MA204: Mathematics IV

Partial Differential Equation (Fourier Series)

Fourier Series

The theory of Fourier series had its historical origin in the middle of the eighteenth century, when several mathematicians were studying the vibrations of stretched strings.

For the case of a string stretched between the points $x = 0$ and $x = \pi$, Daniel Bernoulli (in 1753) gave the solution of the wave equation $u_{tt} = c^2 u_{xx}$ as a series of the form

$$u(x,t) = b_1 \sin x \cos ct + b_2 \sin 2x \cos 2ct + \dots .$$

Note that

(1) A typical term of this series, $b_n \sin nx \cos nct$, is a solution of the wave equation.

(2) Further, every finite sum of such terms is a solution.

(3) The series given by Bernoulli will also be a solution if term-by-term differentiation of the series is justified.

Fourier Series

When $t = 0$, the series of Bernoulli reduces to

$$u(x, 0) = b_1 \sin x + b_2 \sin 2x + \dots$$

If the initial condition is $u = u(x, 0) = \phi(x)$, then we should have

$$\phi(x) = b_1 \sin x + b_2 \sin 2x + \dots .$$

Analysing the the shapes and motion of the string at different positions, Bernoulli arrived at an idea that has had very far-reaching influence on the history of mathematics and physical science, namely, the possibility that any function can be expanded in a trigonometric series.

However, D'Alembert (in 1747) and Euler (in 1748) rejected Bernoulli's idea, and for essentially the same reason.

Fourier Series

The controversy bubbled on for many years, and in the absence of mathematical proofs, no one could convince anyone else to his way of thinking.

In 1807 the French physicist-mathematician Fourier announced in this connection that an arbitrary function $f(x)$ can be represented in the form of trigonometric series.

He supplied no proofs, but instead heaped up the evidence of many solved problems and many convincing specific expansions-so many, indeed, that the mathematicians of the time began to spend more effort on proving, rather than disproving, his conjecture.

The first major result of this shift in the winds of opinion was the classical paper of Dirichlet in 1829, in which he proved with full mathematical rigor that such series exists.

Fourier Series

Before going to details about Fourier transfer, we here state some basic orthogonality properties of certain trigonometric functions.

Definition

A set of functions $\{f_n(x)\}_{n=1}^{\infty}$ is said to be an orthogonal set on the interval $[a, b]$ if

$$\int_a^b f_n(x) f_m(x) dx = 0, \text{ for } m \neq n.$$

If $m, n = 1, 2, \ldots$, then we have the following results:

(1) $\int_{-L}^{L} \cos \frac{n\pi x}{L} = 0$ and $\int_{-L}^{L} \sin \frac{n\pi x}{L} = 0$

(2) $\int_{-L}^{L} \sin \frac{m\pi x}{L} \cos \frac{n\pi x}{L} = 0$

(3) $\int_{-L}^{L} \sin \frac{m\pi x}{L} \sin \frac{n\pi x}{L} = \begin{cases} 0, & \text{if } m \neq n; \\ L, & \text{if } m = n. \end{cases}$

(4) $\int_{-L}^{L} \cos \frac{m\pi x}{L} \cos \frac{n\pi x}{L} = \begin{cases} 0, & \text{if } m \neq n; \\ L, & \text{if } m = n. \end{cases}$

Fourier Series

For an orthogonal set $\{f_n(x)\}_{n=1}^{\infty}$ in $[a, b]$ and a given function $f(x)$, if we have

$$f(x) = c_1 f_1(x) + c_2 f_2(x) + c_3 f_3(x) + \ldots,$$

then

$$cn = \frac{\int_a^b f(x) f_n(x) dx}{\int_a^b f_n(x)^2 dx} \text{ for } n = 1, 2, \ldots.$$

Since the set of trigonometric functions

$$\{1, \cos x, \sin x, \cos 2x, \sin 2x, \ldots\}$$

is orthogonal on $[-\pi, \pi]$, and we have

$$f(x) = c_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx),$$

then all coefficients c_0, a_n and b_n can be determined uniquely.

Fourier Series

An infinite series of the form

$$\frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx),$$

where

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \cos nx dx, n = 0, 1, 2, \ldots,$$

and

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \sin nx dx, n = 1, 2, \ldots,$$

is called a Fourier Series in $[-\pi, \pi]$.

Terms a_n and b_n are called Fourier coefficients.

Fourier Series

An infinite series of the form

$$\frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}),$$

where

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}dx, n = 0, 1, 2, \ldots,$$

and

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}dx, n = 1, 2, \ldots,$$

is called a Fourier Series in $[-L, L]$.

Note: If a function $f(x)$ is piecewise continuous, periodic with period $2L$, and has a finite number of relative maxima and minima in $[-L, L]$, then $f(x)$ has a Fourier series representation for every x in $[-L, L]$.

Fourier Series

Problem: Find Fourier series expansion for the following functions in the given region.

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(2) $f(x) = \begin{cases} 1, & \text{if } x > 0; \\ -1, & \text{if } x < 0; \end{cases}$ and region $[-\pi, \pi]$.

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(2) $f(x) = \begin{cases} 1, & \text{if } x > 0; \\ -1, & \text{if } x < 0; \end{cases}$ and region $[-\pi, \pi]$.

If $f(x)$ is an odd function in $[-L, L]$, then the Fourier series expansion of $f(x)$ has only sine terms.

If $f(x)$ is an even function in $[-L, L]$, then the Fourier series expansion of $f(x)$ has only cosine terms.

Thank you

Thank You!!