MA102 : Multivariable Variable Calculus

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Syllabus:

1. Vectors and the Geometry of Space
2. Vector-Valued Functions and Motion in Space
3. Partial Derivatives
4. Multiple Integrals
5. Stokes and divergence theorem

Book : George B. Thomas, Ross L. Finney - Calculus and Analytic Geometry, Ninth Edition (1998, Addison Wesley)

Weightage : 50%

Evaluation:

Quiz - 1 : weightage : 10%
Midsem : weightage : 40%

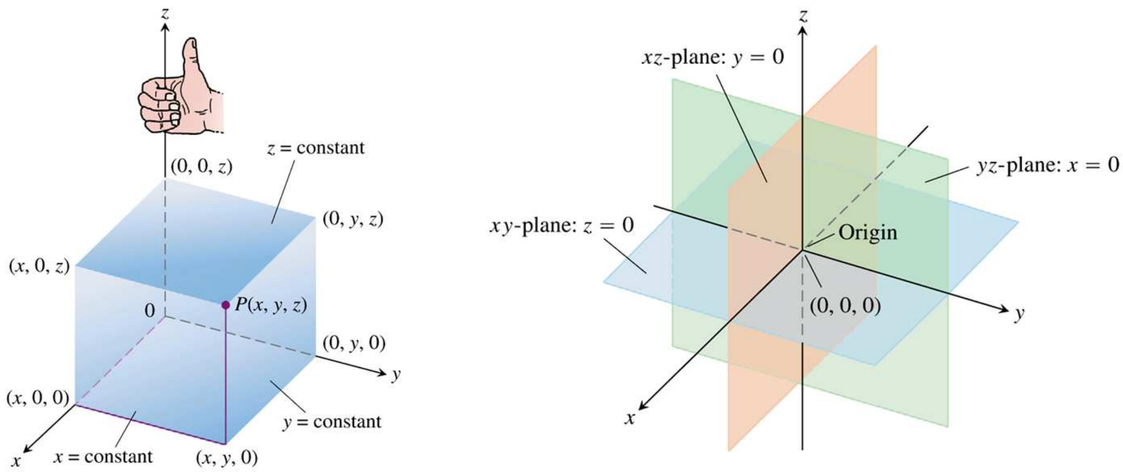
Vectors and the Geometry of Space

Three-Dimensional Coordinate Systems

Note. You have never legitimately been into three dimensional space in your calculus career. In this section, we legitimately mathematically enter three dimensions (physically, you lived there your whole life)!

Definition. We introduce *three-dimensional Cartesian coordinates*, (x, y, z), by considering three mutually orthogonal (i.e., perpendicular) coordinate axes, the x-axis, the y-axis, and the z-axis. We do so in such a way as to determine a *right-hand* coordinate system. If you curl the fingers of your right hand from the positive x-axis to the positive y-axis, then your thumb will point in the direction of the positive z-axis. Such a system determines three coordinate planes, the xy-plane, the xz-plane, and the yz-plane. For point $P(x, y, z)$, coordinate x represents the distance of P from the yz-plane, coordinate y represents the distance of P from the xz-plane, and coordinate z represents the distance of P from the xy-plane. The coordinate planes divide three-dimensional space into eight *octants*, depending on the signs of the coordinates of the points in that octant.

The *first octant* contains all points with positive coordinates.



Figures 1

Note. It follows from the Pythagorean Theorem that distance is measured in three-dimensional space between points $P_x(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ is

$$P_1P_2 = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}.$$

It follows from this formula for distance that the formula for a sphere of radius a and center (x_0, y_0, z_0) is

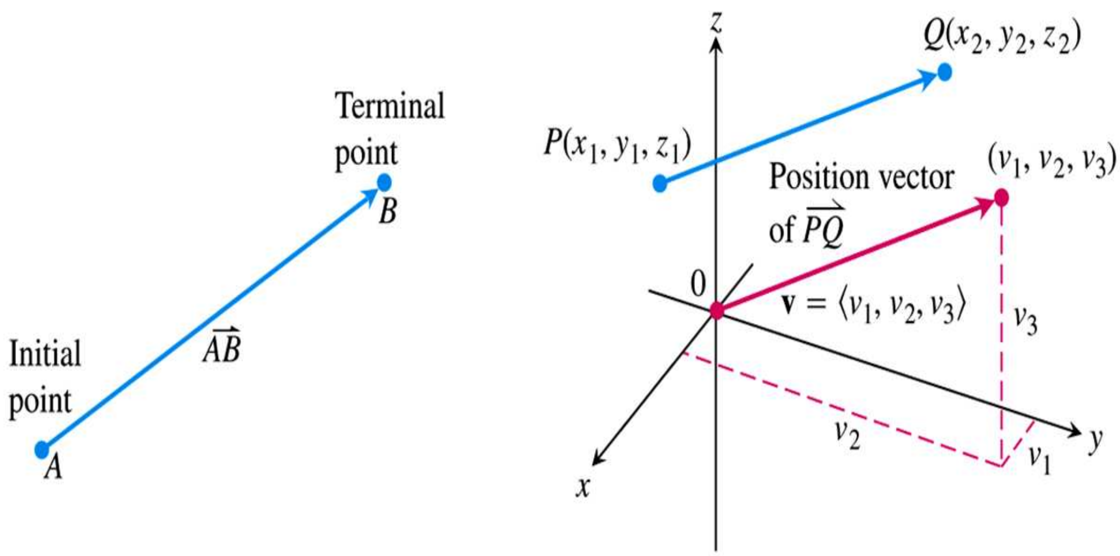
$$(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = 1^2.$$

Vectors and the Geometry of Space

Vectors

Note. Several physical quantities are represented by an entity which involves both magnitude and direction. Examples of such entities are force, velocity, acceleration (and some-times position).

Definition. The *vector* from point A to point B is the directed line segment from A to B and is denoted $\vec{AB}$. Point A is the *initial point* and point B is the *terminal point* of vector $\vec{AB}$.



Figures 2

Note. Though not yet defined, a vector will only have *magnitude* and *direction*. It will not have a *position*! Geometrically, think of a vector as an arrow which can be translated around, but which can be neither stretched nor rotated. If we translate a vector so that its initial point is at the origin of a Cartesian coordinate system, then the vector is said to be in *standard position* (see Figure 2 above).

Definition. When a vector is in standard position, it will then have as its terminal point, some point (v_1, v_2, v_3) (or some point (v_1, v_2) if the vector is in two-dimensions). The *component form* of this vector is then $\langle v_1, v_2, v_3 \rangle$ (or $\langle v_1, v_2 \rangle$ if the vector is in two-dimensions). The numbers v_1, v_2, and v_3 are the *components* of $\mathbf{v}$. *In these notes* (and in the text), we will use bold-faced fonts to represent vectors. For example, we represent vector $\langle v_1, v_2, v_3 \rangle$ as $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$. *On the whiteboard* we use a little arrow over the letter which represents the vector: $\vec{v} = \langle v_1, v_2, v_3 \rangle$.

Note. It now follows that the vector from point $P(x_1, y_1, z_1)$ to point $Q(x_2, y_2, z_2)$ is $\mathbf{v} = \langle x_2 - x_1, y_2 - y_1, z_2 - z_1 \rangle$.

Note. Notice that there is a vital difference between a vector and a point!!! A vector has a magnitude and direction, but no position! A point has a position, but neither magnitude nor direction! Hence, we **must** have a notation which distinguishes between the two. That is why we use parentheses to represent points (the *point* (x, y, z)) and angled brackets to represent vectors (the *vector* $\langle x, y, z \rangle$).

Definition. Two vectors are *equal* if they have the same component form. The *magnitude* (or *length*) of vector $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ is

$$|\mathbf{v}| = \sqrt{v_1^2 + v_2^2 + v_3^2}.$$

Notice that this is the distance between endpoints of vector $\mathbf{v}$.

Note. We now explore the *algebraic* properties of vectors. You saw this more formally in Linear Algebra class.

Definition. Let $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ be vectors with k a scalar (i.e., number). Then define:

Vector Addition: $\mathbf{u} + \mathbf{v} = \langle u_1 + v_1, u_2 + v_2, u_3 + v_3 \rangle$,

Scalar Multiplication: $k\mathbf{u} = \langle ku_1, ku_2, ku_3 \rangle$.

Of course, similar definitions hold when the vector is two-dimensional.

Note. The definition of vector addition can be illustrated *geometrically* in terms of the following diagram (Figure 3). These diagrams illustrate the fact that vectors follow a *parallelogram law* of addition. The vector $\mathbf{u} + \mathbf{v}$ is called the *resultant vector* of the vector addition. We define the *difference* $\mathbf{u} - \mathbf{v} = \mathbf{u} + (-1\mathbf{v})$.

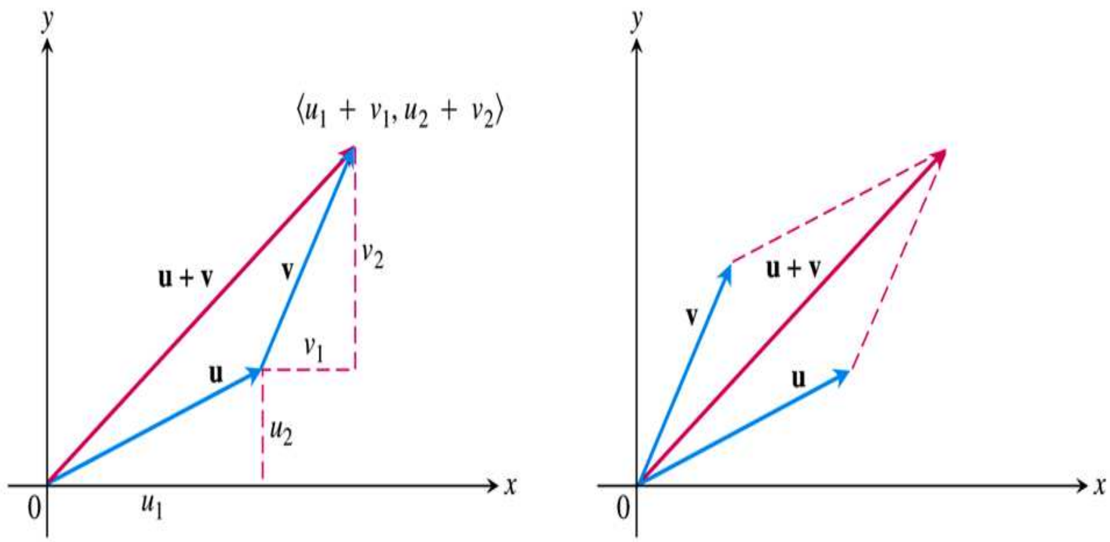


Figure 3

Note. Scalar addition can be illustrated *geometrically* in terms of the following diagram (Figure 4). Notice that the scalar stretches (or shrinks) the magnitude of the original vector and if the scalar is negative, then it reverses the direction of the original vector.

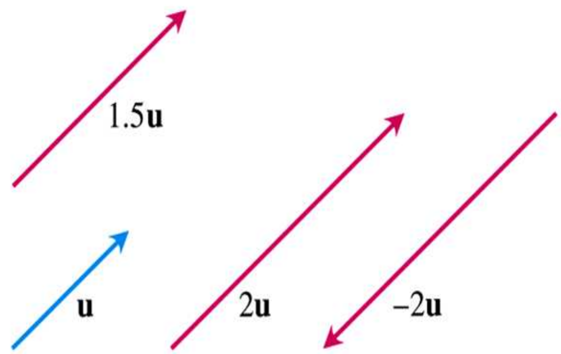


Figure 4

Theorem (Properties of Vector Operations). Let $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ be vectors and a, b be scalars. Then

1. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$ (Commutative Property of Vector Addition).

2. $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$ (Associative Property of Vector Addition).

3. $\mathbf{u} + \mathbf{0} = \mathbf{u}$ (The Zero Vector $\mathbf{0}$ is the Additive Identity under Vector Addition).

4. $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$ (The Additive Inverse of Vector $\mathbf{u}$ is $-\mathbf{u}$ under Vector Addition).

5. $0\ \mathbf{u} = \mathbf{0}$ (Behavior of Scalar 0 in Scalar Multiplication).

6. $1\ \mathbf{u} = \mathbf{u}$ (Behavior of Scalar 1 in Scalar Multiplication).

7. $a\ (b\mathbf{u}) = (ab)\mathbf{u}$ (Associativity of Scalar Multiplication).

8. $a\ (\mathbf{u} + \mathbf{v}) = a\mathbf{u} + a\mathbf{v}$ (Distribution of Scalar Multiplication over Vector Addition).

9. $(a + b)\mathbf{u} = a\mathbf{u} + b\mathbf{u}$ (Distribution of Scalar Addition over Scalar Multiplication).

Definition. A vector $\mathbf{v}$ of length 1 is called a *unit vector*. The three *standard unit vectors* are:

$$\mathbf{i} = \langle 1, 0, 0 \rangle, \quad \mathbf{j} = \langle 0, 1, 0 \rangle, \quad \mathbf{k} = \langle 0, 0, 1 \rangle.$$

Note. Any vector $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ can be written as a *linear combination* of the standard unit vectors as

$$\mathbf{v} = \langle v_1, v_2, v_3 \rangle = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}.$$

The use of the word *linear* here is the same as its use in the class titled "*Linear* Algebra." We call v_1 the $\mathbf{i}$-*component*, v_2 the $\mathbf{j}$-*component*, and v_3 the $\mathbf{k}$-*component*. In component form, the vector from point $P_1(x_1, y_1, z_1)$ to point $P_2(x_2, y_2, z_2)$ is

$$\overrightarrow{P_1P_2} = (x_2 - x_1)\mathbf{i} + (y_2 - y_1)\mathbf{j} + (z_2 - z_1)\mathbf{k}.$$

In standard position, this vector has its tail at the origin and its head at the point $(x_2 - x_1, y_2 - y_1, z_2 - z_1)$.

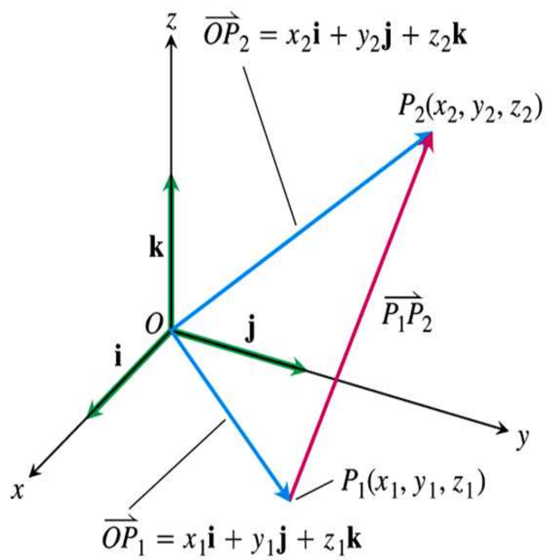


Figure 5

Definition. The *direction* of nonzero vector $\mathbf{v}$ is the unit vector $\mathbf{v}/|\mathbf{v}|$.

Definition. The *midpoint* M of the line segment joining points $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ is the point

$$M = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2} \right).$$

Vectors and the Geometry of Space

The Dot Product

Note. In this section we introduce an operation which can be performed on two vectors. The operation is called *dot product*, or sometimes *inner product* or *scalar product*. We use this product to measure angles between vectors.

Theorem 1. Angle Between Two Vectors. The angle θ between two nonzero vectors $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ is given by

$$\theta = \cos^{-1}\left(\frac{u_1v_1 + u_2v_2 + u_3v_3}{|\mathbf{u}|\,|\mathbf{v}|}\right).$$

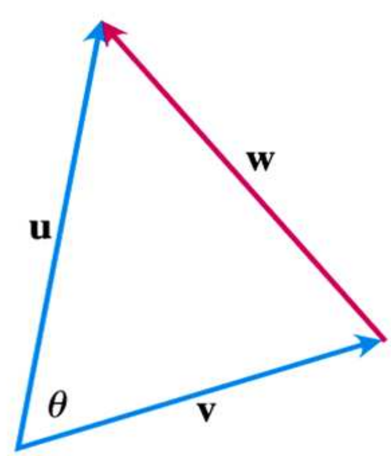


Figure 6

Proof. Referring to Figure 6 above, we have by the Law of Cosines that

$$|\mathbf{w}|^2 = |\mathbf{u}|^2 + |\mathbf{v}|^2 - 2|\mathbf{u}|\,|\mathbf{v}|\cos\theta, \text{ or}$$

$$2|\mathbf{u}|\,|\mathbf{v}|\cos\theta = |\mathbf{u}|^2 + |\mathbf{v}|^2 - |\mathbf{w}|^2.$$

In terms of components, $\mathbf{w} = \mathbf{u} - \mathbf{v} = \langle u_1 - v_1, u_2 - v_2, u_3 - v_3\rangle$. So

$$\begin{aligned}
|\mathbf{u}|^2 &= u_1^2 + u_2^2 + u_3^2 \\
|\mathbf{v}|^2 &= v_1^2 + v_2^2 + v_3^2 \\
|\mathbf{w}|^2 &= (u_1 - v_1)^2 + (u_2 - v_2)^2 + (u_3 - v_3)^2 \\
&= u_1^2 - 2u_1v_1 + v_1^2 + u_2^2 - 2u_2v_2 + v_2^2 + u_3^2 - 2u_3v_3 + v_3^2.
\end{aligned}$$

These equations combine to give

$$|\mathbf{u}|^2 + |\mathbf{v}|^2 - |\mathbf{w}|^2 = 2(u_1v_1 + u_2v_2 + u_3v_3).$$

Therefore,

$$2|\mathbf{u}|\,|\mathbf{v}|\cos\theta = |\mathbf{u}|^2 + |\mathbf{v}|^2 - |\mathbf{w}|^2 = (2(u_1v_1 + u_2v_2 + u_3v_3) \text{ and}$$

$$\cos\theta = \frac{u_1v_1 + u_2v_2 + u_3v_3}{|\mathbf{u}|\,|\mathbf{v}|}.$$

Since $\theta \in [0, \pi)$, we have

$$\theta = \cos^{-1}\left(\frac{u_1v_1 + u_2v_2 + u_3v_3}{|\mathbf{u}|\,|\mathbf{v}|}\right).$$

Definition. The *dot product* of two vectors $\mathbf{u} = \langle u_1, u_2, v_2 \rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ is defined as

$$\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2 + u_3 v_3.$$

If the vectors only have two components, then the dot product is similarly defined.

Note. Notice that the dot product of two vectors is *not* a vector, but a scalar (and that's why the dot product is sometimes called a "scalar product"). In terms of dot products, the angle θ between vectors $\mathbf{u}$ and $\mathbf{v}$ are

$$\theta = \cos^{-1}\left(\frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{u}|\,|\mathbf{v}|}\right).$$

Note. We will be particularly interested in the situation when vectors are perpendicular. That is, when the angle between the vectors is $\pi/2$. Since $\cos(\pi/2) = 0$, we have the following definition.

Definition. Vectors $\mathbf{u}$ and $\mathbf{v}$ are *orthogonal* (or *perpendicular*) if an only if $\mathbf{u} \cdot \mathbf{v} = 0$.

Theorem. Properties of the Dot Product. If $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are any vectors and c is a scalar, then

1. $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$ (Commutative Property of Dot Product).

2. $(c\mathbf{u}) \cdot \mathbf{v} = \mathbf{u} \cdot (c\mathbf{v}) = c(\mathbf{u} \cdot \mathbf{v})$ (Distribution of scalar Multiplication through Dot Product).

3. $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$ (Distribution of Dot Product over Vector Addition).

4. $\mathbf{u} \cdot \mathbf{u} = |\mathbf{u}|^2$.

5. $\mathbf{0} \cdot \mathbf{u} = 0$.

Each of these properties is easily verified by computations with the components of the vectors.

Note. In applications (and theoretical problems), it is often desired to find the "piece" of a vector which goes in a certain direction. That is, we desire to find the *projection* of one vector $\mathbf{u}$ onto another $\mathbf{v}$ (where the projection is denoted $\text{proj}_{\mathbf{v}}(\mathbf{u})$), as illustrated in Figure **7** below.

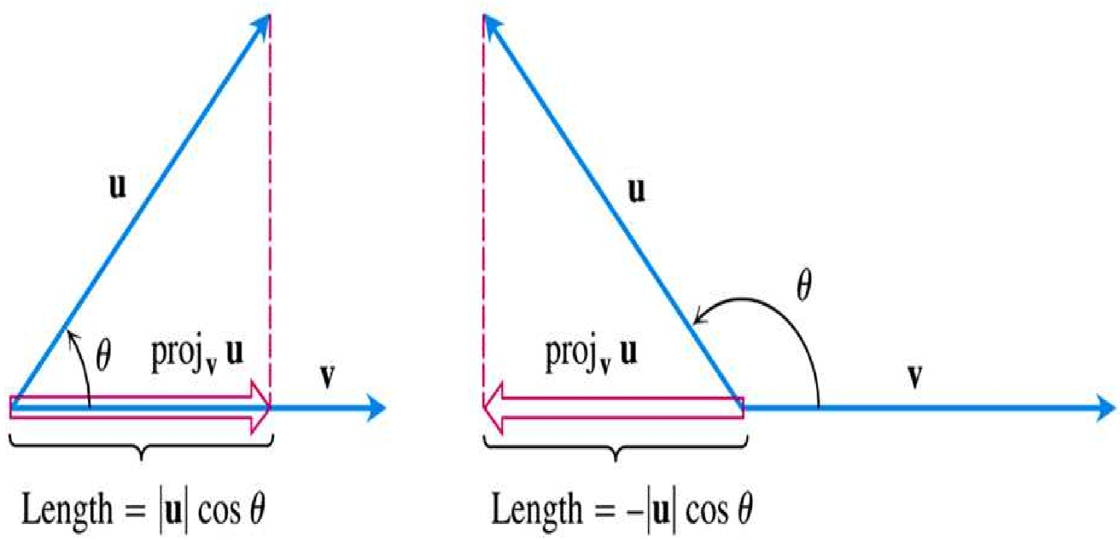


Figure **7**

Definition. The vector projection of $\mathbf{u}$ onto $\mathbf{v}$ is the vector

$$\text{proj}_{\mathbf{v}}(\mathbf{u}) = \left(\frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{v}|^2}\right)\mathbf{v}.$$

Note. Notice that the projection of $\mathbf{u}$ onto $\mathbf{v}$ is the vector with scalar component $\dfrac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{v}|} = |\mathbf{u}|\cos\theta$ (see Figure **7**) and direction $\mathbf{v}/|\mathbf{v}|$.

Vectors and the Geometry of Space

The Cross Product

Note. In this section we introduce an operation which can be performed on two vectors, each with three components. The operation is called *cross product*, or sometimes *vector product*. We use this product to find volumes and to construct planes in three dimensions.

Notes. Two non-parallel vectors $\mathbf{u}$ and $\mathbf{v}$ (i.e., $\mathbf{u}$ is not a scalar multiple of $\mathbf{v}$) determine (in three-dimensions) a plane. If we make the tails of the vectors coincide, then we have three points: (1) the point at the tails of the vectors, (2) the head of vector $\mathbf{u}$, and (3) the head of vector $\mathbf{v}$. Three points determine a plane (just as two points determine a line). We now choose unit vector $\mathbf{n}$ to be perpendicular to both vectors $\mathbf{u}$ and $\mathbf{v}$ and so that vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{n}$ form a right-hand coordinate system:

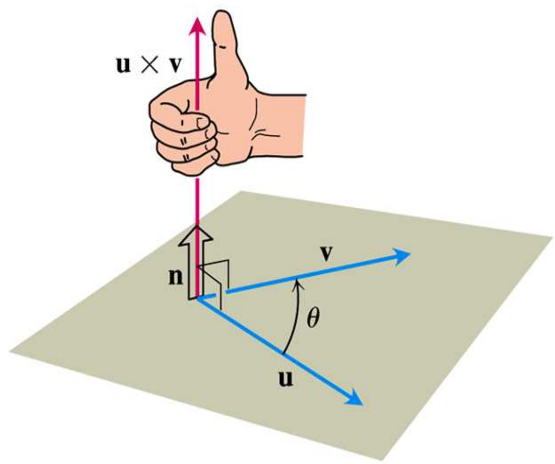


Figure 8

Definition. The *cross product* of two non-parallel vectors $\mathbf{u}$ and $\mathbf{v}$ is defined as

$$\mathbf{u} \times \mathbf{v} = (|\mathbf{u}|\,|\mathbf{v}|\sin\theta)\mathbf{n},$$

where vector $\mathbf{n}$ is the unit vector mentioned above. If vectors $\mathbf{u}$ and $\mathbf{v}$ are parallel, then we define $\mathbf{u} \times \mathbf{v} = \mathbf{0}$.

Theorem. Properties of the Cross Product. If $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are any vectors and r, s are scalars, then

1. $(r\mathbf{u}) \times (s\mathbf{v}) = (rs)(\mathbf{u} \times \mathbf{v})$ (Distribution of Scalar Multiplication over Cross Product).

2. $\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = \mathbf{u} \times \mathbf{v} + \mathbf{u} \times \mathbf{w}$ (Distribution of Cross Product over Vector Addition).

3. $\mathbf{v} \times \mathbf{u} = -(\mathbf{u} \times \mathbf{v})$ (Anticommutivity of Cross Product).

4. $(\mathbf{v} + \mathbf{w}) \times \mathbf{u} = \mathbf{v} \times \mathbf{u} + \mathbf{w} \times \mathbf{u}$ (Distribution of Cross Product over Vector Addition).

5. $\mathbf{0} \times \mathbf{u} = \mathbf{0}$.

6. $\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \cdot \mathbf{w})\mathbf{v} - (\mathbf{u} \cdot \mathbf{v})\mathbf{w}$.

Note. This operation of taking a cross product of two vectors is likely the first operation you have encountered which is not commutative. Verification of the claims of this theorem are hard to establish, since we do not yet have a way to compute cross products (but we soon will). However, we can graphically convince ourselves of the validity of the following.

Theorem. The standard unit vectors satisfy the following cross product relationships:

$$\mathbf{i} \times \mathbf{j} = -(\mathbf{j} \times \mathbf{i}) = \mathbf{k}$$

$$\mathbf{j} \times \mathbf{k} = -(\mathbf{k} \times \mathbf{j}) = \mathbf{i}$$

$$\mathbf{k} \times \mathbf{i} = -(\mathbf{i} \times \mathbf{k}) = \mathbf{j}$$

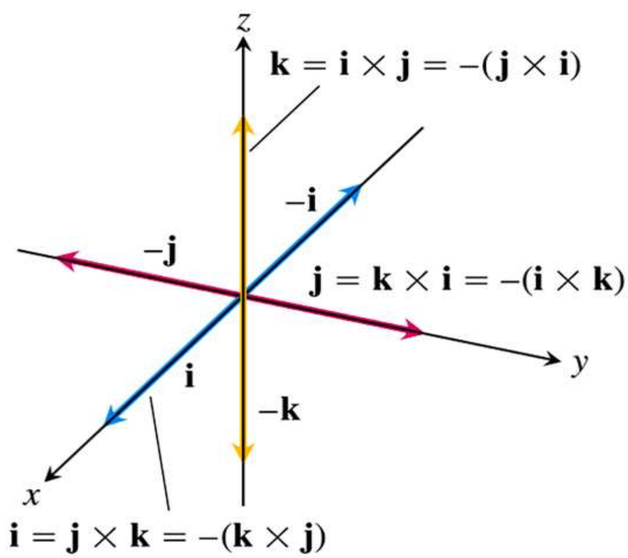


Figure 9

Note. We are now finally ready to compute $\mathbf{u} \times \mathbf{v}$ in terms of the components of $\mathbf{u}$ and $\mathbf{v}$. Suppose $\mathbf{u} = \langle u_1, u_2, u_3 \rangle = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}$ and $\mathbf{v} = \langle v_1, v_2, v_3 \rangle = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$. The computation is based on the cross products of the standard unit vectors and properties of cross product:

$$\begin{aligned}
\mathbf{u} \times \mathbf{v} &= (u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}) \times (v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}) \\
&= u_1v_1\mathbf{i} \times \mathbf{i} + u_1v_2\mathbf{i} \times \mathbf{j} + u_1v_3\mathbf{i} \times \mathbf{k} \\
&\quad + u_2v_1\mathbf{j} \times \mathbf{i} + u_2v_2\mathbf{j} \times \mathbf{j} + u_2v_3\mathbf{j} \times \mathbf{k} \\
&\quad + u_3v_1\mathbf{k} \times \mathbf{i} + u_3v_2\mathbf{k} \times \mathbf{j} + u_3v_3\mathbf{k} \times \mathbf{k} \\
&= (u_2v_3 - u_3v_2)\mathbf{i} - (u_1v_3 - u_3v_1)\mathbf{j} + (u_1v_2 - u_2v_1)\mathbf{k}.
\end{aligned}$$

This computation can be remembered by computing what the book calls a *symbolic determinant* (determinants are explored in more detail in the Linear Algebra class):

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \mathbf{k}$$

$$= (u_2v_3 - u_3v_2)\mathbf{i} - (u_1v_3 - u_3v_1)\mathbf{j} + (u_1v_2 - u_2v_1)\mathbf{k}.$$

Note. We know that two non-parallel vectors, $\mathbf{u}$ and $\mathbf{v}$, determine a plane. If we place these vectors with their tails at the same point, then we can use them to determine a parallelogram (see Figure 10 below). The area of this parallelogram is

$$|\mathbf{u} \times \mathbf{v}| = |\mathbf{u}|\,|\mathbf{v}|\,|\sin\theta|\,|\mathbf{n}| = |\mathbf{u}|\,|\mathbf{v}|\,|\sin\theta|.$$

We can use this fact to find the area of a triangle determined by three points.

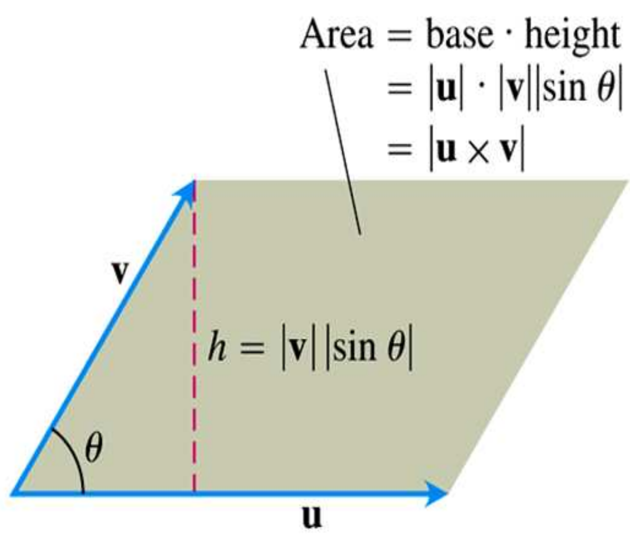


Figure 10

Definition. The *scalar triple product* of vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ (in order) is

$$(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w} = |\mathbf{u} \times \mathbf{v}|\,|\mathbf{w}|\,\cos\theta.$$

Note. We see from its definition that the scalar triple product represents the volume of the parallelepiped determined by $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$.

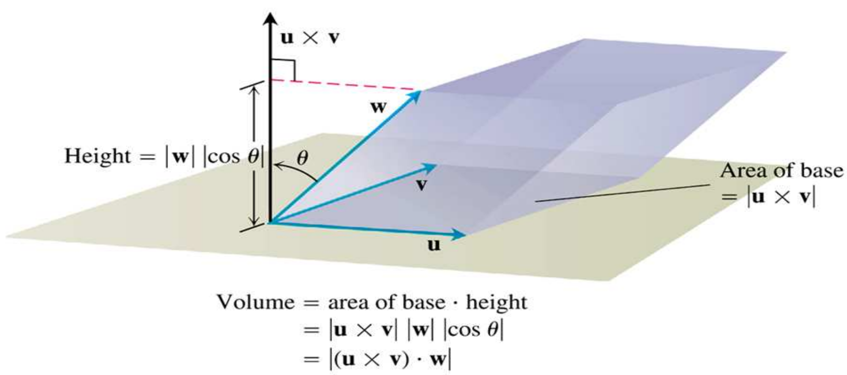


Figure 11

Note. We can verify computationally that:

$$(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w} = (\mathbf{w} \times \mathbf{u}) \cdot \mathbf{v} = (\mathbf{v} \times \mathbf{w}) \cdot \mathbf{u} \text{ and}$$

$$(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w} = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}.$$

Vectors and the Geometry of Space

Lines and Planes in Space

Note. In the plane, a line is determined by a point and a number giving the slope of the line. In space, a line is determined by a point and a *vector* giving the direction of the line.

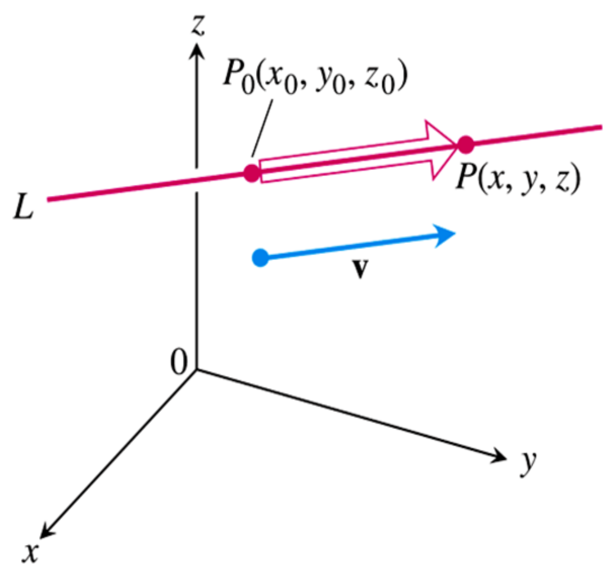


Figure 12

Definition. The *vector equation for the line* L through $P_0(x_0, y_0, z_0)$ parallel to $\mathbf{v}$ is

$$\mathbf{r}(t) = \mathbf{r}_0 + t\mathbf{v}, \quad t \in (-\infty, \infty),$$

where $\mathbf{r}$ is the position vector of a point $P(x, y, z)$ on L and $\mathbf{r}_0$ is the position vector of $P_0(x_0, y_0, z_0)$ (and so $\mathbf{r}_0 = \langle x_0, y_0, z_0 \rangle$).

Definition. The *standard parametrization of the line* through $P_0(x_0, y_0, z_0)$ parallel to $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$ is

$$x = x_0 + tv_1, \ \ y = y_0 + tv_2, \ \ z = z_0 + tv_3, \ \ t \in (-\infty, \infty).$$

Note. To find the distance from a point S to a line that passes through a point P parallel to a vector $\mathbf{v}$, we find the absolute value of the scalar component of $\vec{PS}$ in the direction of a vector normal to the line. As given in Figure 13 below, this value is $|\vec{PS}| \sin\theta = \dfrac{|\vec{PS} \times \mathbf{v}|}{|\mathbf{v}|}$.

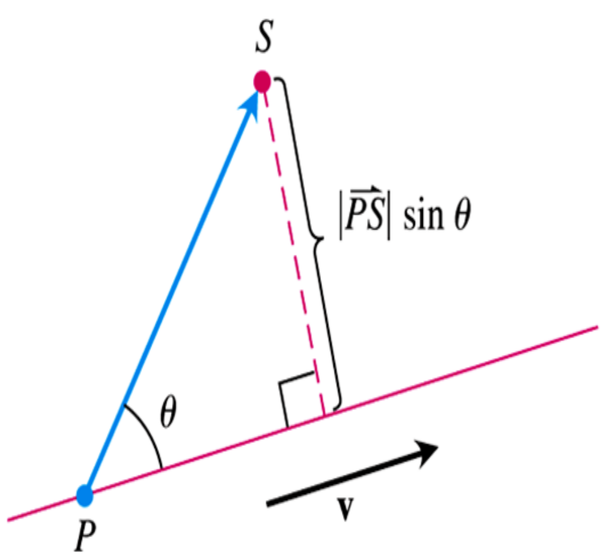


Figure 13

Note. Suppose that plane M passes through a point $P_0(x_0, y_0, z_0)$ and is normal to the nonzero vector $\mathbf{n} = A\mathbf{i} + B\mathbf{j} + C\mathbf{k}$. Then M is the set of all points $P(x, y, z)$ for which $\overrightarrow{P_0P}$ is orthogonal to $\mathbf{n}$. Thus, the dot product $\mathbf{n} \cdot \overrightarrow{P_0P} = 0$. This yields

$$(A\mathbf{i} + B\mathbf{j} + C\mathbf{k}) \cdot ((x - x_0)\mathbf{i} + (y - y_0)\mathbf{j} + (z - z_0)\mathbf{k}) = 0$$

or

$$A(x - x_0) + B(y - y_0) + C(z - z_0) = 0.$$

So the equation of a plane is determined by a point P_0 and a normal vector $\mathbf{n}$.

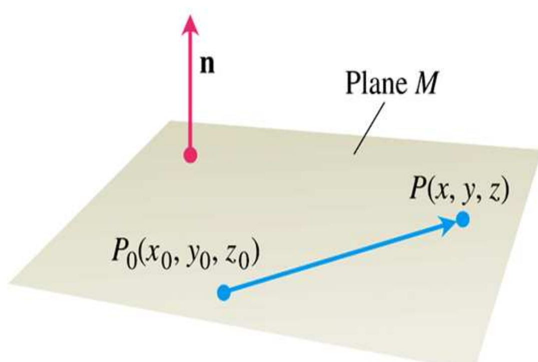


Figure 14

Note. If P is a point on a plane with normal vector $\mathbf{n}$, then the distance from any point S to the plane is the length of the vector projection of $\vec{PS}$ onto $\mathbf{n}$. This distance is $|\text{proj}_{\mathbf{n}}\vec{PS}| = \left|\vec{PS} \cdot \frac{\mathbf{n}}{|\mathbf{n}|}\right|$.

Vectors and the Geometry of Space

Cylinders and Quadric Surfaces

Definition. A *cylinder* is a surface that is generated by moving a straight line along a given planar curve while holding the line parallel to a given fixed line. The curve is called a *generating curve* for the cylinder.

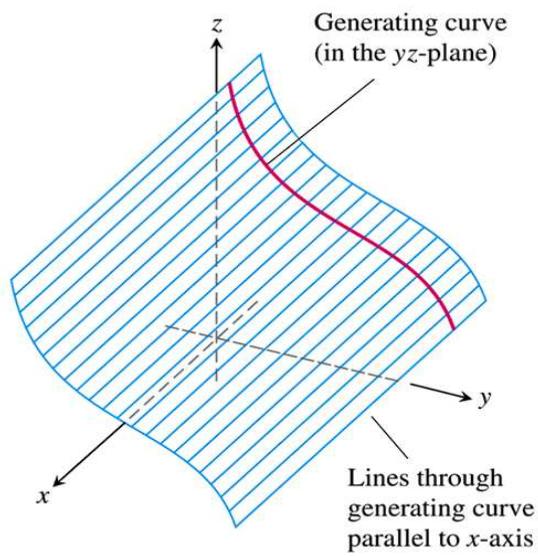


Figure 15

Definition. A *quadric surface* is the graph in space of a second-degree equation in x, y, and z. We focus on the special equation $Ax^2 + By^2 + Cz^2 + Dz = E$ where A, B, C, D, and E are constants. The basic quadric surfaces are ellipsoids, paraboloids, elliptical cones, and hyperboloids.

Example. Consider the ellipsoid $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}+\dfrac{z^2}{c^2}=1.$

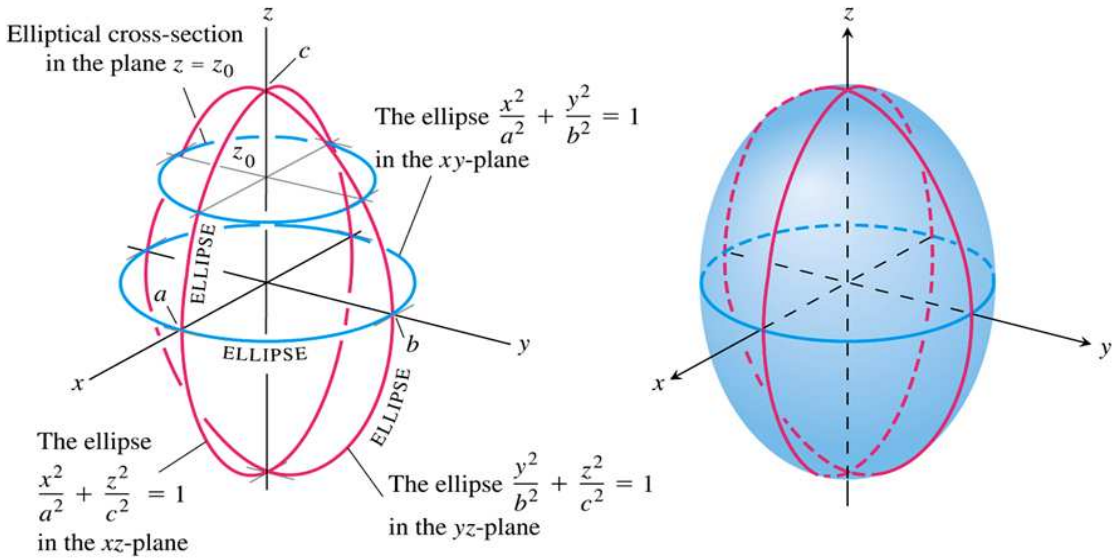


Figure 16

Example. Consider the hyperbolic paraboloid
$\dfrac{y^2}{b^2}-\dfrac{x^2}{a^2}=\dfrac{z}{c},\ c>0.$

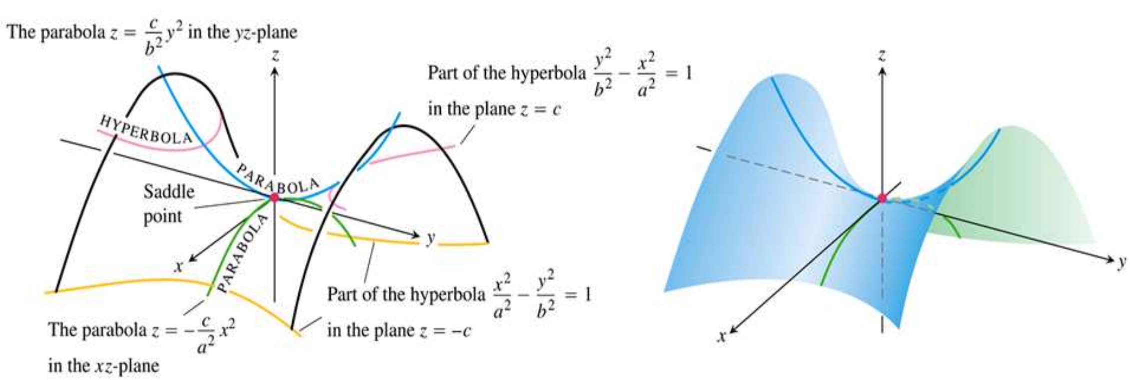
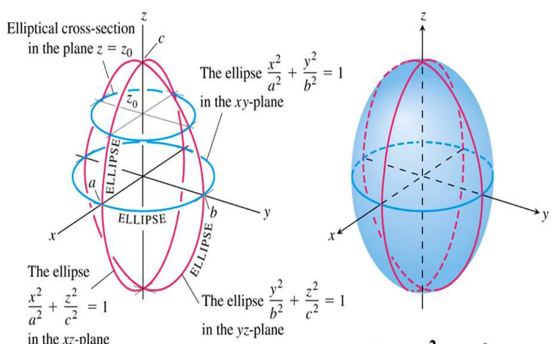
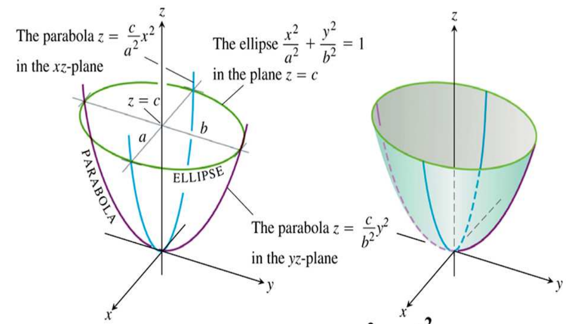


Figure 17

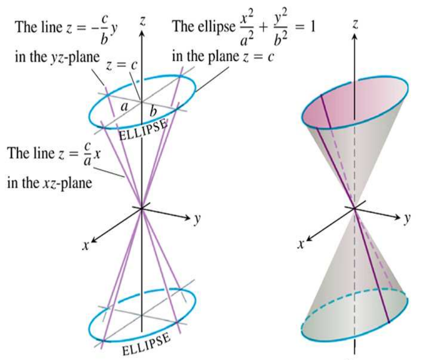
Note. Gives the graphs of several quadric surfaces.



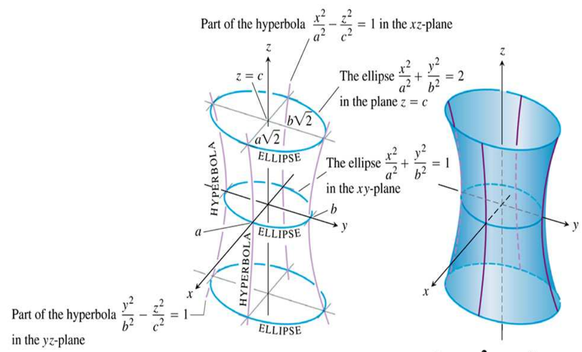
ELLIPSOID $\quad \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1$



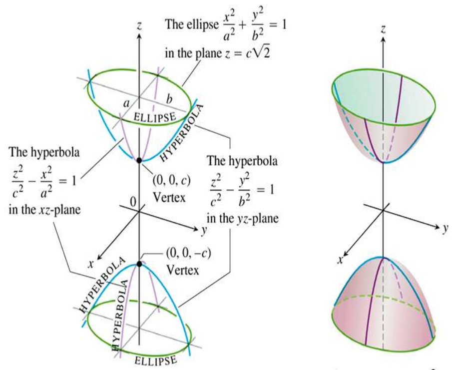
ELLIPTICAL PARABOLOID $\quad \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = \dfrac{z}{c}$



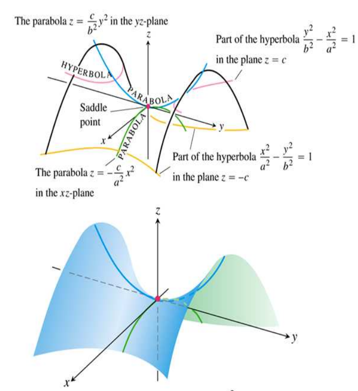
ELLIPTICAL CONE $\quad \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = \dfrac{z^2}{c^2}$



HYPERBOLOID OF ONE SHEET $\quad \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} - \dfrac{z^2}{c^2} = 1$



HYPERBOLOID OF TWO SHEETS $\quad \dfrac{z^2}{c^2} - \dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$



HYPERBOLIC PARABOLOID $\quad \dfrac{y^2}{b^2} - \dfrac{x^2}{a^2} = \dfrac{z}{c}, \quad c > 0$

Vector-Valued Functions and Motion in Space

Curves in Space and Their Tangents

Note. When a particle moves through space during a time interval I, we think of the particle's coordinates as functions defined on I:

$$x = f(t),\ y = g(t),\ z = h(t),\ t \in I.$$

The points $(x, y, z) = (f(t), g(t), h(t))$, $t \in I$, make up the *curve* in space that we call the particle's *path*. The above equations *parametrize* the curve. A curve in space can also be represented in vector form. The vector

$$\mathbf{r}(t) = \vec{OP} = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$$

from the origin to the particle's *position* $P(f(t), g(t), h(t))$ at time t.

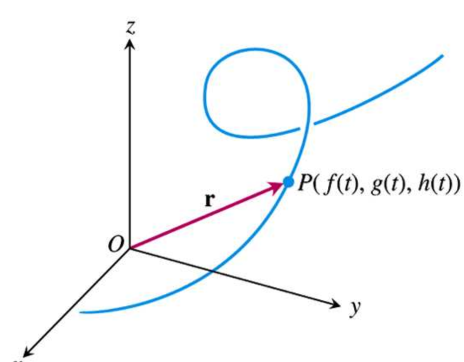


Figure 1

Example. Consider the vector function $\mathbf{r}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j} + t\mathbf{k}$. This curve is a helix.

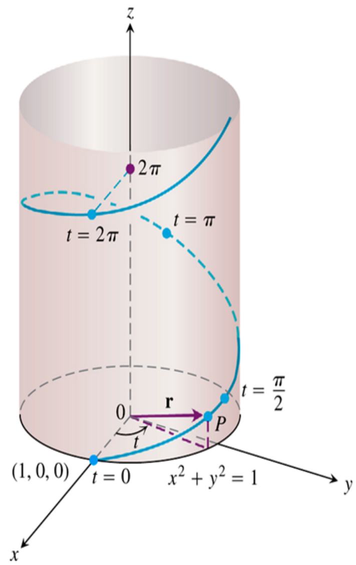


Figure 2

Definition. Let $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$ be a vector function defined on an open interval containing t_0 except possibly at t_0 itself, and let $\mathbf{L}$ a vector. We say that $\mathbf{r}$ has *limit* $\mathbf{L}$ as t approaches t_0 and write $\lim_{t \to t_0} \mathbf{r}(t) = \mathbf{L}$ if, for every number $\epsilon > 0$, there exists a corresponding number $\delta > 0$ such that

$$|\mathbf{r}(t) - \mathbf{L}| < \epsilon \quad \text{whenever} \quad 0 < |t - t_0| < \delta.$$

Definition. A vector function $\mathbf{r}(t)$ is *continuous at a point* $t = t_0$ in its domain if $\lim_{t \to t_0} \mathbf{r}(t) = \mathbf{r}(t_0)$. The function is *continuous* if it is continuous at every point in its domain.

Definition. The vector function $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$ has a *derivative* at t if f, g, and h have derivatives at t. The derivative is the vector function

$$\mathbf{r}'(t) = \frac{d\mathbf{r}}{dt} = \lim_{\Delta t \to 0} \frac{\mathbf{r}(t+\Delta t) - \mathbf{r}(t)}{\Delta t} = \frac{df}{dt}\mathbf{i} + \frac{dg}{dt}\mathbf{i} + \frac{dh}{dt}\mathbf{k}.$$

The curve traced by $\mathbf{r}$ is *smooth* if $d\mathbf{r}/dt$ is continuous and never $\mathbf{0}$, that is, if f, g, and h have continuous first derivatives that are not simultaneously 0.

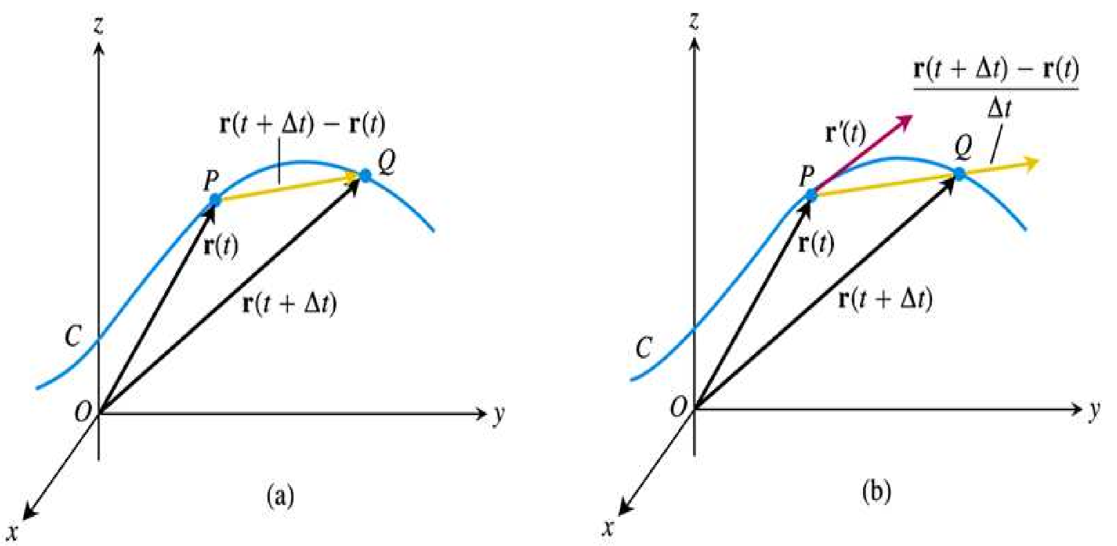


Figure 3

Definition. The vector $\mathbf{r}'(t)$, when different from $\mathbf{0}$, is defined to be the vector *tangent* to the curve at P. The *tangent line* to the curve at a point $(f(t_0), g(t_0), h(t_0))$ is defined to the the line through the point parallel to $\mathbf{r}'(t_0)$.

Definition. If $\mathbf{r}$ is the position vector of a particle moving along a smooth curve in space, then $\mathbf{v}(t) = \dfrac{d\mathbf{r}}{dt}$ is the particle's *velocity vector*, tangent to the curve. At any time, the direction of $\mathbf{v}$ is the *direction of motion*, the magnitude of $\mathbf{v}$ is the particle's *speed*, and the derivative $\mathbf{a} = d\mathbf{v}/dt$, when it exists, is the particles *acceleration vector*. In summary,

1. Velocity is the derivative of position: $\mathbf{v} = \dfrac{d\mathbf{r}}{dt}$.

2. Speed is the magnitude of velocity: Speed $= |\mathbf{v}|$.

3. Acceleration is the derivative of velocity: $\mathbf{a} = \dfrac{d\mathbf{v}}{dt} = \dfrac{d^2\mathbf{r}}{dt^2}$.

4.T he unit vector $\mathbf{v}/|\mathbf{v}|$ is the direction of motion at time t.

Theorem. Differentiation Rules for Vector Functions.

Let $\mathbf{u}$ and $\mathbf{v}$ be differentiable vector functions of t, $\mathbf{C}$ a constant vector, c any scalar, and f any differentiable scalar function.

1. *Constant Function Rule:* $\dfrac{d}{dt}[\mathbf{C}] = \mathbf{0}$.

2. *Scalar Multiple Rules:* $\dfrac{d}{dt}[c\mathbf{u}(t)] = c\mathbf{u}'(t)$.

$$\frac{d}{dt}[f(t)\mathbf{u}(t)] = [f'(t)](\mathbf{u}(t)) + (f(t))[\mathbf{u}'(t)].$$

3. *Sum Rule:* $\dfrac{d}{dt}[\mathbf{u}(t)+\mathbf{v}(t)] = \mathbf{u}'(t)+\mathbf{v}'(t)$.

4. *Difference Rule:* $\dfrac{d}{dt}[\mathbf{u}(t)-\mathbf{v}(t)] = \mathbf{u}'(t)-\mathbf{v}'(t)$.

5. *Dot Product Rule:* $\dfrac{d}{dt}[\mathbf{u}(t)\cdot\mathbf{v}(t)] = [\mathbf{u}'(t)]\cdot(\mathbf{v}(t))+(\mathbf{u}(t))\cdot[\mathbf{v}'(t)]$.

6. *Cross Product Rule:* $\dfrac{d}{dt}[\mathbf{u}(t)\times\mathbf{v}(t)] = [\mathbf{u}'(t)]\times(\mathbf{v}(t))+(\mathbf{u}(t))\times[\mathbf{v}'(t)]$.

7. *Chain Rule:* $\dfrac{d}{dt}[\mathbf{u}(f(t))] = f'(t)\mathbf{u}'(f(t))$.

Proof of the Dot Product Rule.

Suppose that $\mathbf{u} = u_1(t)\mathbf{i}+u_2(t)\mathbf{j}+u_3(t)\mathbf{k}$ and $\mathbf{v} = v_1(t)\mathbf{i}+v_2(t)\mathbf{j}+v_3(t)\mathbf{k}$.

Then

$$\begin{aligned}
\frac{d}{dt}[\mathbf{u}\cdot\mathbf{v}] &= \frac{d}{dt}[u_1v_1+u_2v_2+u_3v_3] \\
&= [u_1'](v_1)+(u_1)[v_1']+[u_2'](v_2)+(u_2)[v_2']+[u_3'](v_3)+(u_3)[v_3'] \\
&= [u_1'](v_1)+[u_2'](v_2)+[u_3'](v_3)+(u_1)[v_1']+(u_2)[v_2']+(u_3)[v_3'] \\
&= \mathbf{u}'\cdot\mathbf{v}+\mathbf{u}\cdot\mathbf{v}'.
\end{aligned}$$

Proof of the Cross Product Rule.

This proof resembles the Product Rule from Calculus 1. By definition,

$$\frac{d}{dt}[\mathbf{u}\times\mathbf{v}] = \lim_{h\to 0}\frac{\mathbf{u}(t+h)\times\mathbf{v}(t+h)-\mathbf{u}(t)\times\mathbf{v}(t)}{h}.$$

This leads to

$$\frac{d}{dt}[\mathbf{u}\times\mathbf{v}] = \lim_{h\to 0}\frac{\mathbf{u}(t+h)\times\mathbf{v}(t+h)-\mathbf{u}(t)\times\mathbf{v}(t)}{h}$$

$$
\begin{aligned}
&= \lim_{h\to 0} \frac{\mathbf{u}(t+h)\times\mathbf{v}(t+h)-\mathbf{u}(t)\times\mathbf{v}(t+h)+\mathbf{u}(t)\times\mathbf{v}(t+h)-\mathbf{u}(t)\times\mathbf{v}(t)}{h}\\
&= \lim_{h\to 0}\left[\frac{\mathbf{u}(t+h)-\mathbf{u}(t)}{h}\times\mathbf{v}(t+h)+\mathbf{u}(t)\times\frac{\mathbf{v}(t+h)-\mathbf{v}(t)}{h}\right]\\
&= \lim_{h\to 0}\frac{\mathbf{u}(t+h)-\mathbf{u}(t)}{h}\times\lim_{h\to 0}\mathbf{v}(t+h)+\lim_{h\to 0}\mathbf{u}(t)\times\lim_{h\to 0}\frac{\mathbf{v}(t+h)-\mathbf{v}(t)}{h}\\
&= [\mathbf{u}'(t)]\times(\mathbf{v}(t))+(\mathbf{u}(t))\times[\mathbf{v}'(t)].
\end{aligned}
$$

We have used the fact that the limit of a product is the product of the limits and that $\mathbf{v}$ is continuous and hence $\lim \mathbf{v}(t+h) \underset{h\to 0}{=} \mathbf{v}(t)$.

Vector-Valued Functions and Motion in Space

Integrals of Vector Functions; Projectile Motion

Definition. A differentiable vector function $\mathbf{R}(t)$ is an *antiderivative* of a vector function $\mathbf{r}(t)$ on in interval I if $d\mathbf{R}/dt = \mathbf{r}$ at each point of I. The *indefinite integral* of $\mathbf{r}$ with respect to t is the **set** of all antiderivatives of $\mathbf{r}$, denoted by $\int \mathbf{r}(t)\,dt$. If $\mathbf{R}$ is any antiderivative of $\mathbf{r}$, then

$$\int \mathbf{r}(t)\,dt = \{\mathbf{R} \mid \mathbf{R}'(t) = \mathbf{r}(t)\} = \mathbf{R}(t) + \mathbf{C}.$$

Note. Whereas **antiderivatives are functions**, **indefinite integrals are sets**—indefinite integrals are sets of antiderivatives. We will use set notation sometimes, but often will abbreviate the set notation with the "+$\mathbf{C}$" which is similar to how indefinite integrals were dealt with in Calculus 1. Also similar to Calculus 1, we see in the following definition that **definite integrals are numbers**.

Definition. If the components of $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$ are integrable over $[a, b]$, then so is $\mathbf{r}$, and the *definite integral* of $\mathbf{r}$ from a to b is

$$\int_a^b \mathbf{r}(t)\,dt = \left(\int_a^b f(t)\,dt\right)\mathbf{i} + \left(\int_a^b g(t)\,dt\right)\mathbf{j} + \left(\int_a^b h(t)\,dt\right)\mathbf{k}.$$

Note. Suppose an object (a "projectile") is given an initial velocity $\mathbf{v}_0$ and is then only acted on by the force of gravity (so we ignore frictional drag, for example). We assume that $\mathbf{v}_0$ makes an angle α with the horizontal.

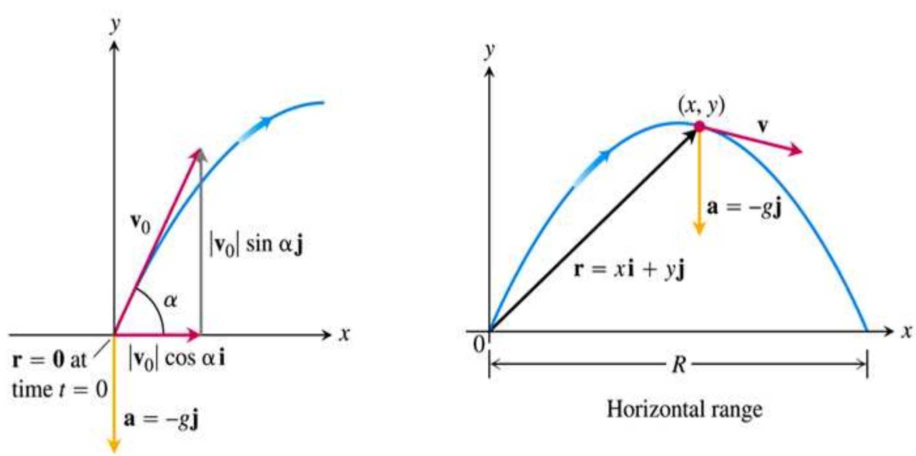


Figure 4

Then

$$\mathbf{v}_0 = (|\mathbf{v}_0| \cos \alpha)\mathbf{i} + (|\mathbf{v}_0| \sin \alpha)\mathbf{j} = (v_0 \cos \alpha)\mathbf{i} + (v_0 \sin \alpha)\mathbf{j}.$$

Vector-Valued Functions and Motion in Space

Arc Length in Space

Definition. The *length* of a smooth curve $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$, for $t \in [a, b]$, that is traced exactly once as t increases from $t = a$ to $t = b$, is

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt = \int_a^b |\mathbf{v}|\, dt,$$

where $|\mathbf{v}| = |d\mathbf{r}/dt|$.

Note. If we choose a base point $P(t_0)$ on a smooth curve C parametrized by t, each value of t determines a point $P(t) = (x(t), y(t), z(t))$ on C and a "directed distance

$$s(t) = \int_{t_0}^t |\mathbf{v}(\tau)|\, d\tau = \int_{t_0}^t \sqrt{(x'(\tau))^2 + (y'(\tau))^2 + (z'(\tau))^2}\, d\tau,$$

measured along C from the base point. We call s an *arc length parameter* for the curve. In the next section, we will parametrize curves in terms of arclength in order to describe the shape of a curve in the sense of "curvature." The idea of curvature is to reflect how much a curve changes direction. This will be reflected in the acceleration vector, but only the component of the acceleration vector which reflects a change in direction.

This is why we will want to parametrize in terms of arc length, so as to traverse the curve at a uniform speed and hence the only acceleration will be in terms of change of direction.

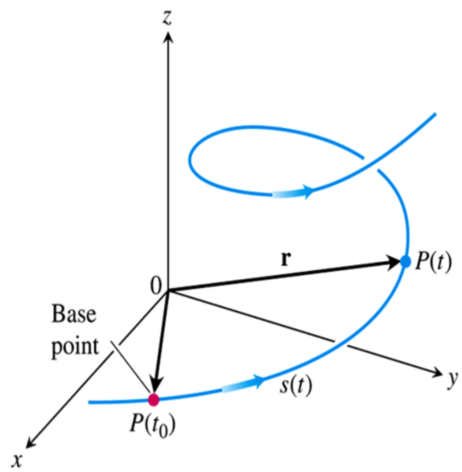


Figure 5

Definition. The *unit tangent vector* to a curve $\mathbf{r}$ is $\mathbf{T} = \mathbf{v}/|\mathbf{v}|$.

Vector-Valued Functions and Motion in Space

Curvature and Normal Vectors of a Curve

Note. The rate at which $\mathbf{T}$ turns per unit of length along the curve is called the *curvature*. The symbol for curvature is kappa, κ. When a particle is traveling though space (or in a plane), then it can undergo an acceleration in two distinct ways: (1) an acceleration in the direction of travel, and (2) an acceleration which changes direction. This first type of acceleration does not reflect the shape of the curve, but how the curve is parametrized. By parametrizing the curve in terms of arc length, this type of acceleration becomes zero. Hence all of the acceleration is reflected in the second type of acceleration—the acceleration which results in a change of direction of the curve.

Definition. If $\mathbf{T}$ is the unit vector of a smooth curve, the *curvature* function of the curve is

$$\kappa = \left|\frac{d\mathbf{T}}{ds}\right|.$$

Note. If $|d\mathbf{T}/ds|$ is large, $\mathbf{T}$ turns sharply as the particle passes through P, and the curvature at P is large. If $|d\mathbf{T}/ds|$ is close to zero, $\mathbf{T}$ turns more slowly and the curvature at P is smaller. As we will see, the curvature of a circle of radius r is $\kappa = 1/r$.

Note. If a smooth curve $\mathbf{r}(t)$ is already given in terms of some parameter t other than the arc length s, we can calculate curvature using the Chain Rule:

$$\kappa = \left|\frac{d\mathbf{T}}{ds}\right| = \left|\frac{d\mathbf{T}}{dt}\frac{dt}{ds}\right| = \frac{1}{|ds/dt|}\left|\frac{d\mathbf{T}}{dt}\right| = \frac{1}{|\mathbf{v}|}\left|\frac{d\mathbf{T}}{dt}\right|.$$

Example. Consider the circle of radius a: $\mathbf{r}(t) = (a\cos t)\mathbf{i} + (a\sin t)\mathbf{j}$. Show that the curvature is $\kappa = 1/a$.

Definition. At a point where $\kappa \neq 0$, the *principal unit normal vector* for a smooth curve in the plane is

$$\mathbf{N} = \frac{1}{\kappa}\frac{d\mathbf{T}}{ds}.$$

Note. The vector $d\mathbf{T}/ds$ points in the direction in which $\mathbf{T}$ turns as the curve bends.

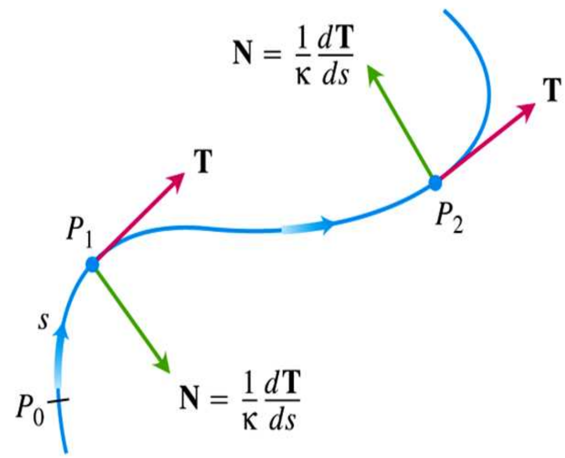


Figure 6

Note. If $\mathbf{r}(t)$ is not parametrized in terms of s, we can use the Chain Rule to calculate $\mathbf{N}$ in terms of t:

$$\mathbf{N} = \frac{d\mathbf{T}/ds}{\kappa} = \frac{d\mathbf{T}/ds}{|d\mathbf{T}/ds|} = \frac{(d\mathbf{T}/dt)(dt/ds)}{|(d\mathbf{T}/dt)(dt/ds)|}$$

$$= \frac{(d\mathbf{T}/dt)(dt/ds)}{|(d\mathbf{T}/dt)||(dt/ds)|} = \frac{(d\mathbf{T}/dt)}{|(d\mathbf{T}/dt)} \text{ (since } dt/ds > 0).$$

Definition. The *circle of curvature* or *osculating circle* at a point P on a plane curve where $\kappa \neq 0$ is the circle in the plane of the curve that

1. is tangent to the curve at P (has the same tangent line the curve has),

2. has the same curvature the curve has at P, and

3. lies toward the concave or inner side of the curve.

The *radius of curvature* of the curve at P is the radius of the circle of curvature, denoted ρ. The *center of curvature* of the curve at P is the center of the circle of curvature.

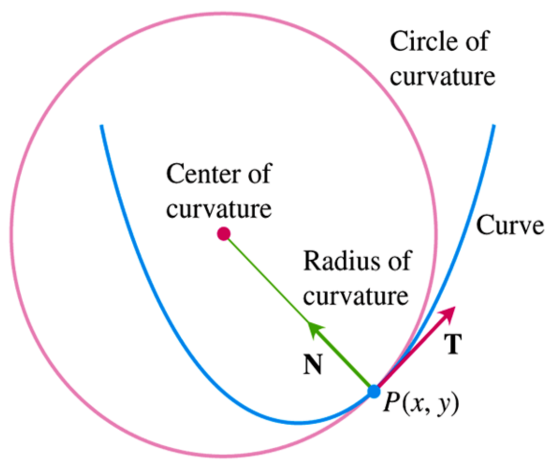


Figure 7

Example. Find and graph the osculating circle of the parabola $y = x^2$ at the origin. The solution is:

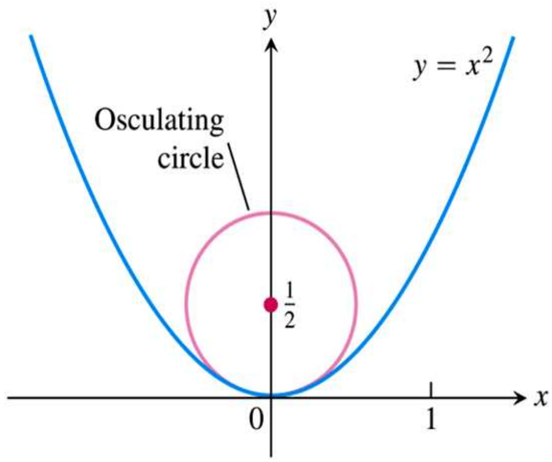


Figure 8

Note. We use the same formulae above, even for curves in three dimensions, instead of two.

Vector-Valued Functions and Motion in Space

Tangential and Normal Components of Acceleration

Note. If we let $\mathbf{r}(t)$ be a position function and interpret this as the movement of a particle as a function of time, then the unit tangent vector $\mathbf{T}$ represents the *direction* of travel of the particle and the principal unit vector $\mathbf{N}$ indicates the direction the path is turning into. Since both of these vectors are unit vectors, it is their *direction* that contains information. For any fixed time t, acceleration is a linear combination of $\mathbf{T}$ and $\mathbf{N}$: $\mathbf{a} = a_T\mathbf{T} + a_N\mathbf{N}$ for some a_T and a_N.

Definition. Define the *unit binormal vector* as $\mathbf{B} = \mathbf{T} \times \mathbf{N}$.

Note. Notice that since $\mathbf{T}$ and $\mathbf{N}$ are orthogonal unit vectors, then $\mathbf{B}$ is in fact a unit vector. Changes in vector $\mathbf{B}$ reflect the tendency of the motion of the particle with position function $\mathbf{r}(t)$ to 'twist' out of the plane created by vectors $\mathbf{T}$ and $\mathbf{N}$. Also notice that vectors $\mathbf{T}$, $\mathbf{N}$, and $\mathbf{B}$ define a moving right-hand vector "frame." This frame is called the

Frenet frame or the **TNB** *frame.*

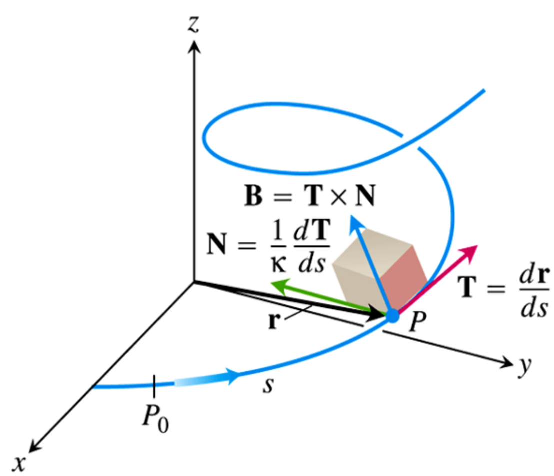


Figure 9

Note. As commented above, we can write $\mathbf{a} = a_T\mathbf{T} + a_N\mathbf{N}$ for some a_T and a_N. We want to find formulae for a_T and a_N. By the Chain Rule,

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = \frac{d\mathbf{r}}{ds}\frac{ds}{dt} = \mathbf{T}\frac{ds}{dt}.$$

So acceleration is

$$\begin{aligned}
\mathbf{a} &= \frac{d\mathbf{v}}{dt} = \frac{d}{dt}\left[\mathbf{T}\frac{ds}{dt}\right] = \frac{d^2s}{dt^2}\mathbf{T} + \frac{ds}{dt}\frac{d\mathbf{T}}{dt} \\
&= \frac{d^2s}{dt^2}\mathbf{T} + \frac{ds}{dt}\left(\frac{d\mathbf{T}}{ds}\frac{ds}{dt}\right) = \frac{d^2s}{dt^2}\mathbf{T} + \frac{ds}{dt}\left(\kappa\mathbf{N}\frac{ds}{dt}\right) \\
&= \frac{d^2s}{dt^2}\mathbf{T} + \kappa\left(\frac{ds}{dt}\right)^2\mathbf{N}.
\end{aligned}$$

(Recall that $\dfrac{d\mathbf{T}}{ds} = \kappa\mathbf{N}$.)

Definition. If the acceleration vector is written as $\mathbf{a} = a_T\mathbf{T} + a_N\mathbf{N}$, then

$$a_T = \frac{d^2s}{dt^2} = \frac{d}{dt}[|\mathbf{v}|] \text{ and } a_N = \kappa\left(\frac{ds}{dt}\right)^2 = \kappa|\mathbf{v}|^2$$

are the *tangential* and *normal* scalar components of acceleration. (Recall that s is arclength and so ds/dt is the rate at which arclength is traversed with respect to time. That is, ds/dt is speed: $ds/dt = |\mathbf{v}|$.)

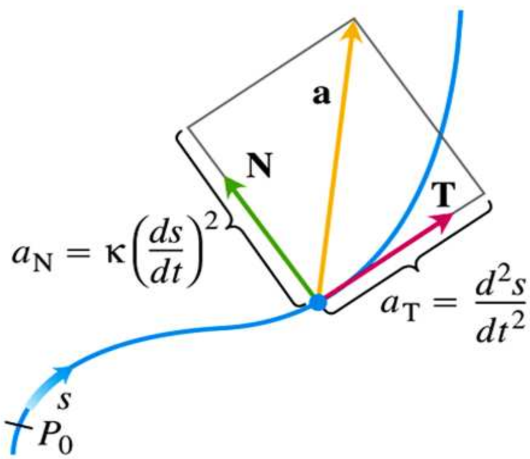


Figure 10

Note. If we are given the position function $\mathbf{r}(t)$, then a_T is easy to find (just calculate $\frac{d}{dt}\left[\left|\frac{d\mathbf{r}}{dt}\right|\right]$). But the computation of a_N seems to require us to find curvature κ. But there is a quicker way. Since $\mathbf{a} = a_T\mathbf{T} + a_N\mathbf{N}$ and $\mathbf{T}$ and $\mathbf{N}$ are orthogonal, then $|\mathbf{a}|^2 = a_T^2 + a_N^2$. Therefore we can solve to a_N and find that: $a_N = \sqrt{|\mathbf{a}|^2 - a_T}$.

Note. We have commented that changes in the binormal vector $\mathbf{B}$ reflect the tendency of the motion of the particle with position function $\mathbf{r}(t)$ to 'twist' out of the plane created by vectors $\mathbf{T}$ and $\mathbf{N}$. This twisting is called *torsion*. We are interested in how $\mathbf{B}$ changes with respect to arclength s:

$$\frac{d\mathbf{B}}{ds} = \frac{d[\mathbf{T} \times \mathbf{N}]}{ds} = \frac{\mathbf{T}}{ds} \times \mathbf{N} + \mathbf{T} \times \frac{d\mathbf{N}}{ds} = \mathbf{0} + \mathbf{T} \times \frac{d\mathbf{N}}{ds} = \mathbf{T} \times \frac{d\mathbf{N}}{ds}$$

since $d\mathbf{T}/ds$ is parallel to $\mathbf{N}$.

We need a quick result concerning vector functions of constant magnitude: **Lemma.** If $\mathbf{r}(t)$ is a vector function such that $|\mathbf{r}(t)| = c$ for some constant c, then $\mathbf{r}(t)$ and $\mathbf{r}'(t)$ are orthogonal. The proof is computational:

$$\begin{aligned}
\mathbf{r}(t) \cdot \mathbf{r}(t) &= |\mathbf{r}(t)|^2 = c^2 \\
\frac{d}{dt}[\mathbf{r}(t) \cdot \mathbf{r}(t)] &= \frac{d}{dt}[c^2] \\
\mathbf{r}'(t) \cdot \mathbf{r}(t) + \mathbf{r}(t) \cdot \mathbf{r}'(t) &= 0 \\
2\mathbf{r}'(t) \cdot \mathbf{r}(t) &= 0.
\end{aligned}$$

Since $\mathbf{r}'(t) \cdot \mathbf{r}(t) = 0$, the vectors are orthogonal.

Returning to $\mathbf{B}$, we know from above that $d\mathbf{B}/ds$ is orthogonal to $\mathbf{T}$ since it is the cross product of vector $\mathbf{T}$ and another vector. Since $\mathbf{B}$

is always a unit vector, then by Lemma $d\mathbf{B}/ds$ is also orthogonal to $\mathbf{B}$. Therefore $d\mathbf{B}/ds$ must be a multiple of vector $\mathbf{N}$. We define the *torsion* τ with the formula $\dfrac{d\mathbf{B}}{ds} = -\tau\mathbf{N}$. We can compute τ as follows:

$$\frac{d\mathbf{B}}{ds} \cdot \mathbf{N} = -\tau\mathbf{N} \cdot \mathbf{N} = -\tau(1) = -\tau$$

and so $\tau = -\dfrac{d\mathbf{B}}{ds} \cdot \mathbf{N}$. As the book states, the curvature $\kappa = |d\mathbf{T}/ds|$ can be thought of as the rate at which the normal plane turns as the point P moves along its path. The torsion $\tau = -(d\mathbf{B}/ds) \cdot \mathbf{N}$ is the rate at which the osculating plane turns about $\mathbf{T}$ as P moves along the curve. "Torsion measures how the curve twists. ...In a more advanced course it can be shown that a space curve is a helix if and only if it has constant nonzero curvature and constant nonzero torsion."

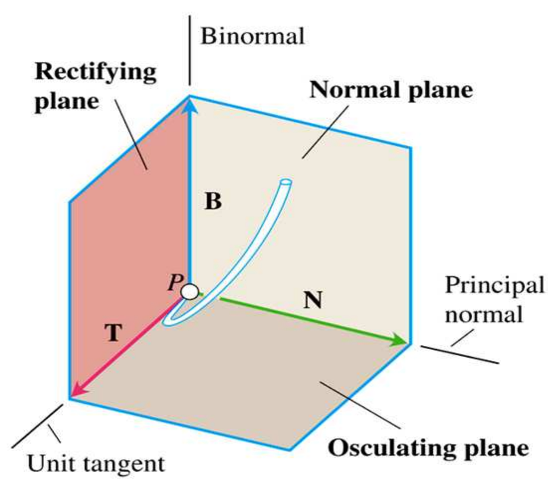


Figure 11

Note. Consider a position function $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$. It can be shown ("in more advanced texts") that torsion can be computed as

$$\tau = \frac{\begin{vmatrix} \dot{x} & \dot{y} & \dot{z} \\ \ddot{x} & \ddot{y} & \ddot{z} \\ \dddot{x} & \dddot{y} & \dddot{z} \end{vmatrix}}{|\mathbf{v} \times \mathbf{a}|^2}$$

where $\mathbf{v} \times \mathbf{a} \neq \mathbf{0}$ and the dots indicate (as is tradition in physics) deriva-

tives with respect to time t: $\dot{x} = dx/dt$. So the first row of the matrix consists of the components of velocity $\mathbf{r}'(t) = \mathbf{v}$, the second row consists of components of acceleration $\mathbf{r}''(t) = \mathbf{a}$ and the third row consists of components of jerk $\mathbf{r}'''(t)$.

Note. In summary, we have the following formulae:

Position: $\mathbf{r}(t) = \mathbf{r}$

Unit tangent vector: $\mathbf{T} = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \frac{\mathbf{v}}{|\mathbf{v}|}$

Principal unit normal vector: $\mathbf{N} = \frac{d\mathbf{T}/dt}{|d\mathbf{T}/dt|}$

Binormal vector: $\mathbf{B} = \mathbf{T} \times \mathbf{N}$

Curvature: $\kappa = \left|\frac{d\mathbf{T}}{ds}\right| = \frac{|\mathbf{v}\times\mathbf{a}|}{|\mathbf{v}|^3}$

$$\text{Torsion: } \tau = -\frac{d\mathbf{B}}{ds} \cdot \mathbf{N} = \frac{\begin{vmatrix} \dot{x} & \dot{y} & \dot{z} \\ \ddot{x} & \ddot{y} & \ddot{z} \\ \dddot{x} & \dddot{y} & \dddot{z} \end{vmatrix}}{|\mathbf{v} \times \mathbf{a}|^2} = -\frac{1}{|\mathbf{v}|}\left(\frac{d\mathbf{B}}{dt} \cdot \mathbf{N}\right)$$

Tangential and normal scalar components of acceleration:

$\mathbf{A} = a_T \mathbf{T} + a_N \mathbf{N}$

where $a_T = \frac{d}{dt}[|\mathbf{v}|]$ and $a_N = \kappa|\mathbf{v}|^2 = \sqrt{|\mathbf{a}^2 - a_T}$.