# MA204: Mathematics IV

## Partial Differential Equation (Fourier Transform)

# Fourier Series

The infinite series expansion

$$\frac{a_0}{2} + \sum_{n=1}^{\infty}\{a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\},$$

where

$$a_n = \frac{1}{L}\int_c^{c+L} f(x)\cos\frac{n\pi x}{L}dx$$

and

$$b_n = \frac{1}{L}\int_c^{c+L} f(x)\sin\frac{n\pi x}{L}dx$$

is called Fouries series expansion of $f(x)$ in $[c, c+L]$.

# Fourier Series

The term-by-term differentiation of a Fourier series is not always permissible.

**Theorem**

*Let $f(x) : \mathbb{R} \to \mathbb{R}$ be continuous and $f(x+2L) = f(x)$. Let $f'(x)$ and $f''(x)$ be piecewise continuous on $[-L, L]$. Then, The Fourier series of $f'(x)$ can be obtained from the Fourier series for $f(x)$ by termwise differentiation.*

# Fourier Series

Termwise integration of a Fourier series is permissible under much weaker conditions.

## Theorem

*Let $f(x) : [-L, L] \to \mathbb{R}$ be piecewise continuous function with Fourier series*

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\{a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\}.$$

*Then, for any $x \in [-L, L]$, we have*

$$\int_{-L}^{x} f(x)dx = \int_{-L}^{x}\{\frac{a_0}{2} + \sum_{n=1}^{\infty}\{a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\}\}dx.$$

# Fourier Series

The complex form of Fourier series expansion of a function $f(x)$ is given by

$$f(x) \equiv \sum_{-\infty}^{\infty} c_n e^{\frac{in\pi x}{L}},$$

where

$$c_n = \frac{1}{2}(a_n - ib_n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-\frac{in\pi x}{L}}dx$$

and

$$c_{-n} = \bar{c_n} = \frac{1}{2}(a_n + ib_n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{\frac{in\pi x}{L}}dx.$$

# Fourier Integral

Thus we have the expression

$$f(x) = \frac{1}{\pi}\int_0^\infty \int_{-\infty}^\infty f(t)\cos\{s(t-x)\}dtds,$$

called the Fourier integral representation of $f(x)$.

Another way of defining the Fourier integral of $f(x)$ is

$$f(t) = \int_{-\infty}^\infty \frac{1}{\sqrt{2\pi}} e^{-ist}\left\{\int_{-\infty}^\infty \frac{1}{\sqrt{2\pi}} f(\alpha)e^{is\alpha}d\alpha\right\}ds.$$

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(3) Solve the integral equation $\int_0^\infty F(x) \cos px dx = \begin{cases} 1 - p, & \text{if } 0 \le p \le 1; \\ 0, & \text{if } p > 1. \end{cases}$ Hence deduce that $\int_0^\infty \frac{\sin^2 t}{t^2} dt = \frac{\pi}{2}$.

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(4) Find the Fourier sine and cosine transform of $f(x) = \begin{cases} 1, & \text{if } 0 \leq x < 1; \\ 0, & \text{if } x > 1. \end{cases}$

# Fourier Transform

# Fourier Transform of derivatives

Under the assumption that $U$ and $\frac{\partial U}{\partial x}$ vanish as $x \to \pm\infty$, let $\bar{U}(s,t)$ be the Fourier transform of the function $U(x,t)$, then

(1) $\mathcal{F}\{U_x\} = is\bar{U}(s,t)$
(2) $\mathcal{F}\{U_{xx}\} = -s^2\bar{U}(s,t)$
(3) $\mathcal{F}\{U_t\} = \bar{U}'(s,t)$
(4) $\mathcal{F}\{U_{tt}\} = \bar{U}''(s,t)$

In addition, we have the following results

(1) $\mathcal{F}_s\{U_t\} = \bar{U}_s'$, $\mathcal{F}_s\{U_{tt}\} = \bar{U}_s''$, and $\mathcal{F}_s\{U_{xx}\} = \sqrt{\frac{2}{\pi}}\{sU(0,t) - s^2\bar{U}_s\}$

(2) $\mathcal{F}_c\{U_t\} = \bar{U}_c'$, $\mathcal{F}_c\{U_{tt}\} = \bar{U}_c''$, and

$$\mathcal{F}_c\{U_{xx}\} = \sqrt{\frac{2}{\pi}}\{-U_x(0,t) - s^2\bar{U}_c\}$$

# Fourier Transform

**Problem:** Solve the following PDEs using Fourier transforms:

(1) $u_t = c^2 u_{xx}$ for $t > 0$ with $u(x, 0) = f(x)$ if $|x| < a$ and 0 otherwise.

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# Fourier Transform

# Convolution

A convolution is an integral that expresses the amount of overlap of one function $f_2$ as it is shifted over another function $f_1$.

In other words, the output which produces a third function can be viewed as a modified version of one of the original functions.

## Definition

The convolution of two functions $f_1(t)$ and $f_2(t)$, $-\infty < t < \infty$, is defined as

$$f_1(t)*f_2(t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f_1(\tau)f_2(t-\tau)d\tau = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f_1(t-\zeta)f_2(\zeta)d\zeta = f_2(t)*f_1$$

provided the integral exists for each $t$.

# Convolution

## Theorem

*If $F(s)$ and $G(s)$ are the Fourier transforms of $f(x)$ and $g(x)$ respectively, then the Fourier transform of the convolution $f * g$ is $F(s)G(s)$.*

**Problem:** Find the convolution of $u(t) = \begin{cases} 1 - t, & \text{if } 0 \leq t < 1; \\ 0, & \text{otherwise}; \end{cases}$ and $v(t) = \begin{cases} e^{-t}, & \text{if } t \geq 0; \\ 0, & \text{if } t < 0. \end{cases}$ Moreover find $\mathcal{F}\{u(t) * v(t)\}$.

# Convolution

# Thank you

**Thank You!!**