Probability Analysis of Used Car Dealership Inventory

Introduction

This document provides a comprehensive probabilistic analysis of a used car dealership's inventory, addressing various scenarios involving car brands, colors, conditions, and other characteristics. Each problem is solved using appropriate probability distributions and set theory, with calculations grounded in the provided dataset.

Problem 1: Silver and Ford Cars

A dataset of 2,499 used cars includes 300 silver cars, 1,235 Ford vehicles, and 127 silver Ford cars. Determine the number of cars that are:

a. Silver, Ford, or both.

b. Ford cars that are not silver.

c. Cars that are neither silver nor Ford.

Solution: Let A be the set of silver cars ($|A| = 300$), B the set of Ford cars ($|B| = 1,235$), and $A \cap B$ the set of silver Ford cars ($|A \cap B| = 127$). The total number of cars is 2,499.

a. Using the inclusion-exclusion principle:

$$|A \cup B| = |A| + |B| - |A \cap B| = 300 + 1,235 - 127 = 1,408$$

b. Ford but not silver:

$$|B| - |A \cap B| = 1,235 - 127 = 1,108$$

c. Neither silver nor Ford:

$$\text{Total} - |A \cup B| = 2,499 - 1,408 = 1,091$$

Problem 2: Ford and Texas Listings

A dataset categorizes cars by whether they are Ford and whether they are listed from Texas, with proportions given in the following table:

	From Texas (Yes)	From Texas (No)
Ford (Yes)	0.0540	0.4402
Ford (No)	0.0316	0.4742

Find the probabilities that a randomly selected car is:

a. A Ford.

b. From Texas but not a Ford.

c. Either a Ford or from Texas (or both).

Solution:

a. Probability of being a Ford:

$$P(\text{Ford}) = 0.0540 + 0.4402 = 0.4942$$

b. Probability of being from Texas but not a Ford:

$$P(\text{Texas, not Ford}) = 0.0316$$

c. Probability of being either a Ford or from Texas:

$$P(\text{Ford or Texas}) = 1 - P(\text{not Ford, not Texas}) = 1 - 0.4742 = 0.5258$$

Problem 3: Ranking Car Brands

Three car brands—Ford (1,235 cars), Dodge (432 cars), and Chevrolet (297 cars)—are ranked by an analyst guessing randomly from most to least common. Find:

a. One sample point for this experiment.

b. The sample space.

c. The probability that Ford, the most common brand, is ranked no worse than second.

Solution:

a. A sample point: (Dodge, Ford, Chevrolet).

b. Sample space (all permutations):

$$(\text{Ford, Dodge, Chevrolet}), (\text{Ford, Chevrolet, Dodge}), (\text{Dodge, Ford, Chevrolet}),$$
$$(\text{Dodge, Chevrolet, Ford}), (\text{Chevrolet, Ford, Dodge}), (\text{Chevrolet, Dodge, Ford})$$

c. Favorable outcomes where Ford is ranked 1st or 2nd:

$$(\text{Ford, Dodge, Chevrolet}), (\text{Ford, Chevrolet, Dodge}), (\text{Dodge, Ford, Chevrolet}), (\text{Chevrolet,$$

Probability:

$$P(\text{Ford ranked } \leq 2) = \frac{4}{6} = \frac{2}{3} \approx 0.667$$

Problem 4: Color Selection

From a lot with 49 possible car colors, n cars are randomly selected. Find:

a. An expression for the probability that none of the selected cars is black.

b. The number of cars needed so the probability of at least one black car is at least 0.5.

Solution:

a. Probability a car is not black: $\frac{48}{49}$. For n cars:

$$P(\text{no black}) = \left(\frac{48}{49}\right)^n$$

b. Probability of at least one black car:

$$P(\text{at least one black}) = 1 - \left(\frac{48}{49}\right)^n \geq 0.5$$

Solving:

$$\left(\frac{48}{49}\right)^n \leq 0.5 \implies n \geq \frac{\log 0.5}{\log\left(\frac{48}{49}\right)} \approx 34$$

Problem 5: Ford and Urgent Condition

In a dealership, 49.42% of cars are Fords, 41.22% are in urgent condition, and 16.57% are both. Find the probability that a randomly selected Ford car is:

a. In urgent condition.

b. In non-urgent condition.

Solution: Let $P(\text{Ford}) = 0.4942$, $P(\text{Urgent}) = 0.4122$, $P(\text{Ford} \cap \text{Urgent}) = 0.1657$.

a. Conditional probability:

$$P(\text{Urgent} \mid \text{Ford}) = \frac{P(\text{Ford} \cap \text{Urgent})}{P(\text{Ford})} = \frac{0.1657}{0.4942} \approx 0.3352$$

b. Non-urgent condition:

$$P(\text{Non-Urgent} \mid \text{Ford}) = 1 - 0.3352 = 0.6648$$

Problem 6: Ford or Urgent Condition

Using the same data as Problem 5, find:

a. The probability a car is either a Ford or in urgent condition (or both).

b. The conditional probability a car is both a Ford and in urgent condition, given it is either a Ford or in urgent condition.

Solution:

a. Union probability:

$$P(\text{Ford}\cup\text{Urgent}) = P(\text{Ford})+P(\text{Urgent})-P(\text{Ford}\cap\text{Urgent}) = 0.4942+0.4122-0.1657 = 0.740$$

b. Conditional probability:

$$P(\text{Ford} \cap \text{Urgent} \mid \text{Ford} \cup \text{Urgent}) = \frac{P(\text{Ford} \cap \text{Urgent})}{P(\text{Ford} \cup \text{Urgent})} = \frac{0.1657}{0.7407} \approx 0.2237$$

Problem 7: Identifying Dodge Vehicles

Given: 17.29% of cars are Dodge, 15.97% of Dodge cars are black, 78.37% of non-Dodge cars are not black, and 20.65% of cars are black. Find the probability a black car is a Dodge and assess if color is a good indicator.

Solution:

$$P(\text{Dodge} \cap \text{Black}) = P(\text{Dodge}) \cdot P(\text{Black} \mid \text{Dodge}) = 0.1729 \cdot 0.1597 = 0.0276$$

$$P(\text{Dodge} \mid \text{Black}) = \frac{P(\text{Dodge} \cap \text{Black})}{P(\text{Black})} = \frac{0.0276}{0.2065} \approx 0.1337$$

With only a 13.37% chance, black color is not a reliable indicator for Dodge vehicles.

Problem 8: Brand Matching

A trainee randomly assigns Ford, Dodge, and Chevrolet labels to three cars. Let Y be the number of correct matches. Find the probability distribution of Y.

Solution: Total permutations: $3! = 6$. Outcomes:

$$P(Y=0) = \frac{2}{6} = \frac{1}{3}, \quad P(Y=1) = \frac{3}{6} = \frac{1}{2}, \quad P(Y=2) = 0, \quad P(Y=3) = \frac{1}{6}$$

Problem 9: Black Cars in Sample

For a sample of 20 cars with $P(\text{black}) = 0.2065$, find the probabilities of:

a. Exactly 10 black cars.

b. At least 12 black cars.

c. 8 to 15 black cars.

d. At most 16 black cars.

Solution: Let $X \sim \text{Binomial}(20, 0.2065)$.

a. $P(X = 10) \approx 0.0026$

b. $P(X \geq 12) \approx 0.0008$

c. $P(8 \leq X \leq 15) \approx 0.0383$

d. $P(X \leq 16) \approx 1.0$

Problem 10: Geometric Distribution for Black Cars

An inspector checks cars until a black car is found ($p = 0.2065$). Find:

a. Probability the third car is the first black car.

b. Probability no black car in 10 tries.

c. Expected number and variance of tries.

Solution: Let $X \sim \text{Geometric}(0.2065)$.

a. $P(X = 3) = (1 - 0.2065)^2 \cdot 0.2065 \approx 0.13$

b. $P(X > 10) \approx 0.099$

c. $E[X] = \frac{1}{0.2065} \approx 4.84$, $\text{Var}(X) = \frac{1-0.2065}{0.2065^2} \approx 18.61$

Problem 11: Red Cars

For red cars ($p = 0.0768$), find the expected number and variance of cars inspected until the first red car.

Solution:

$$E[X] = \frac{1}{0.0768} \approx 13.02, \quad \text{Var}(X) = \frac{1 - 0.0768}{0.0768^2} \approx 156.39$$

Problem 12: Defective Cars

From 2,499 cars, 163 have non-clean titles. For 5 randomly selected cars, find:

a. Probability all have clean titles.

b. Expected repair cost and variance ($\$500$ per repair).

Solution: Let $X \sim$ Hypergeometric$(M = 2499, n = 2336, N = 5)$.

a. $P(X = 5) = \frac{\binom{2336}{5}}{\binom{2499}{5}} \approx 0.7135$

b. Expected defectives: $\mu = 5 \cdot \frac{163}{2499} \approx 0.3262$. Cost: $0.3262 \cdot 500 \approx 163.07$. Variance: $0.3044 \cdot 500^2 \approx 76092.52$.

Problem 13: Shipment of Cars

For 4 randomly selected cars, find the probability of:

a. No defective cars.

b. At least one defective car.

Solution: Let $X \sim$ Hypergeometric$(M = 2499, n = 2336, N = 4)$.

a. $P(X = 4) = \frac{\binom{2336}{4}}{\binom{2499}{4}} \approx 0.7634$

b. $P(X \geq 1) = 1 - 0.7634 = 0.2366$

Problem 14: Red Car Scans

An automated tool flags red cars ($p = 0.0768$). Find:

a. Probability first red car on third scan.

b. Probability third red car on seventh scan.

c. Assumptions.

d. Mean and variance for three red cars.

Solution:

a. Geometric: $P(X = 3) = (1 - 0.0768)^2 \cdot 0.0768 \approx 0.0655$

b. Negative Binomial: $P(X = 7) = \binom{6}{2} \cdot 0.0768^3 \cdot (1 - 0.0768)^4 \approx 0.0049$

c. Assumptions: Independent scans, constant probability.

d. $E[X] = \frac{3}{0.0768} \approx 39.05$, Var$(X) = \frac{3(1-0.0768)}{0.0768^2} \approx 469.17$

Problem 15: Poisson Distribution for Red Cars

Red cars appear at a rate of 7.68 per 100 listings. Find probabilities for:

a. No more than three red cars.

b. At least two red cars.

c. Exactly five red cars.

Solution: Let $X \sim \text{Poisson}(7.68)$.

a. $P(X \leq 3) \approx 0.0524$

b. $P(X \geq 2) = 1 - P(X \leq 1) \approx 0.996$

c. $P(X = 5) \approx 0.1027$

Problem 16: Tchebysheff's Theorem

Car prices have mean \$18,767.67 and standard deviation \$12,116.09. Find a lower bound for cars priced between \$6,535.49 and \$31,000.85.

Solution: For $k = 2$, Tchebysheff's Theorem gives:

$$P(|X - 18767.67| < 2 \cdot 12116.09) \geq 1 - \frac{1}{2^2} = 0.75$$

$$0.75 \cdot 2499 \approx 1874 \text{ cars}$$

Problem 17: Lot Acceptance

For a sample of 5 cars, the lot is accepted if no defectives are found. Calculate $P(\text{accept})$ for $p = 0.0, 0.1, 0.3, 0.5, 1.0$, and describe the OC curve.

Solution:

$$P(Y = 0) = (1 - p)^5$$

- $p = 0.0$: 1.0000
- $p = 0.1$: 0.5905
- $p = 0.3$: 0.1681
- $p = 0.5$: 0.0313
- $p = 1.0$: 0.0000

The OC curve shows a sharp decline in acceptance probability as p increases, indicating a strict sampling plan.

Problem 18: Silver Car Inspection

Five cars, one silver, are inspected until the silver car is found. Let Y be the trial number. Find:

a. Probability function.

b. Distribution function.

c. $P(Y = 3)$, $P(Y \leq 3)$, $P(Y < 3)$.

Solution:

a. $P(Y = k) = \frac{1}{5}$, $k = 1, 2, 3, 4, 5$

b. $F(Y) = P(Y \leq k) = \frac{k}{5}$

c. $P(Y = 3) = 0.2$, $P(Y \leq 3) = 0.6$, $P(Y < 3) = 0.4$

Problem 19: Piecewise

For a $F(y)$ defined piecewise, find the mean and variance of Y.

Solution: PDF:

$$f(y) = \begin{cases} \frac{1}{10}, & 0 < y < 2 \\ \frac{2(y-1)}{9}, & 2 \leq y < 4 \\ 0, & \text{otherwise} \end{cases}$$

$$E[Y] \approx 3.0148, \quad \text{Var}(Y) \approx 0.3627$$

Problem 20: Triangular Distribution

For a symmetric triangular distribution, find the mean and variance of Y.

Solution:

$$f(y) = \begin{cases} \frac{y}{4}, & 0 < y < 2 \\ \frac{4-y}{4}, & 2 \leq y < 4 \\ 0, & \text{otherwise} \end{cases}$$

$$E[Y] = 2.0, \quad \text{Var}(Y) = 0.6667$$

Problem 21: Uniform Price Distribution

Car prices are uniform between \$2,000 and \$42,030. Find probabilities for prices:

a. Below \$15,000.

b. Above \$25,000.

Solution:

a. $P(X < 15000) = \frac{13000}{40030} \approx 0.3248$

b. $P(X > 25000) = \frac{17030}{40030} \approx 0.4254$

Problem 22: Exponential Mileage

Mileage follows an exponential distribution with mean 48,922.93 miles. Find:

a. Probability of mileage over 100,000 miles.

b. Mileage where 1% of cars exceed.

Solution:

$$P(X > 100000) \approx 0.1295, \quad x_{0.01} \approx 225298.41 \text{ miles}$$

Problem 23: Joint Probability of Ford and Chevrolet

For two randomly selected cars, let Y_1 be the number of Fords and Y_2 the number of Chevrolets. Find:

a. Joint probability function.

b. $F(1,0) = P(Y_1 \leq 1, Y_2 \leq 0)$.

Solution:

Y_1	Y_2	$P(Y_1, Y_2)$
0	0	0.1513
0	1	0.0920
0	2	0.0139
1	0	0.3814
1	1	0.1180
2	0	0.2434

$$F(1,0) = 0.1513 + 0.3814 = 0.5327$$

Problem 24: Normalized Price and Mileage

For uniform Y_1, Y_2 over $[0,1] \times [0,1]$, find:

a. $P(Y_1 < 0.5, Y_2 > 0.75)$.

b. $P(Y_1 + Y_2 \leq 1)$.

Solution:

a. $P(Y_1 < 0.5, Y_2 > 0.75) \approx 0.0149$

b. $P(Y_1 + Y_2 \leq 1) \approx 0.9735$

Problem 25: Chevrolet Model Year and Mileage

For $f(y_1, y_2) = 6(1 - y_2)$, $0 \leq y_1 \leq y_2 \leq 1$, find $P(Y_2 \leq 0.5 \mid Y_1 \leq 0.6)$.

Solution:

$$P(Y_2 \leq 0.5 \mid Y_1 \leq 0.6) \approx 0.5$$

Problem 26: Independence of Price and Mileage

For $f(y_1, y_2) = y_1 + y_2$, check if Y_1 and Y_2 are independent.

Solution: Mean squared error between joint and marginal product distributions:

$$7.3 \times 10^{-5}$$

This suggests approximate independence.