions **A1** and **A2**, and $\{f_t\}_{t=1}^T$ being strongly convex:

$$\text{BD-Reg}_T \leq \mathcal{O}\left(1 + \min\{S_{1,T}, S_{2,T}\}\right),$$

where $S_{p,T} := P_{p,T} + Y_{p,T}$ with

$$P_{p,T} := \sum_{t=2}^T \|\mathbf{x}_{t-1}^* - \mathbf{x}_t^*\|^p, \quad \text{and} \quad Y_{p,T} := \sum_{t=2}^T \|\mathbf{y}_{t-1}^*(\mathbf{x}_{t-1}^*) - \mathbf{y}_t^*(\mathbf{x}_t^*)\|^p.$$

- **(Convex)** Under Assumptions **A1** and **A2**, and the convexity of $\{f_t\}_{t=1}^T$:

$$\text{BD-Reg}_T \leq \mathcal{O}\left(1 + S_{1,T} + Y_{2,T}\right).$$

- **(Nonconvex)** Under Assumptions **A1** and **A3**, and $\mathcal{X} \equiv \mathbb{R}^{d_1}$:

$$\text{BL-Reg}_T \leq \mathcal{O}\left(\frac{T}{W} + H_{1,T} + H_{2,T}\right),$$

where $H_{p,T} := \sum_{t=2}^T \sup_{\mathbf{x} \in \mathbb{R}^{d_1}} \|\mathbf{y}_{t-1}^*(\mathbf{x}) - \mathbf{y}_t^*(\mathbf{x})\|^p$.

- **OAGD achieves a problem-dependent dynamic regret bound.**
- **OAGD can achieve a sublinear local regret bound given $H_{p,T} = o(T)$ and $W = o(T)$.**

Time-Averaged Function and its Hypergradient

$$F_{t,u}(\mathbf{x}, \mathbf{y}) := (1/W) \sum_{i=0}^{w-1} u_i f_{t-i}(\mathbf{x}, \mathbf{y}). \quad (\text{TAF})$$

Here, $W = \sum_{i=0}^{w-1} u_i$, $f_t \equiv 0$ for $t \leq 0$, $\{u_i\}_{i=0}^{w-1}$ is a positive decreasing sequence with $u_0 = 1$ and $\mathbf{M}_t(\mathbf{x}, \mathbf{y})$ satisfies

$$\nabla_{\mathbf{xy}}^2 g_t(\mathbf{x}, \mathbf{y}) + \mathbf{M}_t(\mathbf{x}, \mathbf{y}) \nabla_{\mathbf{y}}^2 g_t(\mathbf{x}, \mathbf{y}) = 0.$$

Time-Averaged Hypergradient:

$$\tilde{\nabla} F_{t,u}(\mathbf{x}, \mathbf{y}) := (1/W) \sum_{i=0}^{w-1} u_i \tilde{\nabla} f_{t-i}(\mathbf{x}, \mathbf{y}),$$

where $\tilde{\nabla} f_t(\mathbf{x}, \mathbf{y}) := \nabla_{\mathbf{x}} f_t(\mathbf{x}, \mathbf{y}) + \mathbf{M}_t(\mathbf{x}, \mathbf{y}) \nabla_{\mathbf{y}} f_t(\mathbf{x}, \mathbf{y})$.

OAGD Algorithm for Bilevel Regret Minimization

- **REQUIRE:** Initial values $(\mathbf{x}_1, \mathbf{y}_1) \in \mathcal{X} \times \mathbb{R}^{d_2}$; parameters $w, T, K_1, K_2, \dots, K_T \in \mathbb{N}$; stepsizes $\{(\alpha_t, \beta_t) \in \mathbb{R}_{++}^2\}_{t=1}^T$; and weights $\{u_i\}_{i=0}^{w-1}$ with $1 = u_0 \geq u_1 \geq \dots \geq u_{w-1} > 0$.

FOR $t = 1$ **to** T
 Set $\mathbf{z}_t^1 \leftarrow \mathbf{y}_t$
 FOR $k = 1$ **to** K_t
 Update $\mathbf{z}_t^{k+1} \leftarrow \mathbf{z}_t^k - \beta_t \nabla_{\mathbf{z}} g_t(\mathbf{x}_t, \mathbf{z}_t^k)$
 ENDFOR
 Update $\mathbf{y}_{t+1} \leftarrow \mathbf{z}_t^{K_t+1}$
 Update $\mathbf{x}_{t+1} \leftarrow \Pi_{\mathcal{X}}[\mathbf{x}_t - \alpha_t \tilde{\nabla} F_{t,u}(\mathbf{x}_t, \mathbf{y}_{t+1})]$
ENDFOR

Experiments

- **Online Parametric Loss Tuning for Imbalanced Data:** OAGD trains on MNIST with class sampling at 0.6^i ($i = 0$ to 9). Uses 128-sample batches; for $w > 1$, merges with $w - 1$ previous batches. Define training and validation losses, respectively, as

$$g(\mathbf{x}_{t-1}, \mathbf{y}_t; \mathcal{D}_t^{\text{tr}}) = -\log \frac{e^{\gamma_{b_t}[\mathbf{y}_t(\mathbf{a}_t)]_{b_t} + \Delta_{b_t}}}{\sum_{j=1}^J e^{\gamma_j[\mathbf{y}_t(\mathbf{a}_t)]_j + \Delta_j}}, \quad f(\mathbf{y}_t(\mathbf{x}_t); \mathcal{D}_t^{\text{val}}) = -u_{b_t} \log \frac{e^{\gamma_{b_t}[\mathbf{a}_t]_{b_t}}}{\sum_{j=1}^J e^{\gamma_j[\mathbf{a}_t]_j}}.$$

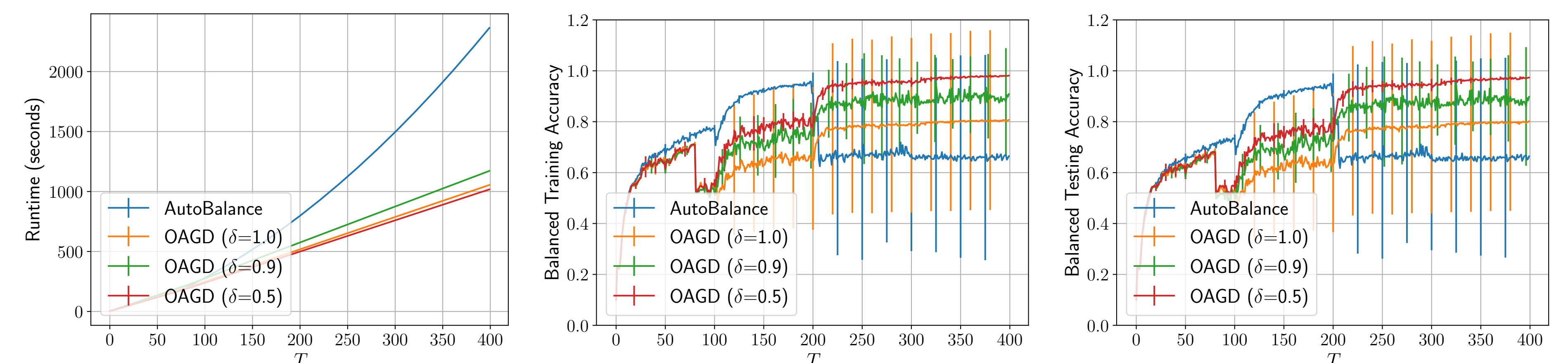


Figure: Performance comparison (mean±std) on loss tuning for imbalanced MNIST data across five runs.

- **Online Meta Learning:** Uses FC100 for a 5-way 5-shot classification. Each timestep features 25 train/test samples. If $w > 1$, OAGD includes data from $w - 1$ previous tasks.

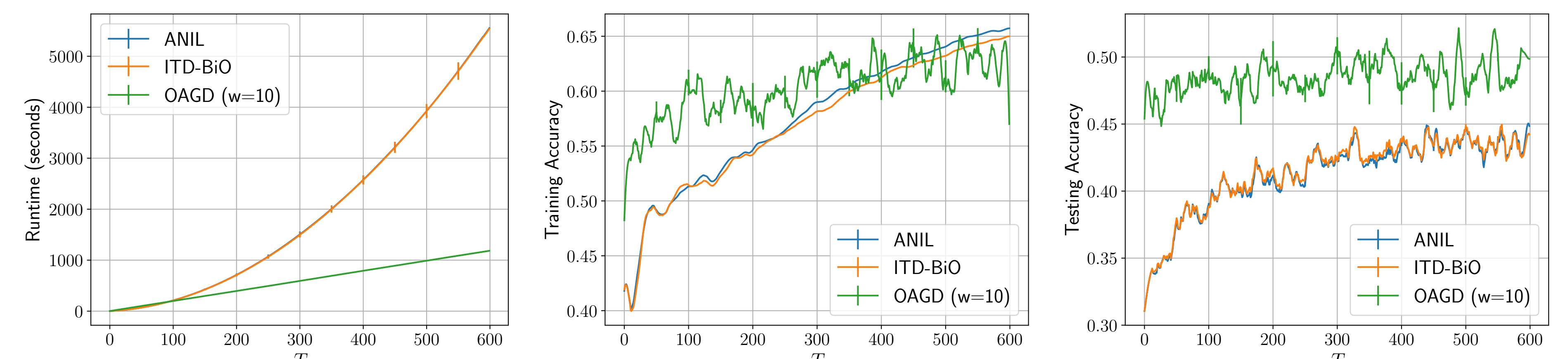


Figure: Performance comparison (mean±std) on meta-learning for FC100 data across five runs.