EE558 - Digital Communications

Lecture 2: Review of Signals and Systems

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Leadership Starts Here

Signals and Systems

$$\begin{array}{c} \text{Input} \\ \text{signal} \\ x[n] \end{array} \longrightarrow \boxed{\text{System}} \longrightarrow \begin{array}{c} \text{Output} \\ \text{signal} \\ y[n] \end{array}$$

- Signal
  - Applied to something that conveys information
  - Represented as a function of one or more independent variables
  - Continuous-time *vs.* Discrete-time
  - Continuous-amplitude *vs.* Discrete-amplitude

- System: A transformation or operator that maps a input sequence into an output sequence

$$y[n] = T\big(x[n]\big) \quad \text{or} \quad y(t) = T\big(x(n)\big).$$

Signals

- Discrete-time signal $x[n]$

$$E_\infty = \sum_{n=-\infty}^{\infty} \left|x[n]\right|^2, \qquad P = \lim_{T\to\infty} \frac{1}{2T} \sum_{n=-T}^{T} \left|x[n]\right|^2 \tag{1}$$

- Some signals have infinite average power, energy or both
- A signal is called an **energy signal** if $E_\infty < \infty$
- A signal is called an **power signal** if $0 < P_\infty < \infty$
- A signal can be an energy signal, a power signal, or neither type
- A signal cannot be both an energy signal or a power signal
- Examples: $x[n] = 1$, $x[n] = \sin n$, $x[n] = n$

Some Examples

- Time shift: $x[n - n_0]$
- Time reversal: $x[-n]$
- Time scaling: $x[an]$
- Periodic signal with period N: $x[n] = x[t + N]$
- Even signal: $x[-n] = x[n]$
- Odd signal: $x[-n] = -x[n]$
- Exponential signal: $x[n] = C\mathrm{e}^{an}$
  - Real-valued exponential vs Complex exponential
  - Growing or decaying?
  - Periodic or aperiodic?
- Real sinusoidal signal: $x[n] = A\cos(\omega n + \phi)$

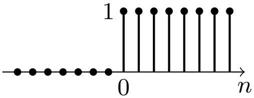
Unit Step Function and Unit Impulse

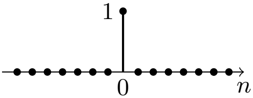
- Unit step function

$$u[n] = \begin{cases} 0, & n < 0 \\ 1, & n > 0 \end{cases}$$

- Unit impulse function

$$\delta[n] = u[n] - u[n-1], \qquad u[n] = \sum_{m=-\infty}^{n} \delta[m]$$





- Some properties:
  - $\sum_{n-\infty}^{\infty} x[n]\delta[n-n_0] = x[n_0]$: sifting property
  - $x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k]$: signal decomposition

Linearity

- Input-output relationship: $y_i[n] = T\big(x_i[n]\big)$
- A system is linear if
  - $T\big(ax[n]\big) = aT\big(x[n]\big)$
  - $T\big(x_1[n] + x_2[n]\big) = T\big(x_1[n]\big) + T\big(x_2[n]\big)$
  - or $y[n] = T\big(a_1x_1[n] + a_2x_2[n]\big) = a_1y_1[n] + a_2y_2[n]$.
- <u>Examples:</u> linear or not
  1. Time scaler: $y[n] = x[2n]$
  2. Amplifier: $y[n] = 2x[n] + 1$
  3. Accumulator: $y[n] = \displaystyle\sum_{k=-\infty}^{n} x[k]$
  4. Squarer: $y[n] = x^2[n]$

Causality and Stability

- **Causality:** Output only depends on values of the input at only the present and past times
- Examples: casual or not
  1. Time scaler: $y[n] = x[2n]$ and $y[n] = x[n/2]$
  2. $y[n] = \sin\big(x[n]\big)$
- **Stability:** Small input lead to responses that do diverge

$$\big|x[n]\big| \leq B \text{ for some } B < \infty \longrightarrow \big|y[n]\big| < \infty$$

- Examples: stable or not
  1. $y[n] = nx[n]$
  2. $y[n] = \mathrm{e}^{x[n]}$
  3. $y[n] = y[n-1] + x[n]$

Time-Invariance

- Time-invariant system: characteristics of the system are fixed over time

$$y[n] = T\big(x[n]\big) \quad \longrightarrow \quad y[n-n_0] = T\big(x[n-n_0]\big)$$

- Examples: Time-invariant or not
  1. $y[n] = \sin x[n]$
  2. $y[n] = nx[n]$
  3. $y[n] = x[2n]$

- Linear time-invariant (LTI) system: good model for many real-life systems

- Examples: LTI or not
  1. $y[n] = \dfrac{1}{2n_0} \displaystyle\sum_{k=n-n_0}^{n+n_0} x[k]$

Response in LTI Systems

$$x[n] = \delta[n] \longrightarrow \boxed{\text{System}} \longrightarrow y[n] = h[n]$$

- Impulse response: Response to a unit impulse
- Any signal can be expressed as a sum of impulses

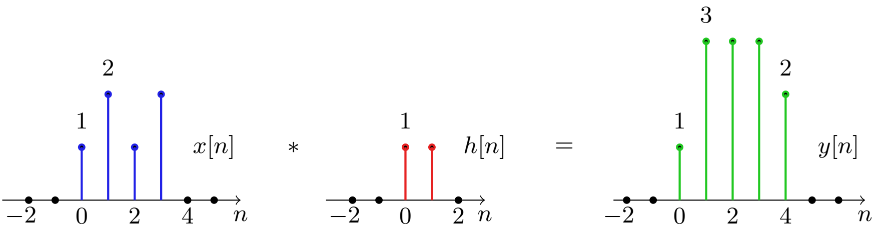
$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k]$$

- LTI system: $\delta[n-k] \to h[n-k]$
- Output signal:

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

Convolution Operation

- Convolution operation: $y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$
- Commutative: $x[n] * h[n] = h[n] * x[n]$
- Associative: $x[n] * (h_1[n] * h_2[n]) = (x[n] * h_1[n]) * h_2[n]$
- Distributive: $x[n] * (h_1[n] + h_2[n]) = x[n] * h_1[n] + x[n] * h_2[n]$
- Examples: Flip, shift, multiply and add



LTI System Properties and Impulse Response

- Any LTI system can described by its impulse response
- Memoryless: $h[n] = a\delta[n]$
- Causal: $h[n] = 0, \ \forall n < 0$
- Stable: $\displaystyle\sum_{n=-\infty}^{\infty} \big|h[n]\big| < \infty$

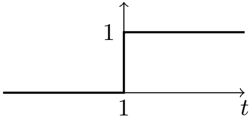
Continuous time Signals

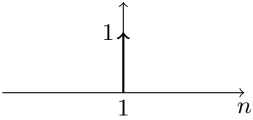
- Unit step function

$$u(t) = \begin{cases} 0, & t < 0 \\ 1, & t > 0 \end{cases}$$

- Unit impulse function or Dirac delta function

$$\delta(t) = \frac{\mathrm{d}u(t)}{\mathrm{d}t}, \qquad u(t) = \int_{-\infty}^{t} \delta(\tau)\mathrm{d}\tau$$





- $\delta(t) = 0$ for $t \neq 0$
- $\delta(t)$ in unbounded at $t = 0$
- $\int_{-\infty}^{\infty} \delta(t)\mathrm{d}t = 1$ and $\int_{-\infty}^{\infty} x(t)\delta(t - t_0)\mathrm{d}t = x(t_0)$: sifting property

Response in LTI Systems

$$x(t) = \delta(t) \longrightarrow \boxed{\text{System}} \longrightarrow y(t) = h(t)$$

- Impulse response: Response to a unit impulse
- Any continuous-time signal can be expressed as

$$x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)\mathrm{d}\tau$$

- LTI system: $\delta(t-\tau) \to h(t-\tau)$
- Output signal:

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)\mathrm{d}\tau \triangleq x(t) * h(t)$$

- <u>Examples:</u> $x(t) = \mathrm{e}^{-at}u(t)$, $h(t) = u(t)$. Then, $y(t) = \frac{1}{-a}\left[1 - \mathrm{e}^{-at}\right]$.

Response to Complex Exponentials

- Input signal: $x(t) = \mathrm{e}^{st}$
- Output signal:

$$y(t) = \mathrm{e}^{st} \int_{-\infty}^{\infty} h(\tau)\mathrm{e}^{-s\tau}\mathrm{d}\tau = H(s)\mathrm{e}^{st}$$

- $H(s)$ at s: eigenvalue associated with the eigenfunction e^{st}
- Input signal: $x[n] = z^n$
- Output signal:

$$y[n] = z^n \sum_{k=-\infty}^{\infty} h[k]z^{-k} = H(z)z^n$$

- $H(z)$ at z: eigenvalue associated with the eigenfunction z^n
- Why is eigenfunction is important?
- Can any signal be represented as a summation of complex exponentials?

Fourier Series I

- Periodic signal with period T: $x(t) = x(t+T)$
- $\omega_0 = 2\pi/T$ is called the *"angular fundamental frequency"*
- $f_0 = 1/T$ is called the *"fundamental frequency"*
- Harmonically related complex exponentials: $\Phi_k(t) = \mathrm{e}^{\mathrm{j}k\omega_0 t}$
- Assume a periodic signal $x(t)$ can be represented as

$$\text{Synthesis form :} \quad x(t) = \sum_{k=-\infty}^{\infty} a_k \mathrm{e}^{\mathrm{j}k\omega_0 t}$$

- Coefficients a_k's

$$\text{Analysis form:} \quad a_k = \frac{1}{T}\int_T x(t)\mathrm{e}^{-\mathrm{j}k\omega_0 t}\mathrm{d}t$$

Fourier Series II

- Fourier Analysis using fundamental frequency $f_0 = \omega_0/(2\pi)$
  - Synthesis form:

$$x(t) = \sum_{k=-\infty}^{\infty} a_k \mathrm{e}^{\mathrm{j}k2\pi f_0 t}$$

  - Analysis form:

$$a_k = \frac{1}{T}\int_{-T/2}^{T/2} x(t)\mathrm{e}^{-\mathrm{j}k2\pi f_0 t}\mathrm{d}t$$

- Parseval's theorem

$$\frac{1}{T}\int_{-T/2}^{T/2} x^2(t)\mathrm{d}t = \sum_{k=-\infty}^{\infty} |c_k|^2$$

- Examples: A periodic square wave

Fourier Transform

- A periodic square wave & Fourier Coefficients

$$x(t) = \begin{cases} 1, & |t| < T_1 \\ 0, & T_1 < |t| < T/2 \end{cases}, \qquad a_k = \frac{2\sin(k\omega_0 T_1)}{k\omega_0 T}$$

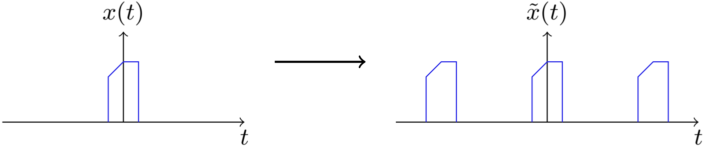
- Envelop function

$$Ta_k = \left. \frac{2\sin \omega T_1}{\omega} \right|_{\omega = k\omega_0}$$

- Fourier series coefficients and their envelop with different values of T with T_1 fixed
- $T \to \infty$: Fourier series coefficients approaches the envelope function.

Fourier Transform

- Aperiodic signal: can be treated as a periodic signal with $T \to \infty$
- The envelop function is called the Fourier Transform
- Derivations of Fourier Transform
  - Period padding for a aperiodic signal $x(t)$ with finite duration



Fourier Transform II

- Express $\tilde{x}(t)$ using Fourier Series

$$\tilde{x}(t) = \sum_{k=-\infty}^{\infty} a_k \mathrm{e}^{\mathrm{j}k\omega_0 t}$$

where the Fourier Series coefficients are

$$a_k = \frac{1}{T}\int_T \tilde{x}(t)\mathrm{e}^{\mathrm{j}k\omega_0 t}\mathrm{d}t = \frac{1}{T}\int_{-\infty}^{\infty} x(t)\mathrm{e}^{\mathrm{j}k\omega_0 t}\mathrm{d}t$$

Define $X(\mathrm{j}\omega) = \int_{-\infty}^{\infty} x(t)\mathrm{e}^{-\mathrm{j}\omega t}\mathrm{d}t$: Analysis Equation of Fourier Transform, then $a_k = \frac{1}{T}X(j\omega)$. Thus,

$$\tilde{x}(t) = \sum_{k=-\infty}^{\infty} \frac{1}{T}X(\mathrm{j}k\omega_0)\mathrm{e}^{\mathrm{j}k\omega_0 t} = \sum_{k=-\infty}^{\infty} \frac{1}{2\pi}X(\mathrm{j}k\omega_0)\mathrm{e}^{\mathrm{j}k\omega_0 t}\omega_0$$

Fourier Transform III

- As $T \to \infty$, $\omega_0 \to 0$

$$\lim_{\omega_0 \to 0} \sum_{k=-\infty}^{\infty} \frac{1}{2\pi} X(\mathrm{j}k\omega_0) \mathrm{e}^{\mathrm{j}k\omega_0 t} \omega_0 = \int_{-\infty}^{\infty} \frac{1}{2\pi} X(\mathrm{j}\omega) \mathrm{e}^{\mathrm{j}\omega t} \mathrm{d}\omega$$

  As $\tilde{x}(t) \to x(t)$, Synthesis Equation of Fourier Transform of $x(t)$:

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\mathrm{j}\omega) \mathrm{e}^{\mathrm{j}\omega t} \mathrm{d}\omega$$

- Fourier Transform can be applied to periodic and aperiodic signals. Fourier Series can only be applied to periodic signals

- Examples: $x(t) = \mathrm{e}^{-at} u(t)$ for $a > 0$

Properties of Fourier Transform I

- Linearity: if $x_1(t) \longleftrightarrow X_1(\mathrm{j}\omega)$ and $x_2(t) \longleftrightarrow X_2(\mathrm{j}\omega)$

$$a_1 x_1(t) + a_2 x_2(t) \longleftrightarrow a_1 X_1(\mathrm{j}\omega) + a_2 X_2(\mathrm{j}\omega)$$

- Time shifting: $x(t - t_0) \longleftrightarrow \mathrm{e}^{-\mathrm{j}\omega t_0} X(\mathrm{j}\omega)$
- Conjugate: $x^*(t) \longleftrightarrow X^*(-\mathrm{j}\omega)$
- Differentiation and Integration:

$$\frac{\mathrm{d}}{\mathrm{d}t} x(t) \longleftrightarrow \mathrm{j}\omega X(\mathrm{j}\omega)$$

$$\int_{-\infty}^{t} x(\tau)\mathrm{d}\tau \longleftrightarrow \frac{1}{\mathrm{j}\omega} X(\mathrm{j}\omega) + \pi X(0)\delta(\omega)$$

- Time scaling: $x(at) \longleftrightarrow \frac{1}{|a|} X\left(\frac{\mathrm{j}\omega}{a}\right)$

Properties of Fourier Transform II

- Parseval Equality: $\int_{-\infty}^{\infty} \left|x(t)\right|^2 \mathrm{d}t = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left|X(\mathrm{j}\omega)\right|^2 \mathrm{d}\omega$
- Duality: Suppose $x(t) \longleftrightarrow X(\mathrm{j}\omega)$ and $y(t) \longleftrightarrow Y(\mathrm{j}\omega)$. If $y(t)$ has the shape of $X(\mathrm{j}\omega)$, then $Y(\mathrm{j}\omega)$ has the shape of $x(t)$
  Example: $\delta(t) \longleftrightarrow 1$
- Convolution: $x(t) * h(t) \longleftrightarrow X(\mathrm{j}\omega)H(\mathrm{j}\omega)$
- Multiplication: $x(t)h(t) \longleftrightarrow \frac{1}{2\pi}X(\mathrm{j}\omega) * H(\mathrm{j}\omega)$
- Fourier Transform can often be denoted as $X(f)$ instead of $X(\mathrm{j}\omega)$

$$X(f) = \int_{-\infty}^{\infty} x(t)\mathrm{e}^{-\mathrm{j}2\pi f t}\mathrm{d}t$$

$$x(t) = \int_{-\infty}^{\infty} X(f)\mathrm{e}^{\mathrm{j}2\pi f t}\mathrm{d}f$$

Frequency Transfer Function

- LTI system: $y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)\mathrm{d}\tau$
- Fourier transform: $Y(f) = X(f)H(f)$
- Fourier transform of the impulse response function

$$H(f) = \int_{-\infty}^{\infty} h(t)\mathrm{e}^{-\mathrm{j}2\pi f t}\mathrm{d}t$$

  is called *frequency transfer function* or the *frequency response*
- $H(f) = \big|H(f)\big|\mathrm{e}^{\mathrm{j}\theta(f)}$
  - $\big|H(f)\big|$: magnitude response
  - $\theta(f)$: phase response
- <u>Examples:</u> $x(t) = A\cos 2\pi f_0 t$, output will be

$$y(t) = A\big|H(f_0)\big|\cos\left[2\pi f_0 t + \theta(f_0)\right]$$

Distortionless Transmission

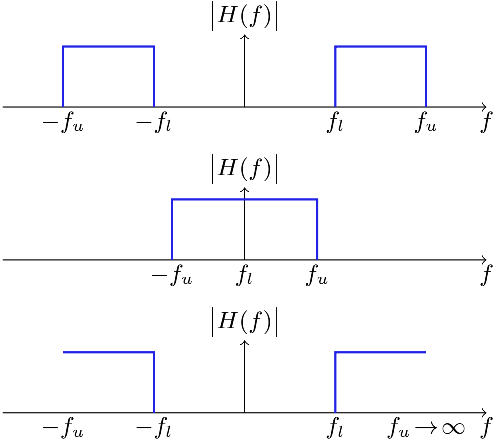
- Ideal system with constant delay and amplifier $y(t) = Kx(t - t_0)$
- Fourier Transform from both sides: $Y(f) = KX(f)\mathrm{e}^{-\mathrm{j}2\pi f t_0}$
- Transfer function

$$H(f) = K\mathrm{e}^{-\mathrm{j}2\pi f t_0}$$

- Ideal distortionless transmission: constant magnitude response and its phase shift must be linear with frequency
- In practice, a signal will be distorted by some parts of a system
- Phase or amplitude correction (equalization) may be required for correction

Ideal Filter

- No ideal network exists: $\left|H(f)\right| = K,\ \forall f \longrightarrow$ infinite bandwidth
- Truncated network: all frequencies in $\left[f_l, f_u\right]$ without distortion
- Passband: $f_l < f < f_u$, bandwidth $W_f = f_u - f_l$



Ideal Bandpass Filter

- Constant magnitude response

$$\left|H(f)\right| = \begin{cases} 1 & \text{for } |f| < f_u \\ 0 & \text{for } |f| \geq f_u \end{cases}$$

- Linear phase response: $\mathrm{e}^{-\mathrm{j}\theta(f)} = \mathrm{e}^{-\mathrm{j}2\pi f t_0}$

- Impulse response of the ideal low-pass filter

$$\begin{aligned} h(t) = \mathcal{F}^{-1}\left\{H(f)\right\} &= \int_{-\infty}^{\infty} H(f)\mathrm{e}^{\mathrm{j}2\pi f t}\mathrm{d}f \\ &= 2f_u \frac{\sin 2\pi f_u(t-t_0)}{2\pi f_u(t-t_0)} \end{aligned}$$

- What is wrong with this impulse response function?

- Realizable filters: Butterworth filter, Raised-cosine filter, etc