MA204: Mathematics IV

Partial Differential Equation (Heat, Wave, Laplace Equation)

Laplace Equation

The general form of an n-dimensional Laplace equation is

$$\nabla^2 u = u_{x_1 x_1} + u_{x_2 x_2} + \ldots + u_{x_n x_n} + = 0.$$

Using the separation of variables, we see that the solution of the two dimensional Laplace equation $u_{xx} + u_{yy} = 0$ is given by

$$u(x,y) = \begin{cases} (c_1 e^{px} + c_2 e^{-px})(c_3 \cos py + c_4 \sin py), & \text{if } \lambda = p^2 \text{ with real } p; \\ (c_1 x + c_2)(c_3 y + c_4), & \text{if } \lambda = 0; \\ (c_1 \cos px + c_2 \sin px)(c_3 e^{py} + c_4 e^{-py}), & \text{if } \lambda = -p^2 \text{ with real } p. \end{cases}$$

Under the initial and boundary conditions, one of these cases will proved the exact solution for a problem following the Laplace equation model.

Laplace Equation

With the change of change of variables from cartesian to polar, the Laplace equation $u_{xx} + u_{yy} = 0$ takes the form

$$u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta} = 0.$$

With the change of change of variables from cartesian to cylindrical, the Laplace equation $u_{xx} + u_{yy} + u_{zz} = 0$ takes the form

$$u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta} + u_{zz} = 0.$$

Using the separation of variables, these transformed equations can also be solved to get solution for the Laplace equation.

Heat Equation

The general form of a heat equation is

$$u_t = k\nabla^2 u + r,$$

where k is normalized conductivity called thermal diffusibility and r is source term.

A important class of solution of the heat equation are the steady-state solutions. In this, u is considered to be independent of t, i.e., $\frac{\partial u}{\partial t} = 0$.

Thus we have the heat equation in the Poisson's form

$$k\nabla^2 u + r = 0.$$

In addition, if $r = 0$, then u satisfies the Laplace equation

$$\nabla^2 u = 0.$$

Heat Diffusion Equation

In the absence of source term, the heat equation reduces to

$$u_t = k\nabla^2 u.$$

This is called **Fourier heat conduction equation** or **heat diffusion equation**.

For the one dimensional diffusion equation $u_t = ku_{xx}$, we have the following solutions:

(1) $u(x,t) = \frac{1}{\sqrt{4\pi kt}} e^{\frac{-(x-\zeta)^2}{4kt}}$ for a constant ζ.

(2) $u(x,t) =$
$$\begin{cases} (c_1 e^{px} + c_2 e^{-px})c_3 e^{kp^2 t}, & \text{if } \lambda = p^2 \text{ with real } p; \\ (c_1 x + c_2)c_3, & \text{if } \lambda = 0; \\ (c_1 \cos px + c_2 \sin px)c_3 e^{-kp^2 t}, & \text{if } \lambda = -p^2 \text{ with real } p. \end{cases}$$

(3) Solutions can also be obtained by changing the variables to cylindrical and spherical coordinates, followed by separation of variables.

Wave Equation

The wave equation for a wave with speed c without any external force is given by

$$u_{tt} = c^2\nabla^2 u.$$

The solution of a wave equation is called a **wave function**.

If $f =$ vertical force per unit length at point x, at time t, then the wave equation becomes

$$u_{tt} - c^2\nabla^2 u = F,$$

where $F(x,t) = \frac{1}{\rho} f(x,y)$ with ρ as the mass per unit length of the string.

Another PDE related to the wave equation is

$$u_{tt} + 2\gamma u_t - c^2\nabla^2 u = F,$$

where γ is areal positive constant.

Wave Equation

For the one dimensional wave equation $u_{tt} = c^2 u_{xx}$, we have the canonical form

$$u_{\zeta\eta} = 0,$$

with the solution

$$u(x,t) = \phi(\zeta) + \chi(\eta) = \phi(x - ct) + \chi(x + ct),$$

where ϕ and χ are arbitrary functions.

Under the initial conditions $u(x,0) = f(x)$ and $u_t(x,0) = g(x)$, the solution for the one dimensional wave equation has the D'Alembert's solution

$$u(x,t) = \frac{1}{2}\{f(x+ct) + f(x-ct)\} + \frac{1}{2c}\int_{x-ct}^{x+ct} g(p)dp.$$

Wave Equation

Using the separation of variables, the solution for $u_{tt} = c^2 u_{xx}$ can also be obtained as

$$u(x,t) = \begin{cases} (c_1 e^{px} + c_2 e^{-px})(c_3 e^{cpt} + c_4 e^{-cpt}), & \text{if } \lambda = p^2 \text{ with real} \\ (c_1 x + c_2)(c_3 t + c_4), & \text{if } \lambda = 0; \\ (c_1 \cos px + c_2 \sin px)(c_3 \cos cpt + c_4 \sin cpt), & \text{if } \lambda = -p^2 \text{ with rea} \end{cases}$$

Based on the initial and boundary conditions on the PDE, one can choose the appropriate solution for the given PDE.

Another method is used for wave equations to get another kind of solution, called periodic solutions (in cylindrical and spherical coordinates) of the form

$$u = F(r)e^{iwt}.$$

Thank you

Thank You!!