MA204: Mathematics IV

Partial Differential Equation (First Order PDE)

Introduction

We now move our attention to the quasi-linear PDE given by the equation

$$a(x,y,z)z_x + b(x,y,z)z_y = c(x,y,z). \tag{1}$$

Method of characteristics: Following the method of characteristics for semilinear equations, we obtain the characteristic equations for (1) as

$$\frac{dx}{dt} = a(t),$$
$$\frac{dy}{dt} = b(t),$$
$$\frac{dz}{dt} = c(t),$$

along a solution curve $C : \vec{r}(t)$ for the integral surface $F(x,y,z) = 0$.

Method of Lagrange

Theorem

If $u = u(x, y, z)$ and $v = (x, y, z)$ are two given functions of x, y, and z and if $F(u, v) = 0$, where F is an arbitrary function of u and v, then $z = z(x, y)$ satisfies a first order PDE

$$\frac{\partial(u,v)}{\partial(y,z)} z_x + \frac{\partial(u,v)}{\partial(z,x)} z_y = \frac{\partial(u,v)}{\partial(x,y)},$$

where

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix}.$$

Proof.

An idea of the proof will be discussed in the class. □

Method of Lagrange

Theorem

The general solution of the quasi-linear equation

$$a(x,y,z)z_x + b(x,y,z)z_y = c(x,y,z)$$

is given by

$$F(u(x,y,z), v(x,y,z)) = 0,$$

where F is an arbitrary function of u and v with $u(x,y,z) = c_1$ and $v(x,y,z) = c_2$ are two linearly independent solutions of the equations

$$\frac{dx}{a(x,y,z)} = \frac{dy}{b(x,y,z)} = \frac{dz}{c(x,y,z)}.$$

Proof.

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Problem

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(a) $(x^2 - y^2 - z^2)z_x + 2xyz_y = 2xz$

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(c) $xzz_x + yzz_y = xy$

Integral surface passing through a given curve

We have already a method to find the general solution or integral surface of a quasilinear PDE using the method of Lagrange.

In certain cases, we need to find an integral surface for a PDE passigng through a particular curve.

Suppose the general solution for the quasilinear PDE

$$a(x,y,z)z_x + b(x,y,z)z_y = c(x,y,z)$$

is $F(u(x,y,z), v(x,y,z)) = 0$, where $u(x,y,z) = c_1$ and $v(x,y,z) = c_2$ are two linearly independent solutions of

$$\frac{dx}{a(x,y,z)} = \frac{dy}{b(x,y,z)} = \frac{dz}{c(x,y,z)}.$$

Suppose we want to find the integral curve for the given PDE passing through the curve C given by the parametric equation

$$x(0) = x(\tau), y(0) = y(\tau), \text{ and } z(0) = z(\tau).$$

Integral surface passing through a given curve

Thus we must have

$$u(x(\tau), y(\tau), z(\tau)) = c_1 \text{ and } v(x(\tau), y(\tau), z(\tau)) = c_2.$$

We then eliminate the parameter τ from these two equations, and obtain an relation of the form $F(c_1, c_2) = 0$.

Finally, we replace the constants c_1 and c_2 from the expressions of the general solution of the given PDE.

Problem: Find the equation of the integral surface for the PDE $2y(z-3)z_x + (2x-z)z_y = y(2x-3)$ passing through the circle $x^2 + y^2 = 2x, z = 0$.

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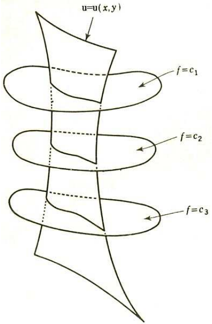
Problem: Find the integral surface for the PDE $yz_x + xz_y = z - 1$ passing through the curve $x^2 + y^2 = z, y = 2x$.

Surfaces orthogonal to a given system of surfaces

We now talk about an application of 1st order PDE in finding orthogonal surfaces to a given system of surfaces.



Note that $f(x, y, z) = c$ is the given family of surfaces, and $u = u(x, y)$ is orthogonal surface to the given family of surfaces.

Surfaces orthogonal to a given system of surfaces

Suppose a one pararmeter family of surfaces is given by the equation

$$f(x, y, z) = c. \tag{2}$$

We want to find a system of surfaces which cut each of the surface of (2) at a right angle.

Let the system of surfaces which cut each of (2) at a right angle be

$$z = \phi(x, y) \text{ or } F(x, y, z) = \phi(x, y) - z. \tag{3}$$

Since both the surfaces (2) and (3) intersect orthogonally, at a point of intersection (x, y, z), we must have that their respective normals are perpendicular.

As a result, we have

$$\nabla f \cdot \nabla F = f_x F_x + f_y F_y + f_z F_z = 0 \text{ or } f_x z_x + f_y z_y = f_z. \tag{4}$$

Note that (4) is a quasilinear PDE, which can be solved for F using the method of Lagrange.

Problem

Problem: Find the system of surfaces orthogonal to the family of surfaces given by $x(x^2 + y^2 + z^2) = cy^2$.

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Problem: Find the family of surfaces passing through the hyperbola $x^2 - y^2 = a^2, z = 0$ and orthogonal to the family of surfaces given by $z = cxy(x^2+y^2)$.

Thank you

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