MA 102
Calculus
Tutorial–4

(1) Verify both forms of Green's Theorem for the vector field $F(x, y) = (x - y)i + xj$ and the region R bounded by the unit circle $C\colon r(t) = \cos(t)i + \sin(t)j, 0 \le t \le 2\pi$.
Ans: We have, $M = x - y = \cos t - \sin t$; $N = x = \cos t$, and $dx = -\sin t dt, dy = \cos t dt$. For the Normal Form of Green's theorem, we have $\int_c (M dy - N dx) = \int_0^{2\pi} \cos^2 t dt = \pi = \iint_R (\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}) dx\ dy$. For the Tangential Form of Green's theorem, we have $\int_c (M dx + N dy) = \int_0^{2\pi} 1 - \frac{\sin 2t}{2} dt = 2\pi = \iint_R (\frac{\partial N}{\partial x} + \frac{\partial m}{\partial y}) dx\ dy$.

(2) Use Stokes' Theorem to evaluate $\oint_C \vec{F} \cdot \vec{dr}$ if $F = xz\ i + xy\ j + 3xz\ k$ and C is the boundary of the portion of the plane $2x + y + z = 2$ in the first octant, traversed counter clock wise as viewed from above.
Ans: The plane is the level surface $f(x, y, z) = 2$ of the function $f(x, y, z) = 2x + y + z$. The unit vector $n = \frac{\nabla f}{|\nabla f|} = \frac{1}{\sqrt{6}(2i + j + k)}$. $Curl(F) = (x - 3z)j + yk = (7x + 3y - 6)j + yk$. $Curl(F) \cdot n = \frac{1}{\sqrt{6}}(7x + 4y - 6)$. Let $r(u, v) = ui + vj + (2 - 2u - v)k$, $dS = |r_u \times r_v| dudv$. Hence, $\oint_C \vec{F} \cdot \vec{dr} = \int_0^1 \int_0^{2-2x} Curl(F) \cdot ndS = -1$.

(3) Find the flux of $F = xy\ i + yz\ j + xz\ k$ outward through the surface of the cube cut from the first octant by the planes $x = 1$,$y = 1$ and $z = 1$.
Ans: $Div(F) = x + y + z$. The Flux $= \int_0^1 \int_0^1 \int_0^1 Div(F) dV = \frac{3}{2}$.

(4) Evaluate the line integral $\oint_C \vec{F} \cdot \vec{dr}$ when $\vec{F} = \langle z^2, y^2, xy \rangle$, C is the triangle defined by $(1, 0, 0)$, $(0, 1, 0)$, and $(0, 0, 2)$, and C is traversed counter clockwise as viewed from the origin.
Ans: Vector from $(1, 0, 0)$ to $(0, 1, 0)$ is $v_1 = (-1, 1, 0)$, and vector from $(1, 0, 0)$ to $(0, 0, 2)$ is $v_2 = (-1, 0, 2)$. So the normal to the plane S is $v = v_1 \times v_2 = (2, 2, 1)$. Hence $n = \frac{v}{|v|} = \frac{1}{3}(2, 2, 1)$ but we take $n = -\frac{1}{3}(2, 2, 1)$ because of the orientation of S. From the normal v and the point $(1, 0, 0)$, we get the equation of the plane S is $2x + 2y + z = 2$, and so $dS = 3dA$. Also, $Curl\ F = (x, 2z - y, 0)$. Therefore, $\oint_C \vec{F} \cdot \vec{dr} = \iint_S (Curl\ F).n\ dS = -\int_0^1 \int_0^{1-x} (8 - 6x - 10y) dydx = -\frac{4}{3}$.

(5) Evaluate the flux integral $\iint_S Curl(F) \cdot \vec{n}\ dS$ where $F = \langle 2z - y, x - z, y - x \rangle$ and S is the portion of the sphere $x^2 + y^2 + z^2 = 9$ with $z \ge y$ and $\vec{n}$ points away from the origin.
Ans: The boundary C of S is the circle obtained by intersecting the sphere with the plane $z = y$, but we will write C as the boundary of a disc D in the plane $y = z$. By Stoke's Theorem, we have, $\iint_S (Curl\ F).n\ d\sigma = \oint_C \vec{F} \cdot \vec{dr} = \iint_D (Curl\ F).n_2\ d\sigma$ where n_2 is the normal to the disc D, i.e., to the plane $y = z$ and $n_2 = \frac{1}{\sqrt{2}}(0, -1, 1)$. Also, $Curl\ F = 2i + 3j + 2k$, and $Curl\ F.n_2 = -\frac{1}{\sqrt{2}}$. Therefore, $\iint_D (Curl\ F).n_2\ d\sigma = \iint_D -\frac{1}{\sqrt{2}} d\sigma = -\frac{1}{\sqrt{2}} Area(D) = -\frac{9}{\sqrt{2}}\pi$.

(6) Let S be the surface of the cube $D\colon 0 \le x \le 1, 0 \le y \le 1, 0 \le z \le 1$ and $F = (e^x + z)i + (y^2 - x)j - xe^y k$. Compute the outward flux $\iint_S F \cdot \vec{n}\ dS$.
Ans: $\iint_S F \cdot \vec{n}\ dS = \iiint_D div\ F\ dV = \int_0^1 \int_0^1 \int_0^1 (e^x + 2y) dxdydz = e$.

(7) Use the divergence theorem to find the outward flux $\iint_S F \cdot \vec{n}\ dS$ of the vector field $F = x^3 i + y^3 j + z^3 k$ with D the region bounded by the sphere $S\colon x^2 + y^2 + z^2 = a^2$.
Ans: $\iint_S F \cdot \vec{n}\ dS = \iiint_D div\ F\ dV = \iiint_D (3x^2 + 3y^2 + 3z^2)\ dV = \int_0^{2\pi} \int_0^{\pi} \int_0^a (3\rho^2)\rho^2 \sin\phi\ d\rho\ d\phi\ d\theta = \frac{12\pi}{5} a^5$.