MA204: Mathematics IV

Complex Analysis: Some Elementary Functions

Gautam Kalita
IIIT Guwahati

Exponential function

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$$e^{iy} = \cos y + i \sin y.$$

If $z = x + iy$, then the exponential function is defined as

$$e^z = e^{x+iy} = e^x e^{iy} = e^x(\cos y + i \sin y).$$

As a result,

$$|e^z| = e^x \text{ and } \arg(e^z) = y + 2n\pi \text{ for } n \in \mathbb{Z}.$$

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Properties:

(a) $e^{z_1+z_2} = e^{z_1}e^{z_2}$.

(b) $e^z \neq 0$ for all $z \in \mathbb{Z}$.

(c) e^z is an entire function and $\frac{d}{dz}e^z = e^z$.

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Question: Is e^{z} a bijection?

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Problem: Find complex values of w for which $e^{z+w} = e^z$. Can you say something about period of the exponential function?

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Problem: Give conditions on z such that e^z is (a) real (b) purely imaginary.

Complex Logarithm

It is easy to see that the exponential function $e^z : \mathbb{C} \to \mathbb{C}$ is neither one-one as $e^{z+2ni\pi} = e^z$ nor onto. Thus the inverse of the function does not exist in the complex plane $\mathbb{C}$.

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However, the exponential function $e^z : \mathbb{C} \to \mathbb{C} - \{0\}$ is on-to. For any $z \in \mathbb{C} - \{0\}$, we have $e^w = z$. As a result, $w := \log z = \ln|z| + i\,\arg(z)$.

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Clearly, the complex logarithm $\log z$ is not well defined. It is multi-valued assignment.

The **principal value** of the logarithm is defined as

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Note that Log z is well defined single valued function.

Complex logarithm

Note:

(a) If $z \neq 0$, then $e^{\log z} = z$. What about of $\log(e^z)$?
(b) The function Log z is not continuous on the negative real axis.
(c) The function Log z is analytic everywhere except on the negative real axis and at zero.
(d) Log $(z_1 z_2) =$Log z_1+Log z_2. What if Log is replaced by log?
(e) For $z_2 \neq 0$, we have Log $(\frac{z_1}{z_2}) =$Log z_1-Log z_2. What if Log is replaced by log?

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Question: Find valued of $\log(1+i)$, Log $(-i)$.

Branch, Branch cut, and Branch point

Branch of a multiple valued function: Let F be a multiple valued function defined on a domain D. A function f is said to be a branch of the multiple valued function F if in a domain $D_0 \subset D$ if $f(z)$ is single valued and analytic in D_0.

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Branch Point: Any point that is common to all branch cuts is called a branch point.

Complex exponent

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The **principal branch** or the **principal value** of the complex exponent z^w is given by

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Problem: Find values of i^{-i}, $(1+i)^i$, and $i^{(1-i)}$.

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Problem: Discuss analyticity for the function $f(w) = z^w$ and $g(z) = z^w$. Find derivative of the functions.

Trigonometric function

Note that $e^{ix} = \cos x + i\sin x$ and $e^{-ix} = \cos x - i\sin x$. Thus, we have

$$\sin x = \frac{e^{ix} - e^{-ix}}{2i} \text{ and } \cos x = \frac{e^{ix} + e^{-ix}}{2}.$$

As a result, it is natural to extend the definition as follows:

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Thus we can define $\tan, \cot, \sec$, and csc function in the usual way.

Trigonometric function

Problem: Show that $1 + \tan^2 z = \sec^2 z$.

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Problem: Find values of z for which (a) $\sin z = -2$ (b) $\cos z = k$

Hyperbolic trigonometric function

One can define the hyperbolic sine and cosine functions as

$$\sinh z = \frac{e^z - e^{-z}}{2} \text{ and } \cosh z = \frac{e^z + e^{-z}}{2}.$$

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$\cos^2 z + \sin^2 z = 1$	$\cosh^2 z - \sinh^2 z = 1$
$\sin(-z) = -\sin z, \cos(-z) = \cos z$	
$\sin(z + 2k\pi) = \sin z, \cos(z + 2k\pi) = \cos z$	
$\sin z = 0$ iff $z = n\pi$	
$\cos z = 0$ iff $z = (2n+1)\frac{\pi}{2}$	
$\frac{d}{dz} \sin z = \cos z, \frac{d}{dz} \cos z = -\sin z$	

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$$\cos z = \cos(x + iy) = \cos x \cosh y - i \sin x \sinh y$$
$$\sin z = \sin(x + iy) = \sin x \cosh y + i \cos x \sin hy$$

Hyperbolic triginometric function

Problem: Solve (a) $\sin z = \cosh 4$ (b) $\sinh z = -i$ (c) $\cosh z = -2$.

Problem: Find values of (a) $\tan^{-1}(1+i)$ (b) $\sinh^{-1}(-i^{i})$.

Thank You

Any Question!!!