MA204: Mathematics IV

Partial Differential Equation (First Order PDE)

Introduction

We have seen that a relation of the form

$$F(x, y, z, a, b) = 0 \tag{1}$$

give rise to a PDE of first order given by

$$f(x, y, z, z_x, z_y) = 0. \tag{2}$$

Thus a relation (1) containing two arbitrary constants is a **complete solution or complete integral** of the PDE (2).

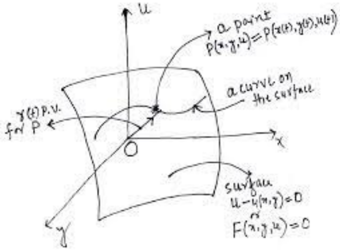
Definition (Solution of a first order PDE)

A function $z = z(x, y)$ or $F(x, y, z) = 0$ is a solution to the first order PDE (2) if z and its partial derivatives appearing in the PDE satisfy the PDE identically for (x, y) in some region $\Omega \subseteq \mathbb{R}^2$.

A solution of a PDE (2) in the form $F(x, y, z) =$ or $z = z(x, y)$, geometrically represents a surface in $\mathbb{R}^3$, is called a **integral surface or solution surface**.

Introduction

In other words, any point (x, y, z) on the integral surface will satisfy the first order PDE (2).



A curve on the solution surface of the PDE (2) is called a **solution curve** to the PDE (2).

Any point (x, y, z) on the solution curve of (2) also satisfies the first order PDE (2).

Semilinear equations

We consider the first order semilinear PDE given by

$$a(x,y)\frac{\partial z}{\partial x} + b(x,y)\frac{\partial z}{\partial y} = c(x,y,z) \tag{3}$$

Suppose C is a solution curve to the PDE (3).

We use the curve C for (3) to reduce it to a simple equation.

In fact, we reduce the PDE (3) into an ODE along C, and then find solution for the given equation.

This process or method to obtain solution of the semilinear PDE (3) is known as 'METHOD of CHARACTERISTICS'.

Method of Characteristics

In this method, we consider the parametric equation for the solution curve C given by

$$x = x(t), \quad y = y(t), \quad z = z(t),$$

with $t \in I$, I is a suitable interval.

Thus a moving particle whose position at time t is $(x, y, z) = (x(t), y(t), z(t))$ traces out the curve C.

As a result, the semilinear PDE (3) reduces to

$$a(t)\frac{\partial z}{\partial x} + b(t)\frac{\partial z}{\partial y} = c(t) \tag{4}$$

along the curve C.

Again, the chain rule yields

$$\frac{dz}{dt} = z_x \frac{dx}{dt} + z_y \frac{dy}{dt}. \tag{5}$$

Method of Characteristics

Comparing (4) and (5), we obtain the system of ODEs given by

$$\frac{dx}{dt} = a(t),$$
$$\frac{dy}{dt} = b(t),$$
$$\frac{dz}{dt} = c(t),$$

which are called **characteristic equations** for the PDE

$$a(x,y)z_x + b(x,y)z_y = c(x,y,z).$$

The solution curves of the characteristic equation are the **characteristic curves** for

$$a(x,y)z_x + b(x,y)z_y = c(x,y,z).$$

Geometry of the method of characteristics

Let $z = z(x, y)$ or $F(x, y, z) = 0$ be the integral surface for the semilinear PDE (3), and C a solution curve (characteristic curve) on the integral surface.

Let $P(x, y, z) = P(x(t), y(t), z(t))$ be a point on the curve C with

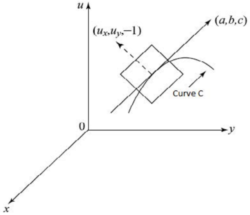
$$\overline{OP} = \vec{r} = \vec{r}(t) = (x(t), y(t), z(t)).$$

Then the tangent vector to the curve C at P is

$$\vec{r}'(t) = (x'(t), y'(t), z'(t)).$$

Again, the normal to the surface $F(x, y, z) = 0$ is given by

$$\nabla F = (z_x, z_y, -1).$$



Geometry of the method of characteristics

Thus, we must have

$$(a(t), b(t), c(t) \cdot (z_x, z_y, -1) = 0$$

along the curve C.

This implies that $(a(t), b(t), c(t))$ is a tangent vector to C at $P(x(t), y(t), z(t))$.

We already have a tangent vector to the curve C at P as

$$\vec{r}'(t) = (x'(t), y'(t), z'(t)).$$

As a result,

$$\frac{dx}{dt} = a(t) \Rightarrow x(t) = x(0) + \int_0^t a(t)dt,$$
$$\frac{dy}{dt} = b(t) \Rightarrow y(t) = y(0) + \int_0^t b(t)dt,$$
$$\frac{dz}{dt} = c(t) \Rightarrow z(t) = z(0) + \int_0^t c(t)dt.$$

Method of Characteristics

For the method characteristics, we note the following points:

(a) Thus location of point $P(x, y, z) = P(x(t), y(t), z(t))$ on the solution curve C is known completely provided the point $(x(0), y(0), z(0))$ is given.

(b) Any point $(x(0), y(0), u(0))$ on the solution surface will be called as a initial point.

(c) For n numbers of initial points P_n on the integral surface, we determine n numbers of solution curves. However, this is not enough to generate the whole solution surface.

(d) Suppose Γ is a curve on the integral surface passing through all initial points, then we call Γ an initial curve on the integral surface.

(e) Considering each characteristic curve passing through the points on initial curve, we can construct the integral surface.

(f) The solution curve passes through $(x(0), y(0), z(0))$ intersect the initial curve at $(x(0), y(0), z(0))$ as the initial curve passes through all initial points.

Initial curve and solution curve

Let the parametric form of initial curve Γ be given as

$$\begin{aligned}\Gamma &= \{(x, y, z) \in S : S \text{ is the solution surface}\} \\ &= \{(x_0(\tau), y_0(\tau), z_0(\tau)) : \text{ for some } \tau \text{ belongs to an interval } J\}.\end{aligned}$$

Thus

$$x(0) = x_0(\tau), y(0) = y_0(\tau), z(0) = z_0(\tau).$$

Thus the solution curve C is given by

$$\begin{aligned}x(t) &= x_0(\tau) + \int_0^t a(t)dt = x(t, \tau), \\ y(t) &= y_0(\tau) + \int_0^t b(t)dt = y(t, \tau), \\ z(t) &= z_0(\tau) + \int_0^t c(t)dt = z(t, \tau).\end{aligned}$$

Problem

Problem (Cauchy's problem or IVP for first-order linear PDEs) Find integral surface of the PDE

$$az_x + bz_y = cz + d$$

containing a initial curve

$$\Gamma = \{(x_0(\tau), y_0(\tau), z_0(\tau)) : \tau \in J\}.$$

Problem

Problem: Determine the solution the following IVP

$$u_y + cu_x = 0, u(x,0) = f(x),$$

where $f(x)$ is a given function and c is a constant.

Problem

Problem: Find the general integral of the PDE

$$y^2 z_x - xyz_y = x(z - 2y).$$

Thank you

Thank You!!