MA204: Mathematics IV

Partial Differential Equation (First Order Non-linear PDE)

Introduction

We now move our attention to the first order non-linear partial differential equation

$$F(x, y, z, z_x, z_y) = 0, \tag{1}$$

where F is non-linear in z_x or z_y.

We have seen that the PDE (1) can be obtained from a two parameter family of surfaces given by

$$f(x, y, z, a, b) = 0. \tag{2}$$

Then (2) is called a **complete integral** of (1). A **particular integral** of (1) is obtained by putting particular values of a and b.

For an arbitrary function ϕ, the one-parameter subsystem

$$f(x, y, z, a, \phi(a)) = 0$$

of (2), and we form its envelope, then the envelope is called the **general solution** of (1).

The envelope of (2), if exists, is called the **singular solution** of (1).

Introduction

We explain the kinds of solutions for the PDE

$$z^2(1 + z_x^2 + z_y^2) = 1.$$

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We here mainly discuss two methods to solve a first order non-linear PDE (1). These methods are

1. **Cauchy's method of characteristics:** Its a geometrical method.
2. **Charpit's method:** Its a method based on compatible system of first order PDE and the method Lagrange.

Cauchy's method of characteristics

Let $z = z(x, y)$ be the integral surface S for the PDE (1) in (x, y, z)-space.

Then the direction ratios of the normal to the surface S is given by $(p = z_x, q = z_y, -1)$.

At any point $P(x_0, y_0, z_0)$ on the integral surface, since $F(x_0, y_0, z_0, p_0, q_0)$ is not necessarily linear, the tangent planes to integral surfaces through P form a family of planes enveloping a conical surface called **Monge Cone** with P as its vertex.

Thus we find the surfaces which touch the Monge Cone at each point along a generator to find the integral surfaces for the given PDE.

As a result, we need to find the Monge cone and the generator of the Monge Cone of the given PDE (1).

Cauchy's method of characteristics

The equation of the tangent plane to the integral surface $z = z(x, y)$ of (1) at $P(x_0, y_0, z_0)$ is

$$(x - x_0)p + (y - y_0)q = (z - z_0). \tag{3}$$

Again, the given PDE (1) can be written as

$$q = q(x_0, y_0, z_0, p). \tag{4}$$

The lines of contact between the tangent planes of the integral surface and the corresponding Monge cone at each point on the integral surface is the generator of the Monge cone.

These lines define a direction field of the integral surface S, called characteristics directions or Monge directions on S.

Cauchy's method of characteristics

The Monge cone can be obtained by eliminating p from (3), (4), and

$$\frac{x - x_0}{F_p} = \frac{y - y_0}{F_q}. \tag{5}$$

Note that (5) is obtained by eliminating $\frac{dq}{dp}$ from

$$(x - x_0) + (y - y_0)\frac{dq}{dp} = 0$$

and

$$\frac{dF}{dp} = \frac{\partial F}{\partial p} + \frac{\partial F}{\partial q}\frac{dq}{dp} = 0.$$

Note that (3) and (5) define the generator for the Monge cone, and solving them, we obtain

$$\frac{x - x_0}{F_p} = \frac{y - y_0}{F_q} = \frac{z - z_0}{pF_p + qF_q}$$
$$\Rightarrow \frac{dx}{F_p} = \frac{dy}{F_q} = \frac{dz}{pF_p + qF_q}.$$

Cauchy's method of characteristics

As a result, we obtain the characteristics equations as

$$\frac{dx}{dt} = F_p, \frac{dy}{dt} = F_q, \frac{dz}{dt} = pF_p + qF_q,$$

$$\frac{dp}{dt} = -F_x - pF_z, \text{ and } \frac{dq}{dt} = -F_y - qF_z.$$

Thus we can find the characteristics curve $x = x(t), y = y(t), z = z(t)$ and the equation of the plane (3), together called characteristic strip, without the knowledge of the integral surface $z = z(x, y)$.

As a result, we obtain the equation of the integral surface $z = z(x, y)$ by relating x, y, and z.

Problem: Find the integral surface for the first order nonlinear equation $z = \frac{p^2+q^2}{2} + (p - x)(q - y)$ which passes through the x-axis.

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Problem: Find the characteristics of the equation $pq = z$, and determine the integral surface which passes through the parabola $x = 0, y^2 = z$.

Compatible system of first order PDEs

Definition

A system of two first-order PDEs

$$f(x, y, u, p, q) = 0 \text{ and } g(x, y, u, p, q) = 0$$

are said to be compatible if they have a common solution.

For a PDE

$$f(x, y, u, p, q) = 0,$$

if $z = z(x, y)$ gives the integral surface, then

$$dz = pdx + qdy.$$

As a result, we expect that the given PDE is solvable for p and q, and

$$\int pdx \text{ and } \int qdy$$

are possible to evaluate to find z.

Compatible system of first order PDEs

Theorem

The system of PDEs

$$f(x, y, z, p, q) = 0 \text{ and } g(x, y, z, p, q) = 0$$

are compatible on a domain D if

$$J = \frac{\partial(f, g)}{\partial(p, q)} = \begin{vmatrix} f_p & f_q \\ g_p & g_q \end{vmatrix} \neq 0 \text{ on } D$$

with p, q being explicitly solvable from the PDEs as $p = \phi_1(x, y, z)$ and $q = \phi_2(x, y, z)$ so that

$$du = \phi_1(x, y, z)dx + \phi_2(x, y, z)dy$$

is integrable.

Compatible system of first order PDEs

Theorem

A necessary and sufficient condition for the integrability of the equation

$$du = \phi_1(x, y, z)dx + \phi_2(x, y, z)dy$$

is

$$[f, g] = \frac{\partial(f, g)}{\partial(x, p)} + \frac{\partial(f, g)}{\partial(y, q)} + p\frac{\partial(f, g)}{\partial(z, p)} + q\frac{\partial(f, g)}{\partial(z, q)} = 0.$$

Problem: Check if the following PDEs are compatible. If so, find their solutions.

(a) $p = 5x - 7y$ and $q = 6x + 8y$.

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(b) $xp = yq$ and $z(xp + yq) = 2xy$.

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(a) $p = 5x - 7y$ and $q = 6x + 8y$.

(b) $xp = yq$ and $z(xp + yq) = 2xy$.

(c) $xp - yq = x$ and $x^2p + q = xz$.

Compatible system of first order PDEs

For the compatibility of $f(x, y, z, p, q) = 0$ and $g(x, y, z, p, q) = 0$, it is not necessary that every solution of one equation is solution of the other one and vice verse.

For example, the equations

$$f = xp - yq - x = 0 \text{ and } g = x^2 p + q - xz = 0$$

are compatible.

The equations have common solutions $z = x + (1 + xy)$. But $z = x(y + 1)$ is a solution of f but not of g.

Charpit's method

The Charpit's method is a general method for finding the complete integral of a nonlinear PDE of first-order of the form

$$f(x, y, z, p, q) = 0.$$

The method uses the concept of compatible PDEs to find complete integral of the given PDE.

Thus the main task in this method is to find a PDE

$$g(x, y, z, p, q) = 0$$

which is compatible with the given PDE $f(x, y, z, p, q) = 0$.

Thus we must have

$$[f, g] = \frac{\partial(f, g)}{\partial(x, p)} + \frac{\partial(f, g)}{\partial(y, q)} + p\frac{\partial(f, g)}{\partial(z, p)} + q\frac{\partial(f, g)}{\partial(z, q)} = 0.$$

Charpit's method

Expanding, we obtain

$$f_p g_x + f_q g_y + (pf_p + qf_q)g_z - (f_x + pf_z)g_p - (f_y + qf_z)g_q = 0,$$

which is a linear PDE.

The above equation is solved to obtain g using the following auxiliary equations, called Charpit's equations,

$$\frac{dx}{f_p} = \frac{dy}{f_q} = \frac{dz}{pf_p + qf_q} = \frac{dp}{-f_x - pf_z} = \frac{dq}{-f_y - qf_z}.$$

Once we obtain the compatible PDEs $f(x, y, z, p, q) = 0$ and $g(x, y, z, p, q) = 0$, we solve them for p and q.

Finally, plugging these in

$$dz = pdx + qdy,$$

and then integrating we obtain the complete solution for the PDE.

Charpit's method

Problem: Solve the following PDEs :

(a) $z = px + qy + p^2 + q^2$.

(b) $p^2 - y^2 q = y^2 - x^2$.

(c) $(p^2 + q^2)y = qz$.

Some particular cases

There are particular forms of 1st order nonlinear PDEs for which the complete solution can be deduced without much computation.

1. **f is a function of p and q only:**

2. **f does not contain the independent variables x and y, i.e., f is a function of p, q and z only:**

Some particular cases

3. **Clairaut form** $z = px + qy + h(p, q)$**:**

4. **Variable separable, i.e.,** $f_1(x, p) = f_2(y, q)$**:**

Thank you

Thank You!!