MA204: Mathematics IV

Complex Analysis: Introduction and basic terminologies

Gautam Kalita
IIIT Guwahati

Introduction

For the set of real numbers $\mathbb{R}$, we have the following properties:

(a) $\mathbb{R}$ is group with respect to addition. $\mathbb{R}^{\times}$ is a group with respect to multiplication. In fact, $\mathbb{R}$ is a field.

(b) $\mathbb{R}$ is a one dimensional vector space over $\mathbb{R}$.

(c) $\mathbb{R}$ is ordered complete.

However, the algebraic structure of $\mathbb{R}$ has some limitations.

The quadratic equation $x^2 + 1 = 0$ does not have any root in real numbers. Let i (iota) be a root of the equation.

To address this issue of real numbers, the complex number system is introduced.

Introduction

Definition

The smallest field containing the real field and i is called the complex field. We denote this field by $\mathbb{C}$.

Note that

$$\mathbb{C} \equiv \mathbb{R}[x]/\langle x^2+1\rangle \equiv \mathbb{R}[i] = \{a+ib : a, b \in \mathbb{R}\},$$

where $\langle x^2+1\rangle = \{a(x)(x^2+1) : a(x) \in \mathbb{R}[x]\}$ with $\mathbb{R}[x]$ being the set of all polynomials over $\mathbb{R}$, is the set of all residues when a polynomial over $\mathbb{R}$ is divided by x^2+1.

It is easy to see that

(a) $\mathbb{C}$ is group with respect to addition. $\mathbb{C}^\times$ is a group with respect to multiplication. In fact, $\mathbb{C}$ is a field.

(b) $\mathbb{C}$ is a two dimensional vector space over $\mathbb{R}$.

Complex numbers

Thus we can write

$$\mathbb{C} = \{a + ib : a, b \in \mathbb{R} \text{ and } i^2 + 1 = 0\}.$$

Any element $z = a + ib \in \mathbb{C}$ is called a complex number with a and b being the real and imaginary parts of z, respectively. We denote

$$\mathrm{Re}(z) = a \text{ and } \mathrm{Im}(z) = b.$$

Operations: Let $z_1 = a_1 + ib_1$ and $z_2 = a_2 + ib_2$. Then

(1) Equality: $z_1 = z_2$ if and only if $a_1 = a_2$ and $b_1 = b_2$.

(2) Addition and subtraction: $z_1 \pm z_2 = (a_1 \pm a_2) + i(b_1 \pm b_2)$.

(3) Multiplication and division:

$$z_1 z_2 = (a_1 a_2 - b_1 b_2) + i(a_1 b_2 + a_2 b_1)$$

and

$$\frac{z_1}{z_2} = \frac{a_1 + ib_1}{a_2 + ib_2} = \frac{a_1 a_2 + b_1 b_2}{a_2^2 + b_2^2} + i\frac{a_2 b_1 - a_1 b_2}{a_2^2 + b_2^2}.$$

Denoting 1 by $(1, 0)$ and i by $(0, 1)$, one can identify $\mathbb{C}$ with $\mathbb{R}^2$. As a result, a complex number $z = a + ib$ can be identified as an ordered pair (a, b) in the complex plan or Argand plan.

Question: Are $\mathbb{C}$ and $\mathbb{R}^2$ same?

Properties

Properties:

(a) Addition and multiplication of complex numbers is commutative.
(b) Addition and multiplication of complex numbers is associative.
(c) Multiplication of complex numbers is distributive over addition of complex numbers.
(d) Additive identity and multiplicative identity exist in complex numbers.
(e) Additive inverse for all complex numbers and multiplicative inverse for non-zero complex numbers exist.

Exercise:

(a) $\text{Re}(iz) = -\text{Im}(z)$ and $\text{Im}(iz) = \text{Re}(z)$.
(b) Solve $z^2 - 2z + 2 = 0$ directly as well as converting to Cartesian coordinates.
(c) Express $\frac{5i}{(1-i)(2-i)(3-i)}$ in the $a + ib$ form.

Conjugate of a complex number

Let $z = a + ib$ be a complex number, then the complex number $a - ib$, denoted by $\bar{z} = a - ib$, is called the conjugate of z.

The conjugate of a complex number has the following properties:

(a) $\overline{(z_1 \pm z_2)} = \bar{z}_1 \pm \bar{z}_2$.

(b) $\overline{z_1 z_2} = \bar{z}_1 \bar{z}_2$ and $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\bar{z}_1}{\bar{z}_2}$.

(c) $\bar{\bar{z}} = z$ and $\overline{\alpha z} = \alpha \bar{z}$.

(d) $\mathrm{Re}(\bar{z}) = \mathrm{Re}(z)$ and $\mathrm{Im}(\bar{z}) = -\mathrm{Im}(z)$.

(e) $\mathrm{Re}(z) = \frac{z+\bar{z}}{2}$ and $\mathrm{Im}(z) = \frac{z-\bar{z}}{2i}$.

Geometrically, the conjugate of a complex number is reflection of the complex number along the real axis.

Modulus

Let $z = a + ib$ be a complex number. The modulus or absolute value of the complex number z, denoted by $|z|$, is a non-negative real number defined as

$$|z| = \sqrt{a^2 + b^2}.$$

Note that $|z|$ is identical to the usual distance of (a, b) from the origin in the Euclidean space. Thus we can define a metric or distance function $d : \mathbb{C} \times \mathbb{C} \to \mathbb{R}$ on $\mathbb{C}$ as

$$d(z, w) = |z - w|.$$

As a result, the topological notions of $\mathbb{C}$ coincide with those of $\mathbb{R}^2$.

Properties:

(a) $|z| = |\bar{z}|$ and $|z|^2 = z\bar{z}$.

(b) $|z_1 z_2| = |z_1||z_2|$ and $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$ if $z_2 \neq 0$.

(c) $|z_1 + z_2| \leq |z_1| + |z_2|$ and $||z_1| - |z_2|| \leq |z_1 - z_2|$.

Modulus

Exercise:

(a) Find modulus and argument for the complex numbers $z = \frac{-1+3i}{2-i}$ and $z = (1-i)(2i-3)$.

(b) Describe the geometrical figures satisfying $|z-4i| + |z+4i| = 10$, $z^2 + \bar{z}^2 = 2$, $\mathrm{Re}(\bar{z} - i) = 2$, and $|z-1+i| < 1$.

(c) Show that $\sqrt{2}|z| \geq |\mathrm{Re}(z)| + |\mathrm{Im}(z)|$.

(d) If z lies on the circle $|z| = 2$, then show that $|\frac{1}{z^4 - 4z^2 + 3}| \leq \frac{1}{3}$.

Argument

Let $\vec{r}$ denotes the position vector of a complex number $z = a + ib$ in the Argand plan, then the angles made by $\vec{r}$ with the positive direction of the real axis are called argument of the complex number z, denoted by $\arg(z)$. Note that $|\vec{r}| = r = |z|$.

The value of $\arg(z)$ which satisfies the condition $-\pi < \arg(z) \leq \pi$ is called principal argument of z, and it is denoted by $\mathrm{Arg}(z)$.

For the complex number $z = a + ib$, we have $\mathrm{Arg}(z) = \tan^{-1}(b/a)$ or $\tan^{-1}(b/a) + \pi$ or $\tan^{-1}(b/a) - \pi$.

Moreover $\arg(z) = \mathrm{Arg}(z) \pm 2n\pi$ for all $n \in \mathbb{Z}$.

Polar representation

For any non zero complex number z, it is easy to see that $\frac{z}{|z|}$ lies on an unit circle centered at the origin.

Any point on an unit circle centered at the origin is represented by $(\cos\theta, \sin\theta) = \cos\theta + i\sin\theta$ for some $\theta \in [0, 2\pi]$.

As a result, we must have

$$\frac{z}{|z|} = \cos\theta + i\sin\theta,$$

where $\theta =$arg$\left(\frac{z}{|z|}\right) =arg(z)$. Thus

$$z = r\cos\theta + ir\sin\theta,$$

where $r = |z|$ and $\theta =$arg(z). This expression is called polar representation of z.

Exponential form

The famous Euler's formula states that

$$e^{i\theta} = \cos\theta + i\sin\theta,$$

and hence

$$z = re^{i\theta},$$

where $r = |z|$ and $\theta = \arg(z)$. This is called exponential form of z.

Properties:

(a) $\arg(z_1 z_2) = \arg(z_1) + \arg(z_2)$ and $\arg\left(\frac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2)$

(b) $z^n = \{r\cos\theta + ir\sin\theta\}^n = r^n\cos(n\theta) + ir^n\sin(n\theta)$. [De Moiver's Theorem]

Exercise:

(a) Express $-1+i$, $-1-i\sqrt{3}$, $(1+i)^7$, and i in the polar and exponential form.

(b) Prove/disprove $\text{Arg}(z_1 z_2) = \text{Arg}(z_1) + \text{Arg}(z_2)$ and $\text{Arg}\left(\frac{z_1}{z_2}\right) = \text{Arg}(z_1) - \text{Arg}(z_2)$.

Roots of complex numbers

Given a nonzero complex number z_0 and a natural number $n \in \mathbb{N}$, find all distinct complex numbers w such that $z_0 = w^n$ or $w = z_0^{\frac{1}{n}}$.

Note that $z_0 = |z_0|(\cos\theta + i\sin\theta) = |z_0|(\cos(\theta + 2k\pi) + i\sin(\theta + 2k\pi))$ for any $k \in \mathbb{Z}$.

As a result

$$\begin{aligned} w = z_0^{\frac{1}{n}} &= \{|z_0|(\cos(\theta + 2k\pi) + i\sin(\theta + 2k\pi))\}^{\frac{1}{n}} \\ &= |z_0|^{\frac{1}{n}} \left\{\cos\left(\frac{\theta + 2k\pi}{n}\right) + i\sin\left(\frac{\theta + 2k\pi}{n}\right)\right\} \text{ for } k \in \mathbb{Z}. \end{aligned}$$

Finally, the distinct values of w are obtained for $k = 0, 1, 2, \ldots (n-1)$.

Exercise:

(a) Find values of $i^{\frac{1}{4}}$, $(-1)^{\frac{1}{6}}$, and $(1 - \sqrt{3})^{\frac{1}{3}}$. Represent them geometrically.

Note: The values of w satisfying $z_0 = w^n$ or $w = z_0^{\frac{1}{n}}$ form vertices of a regular n-polygon inscribed on a circle of radius $|z_0|^{\frac{1}{n}}$ and centered at the origin.

Thank You

Any Question!!!