MA204: Mathematics IV

Complex Analysis: Topology on Complex numbers

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Metric or distance function

Recall that the modulus or absolute value of complex numbers naturally extend the concept of absolute value of real numbers.

Using the absolute value of complex numbers, we define the metric or distance function on complex numbers $d : \mathbb{C} \times \mathbb{C} \to \mathbb{R}$ given by

$$d(z, w) = |z - w| = \sqrt{(a_1 - a_2)^2 + (b_1 - b_2)^2},$$

where $z = a_1 + ib_1$ and $w = a_2 + ib_2$.

Considering this metric for discussion on complex numbers, we introduce certain basic terms which shall be used throughout the course.

Terminologies

(a) **Open disc/Open ball/neighborhood:** Let $z_0 \in \mathbb{C}$ and $r > 0$ a real number, then the Open disc/Open ball/neighborhood centered at z_0 with radius r, denoted by $B_r(z_0)$, is defined as

$$B_r(z_0) := \{z \in \mathbb{C} : |z - z_0| < r\}.$$

If we remove the center z_0 from this neighborhood, then the resulted set is called a deleted neighborhood of z_0 with radius r. In other words, $B_r(z_0) - \{z_0\}$ is the **deleted neighborhood** of z_0 with radius r.

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Note that $\mathrm{Int}(S) \cup \mathrm{Bd}(S) \cup \mathrm{Ext}(S) = \mathbb{C}$.

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(b) **Connected Set and convex set:** An open set $S \subseteq \mathbb{C}$ is said to be (connected) (convex) if each pair of points z_1 and z_2 in S can be joined by a (polygonal line consisting of a finite number of line segments joined end to end) (straight line) that lies entirely in S.

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(g) **Closure of a Set:** The closure of a set $S \subseteq \mathbb{C}$, denoted by $\bar{S}$, defined by the set S together with all its limit points. Thus $\bar{S} = S \cup S'$.

Problem

Problem:

(1) Find interior points, exterior points, boundary points, limit points, and closure of the following sets. State if they are open, closed, bounded, connected, domain, and region

(a) $\{z \in \mathbb{C} : \mathrm{Re}(z) > 2\}$
(b) $\{z \in \mathbb{C} : \mathrm{Im}(z) \leq 1\}$
(c) $\{z \in \mathbb{C} : |z - 2 + 3i| > 2\}$
(d) $\{z \in \mathbb{C} : |z - 4| \leq |z|\}$
(e) $\{z \in \mathbb{C} : |z| < 1 \text{ or } |z - 2i| < 1\}$

(2) A finite subset of complex numbers can not have any limit point.

Properties

Properties:

(a) A set S is closed if and only if S^c is open.

(b) $\mathbb{C}$ and ϕ are both open and closed sets in $\mathbb{C}$.

(c) Finite intersection of open sets in $\mathbb{C}$ is open in $\mathbb{C}$.

(d) Arbitrary union of open sets in $\mathbb{C}$ is open in $\mathbb{C}$.

Thank You

Any Question!!!