MA204: Mathematics IV

Partial Differential Equation (Second Order Linear PDE)

Introduction

The general form of a second order PDE is given by

$$F(x, y, z, z_x, z_y, z_{xx}, z_{xy}, z_{yy}) = 0.$$

In the broad sense, a second order PDE can be divided into two categories

Linear Second Oder PDE

Non-Linear Second Oder PDE

We here only concern about the second order linear PDEs.

Consider the second order linear equation in two independent variables

$$Az_{xx} + Bz_{xy} + Cz_{yy} + Dz_x + Ez_y + Fz + G = 0.$$

A function $z = z(x, y)$ that satisfies the above PDE is called a solution of the PDE.

Second order linear PDE

Considering

$$T(z) = Az_{xx} + Bz_{xy} + Cz_{yy} + Dz_x + Ez_y + Fz,$$

we can write the PDE in the operator form as

$$(T(z))(x,y) = -G = f(x,y).$$

One can easily note that the operator $T(z)$ is linear in z.

(1) Any linear combination of solutions of the linear PDE $(T(z))(x,y) = 0$ is also a solution.

(2) A solution $z = z(x,y)$ to the PDE $(T(z))(x,y) = 0$ is called general solution if it contains two arbitrary functions.

(3) If z_g is a general solution to $(T(u))(x,y) = 0$ and z_p is a particular solution to $(T(z))(x,y) = f(x,y)$, then $z_g + z_p$ is also a general solution to the PDE $(T(z))(x,y) = f(x,y)$.

Second order linear PDE

One-dimensional wave equation:

$$z_{tt}(x,t) = c^2 z_{xx}(x,t),$$

where c represents a physical quantity.

If $f(x,t) =$ vertical force per unit length at point x, at time t, then the wave equation becomes

$$z_{tt}(x,t) - c^2 z_{xx}(x,t) = F(x,t),$$

where $F(x,t) = \frac{1}{\rho} f(x,y)$ with ρ as the mass per unit length of the string.

Second order linear PDE

For the second order linear PDE

$$Az_{xx} + Bz_{xy} + Cz_{yy} + Dz_x + Ez_y + Fz = f(x, y),$$

the expression

$$Lz := Az_{xx} + Bz_{xy} + Cz_{yy},$$

containing the second derivatives, is called the **Principal part** of the equation.

The classification of the second order linear PDEs is based on this principal part of the PDE.

The classification of the quadratic equation of the form

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

as elliptic, parabolic, and hyperbolic based on the values of the discriminant $B^2 - 4AC$ as negative, zero, and positive, respectively.

The classification of the second order linear PDEs is motivated by this classification of quadratic equations.

Second order linear PDE

We first note that the coefficients A, B, C, D, E, F of the second order linear PDE

$$Az_{xx} + Bz_{xy} + Cz_{yy} + Dz_x + Ez_y + Fz = f(x, y).$$

may not be constants, but functions of x, y.

Thus at a point (x, y), the above equation is said to be

Hyperbolic if $B^2(x, y) - 4A(x, y)C(x, y) > 0$
Parabolic if $B^2(x, y) - 4A(x, y)C(x, y) = 0$
Elliptic if $B^2(x, y) - 4A(x, y)C(x, y) < 0$

For example,

Wave Equation: $z_{tt} - z_{xx} = 0$ is hyperbolic.
Laplace's Equation: $z_{xx} + z_{yy} = 0$ is elliptic.
Heat Equation: $z_t = z_{xx}$ is parabolic.

Problem

Problem: Classify the following second order linear PDEs:

(a) $2\frac{\partial^2 z}{\partial x^2} + 4\frac{\partial^2 z}{\partial x \partial y} + 3\frac{\partial^2 z}{\partial y^2} = 2$

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(c) $z_{xx} + 2z_{xy} + z_{yy} + 2z_x + 2z_y + z = 0.$

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(b) $r + 4s + 4t = 0$

(c) $z_{xx} + 2z_{xy} + z_{yy} + 2z_x + 2z_y + z = 0.$

(d) $2y^2 r - 2xys + x^2 t = \frac{y^2 p}{x} + \frac{x^2 q}{y}$

Second order linear PDE

Methods and techniques for solving PDEs:

(1) **Change of coordinates:** A PDE can be converted to an ODE or to an easier PDE by changing the coordinates of the problem.

(2) **Separation of variables:** A PDE in n independent variables is reduced to n ODEs.

(3) **Integral transforms:** A PDE in n independent variables is reduced to one in $(n-1)$ independent variables. Hence, a PDE in two variables could be changed to an ODE.

(4) **Numerical Methods:**

Change of coordinates: Canonical Transformations

In this method, we change the variables as

$$\zeta = \zeta(x, y) \text{ and } \eta = \eta(x, y)$$

in such a way that ζ and η are continuously differentiable and the Jacobian

$$J = \frac{\partial(\zeta, \eta)}{x, y} = \begin{vmatrix} \zeta_x & \zeta_y \\ \eta_x & \eta_y \end{vmatrix} \neq 0.$$

We then compute

$$\begin{aligned}
z_x &= z_\zeta \zeta_x + z_\eta \eta_x \\
z_y &= z_\zeta \zeta_y + z_\eta \eta_y \\
z_{xx} &= z_{\zeta\zeta} \zeta_x^2 + 2 z_{\zeta\eta} \zeta_x \eta_y + z_{\eta\eta} \eta_x^2 + z_\zeta \zeta_{xx} + z_\eta \eta_{xx} \\
z_{yy} &= z_{\zeta\zeta} \zeta_y^2 + 2 z_{\zeta\eta} \zeta_y \eta_x + z_{\eta\eta} \eta_y^2 + z_\zeta \zeta_{yy} + z_\eta \eta_{yy} \\
z_{xy} &= z_{\zeta\zeta} \zeta_x \zeta_y + z_{\zeta\eta} (\zeta_x \eta_y + \zeta_y \eta_y) + z_{\eta\eta} \eta_x \eta_y + z_\zeta \zeta_{xy} + z_\eta \eta_{xy}.
\end{aligned}$$

Change of coordinates: Canonical Transformations

Substituting these values into the equation

$$Az_{xx} + Bz_{xy} + Cz_{yy} + Dz_x + Ez_y + Fz = f(x,y)$$

to obtain a PDE in the variables ζ and η as

$$\bar{A}z_{\zeta\zeta} + \bar{B}z_{\zeta\eta} + \bar{C}z_{\eta\eta} + \bar{D}z_{\zeta} + \bar{E}z_{\eta} + Fz = \bar{f}(\zeta,\eta),$$

where

$$\begin{aligned}
\bar{A} =& A\zeta_x^2 + B\zeta_x\zeta_y + C\zeta_y^2 \\
\bar{B} =& 2A\zeta_x\eta_x + B(\zeta_x\eta_y + \zeta_y\eta_x) + 2C\zeta_y\eta_y \\
\bar{C} =& A\eta_x^2 + B\eta_x\eta_y + C\eta_y^2 \\
\bar{D} =& A\zeta_{xx}B\zeta_{xy} + C\zeta_{yy} + D\zeta_x + E\zeta_y \\
\bar{E} =& A\eta_{xx} + B\eta_{xy} + C\eta_{yy} + D\eta_x + E\zeta_y \\
\bar{F} =& F \text{ and } \bar{f} = f.
\end{aligned}$$

Change of coordinates: Canonical Transformations

It is easy to note that

$$\begin{pmatrix} 2\bar{A} & \bar{B} \\ \bar{B} & 2\bar{C} \end{pmatrix} = \begin{pmatrix} \zeta_x & \zeta_y \\ \eta_x & \eta_y \end{pmatrix} \begin{pmatrix} 2A & B \\ B & 2C \end{pmatrix} \begin{pmatrix} \zeta_x & \zeta_y \\ \eta_x & \eta_y \end{pmatrix},$$

and hence

$$\bar{B}^2 - 4\bar{A}\bar{C} = J^2(B^2 - 4AC).$$

Thus the form of the PDE remains invariant even after coordinate transformation.

The transformation given by $\zeta = \zeta(x, y)$ and $\eta = \eta(x, y)$ is called **canonical transformation or characteristics** and the reduced form of the PDE is called **canonical form**.

Canonical forms

Thus a second order linear PDE takes one of the forms canonical forms:

(1) **Hyperelliptic case:**

(a) $z_{\zeta\zeta} - z_{\eta\eta} = \phi(\zeta, \eta, z, z_\zeta, z_\eta)$

(b) $z_{\zeta\eta} = \phi(\zeta, \eta, z, z_\zeta, z_\eta)$

(2) **Elliptic case:** $z_{\zeta\zeta} + z_{\eta\eta} = \phi(\zeta, \eta, z, z_\zeta, z_\eta)$

(3) **Parabolic case:**

(a) $z_{\zeta\zeta} = \phi(\zeta, \eta, z, z_\zeta, z_\eta)$

(b) $z_{\eta\eta} = \phi(\zeta, \eta, z, z_\zeta, z_\eta)$

Hyperbolic PDE

Hyperbolic PDEs: $\bar{B}^2 - 4\bar{A}\bar{C} > 0$

A natural choice is $\bar{A} = 0$ and $\bar{C} = 0$.

Thus we obtain the algebraic equations

$$A\frac{\zeta_x}{\zeta_y}^2 + B\frac{\zeta_x}{\zeta_y} + C = 0$$

and

$$A\frac{\eta_x}{\eta_y}^2 + B\frac{\eta_x}{\eta_y} + C = 0.$$

Hyperbolic PDE

Solving these two equations for $\frac{\zeta_x}{\zeta_y}$ and $\frac{\eta_x}{\eta_y}$, and then using $d\zeta = \zeta_x dx + \zeta_y dy$ and $d\eta = \eta_x dx + \eta_y dy$, respectively, we obtain the ODEs

$$\frac{dy}{dx} = -\frac{\zeta_x}{\zeta_y} \text{ and } \frac{dy}{dx} = -\frac{\eta_x}{\eta_y}.$$

Thus we obtain two families of characteristics $\zeta(x, y) = c_1$ and $\eta(x, y) = c_2$.

Substituting these in the transformed PDE, we obtain a canonical form for the hyperbolic PDE.

Problem

Ex: Write the canonical form pf $z_{xx} - z_{yy} = 0$, and then find the solution.

Parabolic PDE

Parabolic PDEs: $\bar{B}^2 - 4\bar{A}\bar{C} = 0$

A natural choice is $\bar{B} = 0$ and either $\bar{A} = 0$ or $\bar{C} = 0$.

Thus we obtain the algebraic equations

$$A\frac{\zeta_x}{\zeta_y}^2 + B\frac{\zeta_x}{\zeta_y} + C = 0$$

or

$$A\frac{\eta_x}{\eta_y}^2 + B\frac{\eta_x}{\eta_y} + C = 0.$$

Parabolic PDE

Solving the equation for $\frac{\zeta_x}{\zeta_y}$ (or $\frac{\eta_x}{\eta_y}$), and then using $d\zeta = \zeta_x dx + \zeta_y dy$ (or $d\eta = \eta_x dx + \eta_y dy$), we obtain the ODE

$$\frac{dy}{dx} = -\frac{\zeta_x}{\zeta_y} (\text{ or } \frac{dy}{dx} = -\frac{\eta_x}{\eta_y}).$$

Thus we obtain a family of characteristics as $\zeta(x, y) = c_1$ (or $\eta(x, y) = c_2$).

The other family of characteristics η (or ζ) is choosen any arbitrary function of x and y such that $J = \frac{\partial(\zeta,\eta)}{\partial(x,y)} \neq 0$.

Substituting these in the transformed PDE, we obtain a canonical form for the parabolic PDE.

Problem

Ex: Write the canonical form of $y^2 z_{xx} - 2xyz_{xy} + x^2 z_{yy} = \frac{y^2}{x} z_x + \frac{x^2}{y} z_y$, and then find the solution.

Elliptic PDE

Elliptic PDEs: $\bar{B}^2 - 4\bar{A}\bar{C} < 0$. In this case the algebraic equation

$$Ax^2 + Bx + C = 0$$

provide us the two characteristic equations

$$\frac{dy}{dx} = \frac{B - \sqrt{B^2 - 4AC}}{2A} \text{ and } \frac{dy}{dx} = \frac{B + \sqrt{B^2 - 4AC}}{2A}.$$

Note that the characteristic equations give us complex conjugate coordinates as ζ and η.

As a result, we introduce new real variables

$$\alpha = \frac{\zeta + \eta}{2} \text{ and } \beta == \frac{\zeta + \eta}{2}.$$

Under the transformation $(x, y) \to (\alpha, \beta)$, the canonical form for elliptic PDEs is obtained by

$$z_{\alpha\alpha} + z_{\beta\beta} = \phi(\alpha, \beta, z, z_\alpha, z_\beta).$$

Problem

Ex: Write the canonical form of of the PDE $y^2 z_{xx} + x^2 z_{yy} = 0$.

Elliptic PDE

Note that method of characteristics is not found suitable for elliptic equations since even after using the transformation, that is, the characteristics (in new variables), the equation gets reduced to Laplace's equation form only.

In other words, the given equation gets reduced marginally only with two double derivatives still remaining.

For this reason we will not apply method of characteristics to elliptic equations.

Thank you

Thank You!!