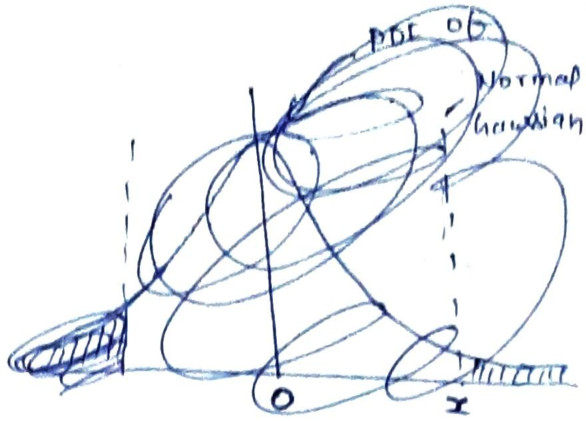
1. Q - function :-

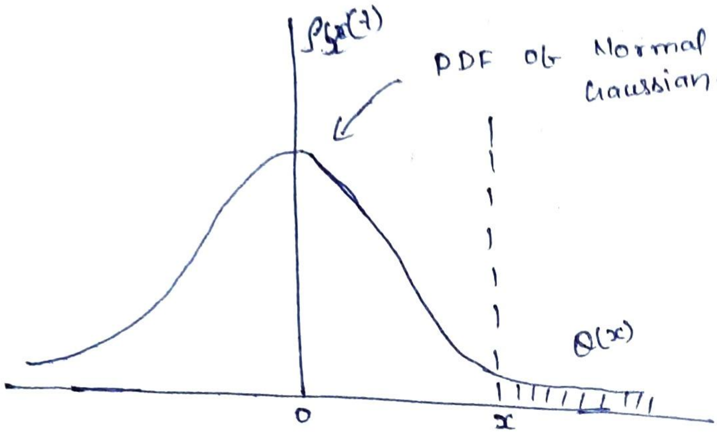
$$Q(x) = \int_{x}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}} \, dt$$

$\mathcal{N}(0,1)$

mean variance

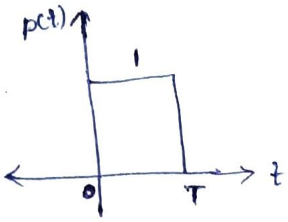


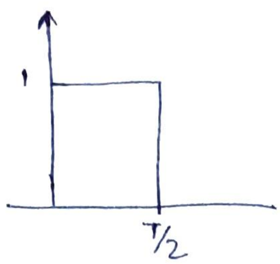
It is tail probability.



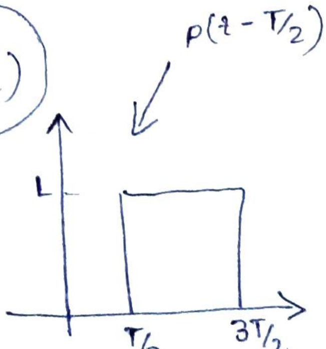
$$\boxed{Q(x) = \frac{1}{2} \operatorname{erfc}\left(\frac{x}{\sqrt{2}}\right).}$$

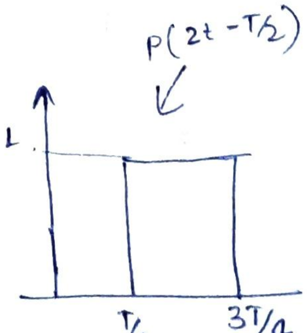
2. Signal time Shifting and Scaling





$p(2t - T/2)$

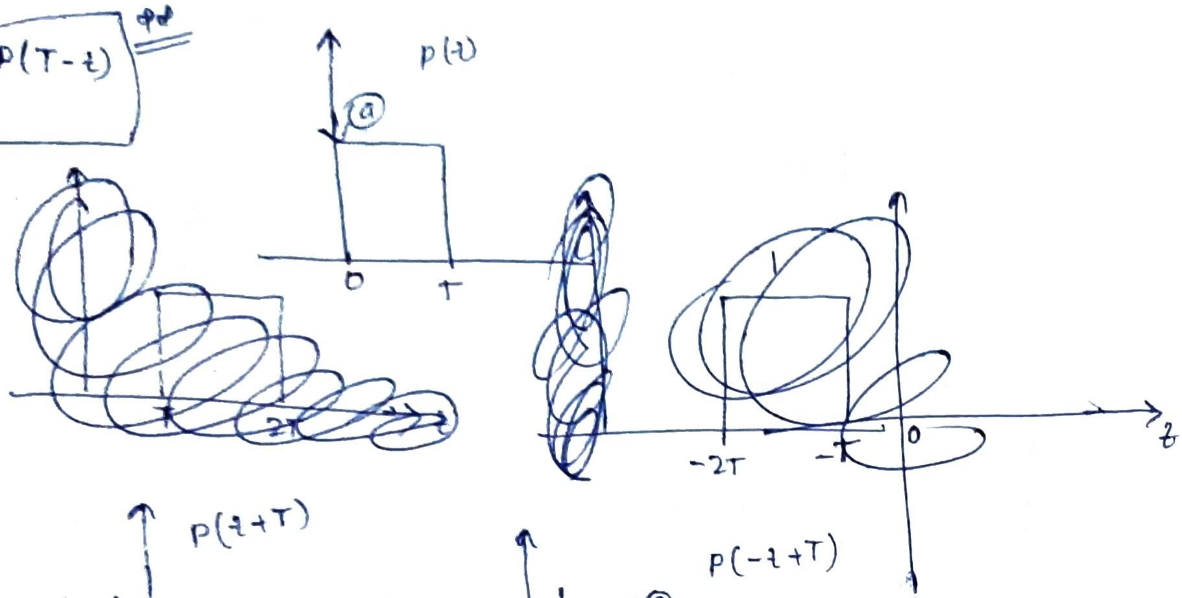


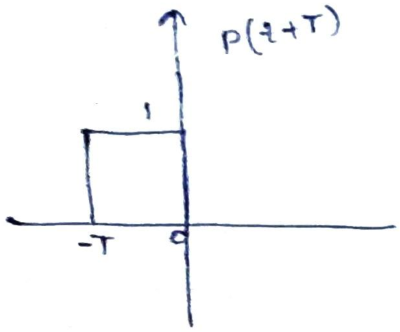


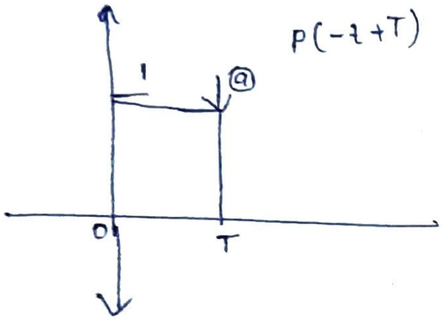
$p(2t - T/2) = p(2(t - T/4))$

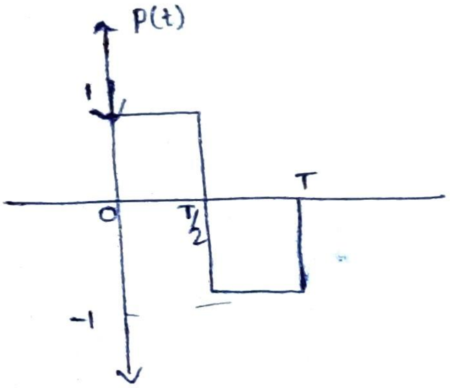
① Shifting
② Scaling — Rule.

$\boxed{p(T-t)}$

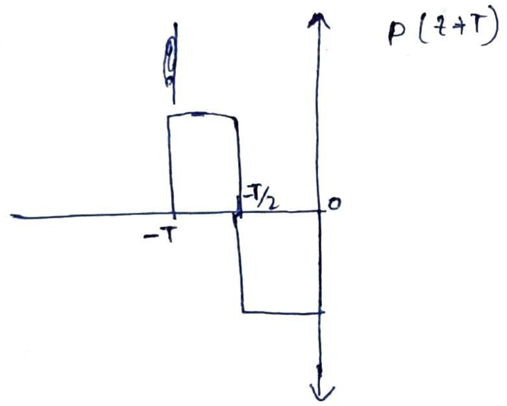


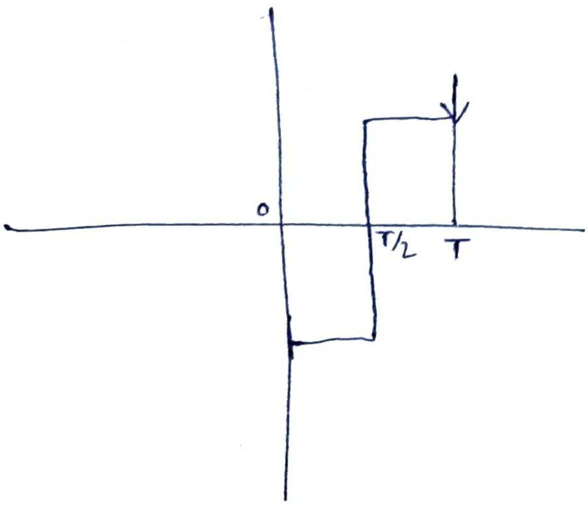






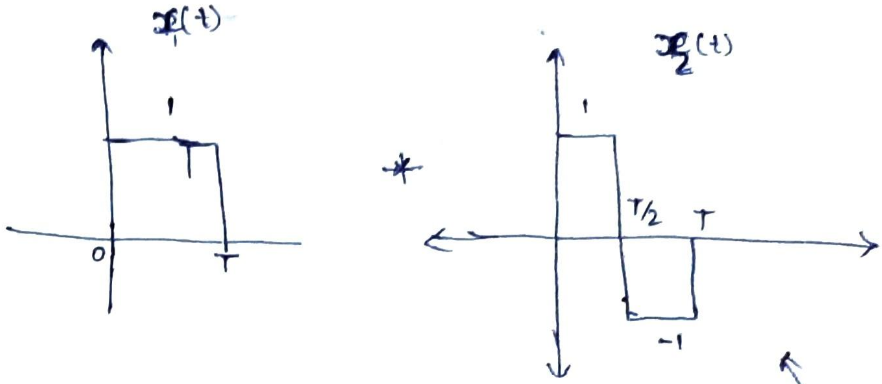
→

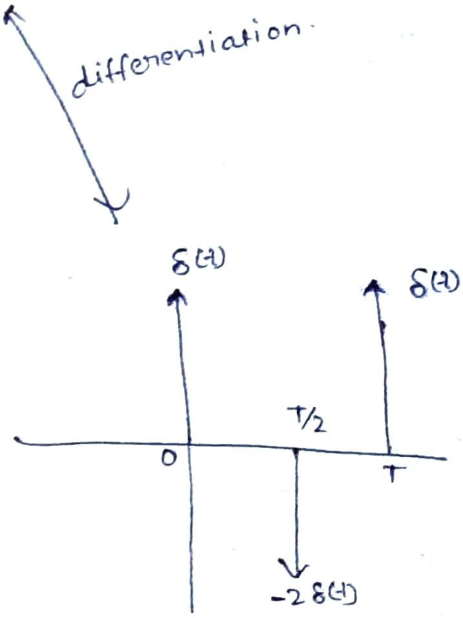




$p(-t+T)$.

Symmetric about y-axis
✓
$p(t) = p(-t+T)$.





$x_1(t) * x_2(t) = ??$

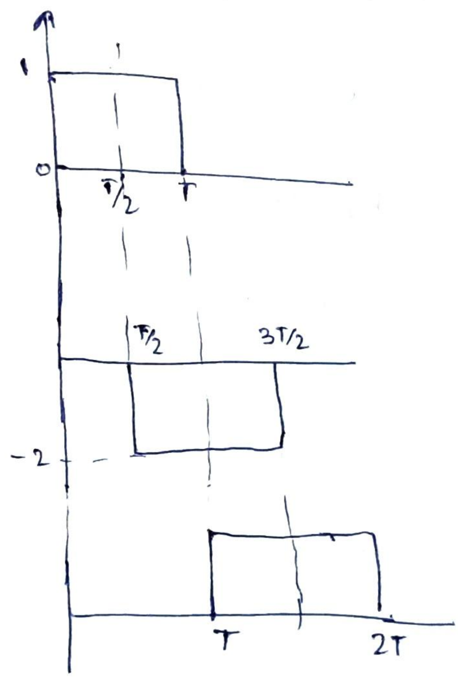
$$\delta(t-\tau) * x(t) = x(t-\tau)$$
$$x(t) * y(t) = \int x(t) * \frac{d}{du} y(u) \Big|_{u=t} dt$$

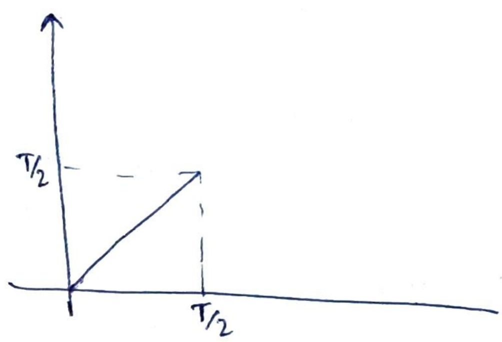
$x_1(t) * x_2(t) =$

$= x_1(t) * \delta(t) + x_1(t) * \left(-2\,\delta(t-T/2)\right)$

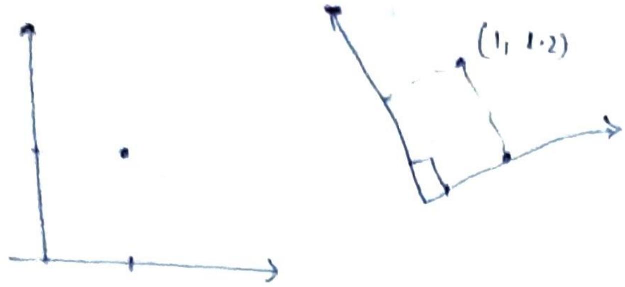
$+ x_1(t) * \delta(t-T).$

$= x_1(t) - 2\,x_1(t-T/2) + x_1(t-T)$



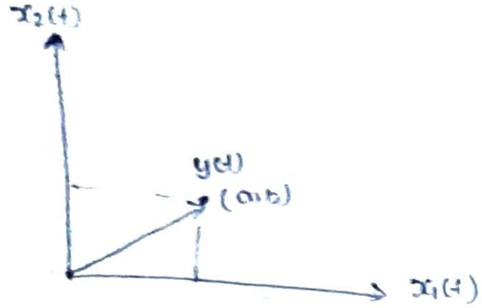


Vectors : $\{1, 1.2\}$



$y(t) = a\, x_1(t) + b\, x_2(t)$

Basis dimension / kernal.



Statistics

$X \rightarrow f_X(x)$

$X \downarrow x$

random Variable

$\boxed{Y = \pm a + X}$ — $f_Y(y) = f_X(y \mp a)$

new Variable

$f_Y(y) = ??$

$F_Y(y) = P_r[Y \le y]$

CDF of Y

$= P_r[a + X \le y]$

$= P_r[X \le y - a]$.

$F_Y(y) = F_X(y-a)$

differential

$f_Y(y) = f_X(y-a)$.

$Y = aX$

$F_Y(y) = P_r[Y \le y]$

$= P_r[aX \le y]$

$= P_r[X \le y/a]$

$F_Y(y) = F_X(y/a)$.

$f_Y(y) = \frac{1}{|a|} f_X(y/a)$.

Anteena length $\lambda/2$ to $\lambda/4$; 10 Hz to 3.5 kHz.

← Bonds → birds.

$$\lambda = \frac{c}{f} = \frac{\cancel{3} \times 10^8}{\cancel{3} \times 10^3}$$

$= 10^5$ m.

$= 100$ km

(50 to 25) km

Anteena length.

Sensitivity in device

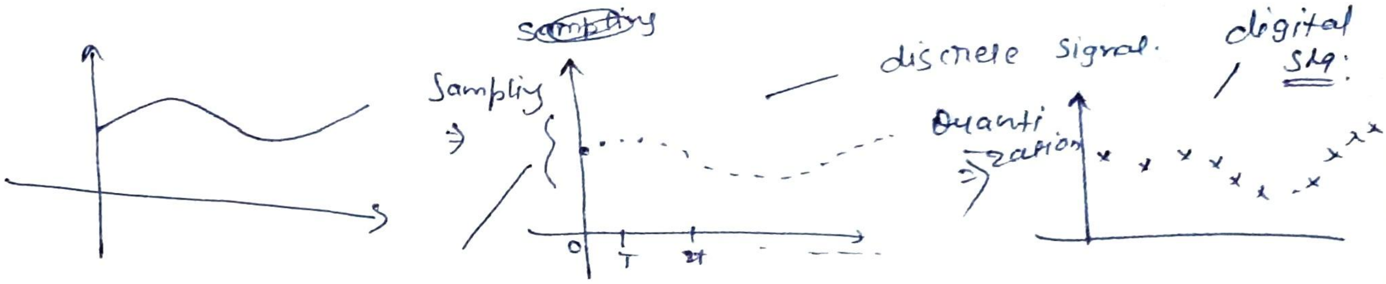
SNR can be maximize. by.

1. Reduce noise
2. Increase signal power

Continous Signal → Continous in time and value.

discrete Signal → Values are continuous.

digital Signal →



CTS.

Continous Sig + discrete Sig } ⇒ Analog. Sig

Baseband Signal :- near to origin. (message signal)

passband Signal :- high frequency. (modulated signal).

03/08/23

Channel

wired — In this course

wireless

- RF
- Visible light Communication,
- IR communication.

Channel

wired — wireless

wired: Narrowband — wideband

Narrowband: $(B.W)_{msg} \ll (B.W)_{chanel}$

wideband: $(B.W)_{msg} >> (B.W)_{channel}$.

MATLAB

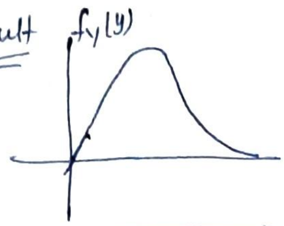
$Y = X_1 + X_2 + X_3 + X_4 \ldots$

uniformly distributed random variable.

$[0,1]$

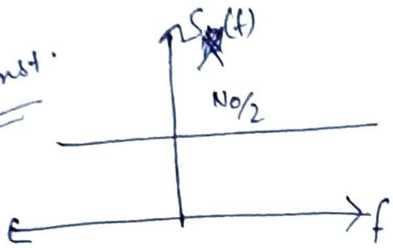
plot $f_Y(y)$ — PDF of Y.

result



White noise
In all frequency, same noise colour

freq vs PSD = const.

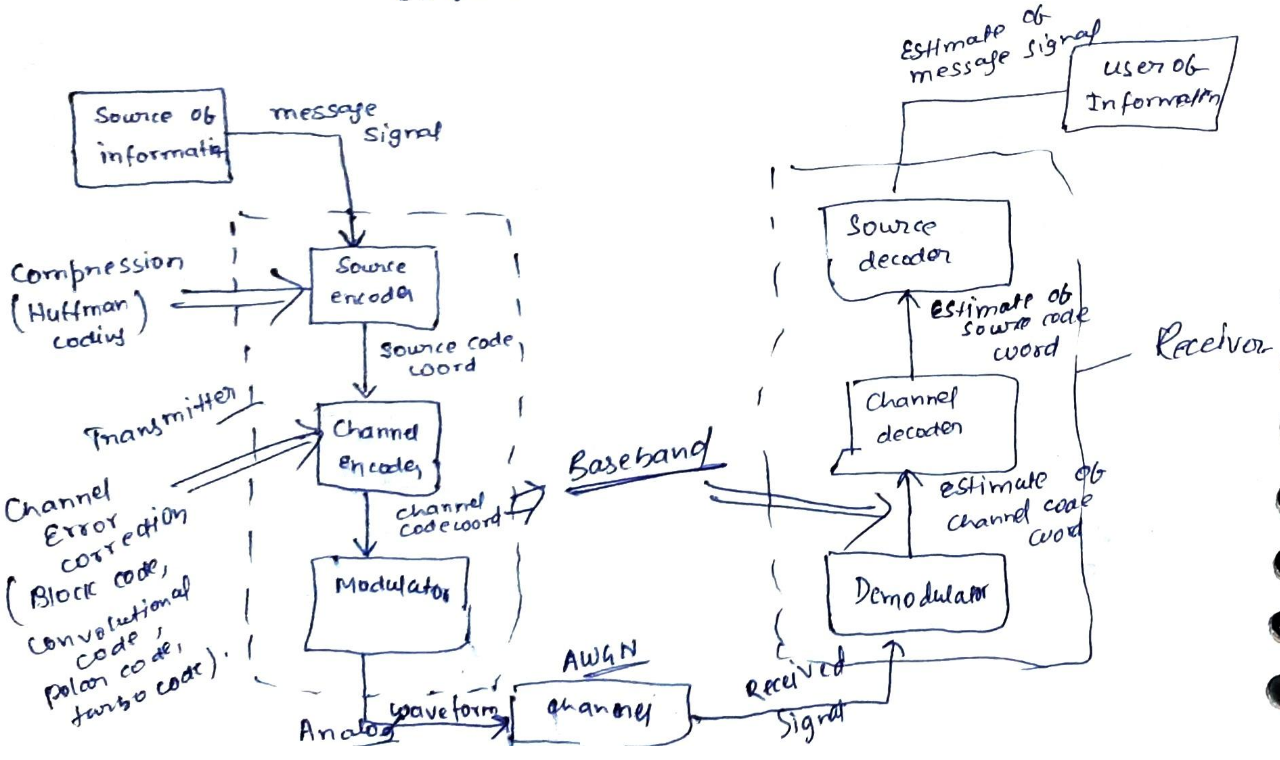


random variables
mean = 0
Variance = $\frac{No}{2}$.

$\frac{No}{2}$ power because (consider +ve half frequency).

07/08/23

Block-diagram
(Digital communication)



Digital Modulation Techniques

Modulator :- Freq upscalar.

S(t) :- modulated waveform.

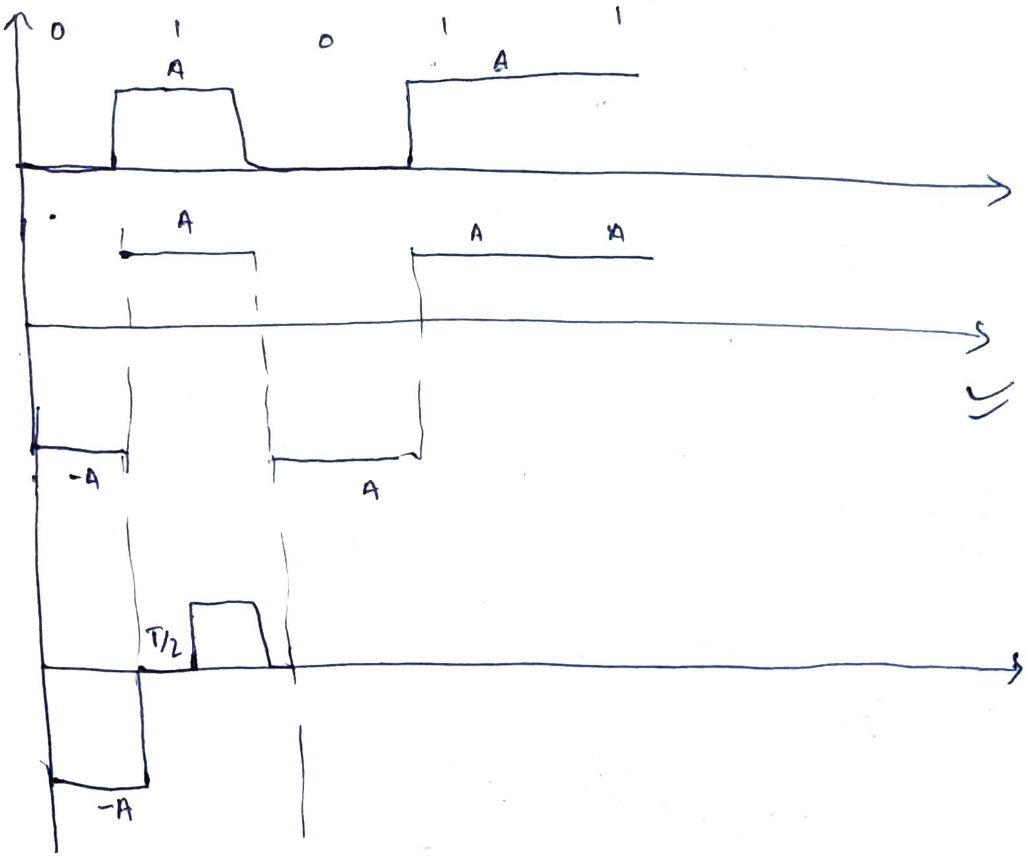
Baseband Modulation Scheme.

M. Imp.

⇒ Low pass communication.

0	1	0	1	1	- - - -	
↓	↓	↓	↓	↓		
0V	5V	0V	5V	5V	- ———	ON-Off Signaly
-AV	AV	- - - -			— —	← NRZ encoding

non-return to zero.



less dc component.

- to minize dc component ⇒ different coding Scheme introduced.

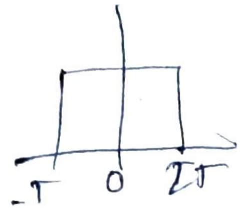
It doesn't contain any information.

Note :- In this course, we are going to use NRZ line codes.

(H.W)

① How many line codes are there?

② What is most effective line codes in terms DC component.

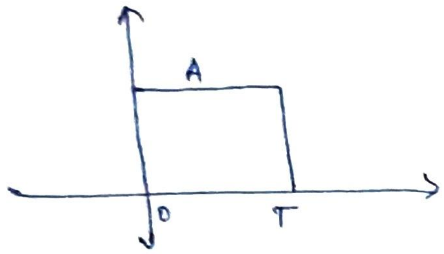


rect (2t/4T)

rect (2t ± T).

0 → -AV

1 → +AV



$$= A \int_0^T e^{-j2\pi ft}\, dt.$$

$$A \left[\frac{e^{-j2\pi ft}}{-j2\pi f} \right]_0^T$$

$$= \frac{A}{-j2\pi f} \cdot \left[e^{-j2\pi fT} - 1 \right]$$

$$= \frac{A}{j2\pi f} \left[1 - e^{-j2\pi fT} \right]$$

$$= 2\cancel{j}\, \frac{A}{\cancel{j}2\pi f}\, e^{-j2\pi fT/2} \left[\frac{e^{j2\pi fT/2} - e^{-j2\pi fT/2}}{2j} \right]$$

$$= \frac{2A}{2\pi f}\, e^{-j2\pi fT/2} \sin(2\pi f T/2).$$

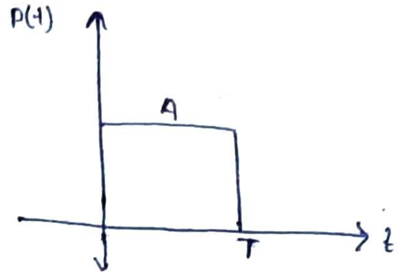
$$\frac{\sin \pi x}{\pi x} = \text{sinc}(\pi)$$

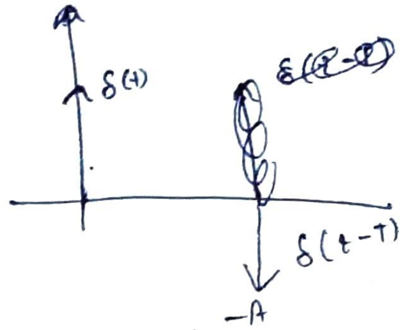
$$= AT\, \text{Sinc}(fT)\, e^{-j2\pi f\frac{T}{2}}.$$

→

09/08/23 (T)

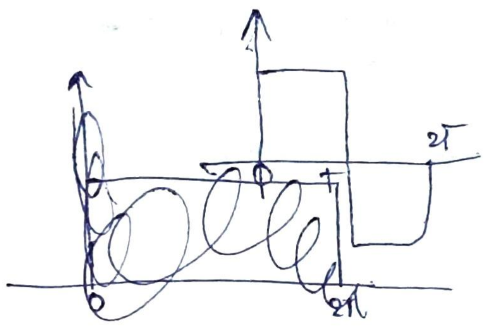
Q1. Convolve $p(t)$ with itself.



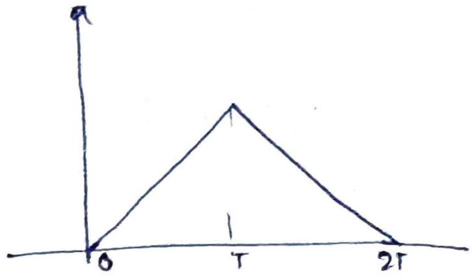


$$p(t) * p(t) = p(t) * \delta(t) + p(t) * \delta(t-T)$$

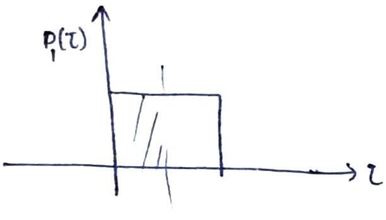
$$= p(t) + p(t-T)$$

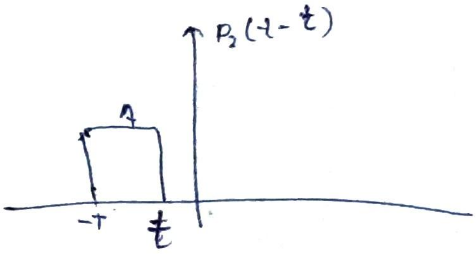


Integration



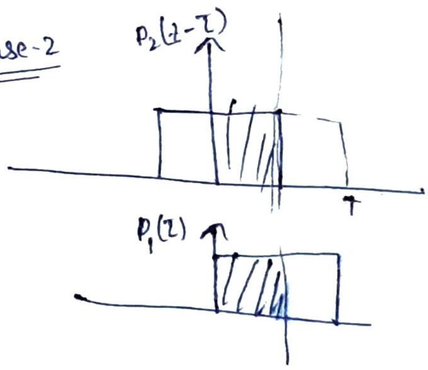
① $t < 0$



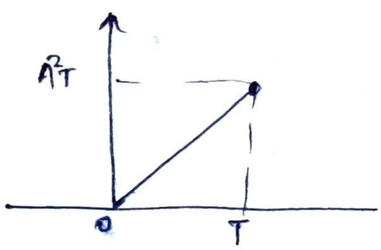


common = 0.

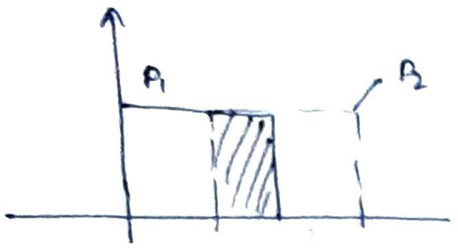
Case-2

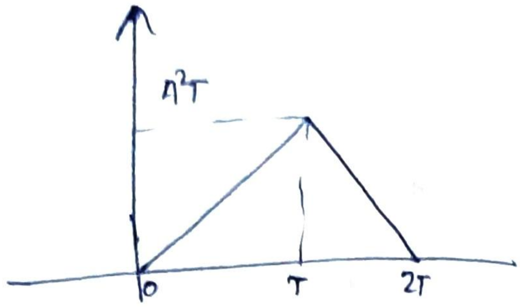


$0 \leq t \leq T$



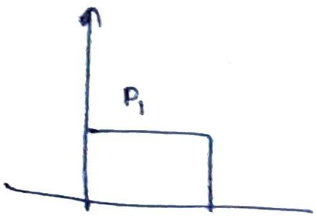
Case-3

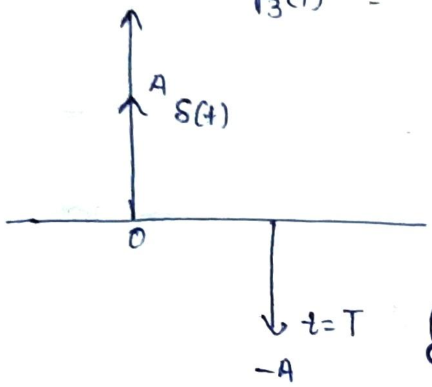




Method-2

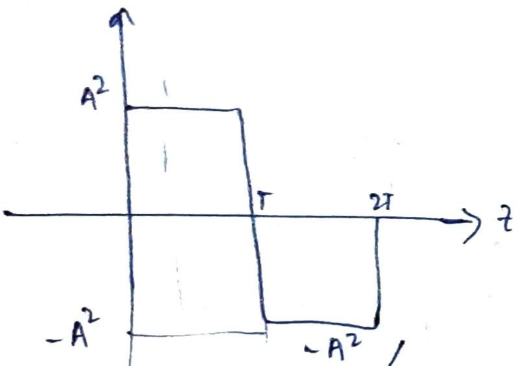
$P_3(t) = P_2'(t)$



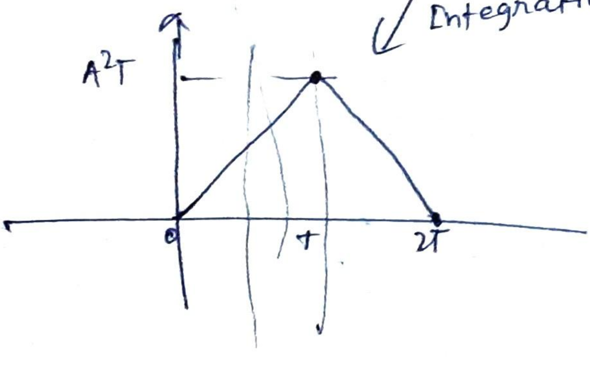


$\delta(-t+T)$

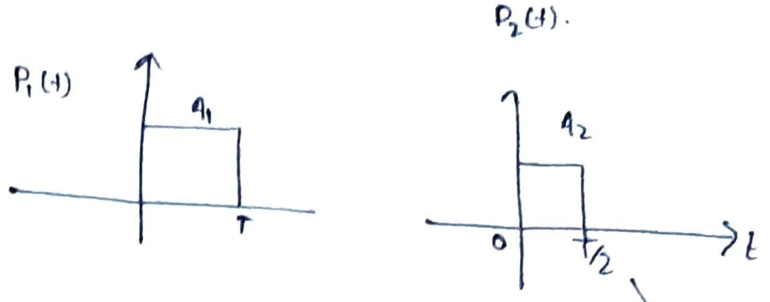
$P_1(t) * P_3(t) = P_1(t) * A\,\delta(t) + P_1(t) * A\,\delta(t-T)$

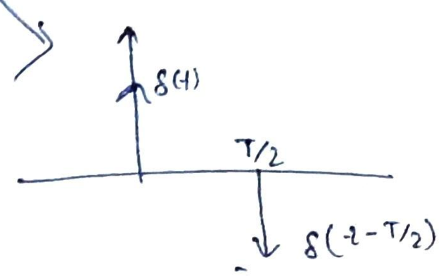


Integration



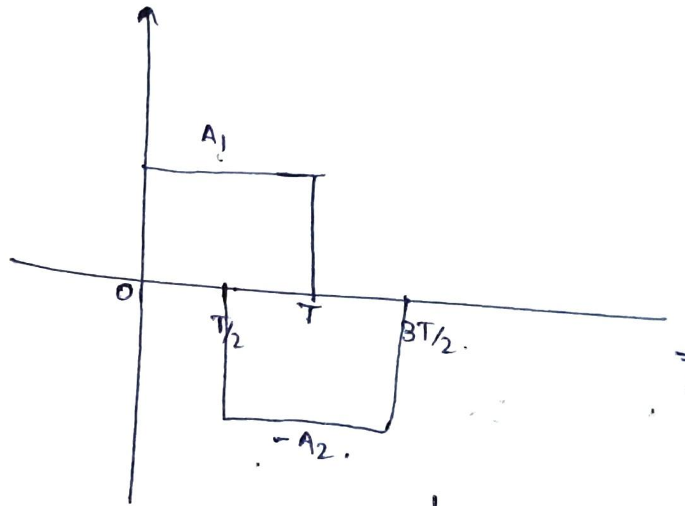
Q2.

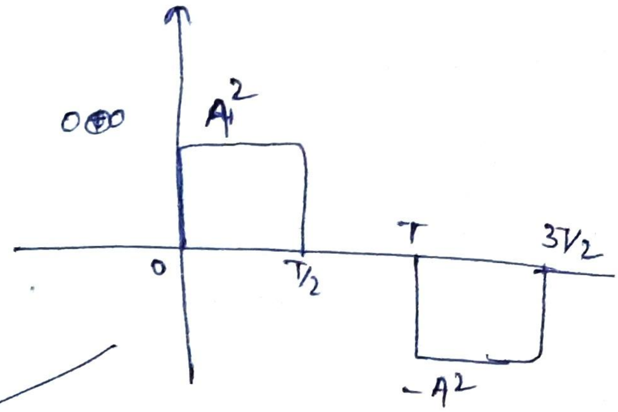


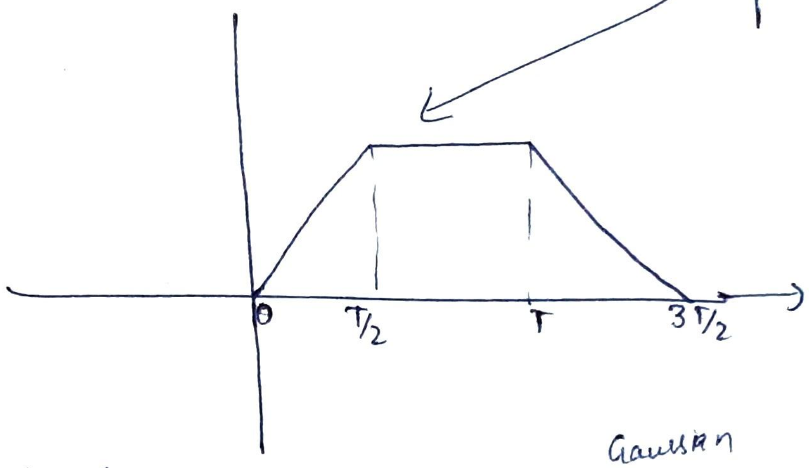


$P_1(t) * A_1\delta(t) + P_2(t) * -A_2\,\delta(t - T/2)$

$A_1 \; P_1(t) + A_2 \; P_1(t - T/2)$







Q. If X ~ N(0, 1) ; If Y is another Gaussian r.v with mean

Gaussian and Variance $\frac{N_0}{2}$: what is the relation-ship b/w X and Y.?

ans:

$$Y = aX + b$$

$$E[Y] = a\,E[X] + E[b]$$

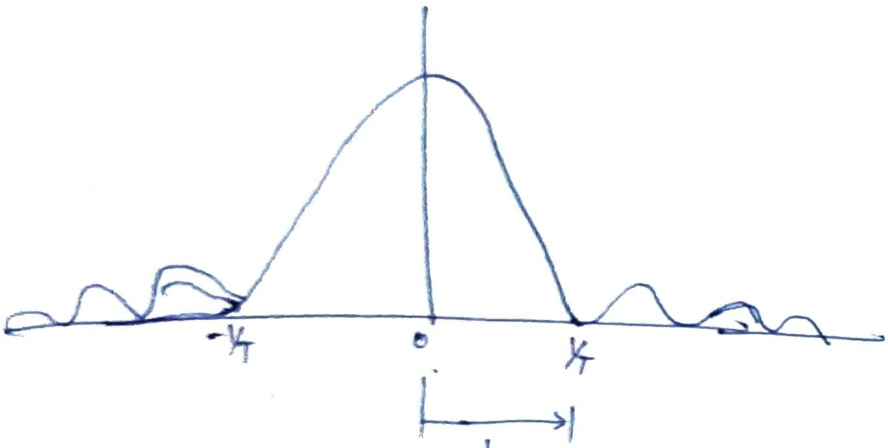
$$= a\cdot 0 + b \qquad E[Y] = b$$

$$Var[Y] = E[(Y-\mu)^2]$$

$$a = \sqrt{\frac{N_0}{2}}$$

09/08/23

→ we want to transmit more bit per unit of freq (Hz).



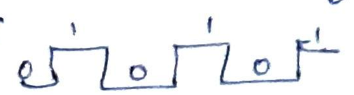
Ideal B.W of Sinc funⁿ is '∞'.

⇒ we will assume the channel is ideal i.e the channel has '∞' Bandwidth.

$x(t) = s(t) + n(t)$

Random process (AWGN).

Base band

$s(t) =$ () × (carrier) $+ n(t)$.

DMS (Discrete memoryless source). Probability of 0 & 1 are independent

iid.

inid — Independent non-identical distribution.

Q
PSD
Autocorrelation fn
Exam

PSD - fourier representation of random process.

PSD of Binary wave

$b_n \rightarrow (1 \mid 0 \mid 1 \mid 0 \mid 0 \mid 1 \mid 1 \mid - - -) \; p(t).$

$T \quad 2T \quad 3T \quad 4T \quad 5T \quad 6T \quad 7T$

PSD of $P_b(t)$ is defined as

Autocorrelation.

$$S_b(f) = F.T \{ R_b(\tau) \}$$

$$= F.T. \{ E [P_b(t) \cdot P_b(t+\tau)] \}$$

$$R_b(\tau) = R_b(-\tau).$$

Assignment

Q. Correlation and Independency is not same. ??

random process :- sequence of random variable associated with time.

Q. WSS process is also strict sense Gaussian process. Proof:

14/08/23

⇒ To get PSD of a random process, we need to Evaluate $R_x(\tau) = E[x(t) \cdot x(t+\tau)]$.

$$= E[x(t) \cdot x(t-\tau)] \_\_ R_x(-\tau).$$

H.W Example 3.4 | H.W example 3.5 | Example - 3.6

$x(t) = A \cos(2\pi f_0 t + \theta)$ where θ is a random variable uniformly distributed on $[0, 2\pi]$. This process is both stationary and ergodic. proof

$R_w(\tau) = \frac{N_0}{2}\,\delta(t)$

Auto-correlation function.

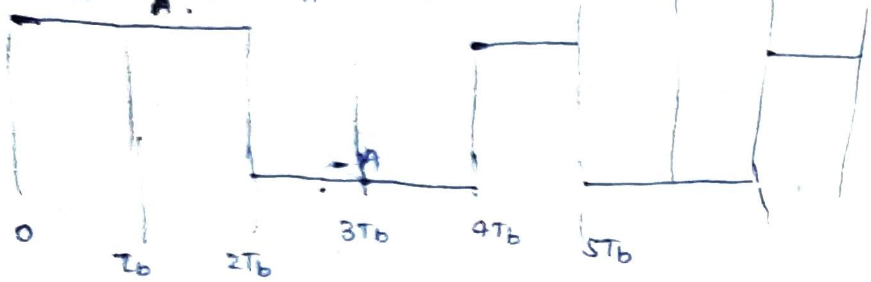
Line code

① NRZ

S_m	1	1	0	0	1	0	0	1
b_n	A	A	-A	-A	A	-A	-A	A

we define b_n according to S_m.

NRZ



Since the NRZ waveform is random process, we need to evaluate its PSD to determine its frequency spectrum.

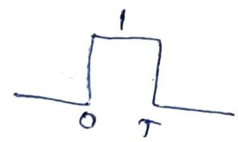
$R_x(\tau) = E[x(t)\cdot x(t+\tau)]$

$x(t) = \begin{cases} A & 0 \le t \le 2T_b \\ -A & 2T_b \le t \le 4T_b \end{cases}$

$P_r(0) = P_r(1) = \frac{1}{2}.$

$x(t) = \sum_{n=-\infty}^{\infty} b_n\, p(t - nT_b)$

$p(t) =$



$R'_x(\tau) = \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} E\left[\underbrace{b_n b_m}_{\text{random variable}}\, p(t - nT_b)\, p(t - mT_b + \tau)\right]$

$$= \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} E[b_n b_m]\ p(t - nT_b)\ p(t - mT_b - \tau)$$

$x \cdot \frac{1}{2}$

$x \cdot \frac{1}{2}\, dx$

$\int_{-\infty}^{\infty} \frac{x^2}{2}$

$\frac{1}{4} \quad x^2$

$\underbrace{b_n = b_m}$, $b_n \neq b_m$

Probability $= 1/4$ ✓   for $m \neq n$

NRZ $\quad E[b_n b_m] = 0 \quad = \frac{1}{4}A^2 + \frac{1}{4}A^2 + \frac{1}{4}(-A^2) + \frac{1}{4}(-A^2).$

for $m = n$,

Tutorial 16/08/23

Q. Consider a sinusoidal process $X(t) = A\cos(2\pi f_c t)$ where the frequency f_c is constant and the amplitude A is uniformly distributed

$$f_A(a) = \begin{cases} 1 & 0 \leq a \leq 1 \\ 0 & \text{o.w.} \end{cases}$$

Determine whether or not this process is strictly stationary.

Soln:-

$E_A[X(t)] = \cos(2\pi f_c t) \cdot E_A[A]$

Expectation is determined on the random variable

$= \cos(2\pi f_c t) \cdot \int_0^1 a\, f_A(a)\, da$

$= \cos(2\pi f_c t) \int_0^1 a\, da = \frac{1}{2}\cos(2\pi f_c t)$

$$\boxed{\begin{array}{lll} 1^{st} \text{ order stationary} & \longrightarrow & \text{means should be constant.} \\ 2^{nd} \text{ order} \quad '' & \longrightarrow & \text{Auto-correlation.} \end{array}}$$

$t = \frac{1}{f_c}$; $\mu_X(t) = \frac{1}{2} \times 1 = \frac{1}{2}$

$t = \frac{1}{2f_c}$; $\mu_X(t) = \frac{1}{2} \times -1 = -\frac{1}{2}$

$\Big\}$ depends on t_1, not $t_1, t_1 + \tau$.

Method. 2

$0 \le a \le 1$

$$f_{x(t_1)}(x_1) = \begin{cases} \dfrac{1}{\cos 2\pi f_c t_1} & 0 \le x(t) \le \cos 2\pi f t_1 \\ 0 & \text{o.w} \end{cases}$$

Q. Consider a random process $X(t)$ defined by $X(t) = \sin(2\pi f_c t)$ in which the frequency f_c is random variable uniformly distributed over the interval $[0, \omega]$. Show that $X(t)$ is non-stationary?

Hint: examine specific sample function for the freq.

$f = \frac{\omega}{4}, \frac{\omega}{2}, \omega.$

$\rightarrow f = 4/\omega, 2/\omega, 1/\omega$

Sol^n:-

$$E_{f_c}(X(t)) = E[x(t)]$$

$$= E\{\sin(2\pi f_c t)\}.$$

$$= \int_0^{\omega} f_{f_c}(\omega) \sin(2\pi f_c t)\, df_c$$

$$\int_0^{\omega} k\, df = 1$$

$$k\{\omega\}_0^{\omega} = 1$$

$$k = 1/\omega.$$

$\sin(2\pi r$

$$= \frac{1}{\omega \cdot 2\pi t}(1 - \cos 2\pi \omega t).$$

1.3

Q. A random process $X(t)$ is defined by $X(t) = A\cos 2\pi f_c t$ where A is gaussian r.v with $A \sim \{0, \sigma_A^2\}$. This random process is applied to ideal integrator

$$Y(t) = \int_0^t X(\tau)\, d\tau.$$

(a) Determine the PDF of the output random process $y(t)$ $f_{Y(t_k)}(y_k)$.

Solⁿ: $Y(t_k)$ is a Gaussian R.V.

R.V because we are taking Sampling.

$$Y(t_k) = \int_0^{t_k} x(\tau)\, d\tau)$$

$$E_A[Y(t_k)] = E[A] \cos 2\pi f_c t_k . = 0$$

$$Var[Y(t_k)] = E[(A-0)^2] \cos 2\pi f_c t_k$$

$$= E[A^2] \cos^2 2\pi f_c t_c$$

$$= \sigma_A^2 \cos^2 2\pi f_c t_k .$$

$$Y(t_k) \sim N(0, \cos^2 2\pi f_c t_k \; \sigma_A^2)$$

Y_{t_k} is not Stationary

not erogidicity

16/08/23
class

Line code
NRZ

$$R'_X(\tau) = \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} E[b_n b_m] \; P(t - nT_b) \; P(t - mT_b - \tau).$$

$b_n = b_m$, $b_n \neq b_m$

$$E[b_n b_m] = 0 = \tfrac{1}{4} A^2 + \tfrac{1}{4} A^2 + \tfrac{1}{4}(-A^2) + \tfrac{1}{4}(-A^2)$$

for, m=n

$$E[b_n b_m] = \tfrac{1}{2} \cdot A \cdot A + \tfrac{1}{2} (-A)(-A)$$

$$= A^2.$$

$P_r(0,0) = P_r(0) = 1/2$

$P_r(0,1) = 0$

$P_r(1,0) = 0.$

Comment :-

$R'(\tau)$ is a cyclo stationary process.

H.W $R'_x(\tau) = R'_x(\tau \pm KT).$

Prove

$R'_x(\tau)$ is periodic Signal.

We never define energy of periodic Signal.

$$R_x(\tau) = \underbrace{\frac{1}{T_b}\int_0^{T_b} R_x'(t, t+\tau)\, dt}_{\text{Average over one time period.}}$$

$$R_x(\tau) = \frac{1}{T_b}\int_0^{T_b} \sum_n \sum_m E[b_n b_m]\; p(t-nT)\; p(t-mT-\tau)\, dt$$

$$= \frac{1}{T_b} \sum_n \sum_m E[b_n b_m] \underbrace{\int_0^{T_b} p(t-nT)\; p(t-mT-\tau)\, dt}_{A}$$

$$A = \int_0^{T_b} p(t-nT_b)\; p(t-mT_b+\tau)\, dt$$

$$t - mT = \lambda$$

$$dt = d\lambda$$

$$A = \int_{-mT_b}^{-T_b(m-1)} p(\lambda + (m-n)T_b)\quad p(\lambda+\tau)\, d\lambda$$

Now,

$$R_x(\tau) = \frac{1}{T_b} \sum_n \left[\sum_{m=-\infty}^{\infty} E[b_n b_m] \int_{-mT_b}^{-(m-1)T_b} p(\lambda + (m-n)T_b)\; p(\lambda+\tau)\, d\lambda \right.$$

let $m-n = K$

$$= \frac{1}{T_b} \sum_{K=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \underbrace{E[b_m b_{m+K}]}_{R_b(K)} \int_{-mT_b}^{-(m-1)T_b} p(\lambda - KT_b)\; p(\lambda+\tau)\, d\lambda$$

$$= \frac{1}{T_b} \sum_{K=-\infty}^{\infty} R_b(K) \underbrace{\sum_{m=-\infty}^{\infty} \int_{-mT_b}^{-(m-1)T_b}} p(\lambda - KT_b)\quad p(\lambda+\tau)\, d\lambda$$

$$= \frac{1}{T_b} \sum_{K=-\infty}^{\infty} R_b(K) \int_{-\infty}^{\infty} p(\lambda - KT_b)\; p(\lambda+\tau)\, d\tau$$

$$S_x(f) = F.T\,[R_x(\tau)]$$

$$S_x(f) = \frac{1}{T_b}\sum_{k=-\infty}^{\infty} R_b(k)\; F.T\left[\int_{-\infty}^{\infty} p(\alpha - kT_b)\, p(\alpha+\tau)\, d\alpha\right]$$

Proof ??

$\Downarrow$

$$|p(t)|^2$$

Hint :- you need to apply the relationship b/w Correlation and convolution

$\Downarrow$

Wiener - Kinchin theorem — H.W

$$\boxed{\int_{-\infty}^{\infty} p(\alpha+\tau)\, p(\alpha)\, d\alpha = |P(f)|^2}$$

$$p(-(-\alpha)+\tau)\; p(\alpha)\, d\alpha.$$

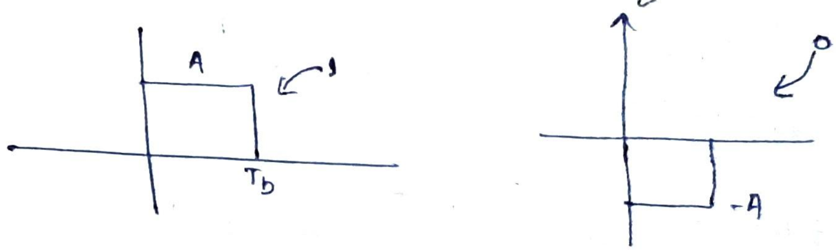
Rem:

$$\boxed{S_x(f) = \frac{1}{T_b}\sum_{k=-\infty}^{\infty} R_b(k)\; |P(f)|^2\, e^{-j2\pi f k T_b}.}$$

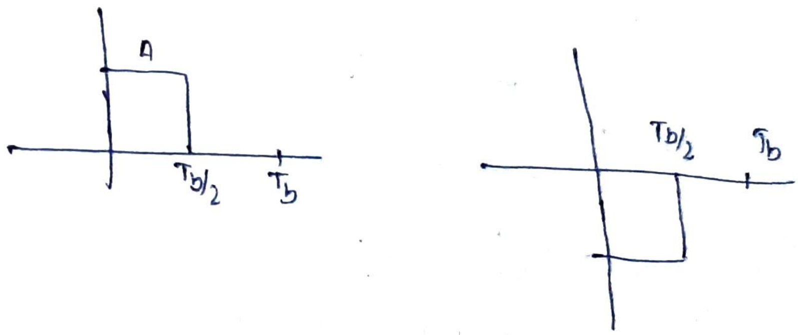
Q. Assignment

Q. PSD of NRZ, RZ, ON-OFF; } — Proof and also plot in Matlab.

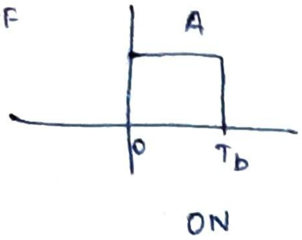
NRZ

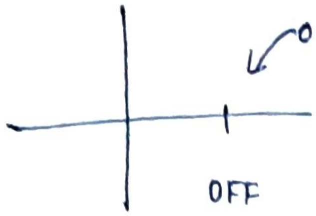


RZ

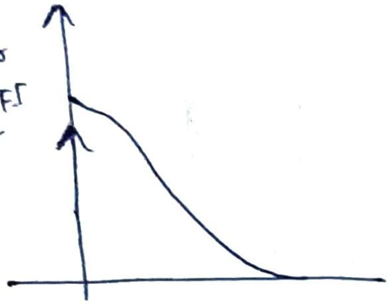


ON.OFF





PSD of ON.OFF



Q. Derive PSD for all the cases given above and plot PSD vs. freq.

Vector Representation of Signals

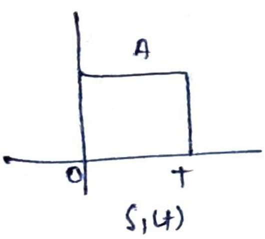
Vectors :-

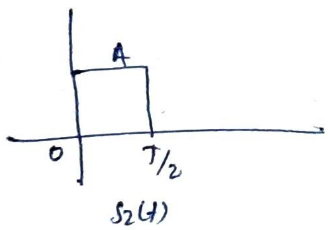
$|i| = 1$
$|j| = 1$
$\langle i \cdot j \rangle = 0$
} — Basis fn

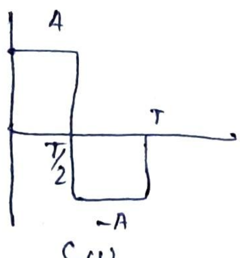
$\frac{1}{\sqrt{6}} \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$; $\frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

→ Now the two are basis function } ⇒ Qualified Basis fn.

we take orthogonal because we don't have any common b/w two axis. i.e projection.



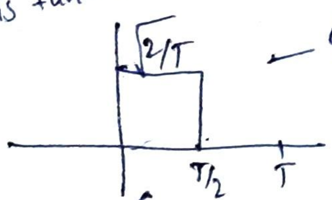




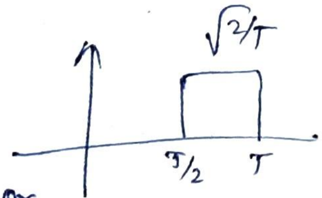
Minimum no. of basis function

Energy should be 1.

Basis function —

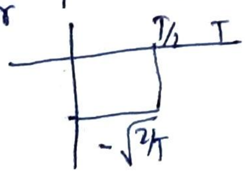


— $\phi_1(t)$



— $\phi_2(t)$

Or



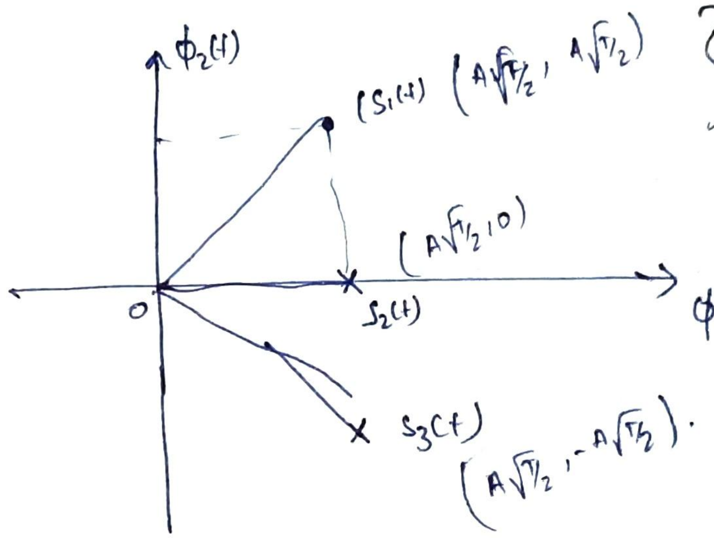
energy = $\int (\ )^2 dt = 1$

$s_1(t) = A\sqrt{T/2}\,\phi_1(t) + A\sqrt{T/2}\,\phi_2(t)$ | Reference, $\phi_1(t), \phi_2(t)$

$s_2(t) = A\sqrt{T/2}\,\phi_1(t) + 0\,\phi_2(t).$

$s_3(t) = A\sqrt{T/2}\,\phi_1(t) - A\sqrt{T/2}\,\phi_2(t)$

constellation of s_1, s_2 & s_3.



Vector representation of signal
⇓
not unique
⇓
Because basis fn is not unique.

$d_{s_1(t)} = A\sqrt{T}$

$d_{s_2(t)} = A\sqrt{T/2}$

$d_{s_3(t)} = A\sqrt{T}.$

Squaring
$$\left.\begin{aligned} d^2_{s_1(t)} &= A^2T \\ d^2_{s_2(t)} &= A^2T/2 \\ d^2_{s_3(t)} &= A^2T. \end{aligned}\right\} \rightarrow$$

Square of distance
⇓
Energy of signal

$d_{12} = d_{21} = A\sqrt{T/2}$

$d_{23} = d_{32} = A\sqrt{T/2}$

Dot product of two signals

$s_1(t)\ ,\ s_2(t)$

projection of s_1 on s_2.

$$\langle s_1(t), s_2(t)\rangle = \int_{-\infty}^{\infty} s_1(t)\, s_2(t)\, dt$$

$$= \langle \underline{s_1}, \underline{s_2} \rangle \rightarrow \text{vector}$$

Similarly we can visualize Correlation & convolution as projection

$\underline{s_1} = \begin{bmatrix} s_{11} \\ s_{12} \end{bmatrix}.$

$\underline{s_2} = \begin{bmatrix} s_{21} \\ s_{22} \end{bmatrix}$

H.W

$$\langle s_1(t), s_2(t)\rangle = \langle \underline{s_1}, \underline{s_2}\rangle$$
$$= s_{11}s_{21} + s_{12}s_{22}.$$

Energy of $S_1(t)$

$$E_{S_1(t)} = \int_{-\infty}^{\infty} S_1^2(t)\, dt = \langle S_1(t), S_1(t) \rangle$$

$$= S_{11}^2 + S_{12}^2$$

$$= S_1^T S_1$$

Distance of Signal $S_1(t)$ and $S_2(t)$

$$d_{12} = d_{21} = \int_{-\infty}^{\infty} \{S_1(t) - S_2(t)\}^2\, dt$$ — Prove

$$= \sqrt{(S_{11} - S_{21})^2 + (S_{12} - S_{22})^2}$$

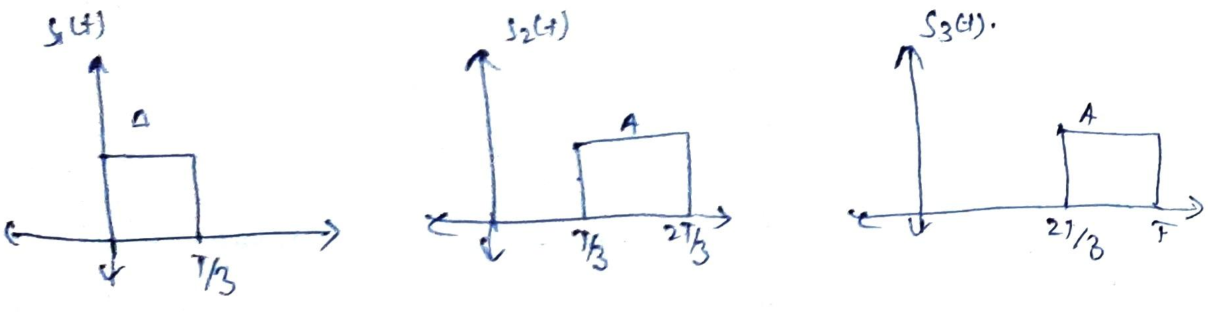
$$\langle S_1(t), S_2(t) \rangle = |S_1(t)|\, |S_2(t)| \cos\theta.$$

The minimum Number of basis function

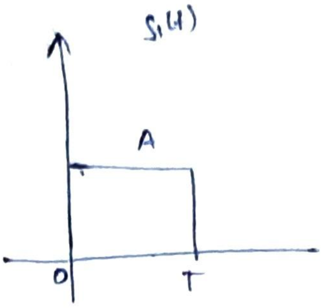
Max^m no. of minimum basis function = M.

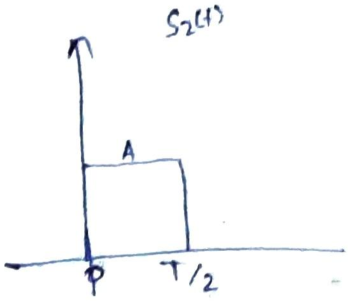
For an M number arbitrary signals, we require maxime M no. of minimum basis function.

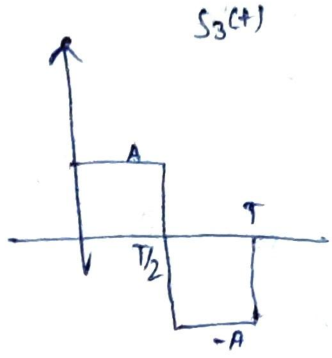
$$\boxed{\text{The no. of Basis function required} \le M.}$$



Gram Schmitt' Orthogonalization

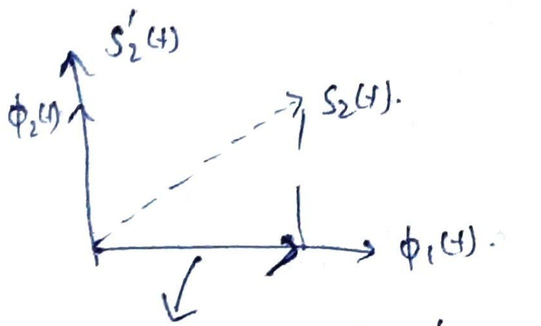






Based on $s_1(t)$

$$\phi_1(t) = \frac{s_1(t)}{|s_1(t)|}$$



$$s_2(t) = s_2'(t) + \langle s_2(t), \phi_1(t) \rangle \phi_1(t)$$

$\langle s_2(t), \phi_1(t) \rangle \phi_1(t)$

projection of $s_2(t)$ on $\phi_1(t)$.

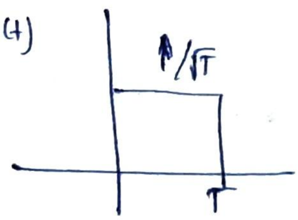
$$\boxed{s_2'(t) = s_2(t) - \langle s_2(t), \phi_1(t) \rangle \phi_1(t).}$$

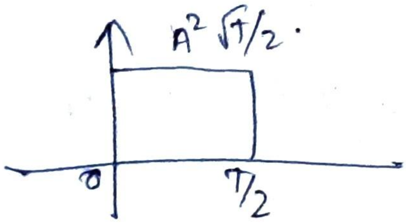
$$\phi_2(t) = \frac{s_2'(t)}{|s_2'(t)|}$$

$$s_3'(t) = s_3(t) - \langle s_3(t), \phi_1(t) \rangle \phi_1(t) - \langle s_3(t), \phi_2(t) \rangle \phi_2(t).$$

$$\phi_3(t) = \frac{s_3'(t)}{|s_3'(t)|}.$$

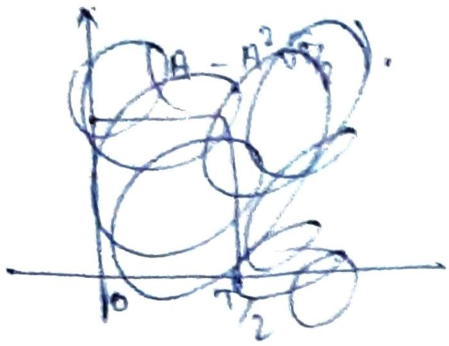
$\phi_1(t)$





$$\langle s_2(t), \phi_1(t) \rangle \phi_1(t) = \frac{A^2}{\sqrt{T}} \cdot T/2 = \frac{A^2\sqrt{T}}{2}. \quad 0 \le t \le T/2$$

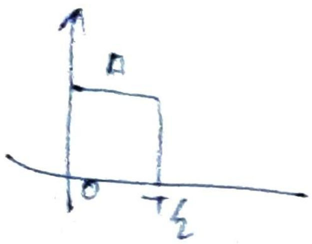
$$0 \quad \text{o.w.}$$



$$\left(A - A^2\sqrt{T/2}\right) x \times T/2 = 1$$

$$x = \frac{2}{T}\left(A - A^2\sqrt{T/2}\right)$$

$$\times\ A\sqrt{T/2} = A/2.$$



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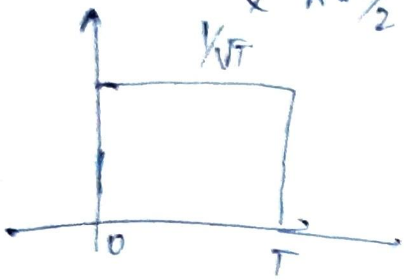
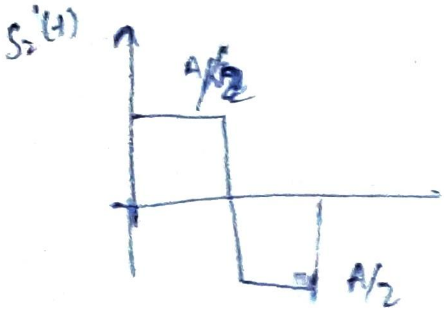
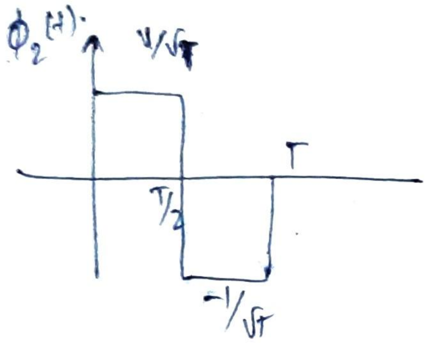






$$s_3'(t) = s_3(t) - \langle s_3(t), \phi_1(t) \rangle \phi_1(t) - \langle s_3(t), \phi_2(t) \rangle \phi_2(t).$$

23/08/23

Tutorial book

Q.1.3

$$Y(t) = \frac{A}{2\pi f_c}\left(\sin(2\pi f_c t)\right).$$

$$\mathrm{Var}(y(t_k)) = \sigma_A^2 \frac{\sin^2 2\pi f_c t}{(2\pi f_c)^2}$$ — variance will depend on t_k.

⇓

no stationary

⇓

non erogodic.

Q.4

$z(t) = \underbrace{X}_{N.S} \cos(2\pi t) + \underbrace{Y}_{N.S} \sin(2\pi t)$

Joint PDF $\longrightarrow$ depend on time difference.

$z(t_1) = X\cos(2\pi t_1) + Y\sin(2\pi t_1)$

$E[z(t)] = \cancel{E[X]}^{0} \cdot \cos 2\pi t + \cancel{E[Y]}^{0} \sin(2\pi t)$

$= 0.$

$Var(z(t)) = E[X^2]\cos^2 2\pi t_1 + E[Y^2]\sin^2(2\pi t_1) + \cancel{E[XY]}^{0}\, 2\sin 2\pi t \cos 2\pi t$

$= 1 \cdot \cos^2 2\pi t_1 + 1 \cdot \sin^2 2\pi$

$E[(z(t_k) - E[z(t_k)])^2] = 1.$

$E[(z(t_k))^2].$

$$\text{Covariance matrix} = \begin{bmatrix} \sigma_x^2 & \rho\sigma_x\sigma_y \\ \rho\sigma_x\sigma_y & \sigma_y^2 \end{bmatrix}.$$

$$cov(z(t_1), z(t_2)) = \begin{bmatrix} \sigma^2_{z(t_1)} & \rho\,\sigma_{z(t_1)} \\ & \end{bmatrix}$$

$$\rho = \cos 2\pi(t_1 - t_2) = \frac{cov(z(t_1), z(t_2))}{\sigma_{z(t_1)} \cdot \sigma_{z(t_2)}}$$

$$f_{z_1,z_2}(z_1, z_2) = \left(\frac{1}{\sqrt{2\pi}}\right)^2 \frac{1}{\sqrt{det(cov)}} \, Exp\left(\frac{-z^T(cov)^{-1}z}{2}\right)$$

$$\mathrm{Cov}(\ ) = \begin{bmatrix} 1 & \cos 2\pi(t_1-t_2) \\ \cos 2\pi(t_1-t_2) & 1 \end{bmatrix}.$$

$$= 1 - \cos^2 2\pi(t_1-t_2).$$

$$f_{z_1 z_2}(z_1, z_2) = \frac{1}{2\pi\sqrt{1-\cos^2 2\pi(t_1-t_2)}} \mathrm{Exp}\left\{ \frac{-1}{2\sin^2(2\pi(t_1-t_2))} \left\{ z_1^2 - 2\cos 2\pi(t_1-t_2) z_1 z_2 + z_2^2 \right\} \right\}.$$

$$\left\{ z_1^2 - 2\cos 2\pi(t_1-t_2)\, z_1 z_2 + z_2^2 \right\}.\ \Big]$$

$z(t)$ is stationary since $z_1(t)$ is independent of t_1 & t_2.

<u>Q.</u> Consider a random process $X(t) = A\cos(2\pi f_c t + \phi)$, ϕ is uniformly distributed with $(0, 2\pi)$ check whether $X(t)$ is ergodic or not.

<u>Soln:</u>

$$E[x(t)] = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} x(t)\,dt.$$

Ensemble Avg.

time Average.

$$E[x(t)\,x(t+\tau)] = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} x(t)\,x(t+\tau)\,dt.$$

$$E[x(t)] = \int_{-\infty}^{\infty} \frac{1}{2\pi} \times A \cos(2\pi f_c t + \phi)\, d\phi$$

$$= \frac{A}{2\pi} \left[\sin(2\pi f_c t + \phi \right]_0^{2\pi}$$

$$= \frac{A}{2\pi} \sin(2\pi f_c t + 2\pi) - \sin(2\pi f_c t)$$

$$= 0.$$

$$\lim_{T\to\infty} \frac{1}{2\pi} \int_{-T}^{T} x(t)\, dt = \lim_{T\to\infty} \frac{1}{2\pi} \int_{-T}^{T} A \cos(2\pi f_c t + \phi)\, dt.$$

$$= \lim_{T\to\infty} \frac{1}{2\pi} A \left[\sin(2\pi f_c t + \phi) \right]_{-T}^{T}$$

$$A \{ \sin(2\pi f_c T + \phi) - \sin(2\pi f_c (-T) + \phi) \}$$

$$= 0.$$

$$E[x(t)\, x(t+\tau)] = \int_{-\infty}^{\infty} \frac{1}{2\pi} A \cos(2\pi f_c t + \phi) A \cos(2\pi f_c (t + \tau) + \phi).\ d\phi$$

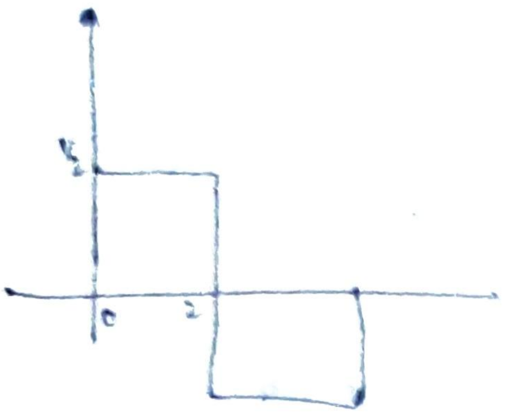
$$= \frac{A^2}{2} \cos(2\pi f_c \tau)$$

$$\lim_{T\to\infty} \frac{1}{2\pi} \int_{-T}^{T} x(t)\, x(t+\tau)\, dt = \frac{A^2}{2} \cos(2\pi f_c \tau).$$

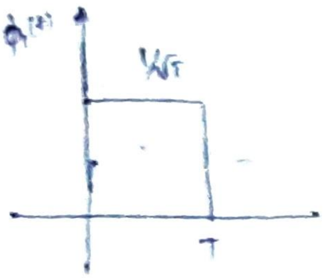
Q. 5.1

$$x(t) = \begin{cases} 1 & 0 \le t \le 1 \end{cases}$$

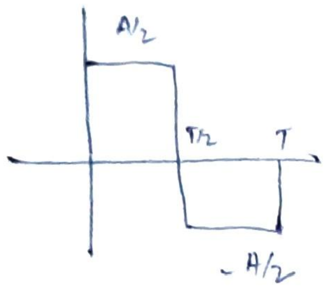
$\frac{1/\sqrt{2} \cdot 1/\sqrt{T}}{2/\sqrt{T}}$ $A \times \sqrt{T}/2$



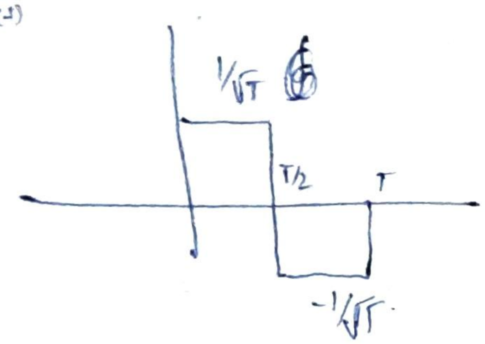
23/08/23
class



$s_2'(t) =$



$\phi_2(t) = \frac{s_2'(t)}{|s_2'(t)|}$

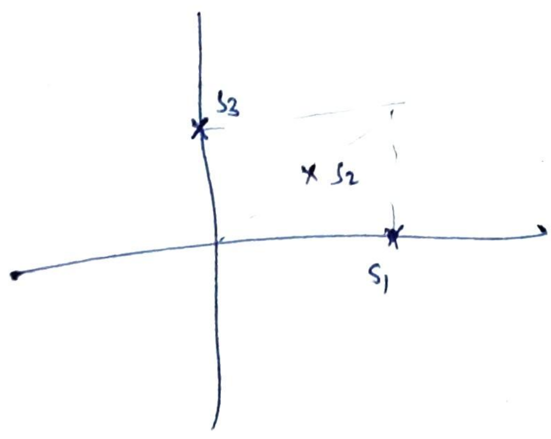


$\phi_3(t) = 0$

$s_1(t) = A\sqrt{T}\,\phi_1(t) + 0\,\phi_2(t)$.

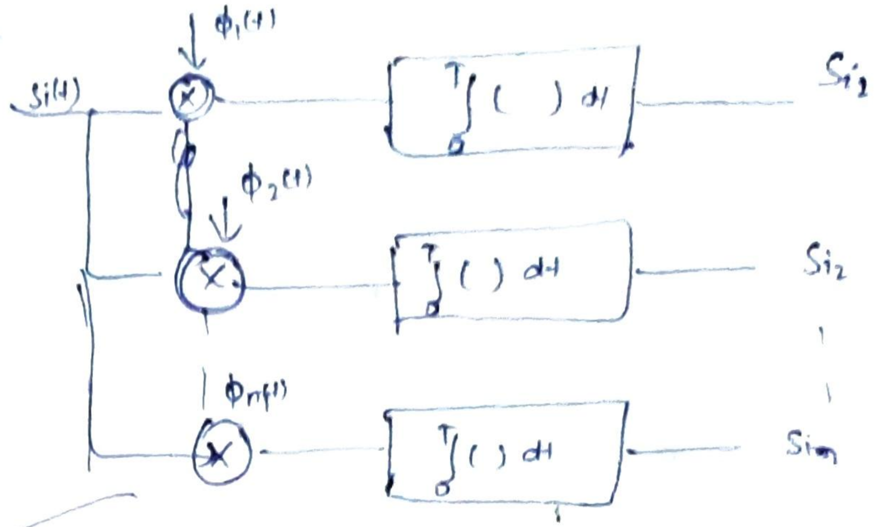
$s_2(t) = (\phi_1(t) + \phi_2(t))\,\frac{A\sqrt{T}}{2}$

$s_3(t) = 0\,\phi_1(t) + A\sqrt{T}\,\phi_2(t)$



~~AWGN Channel Model~~

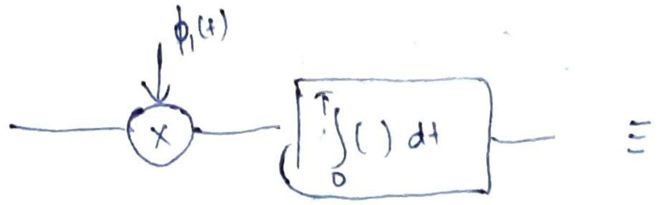
Receiver



projection of s_i on ϕ_1.

$$s_i(t) = s_{i1}\,\phi_1(t) + s_{i2}\,\phi_2(t).$$

$$\int_0^T s_i(t)\,\phi_1(t) = s_{i1}\int_0^T \phi_1(t)\,\phi_1(t)\,dt + s_{i2}\underbrace{\int_0^T \phi_2(t)\,\phi_1(t)\,dt}_{0}$$



$\equiv$

AWGN Channel Model

$$s_i(t) \longrightarrow \underset{\uparrow\, n(t)}{\otimes} \longrightarrow x(t) = s_i(t) + n(t).$$

$n(t)$ = Gaussian noise process with zero mean and PSD $\frac{N_0}{2}$.

$\Downarrow$

due to brownian motion

$$x(t) = \sum_{m=1}^{M} \phi_m(t)\, s_{im} + n(t).$$

$$= \sum_{m=1}^{M} \phi_m(t)\, x_{im}$$

$x_{im} = \langle x(t), \phi_m(t) \rangle$ — projection of $x(t)$ on $\phi_m(t)$.

$= \langle \left(\sum_{n=1}^{M} s_{in} \phi_n(t) + n(t) \right), \phi_m(t) \rangle$.

$= \langle \sum_{n=1}^{M} s_{in} \phi_n(t), \phi_m(t) \rangle + \langle n(t), \phi_m(t) \rangle$

$$\boxed{x_{im} = s_{im} + \hat{n}}$$

constant
$\Downarrow$
not dependent on time.

All projection expect $m=n$ will be zero.

Let $\hat{n} = n_m$.

$f_{x_{im}/s_{im}}(x_{im})$

$E(\hat{n}) = 0 = E\left[\underbrace{\int_0^T n(t)\, \phi_m(t)\, dt}_{\hat{n}} \right]$

$= \int_0^T E[n(t)]\, \phi_m(t)\, dt$ (with $E[n(t)] = 0$)

$= 0.$

$E[\hat{n}^2] = E\left[\left[\int_0^T n(t)\, \phi_m(t)\, dt \right]^2 \right]$

$= E\left[\int_0^T n(t)\, \phi_m(t)\, dt \int_0^T n(u)\, \phi_m(u)\, du \right]$

$= E\left[\int_0^T \int_0^T n(t)\, n(u)\, \phi_m(t)\, \phi_m(u)\, dt\, du \right].$

$= \int_0^T \int_0^T E[n(t) \cdot n(u)]\, \phi_m(t)\, \phi_m(u)\, dt\, du$

$$= \int_0^T \int_0^T R_n(t-u)\, \phi_m(t)\, \phi_m(u)\, dt\, du$$

F.T. $\quad S_n(f) = N_0/2 \quad$ [∵ Given].

$S_n(f) \xrightarrow{\text{I.F.T}} \frac{N_0}{2}\,\delta(\tau)$ — Autocorrelation.

$$= \int_0^T \int_0^T \boxed{\frac{N_0}{2}\,\delta(t-u)}\, \phi_m(t)\, \phi_m(u)\, dt\, du$$

$$= \frac{N_0}{2} \int_0^T \int_0^T \delta(t-u)\, \phi_m(t)\, \phi_m(u)\, dt\, du.$$

$$\boxed{\int \delta(t-u)\, y(t)\, dt = y(u).}$$

$$= \int_0^T \frac{N_0}{2}\, \phi_m(u)\, \phi_m(u)\, du$$

$$= \frac{N_0}{2} \int_0^T \phi_m^2(u)\, dt \qquad \phi_m^2(u) = 1 \text{ — orthonormal}$$

$$= \frac{N_0 T}{2}$$

x_i - mean S_{in}, Variance $\frac{N_0}{2}$.

$$x_{ip} = s_{ip} + n_p \qquad n_p \sim \mathcal{N}\left(0, \frac{N_0}{2}\right).$$

Derive ⇒ n_p and n_m (i.e $m \neq p$) are independent & identically

iid

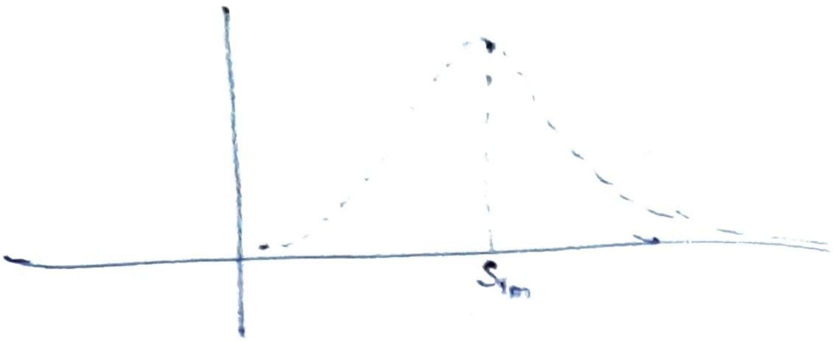
$$X_{im} \,\Big|\, s_{im} \sim \mathcal{N}\left(s_{im}, \frac{N_0}{2}\right).$$

If X is gaussian with mean $= 0$, var $= \sigma^2$

$Y = a + X$

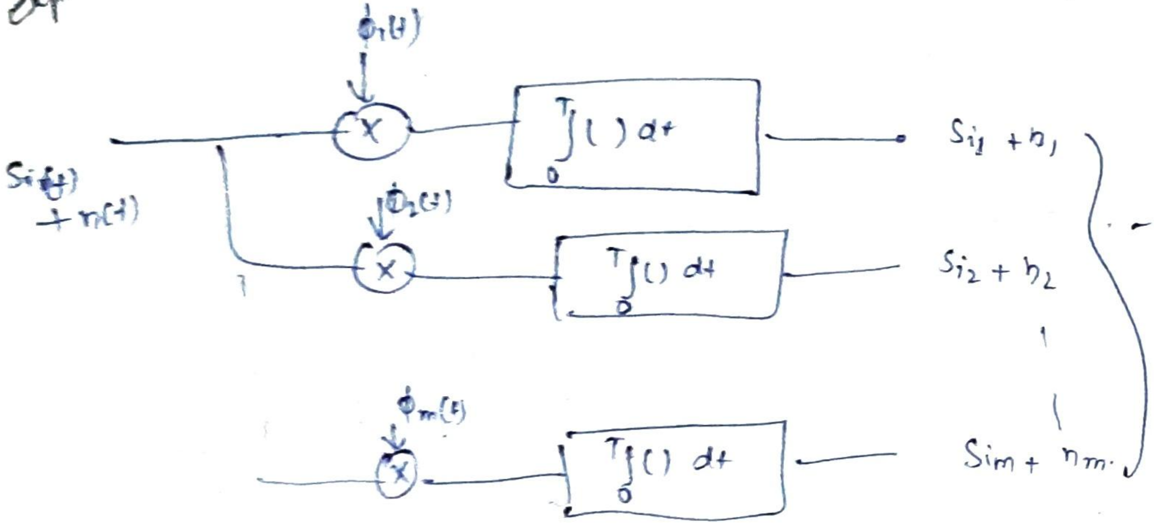
$\text{var}(Y) = \text{var}(X) = \sigma^2$

$\text{mean}(Y) = a$



$$\boxed{n(t) = \sum_{m=-\infty}^{\infty} \phi_m(t)\, n_m} = \sum_{\substack{m=-\infty \\ m \neq 1,2}}^{\infty} \phi_m(t)\, n_m + \sum_{m=1}^{2} \phi_m(t)\, n_m$$

General ways to express noise.

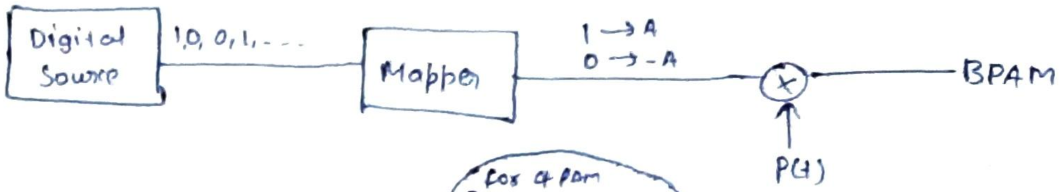


– discretization / time to scalar value

Correlation Receiver

Baseband Digital Modulation

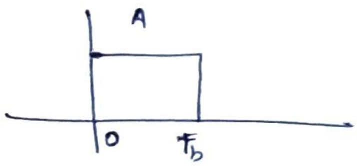
Binary
Pulse Amplitude Modulation (BPAM)



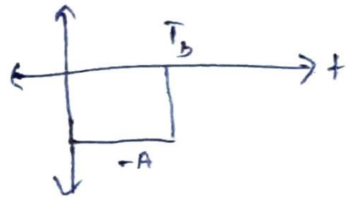
For 4 PAM
00 → -2A
01 → -A
10 → A
11 → 2A

Standard PAM

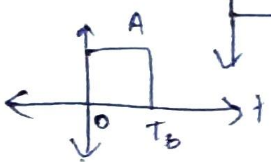
$p(t) =$



$s_1(t)\big|_0 = -p(t)$:

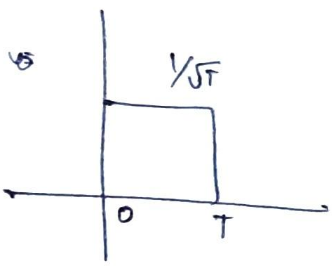


$s_2(t)\big|_1 = p(t)$:



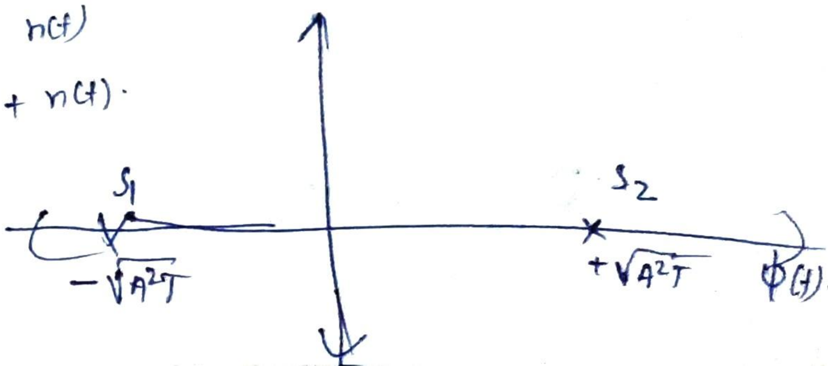
PAM - 1 dimension modulation technique.

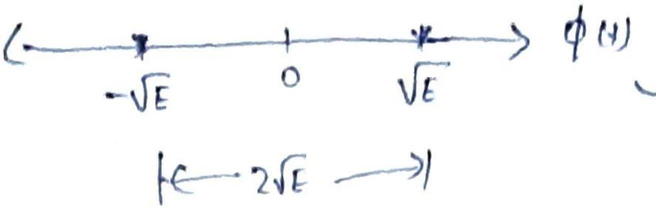
$\phi(t)$ vs



$$X(t)\big|_{s_1(t)} = s_1(t) + n(t)$$
$$= -A\sqrt{T}\,\phi_1(t) + n(t)$$
$$= -\sqrt{A^2T}\,\phi_1(t) + n(t).$$
$$= -\sqrt{E}\,\phi_1(t) + n(t)$$

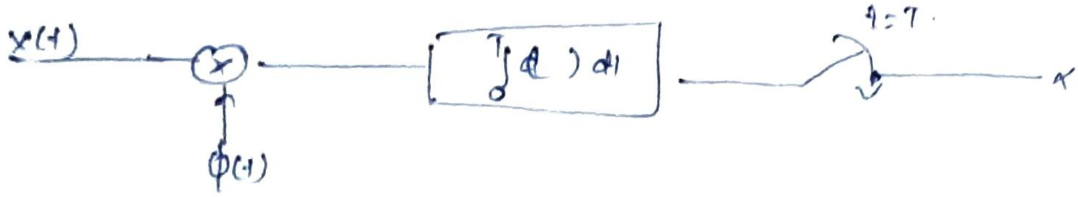
$$x(t)\big|_{s_2(t)} = s_2(t) + n(t)$$
$$= A\sqrt{T}\,\phi_1(t) + n(t)$$
$$= \sqrt{E}\,\phi_1(t) + n(t).$$





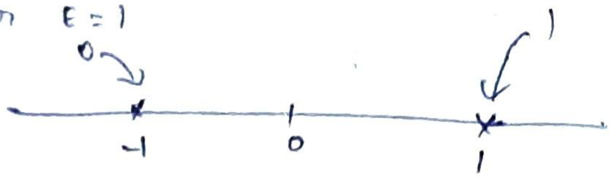
without Noise

noise will be highly concentra. at mean.



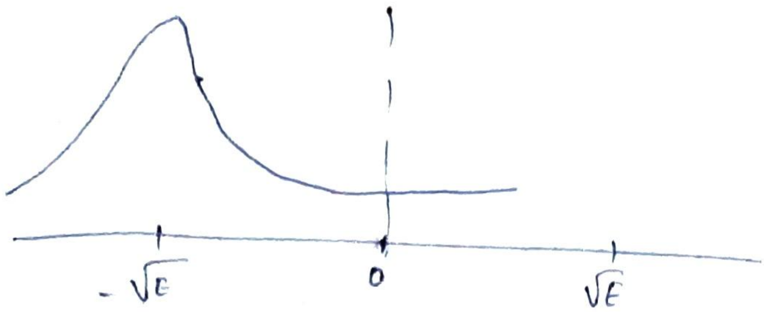
Intituitive Receiver

Consider $E = 1$



E_1 = when you transmit -1 V & receive > 0V

E_2 = when you transmit 1V & Receive < 0V.



$E|_0 \Rightarrow x|_0 > 0$.

$P_r[x|_0 > 0] = \int_0^\infty f_{x|0}(x|0)\,dx$

$$P_r\left[x|_0 \geq 0\right] = \int_0^\infty f_{x|0}(x|0)\,dx$$

$$= \int \frac{1}{\sqrt{2\pi \frac{N_0}{2}}} \, e^{-\frac{(x+\sqrt{E})^2}{2 N_0/2}} \, dx$$

$$= \boxed{Q\left(\sqrt{\frac{2E}{N_0}}\right)}.$$

$$Q(y) = \int_y^\infty \frac{1}{\sqrt{2\pi}} e^{-u^2/2} \, du.$$

$$\frac{(x+\sqrt{E})}{\sqrt{\frac{N_0}{2}}} = u.$$

$$\frac{u^2}{2} = \frac{(x+\sqrt{E})^2}{2 N_0/2} = u$$

$$\boxed{P_r(x>0] = Q\left(\sqrt{\frac{2E}{N_0}}\right) = P_r(\varepsilon|_0) = P_r(\varepsilon|_1).}$$

ε - Error

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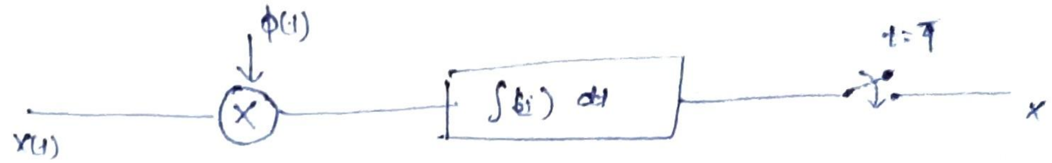
Recap.

Bitwise Receiver for Binary PAM



$$x = s_i + n \quad ; \quad x \sim N\left(s_i, \frac{N_0}{2}\right).$$

0 → -1V

1 → 1V

$$\begin{bmatrix} 0.7 \rightarrow 1 \\ -0.2 \rightarrow 0 \end{bmatrix}$$

$$P_{e|0} = P_{e|s_i(t)} = P_r\left[x|_0 \geq 0\right] = P\left[x|_{s_i(t)} \geq 0\right].$$

Probability of error in transmitting 0.

$$x|_0 \sim \left(\sqrt{E_b}, \frac{N_0}{2}\right)$$

$$= \int_{x=0}^{\infty} \frac{1}{\sqrt{2\pi \frac{N_0}{2}}} \exp\left[-\frac{1}{2\frac{N_0}{2}} (x-(-\sqrt{E_b}))^2\right] dx$$

$$= Q\left(\sqrt{\frac{2E_b}{N_0}}\right);$$

$$\boxed{\frac{x+\sqrt{E_b}}{\sqrt{N_0/2}} = u}$$

$$= \int_{\sqrt{\frac{2E_b}{N_0}}}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{u^2}{2}} du.$$

E_b : bit Energy

$\frac{N_0}{2}$: Noise Energy

$$= Q\left(\sqrt{\frac{E_b}{N_0/2}}\right)$$

$$= Q\left(\sqrt{SNR}\right)$$

$$SNR = \frac{\text{Signal power}}{\text{Noise power}}.$$

SNR $\uparrow$ $\rightarrow$ $Q(\sqrt{SNR}) \downarrow$

↳ tail will further move forward:

Assumption in Intuitive receiver :-

Probability of Generation of 0 and 1 is same

⇓

uniform distribution of the source

$$P_r(0) = P_r(1) = 1/2.$$

ML Desion rule

$$\boxed{P_{e|1} = P_{e|s_2(t)} = Q\left(\sqrt{SNR}\right)}.$$

$$P_e = Pr(0)\, P_{e|0} = Pr(1) \cdot P_{e|1}$$

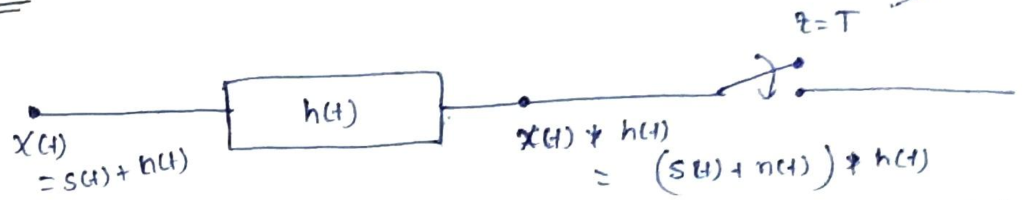
$$= \tfrac{1}{2} Q\left(\sqrt{SNR}\right) + \tfrac{1}{2} Q\left(\sqrt{SNR}\right)$$

$$\boxed{P_e = Q\left(\sqrt{SNR}\right)} = P_{e|0} = P_{e|1}.$$

Probability of error.

Receiver

why sampling at T?



$$= (s(t) + n(t)) * h(t)$$

$$= s(t) * h(t) + n(t) * h(t).$$

$$y(t) = \underbrace{s(t) * h(t)}_{\text{Signal component}} + \underbrace{n(t) * h(t)}_{\text{noise component}}.$$

$$n'(t) = n(t) * h(t) = \int_{0}^{t} n(\tau)\, h(t-\tau)\, d\tau$$

$$E\left[n(t) * h(t)\right] = \int_{0}^{t} E\left[n(\tau)\right] h(t-\tau)\, d\tau = 0$$

$$n = n'(t)\Big|_{t=T} = \int_0^T n(\tau)\, h(T-\tau)\, d\tau$$

$$E[n] = 0$$

$$E[n^2] = E\left[\int_0^T n(\tau)\, h(t-\tau)\, d\tau \int_0^T n(u)\, h(t-u)\, du\right]$$

$$= \int_0^T \int_0^T \frac{N_0}{2} \delta(\tau - u)$$

$$= \int_0^T \int_0^T \frac{N_0}{2} \, \delta(\tau - u) \, h(t-\tau) \, h(t-u) \, d\tau \, du.$$

$$\int_0^T \delta(t - t_0) \, x(t) \, dt = x(t_0)$$

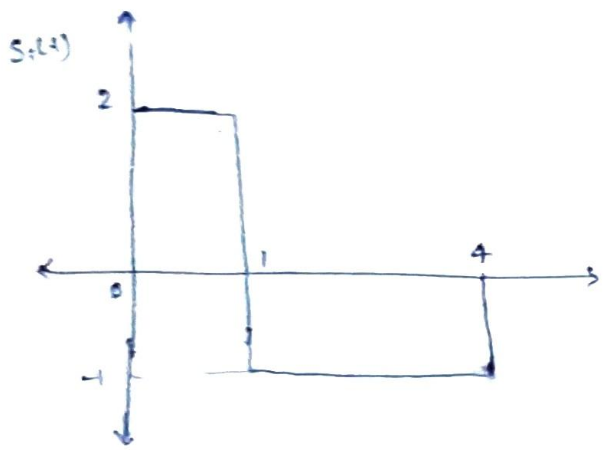
$$= \int_0^T \frac{N_0}{2} \, h(T-u) \, h(T-u) \, du$$

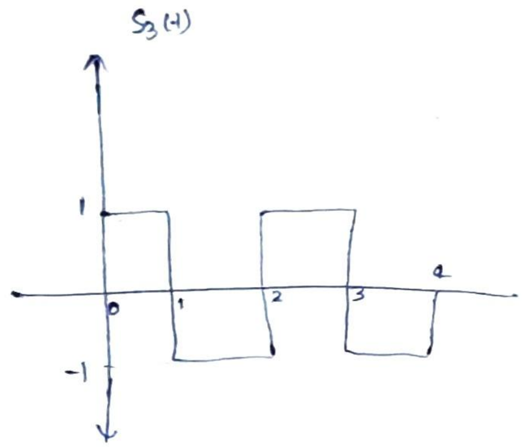
$$= \frac{N_0}{2} E_h$$

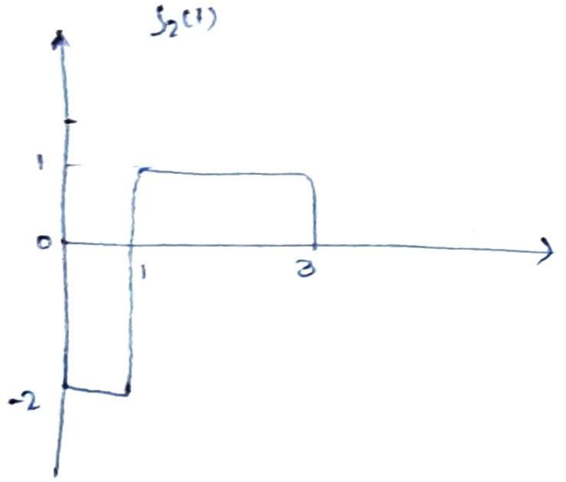
└ Energy of filter.

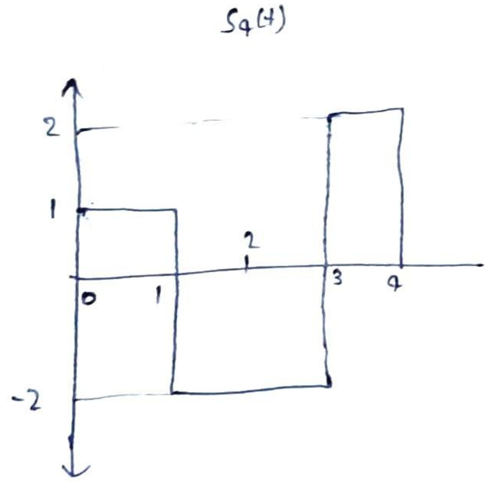
30/08/23

Tutorial









Q. Determine the dimensionality of the waveform and a set of basis function

Q2. Use the basis function to represent the four waveform by s_1 s_2, s_3 s_4.

Q3. Determine the minimum distance b/w pair of vectors -

$$\phi_1(t) = \frac{s_1(t)}{\sqrt{E_1}}$$

$$\int_0^4 s_1(t)^2\,dt = \int_0^1 4 + \int_1^4 1$$

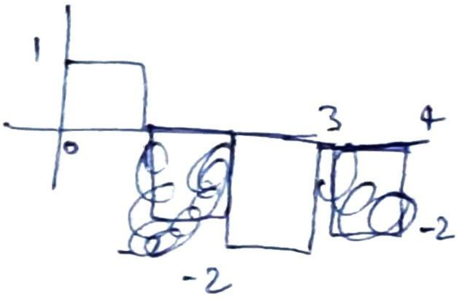
$$= 4 + 1 \times (4-1) = 3.$$

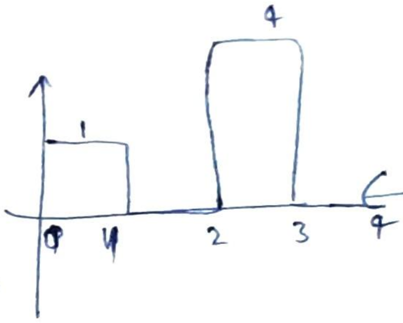
$$= 7.$$

$$= \frac{s_1(t)}{\sqrt{7}}.$$

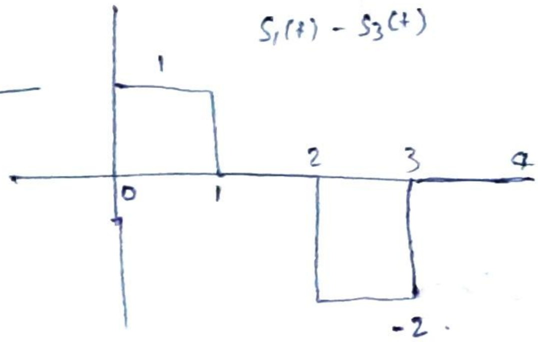
$$\boxed{d_{13} = \sqrt{\int_0^T [s_1(t) - s_3(t)]^2\,dt}}$$

distance b/w 1 & 3





$(\ )^2$



$s_1(t) - s_3(t)$

Area under curve is

$A_{0-1} + A_{1-2} + A_{2-3} + A_{3-4}$

$= 1 + 0 + 4 + 0$

$= 5.$

$d_{13} = \sqrt{5}.$

Q. 5.8 & 5.9

$$s_i(t) = \sum_{m=1}^{M} \phi_m(t)\, s_{im}$$

$$s_k(t) = \sum_{m=1}^{M} \phi_m(t)\, s_{km}$$

$$\int s_i(t) \cdot s_k(t)\,dt = \int \left(\sum_{m=1}^{M} \phi_m(t)\, s_{im}\right)\left(\sum_{l=1}^{M} \phi_l(t)\, s_{kl}\right) dt.$$

$$= \sum_{l=1}^{M} \sum_{m=1}^{M} s_{im}\, s_{kl} \int \phi_m(t)\, \phi_l(t)\, dt$$

Orthogonality and orthonormality property.

$m = l \Rightarrow E$

$m \neq l = 0.$

$$\Rightarrow \sum_{m=1}^{M} \sum_{l=1}^{M} s_{im} s_{rl} \int \phi_m(t) \phi_l(t) \, dt$$

when $m \neq l$

$$\int \phi_m(t) \phi_l(t) \, dt = 0$$

$m = l$ $\int \phi_m(t) \phi_l(t) \, dt = 1$

$$\Rightarrow \boxed{\sum_{m=1}^{M} s_{im} s_{km} = s_i^T s_k}$$

(replace s_i, ...)

Ex:

$s_1(t) = a_{11} \phi_1(t) + a_{12} \phi_2(t)$

$s_2(t) = a_{21} \phi_1(t) + a_{22} \phi_2(t)$

$\langle v_1, v_2 \rangle = |v_1| |v_2| \cos\theta$

$\int s_1 \, dt$

$\langle s_1(t), s_2(t) \rangle = \int s_1(t) \, s_2^*(t) \, dt$

$\langle s_1(t), s_2(t) \rangle = |s_1(t)| |s_2(t)| \cos\theta$

Squaring both side

$\{\langle s_1(t), s_2(t) \rangle\}^2 = |s_1(t)|^2 |s_2(t)|^2 \cos^2\theta$

$\because \{\cos\theta \leq 1 \quad \Rightarrow \cos^2\theta \leq 1\}$.

$\langle s_1(t), s_2(t) \rangle^2 \leq |s_1(t)|^2 |s_2(t)|^2$

$|s_1(t)| = \sqrt{\int s_1^2(t) \, dt}$

$|s_2(t)| = \sqrt{\int s_2^2(t) \, dt}$

$\langle s_1(t), s_2(t) \rangle \le E_1 E_2$

$$E_1 = \int s_1^2(t)\,dt$$

$$E_2 = \int s_2^2(t)\,dt$$

20/8/23

Signal component

$$g(t) * h(t) = \int g(\tau)\, h(t-\tau)\, d\tau = \int h(\tau)\, g(t-\tau)\, d\tau$$

$$\int h(\tau)\, g(t-\tau)\, d\tau = \langle h(\tau), g(t-\tau) \rangle$$ dot product

$$\left(\int h(\tau)\, g(t-\tau)\, d\tau \right)^2 \le \int h^2(\tau) \cdot \int g^2(t-\tau)\, d\tau$$

$$(\langle v_1, v_2 \rangle)^2 = (|v_1||v_2|\cos\theta)^2$$

$$= |v_1|^2 |v_2|^2 \cos^2\theta$$

$\cos^2\theta$

$0 \le \cos^2\theta \le 1$

$$(\langle v_1, v_2 \rangle)^2 \le |v_1|^2 |v_2|^2$$

Cauchy - Schwarz Inequality

Also

The above inequality has equality when $v_1 = \alpha v_2$

$$\left(\int_0^T h(\tau)\, g(T-\tau)\, d\tau \right)^2 \le E_h \int_0^T g^2(T-\tau)\, d\tau$$

Inequality can become equal

when if $\boxed{h(t) = c\, g(T-t)}$; $t \le T$

$K = 1$

$h(\tau) = s(T-\tau)$

$h(t) = s(T-t)$.

matched filter Response

$E_h = E$

for the matched filter SNR

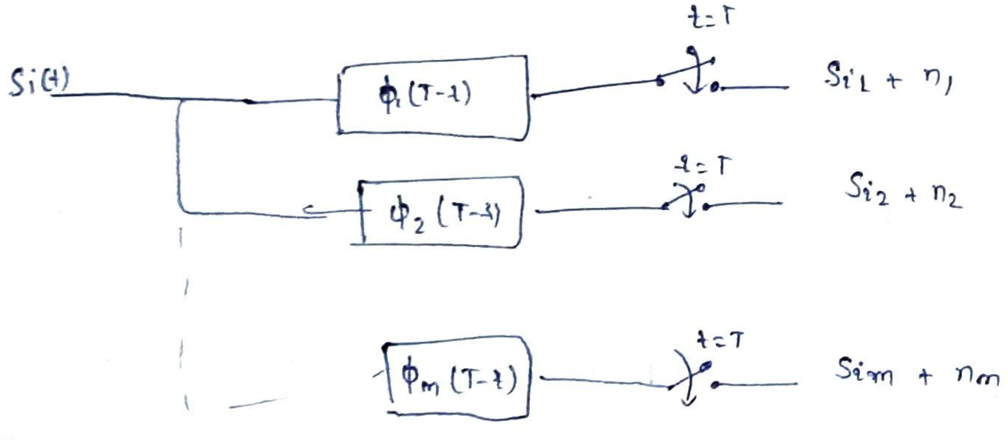
$$SNR = \frac{K E_s^2}{K E_s \frac{N_0}{2}}$$

$$\boxed{SNR = \frac{E_s}{N_0/2}}$$

Same as Intuitive/ receiver.

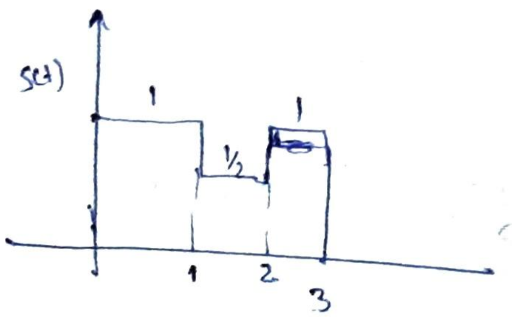
$Y_{corr}(T \pm \Delta t) \neq Y_{matched}(T \pm \Delta t)$

output of correlator receiver



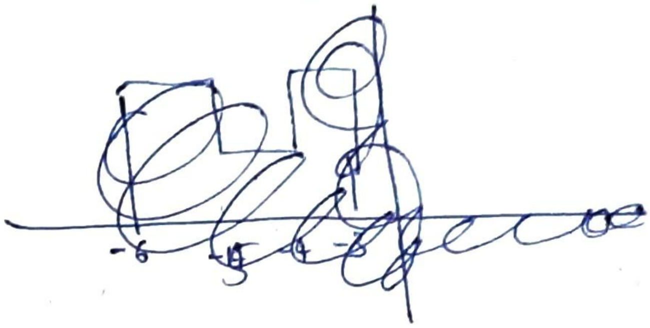
Matched filter Based Receiver.

Q.



$h(t) = s(T-t)$.

$h(t) = s(3 - t) = \underline{\underline{s(t)}}$.

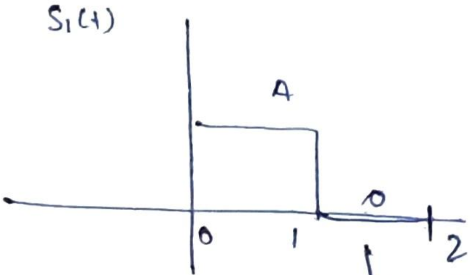


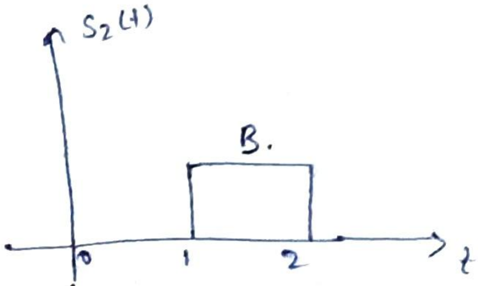
$-3 \quad -2 \quad -1 \quad 0$

$0 \quad 1 \quad 2 \quad 3$

$1 - 2 \quad -1$

$2 - 2 = 0$



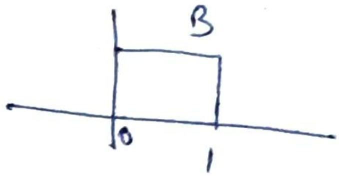


T = 2

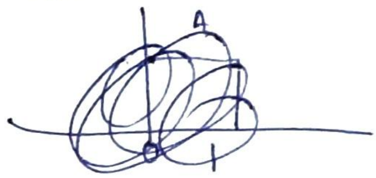
for both

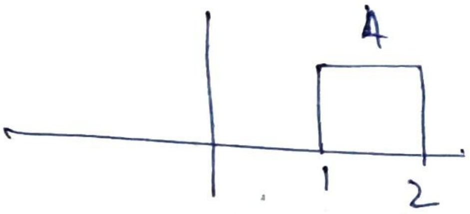
Same time sample

$h_2(t) = h(2 - t)$

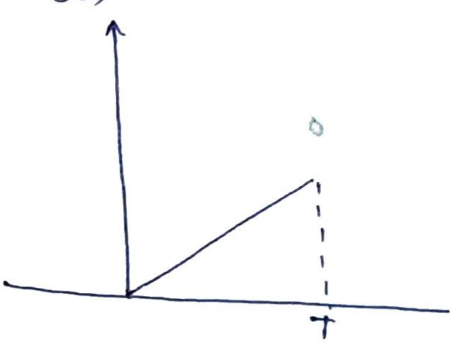


$h(t) = s_1(2 - t)$

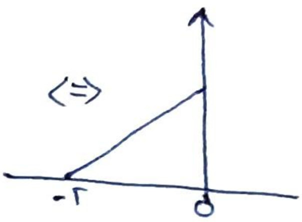




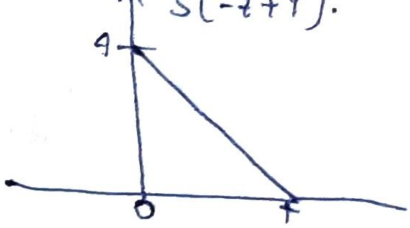
31/08/23



$\Leftrightarrow$



& $\Rightarrow$



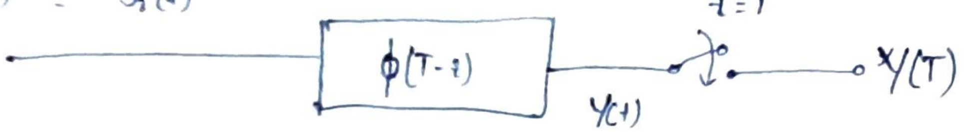
Standard PAM

Matched filter based receiver

0 → $-\sqrt{E}\,\phi(t) = s_1(t)$

1 → $\sqrt{E}\,\phi(t) = s_2(t)$

$x(t) = s_i(t)$

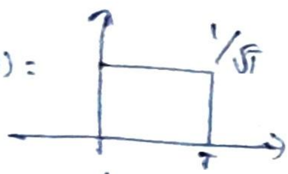


$$y(t) = s_1(t) * \phi(T-t)$$

$$= -\sqrt{E}\,\phi(t) * \phi(T-t)$$

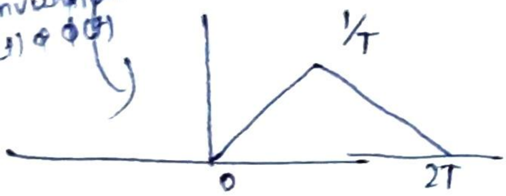
$$= -\sqrt{E}\int \phi(T-\tau)\,\phi(t-\tau)\,d\tau.$$

$\phi(t) =$

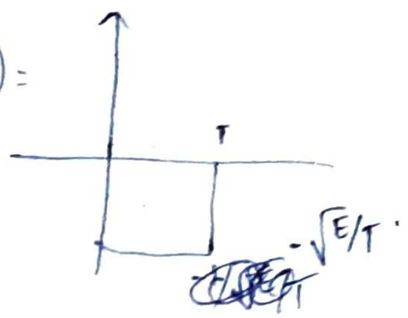


$$= -\sqrt{E}\int \phi(t)$$

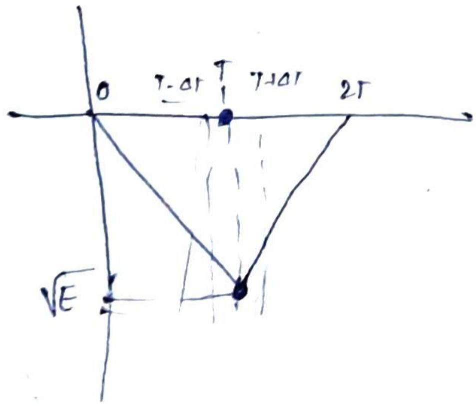
convulation
$\phi(t) * \phi(t)$



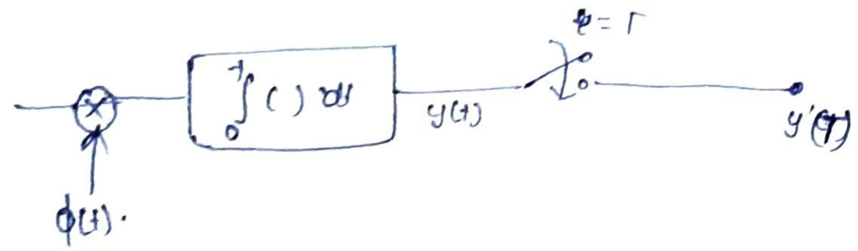
$y(t) =$



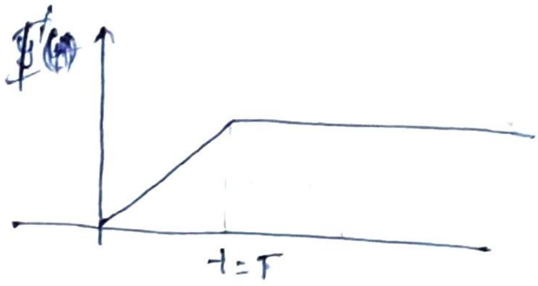
→

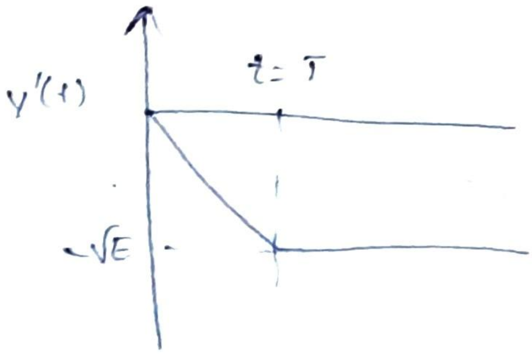


$x(t)$



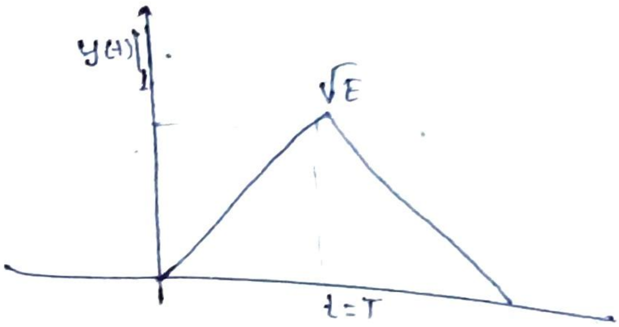
$$y'(t) = -\sqrt{E} \int_0^t \phi(u)\, \phi(v)\, du.$$





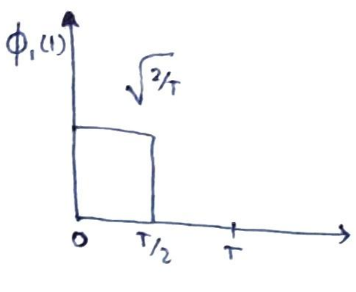
$$\boxed{y(t \pm \Delta t) \neq y'(t \pm \Delta t).}$$

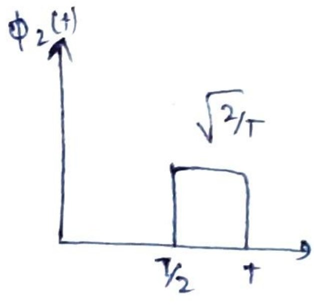
$$y(t)\Big|_1 = y(t)\Big|_{s_2(t)}$$



$$Y(T)\Big|_1 = \sqrt{E} + n.$$

— noise.

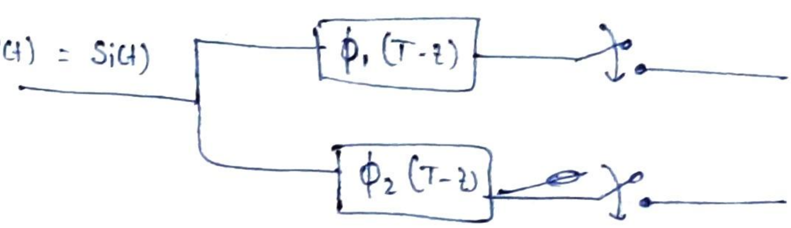




~~Two dimens~~

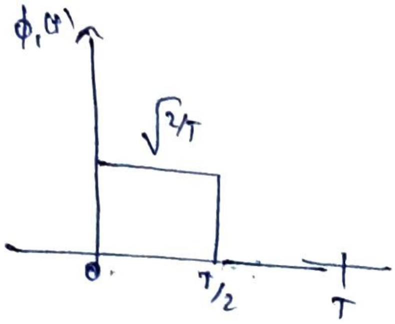
$$\left.\begin{aligned} s_1(t) &= \sqrt{E}\ \phi_1(t) \\ s_2(t) &= \sqrt{E}\ \phi_2(t) \end{aligned}\right\}$$ Two dimension digital filter.

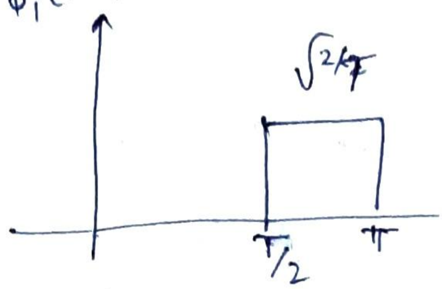
$Y(t) = S_i(t)$



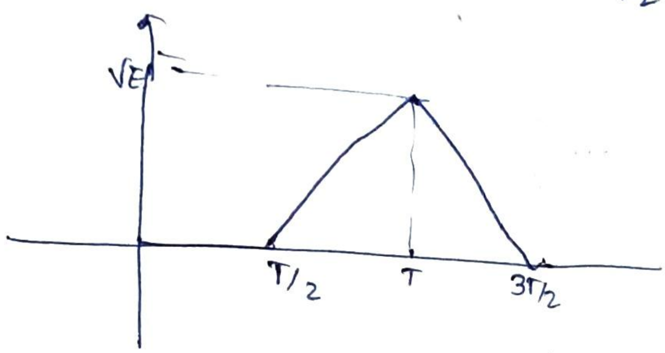
$$y_1(t) = s_1(t) * \phi_1(T-t)$$

$$= \sqrt{E}\ \phi_1(t) * \phi_1(T-t)$$





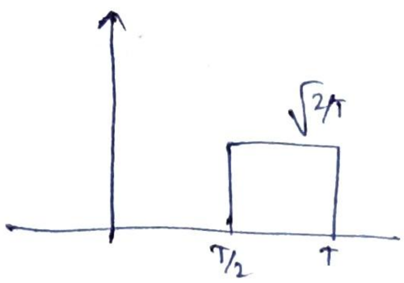
$-T \quad -T/2$



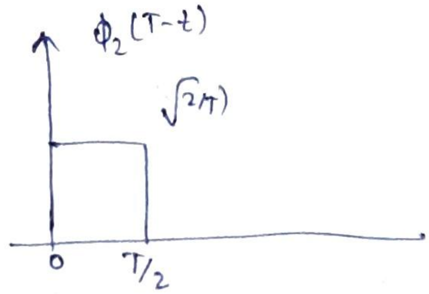
$t = T = \sqrt{E}$

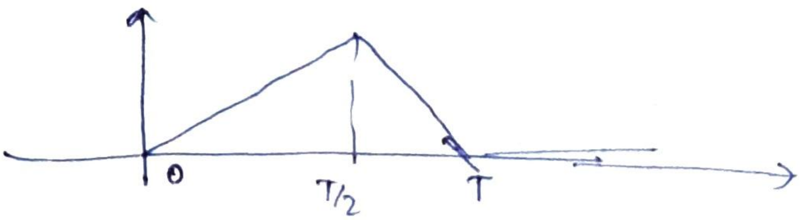
$$y_2(t) = s_2(t) * \phi_2(T-t)$$

$$= \sqrt{E}\ \phi_2(t) * \phi_2(T-t)$$



$*$





at $t = T$, $y_2(T) = 0$

Note:-

Both types of receivers behaves similar at $t = T$.