Control Systems

Subject Code: EC380

Transient & Steady State Response Analysis

Surajit Panja
Associate Professor
Dept. of Electronics and Communication Engineering

Indian Institute of Information Technology Guwahati
Bongora, Guwahati-781015

Introduction

The time response of a control system consists of two parts:

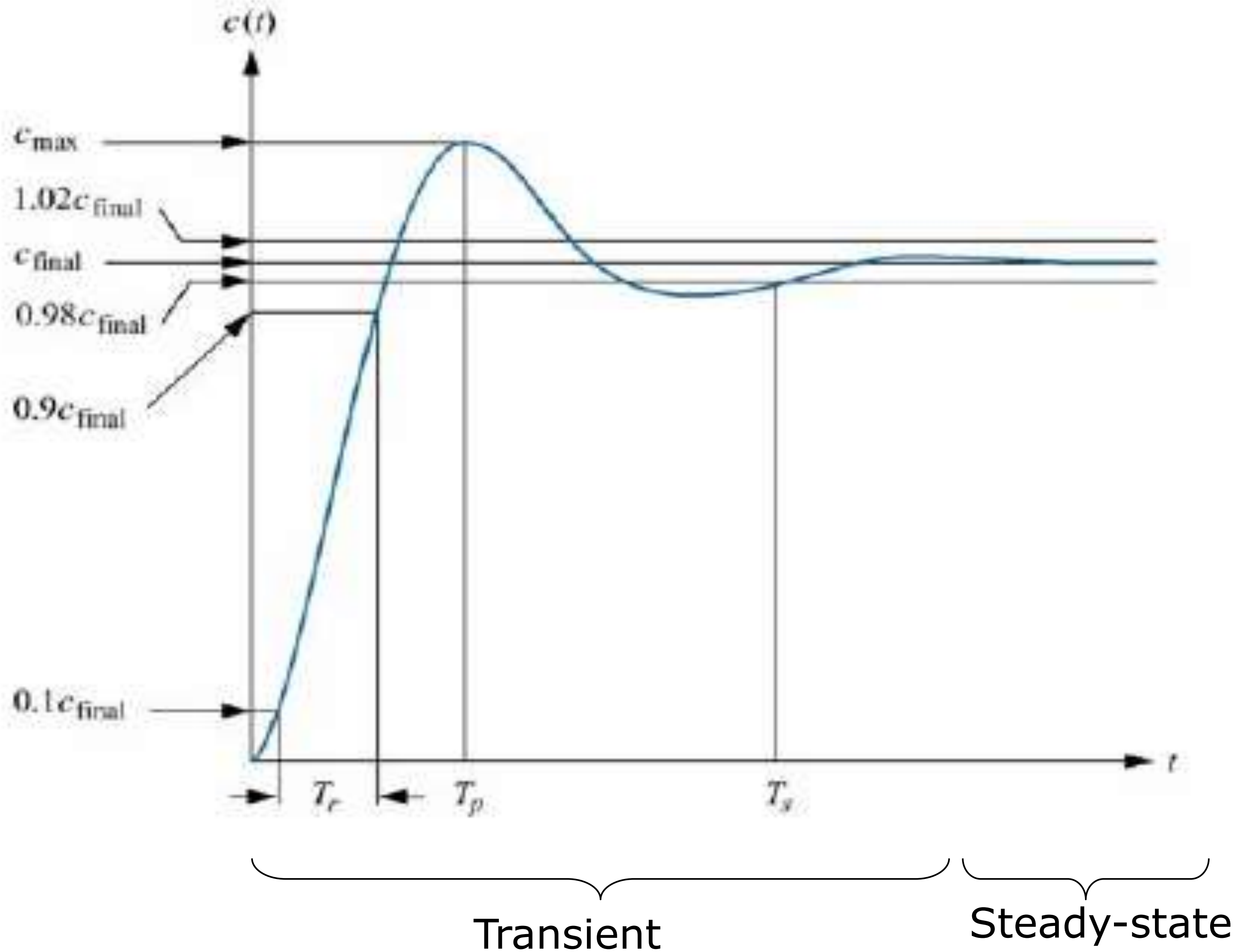
1. Transient response

   - from initial state to the final state – purpose of control systems is to provide a desired response.

2. Steady-state response

   - the manner in which the system output behaves as *t* approaches infinity – the error after the transient response has decayed, leaving only the continuous response.

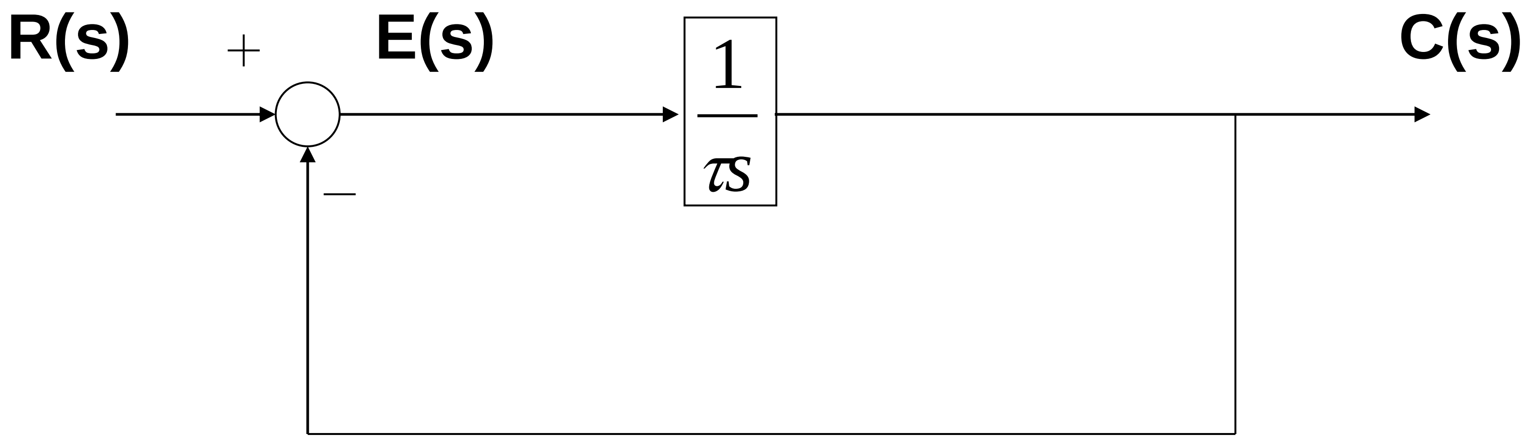
Introduction



Performances of Control Systems

- **Specifications (time domain)**
  - Max OS, settling time, rise time, peak time,
- **Standard input signals used in design**
  - **actual signals unknown**
  - **standard test signals:**
    - **step, ramp, parabola, impulse, etc. sinusoid**
      - **(study freq. response later)**
- **Transient response**
- **Steady-state response**
- **Relate to locations of poles and zeros**

First Order System



Test signal is step function, R(s)=1/s

First – order system

A first-order system without zeros can be represented by the following transfer function

$$\frac{C(s)}{R(s)} = \frac{1}{\tau s + 1}$$

• Given a step input, i.e., $R(s) = 1/s$, then the system output (called **step response** in this case) is

$$C(s) = \frac{1}{\tau s + 1} R(s) = \frac{1}{s(\tau s + 1)} = \frac{1}{s} - \frac{1}{s + \frac{1}{\tau}}$$

First – order system

Taking inverse Laplace transform, we have the step response

$$\boxed{c(t) = 1 - e^{-\frac{t}{\tau}}}$$

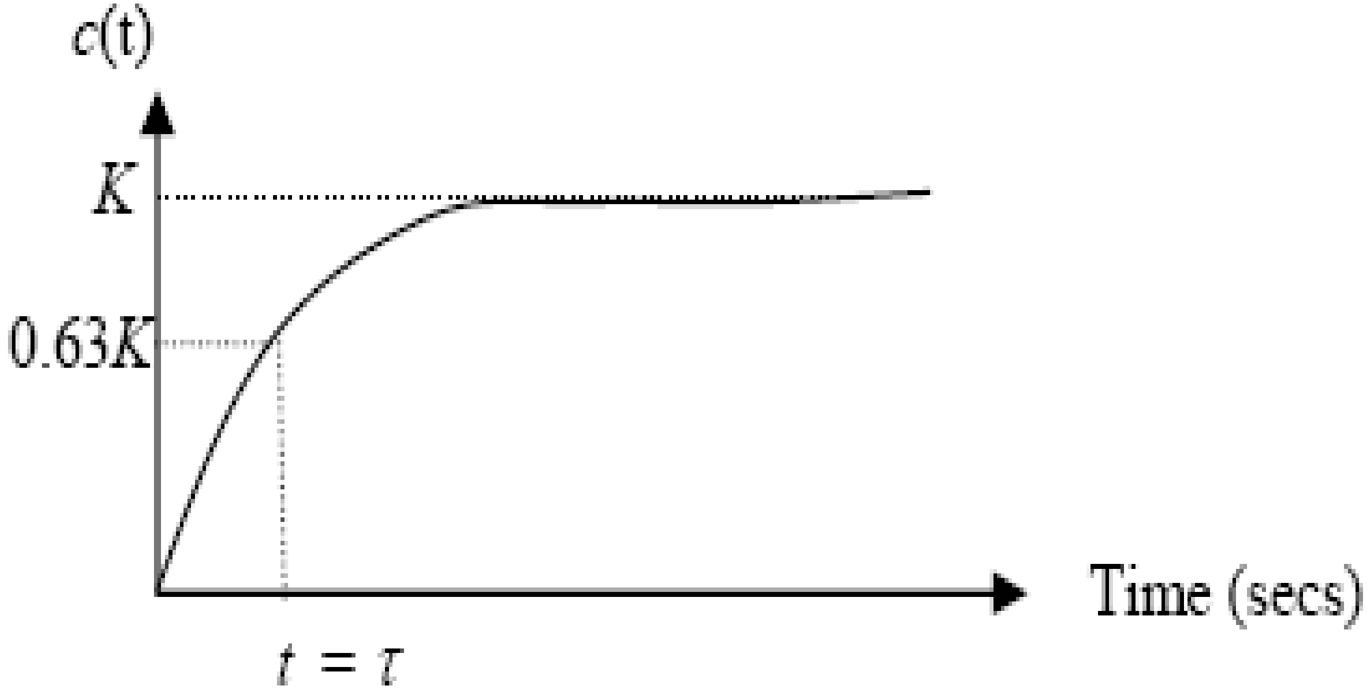
Time Constant: If t= τ , So the step response is

$C(\tau) = (1 - 0.37) = 0.63$

τ is referred to as the **time constant** of the response. In other words, the time constant is the time it takes for the step response to rise to 63% of its final value. Because of this, the time constant is used to measure how fast a system can respond. The time constant has a unit of seconds.

First – order system

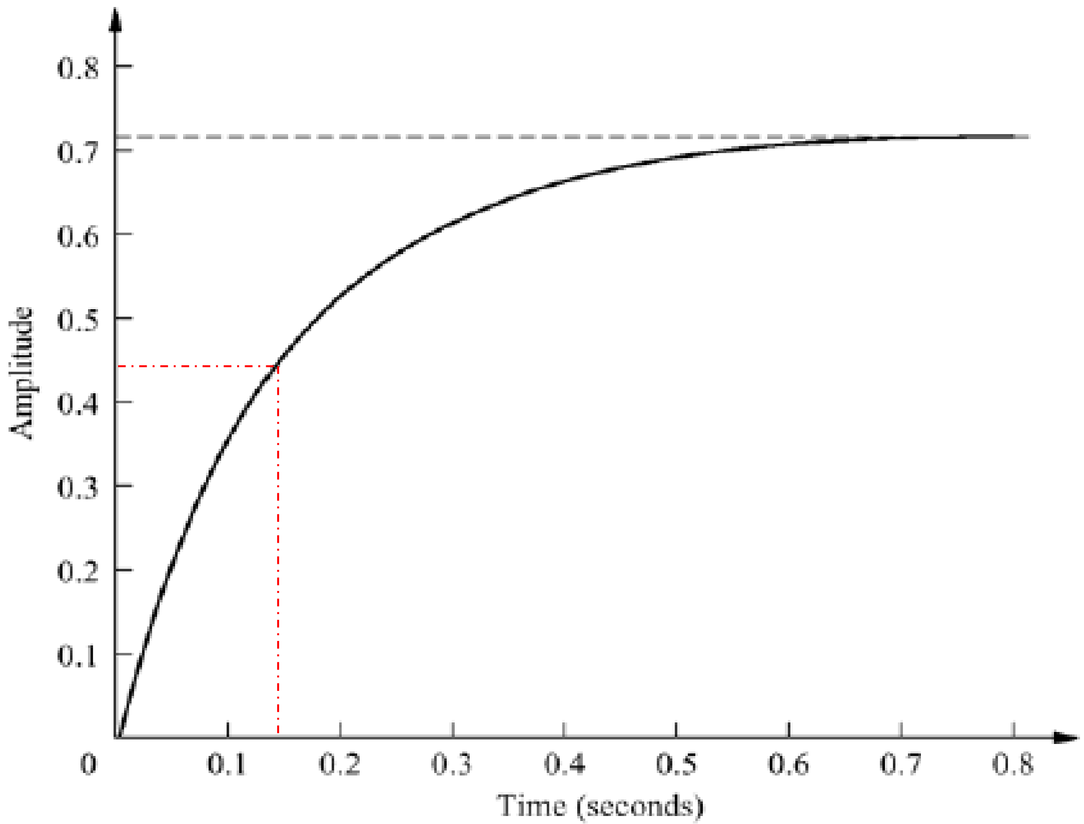
Plot $c(t)$ versus time:



First – order system

Example 1

The following figure gives the measurements of the step response of a first-order system, find the transfer function of the system.



First – order system
Transient Response Analysis

Rise Time T_r:

The rise-time (symbol T_r units s) is defined as the time taken for the step response to go from **10% to 90%** of the final value.

$$\boxed{T_r = 2.31\tau - 0.11\tau = 2.2\tau}$$

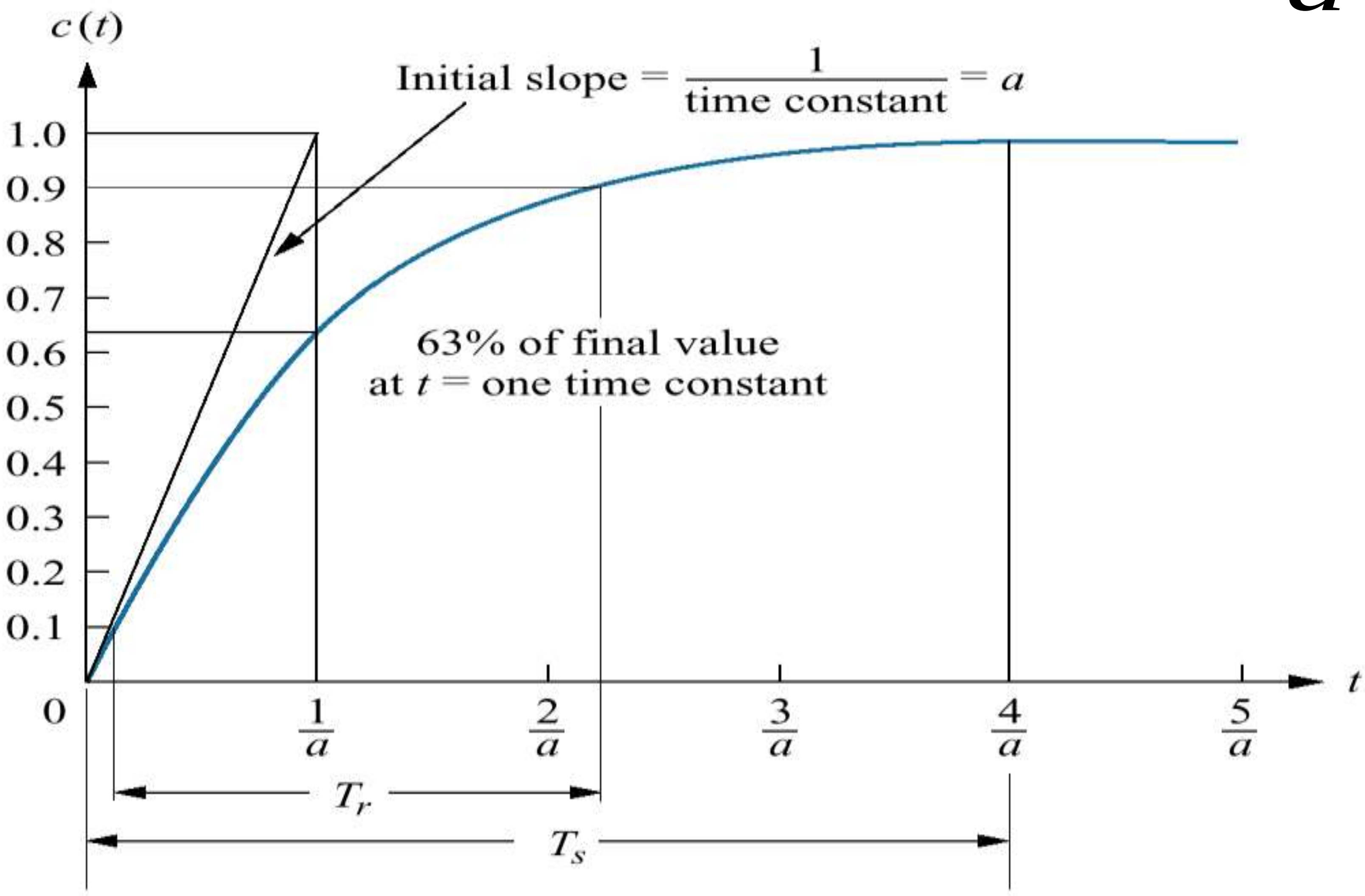
Settling Time T_s:

Defined the settling-time (symbol T_s units s) to be the time taken for the step response to come to within **2% of the final value** of the step response.

$$\boxed{T_s = 4\tau}$$

First – order system

$$\tau = \frac{1}{a}$$



Second – Order System

- *Second-order systems* exhibit a wide range of responses which must be analyzed and described.
  - Whereas for a *first-order system*, varying a single parameter changes the speed of response, changes in the parameters of a *second order system* can change the form of the response.

- *For example*: a second-order system can display characteristics much like a first-order system or, depending on component values, display damped or *pure oscillations* for its *transient response*.

Second – Order System

- A general second-order system is characterized by the following transfer function:

$$G(s) = \frac{b}{s^2 + as + b}$$

- We can re-write the above transfer function in the following form (closed loop transfer function):

$$\boxed{G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}}$$

Second – Order System

$\omega_n \quad (\omega_n = \sqrt{b})$ - referred to as *the un-damped natural frequency* of the second order system, which is the frequency of oscillation of the system without damping.

$\zeta \ (\zeta = \dfrac{a}{2\sqrt{b}})$ - referred to as *the damping ratio* of the second order system, which is a measure of the degree of resistance to change in the system output.

Poles;

$$-\omega_n\zeta + \omega_n\sqrt{\zeta^2 - 1}$$

$$-\omega_n\zeta - \omega_n\sqrt{\zeta^2 - 1}$$

Poles are complex if $\zeta < 1$!

Second – Order System

- According the value of ζ, a second-order system can be set into one of the four categories:

1. *Overdamped* - when the system has two real distinct poles ($\zeta > 1$).
2. *Underdamped* - when the system has two complex conjugate poles ($0 < \zeta < 1$)
3. *Undamped* - when the system has two imaginary poles ($\zeta = 0$).
4. *Critically damped* - when the system has two real but equal poles ($\zeta = 1$).

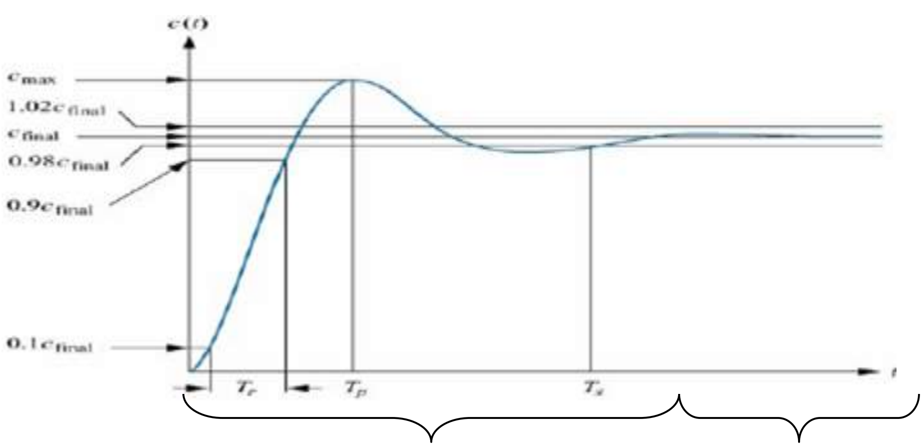
Time-Domain Specification

Given that the closed loop TF

$$T(s) = \frac{C(s)}{R(s)} = \frac{\omega_n^{\,2}}{s^2 + 2\varsigma\omega_n s + \omega_n^{\,2}}$$

The system (2nd order system) is parameterized by ς and ω_n

For 0< ς <1 and $\omega_n > 0$, we like to investigate its response due to a unit step input



Transient Steady State

Two types of responses that are of interest:
(A)Transient response
(B)Steady state response

(A) For transient response, we have 4 specifications:

(a) T_r – rise time = $\dfrac{\pi-\theta}{\omega_n\sqrt{1-\varsigma^2}}$

(b) T_p – peak time = $\dfrac{\pi}{\omega_n\sqrt{1-\varsigma^2}}$

(c) %MP – percentage maximum overshoot = $e^{-\frac{\pi\varsigma}{\sqrt{1-\varsigma^2}}} x100\%$

(d) T_s – settling time (2% error) = $\dfrac{4}{\varsigma\omega_n}$

(B) Steady State Response

(a) Steady State error

Question : How are the performance related to ς and ω_n ?

- Given a step input, i.e., R(s) =1/s, then the system output (or step response) is;

$$C(s) = R(s)G(s) = \frac{\omega_n^2}{s(s^2 + 2\zeta\omega_n s + \omega_n^2)}$$

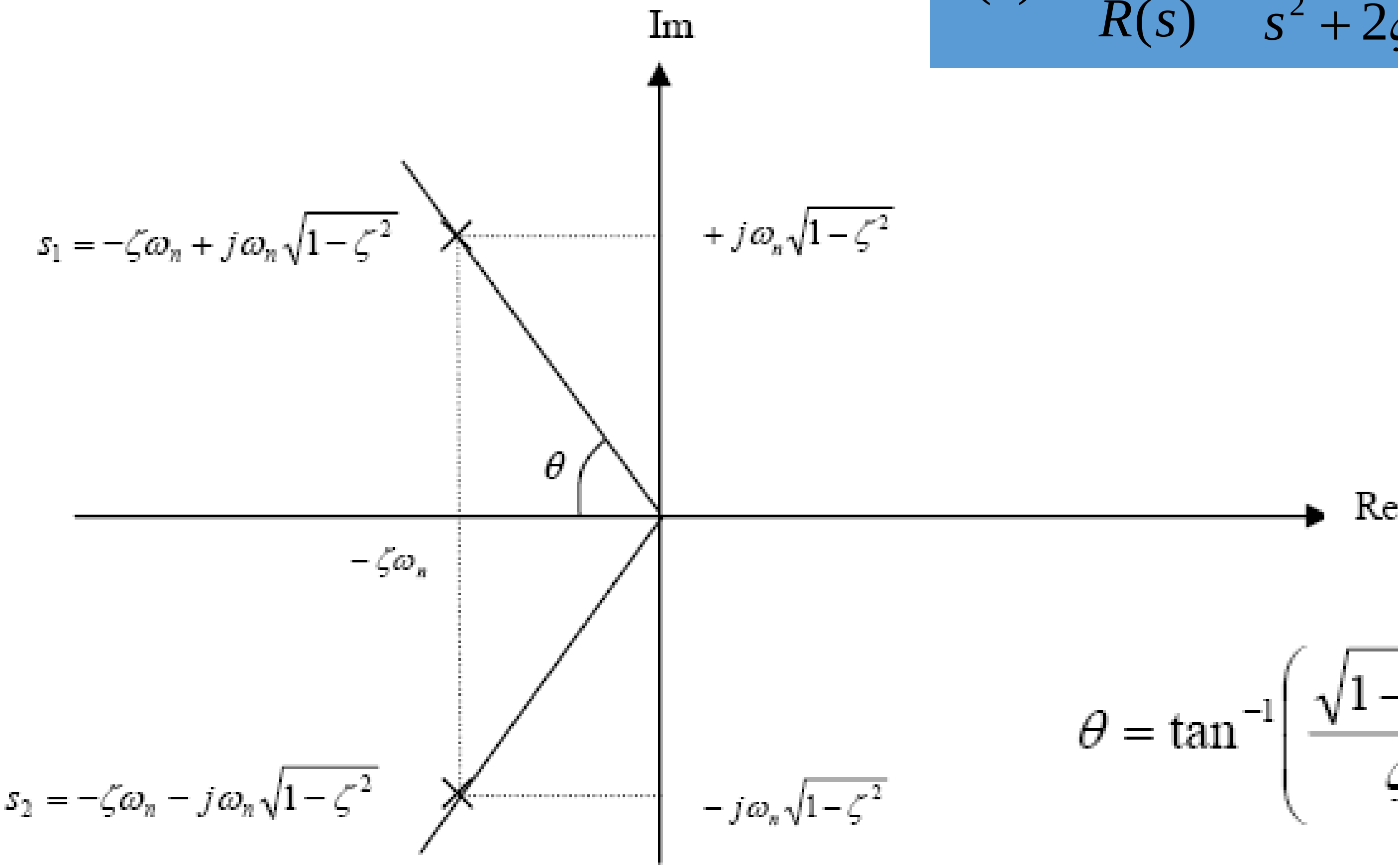
- Taking inverse Laplace transform, we have the step response;

$$c(t) = 1 - \frac{1}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} \sin\left(\omega_n \sqrt{1-\zeta^2}\, t + \theta\right)$$

Where; $\theta = \tan^{-1}\left(\frac{\sqrt{1-\zeta^2}}{\zeta}\right)$. or $\boldsymbol{\theta = \cos^{-1}(\xi)}$

Second – Order System

$$T(s)=\frac{C(s)}{R(s)}=\frac{\omega_n{}^2}{s^2+2\varsigma\omega_n s+\omega_n{}^2}$$



$$\theta=\tan^{-1}\left(\frac{\sqrt{1-\zeta^2}}{\zeta}\right);$$

Mapping the poles into s-plane

Lets re-write the equation for c(t):

Let: $$\beta = \sqrt{1-\xi^2}$$

and

$$\omega_d = \omega_n \sqrt{1-\xi^2}$$

Damped natural frequency

$$\omega_n > \omega_d$$

Thus:

$$c(t) = 1 - \frac{1}{\beta} e^{-\xi\omega_n t} \sin\left(\omega_d t + \theta\right)$$

where $$\theta = \cos^{-1}(\xi)$$

Transient Response Analysis

1) Rise time, T_r. Time the response takes to rise from 0 to 100%

$$c(t)\Big|_{t=T_r} = 1 - \underbrace{\frac{1}{\beta} e^{-\xi\omega_n t}}_{\neq 0} \underbrace{\sin\left(\omega_d t + \theta\right)}_{=0} = 1$$

$$\sin(\omega_d T_r + \theta) = 0$$

$$\omega_d T_r + \theta = \sin^{-1}(0) = \pi$$

$$\boxed{T_r = \frac{\pi - \theta}{\omega_n\sqrt{1-\xi^2}}}$$

Transient Response Analysis

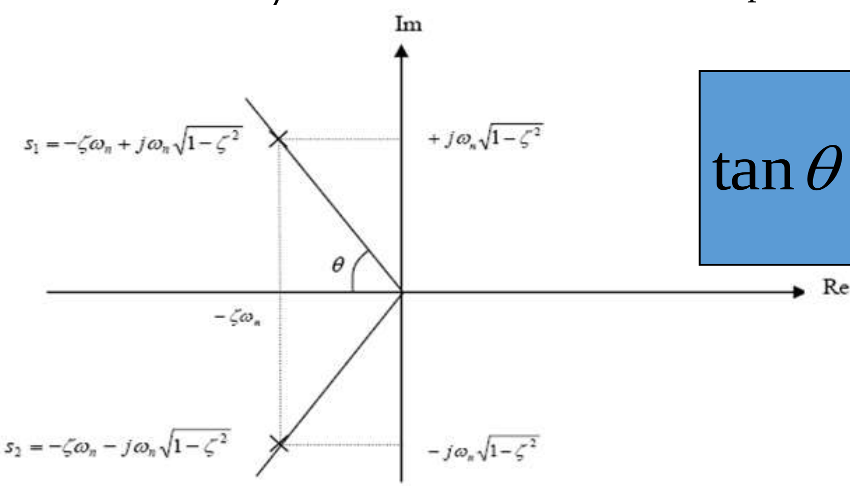
2) Peak time, T_p - The peak time is the time required for the response to reach the first peak, which is given by;

$$\left.\dot{y}(t)\right|_{t=T_p} = 0$$

$$\left.\dot{y}(t)\right|_{t=T_p} = -\frac{1}{\beta}(-\varsigma\omega_n)e^{-\varsigma\omega_n t}\sin(\omega_d t+\theta) - \frac{1}{\beta}e^{-\varsigma\omega_n t}\cos(\omega_d t+\theta)\left[\omega_n\sqrt{1-\varsigma^2}\right] = 0$$

$$\frac{\varsigma\omega_n}{\beta}e^{-\varsigma\omega_n T_p}\sin(\omega_d T_p+\theta) = \frac{\left[\omega_n\sqrt{1-\varsigma^2}\right]}{\beta}e^{-\varsigma\omega_n T_p}\cos(\omega_d T_p+\theta)$$

$$\tan(\omega_d T_p+\theta) = \frac{\sqrt{1-\varsigma^2}}{\varsigma}$$



$$\boxed{\tan\theta = \frac{\sqrt{1-\varsigma}}{\varsigma}}$$

We know that $\tan(\theta) = \tan(\pi + \theta)$

So, $\tan(\omega_d T_p + \theta) = \tan(\pi + \theta)$

From this expression:

$$\omega_d T_p + \theta = \pi + \theta$$

$$\omega_d T_p = \pi$$

$$\boxed{T_p = \frac{\pi}{\omega_d} = \frac{\pi}{\omega_n \sqrt{1 - \varsigma^2}}}$$

Transient Response Analysis

3) Percent overshoot, %OS - The percent overshoot is defined as the amount that the waveform at the peak time overshoots the steady-state value, which is expressed as a percentage of the steady-state value.

$$\%MP \equiv \frac{C(T_p) - C(\infty)}{C(\infty)} x 100\%$$

OR

$$\%OS = \frac{C\max - Cfinal}{Cfinal} \; x \; 100$$

$$\frac{C(T_p)-1}{1}x100\% = -\frac{1}{\beta}e^{-\xi\omega_n t}\sin\left(\omega_d t+\theta\right)x100\%$$

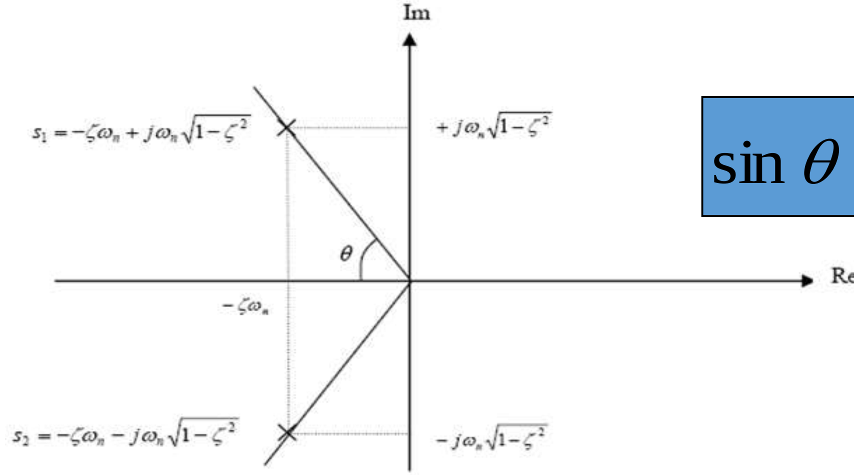
$$= -\frac{1}{\beta}e^{-\xi\omega_n\left[\frac{\pi}{\omega_n\sqrt{1-\varsigma^2}}\right]}\sin\left(\omega_d\left(\frac{\pi}{\omega_d}\right)+\theta\right)x100\%$$

$$= -\frac{1}{\beta}e^{-\frac{\pi\varsigma}{\sqrt{1-\varsigma^2}}}\sin\left(\pi+\theta\right)x100\%$$

$$= \frac{\sin(\theta)}{\beta}e^{-\frac{\pi\varsigma}{\sqrt{1-\varsigma^2}}}x100\% = e^{-\frac{\pi\varsigma}{\sqrt{1-\varsigma^2}}}x100\%$$

From slide 24

$$\beta = \sqrt{1-\xi^2}$$



$$\boxed{\sin\theta = \sqrt{1-\varsigma^2}}$$

Therefore,

$$\boxed{\%MP = e^{-\frac{\pi\varsigma}{\sqrt{1-\varsigma^2}}} x100\%}$$

- For given $\%OS$, the damping ratio can be solved from the above equation;

$$\varsigma = \frac{-\ln\left(\%MP/100\right)}{\sqrt{\pi^2 + \ln^2\left(\%MP/100\right)}}$$

Transient Response Analysis

4) Setting time, T_s - The settling time is the time required for the amplitude of the sinusoid to decay to 2% of the steady-state value.

To find T_s, we must find the time for which c(t) reaches & stays within $\underline{+}$2% of the steady state value, $c_{final.}$ The settling time is the time it takes for the amplitude of the decaying sinusoid in c(t) to reach 0.02, or

$$e^{-\varsigma\omega_n T_s}\frac{1}{\sqrt{1-\varsigma^2}}=0.02$$

Thus,

$$\boxed{T_s=\frac{4}{\varsigma\omega_n}}$$

UNDERDAMPED

Example 2: *Find the natural frequency and damping ratio for the system with transfer function*

$$G(s) = \frac{36}{s^2 + 4.2s + 36}$$

Solution:

Compare with general TF

$$G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

- $\omega_n = 6$
- $\xi = 0.35$

UNDERDAMPED

Example 3: Given the transfer function

$$G(s) = \frac{100}{s^2 + 15s + 100}$$

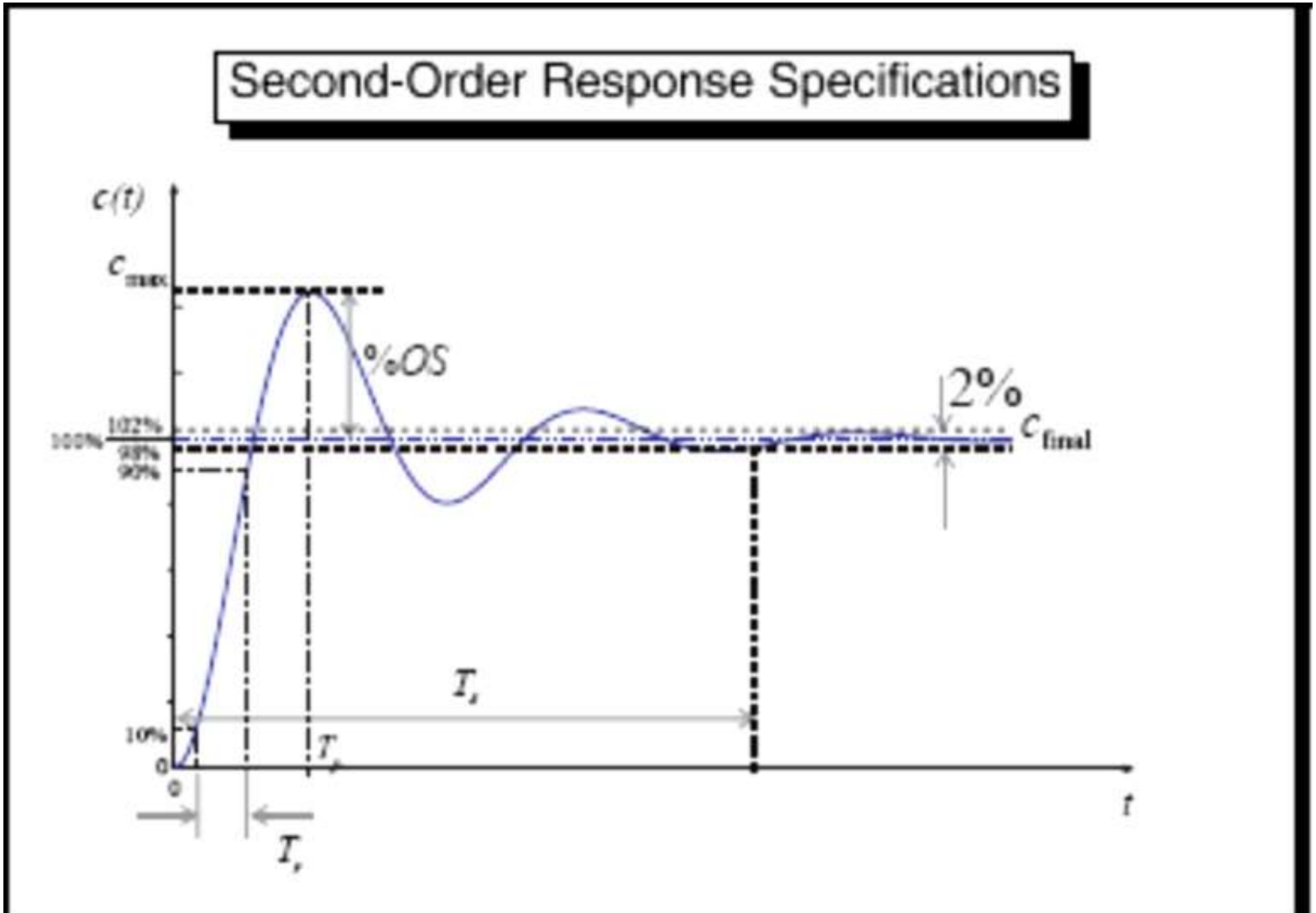
$$find \quad T_s, \%OS, T_p$$

$T_s = 0.533s,\ \%OS = 2.838\%,\ T_p = 0.475s$

Solution:

$$\omega_n = 10 \qquad \xi = 0.75$$

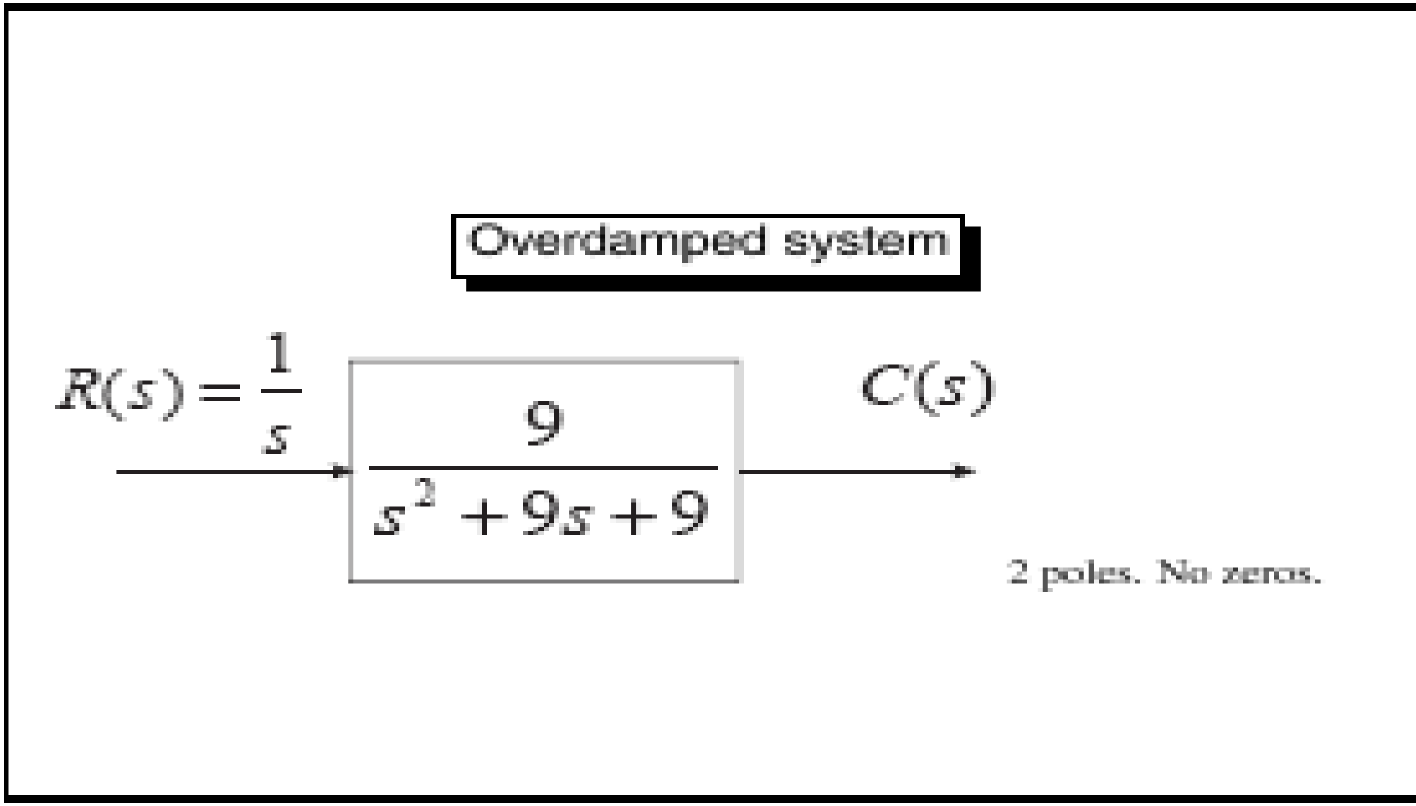
UNDERDAMPED



$$G(s) = \frac{b}{s^2 + as + b}$$

Overdamped Response

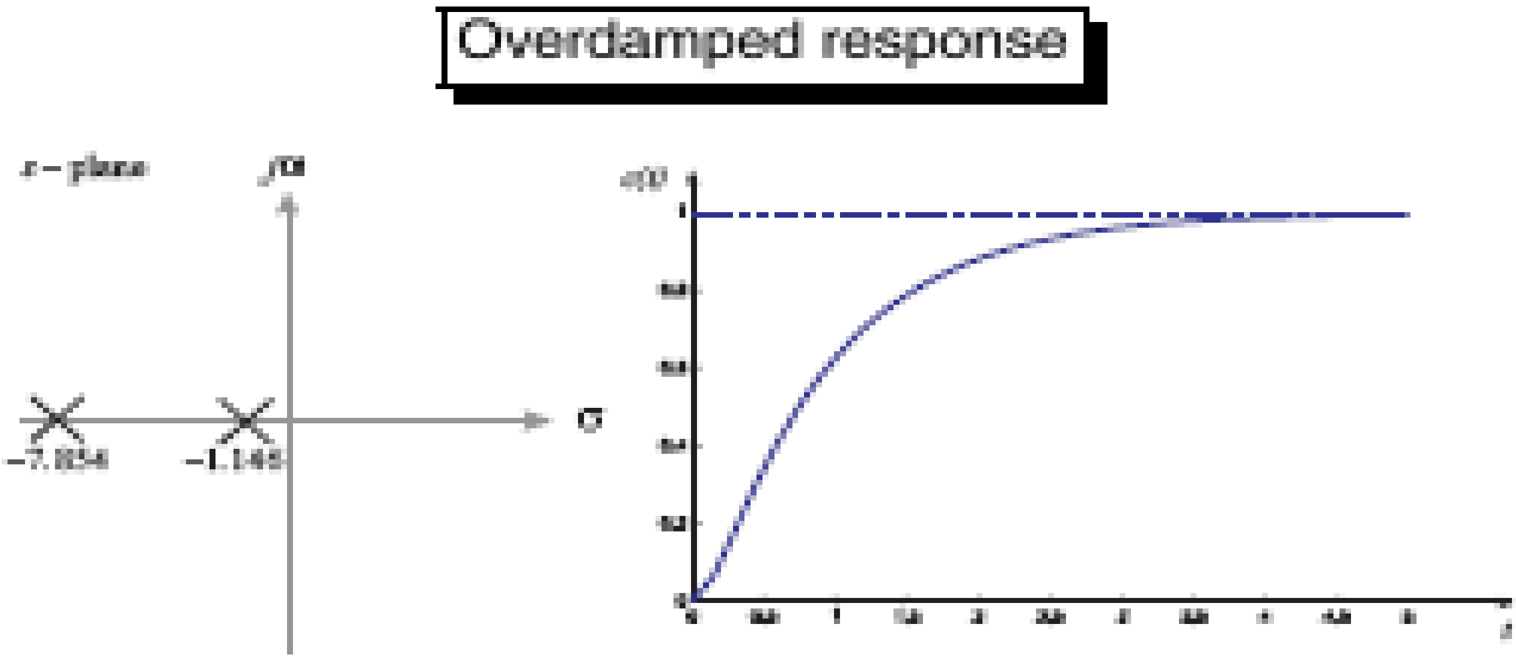
a = 9



$$C(s) = \frac{9}{s(s^2 + 9s + 9)} = \frac{9}{s(s + 7.854)(s + 1.146)}$$

s= 0; s = -7.854; s = -1.146 *(two real poles)*

$$c(t) = K_1 + K_2 e^{-7.854t} + K_3 e^{-1.146t}$$

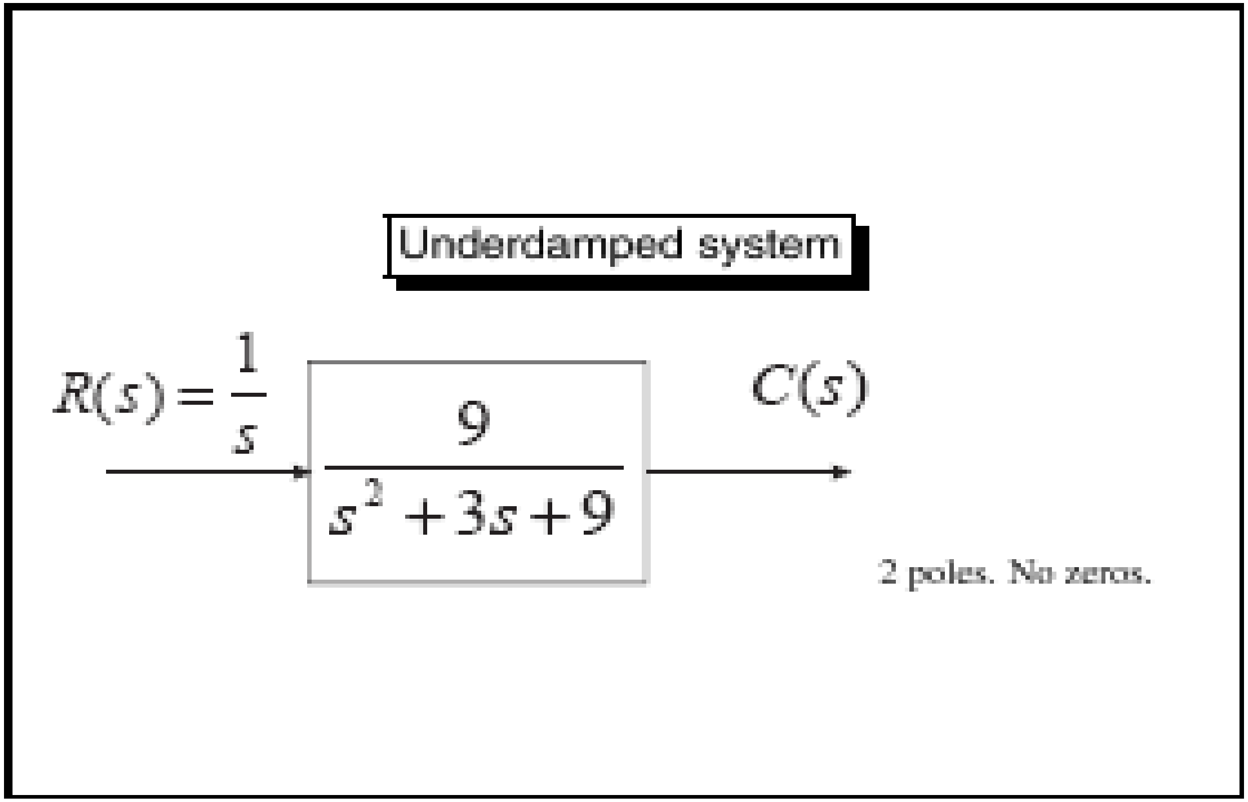


OVERDAMPED RESPONSE !!!

$$G(s) = \frac{b}{s^2 + as + b}$$

Underdamped Response

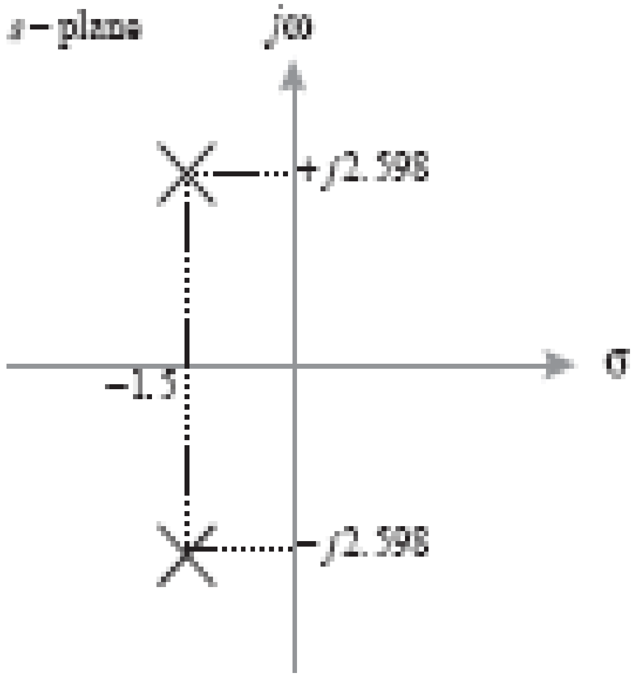
a = 3

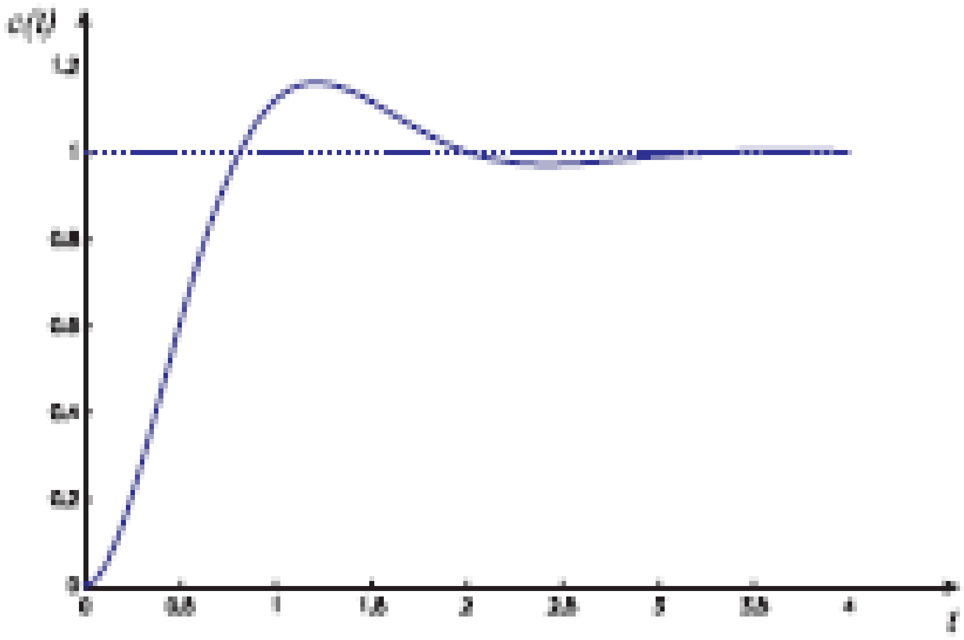


$$c(t) = K_1 + e^{-1.5t}(K_2 \cos 2.598t + K_3 \sin 2.598t)$$

s = 0; s = -1.5 ± j2.598 *(two complex poles)*

Underdamped response



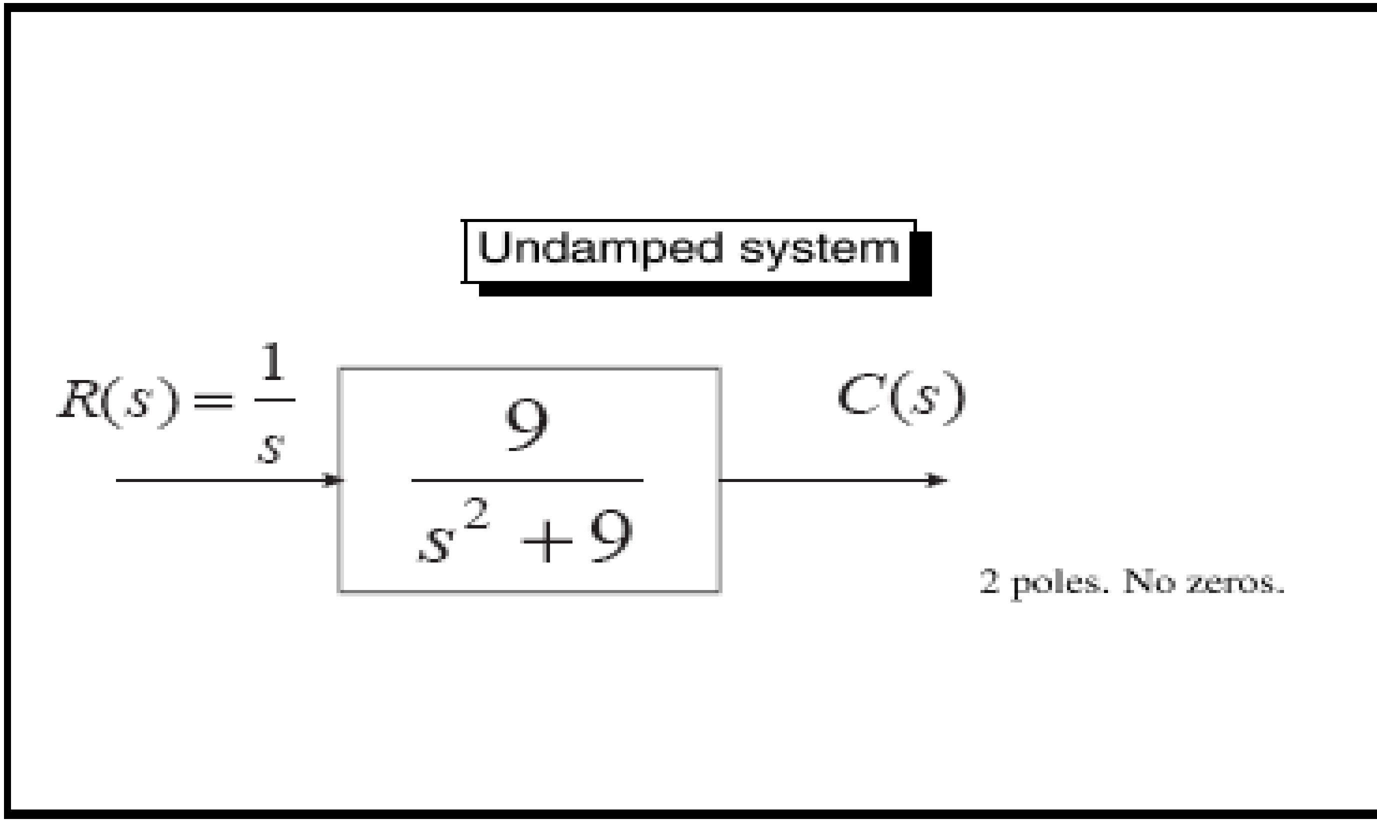


UNDERDAMPED RESPONSE !!!

Undamped Response

$$G(s) = \frac{b}{s^2 + as + b}$$

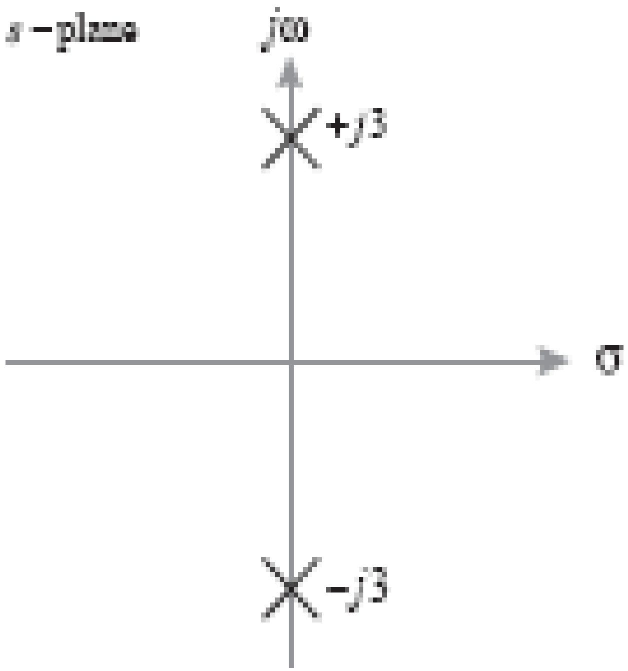
a = 0

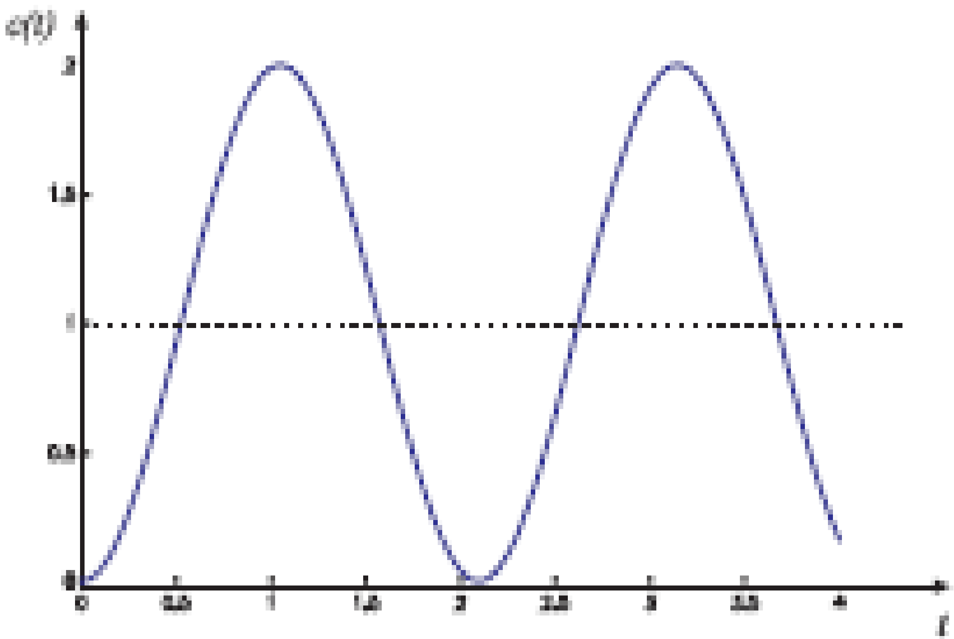


$$c(t) = K_1 + K_2 \cos 3t$$

s = 0; s = ± j3 *(two imaginary poles)*

Undamped response



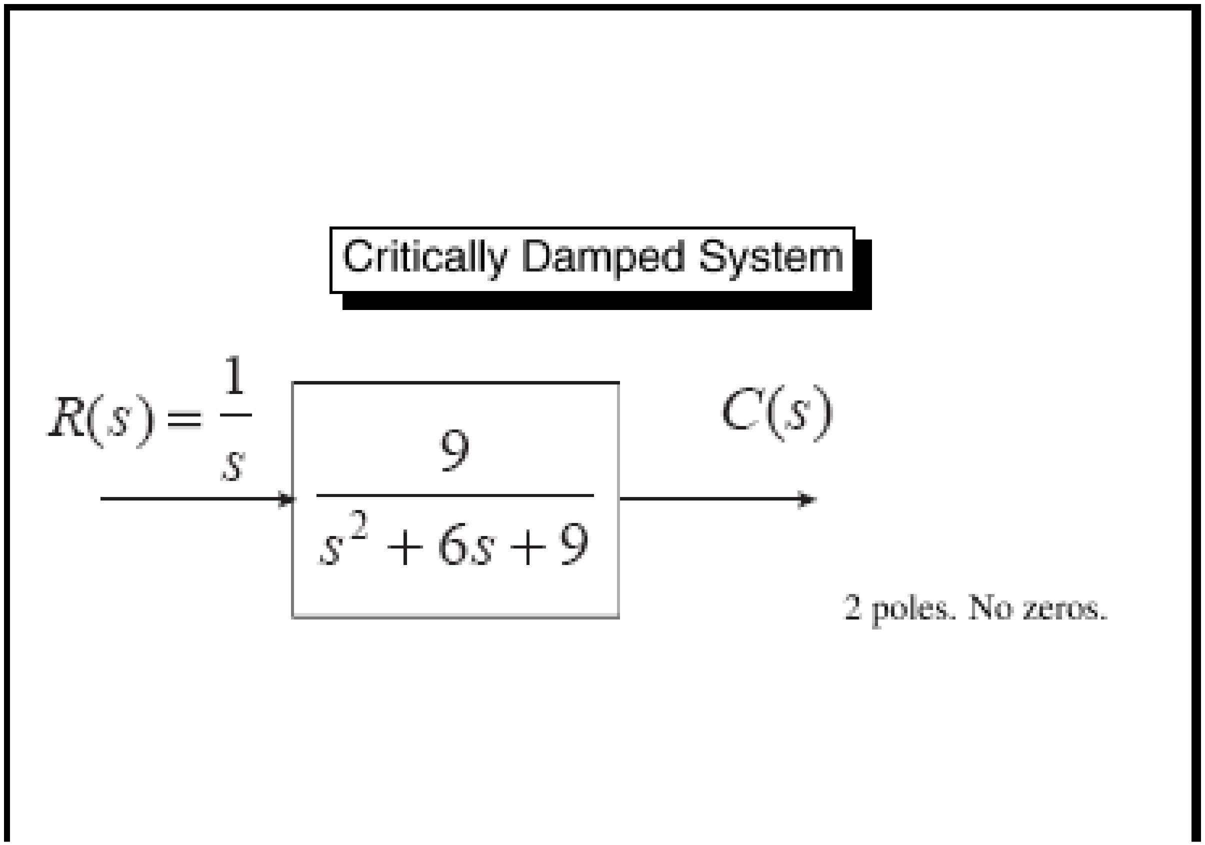


UNDAMPED RESPONSE !!!

Critically Damped System

$$G(s) = \frac{b}{s^2 + as + b}$$

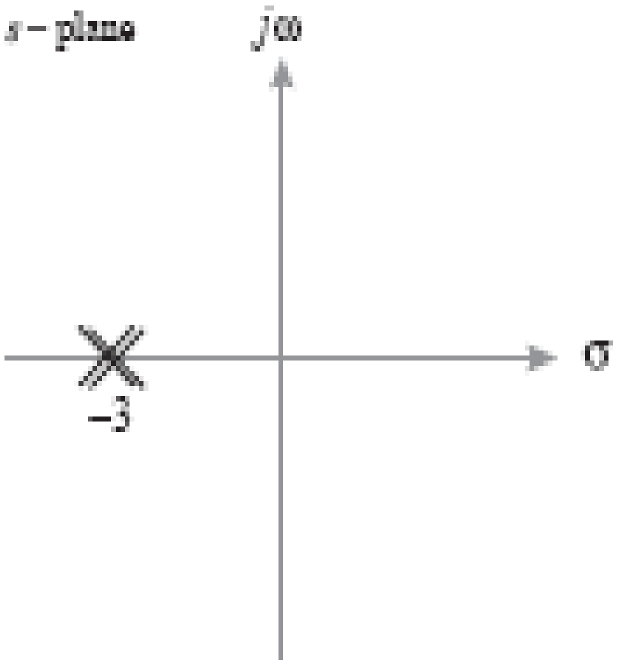
a = 6

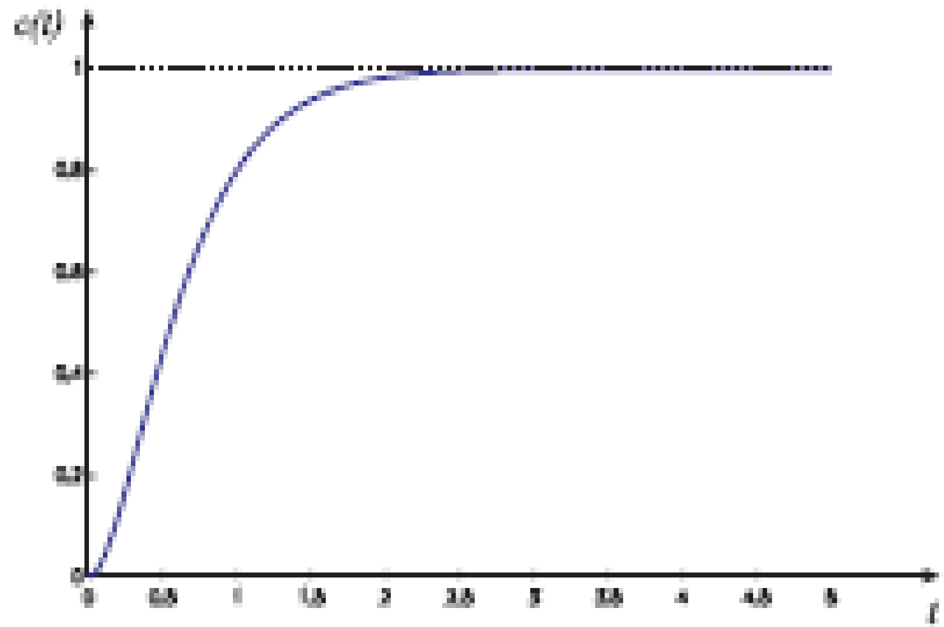


$$c(t) = K_1 + K_2 e^{-3t} + K_3 t e^{-3t}$$

S = 0; s = -3,-3 (*two real and equal poles)*

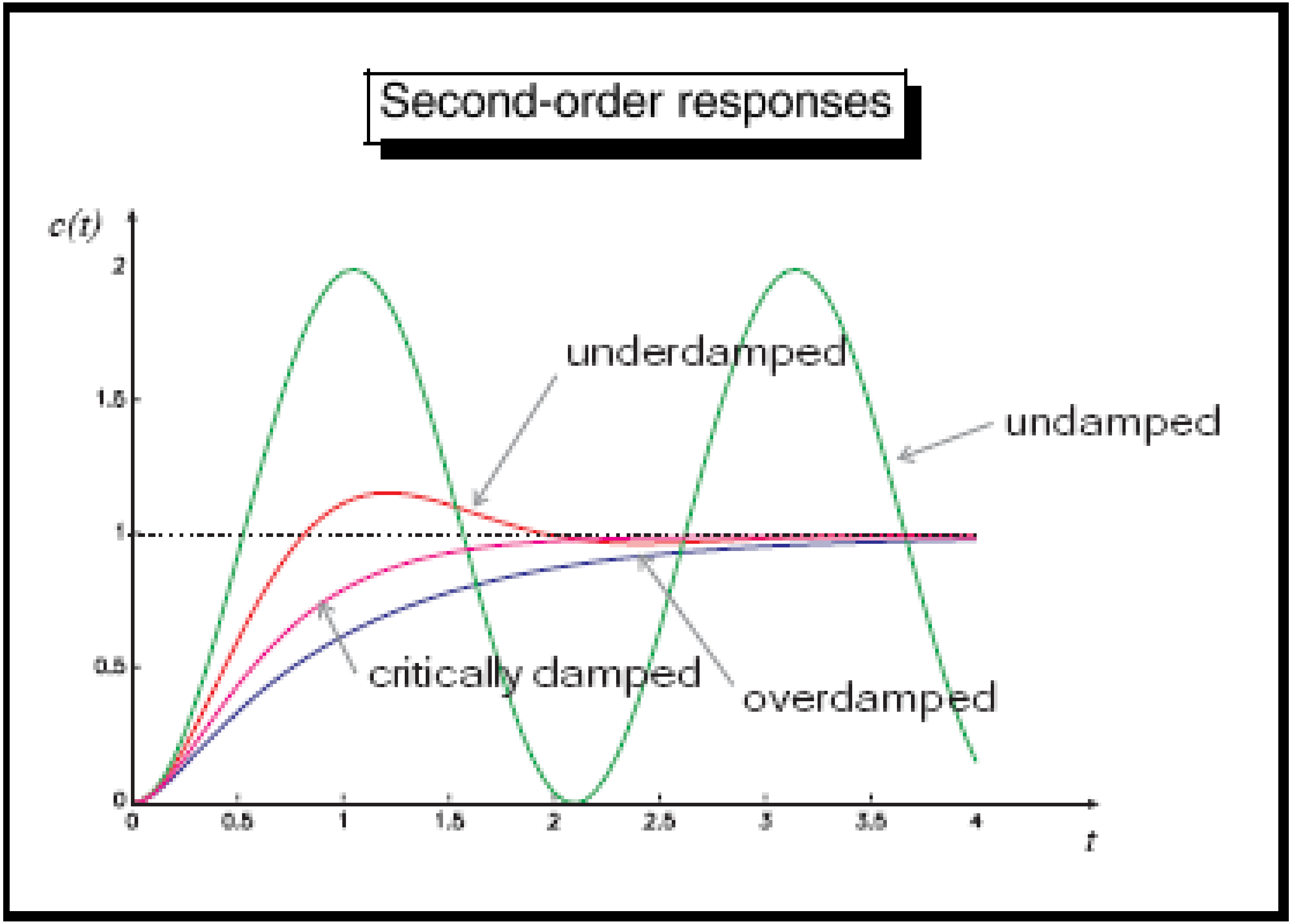
Critically Damped Response

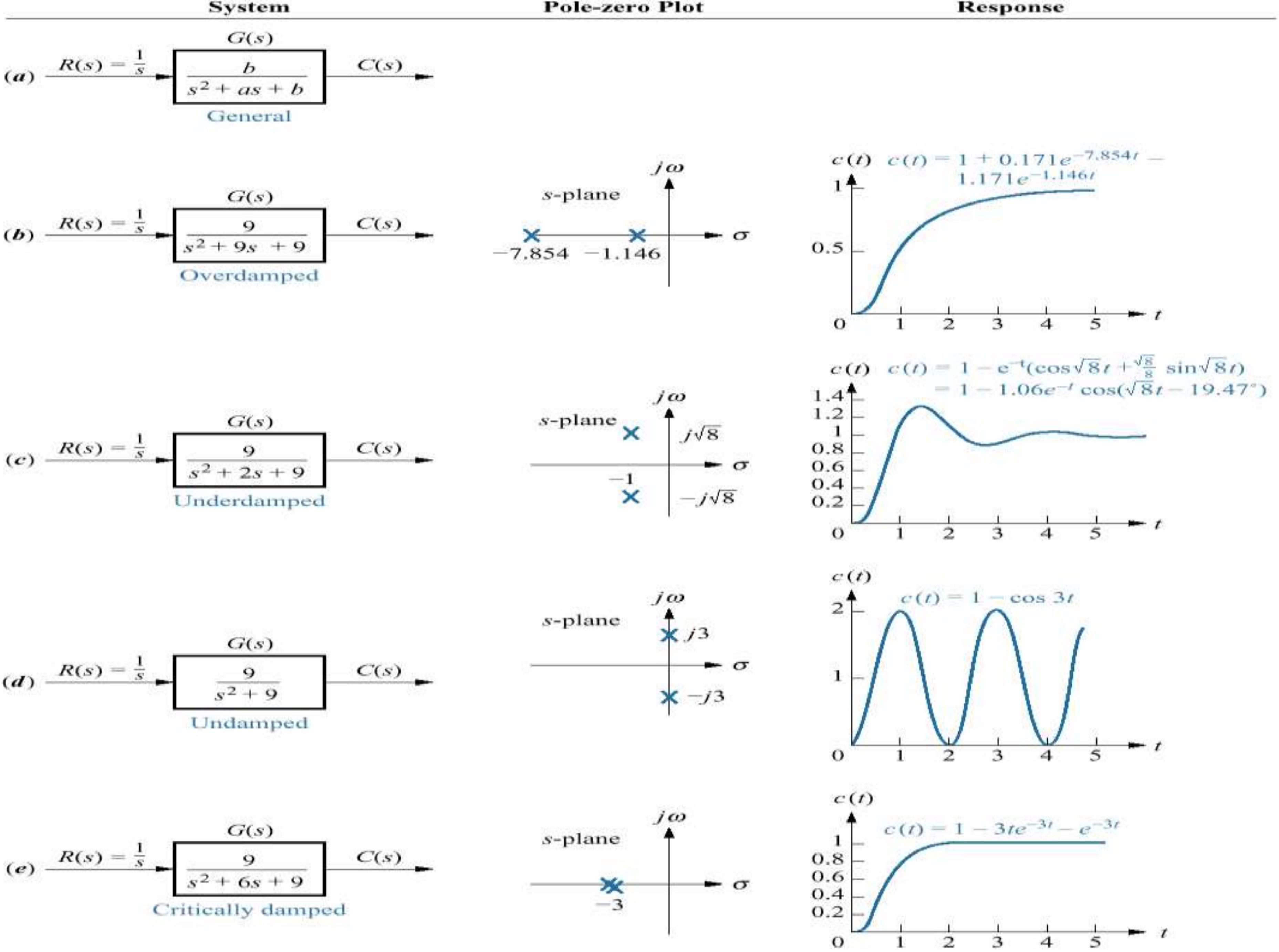




CRITICALLY DAMPED RESPONSE !!!

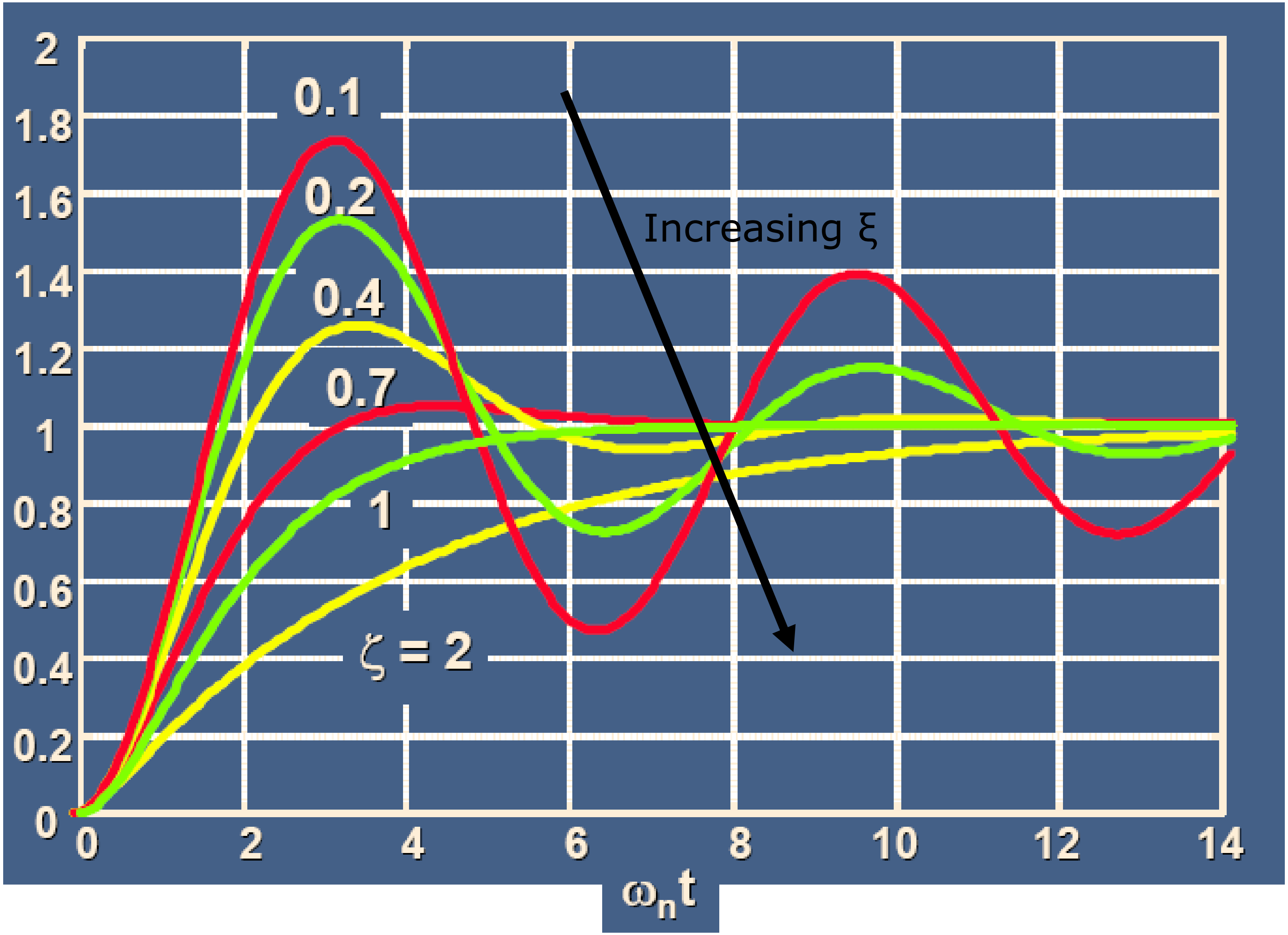
Second – Order System





System
Pole-zero Plot
Response
(a)
$R(s) = \frac{1}{s}$
$G(s)$
$\frac{b}{s^2 + as + b}$
$C(s)$
General
(b)
$R(s) = \frac{1}{s}$
$G(s)$
$\frac{9}{s^2 + 9s + 9}$
$C(s)$
Overdamped
$j\omega$
s-plane
σ
-7.854
-1.146
$c(t)$
$c(t) = 1 + 0.171e^{-7.854t} - 1.171e^{-1.146t}$
(c)
$R(s) = \frac{1}{s}$
$G(s)$
$\frac{9}{s^2 + 2s + 9}$
$C(s)$
Underdamped
$j\omega$
s-plane
$j\sqrt{8}$
-1
$-j\sqrt{8}$
σ
$c(t)$
$c(t) = 1 - e^{-t}(\cos\sqrt{8}t + \frac{\sqrt{8}}{8}\sin\sqrt{8}t)$
$= 1 - 1.06e^{-t}\cos(\sqrt{8}t - 19.47°)$
(d)
$R(s) = \frac{1}{s}$
$G(s)$
$\frac{9}{s^2 + 9}$
$C(s)$
Undamped
$j\omega$
s-plane
$j3$
$-j3$
σ
$c(t)$
$c(t) = 1 - \cos 3t$
(e)
$R(s) = \frac{1}{s}$
$G(s)$
$\frac{9}{s^2 + 6s + 9}$
$C(s)$
Critically damped
$j\omega$
s-plane
-3
σ
$c(t)$
$c(t) = 1 - 3te^{-3t} - e^{-3t}$
t

Effect of different damping ratio, ξ



Second – Order System

Example 4: Describe the nature of the second-order system response via the value of the damping ratio for the systems with transfer function

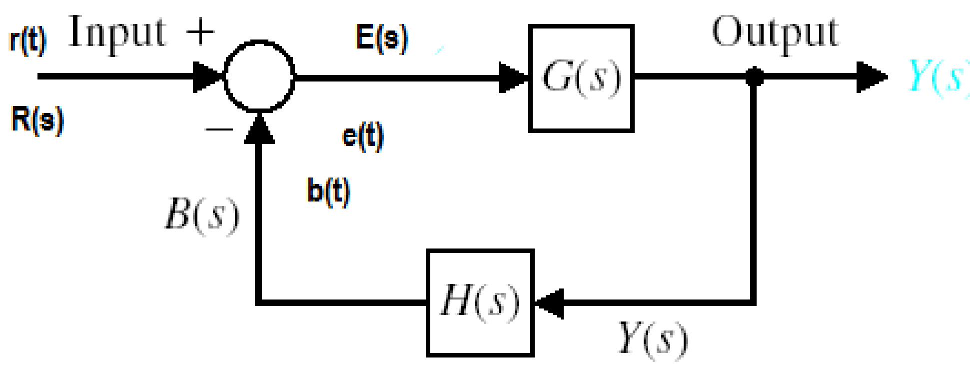
1. $$G(s) = \frac{12}{s^2 + 8s + 12}$$

2. $$G(s) = \frac{16}{s^2 + 8s + 16}$$

3. $$G(s) = \frac{20}{s^2 + 8s + 20}$$

Steady State Error Analysis

Steady State Error Analysis



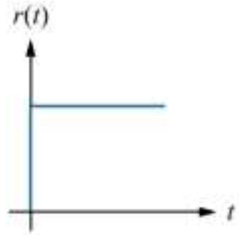
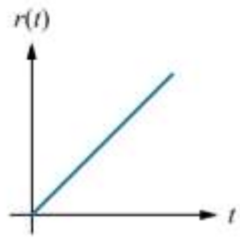
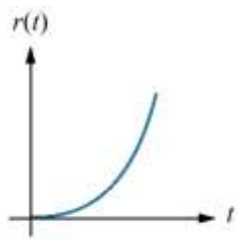
$$Error, e(t) = r(t) - b(t)$$

$$E(s) = R(s) - B(s) = \frac{1}{1 + G(s)H(s)} \times R(s)$$

Steady State Error$=\lim_{t\to\infty} e(t) = \lim_{s\to 0} sE(s)$

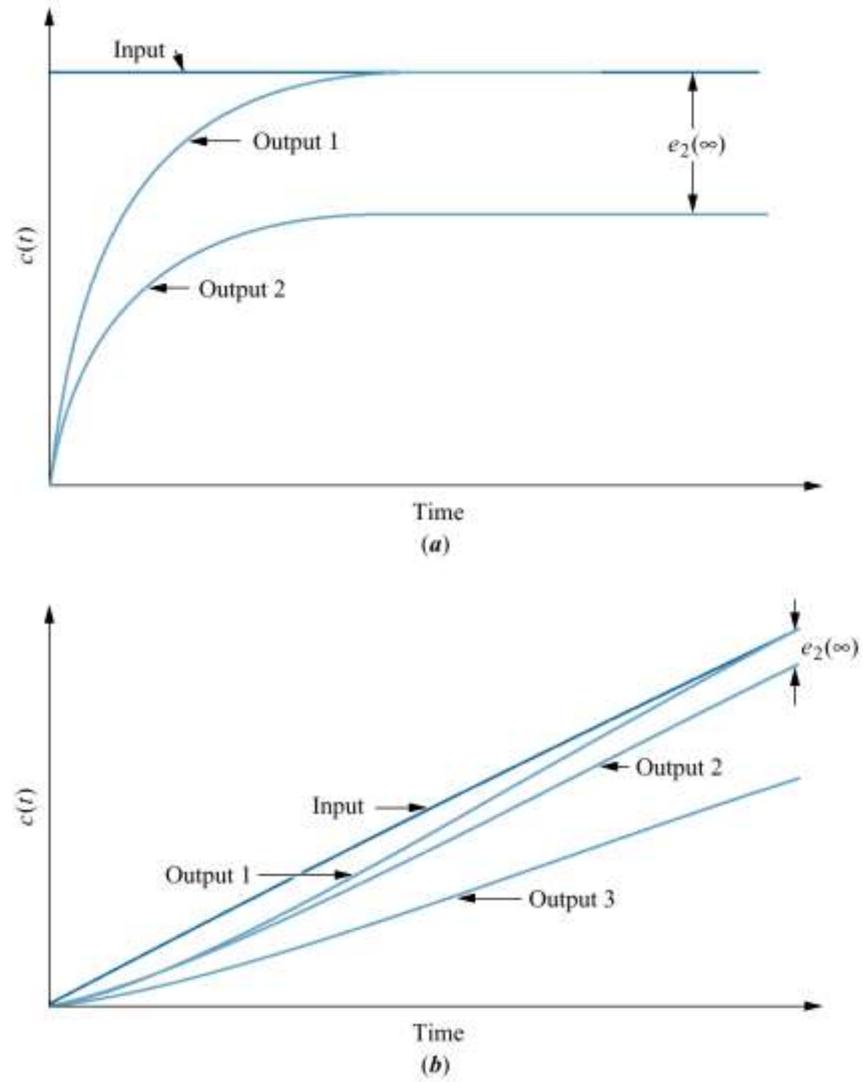
Steady state error analysis

Test waveforms for evaluating steady-state errors of position control systems

Waveform	Name	Physical interpretation	Time function	Laplace transform
	Step	Constant position	1	$\frac{1}{s}$
	Ramp	Constant velocity	t	$\frac{1}{s^2}$
	Parabola	Constant acceleration	$\frac{1}{2}t^2$	$\frac{1}{s^3}$

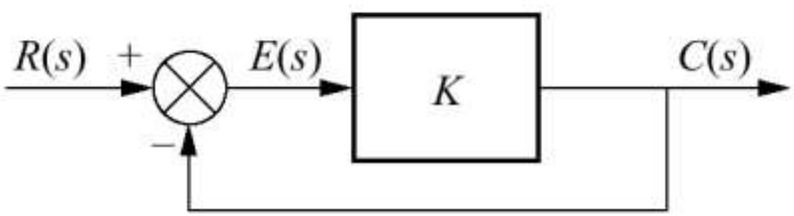
System response to different inputs

Steady-state error:
a. step input;
b. ramp input

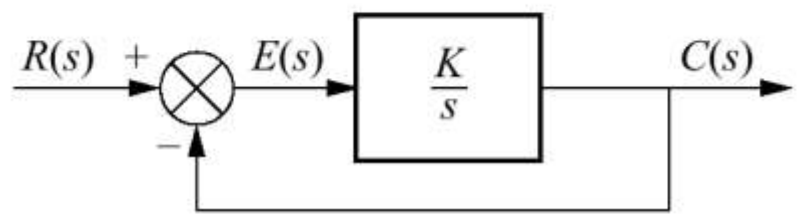


Intuition/motivation for steady state error analysis

System with:
a. finite steady-state error for a step input;
b. zero steady-state error for step input



(a)

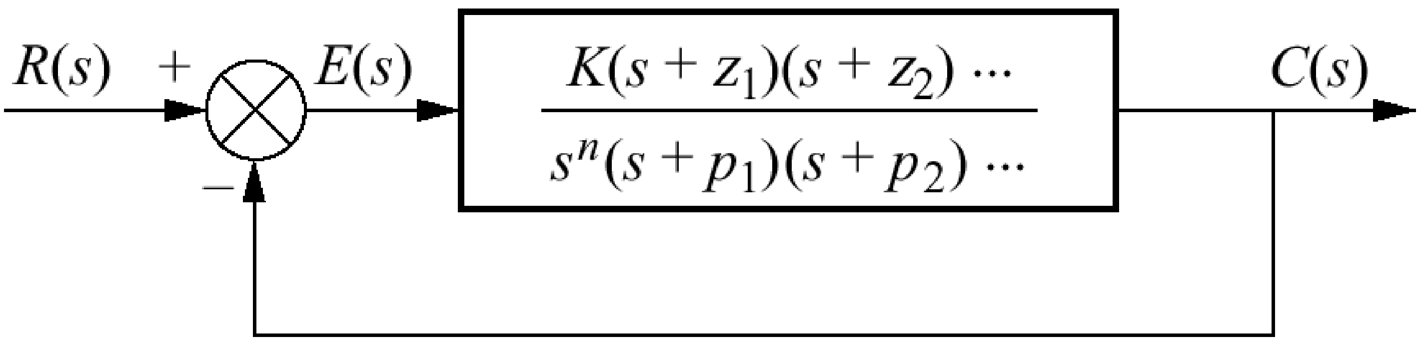


(b)

$E_{steady\ state} = (1/K) C_{steady\ state}$

$E_{steady\ state} = 0$ for $C_{steady\ state} \neq 0$

Feedback control system for defining system type



Type of the system=n

Relationships between input, system type, static error constants, and steady-state errors

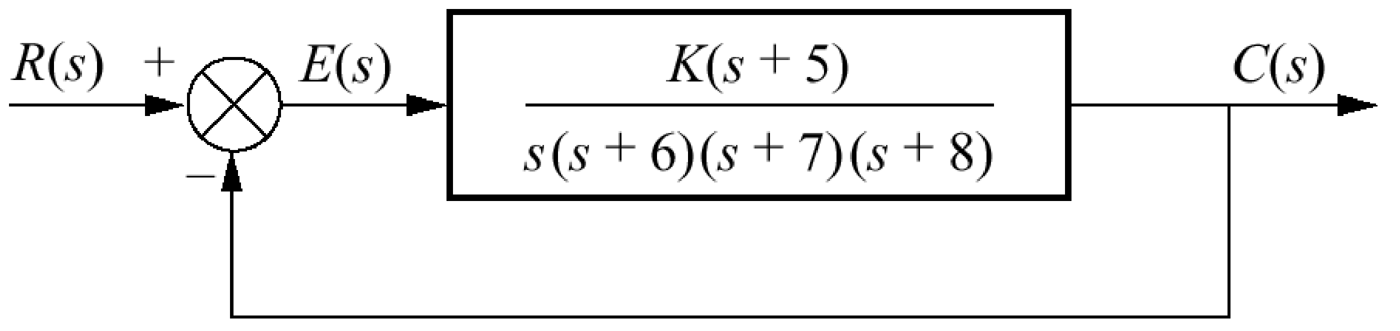
Input	Steady-state error formula	Type 0 Static error constant	Type 0 Error	Type 1 Static error constant	Type 1 Error	Type 2 Static error constant	Type 2 Error
Step, $u(t)$	$\frac{1}{1+K_p}$	$K_p =$ Constant	$\frac{1}{1+K_p}$	$K_p = \infty$	0	$K_p = \infty$	0
Ramp, $tu(t)$	$\frac{1}{K_v}$	$K_v = 0$	∞	$K_v =$ Constant	$\frac{1}{K_v}$	$K_v = \infty$	0
Parabola, $\frac{1}{2}t^2u(t)$	$\frac{1}{K_a}$	$K_a = 0$	∞	$K_a = 0$	∞	$K_a =$ Constant	$\frac{1}{K_a}$

$$Position\ error\ constatnt, K_p = \lim_{s \to 0} G(s)H(s)$$

$$Velocity\ error\ constatnt, K_v = \lim_{s \to 0} sG(s)H(s)$$

$$Accelaration\ \ error\ constatnt, K_a = \lim_{s \to 0} s^2G(s)H(s)$$

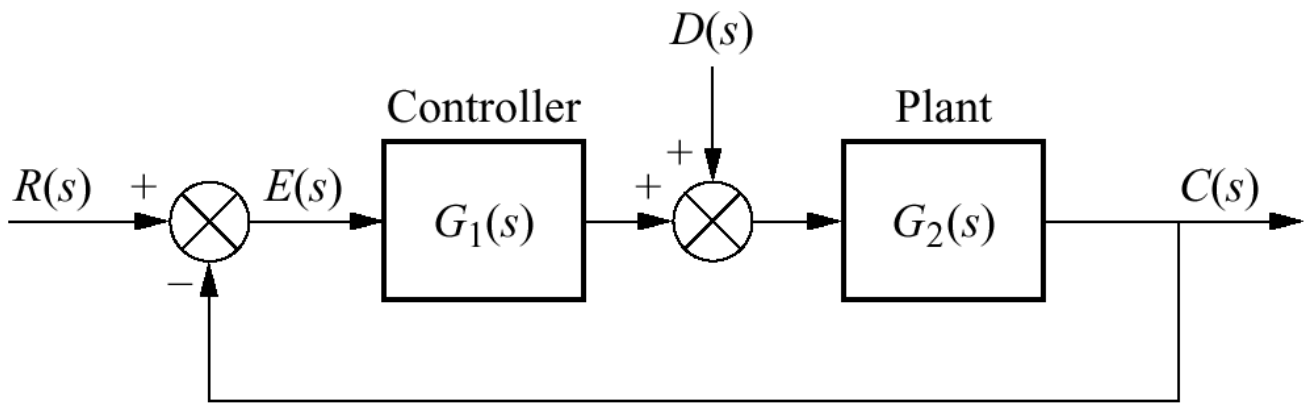
Example 1



Determine position error, velocity error, and acceleration error.

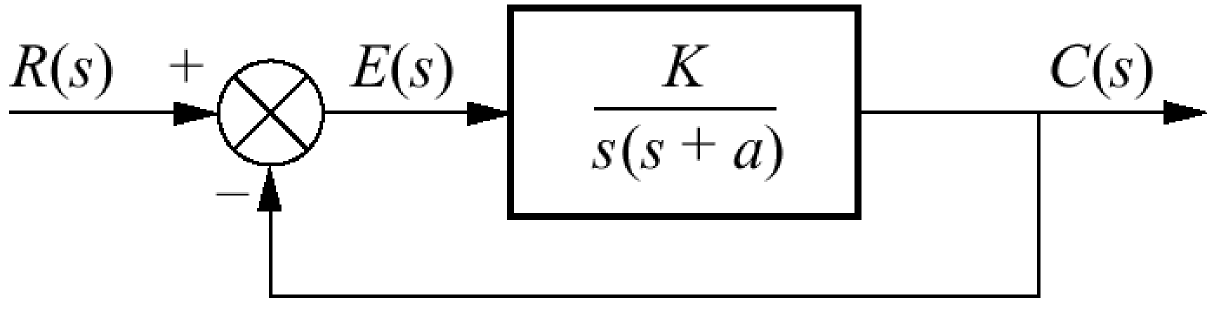
Example-2

Feedback control system showing disturbance



Determine position error, velocity error, and acceleration error.

Example-3



Determine position error, velocity error, and acceleration error.

Thank You