MA204: Mathematics IV

Partial Differential Equation (Boundary and Initial Value Problem)

Boundary and Initial Value Problem

If we denote a region by Ω, typically it is assumed to be an open, connected set with some piecewise smooth boundary $\partial\Omega$.

A boundary condition is then an additional equation that specifies the value of z and/or some of its derivatives on the set $\partial\Omega$.

An initial condition, on the other hand, specifies the value of z and some of its derivatives at some initial time t_0 (often $t_0 = 0$).

Consider the 1D wave equation $u_{tt} = c^2 u_{xx}$ on the region $0 < x < L, 0 < t$ with the boundary conditions

$$u(0,t) = 0, u(L,t) = 0$$

and initial conditions

$$u(x,0) = f(x) \text{ and } u_t(x,0) = g(x).$$

The need of boundary and initial condition for a PDE is to force the solution of the PDE to be unique and well-behaved.

Boundary and initial value problem

We often hear three types of boundary and initial conditions for problems related to physical situations.

(1) **Cauchy conditions:** In this case, for the PDF, the value of the solution z and its normal derivatives are specified along some smooth surface S in the coordinate space of all independent variables. Thus to get a well-posed[1] problem under this condition,

(a) if z is a function of n variables, then the surface S should have dimension $n-1$

(b) if the PDE is of order k, then z and its first $k-1$ normal derivatives must be specified along the normal to S.

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(2) **Neumann condition:** In Neumann condition, the value of the derivatives of z along the normal to surface S are specified.

Problems with Cauchy conditions

Theorem (Existence and Uniqueness Theorem)

If the functions $F, f_0, f_1, \ldots, f_{k-1}$ are analytic near the origin, then there is a neighbourhood of the origin where the following Cauchy initial value problem

$$\frac{\partial^k z}{\partial t^k}(x, y, z, t) = F(x, y, z, t, z_x, z_y, \ldots)$$

with

$$\frac{\partial^j z}{\partial t^j}(x, y, z, 0) = f_j(x, y, z) \text{ for } 0 \leq j < k$$

has a unique analytic solution $z = z(x, y, z, t)$.

Problems with Dirichlet and Neumann conditions

Theorem (Existence and Uniqueness Theorem for Dirichlet condition)

Suppose Ω is an open, bounded, connected region with smooth boundary $\partial\Omega$. Then the Dirichlet problem

$$\nabla^2 u = 0 \ in \ \Omega$$

with

$$u = f \ on \ \partial\Omega$$

has a unique solution for each continuous function f on $\partial\Omega$.

Theorem (Existence and Uniqueness Theorem for Neumann condition)

The Neumann problem

$$\nabla^2 u = 0 \ in \ \Omega$$

with

$$u_\eta = f \ on \ \partial\Omega$$

has a solution for each continuous function f if and only if $\int_{\partial\Omega} f = 0$. In this case, the solution is unique up to an additive constant.

Heat Equation

The general form of a heat equation is

$$u_t = k\nabla^2 u + r,$$

where k is normalized conductivity called thermal diffusibility and r is source term.

A important class of solution of the heat equation are the steady-state solutions. In this, u is considered to be independent of t, i.e., $\frac{\partial u}{\partial t} = 0$.

Thus we have the heat equation in the Poisson's form

$$k\nabla^2 u + r = 0.$$

In addition, if $r = 0$, then u satisfies the Laplace equation

$$\nabla^2 u = 0.$$

Heat Equation

Consider the 1D heat equation $u_t = u_{xx}$ in the positive quadrant $x, t > 0$ under the conditions $u(x, 0) = 0$ and $u(0, t) = 0$.

Problem

Ex: Find the solution of the Dirichlet problem $\nabla^2 u = 0$ with $u(x, b) = u(a, y) = 0, u(0, y) = 0, u(x, 0) = f(x)$.

Problem

Ex: Find the solution of the Neumann problem $\nabla^2 u = 0$ with $u_x(a, y) = u_x(0, y) = 0, u_y(x, 0) = 0, u_y(x, b) = f(x)$.

Thank you

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