# The Three Pillars of Multivariable calculus

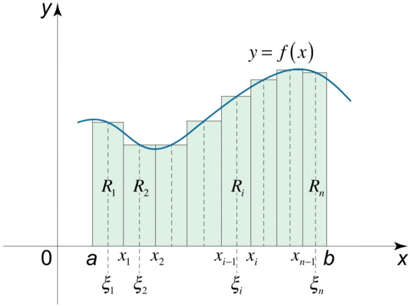
by Gauss, Greens and Stokes

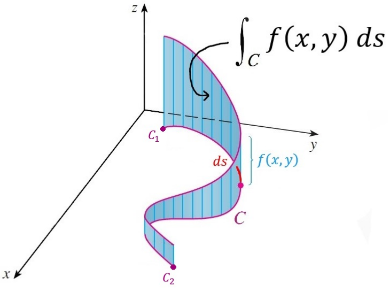
# Prerequisites

- Line integral
- Surface integral
- Multiple integral

# Line integral

Line integral, in a crude way of explaining, is an integral where the integrand is evaluated along a curve.
The following is a comparison between Riemann integral(left) of a function of one variable and line integral(right).



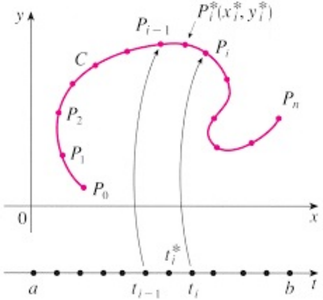
# Definition

Let $C$ be a smooth curve given by the parametric equations

$$x = x(t),\ y = y(t)$$

where $t \in [a, b]$. Divide $[a, b]$ into $n$ equal sub-intervals $[t_{i-1}, t_i]$ of equal width and denote $x_i = x(t_i), y_i = y(t_i)$.

Corresponding subarcs of lengths,say, $\triangle s_1, \triangle s_2, \cdots, \triangle s_n$ can be used to created Riemann sums of the form $\sum_{i=1}^{n} f(x_i^*, y_i^*)\triangle s_i$, where $(x_i^*, y_i^*)$ is any point in the $i$'th sub-interval.




So, if the limit to this sum as $n \to \infty$ exists, then it is called the **line integral** of $f$ along $C$ and is denoted by $\int_C f ds$

# Working formula

Fortunately for smooth curves and continuous functions, this limit always exists and also we have a simpler way to evaluate the line integral

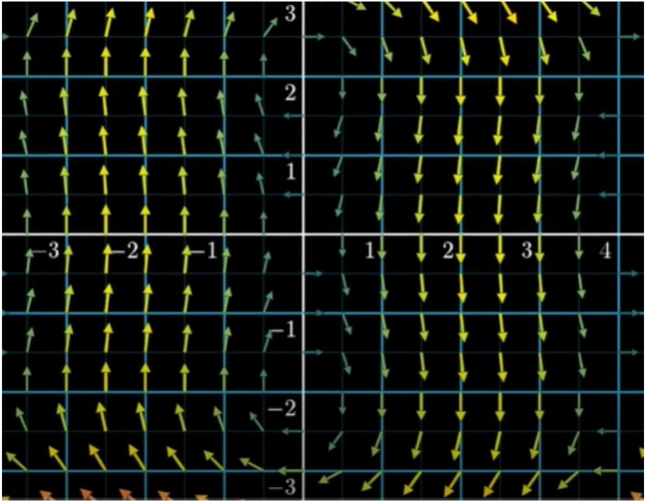
$$\int_C f(x,y)ds = \int_a^b f(x(t),y(t))\sqrt{x'(t)^2 + y'(t)^2}dt$$

This follows from Mean value theorem and

$$\triangle s_i = \sqrt{(x_{i+1} - x_i)^2 + (y_{i+1} - y_i)^2}$$

# Vector field

If every co-ordinate in a plane or space is assigned a vector, it is called a vector field, otherwise scalar field.

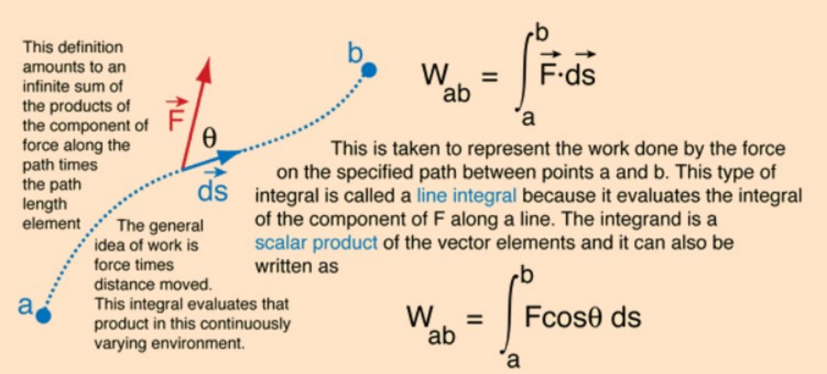


# Line integral in a vector field

Let $\mathbf{F}$ be a vector field in $\mathbb{R}^2$ and $C$ be a smooth curve lying inside it and directed towards $\mathbf{r}$. The line integral of $\mathbf{F}$ along $\mathbf{r}$ is defined as

$$\int_C \mathbf{F}.d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t)).\mathbf{r}'(t)dt$$

Measures the effect of a given field along a path.



# Example 1

Let us evaluate the line integral $\int_C (2 + x^2 y) ds$ using the working formula discussed previously. where $C$ is the upper half of the unit circle.

**Solution:** We first need parametric equations to represent C which in this case is

$$x = cost, y = sint; 0 \leq t \leq \pi$$

$$\begin{aligned}\int_C (2 + x^2 y) ds &= \int_0^\pi (2 + cos^2 tsint)\sqrt{cos^2 t + sin^2 t}dt \\ &= \int_0^\pi (2 + cos^2 tsint) dt \\ &= 2\pi + 2/3\end{aligned}$$

# Example 2

Let us find the work done by the force field $\mathbf{F}(x, y) = x^2\mathbf{i} - xy\mathbf{j}$ in moving a particle along the quarter-circle
$\mathbf{r}(t) = cost\mathbf{i} + sint\mathbf{j}, 0 \leq t \leq \pi/2.$
**Solution** Since $x = cost$ and $y = sint$ , we have

$$\mathbf{F}(x, y) = cos^2 t\mathbf{i} - costsint\mathbf{j}$$

$$\mathbf{r}'(t) = -sint\mathbf{i} + cost\mathbf{j}$$

Therefore the work done is

$$\int_C \mathbf{F}.d\mathbf{r} = \int_0^{\pi/2} (-2cos^2 tsint)dt$$
$$= -\,2/3$$

# Fundamental theorem of line integral

**Theorem**

*Let $C$ be a smooth curve given by the vector function $\mathbf{r}(t), t \in [a, b]$ . Let $f$ be a differentiable function of two or three variables whose gradient vector $\nabla f$ is continuous on $C$ . Then*

$$\int_C \nabla f . d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a))$$

**Proof**

$$\begin{aligned}
\int_C \nabla f . d\mathbf{r} &= \int_a^b \nabla f(\mathbf{r}(t)).\mathbf{r}'(t) dt \\
&= \int_a^b \left( \frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt} \right) dt \\
&= \int_a^b \frac{d}{dt} f(\mathbf{r}(t)) dt \\
&= f(\mathbf{r}(b)) - f(\mathbf{r}(a))
\end{aligned}$$

# Independence of path

The Fundamental theorem of line integral shows that, in a gradient field(where $\mathbf{F} = \nabla f$), the line integral does not depend on the path chosen. It only depends on the initial and final point. Such fields are also known as **conservative fields**.

A consequence of this is that line integrals over a closed loop is zero in a conservative field.

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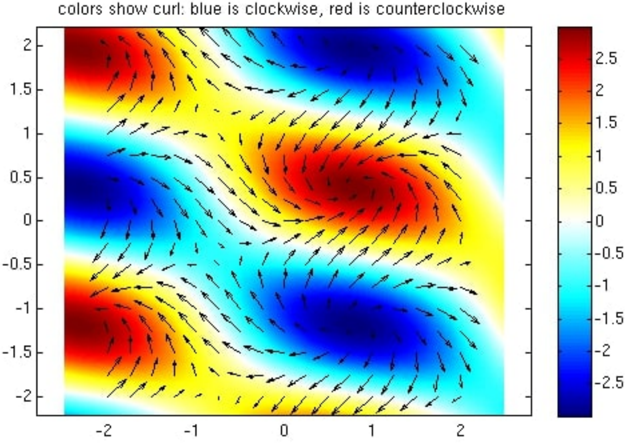
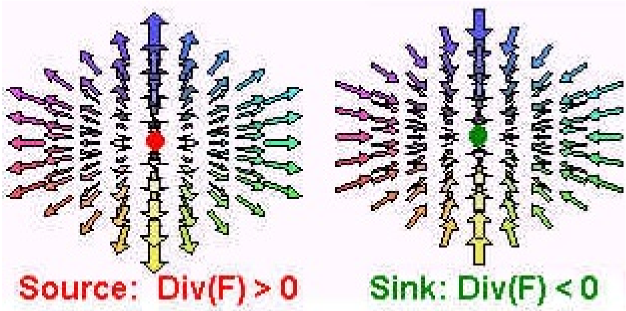
# Curl and its physical significance

Let $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$ be a vector field on $\mathbb{R}^3$ and all their partial derivatives exist. Also, let $\nabla = \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}$. Then, Curl of $\mathbf{F}$ is defined as $\nabla \times \mathbf{F}$. It gives the rotational measure of a vector field.



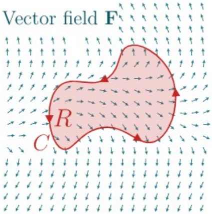
# Divergence and its physical significance

Using the same notations as in curl, divergence of **F** is defined as $\nabla.\mathbf{F}$. It gives the measure of divergence from a point.



# Intuition behind Green's theorem

- Setup:
  - $\mathbf{F}$ is a two-dimensional vector field.
    - $R$ is some region in the $xy$-plane.
    - $C$ is the boundary of that region, oriented **counterclockwise**.



# Green's theorem

The theorem gives a relationship between the line integral around a simple closed curve and a double integral over the plane region bounded by it.

**Theorem**

*Let C be a positively oriented piecewise smooth, simple closed curve in a plane and let D be the region bounded by C. If L and M are functions of (x,y) defined on an open region containing D and having continuous partial derivatives there, then*

$$\int_C (Ldx + Mdy) = \int\int_D \left(\frac{\partial M}{\partial x} - \frac{\partial L}{\partial y}\right) dxdy$$

# Proof

Let D be a region given by

$$D = \{(x, y) = x \in [a, b], y \in [f(x), g(x)]\}$$

where $f, g$ are continuous on $[a, b]$

$$\begin{aligned}
\int\int_D \frac{\partial L}{\partial y} dxdy &= \int_{x=a}^{x=b} \int_{y=f(x)}^{y=g(x)} \frac{\partial L}{\partial y} dxdy \\
&= \int_a^b \left(L(x, g(x)) - L(x, f(x)\right) dx \\
&= \int_a^b L(x, g(x))dx - \int_a^b L(x, f(x))dy \\
&= -\int_{C_3} Ldx - \int_{C_1} Ldx \\
&= -\int_C Ldx
\end{aligned}$$

Similarly, we can prove $\int_C Mdy = \int\int_D \frac{\partial M}{\partial x} dxdy$.

# Green's theorem revisited(Vector form)

Let $\mathbf{F}$=$P\mathbf{i}+Q\mathbf{j}$. So, we know its line integral is

$$\int_C \mathbf{F}.d\mathbf{r} = \int_C Pdx + \int_C Qdy$$

And curl$\mathbf{F}$ is given by $\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\mathbf{k}$. Thus, Green's theorem can be restated as

$$\int_C \mathbf{F}.d\mathbf{r} = \int\int_D (curl\mathbf{F}).\mathbf{k}dA$$

# Application

**Example:** Evaluate

$$\int_C (4x^2 + y - 3)dx + (3x^2 + y^2 - 2)dy$$

around the rectangle $x \in [0, 3], y \in [0, 1]$.

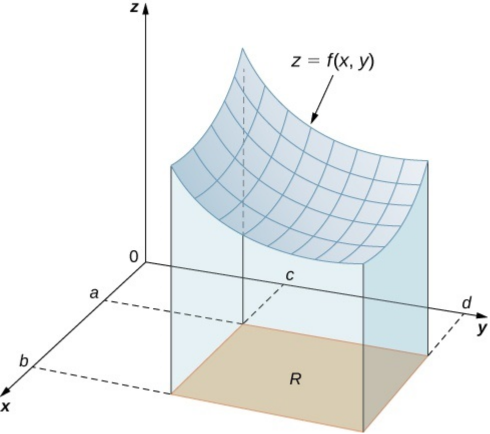
**Solution:** Considering $P = 4x^2 + y - 3$ and $Q = 3x^2 + y^2 - 2$, we evaluate the required line integral using Green's theorem as follows

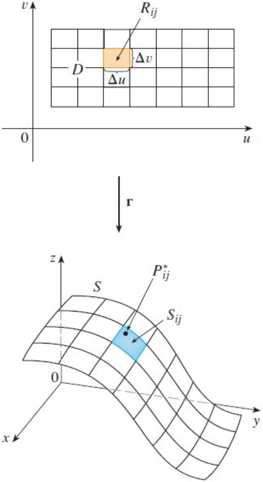
$$\int_{y=0}^{y=1} \int_{x=0}^{x=3} (6x - 1)dxdy = 24$$

The same line integral, if we had avoided the Green's theorem, then we would have to calculate line integral four times(for each side of the rectangle).

# Surface Integral

The idea of surface integral is the same as that of line integral, except, here we integrate a function over a surface in space.



# Definition

Let $f$ be a function of three variables and the surface be given by the vector equation

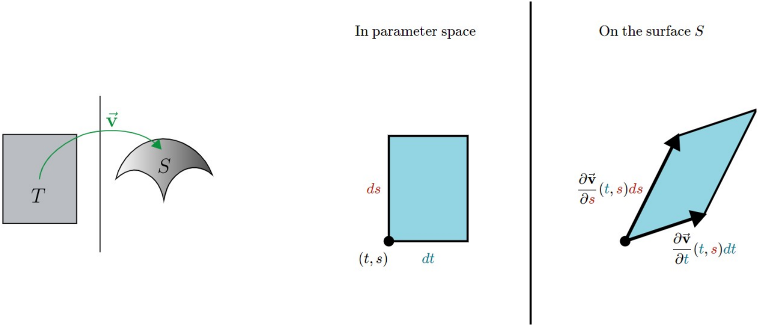
$$\mathbf{r}(u, v) = x(u, v)\mathbf{i} + y(u, v)\mathbf{j} + z(u, v)\mathbf{k}$$

where the parameter domain is the rectangle $R$.
Dividing $R$ into sub-rectangles $R_{ij}$, we get corresponding sub-surfaces $S_{ij}$ with area $\triangle S_{ij}$. Evaluating $f$ at any point $P_{ij}^*$ of $S_{ij}$, we denote the surface integral of $f$ on $S$ as $\int\int_S f(x, y, z)dS$ and define as

$$\int\int_S f(x, y, z)dS = \lim_{m,n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f(P_{ij}^*)\triangle S_{ij}$$

if the limit exists.

# Working formula




$$\underbrace{\iint_S}_{\text{Integral over surface}} f(x,y,z) \overbrace{d\Sigma}^{\text{Area of a tiny piece of } S} = \underbrace{\iint_T}_{\text{Integral in parameter space}} \overbrace{f(\vec{\mathbf{v}}(t,s))}^{\text{See where each point } (t,s) \text{ lands on } S\text{, then evalute } f} \underbrace{\left|\frac{\partial \vec{\mathbf{v}}}{\partial t} \times \frac{\partial \vec{\mathbf{v}}}{\partial s}\right|}_{\text{How much a tiny piece of } T \text{ is scaled after it is mapped onto } S \text{ by } \vec{\mathbf{v}}} \overbrace{dt\,ds}^{\text{Area of a tiny piece of } T}$$

# Surface integral over vector field and oriented surface

If **F** is a continuous vector field defined on an oriented surface $S$ with unit normal vector **n**, then the surface integral of **F** over $S$ is

$$\int\int_S \mathbf{F}.d\mathbf{S} = \int\int_S \mathbf{F}.\mathbf{n}dS$$

This integral is also called the **flux integral** of **F** across $S$.
If $S$ is given by a vector field **R**(u,v), then

$$\int\int_S \mathbf{F}.\mathbf{n}dS = \int\int_S \mathbf{F}.\frac{(\mathbf{R}_u \times \mathbf{R}_v)}{|(\mathbf{R}_u \times \mathbf{R}_v)|}dS$$

or

$$\int\int_D \mathbf{F}.(\mathbf{R}_u \times \mathbf{R}_v)\,dA$$

where $D$ is the domain of the parameter.

**Lemma**

*Let* $\mathbf{R}{:}U \to V \subseteq \mathbb{R}^3$*, where* $U$ *is an open subset of* $\mathbb{R}^2$ *and* $V$ *is an open subset of* $\mathbb{R}^3$*. Suppose,* $\mathbf{R}$ *is* $C^2$ *and let* $\mathbf{F}$ *be a* $C^1$ *vector field defined in* $V$*. Then,*

$$(\mathbf{R}_u \times \mathbf{R}_v).(\nabla \times \mathbf{F}) = (\mathbf{F} \circ \mathbf{R})_u.\mathbf{R}_v - (\mathbf{F} \circ \mathbf{R})_v.\mathbf{R}_u$$

# Stoke's theorem

Let $U$ be any region in $\mathbb{R}^2$ for which Green's theorem holds and let $\mathbf{R}:U \to V \subseteq \mathbb{R}^3$, where $V$ is an open subset of $\mathbb{R}^3$ is a function satisfying $|(\mathbf{R}_u \times \mathbf{R}_v)(u,v)| \neq 0$ for all $(u,v) \in U$ and let $S$ denote the surface

$$S = \{\mathbf{R}(u,v) : (u,v) \in U\}$$
$$\partial S = \{\mathbf{R}(u,v) : (u,v) \in \partial U\}$$

where the orientation on $\partial S$ is consistent with the counter clockwise orientation on $\partial U$.

Let $\mathbf{F}$ be a vector field whose components have continuous partial derivatives on an open region in $\mathbb{R}^3$ that contain S. Then

$$\int_{\partial S} \mathbf{F}.d\mathbf{R} = \int\int_S curl\mathbf{F}.\mathbf{n}dS$$

where $\mathbf{n}$ is the outward normal to S defined by

$$\mathbf{n} = \frac{(\mathbf{R}_u \times \mathbf{R}_v)}{|(\mathbf{R}_u \times \mathbf{R}_v)|}$$

# Proof

Let $C$ be an oriented part of $\partial U$ having parametrization,

$$\mathbf{r}(t) = (u(t), v(t))$$

for $t \in [a, b]$ and letting $\mathbf{R}(C)$ denote the oriented part of $\partial S$ corresponding to $C$.

$$\int_{\mathbf{R}(C)} \mathbf{F}.d\mathbf{R} = \int_a^b \mathbf{F}(\mathbf{R}(u(t), v(t)).(\mathbf{R}_u u'(t) + (\mathbf{R}_v v'(t))dt$$
$$= \int_C ((\mathbf{F} \circ \mathbf{R}).\mathbf{R}_u, (\mathbf{F} \circ \mathbf{R}).\mathbf{R}_v).d\mathbf{r}$$

Since, this holds for each such piece of $\partial U$, it follows

$$\int_{\partial S} \mathbf{F}.d\mathbf{R} = \int_{\partial U} ((\mathbf{F} \circ \mathbf{R}).\mathbf{R}_u, (\mathbf{F} \circ \mathbf{R}).\mathbf{R}_v).d\mathbf{r}$$

By Green's theorem, we get

$$\begin{aligned}
\int_{\partial S} \mathbf{F}.d\mathbf{R} &= \int\int_U \{((\mathbf{F}\circ\mathbf{R}).\mathbf{R}_v)_u - ((\mathbf{F}\circ\mathbf{R}).\mathbf{R}_u))_v\}\, dA \\
&= \int\int_U \{((\mathbf{F}\circ\mathbf{R})_u.\mathbf{R}_v) - ((\mathbf{F}\circ\mathbf{R})_v.\mathbf{R}_u))\}\, dA \\
&= \int\int_U (\mathbf{R}_u \times \mathbf{R}_v).(\nabla \times \mathbf{F}) dA \\
&= \int\int_S \nabla \times \mathbf{F}.\mathbf{n} dS
\end{aligned}$$

The proof is complete.

# Independence of surface

Stoke's theorem implies an interesting fact that for any given vector field $\mathbf{F}(x, y, z)$, $\int\int_D curl\mathbf{F}.\mathbf{n}dS$ will be the same for any surface with the same boundary. These surface integrals involve adding up completely different values at completely different points in space, yet they turn out to be the same simply because they share a boundary.

# Intuition behind divergence theorem

The intuition here is that divergence measures the outward flow of a fluid at individual points, while the flux measures outward fluid flow from an entire region, so adding up the bits of divergence gives the same value as flux.

$$\underbrace{\iiint_V \operatorname{div} \mathbf{F}\, dV}_{\substack{\text{Add up little bits} \\ \text{of outward flow in } V}} = \underbrace{\overbrace{\iint_S \mathbf{F} \cdot \hat{\mathbf{n}}\, d\Sigma}^{\text{Flux integral}}}_{\substack{\text{Measures total outward} \\ \text{flow through } V\text{'s boundary}}}$$

# Divergence theorem

**Theorem**

*let $E$ be a simple solid region and let $S$ be the boundary surface of $E$, given with positive orientation. Let $\mathbf{F}$ be a vector field whose component functions have continuous partial derivatives on an open region that contains $E$. Then*

$$\int\int_S \mathbf{F}.d\mathbf{S} = \int\int\int_E div\mathbf{F}dV$$

# Proof

The proof is based on the assumption that $E$ is a type-1 region. Let $P$, $Q$ and $R$ be the components of $\mathbf{F}$ along the three axes respectively. Then,

$$div\mathbf{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$$

So,

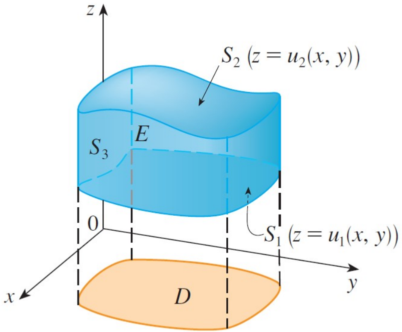
$$\int\int\int_E div\mathbf{F} dV = \int\int\int_E \frac{\partial P}{\partial x} dV + \int\int\int_E \frac{\partial Q}{\partial y} dV + \int\int\int_E \frac{\partial R}{\partial z} dV$$

Let $\mathbf{n}$ be the unit outward normal of S. Then,

$$\int\int_S \mathbf{F}.d\mathbf{S} = \int\int_S P\mathbf{i}.\mathbf{n} dS + \int\int_S Q\mathbf{j}.\mathbf{n} dS + \int\int_S R\mathbf{k}.\mathbf{n} dS$$

We proceed using the fact that $E$ is a type 1 region given by

$$E = \{(x, y, z) : (x, y) \in D, u_1(x, y) \leq z \leq u_2(x, y)\}$$




where $D$ is the projection of $E$ onto the xy plane.

Therefore, by definition of triple integral and Fundamental theorem of calculus, we have

$$\int\int\int_E \frac{\partial R}{\partial z} dV = \int\int_D [R(x,y,u_2(x,y) - R(x,y,u_1(x,y))] dA$$

From the figure, notice that on $S_3$, we have $\mathbf{k}.\mathbf{n}$=0. Thus,

$$\int\int_S R\mathbf{k}.\mathbf{n} dS = \int\int_{S_1} R\mathbf{k}.\mathbf{n} dS + \int\int_{S_2} R\mathbf{k}.\mathbf{n} dS$$

Also, since $S_1 : z = u_2(x,y)$ and $S_2 : z = u_1(x,y)$ we have

$$\int\int_S R\mathbf{k}.\mathbf{n} dS = \int\int_D [R(x,y,u_2(x,y) - R(x,y,u_1(x,y))] dA$$

Comparing ends our proof.

# Example

Find the flux of the field $\mathbf{F} = z\mathbf{i} + y\mathbf{j} + x\mathbf{k}$ over the unit sphere.
**Solution** We first find the divergence of $\mathbf{F}$ and then use divergence theorem on the unit sphere to get

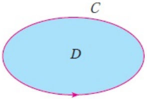
$$\int\int_S \mathbf{F}.d\mathbf{S} = \int\int\int_E div\mathbf{F}dV = \int\int\int_E 1dV$$

which is equal to the volume of the unit sphere.

# Summary

Green's Theorem

$$\iint_D \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA = \int_C P\,dx + Q\,dy$$

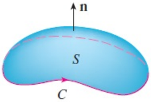



Double integral over a plane region D

Line integral around its plane boundary curve

Stokes' Theorem

$$\iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = \int_C \mathbf{F} \cdot d\mathbf{r}$$

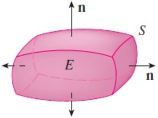



Surface integral over a surface S

Line integral around the boundary curve

Divergence Theorem

$$\iiint_E \operatorname{div} \mathbf{F}\, dV = \iint_S \mathbf{F} \cdot d\mathbf{S}$$




Volume integral of of the divergence over the region inside the surface

surface integral of a vector field

# THANK YOU