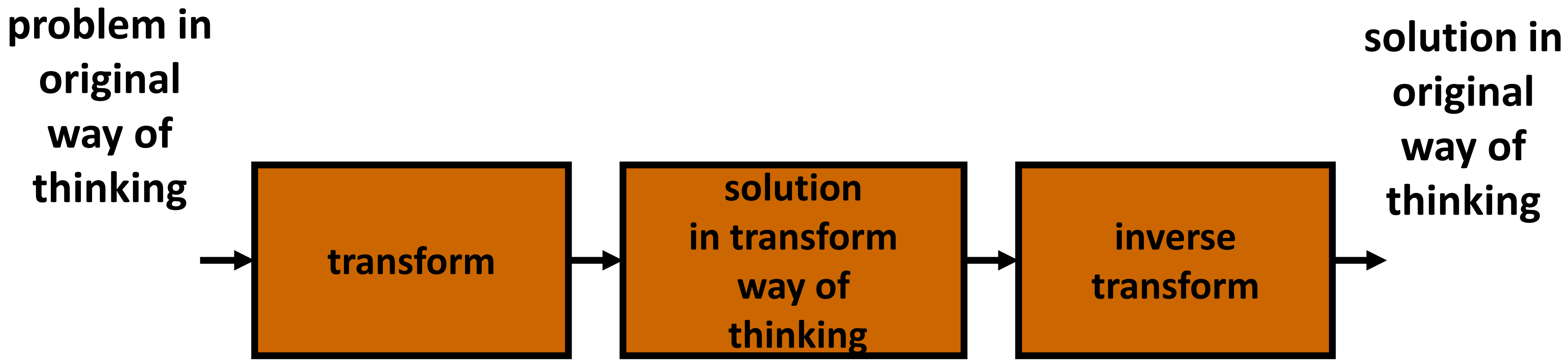
LAPLACE TRANSFORM

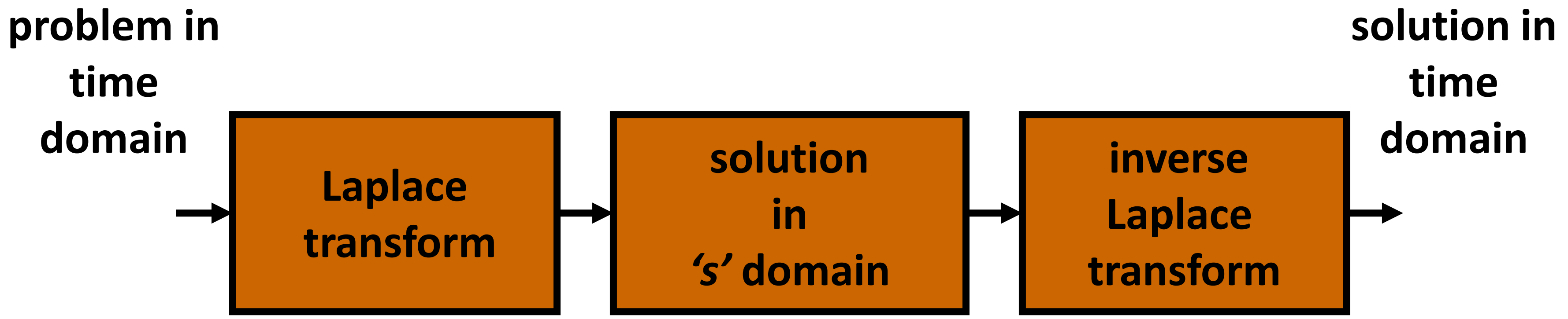
The French Newton
Pierre-Simon Laplace
(23 March 1749 – 5 March 1827)

- Developed mathematics in astronomy, physics, and statistics
- Began work in calculus which led to the Laplace Transform
- Focused later on celestial mechanics
- One of the first scientists to suggest the existence of black holes

Definition

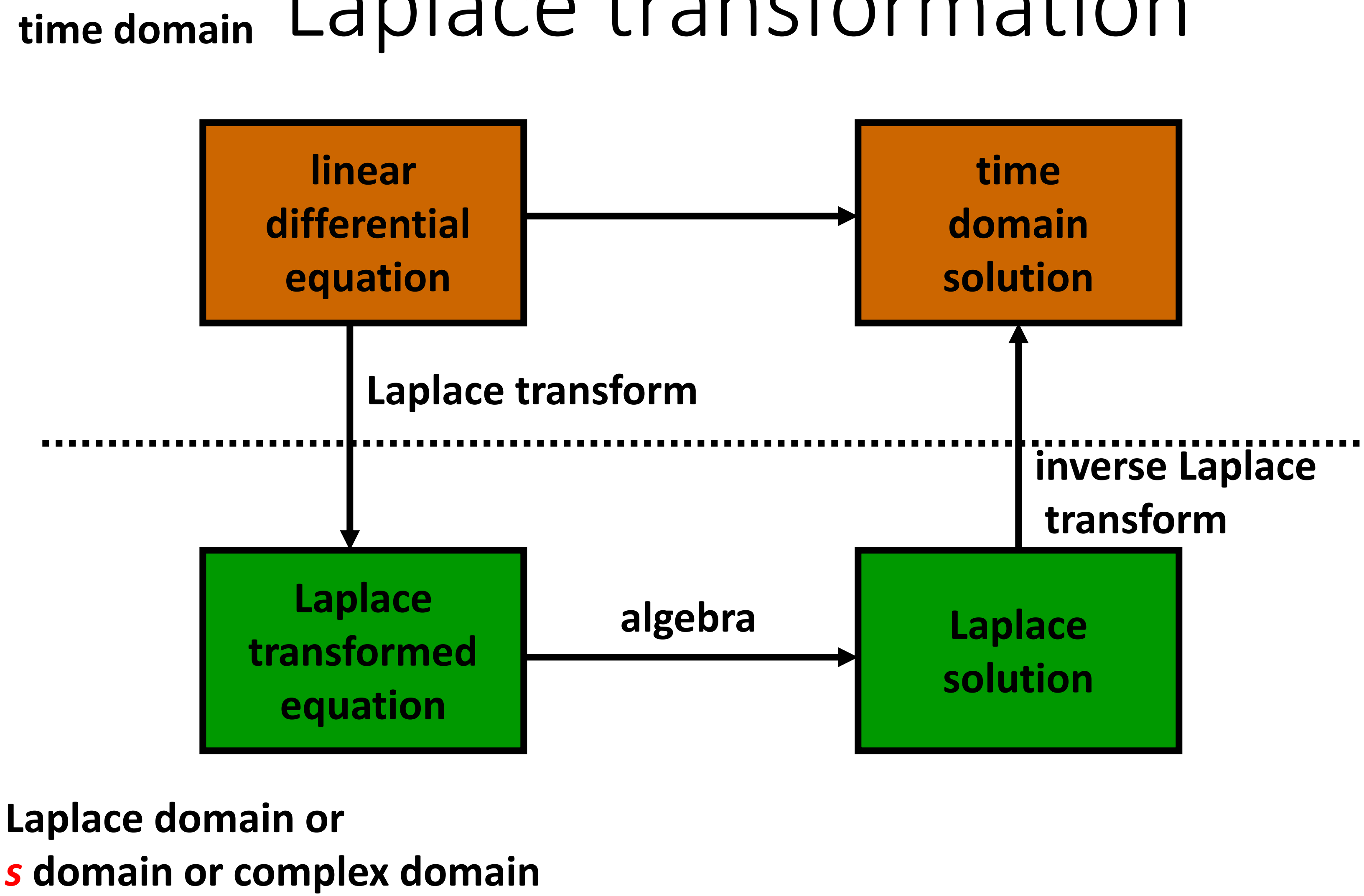
- Transforms: a mathematical conversion from one way of thinking to another to make a problem easier to solve





- Other transforms
  - Fourier
  - z-transform
  - wavelets

Laplace transformation



Laplace Transformation

Let us define

$f(t)$ = a function of time t such that $f(t) = 0$ for $t < 0$

s = a complex variable

$\mathcal{L}$ = an operational symbol indicating that the quantity that it prefixes is to be transformed by the Laplace integral $\int_0^\infty e^{-st}\,dt$

$F(s)$ = Laplace transform of $f(t)$

Then the Laplace transform of $f(t)$ is given by

$$\mathcal{L}[f(t)] = F(s) = \int_0^\infty e^{-st}\,dt[f(t)] = \int_0^\infty f(t)e^{-st}\,dt$$

where, $s = \sigma + j\omega$.

Inverse Laplace Transformation

The reverse process of finding the time function $f(t)$ from the Laplace transform $F(s)$ is called the *inverse Laplace transformation*. The notation for the inverse Laplace transformation is $\mathscr{L}^{-1}$, and the inverse Laplace transform can be found from $F(s)$ by the following inversion integral:

$$\mathscr{L}^{-1}[F(s)] = f(t) = \frac{1}{2\pi j} \int_{c-j\infty}^{c+j\infty} F(s)e^{st}\, ds, \qquad \text{for } t > 0$$

where c, the abscissa of convergence, is a real constant and is chosen larger than the real parts of all singular points of $F(s)$. Thus, the path of integration is parallel to the $j\omega$ axis and is displaced by the amount c from it. This path of integration is to the right of all singular points.

Evaluating the inversion integral appears complicated. In practice, we seldom use this integral for finding $f(t)$. We frequently use the partial-fraction expansion method given

Laplace Transformation

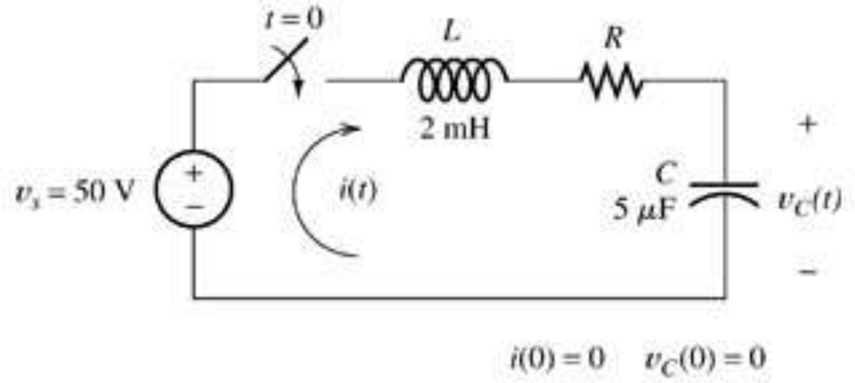
- Convert time-domain functions and operations into s-domain
  - $f(t) \to F(s) \quad (t \in R,\ s \in C)$
  - Linear differential equations (LDE) $\to$ algebraic expression in Complex plane

Time domain relation:

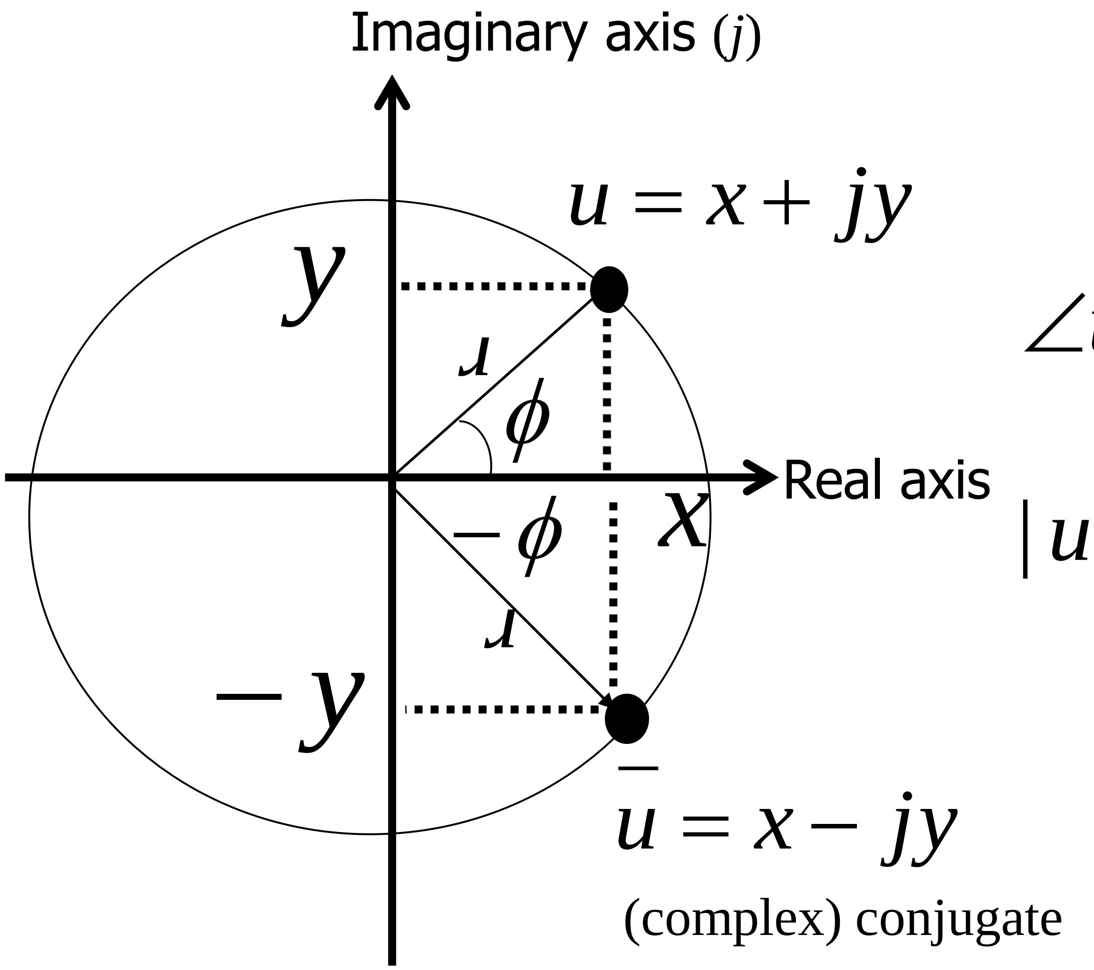
$$LC\frac{d^2v_c}{dt^2} + RC\frac{dv_c}{dt} + v_c = v_s$$

s-domain relation:

$$(LCs^2 + RCs + 1)V_c(s) = V_s(s)$$



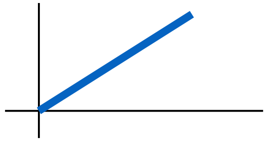
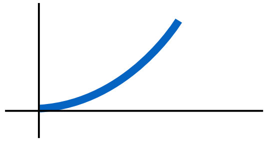
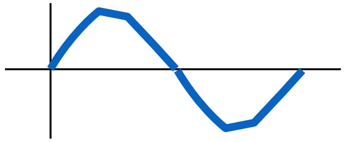
The Complex Plane (review)



$$\angle u \equiv \phi = \tan^{-1}\frac{y}{x}$$

$$|u| \equiv r \equiv |\bar{u}| = \sqrt{x^2 + y^2}$$

Laplace Transforms of Basic Functions

Name	$f(t)$		$F(s)$
Impulse	$f(t) = \delta(t)$		1
Step	$f(t) = u(t)$		$\frac{1}{s}$
Ramp	$f(t) = tu(t)$		$\frac{1}{s^2}$
Exponential	$f(t) = e^{at}u(t)$		$\frac{1}{s-a}$
Sine	$f(t) = \sin(\omega t)u(t)$		$\frac{1}{\omega^2 + s^2}$

Laplace Transform Properties

Addition/Scaling	$L[af_1(t) \pm bf_2(t)] = aF_1(s) \pm bF_2(s)$
Differentiation	$L\left[\dfrac{d}{dt} f(t)\right] = sF(s) - f(0\pm)$
Integration	$L\left[\int f(t)dt\right] = \dfrac{F(s)}{s} + \dfrac{1}{s}\left[\int f(t)dt\right]_{t=0\pm}$
Convolution	$\int_0^t f_1(t-\tau)f_2(\tau)d\tau = F_1(s)F_2(s)$
Initial-value theorem	$f(0+) = \lim_{s\to\infty} sF(s)$
Final-value theorem	$\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s)$

Laplace Transform Properties

Complex shift	e^{at} **f(t)**	**F(s-a)**
Real shift	**f(t - T)**	e^{Ts} **F(as)**
Scaling	**f(t/a)**	**a F(as)**

LAPLACE TRANSFORMS

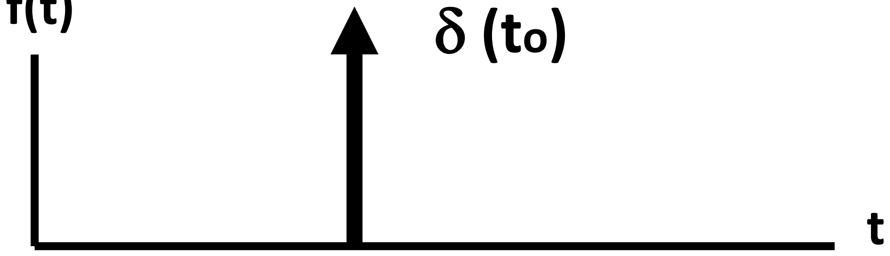
SIMPLE TRANSFORMATIONS

Transforms (1 of 11)

- Shifted Impulse: $f(t)=\delta\,(t-t_o)$

$$F(s) = \int_0^{\infty} e^{-st}\,\delta\,(t-t_o)\;dt$$

$$= e^{-st_o}$$

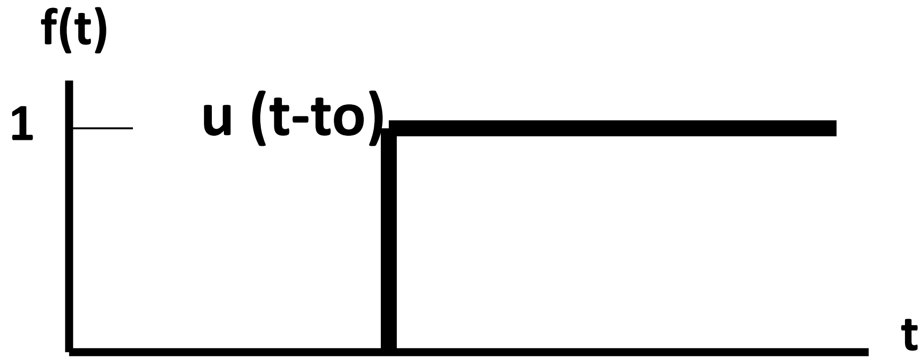


Transforms (2 of 11)

- Step: $f(t)=u\,(t-t_o)$

$$F(s) = \int_{0}^{\infty} e^{-st}\, u\,(t-t_o)\, dt$$

$$= e^{-st_o}/s$$



Transforms (3 of 11)

- Exponential f(t) = e^{-at} u(t)

$$\mathbf{F(s)} = \int_0^{\infty} e^{-st}\, e^{-at}\, dt$$

$$= 1/(s+a)$$

LAPLACE TRANSFORMS

PARTIAL FRACTION EXPANSION

Definition

- Definition: Partial fractions are several fractions whose sum equals a given fraction
- Purpose: Working with transforms requires breaking complex fractions into simpler fractions to allow use of tables of transforms

Partial Fraction Expansions

$$\frac{s+1}{(s+2)(s+3)} = \frac{A}{s+2} + \frac{B}{s+3}$$

$$\frac{s+1}{(s+2)(s+3)} = \frac{A(s+3) + B(s+2)}{(s+2)(s+3)}$$

$$A + B = 1 \qquad 3A + 2B = 1$$

$$\frac{s+1}{(s+2)(s+3)} = \frac{-1}{s+2} + \frac{2}{s+3}$$

- Expand into a term for each factor in the denominator.
- Recombine RHS
- Equate terms in s and constant terms. Solve.
- Each term is in a form so that inverse Laplace transforms can be applied.

Example of Solution of an ODE

$$\frac{d^2y}{dt^2} + 6\frac{dy}{dt} + 8y = 2 \quad y(0) = y'(0) = 0$$

- ODE w/initial conditions

$$s^2 Y(s) + 6sY(s) + 8Y(s) = 2/s$$

- Apply Laplace transform to each term
- Solve for Y(s)

$$Y(s) = \frac{2}{s(s+2)(s+4)}$$

- Apply partial fraction expansion

$$Y(s) = \frac{1}{4s} + \frac{-1}{2(s+2)} + \frac{1}{4(s+4)}$$

- Apply inverse Laplace transform to each term

$$y(t) = \frac{1}{4} - \frac{e^{-2t}}{2} + \frac{e^{-4t}}{4}$$

Laplace Transform Pairs

	$f(t)$	$F(s)$
1	Unit impulse $\delta(t)$	1
2	Unit step $1(t)$	$\frac{1}{s}$
3	t	$\frac{1}{s^2}$
4	$\frac{t^{n-1}}{(n-1)!} \quad (n = 1, 2, 3, \dots)$	$\frac{1}{s^n}$
5	$t^n \quad (n = 1, 2, 3, \dots)$	$\frac{n!}{s^{n+1}}$
6	e^{-at}	$\frac{1}{s+a}$
7	te^{-at}	$\frac{1}{(s+a)^2}$
8	$\frac{1}{(n-1)!} t^{n-1} e^{-at} \quad (n = 1, 2, 3, \dots)$	$\frac{1}{(s+a)^n}$
9	$t^n e^{-at} \quad (n = 1, 2, 3, \dots)$	$\frac{n!}{(s+a)^{n+1}}$
10	$\sin \omega t$	$\frac{\omega}{s^2 + \omega^2}$
11	$\cos \omega t$	$\frac{s}{s^2 + \omega^2}$
12	$\sinh \omega t$	$\frac{\omega}{s^2 - \omega^2}$
13	$\cosh \omega t$	$\frac{s}{s^2 - \omega^2}$
14	$\frac{1}{a}(1 - e^{-at})$	$\frac{1}{s(s+a)}$
15	$\frac{1}{b-a}(e^{-at} - e^{-bt})$	$\frac{1}{(s+a)(s+b)}$
16	$\frac{1}{b-a}(be^{-bt} - ae^{-at})$	$\frac{s}{(s+a)(s+b)}$
17	$\frac{1}{ab}\left[1 + \frac{1}{a-b}(be^{-at} - ae^{-bt})\right]$	$\frac{1}{s(s+a)(s+b)}$

Laplace Transform Pairs

18	$\frac{1}{a^2}(1 - e^{-at} - ate^{-at})$	$\frac{1}{s(s+a)^2}$
19	$\frac{1}{a^2}(at - 1 + e^{-at})$	$\frac{1}{s^2(s+a)}$
20	$e^{-at}\sin\omega t$	$\frac{\omega}{(s+a)^2+\omega^2}$
21	$e^{-at}\cos\omega t$	$\frac{s+a}{(s+a)^2+\omega^2}$
22	$\frac{\omega_n}{\sqrt{1-\zeta^2}}e^{-\zeta\omega_n t}\sin\omega_n\sqrt{1-\zeta^2}\,t \quad (0<\zeta<1)$	$\frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}$
23	$-\frac{1}{\sqrt{1-\zeta^2}}e^{-\zeta\omega_n t}\sin(\omega_n\sqrt{1-\zeta^2}\,t - \phi)$ $\phi = \tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta}$ $(0<\zeta<1,\quad 0<\phi<\pi/2)$	$\frac{s}{s^2+2\zeta\omega_n s+\omega_n^2}$
24	$1-\frac{1}{\sqrt{1-\zeta^2}}e^{-\zeta\omega_n t}\sin(\omega_n\sqrt{1-\zeta^2}\,t + \phi)$ $\phi = \tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta}$ $(0<\zeta<1,\quad 0<\phi<\pi/2)$	$\frac{\omega_n^2}{s(s^2+2\zeta\omega_n s+\omega_n^2)}$
25	$1-\cos\omega t$	$\frac{\omega^2}{s(s^2+\omega^2)}$
26	$\omega t - \sin\omega t$	$\frac{\omega^3}{s^2(s^2+\omega^2)}$
27	$\sin\omega t - \omega t\cos\omega t$	$\frac{2\omega^3}{(s^2+\omega^2)^2}$
28	$\frac{1}{2\omega}t\sin\omega t$	$\frac{s}{(s^2+\omega^2)^2}$
29	$t\cos\omega t$	$\frac{s^2-\omega^2}{(s^2+\omega^2)^2}$
30	$\frac{1}{\omega_2^2-\omega_1^2}(\cos\omega_1 t - \cos\omega_2 t) \quad (\omega_1^2 \neq \omega_2^2)$	$\frac{s}{(s^2+\omega_1^2)(s^2+\omega_2^2)}$
31	$\frac{1}{2\omega}(\sin\omega t + \omega t\cos\omega t)$	$\frac{s^2}{(s^2+\omega^2)^2}$

Properties of Laplace Transform

1	$\mathcal{L}[Af(t)] = AF(s)$
2	$\mathcal{L}[f_1(t) \pm f_2(t)] = F_1(s) \pm F_2(s)$
3	$\mathcal{L}_\pm\left[\frac{d}{dt}f(t)\right] = sF(s) - f(0\pm)$
4	$\mathcal{L}_\pm\left[\frac{d^2}{dt^2}f(t)\right] = s^2F(s) - sf(0\pm) - \dot{f}(0\pm)$
5	$\mathcal{L}_\pm\left[\frac{d^n}{dt^n}f(t)\right] = s^nF(s) - \sum_{k=1}^{n} s^{n-k}\overset{(k-1)}{f}(0\pm)$ where $\overset{(k-1)}{f}(t) = \frac{d^{k-1}}{dt^{k-1}}f(t)$
6	$\mathcal{L}_\pm\left[\int f(t)\,dt\right] = \frac{F(s)}{s} + \frac{1}{s}\left[\int f(t)\,dt\right]_{t=0\pm}$
7	$\mathcal{L}_\pm\left[\int\cdots\int f(t)(dt)^n\right] = \frac{F(s)}{s^n} + \sum_{k=1}^{n}\frac{1}{s^{n-k+1}}\left[\int\cdots\int f(t)(dt)^k\right]_{t=0\pm}$
8	$\mathcal{L}\left[\int_0^t f(t)\,dt\right] = \frac{F(s)}{s}$
9	$\int_0^\infty f(t)\,dt = \lim_{s\to 0} F(s) \qquad \text{if } \int_0^\infty f(t)\,dt \text{ exists}$
10	$\mathcal{L}[e^{-at}f(t)] = F(s+a)$
11	$\mathcal{L}[f(t-\alpha)1(t-\alpha)] = e^{-\alpha s}F(s) \qquad \alpha \ge 0$
12	$\mathcal{L}[tf(t)] = -\frac{dF(s)}{ds}$
13	$\mathcal{L}[t^2f(t)] = \frac{d^2}{ds^2}F(s)$
14	$\mathcal{L}[t^nf(t)] = (-1)^n\frac{d^n}{ds^n}F(s) \qquad (n = 1, 2, 3, \dots)$
15	$\mathcal{L}\left[\frac{1}{t}f(t)\right] = \int_s^\infty F(s)\,ds \qquad \text{if } \lim_{t\to 0}\frac{1}{t}f(t) \text{ exists}$
16	$\mathcal{L}\left[f\left(\frac{1}{a}\right)\right] = aF(as)$
17	$\mathcal{L}\left[\int_0^t f_1(t-\tau)f_2(\tau)\,d\tau\right] = F_1(s)F_2(s)$
18	$\mathcal{L}[f(t)g(t)] = \frac{1}{2\pi j}\int_{c-j\infty}^{c+j\infty} F(p)G(s-p)\,dp$