MA204: Mathematics IV

Partial Differential Equation (Second Order Linear PDE)

Separation of variables

It is a fundamental technique for obtaining solutions of linear partial differential equations.

This means that we look for particular solutions in the form

$$u(x, y) = X(x)Y(y),$$

and try to obtain ordinary differential equations for $X(x)$ and $Y(y)$.

These equations will contain a parameter called the separation constant.

The function $z(x, y)$ is called a separated solution.

Introduction

Let us explain the method of separation of variables for the Laplace equation

$$z_{xx} + z_{yy} = 0.$$

Separation of variables

Usually we think of satisfying a PDE only in a particular region in xyz-space, for instance in a ball of some radius R.

If we denote the region by Ω, typically it is assumed to be an open, connected set with some piecewise smooth boundary $\partial\Omega$.

A boundary condition is then an additional equation that specifies the value of z and some of its derivatives on the set $\partial\Omega$.

For instance, $z = f(x, y)$ or $z_x = g(x, y)$ on $\partial\Omega$ are boundary conditions.

An initial condition, on the other hand, specifies the value of z and some of its derivatives at some initial time t_0 (often $t_0 = 0$).

So the following are examples of initial conditions: $z(x, y, t_0) = f(x, y)$ or $z_t(x, y, t_0) = g(x, y)$ on Ω.

Separation of variables

Let us explain the method of separation of variables with the conditions $z(0, y) = 0, z(L, y) = 0, z(x, 0) = 0$ in the region $0 < x < L, y > 0$ for the Laplace equation

$$z_{xx} + z_{yy} = 0.$$

Separation of variables

Theorem

Let $z(x, y) = v_1(x, y) + iv_2(x, y)$ be a complex-valued solution of the linear PDE

$$\mathcal{L}z = Az_{xx} + Bz_{xy} + Cz_{yy} + Dz_x + Ez_y + Fz = f,$$

where A, B, C, D, E, F, f are real-valued functions of (x, y). Then $v_1(x, y) = Re z(x, y)$ satisfies the PDE $\mathcal{L}z = f$, and $v_2(x, y) = Im z(x, y)$ satisfies the associated PDE $\mathcal{L}z = 0$.

Separation of variables

Ex: Find separated solutions of the PDE

$$z_{xx} - z_t = 0$$

in the form $z(x,t) = e^{\alpha x} e^{iwt}$, where w is real and positive.

Thank you

Thank You!!