MA204: Mathematics IV

Partial Differential Equation (Higher Order PDE)

Introduction

The general form of an nth order PDE with n-independent variables is given by

$$F(x_1, x_2, \ldots, x_n, z, z_{x_1}, z_{x_2}, \ldots, z_{x_n}, z_{x_1 x_1}, z_{x_1 x_2} \ldots z_{x_1 x_2 \ldots x_n}) = 0.$$

We already have the following classifications for 1st order PDEs:

linear/semilinear/quasilinear/nonlinear.

Later, we shall see that the 2nd order PDEs also have the following classifications:

Linear: Parabolic/Hyperbolic/Elliptic
Non-linear:

In general, we have the following classifications for nth order PDEs:

Linear: With constant coefficients/With variable coefficients
Non-linear:

Introduction

Our main concern in this course is to deal with the 2nd order PDEs as such equations mostly appear in practical applications.

However, we here begin with a general basic theory for nth order PDEs. In particular, we concentrate on the nth order linear PDEs.

Linear PDE with constant coefficients: If a_{ij}'s are constants, then an nth order linear PDE with constant coefficients in two independent variables can be written as

$$(a_{00}\frac{\partial^n z}{\partial x^n} + a_{01}\frac{\partial^n z}{\partial x^{n-1}\partial y} + \ldots + a_{0n}\frac{\partial^n z}{\partial^n y}) + (a_{10}\frac{\partial^{n-1} z}{\partial x^{n-1}} + a_{11}\frac{\partial^{n-1} z}{\partial x^{n-2}\partial y}$$
$$+ \ldots + a_{1(n-1)}\frac{\partial^{n-1} z}{\partial^{n-1} y}) + \ldots + (a_{(n-1)0}\frac{\partial z}{\partial x} + a_{(n-1)1}\frac{\partial z}{\partial y}) + a_{n0}z = f(x,y).$$

Denoting $D := \frac{\partial}{\partial x}$ and $D' := \frac{\partial}{\partial y}$, we write the equation as

$$\begin{aligned} F(D,D')z :=& (a_{00}D^n + a_{01}D^{n-1}D' + \ldots a_{0n}D'^n)z \\ &+ \ldots + (a_{(n-1)0}D + a_{(n-1)1}D')z + an0z = f(x,y). \end{aligned}$$

Linear ODE with constant coefficients

If all terms of the expression $F(D, D')$ are of the same order n, then the PDE $F(D, D') = f(x, y)$ is called a **homogeneous** PDE. A homogeneous PDE of order n is

$$F(D, D')z = a_{00}\frac{\partial^n z}{\partial x^n} + a_{01}\frac{\partial^n z}{\partial x^{n-1}\partial y} + \ldots + a_{0n}\frac{\partial^n z}{\partial^n y} = f(x, y).$$

Otherwise the PDE is called **Non-homogeneous**.

Theorem

If z_g is a general solution of the PDE $F(D, D')z = 0$ and z_p is a particular integral of $F(D, D')z = f(x, y)$, then $z_g + z_p$ is a general solution of the PDE $F(D, D')z = f(x, y)$.

Note that z_g is called the complementary function of the PDE $F(D, D') = f(x, y)$ containing n-arbitrary functions if the PDE is of nth order.

CF for linear ODE with constant coefficients

Theorem

If $z_1, z_2, \ldots, z_n$ are solutions of the PDE $F(D, D')z = 0$, then $\sum_{i=1}^{n} c_i z_i$, with c_i being arbitrary constants, is also a solution of $F(D, D')z = 0$.

Thus the problem of finding z_g reduces to find n-linearly independent solutions of the PDE $F(D, D')z = 0$.

For this, we here classify the PDE $F(D, D') = 0$ into two categories:

(a) **Reducible PDE:** If $F(D, D')$ can be expressed as a product of linear factors of the form $aD + bD' + c$ with a, b, c constants. Thus the PDE $F(D, D') = 0$ is reducible if

$$F(D, D') = \prod_{i=1}^{n} (a_i D + b_i D' + c).$$

(b) **Irreducible PDE:** If a PDE $F(D, D') = 0$ is not reducible, then it is called irreducible.

CF for linear ODE with constant coefficients

Theorem

If $aD + bD' + c$ is a factor of $F(D, D')$ with $a \neq 0$, then $z = e^{-\frac{cx}{a}}\phi(bx - ay)$, where ϕ is a real valued function of a variable, is a solution of $F(D, D')z = 0$.

CF for linear ODE with constant coefficients

Theorem

Let $bD' + c$ be a factor of $F(D, D')$ and ϕ is a real valued function of a variable. If $b \neq 0$, then $z = e^{-\frac{cx}{b}}\phi(bx)$ is a solution of $F(D, D')z = 0$.

CF for linear ODE with constant coefficients

Theorem

For $m \leq n$, let $(aD + bD' + c)^m$ is a factor of $F(D, D')$ and $\phi_{1,2}, \ldots, \phi_m$ are arbitrary real valued function of a single variable. If $a \neq 0$, then

$$z = e^{-\frac{cx}{a}} \sum_{j=1}^{m} x^{j-1} \phi_j(bx - ay),$$

and if $a = 0$, then

$$z = e^{-\frac{cx}{b}} \sum_{j=1}^{m} x^{j-1} \phi_j(bx),$$

is a solution of $F(D, D') = 0$.

CF for linear ODE with constant coefficients

Theorem

For a PDE $F(D, D')z = 0$, *a solution is given by*

$$z = \sum_{i=1}^{\infty} c_i e^{a_i x + b_i y},$$

where $F(a_i, b_i) = 0$ *and* c_i *are arbitrary constants.*

Problem

Problem: Find the general solution of the following PDEs:

(a) $\frac{\partial^4 z}{\partial x^4} + \frac{\partial^4 z}{\partial y^4} = 2\frac{\partial^4 z}{\partial x^2 \partial^2 y}$.

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(c) $z_{xx} + 2z_{xy} + z_{yy} + 2z_x + 2z_y + z = 0$.

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(c) $z_{xx} + 2z_{xy} + z_{yy} + 2z_x + 2z_y + z = 0$.

(d) $(2D^4 - 3D^2 D' + D'^2)z = 0$.

PI of linear PDE with constant coefficients

The given PDE $F(D, D')z = f(x, y)$, we can have the particular integral as

$$z = \frac{1}{F(D, D')} f(x, y).$$

We note that

$$\frac{1}{D} f(x, y) := \int f(x, y) dx \text{ and } \frac{1}{D'} f(x, y) := \int f(x, y) dy.$$

As a result, we have

$$z = \frac{1}{F(D, D')} f(x, y) = F(D, D')^{-1} f(x, y)$$

in which we expand $F(D, D')^{-1}$ using binomial theorem.

PI of linear PDE with constant coefficients

There are certain simple cases in which the PI can be obtained very easily.

Case I: If $f(x,y) = e^{ax+by}$, the

$$\text{PI} = \frac{1}{F(D,D')}e^{ax+by} = \frac{1}{F(a,b)}e^{ax+by}.$$

Case II: If $f(x,y) = \sin(ax+by)$ or $\cos(ax+by)$, then

$$\begin{aligned}\text{PI} = \frac{1}{F(D,D')}\sin(ax+by) &= \frac{1}{G(D^2,DD',D'^2)}\sin(ax+by)\\ &= \frac{\sin(ax+by)}{G(-a^2,-ab,-b^2)}\end{aligned}$$

and

$$\begin{aligned}\text{PI} = \frac{1}{F(D,D')}\cos(ax+by) &= \frac{1}{G(D^2,DD',D'^2)}\cos(ax+by)\\ &= \frac{\cos(ax+by)}{G(-a^2,-ab,-b^2)}\end{aligned}$$

PI of linear PDF with constant coefficients

Case III: If $f(x, y) = x^l y^m$, then

$$\text{PI} = \frac{1}{F(D, D')} x^l y^m = F(D, D')^{-1} x^l y^m.$$

Case IV: If $f(x, y) = V(x, y)e^{ax+by}$, then

$$\text{PI} = \frac{1}{F(D, D')} V(x, y)e^{ax+by} = e^{ax+by} \frac{1}{F(D + a, D' + b)} V(x, y).$$

Problem

Problem: Solve the following PDEs :

(a) $(D^2 - D'^2 + D - 1)z = e^{2x+3y}$

(b) $(D - D' - 1)(D - D' - 2)z = \sin(2x + 3y)$

(c) $(D^2 - D' - 1)z = x^2 y$

(d) $(D^2 - D')z = xe^{ax+a^2y}$

(e) $(D^2 - D')z = e^{x+y}$

Thank you

Thank You!!