Control Systems

Subject Code: EC380

Lecture 13-15: Frequency Response

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Frequency Response

- For time domain analysis test signals: step, ramp, etc.
- For frequency domain analysis test signal: Sinusoidal.
- We will examine the transfer function $G(s)$ when $s = j\omega$ and develop methods for graphically displaying the complex number $G(j\omega)$ as ω varies.
- Graphical methods for *frequency response*:

  1) Polar Plot 2) Bode Plots, 3) Nichols chart

- **Methods for Stability analysis:**

  1) Nyquist Plot, 2) Bode Plots, 3) Nichols chart

- **Polar plot will be used for Nyquist plot.**

Frequency Response of a Stable LTI System

$$Asin(\omega t) \longrightarrow \boxed{T(s)} \longrightarrow y(t)$$

For Stable LTI system: $\lim_{t \to \infty} y(t) = A||T(j\omega)||sin(\omega t + \theta)$

Frequency Response of a Stable LTI System

$$A\sin(\omega t) \longrightarrow \boxed{T(s)} \longrightarrow y(t)$$

For example, consider the system $Y(s) = T(s)R(s)$ with $r(t) = A\sin\omega t$. We have

$$R(s) = \frac{A\omega}{s^2 + \omega^2}$$

and

$$T(s) = \frac{m(s)}{q(s)} = \frac{m(s)}{\prod_{i=1}^{n}(s + p_i)},$$

where $-p_i$ are assumed to be distinct poles. Then, in partial fraction form, we have

$$Y(s) = \frac{k_1}{s + p_1} + \cdots + \frac{k_n}{s + p_n} + \frac{\alpha s + \beta}{s^2 + \omega^2}.$$

Taking the inverse Laplace transform yields

$$y(t) = k_1 e^{-p_1 t} + \cdots + k_n e^{-p_n t} + \mathcal{L}^{-1}\left\{\frac{\alpha s + \beta}{s^2 + \omega^2}\right\},$$

where α and β are constants which are problem dependent. If the system is stable, then all p_i have positive real parts and

$$\lim_{t\to\infty} y(t) = \lim_{t\to\infty} \mathcal{L}^{-1}\left\{\frac{\alpha s + \beta}{s^2 + \omega^2}\right\},$$

since each exponential term $k_i e^{-p_i t}$ decays to zero as $t \to \infty$.

In the limit for $y(t)$, it can be shown, for $t \to \infty$ (the steady state),

$$\begin{aligned} y(t) &= \mathcal{L}^{-1}\left[\frac{\alpha s + \beta}{s^2 + \omega^2}\right] \\ &= \frac{1}{\omega}\left|A\omega T(j\omega)\right|\sin(\omega t + \phi) \\ &= A|T(j\omega)|\sin(\omega t + \phi), \end{aligned} \tag{8.1}$$

where $\phi = \angle T(j\omega)$.

Introduction

- **The frequency response of a system is defined as the steady-state response of the system to a sinusoidal input signal.**

- The sinusoid is a unique input signal, and the resulting output signal for a linear system, as well as signals throughout the system, is sinusoidal in the steady-state; it differs form the input waveform only in **amplitude and phase.**

Frequency Response Plots: Polar Plots

The transfer function of a system $G(s)$ can be described in the frequency domain by the relation

$$G(j\omega) = G(s)|_{s=j\omega} = R(\omega) + jX(\omega), \tag{8.8}$$

where

$$R(\omega) = \mathrm{Re}[G(j\omega)] \quad \text{and} \quad X(\omega) = \mathrm{Im}[G(j\omega)].$$

Alternatively, the transfer function can be represented by a magnitude $|G(j\omega)|$ and a phase $\phi(j\omega)$ as

$$G(j\omega) = |G(j\omega)|e^{j\phi(\omega)} = |G(j\omega)|\underline{/\phi(\omega)}, \tag{8.9}$$

where

$$\phi(\omega) = \tan^{-1}\frac{X(\omega)}{R(\omega)} \quad \text{and} \quad |G(j\omega)|^2 = [R(\omega)]^2 + [X(\omega)]^2.$$

The graphical representation of the frequency response of the system $G(j\omega)$ can utilize either Equation (8.8) or Equation (8.9). The **polar plot** representation of the frequency response is obtained by using Equation (8.8). The coordinates of the polar plot are the real and imaginary parts of $G(j\omega)$.

Polar Plot for RC Filter

A simple RC filter is shown in Figure 8.2. The transfer function of this filter is

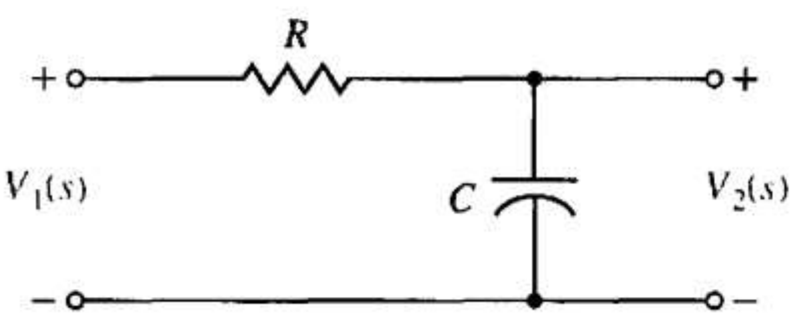
$$G(s) = \frac{V_2(s)}{V_1(s)} = \frac{1}{RCs + 1}, \tag{8.10}$$

and the sinusoidal steady-state transfer function is

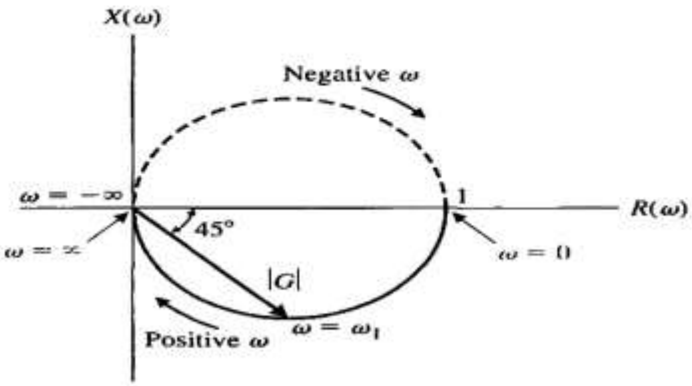
$$G(j\omega) = \frac{1}{j\omega(RC) + 1} = \frac{1}{j(\omega/\omega_1) + 1}, \tag{8.11}$$

where

$$\omega_1 = \frac{1}{RC}.$$



Polar Plot for RC Filter



Then the polar plot is obtained from the relation

$$
\begin{aligned}
G(j\omega) &= R(\omega) + jX(\omega) \\
&= \frac{1 - j(\omega/\omega_1)}{(\omega/\omega_1)^2 + 1} \\
&= \frac{1}{1 + (\omega/\omega_1)^2} - \frac{j(\omega/\omega_1)}{1 + (\omega/\omega_1)^2}.
\end{aligned}
\tag{8.12}
$$

The first step is to determine $R(\omega)$ and $X(\omega)$ at the two frequencies, $\omega = 0$ and $\omega = \infty$. At $\omega = 0$, we have $R(\omega) = 1$ and $X(\omega) = 0$. At $\omega = \infty$, we have $R(\omega) = 0$ and $X(\omega) = 0$. These two points are shown in Figure 8.3. The locus of the real and imaginary parts is also shown in Figure 8.3 and is easily shown to be a circle with the center at $\left(\frac{1}{2}, 0\right)$. When $\omega = \omega_1$, the real and imaginary parts are equal in magnitude, and the angle $\phi(\omega) = -45°$. The polar plot can also be readily obtained from Equation (8.9) as

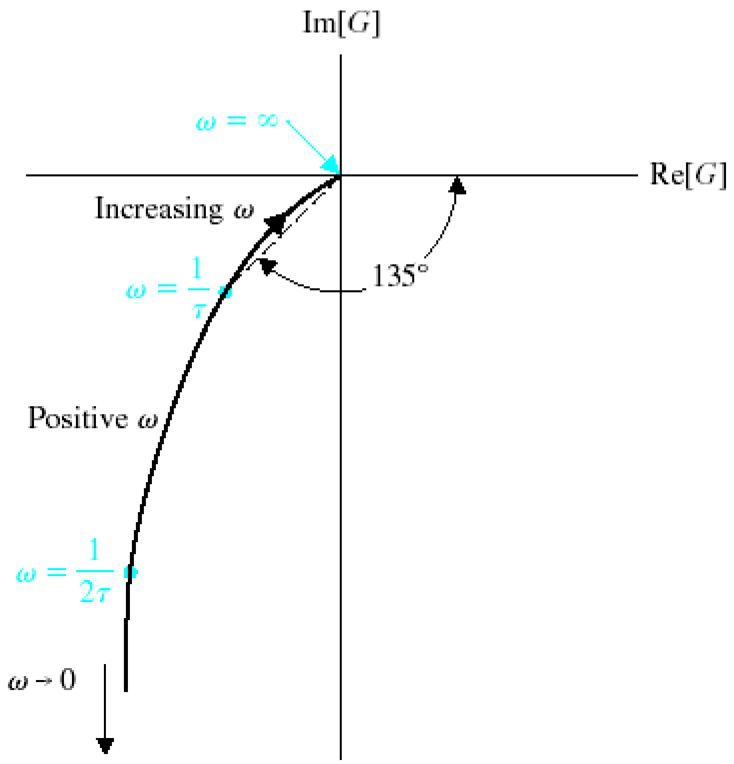
$$G(j\omega) = |G(j\omega)| \angle \phi(\omega), \tag{8.13}$$

where

$$|G(j\omega)| = \frac{1}{[1 + (\omega/\omega_1)^2]^{1/2}} \quad \text{and} \quad \phi(\omega) = -\tan^{-1}(\omega/\omega_1).$$

Hence, when $\omega = \omega_1$, the magnitude is $|G(j\omega_1)| = 1/\sqrt{2}$ and the phase $\phi(\omega_1) = -45°$. Also, when ω approaches $+\infty$, we have $|G(j\omega)| \to 0$ and $\phi(\omega) = -90°$. Similarly, when $\omega = 0$, we have $|G(j\omega)| = 1$ and $\phi(\omega) = 0$. ■

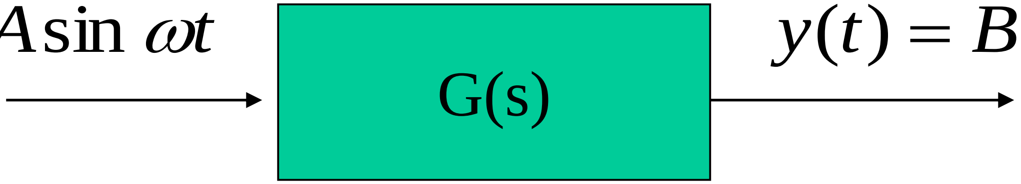
Frequency Response Plots: Polar Plots



Polar plot for $G(j\omega) = K/j\omega(j\omega\tau + 1)$. Note that $\omega = \infty$ at the origin.

Bode Plots

$$r(t) = A\sin \omega t \qquad y(t) = B\sin(\omega t + \phi)$$



Transfer function in frequency domain can be represented as:

$$G(j\omega) = |G(j\omega)|e^{-j\phi} \;\; where \;\; \frac{B}{A} = |G(j\omega)| \text{ and } \phi = \angle G(j\omega)$$

Logarithmic gain, $G_{dB} = 20log_{10}|G(j\omega)|$ dB

$$G(s) = G_1(s)G_2(s)$$

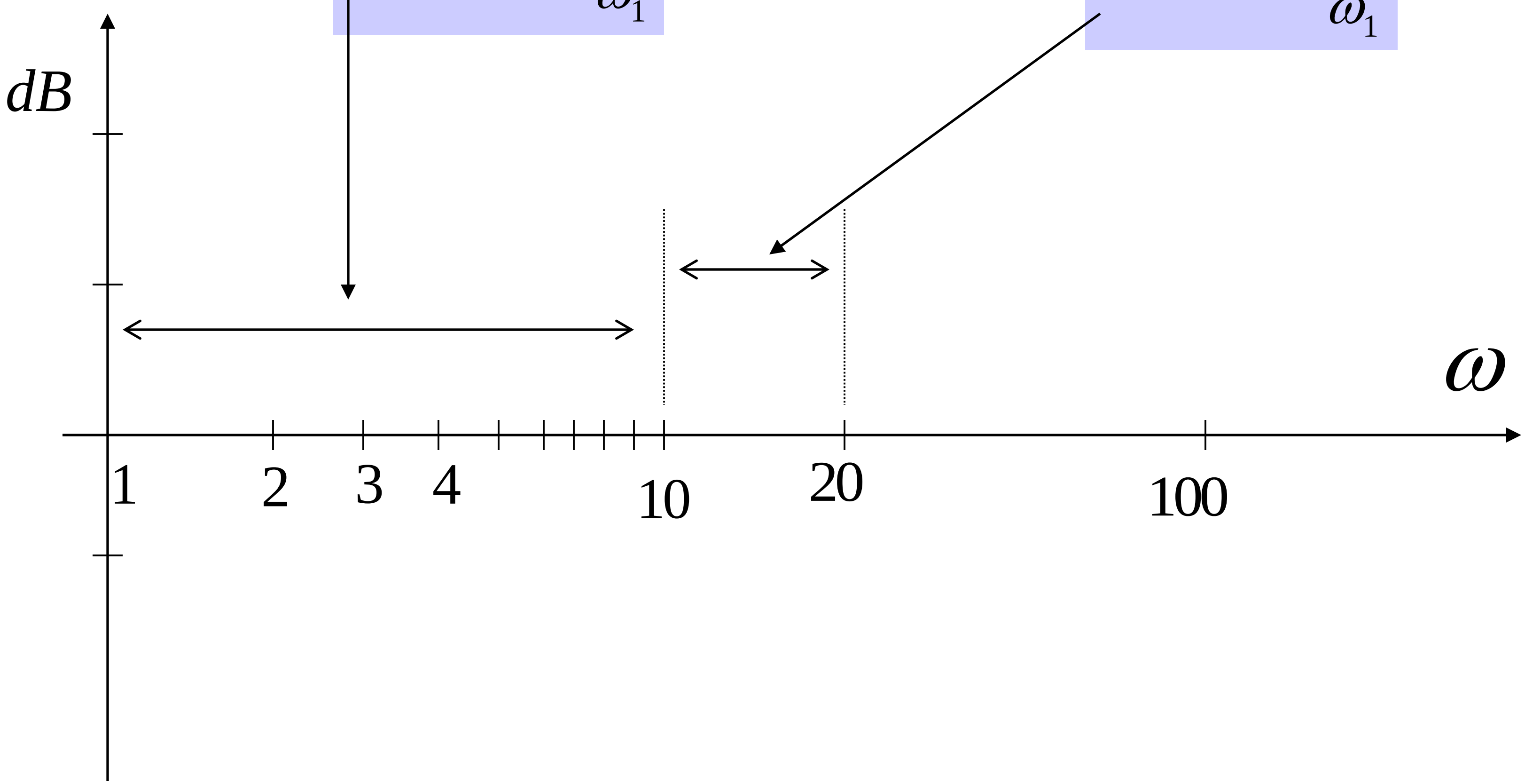
$$G_{dB} = G_{1dB} + G_{2dB}$$

$$\phi = \phi_1 + \phi_2$$

Logarithmic coordinate

Decade : $dec = \log_{10} \frac{\omega_2}{\omega_1}$

Octave : $oct = \log_2 \frac{\omega_2}{\omega_1}$



$$\frac{Y(s)}{R(s)} = \frac{k(s+z_1)(s+z_2)\cdots}{(s+p_1)(s+p_2)(s^2+as+b)\cdots}$$

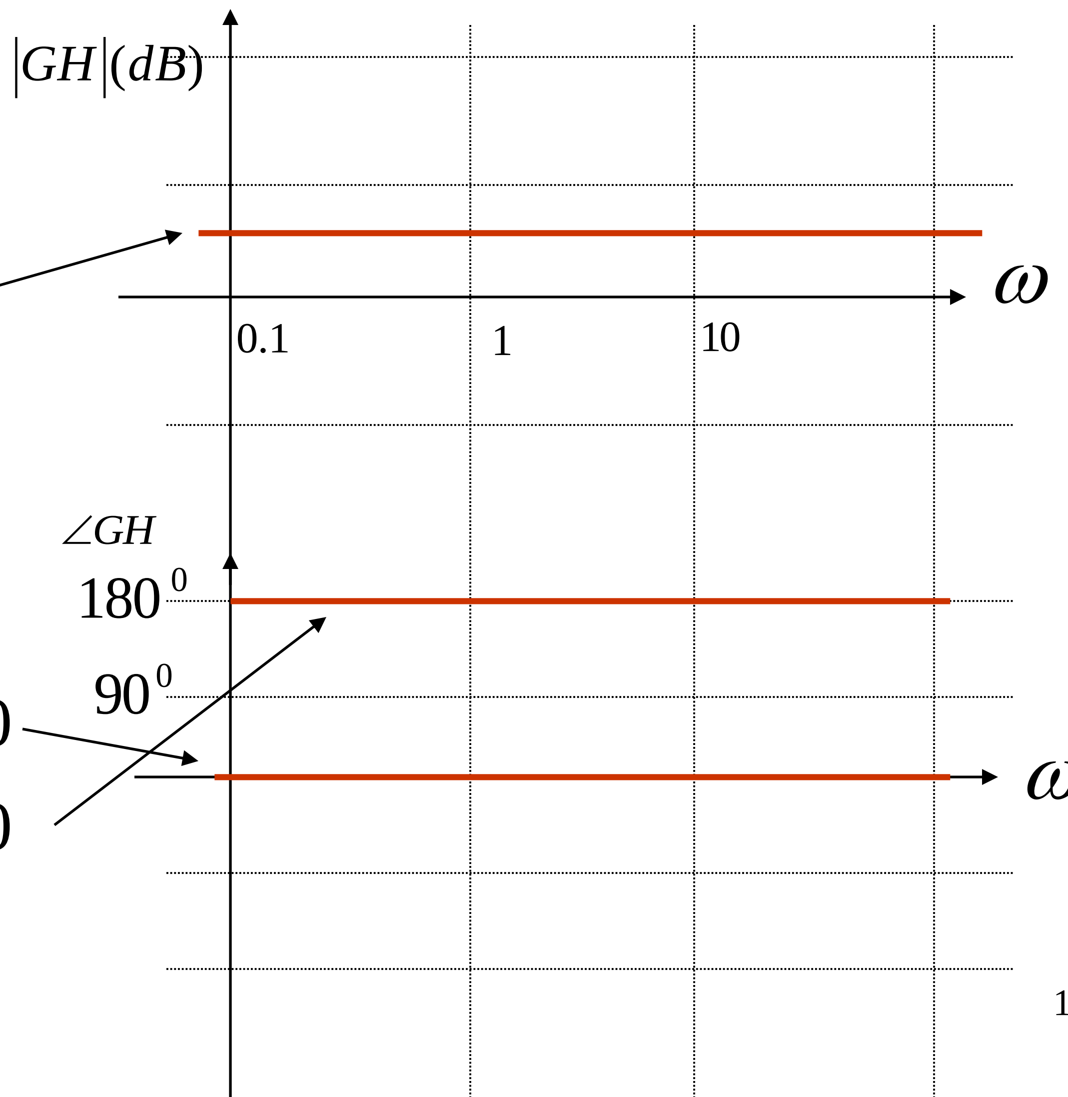
Case $I : k$

Magnitude:

$$|k|_{dB} = 20\log|k|(dB)$$

Phase:

$$\angle k = \begin{cases} 0^o & ,k \succ 0 \\ 180^o & ,k \prec 0 \end{cases}$$



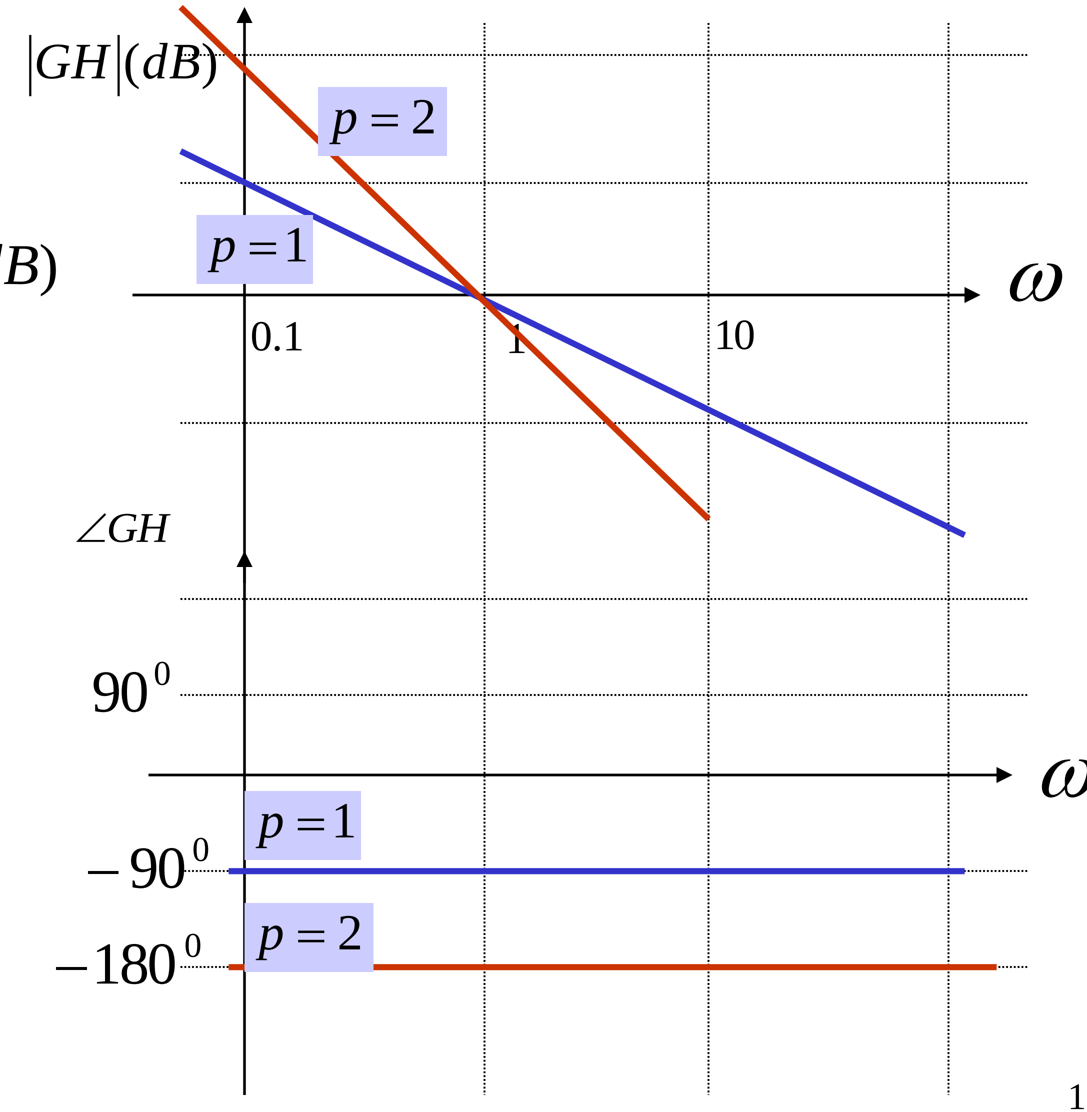
Case II : $\dfrac{1}{s^{p}}$

Magnitude:

$$\left|\frac{1}{(j\omega)^{p}}\right|_{dB} = -20\,p\log\omega(dB)$$

Phase:

$$\angle\frac{1}{(j\omega)^{p}} = (-90^{o})\times p$$



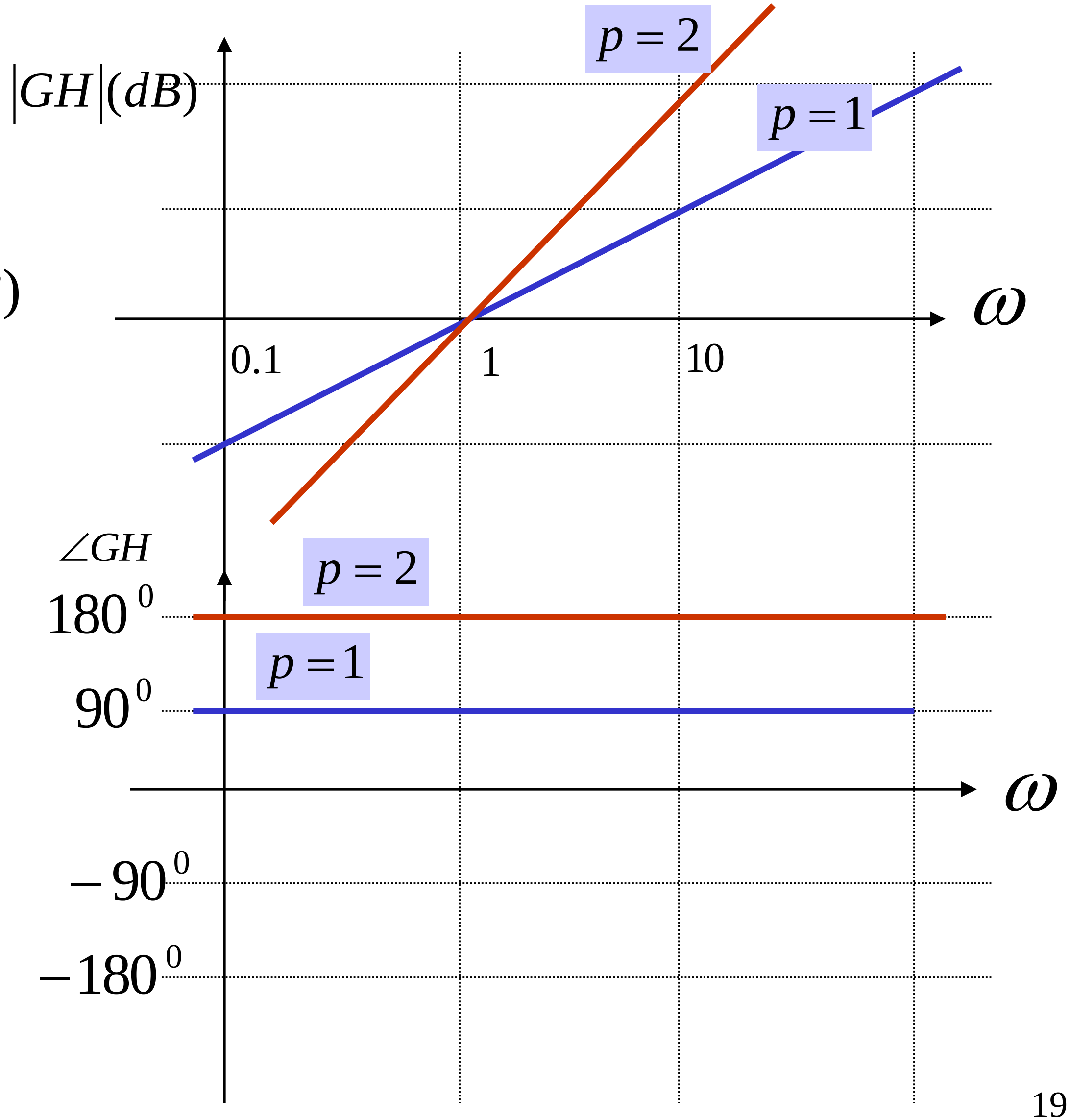
Case $III:$ s^p

Magnitude:

$$\left|(j\omega)^p\right|_{dB} = 20\,p\log\omega(dB)$$

Phase:

$$\angle(j\omega)^p = (90^o)\times p$$



Case IV : $\frac{a}{(s+a)}$ or $(\frac{1}{a}s+1)^{-1}$

$a=1$

Magnitude:

$$\left|(1+j\frac{\omega}{a})^{-1}\right|_{dB} = -20\log\sqrt{1+(\frac{\omega}{a})^2}$$

$$= -10\log[1+(\frac{\omega}{a})^2]$$

$$\omega \prec\prec a \Rightarrow \frac{\omega}{a} \approx 0 \Rightarrow dB = -10\log 1 = 0$$

$$\omega \succ\succ a \Rightarrow 1+j\frac{\omega}{a} \approx \frac{\omega}{a} \Rightarrow dB \approx -20\log\frac{\omega}{a}$$

$$dB = -[20\log\omega - 20\log a]$$

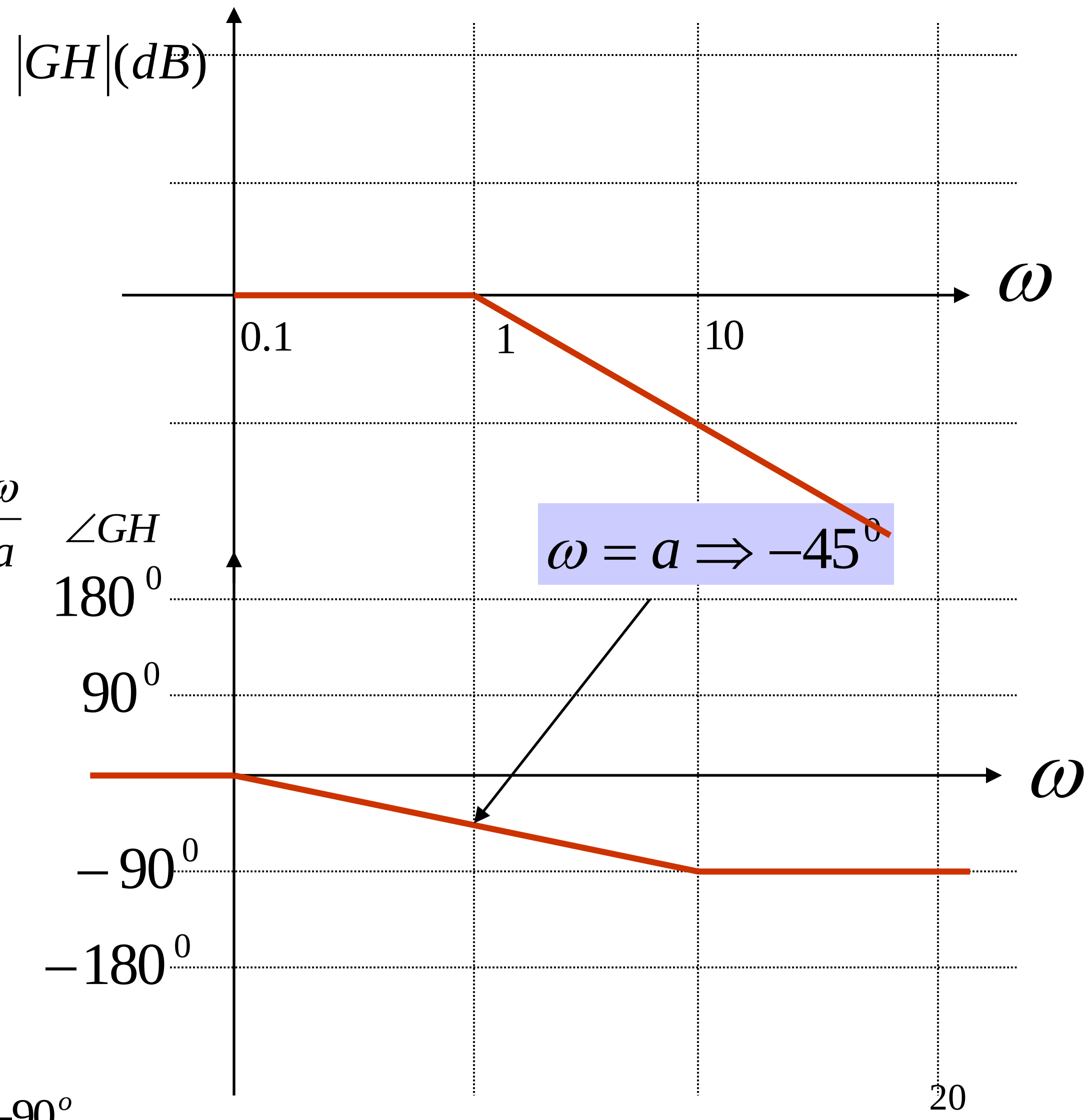
$$\omega = a \Rightarrow 1+j1 \Rightarrow dB = -10\log 2 = -3.01$$

Phase:

$$\angle(1+j\frac{\omega}{a}) = 0^0 - \tan^{-1}\frac{\omega}{a}$$

$$\omega \prec\prec a \Rightarrow \frac{\omega}{a} \approx 0 \Rightarrow \angle GH \approx \tan^{-1}0 = 0^o$$

$$\omega \succ\succ a \Rightarrow \frac{\omega}{a} \approx \infty \Rightarrow \angle GH \approx -\tan^{-1}\infty = -90^o$$



Case V: $\dfrac{(s+a)}{a}$ or $(\dfrac{1}{a}s+1)$

$a=1$

Magnitude:

$$\left|(1+j\frac{\omega}{a})\right|_{dB} = 20\log\sqrt{1+(\frac{\omega}{a})^2}$$

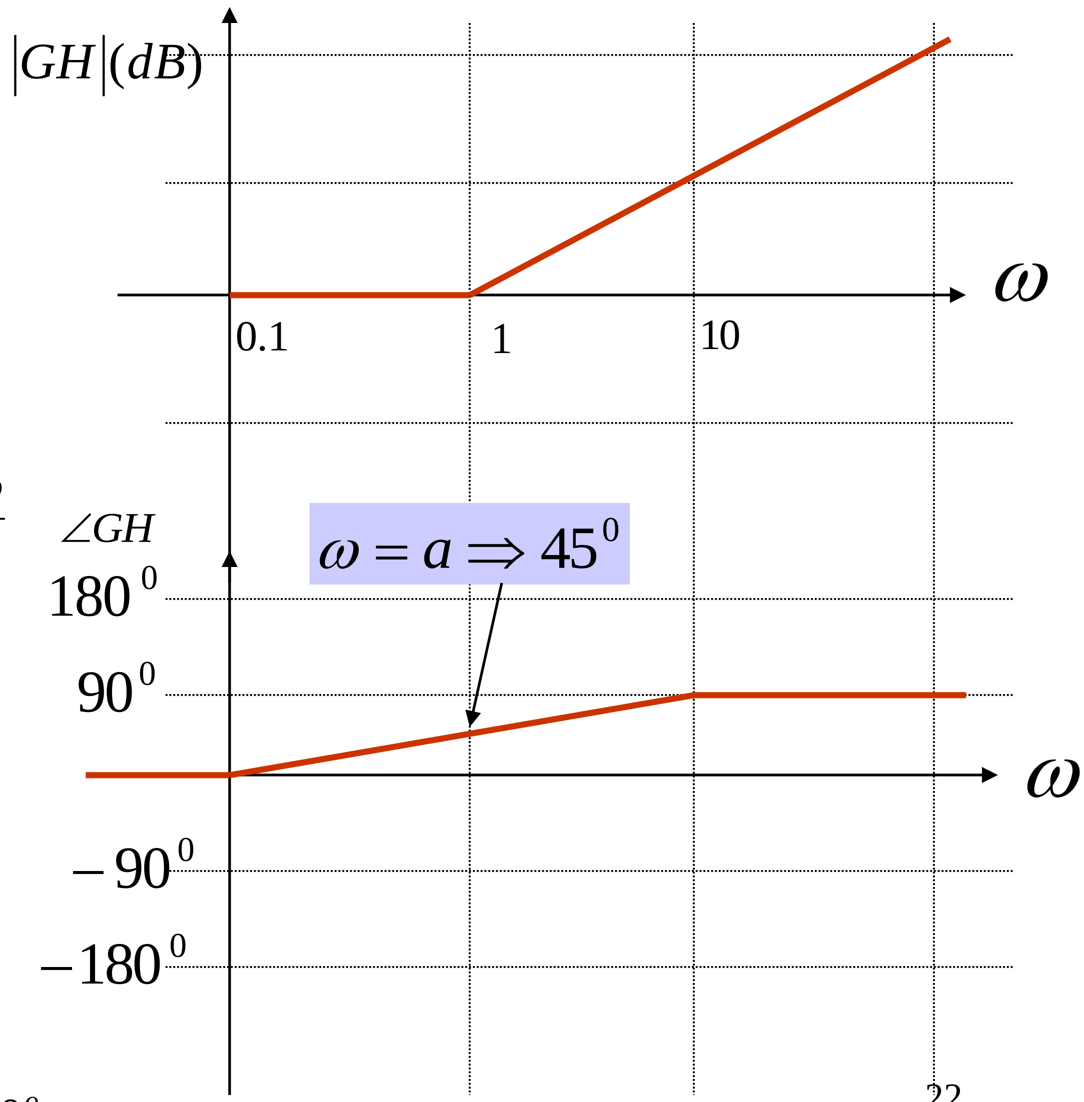
$$=10\log[1+(\frac{\omega}{a})^2]$$

$$\omega \prec\prec a \Rightarrow \frac{\omega}{a} \approx 0 \Rightarrow dB = 10\log 1 = 0$$

$$\omega \succ\succ a \Rightarrow 1+j\frac{\omega}{a} \approx \frac{\omega}{a} \Rightarrow dB \approx 20\log\frac{\omega}{a}$$

$$dB = 20\log\omega - 20\log a$$

$$\omega = a \Rightarrow 1+j1 \Rightarrow dB = 10\log 2 = 3.01$$

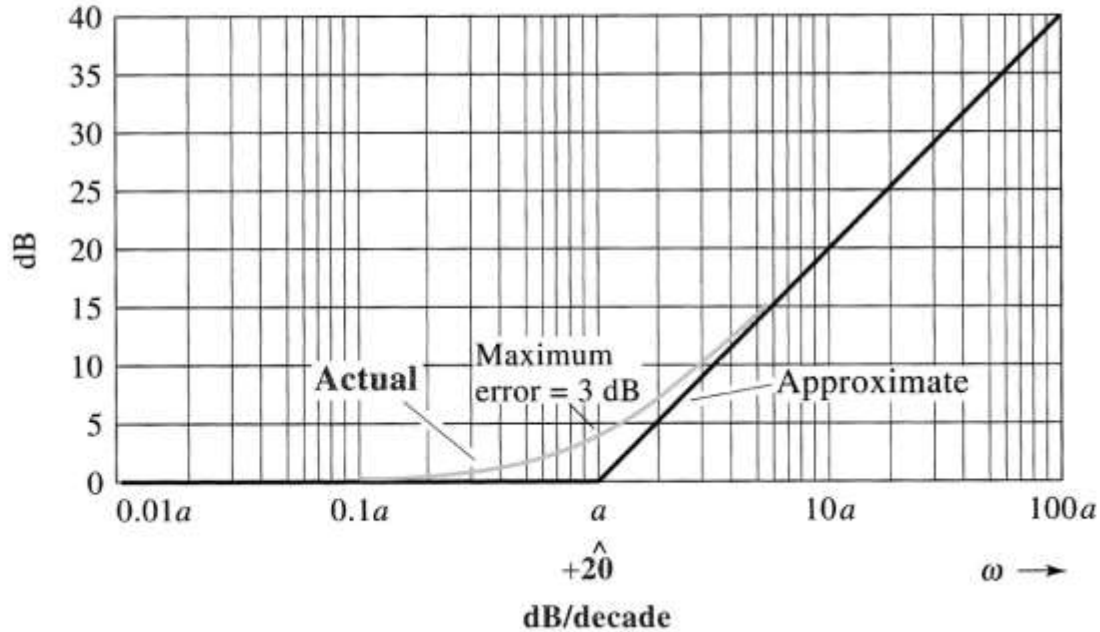


Phase:

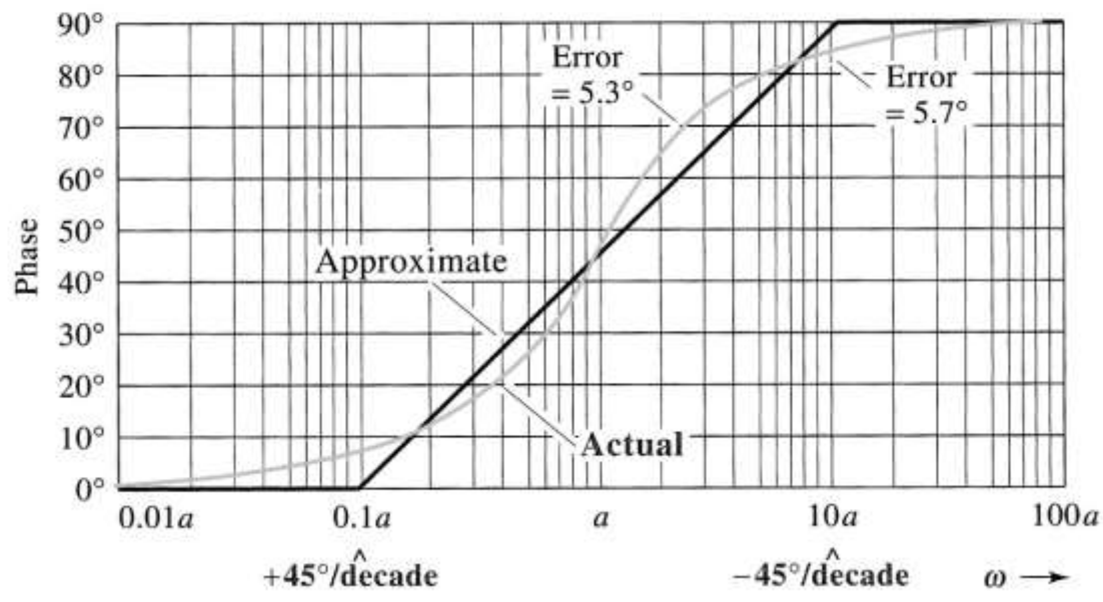
$$\angle(1+j\frac{\omega}{a}) = \tan^{-1}\frac{\omega}{a}$$

$$\omega \prec\prec a \Rightarrow \frac{\omega}{a} \approx 0 \Rightarrow \angle GH \approx \tan^{-1} 0 = 0^o$$

$$\omega \succ\succ a \Rightarrow \frac{\omega}{a} \approx \infty \Rightarrow \angle GH \approx \tan^{-1}\infty = 90^o$$



dB
40
35
30
25
20
15
10
5
0
Actual
Maximum
error = 3 dB
Approximate
0.01a
0.1a
a
10a
100a
+20
dB/decade
ω →



Phase
90°
80°
70°
60°
50°
40°
30°
20°
10°
0°
Error
= 5.3°
Error
= 5.7°
Approximate
Actual
0.01a
0.1a
a
10a
100a
+45°/decade
−45°/decade
ω →

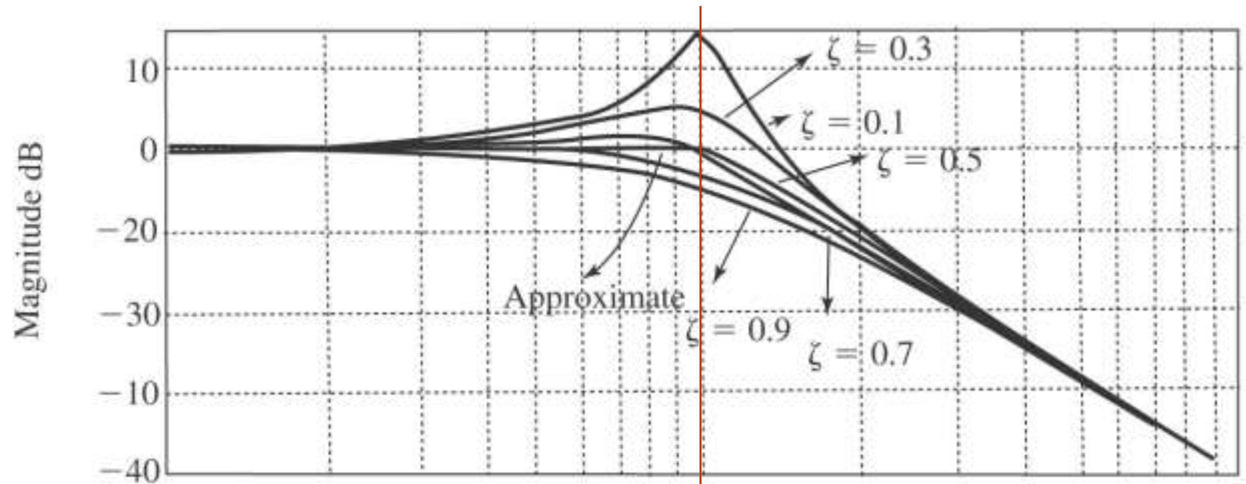
Case VI :

$$T(s)=\frac{\omega_n^2}{s^2+2\xi\omega_n s+\omega_n^2}$$

$$T(j\omega)=\frac{\omega_n^2}{(\omega_n{}^2-\omega^2)+2j\xi\omega_n\omega} \qquad \angle T(j\omega)=-\tan^{-1}\frac{2\xi\omega\omega_n}{(\omega_n{}^2-\omega^2)}$$

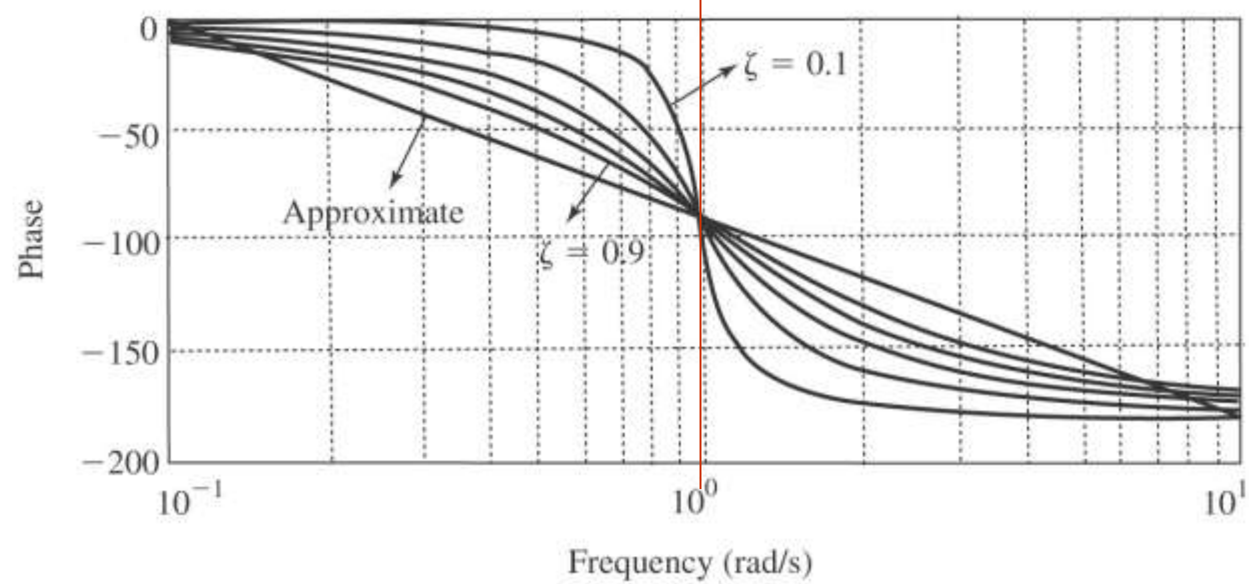
$$T(j\omega)=\frac{1}{(1-(\frac{\omega}{\omega_n})^2)+j2\xi\frac{\omega}{\omega_n}} \qquad \angle T(j\omega)=-\tan^{-1}\frac{2\xi\frac{\omega}{\omega_n}}{1-(\frac{\omega}{\omega_n})^2}$$

$$\left|T(j\omega)\right|=\begin{cases} 0 & ,\frac{\omega}{\omega_n}\prec\prec 1 \\ -20\log(2\xi) & ,\frac{\omega}{\omega_n}=1 \\ -40\log(\frac{\omega}{\omega_n}) & ,\frac{\omega}{\omega_n}\succ\succ 1 \end{cases} \qquad \angle T(j\omega)=\begin{cases} 0^0 & ,\frac{\omega}{\omega_n}\prec\prec 1 \\ -90^0 & ,\frac{\omega}{\omega_n}=1 \\ -180^o & ,\frac{\omega}{\omega_n}\succ\succ 1 \end{cases}$$



(a)

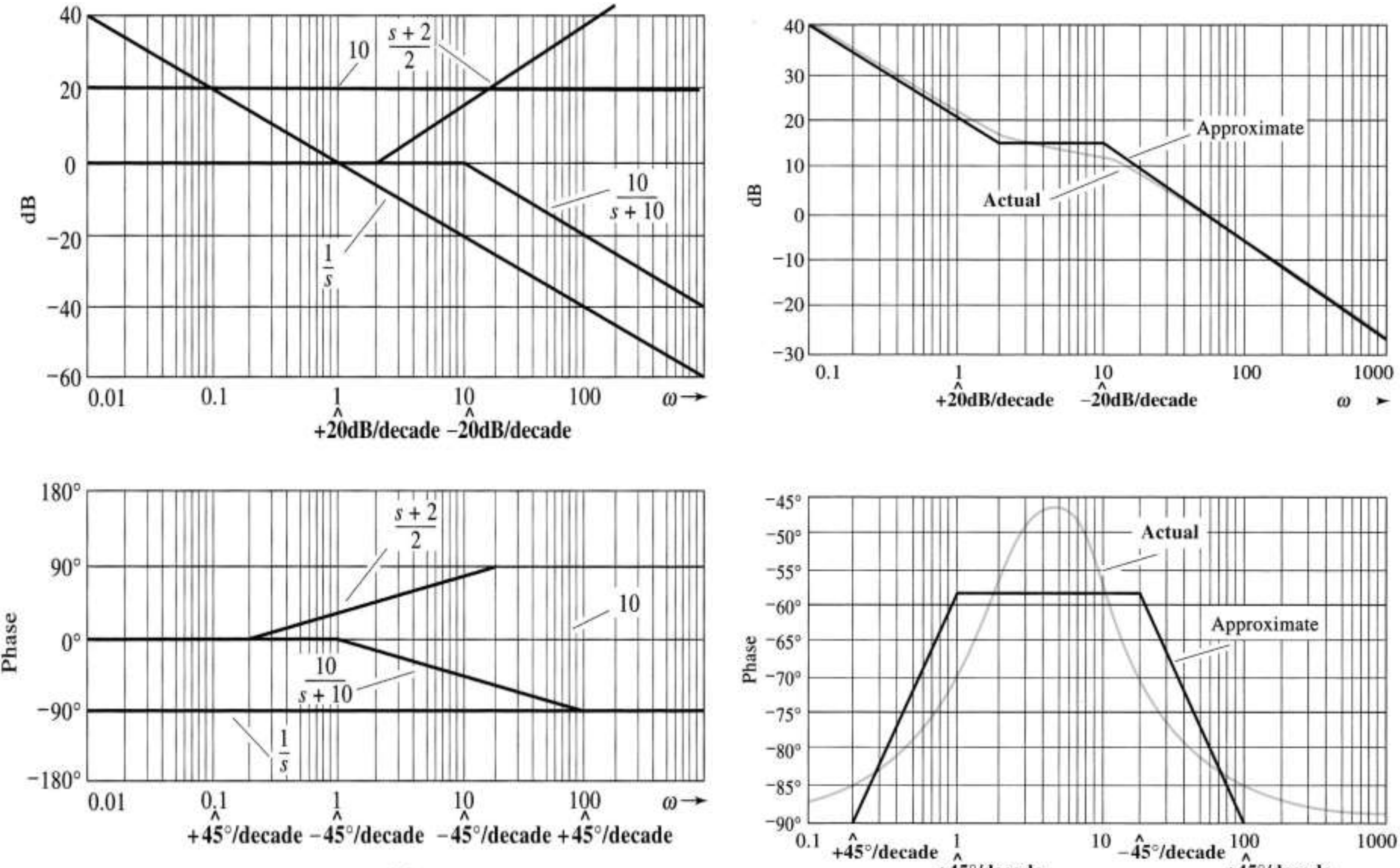
$\omega = \omega_n$



(b)

Example : $T(s)=\dfrac{50(s+2)}{s(s+10)}$

$$\Rightarrow \quad T(s)=10\left(\frac{1}{s}\right)\left(\frac{s+2}{2}\right)\left(\frac{10}{s+10}\right)$$



40
20
0
−20
−40
−60
dB
10
$\frac{s+2}{2}$
$\frac{10}{s+10}$
$\frac{1}{s}$
0.01
0.1
1
10
100
ω →
+20dB/decade −20dB/decade
40
30
20
10
0
−10
−20
−30
dB
Approximate
Actual
0.1
1
10
100
1000
+20dB/decade
−20dB/decade
ω
180°
90°
0°
−90°
−180°
Phase
$\frac{s+2}{2}$
10
$\frac{10}{s+10}$
$\frac{1}{s}$
0.01
0.1
1
10
100
ω →
+45°/decade −45°/decade −45°/decade +45°/decade
−45°
−50°
−55°
−60°
−65°
−70°
−75°
−80°
−85°
−90°
Phase
Actual
Approximate
0.1
1
10
100
1000
+45°/decade
+45°/decade
−45°/decade
+45°/decade

(a)

Frequency Response for Complex Poles

$$T(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

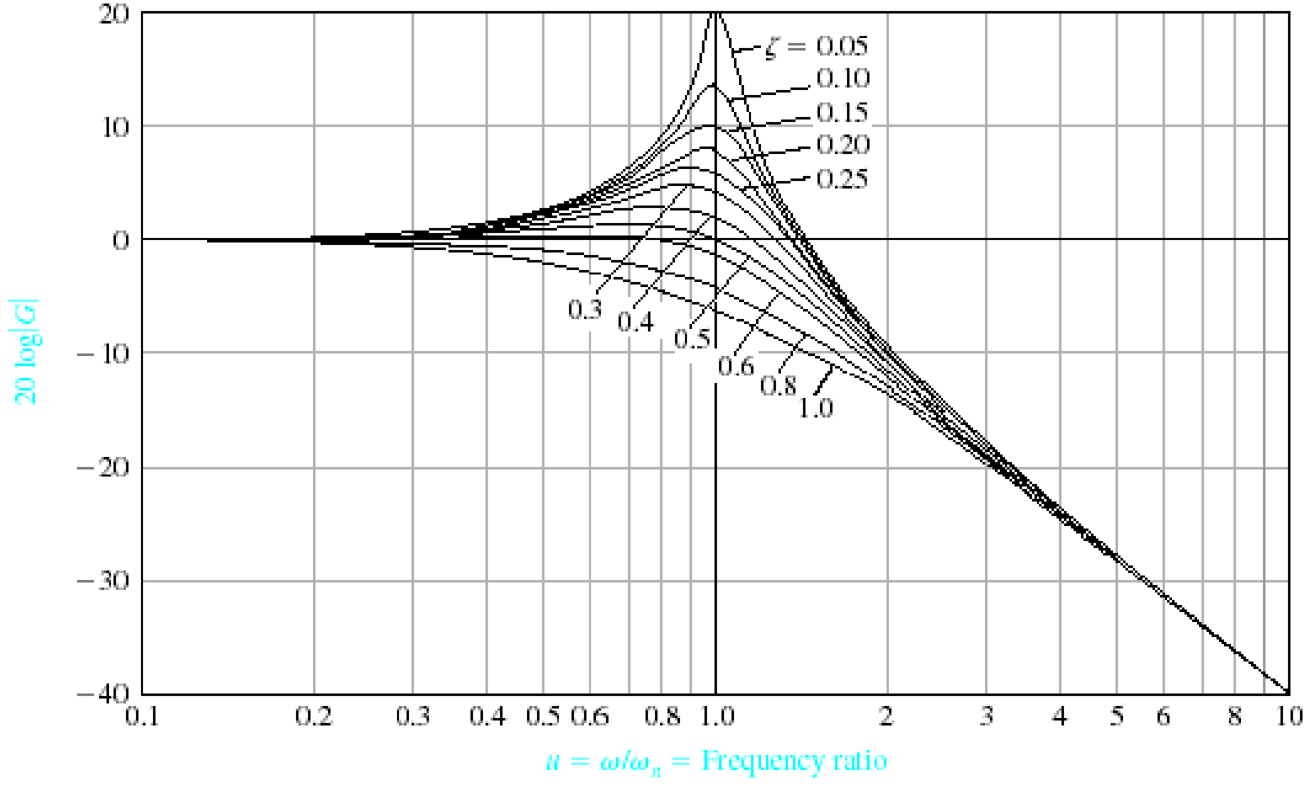
$$T(j\omega) = \frac{\omega_n^2}{\omega_n^2 - \omega^2 + j2\zeta\omega_n\omega}$$

Let, u $= \frac{\omega}{\omega_n}$ $\qquad T(ju) = \frac{1}{(1-u^2) + j2\zeta u}$

Magnitude, M=$|T(ju)|$=$\frac{1}{\sqrt{(1-u^2)^2+(2\zeta u)^2}}$

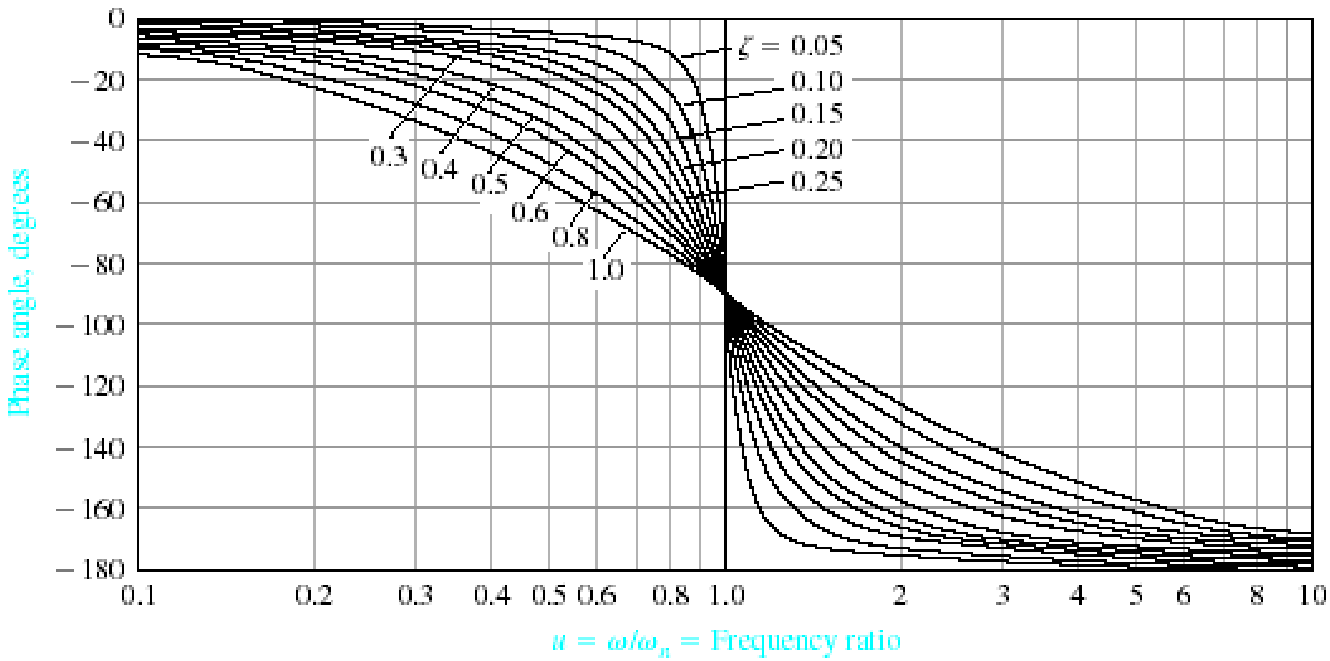
Phase, $\phi =$
$$\begin{cases} -\tan^{-1}\left(\frac{2\zeta u}{1-u^2}\right), & for\ u < 1 \\ -90°, & for\ u = 1 \\ -90° - \tan^{-1}\left(\frac{2\zeta u}{u^2-1}\right), & for\ u > 1 \end{cases}$$

Frequency Response Plots: Bode Plots for Complex Poles



Bode diagram for $G(j\omega) = [1 + (2\zeta/\omega_n)\, j\omega + (j\omega/\omega_n)^2]^{-1}$.

Frequency Response Plots: Bode Plots for Complex Poles



Bode diagram for $G(j\omega) = [1 + (2\zeta/\omega_n)\, j\omega + (j\omega/\omega_n)^2]^{-1}$.

Frequency Response: Resonant Frequency and Resonant Peak

At resonant frequency $\frac{dM}{du} = 0 \Rightarrow u_r = \sqrt{1 - 2\zeta^2}$ for $\zeta < \frac{1}{\sqrt{2}}$

$$\omega_r = \omega_n \cdot \sqrt{1 - 2\cdot\zeta^2} \qquad \zeta < 0.707$$

$$M_{p\omega} = \left|G(\omega_r)\right| = \frac{1}{\left(2\cdot\zeta\cdot\sqrt{1-\zeta^2}\right)} \qquad \zeta < 0.707$$

$M_{p\omega} = Resonant\ Peak$

$\omega_r = Resonant\ Frequency$

$\omega_n = Natural\ Frequency$

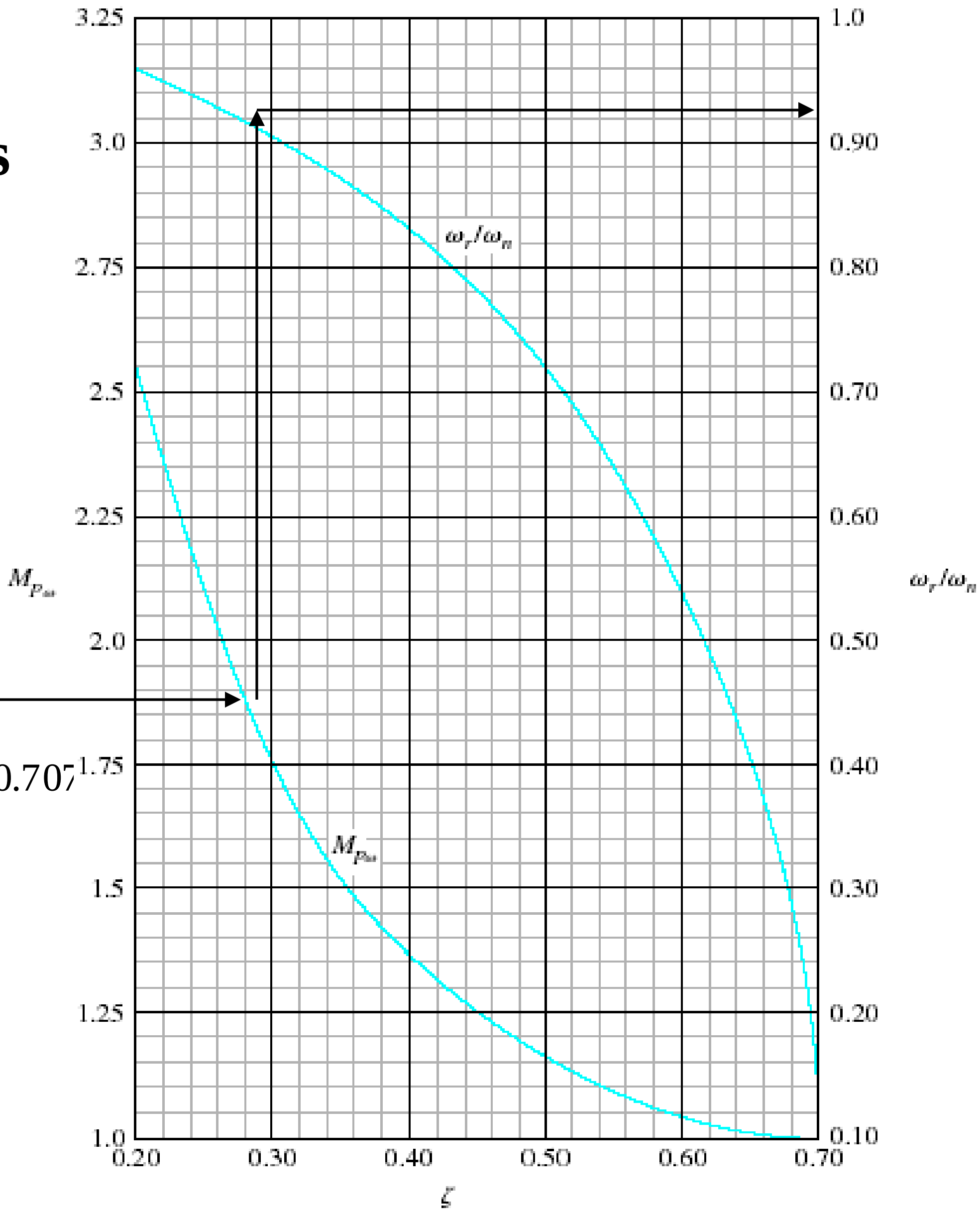
Frequency Response Plots

Bode Plots – Complex Poles

$$\omega_r = \omega_n \cdot \sqrt{1 - 2\cdot\zeta^2} \qquad \zeta < 0.707$$

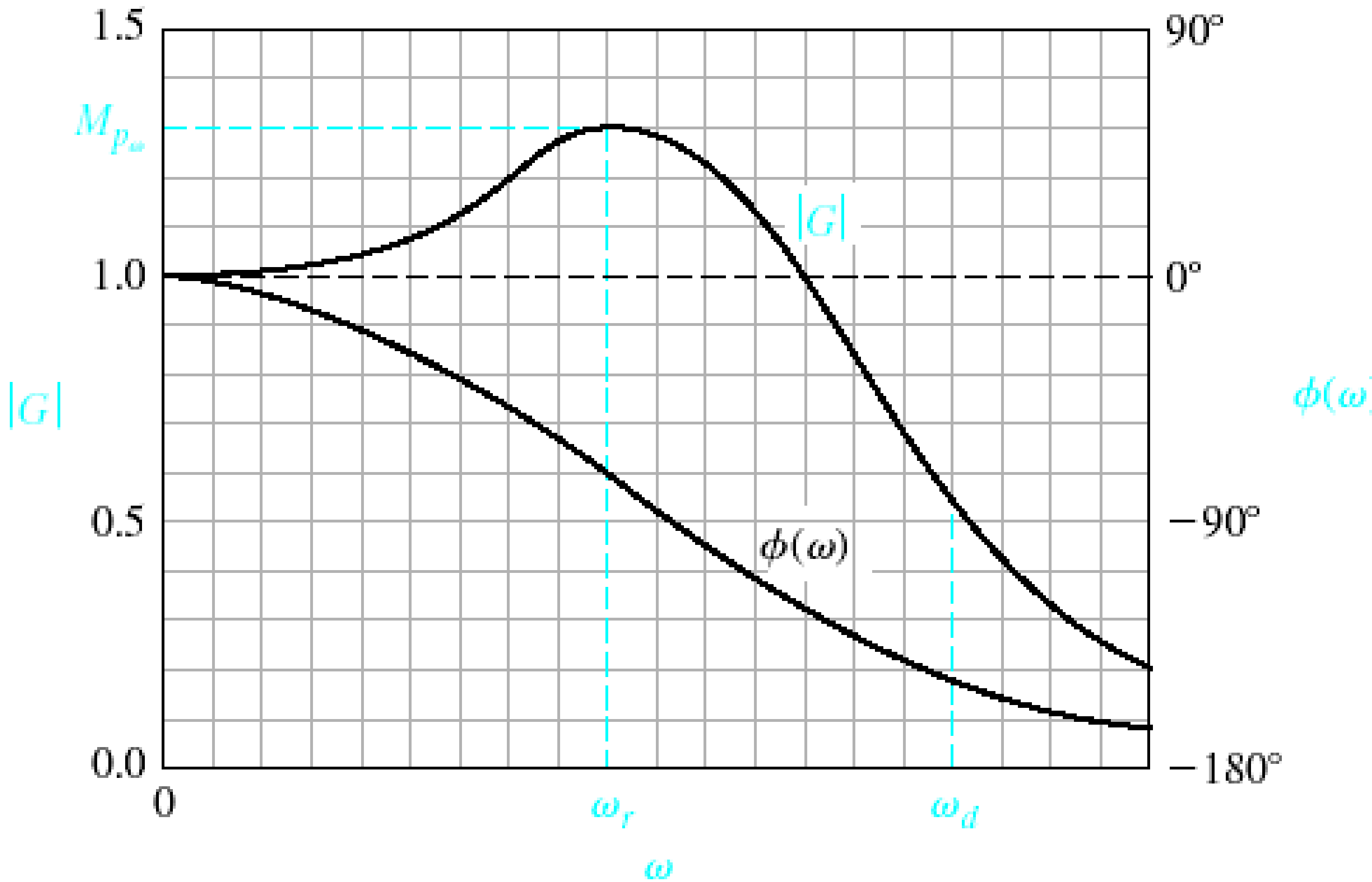
$$M_{p\omega} = \left|G(\omega_r)\right| = \frac{1}{\left(2\cdot\zeta\cdot\sqrt{1-\zeta^2}\right)}$$

$\zeta < 0.707$

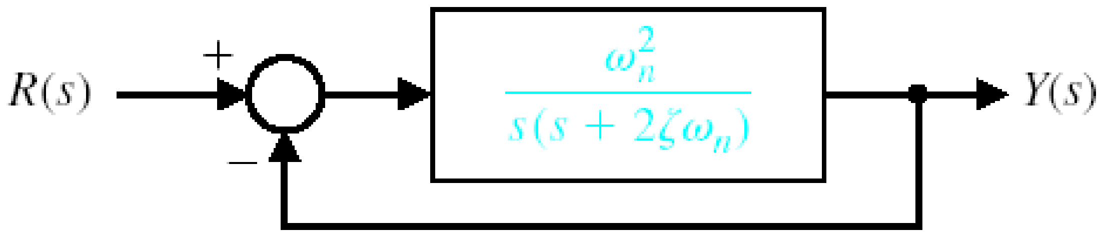


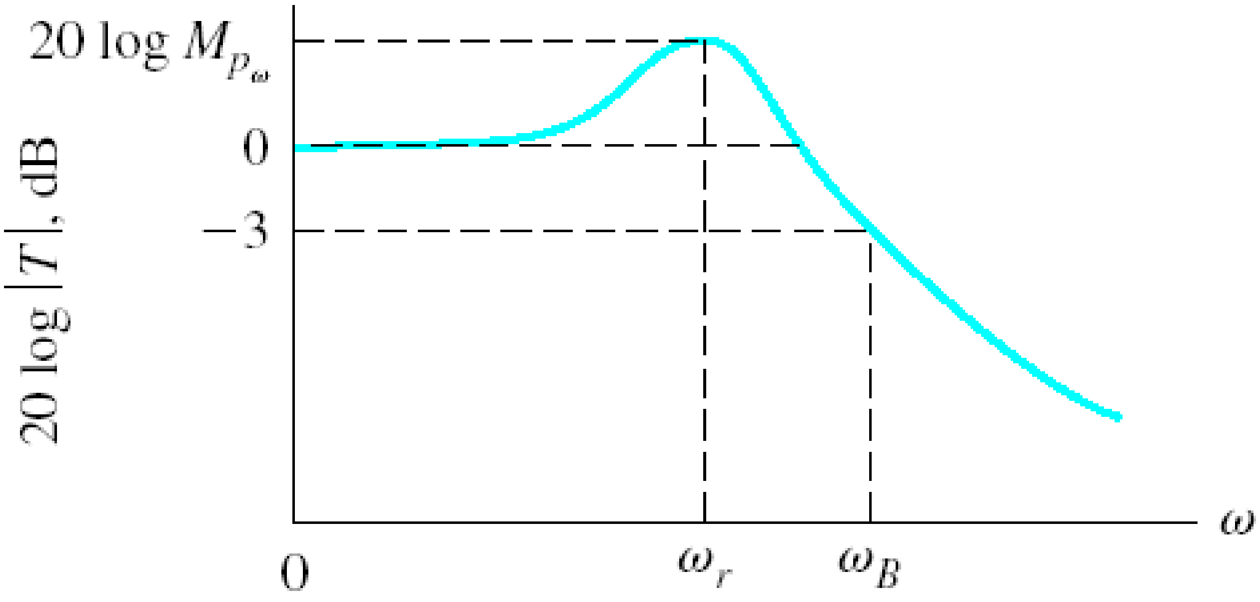
Frequency Response Plots

Bode Plots – Complex Poles

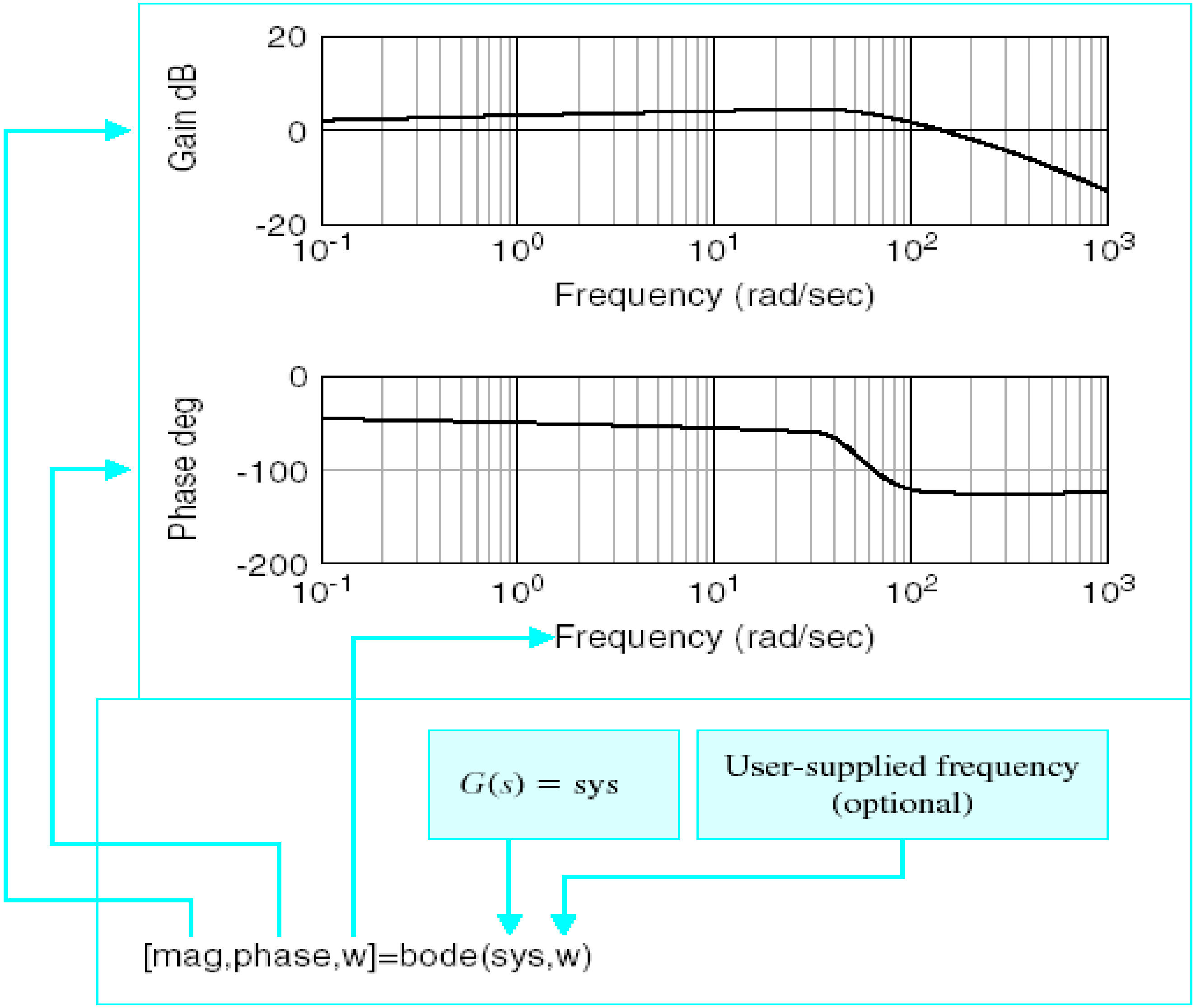


Performance Specification In the Frequency Domain

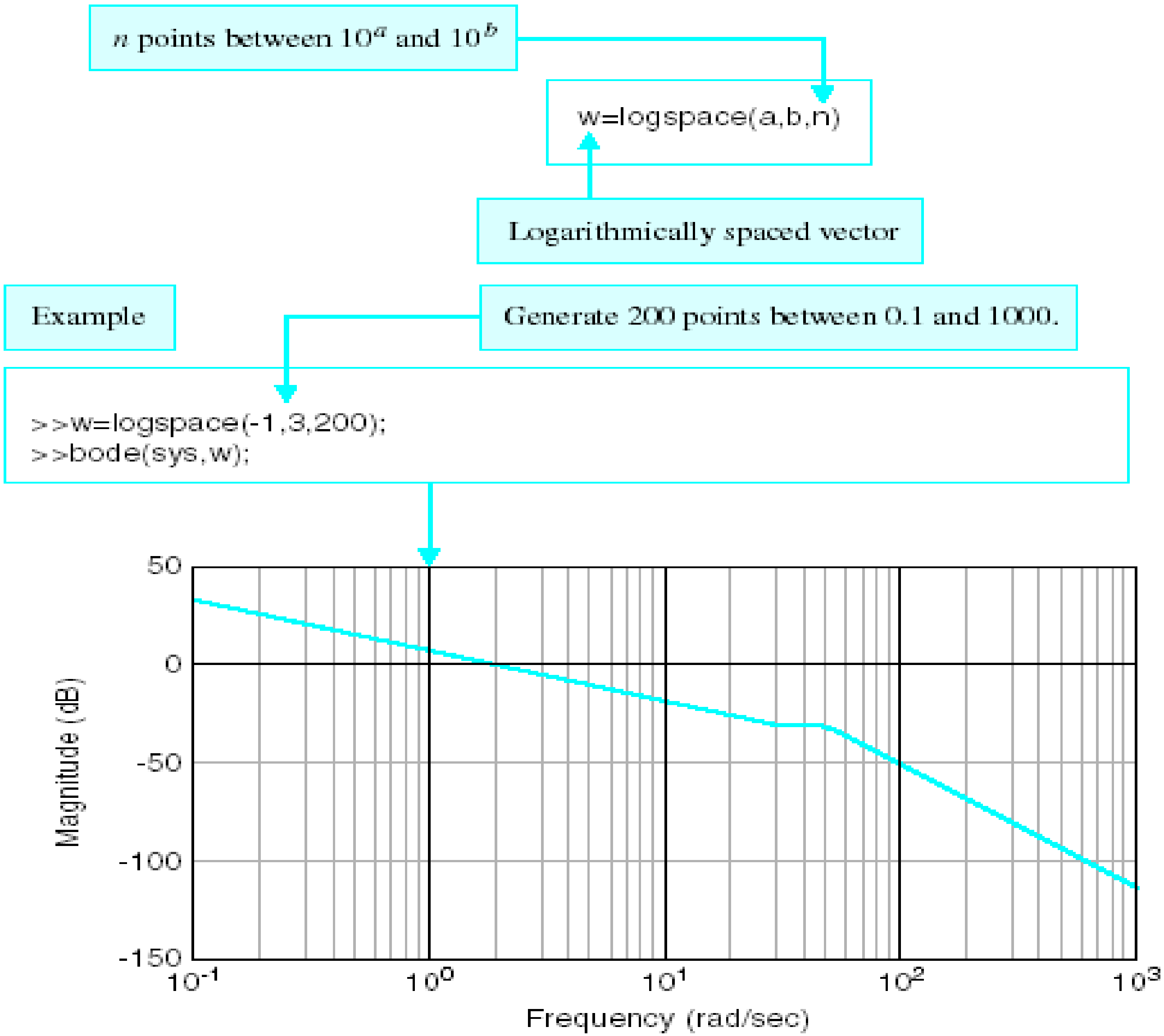




Frequency Response Methods Using MATLAB



Frequency Response Methods Using MATLAB



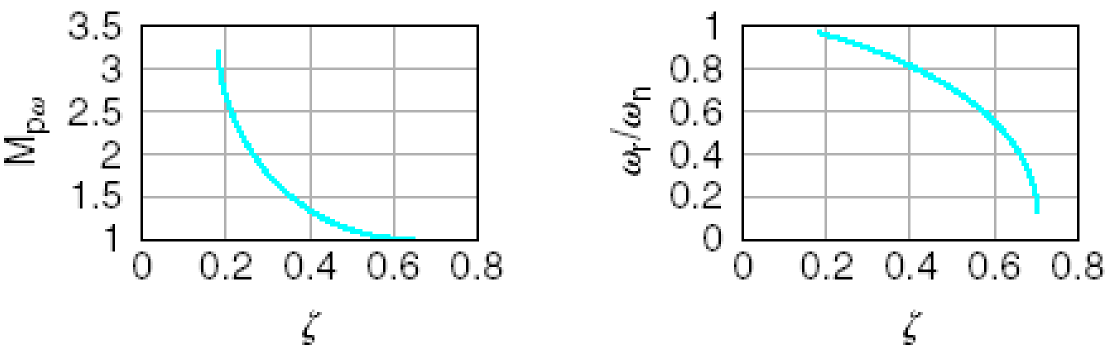
Frequency Response Methods Using MATLAB

```
% Bode plot script for Figure 8.22
%
num=5*[0.1 1];
f1=[1 0]; f2=[0.5 1]; f3=[1/2500 .6/50 1];
den=conv(f1,conv(f2,f3));
%
sys=tf(num,den);
bode(sys)
```

Compute

$$s(1+0.5s)\left(1+\frac{0.6}{50}s+\frac{1}{50^2}s^2\right)$$

Frequency Response Methods Using MATLAB



(a)

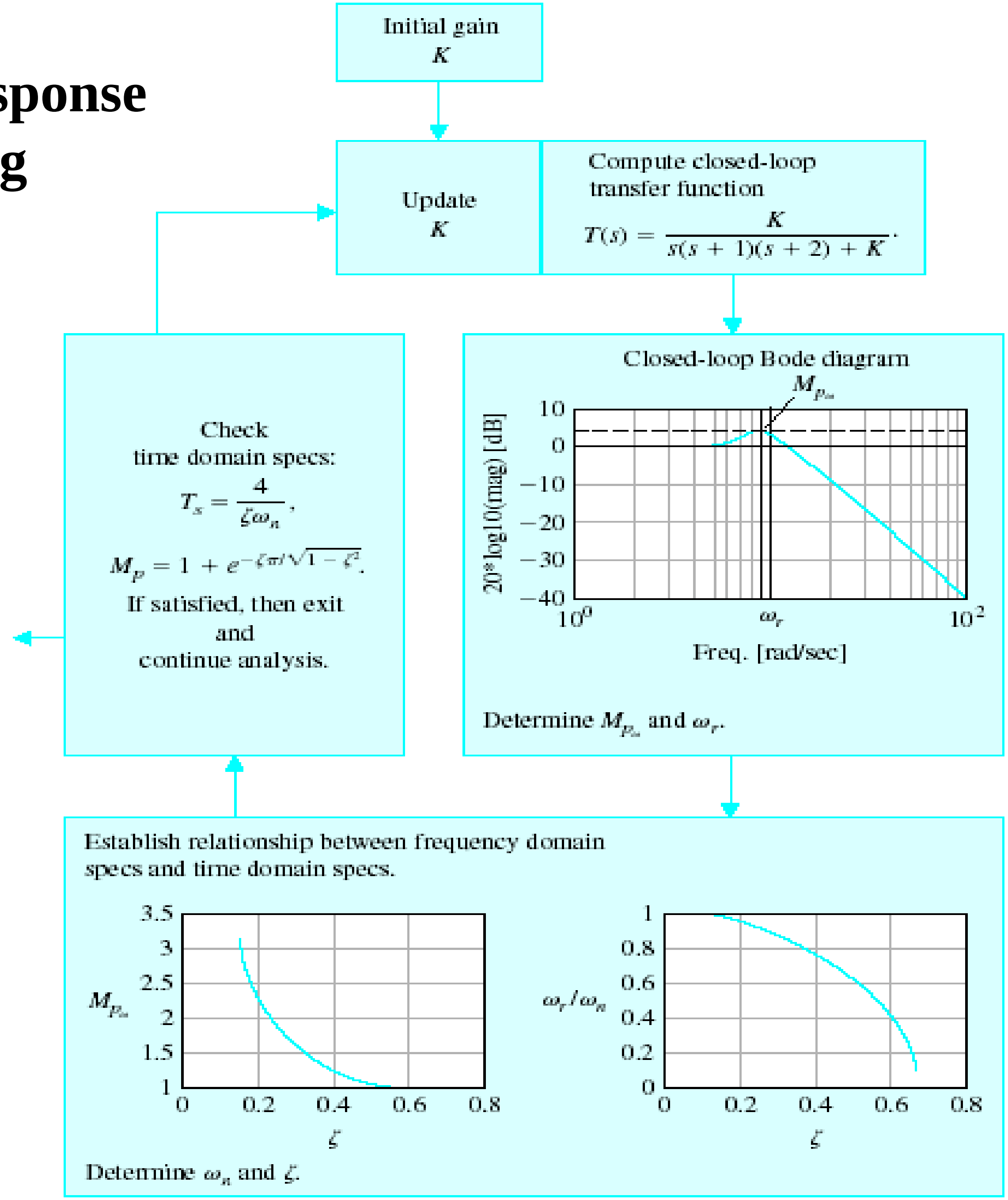
```
zeta=[0.15:0.01:0.7];
wr_over_wn=sqrt(1-2*zeta.^2);
Mp=(2*zeta .* sqrt(1-zeta.^2)).^(-1);
%
subplot(211),plot(zeta,Mp),grid
xlabel('\zeta'), ylabel('M_{p\omega}')
subplot(212),plot(zeta,wr_over_wn),grid
xlabel('\zeta'), ylabel('\omega_r/\omega_n')
```

zeta ranges from 0.15 to 0.70

Generate plots

(a) The relationship between (M_{p_ω}, ω_r) and (ζ, ω_n) for a second-order system. (b) MATLAB script.

Frequency Response Methods Using MATLAB



Frequency Response Methods Using MATLAB

engrave1.m

```
num=[K]; den=[1 3 2 K];
sys=tf(num,den);
w=logspace(-1,1,400);
[mag,phase,w]=bode(sys,w);
[mp,l]=max(mag);wr=w(l);
mp,wr
```

Closed-loop transfer function

Closed-loop Bode diagram

```
>>K=2; engrave1
mp =
     1.8371
wr =
     0.8171
>>
>>
>>
>>zeta=0.29; wn=0.88; engrave2
ts =
     15.6740
po =
     38.5979
```

manual step

Determine ω_n and ζ from Fig. 8.11 using M_{p_ω} and ω_r.

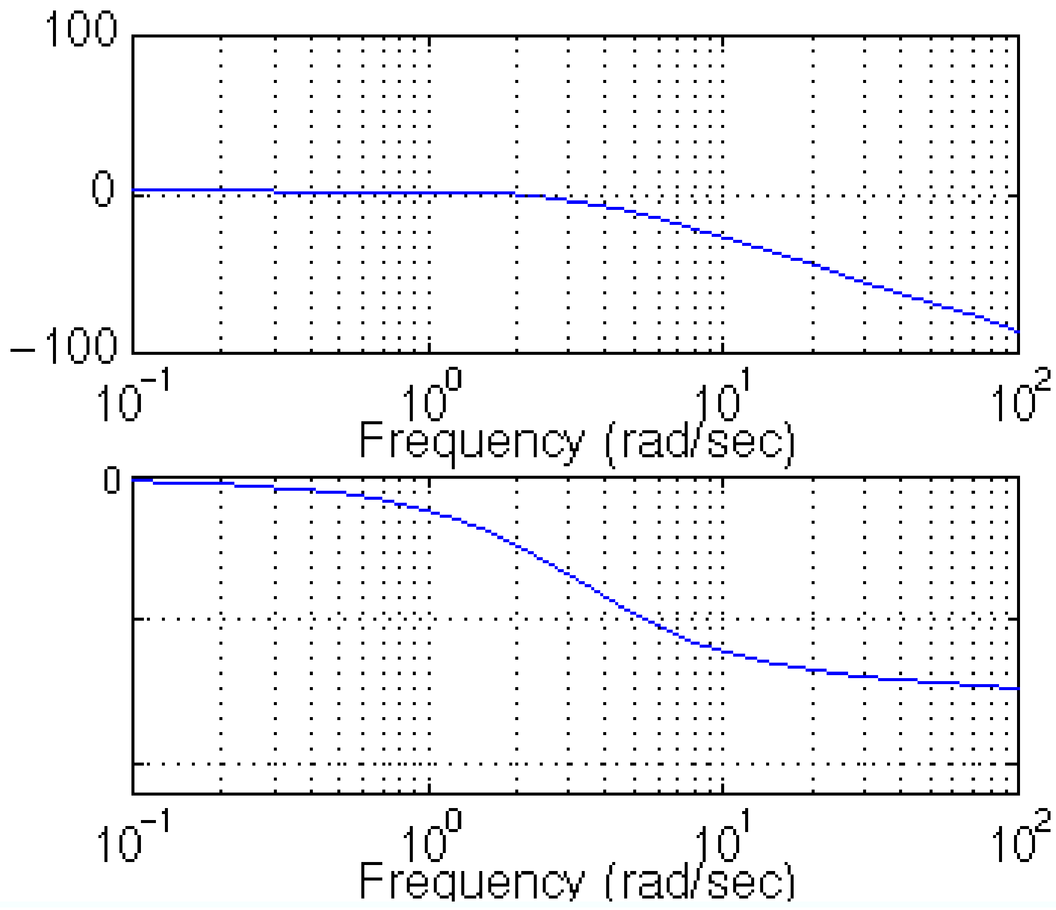
engrave2.m

```
ts=4/zeta/wn
po=100*exp(-zeta*pi/sqrt(1-zeta^2))
```

Check specs and iterate, if necessary.

Bode Plots

Bode plot is the representation of the magnitude and phase of G(j*w) (where the frequency vector w contains only positive frequencies).
To see the Bode plot of a transfer function, you can use the MATLAB
`bode`
command.



For example,

bode(50,[1 9 30 40])

displays the Bode plots for the transfer function:

50 / (s^3 + 9 s^2 + 30 s + 40)

Thank You