MA204: Mathematics IV

Partial Differential Equation (Homogeneous Linear PDE)

Introduction

Recall that if all terms of the expression $F(D, D')$ of a PDE are of the same order n, then the PDE $F(D, D') = f(x, y)$ is called a **homogeneous** PDE.

The general form of a homogeneous linear PDE of order n is

$$F(D, D')z = a_{00}\frac{\partial^n z}{\partial x^n} + a_{01}\frac{\partial^n z}{\partial x^{n-1}\partial y} + \ldots + a_{0n}\frac{\partial^n z}{\partial^n y} = f(x, y).$$

Let the trial solution of the PDE $F(D, D') = 0$ be

$$z = \phi(y + mx).$$

Then we must have

$$a_{00}m^n + a_{01}m^{n-1} + \ldots + a_{0n} = 0,$$

which is called **auxiliary equation** for the given PDE.

It is easy to see that the auxiliary equation for the PDE has n roots with multiplicity.

CF for homogeneous linear PDE

Depending on the nature of the roots of the auxiliary equation of a PDE, we have the following cases:

Case I: Let the roots $m_1, m_2, \ldots, m_n$ of the A.E be distinct. Thus the C.F for the given PDE is obtained as

$$z = \sum_{i=1}^{n} \phi(y + m_i x),$$

where ϕ_i are arbitrary functions.

Case II: If m_i is a root of the A.E with multiplicity k_i such that $\sum_{i=1}^{r} k_i = n$, then the C.F of the PDE is given by

$$z = \sum_{i=1}^{r} \left\{ \sum_{j=1}^{k_i} x^{j-1} \phi_j(y + m_j x) \right\},$$

where ϕ_i are arbitrary functions.

PI for homogeneous linear PDE

Ex: Show that $\frac{1}{D-aD'}f(x,y) = \int f(x, c-ax)dx$, where c is any arbitrary constant.

We first note that

$$\text{PI} = \frac{1}{F(D,D')}f(x,y) = \frac{1}{(D-a_1D')(D-a_2D')\cdots(D-a_nD')}f(x,y)$$

Thus repeatedly applying the above exercise, we can find the PI for the PDE.

Problem

Problem: Solve the following PDEs:

(a) $(D^2 - 6DD' + 9D'^2)z = 12x^2 + 36xy$.

(b) $(D^2 - 4D'^2)z = \frac{4x}{y^2} - \frac{y}{x^2}$.

(c) $(D^2 + DD' - 6D'^2)z = y \cos x$.

Thank you

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