MA204: Mathematics IV

Partial Differential Equation

Introduction

Definition (Differential Equation)

Any equation involving derivative or differential of one or more dependent variables with respect to one or more independent variables is called a differential equation.

For example,

(a) $\frac{dy}{dx} = \sin xy + y^2$

(b) $\frac{d^3y}{dx^3} + x^2\frac{d^2y}{dx^2} + y = x^3$

(c) $\frac{d^2y}{dx^2} = k\{y + \frac{dy}{dx}\}^{\frac{3}{2}}$

(d) $\frac{\partial^2 u}{\partial t^2} = c\frac{\partial^3 u}{\partial x^3} + u$

(e) $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$

(f) $z\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = xyz$

(1) **Ordinary Differential Equation (ODE):** Involve derivatives with respect to only one independent variable.

(2) **Partial Differential Equation (PDE):** Involve partial derivatives with respect to more than one independent variable.

Order and Degree of a PDE

A partial differential equation (PDE) for a function $u : D \subseteq \mathbb{R}^n \to \mathbb{R}$ with $n \geq 2$ is a relation of the form

$$F(x_1, x_2, \ldots, x_n, u, u_{x_1}, u_{x_2}, \ldots, u_{x_1 x_1}, u_{x_1 x_2} \ldots) = 0,$$

where F is a given function of the independent variables $x_1, x_2, \ldots, x_n$; of the unknown function u and of a finite number of its partial derivatives.

Definition (Order of a PDE)

The order a PDE is defined as the order of the highest order partial derivative present in the ODE.

Definition (Degree of a PDE)

The degree a PDE is defined as the power to the highest order partial derivative present in the ODE after rationalizing the equation.

The PDE $\frac{\partial^2 u}{\partial x \partial y} + u(\frac{\partial u}{\partial t})^2 = xy$ is of order 2 and degree 1.

The PDE $\frac{\partial^3 u}{\partial x^3} = \{x(\frac{\partial u}{\partial t})^2 - xy\frac{\partial^2 u}{\partial x \partial y}\}^{\frac{3}{2}}$ is of order 3 and degree 2.

Formation of PDE

Eliminating arbitrary Constants: Let

$$f(x_1, x_2, \ldots, x_k, u, a_1, a_2 \ldots, a_k) = 0 \tag{1}$$

represent a k-parameter family of figure in $\mathbb{R}^k$, where $a_1, a_2 \ldots, a_k$ are arbitrary constants. Differentiating (1) with respect to $x_1, x_2, \ldots, x_k$, we obtain

$$\frac{\partial f}{\partial x_1} + \frac{\partial u}{\partial x_1}\frac{\partial f}{\partial u} = 0, \frac{\partial f}{\partial x_2} + \frac{\partial u}{\partial x_2}\frac{\partial f}{\partial u} = 0, \ldots, \frac{\partial f}{\partial x_k} + \frac{\partial u}{\partial x_k}\frac{\partial f}{\partial u} = 0.$$

Eliminating the arbitrary constants $a_1, a_2, \ldots, a_k$ from the above equations, we get a relation of the form

$$F(x_1, x_2, \ldots, x_k, u, \frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, \ldots, \frac{\partial u}{\partial x_k}) = 0,$$

which is the required PDE with unknown function u and k independent variables $x_1, x_2, \ldots, x_k$ corresponding to (1).

Formation of PDE

Eliminating arbitrary function: Let $u_1, u_2, \ldots, u_k$ be any k functions of independent variables $x_1, x_2, \ldots, x_k$, and u. If F is an arbitrary function of $u_1, u_2, \ldots, u_k$ given by

$$F(u_1, u_2, \ldots, u_k) = 0. \tag{2}$$

Partial differentiation of (2) yields

$$\frac{\partial F}{\partial u_1}\left(\frac{\partial u_1}{\partial x_1} + \frac{\partial u_1}{\partial u}\frac{\partial u}{\partial x_1}\right) + \ldots + \frac{\partial F}{\partial u_k}\left(\frac{\partial u_k}{\partial x_1} + \frac{\partial u_k}{\partial u}\frac{\partial u}{\partial x_1}\right) = 0$$

$$\vdots$$

$$\frac{\partial F}{\partial u_1}\left(\frac{\partial u_1}{\partial x_k} + \frac{\partial u_1}{\partial u}\frac{\partial u}{\partial x_k}\right) + \ldots + \frac{\partial F}{\partial u_k}\left(\frac{\partial u_k}{\partial x_k} + \frac{\partial u_k}{\partial u}\frac{\partial u}{\partial x_k}\right) = 0$$

Thus eliminating the arbitrary constant F, we obtain the required PDE corresponding to (2).

Problem

Problem: Find the PDE corresponding to the following equations:

(a) $z = axe^y + \frac{a^2e^{2y}}{2} + b$
(b) $2z = (ax + y)^2 + b$
(c) $\phi(x + y + z, x^2 + y^2 + z^2) = 0$
(d) $z = f(x^2 - y) + g(x^2 + y)$

First order PDE

If one takes the two parameter family of surfaces in $\mathbb{R}^3$ given by

$$f(x, y, u, a, b) = 0$$

or an arbitrary function in two variables u, v depending on x, y, z given by

$$F(u(x, y, z), v(x, y, z)) = 0,$$

then the PDE obtained by eliminating the arbitrary constant or the arbitrary function from the given equations is always of first order.

The PDE corresponding to

$$x^2 + y^2 + (z - a)^2 = b^2$$

is

$$yz_x - xz_y = 0.$$

The PDE corresponding to

$$z = f(x^2 + y^2)$$

is

$$yz_x - xz_y = 0.$$

Classification of first order PDE

Recall that the general form of a PDE is

$$F(x_1, x_2, \ldots, x_n, u, u_{x_1}, u_{x_2}, \ldots, u_{x_1 x_1}, u_{x_1 x_2} \ldots) = 0,$$

where F is a given function of the independent variables $x_1, x_2, \ldots, x_n$; of the unknown function u and of a finite number of its partial derivatives.

A **first order PDE** in two independent variables x, y and the dependent variable u can be written in the form

$$F\left(x, y, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right) = 0.$$

(a) **Linear PDE:** A PDE is linear if F is linear in u and its derivatives, that is,

$$a(x, y)\frac{\partial u}{\partial x} + b(x, y)\frac{\partial u}{\partial y} = c(x, y)u + d(x, y).$$

Classification of first order PDE

(b) **Semilinear PDE:** A PDE is called semilinear if it is linear in the leading (highest-order) terms, that is,

$$a(x,y)\frac{\partial u}{\partial x} + b(x,y)\frac{\partial u}{\partial y} = c(x,y,u).$$

(c) **Quasi-linear PDE:** A PDE called quasi-linear if F is linear in all the derivatives, that is

$$a(x,y,u)\frac{\partial u}{\partial x} + b(x,y,u)\frac{\partial u}{\partial y} = c(x,y,u).$$

However, the coefficients a, b and c may depend on the independent variables x and y as well as on the unknown u.

(d) **Non-linear PDE:** If F is not linear in the derivatives $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$, then the PDE is said to be nonlinear.

Classification of first order PDE

Linear PDE $\subset$ Semi-linear PDE $\subset$ Quasi-linear PDE $\subset$ PDE

Problem: Classify the following PDE

(a) $xu_x + yu_y = u$

(b) $xu_x + yu_y = u^2$

(c) $u_x + (x+y)u_y = xy$

(d) $uu_x + u_y = 0$

(e) $xu_x^2 + yu_y^2 = 2$

(f) $x^2yu_x + xu_y = y^3u$

(g) $xu_y - yu_x = xu^2$

(h) $u_xu_y = u$

PDE

The main objective of this course is to find solution of a PDE along with initial and/or boundary conditions.

We shall discuss linear, quasi-linear, and nonlinear first-order PDEs involving two independent variables.

We further deal with linear second-order PDEs in two independent variables.

Definition (Solution of a PDF)

For the PDE

$$F(x_1, x_2, \ldots, x_n, u, u_{x_1}, u_{x_2}, \ldots, u_{x_1 x_1}, u_{x_1 x_2} \ldots) = 0,$$

a solution is a function u in the terms of the independent variables $x_1, x_2, \ldots, x_n$ such that it and its derivatives satisfy the given equation in some domain.

Solution of a PDE

For the PDE

$$u_x = 0,$$

we integrate to obtain the solution as

$$u = u(x, y) = c(y)$$

for any arbitrary function c of y.

Problem: Discuss solutions of the equation $u_x = u + c(x, y)$ under the conditions $u(0, y) = 0$.

Problem: Discuss solutions of the equation $u_x = u$ under the conditions (a) $u(x, 0) = 2x$ and (b) $u(x, 0) = e^x$.

Thank you

Thank You!!