Control Systems

Subject Code: EC380

Lecture 2: Mathematical Model of Physical Systems

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Mathematical Models of Systems

Objective: To obtain a quantitative mathematical models of physical systems to design and analyze control systems

Dynamic Model:

a) Ordinary differential equations (linear/nonlinear & Time Invariant /Variant)

b) Transfer functions (Linear Time Invariant (LTI))

c) State-pace(linear/nonlinear & Time Invariant /Variant)

Linearization: To convert a nonlinear model into a linear model using Tailor series expansion technique

Transfer Function: The input–output relationship in Laplace domain for initially rest components and subsystems

Transfer function blocks can be organized into block diagrams or signal-flow graphs to graphically depict the interconnections.

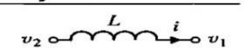
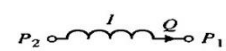
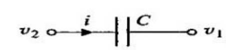
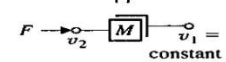
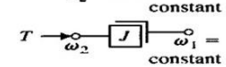
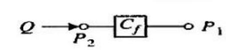
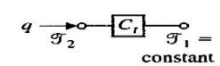
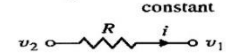
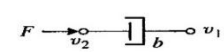
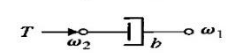
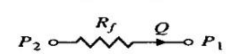
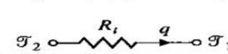
Six Step Approach to Dynamic System Problems

- Define the system and its components
- Formulate the mathematical model and list the necessary assumptions
- Write the differential equations describing the model
- Solve the equations for the desired output variables
- Examine the solutions and the assumptions
- If necessary, reanalyze or redesign the system

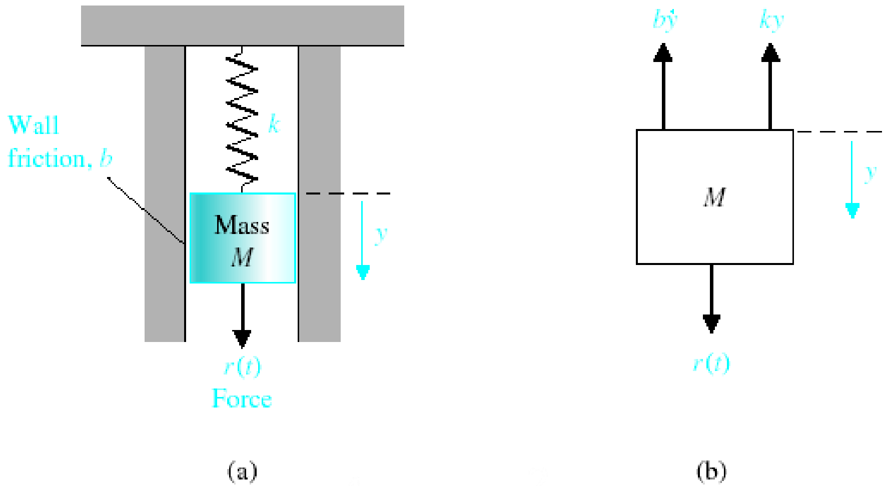
Through- and Across-Variables for Physical Systems

System	Variable Through Element	Integrated Through-Variable	Variable Across Element	Integrated Across-Variable
Electrical	Current, i	Charge, q	Voltage difference, v_{21}	Flux linkage, λ_{21}
Mechanical translational	Force, F	Translational momentum, P	Velocity difference, v_{21}	Displacement difference, y_{21}
Mechanical rotational	Torque, T	Angular momentum, h	Angular velocity difference, ω_{21}	Angular displacement difference, θ_{21}
Fluid	Fluid volumetric rate of flow, Q	Volume, V	Pressure difference, P_{21}	Pressure momentum, γ_{21}
Thermal	Heat flow rate, q	Heat energy, H	Temperature difference, $\mathcal{T}_{21}$	

Governing Differential Equations for Ideal Elements

Type of Element	Physical Element	Governing Equation	Energy E or Power $\mathcal{P}$	Symbol
Inductive storage	Electrical inductance	$v_{21} = L\dfrac{di}{dt}$	$E = \dfrac{1}{2}Li^2$	
	Translational spring	$v_{21} = \dfrac{1}{k}\dfrac{dF}{dt}$	$E = \dfrac{1}{2}\dfrac{F^2}{k}$	
	Rotational spring	$\omega_{21} = \dfrac{1}{k}\dfrac{dT}{dt}$	$E = \dfrac{1}{2}\dfrac{T^2}{k}$	
	Fluid inertia	$P_{21} = I\dfrac{dQ}{dt}$	$E = \dfrac{1}{2}IQ^2$	
Capacitive storage	Electrical capacitance	$i = C\dfrac{dv_{21}}{dt}$	$E = \dfrac{1}{2}C{v_{21}}^2$	
	Translational mass	$F = M\dfrac{dv_2}{dt}$	$E = \dfrac{1}{2}M{v_2}^2$	
	Rotational mass	$T = J\dfrac{d\omega_2}{dt}$	$E = \dfrac{1}{2}J{\omega_2}^2$	
	Fluid capacitance	$Q = C_f\dfrac{dP_{21}}{dt}$	$E = \dfrac{1}{2}C_f{P_{21}}^2$	
	Thermal capacitance	$q = C_t\dfrac{d\mathcal{T}_2}{dt}$	$E = C_t\mathcal{T}_2$	
Energy dissipators	Electrical resistance	$i = \dfrac{1}{R}v_{21}$	$\mathcal{P} = \dfrac{1}{R}{v_{21}}^2$	
	Translational damper	$F = bv_{21}$	$\mathcal{P} = b{v_{21}}^2$	
	Rotational damper	$T = b\omega_{21}$	$\mathcal{P} = b{\omega_{21}}^2$	
	Fluid resistance	$Q = \dfrac{1}{R_f}P_{21}$	$\mathcal{P} = \dfrac{1}{R_f}{P_{21}}^2$	
	Thermal resistance	$q = \dfrac{1}{R_t}\mathcal{T}_{21}$	$\mathcal{P} = \dfrac{1}{R_t}\mathcal{T}_{21}$	

Differential Equation of Physical Systems

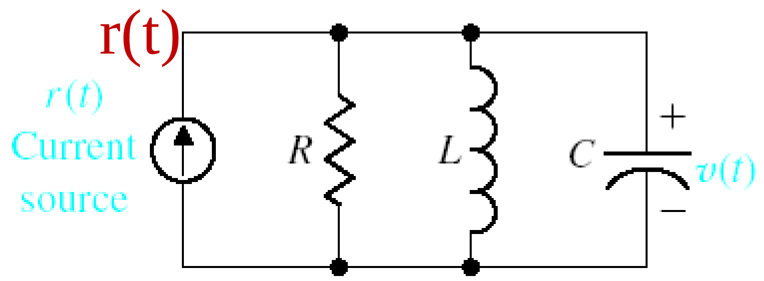


(a) Spring-mass-damper system.
(b) Free-body diagram.

Mathematical Model using Newton's laws for mechanical systems:

$$M\frac{d^2y(t)}{dt^2} + b\,\frac{dy(t)}{dt} + ky(t) = r(t)$$

Differential Equation of Physical Systems



RLC circuit.

Mathematical Model using Kirchhoff’s laws for electrical systems:

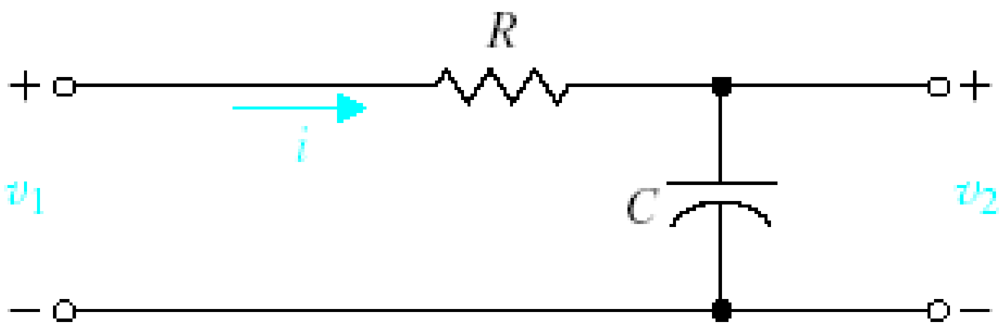
$$\frac{v(t)}{R} + C \cdot \frac{d}{dt} v(t) + \frac{1}{L} \cdot \int_{0}^{t} v(t)\, dt = r(t)$$

Transfer Function

The Transfer function of a LTI system is defined as the ratio of the Laplace transform of the output variable to the Laplace transform of the input variable under the assumption that all initial condition are zero.

Transfer function, $G(s) = \frac{Y(s)}{X(s)}$

The Transfer Function of R-C Circuit



An RC network.

Differential equation Model of the RC circuit using KVL:

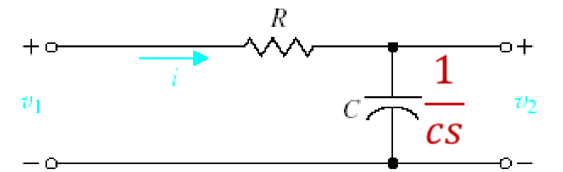
$$Ri + \frac{1}{c}\int_0^t i dt = v_1$$

$$\frac{1}{c}\int_0^t i dt = v_2$$

The Transfer Function of R-C Circuit

$$Ri + \frac{1}{c}\int_0^t i\,dt = v_1$$

$$\frac{1}{c}\int_0^t i\,dt = v_2$$



An RC network.

Taking Laplace Transform of above differential equations:

$$V_1(s) = \left(R + \frac{1}{Cs}\right)\cdot I(s) \qquad Z_1(s) = R$$

$$V_2(s) = \left(\frac{1}{Cs}\right)\cdot I(s) \qquad Z_2(s) = \frac{1}{Cs}$$

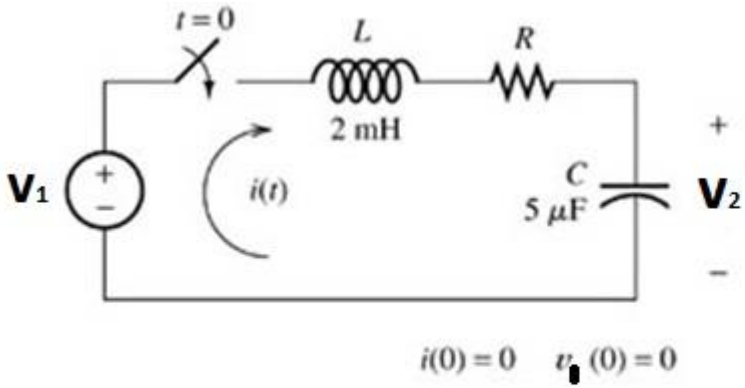
$$\frac{V_2(s)}{V_1(s)} = \frac{\frac{1}{Cs}}{R + \frac{1}{Cs}} = \frac{Z_2(s)}{Z_1(s) + Z_2(s)}$$

Transfer Function, $G(s) = \frac{V_2(s)}{V_1(s)} = \frac{1}{RCs+1}$

The Transfer Function of R-L-C Series Circuit

$$L\frac{di}{dt} + Ri + \frac{1}{c}\int_0^t idt = v_1$$

$$\frac{1}{c}\int_0^t idt = v_2$$



$$LC\frac{d^2v_2}{dt^2} + RC\frac{dv_2}{dt} + v_2 = v_1$$

Transfer Function, $G(s) = \frac{V_2(s)}{V_1(s)} = \frac{1}{LCs^2+RCs+1}$

Transfer Function of n^{th}order systems

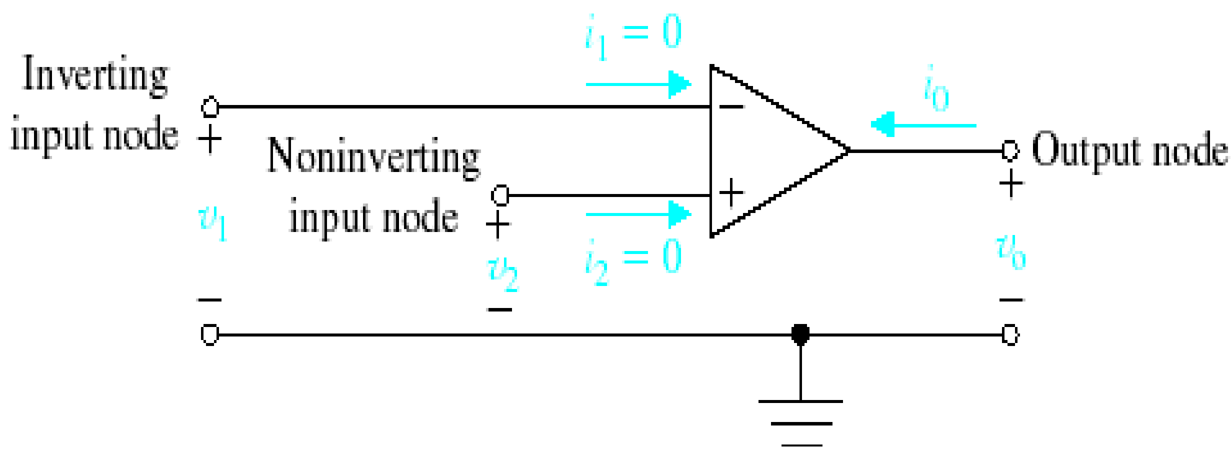
Consider the linear time-invariant system defined by the following differential equation:

$$a_0 \overset{(n)}{y} + a_1 \overset{(n-1)}{y} + \cdots + a_{n-1}\dot{y} + a_n y$$
$$= b_0 \overset{(m)}{x} + b_1 \overset{(m-1)}{x} + \cdots + b_{m-1}\dot{x} + b_m x \qquad (n \geq m)$$

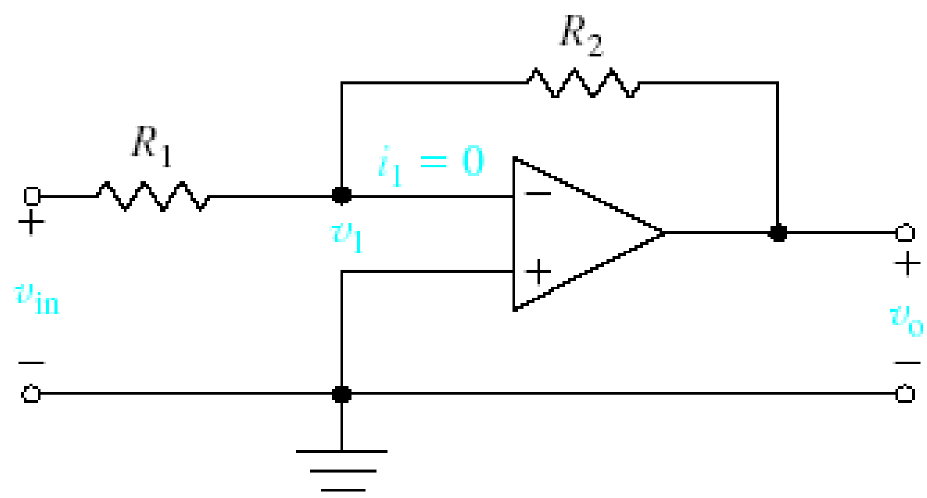
where y is the output of the system and x is the input. The transfer function of this system is the ratio of the Laplace transformed output to the Laplace transformed input when all initial conditions are zero, or

$$\text{Transfer function} = G(s) = \left.\frac{\mathscr{L}[\text{output}]}{\mathscr{L}[\text{input}]}\right|_{\text{zero initial conditions}}$$
$$= \frac{Y(s)}{X(s)} = \frac{b_0 s^m + b_1 s^{m-1} + \cdots + b_{m-1}s + b_m}{a_0 s^n + a_1 s^{n-1} + \cdots + a_{n-1}s + a_n}$$

The Transfer Function of Active Filters

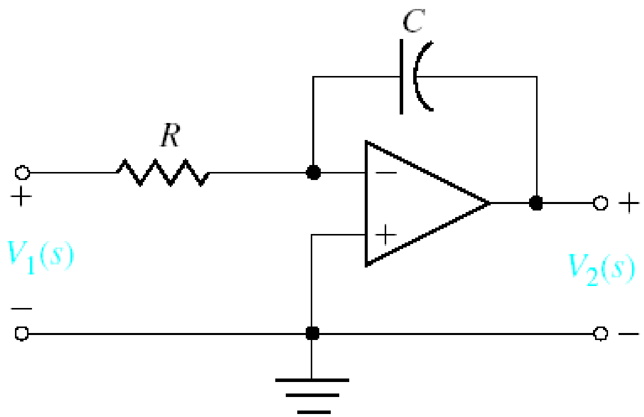


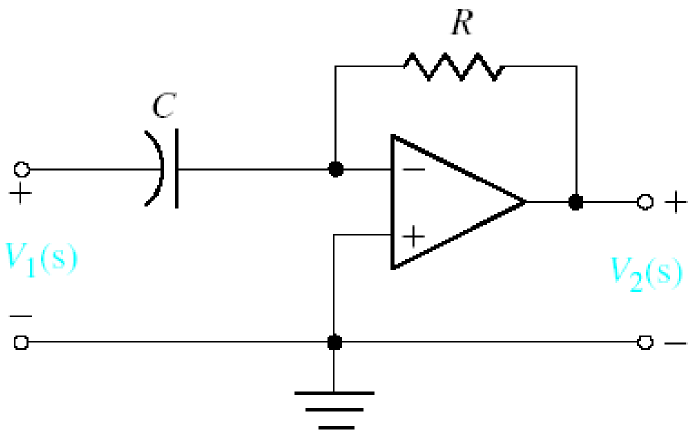
The ideal op-amp



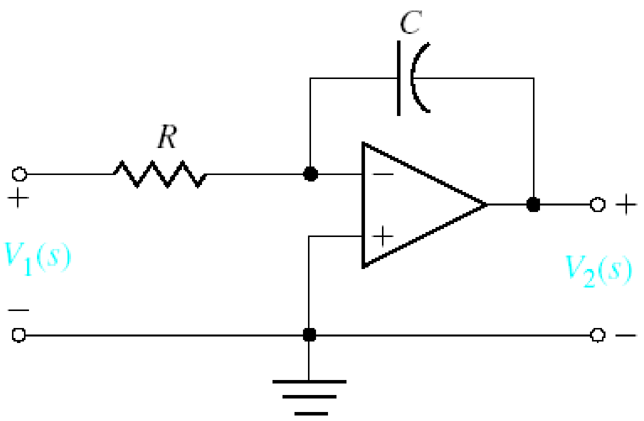
An inverting amplifier operating with ideal conditions.

The Transfer Function of Active Filters

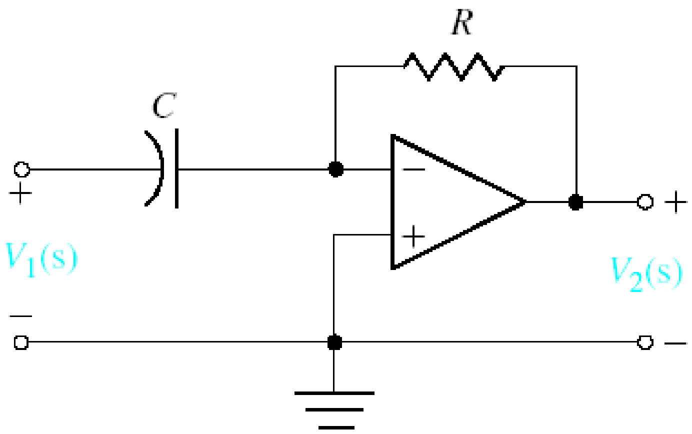




The Transfer Function of Active Filters

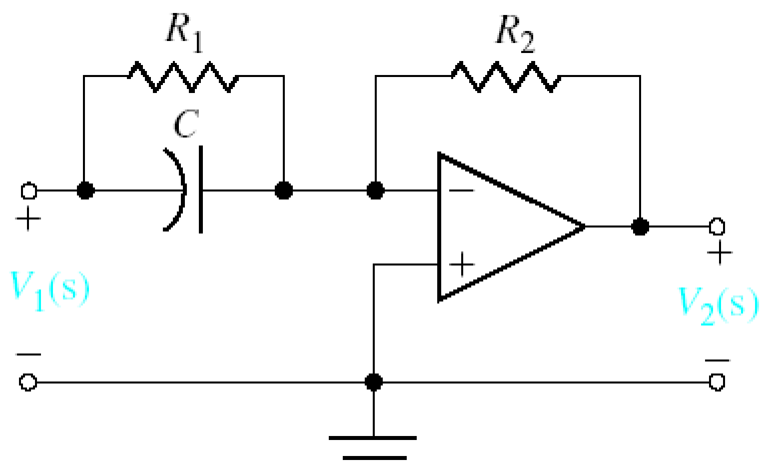


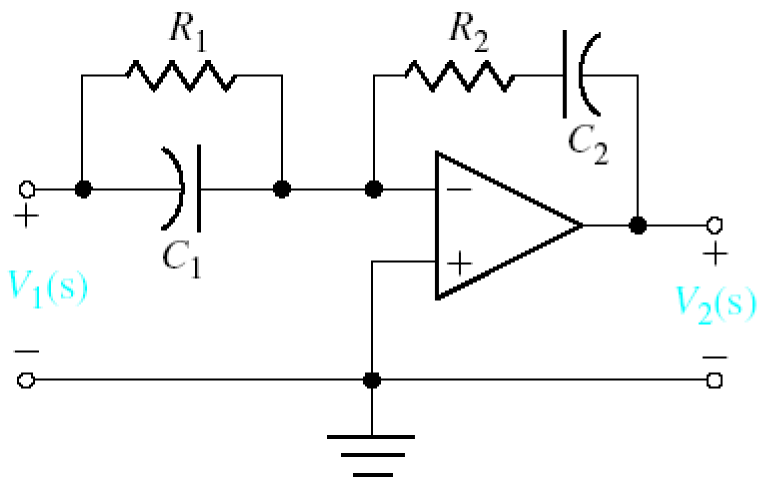
$$\frac{V_2(s)}{V_1(s)} = \frac{-1}{RCs}$$



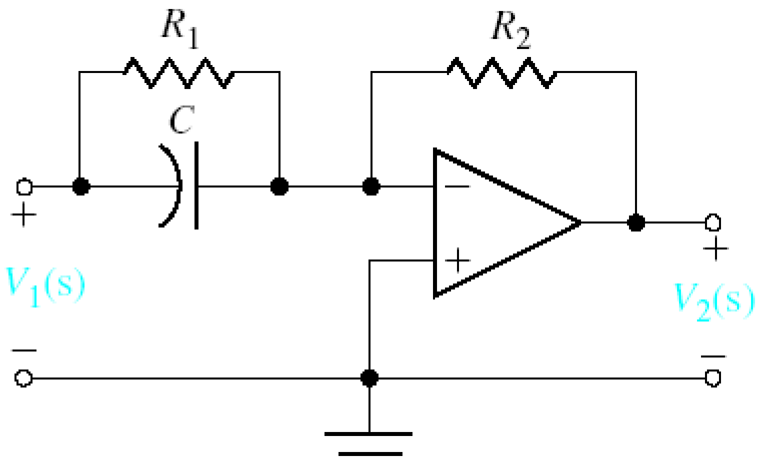
$$\frac{V_2(s)}{V_1(s)} = -RCs$$

The Transfer Function of Active Filters

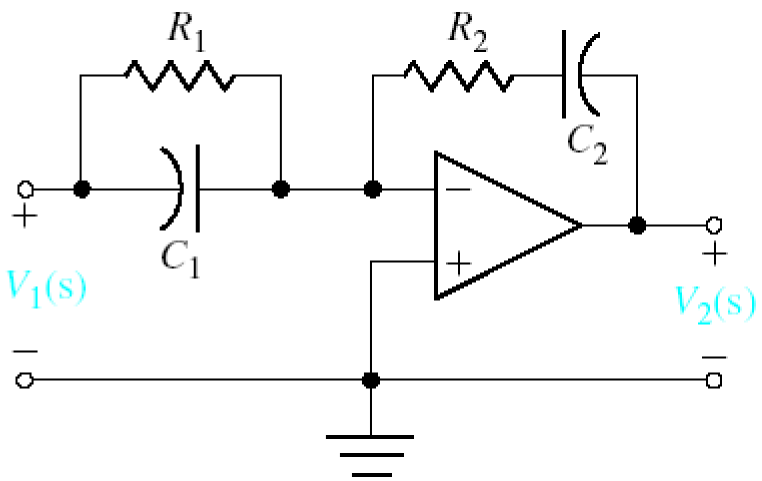




The Transfer Function of Active Filters



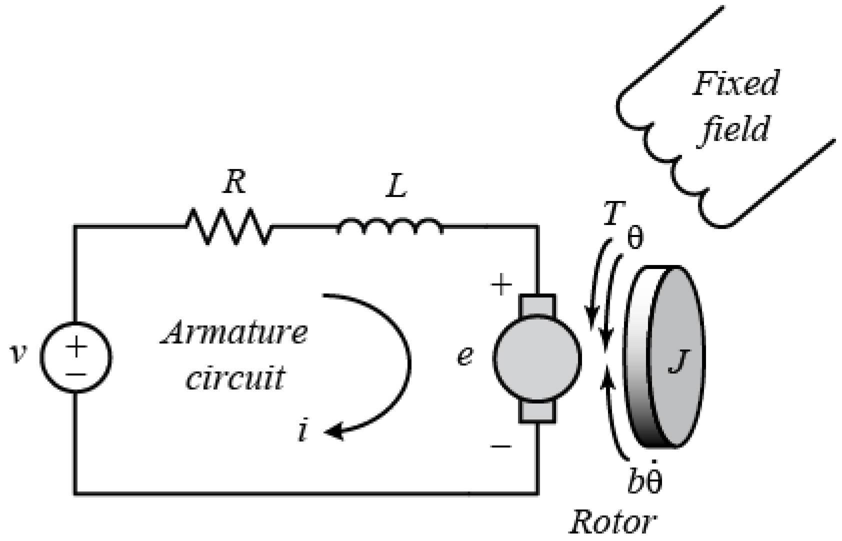
$$\frac{V_2(s)}{V_1(s)} = \frac{-R_2\left(R_1 \cdot C \cdot s + 1\right)}{R_1}$$



$$\frac{V_2(s)}{V_1(s)} = \frac{-\left(R_1 \cdot C_1 \cdot s + 1\right)\left(R_2 \cdot C_2 \cdot s + 1\right)}{R_1 \cdot C_2 \cdot s}$$

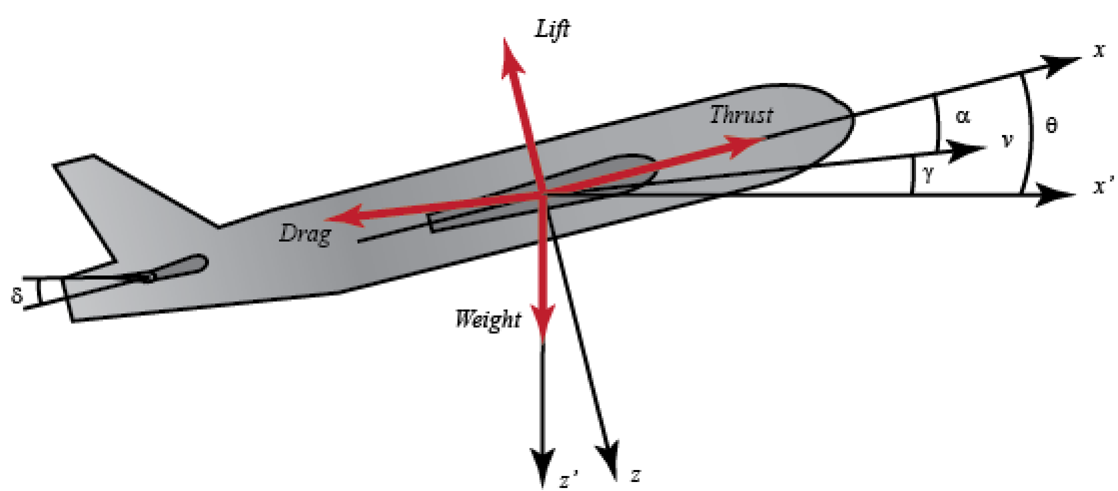
DC Motor Control

The transfer function for a DC motor control system, used in robotics and automation, can be given as: Transfer Function: $G(s) = K / (\tau s + 1)$ Where 'K' is the motor's gain, 'τ' is the time constant, and 's' is the complex frequency variable.



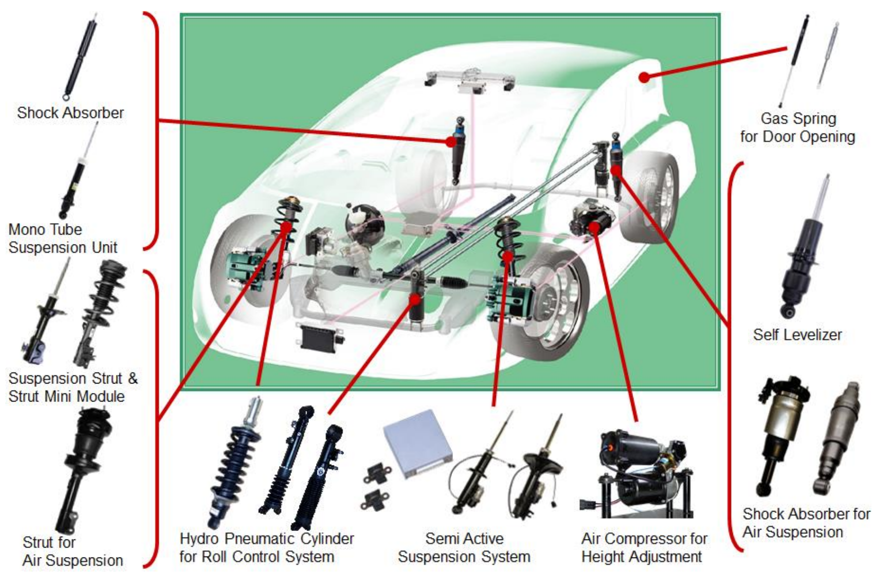
Aircraft Pitch Dynamics

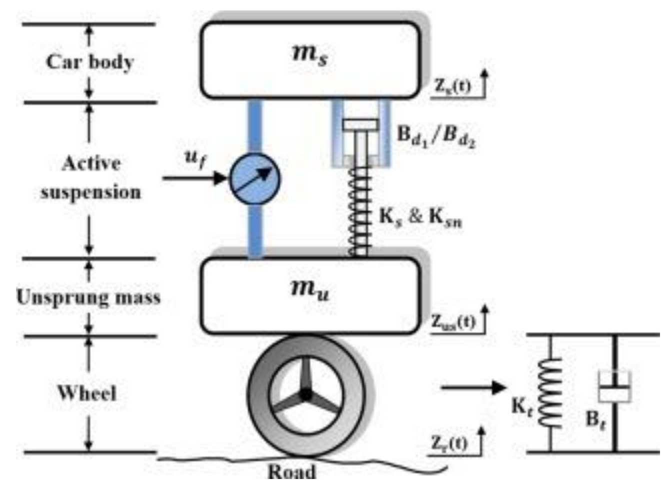
The transfer function for the pitch dynamics of an aircraft, which describes how the pitch angle responds to control inputs, can be given as: Transfer Function: G(s) = K / (Ts^2 + ζs + 1) Where 'K' is the aircraft's gain, 'T' is the time constant, and 'ζ' is the damping ratio.



Active Suspension System

The transfer function for an active suspension system in vehicles, which models the suspension response to road disturbances, can be given as: Transfer Function: G(s) = K / (ms^3 + bs^2 + ks) Where 'm' is the vehicle's mass, 'b' is the damping coefficient, and 'k' is the spring constant.



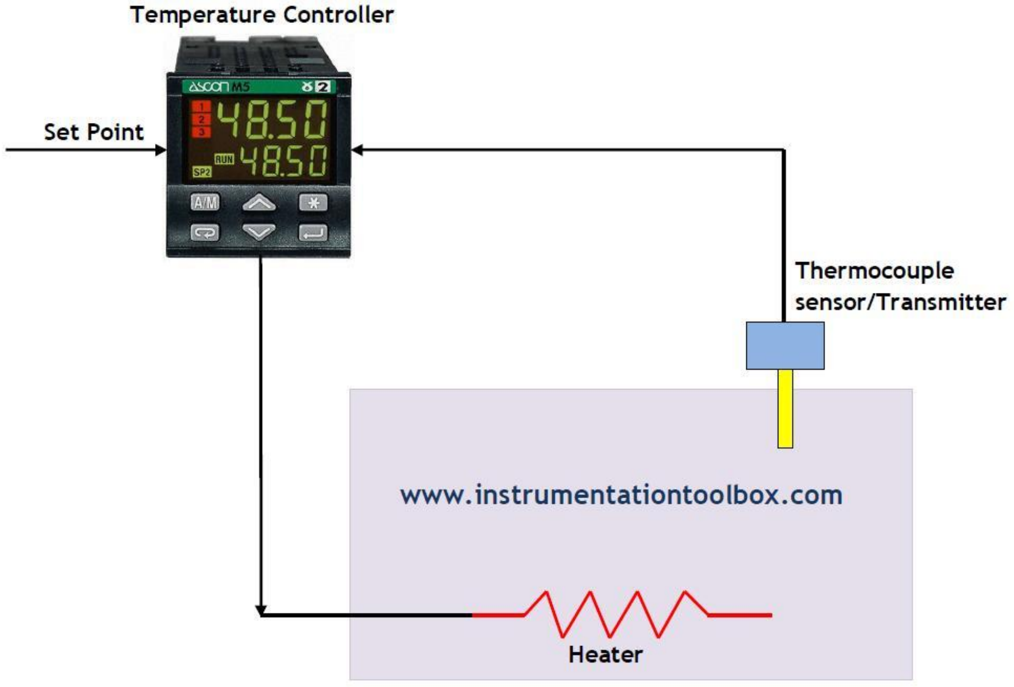


Temperature Control in an Oven

The transfer function for a temperature control system in an oven, which describes how the oven's temperature responds to heating inputs and external disturbances, can be given as:

Transfer Function: $G(s) = K / (\tau s + 1)$

Where 'K' is the oven's gain, 'τ' is the time constant, and 's' is the complex frequency variable.

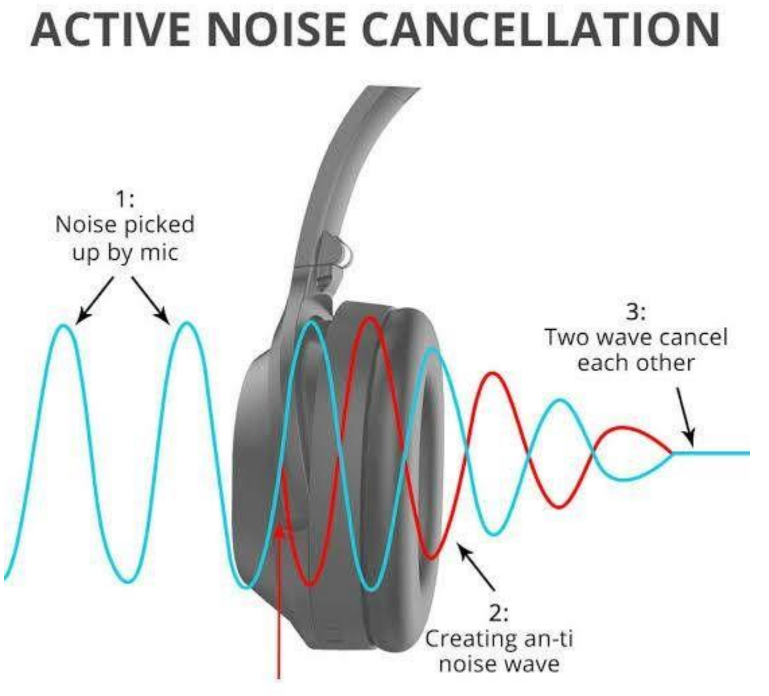


Active Noise Cancellation

The transfer function for an active noise cancellation system, used to reduce unwanted noise in headphones or speakers, can be given as:

Transfer Function: $G(s) = K / (s^2 + 2\zeta\omega_n s + \omega_n^2)$

Where 'K' is the gain, 'ζ' is the damping ratio, 'ω_n' is the natural frequency, and 's' is the complex frequency variable.



Epidemic, Rumour, and Information Diffusion Model

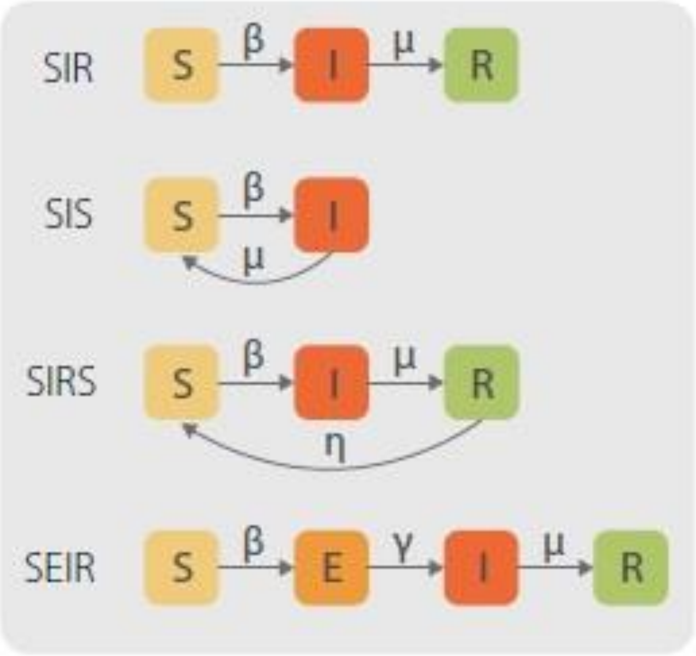
SIS Model:

$$\frac{d\rho^I}{dt} = \beta\rho^I\rho^S - \mu\rho^I$$
$$\frac{d\rho^S}{dt} = -\beta\rho^I\rho^S$$

The number of infectious individuals grows exponentially if

$$\beta - \mu > 0 \quad \Rightarrow \quad R_0 = \frac{\beta}{\mu} > 1,$$

where we have defined the *basic reproduction number* R_0 as the average number of secondary infections caused by a primary case introduced in a fully susceptible population (Anderson and May, 1992).



Reference: R. Pastor-Satorras, C. Castellano, P. V. Mieghem, and A. Vespignani, “Epidemic processes in complex networks,” *Rev. Mod. Phys.*, vol. 87, no. 3, pp. 925–979, 2015.

Important Links for Modern Applications of Control Systems

Journals:

1. **IEEE Control Systems Magazine**
2. Annual Reviews in Control
3. IEEE Transactions on Automatic Control
4. IEEE Transactions on Control of Network Systems
5. IEEE Transactions on Control Systems Technology
6. Automatica
7. IEEE Control Systems Letters

etc.

Conferences:

1. IEEE ACC
2. IEEE CDC
3. IFACWC
4. IEEE ICRA
5. IEEE IROS
6. IEEE ECC
7. IEEE ICC

etc.

Labs:

1. MIT Institute for Data, Systems, and Society
2. http://motion.me.ucsb.edu/
3. Robotics, Aerospace and Information Networks Lab
4. http://magnus.ece.gatech.edu/index_projects.html
5. R**obert** Bosch **Centre** for **Cyber-physical** Systems

etc.