Multiple Integrals

Double and Iterated Integrals over Rectangles

Note. In this section we extend the idea of integral to functions of two variables $f(x, y)$ over a bounded rectangle R in the plane.

Definition. Let $f(x, y)$ be a function defined on a rectangular region $R = \{(x, y) \mid x \in [a, b], y \in [c, d]\}$. Subdivide R into small rectangles using a network of lines parallel to the x- and y-axes. The lines divide R into n rectangular pieces, where the number of pieces n gets large as the width and height of each piece gets small. These rectangles form a *partition* of R. A small rectangular piece of width Δx and height Δy has area $\Delta A = \Delta x \Delta y$. If we number the small pieces partitioning R in some order, then their areas are given by numbers $\Delta A_1, \Delta A_2, \ldots, \Delta A_n$, where

ΔA_k is the area of the kth rectangle.

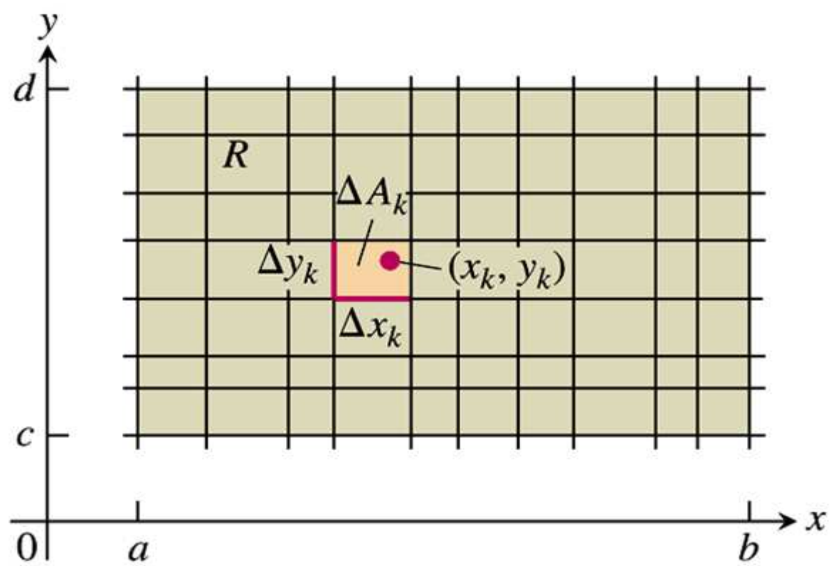


Figure 1

Definition. To form a *Riemann sum* over R, we choose a point (x_k, y_k) in the kth small rectangle, multiply the value of f at the point by the area ΔA_k and add together the products:

$$S_n = \sum_{k=1}^{n} f(x_k, y_k)\Delta A_k.$$

Depending on how we pick (x_k, y_k) in the kth small rectangle, we get different values for S_n.

Note. A Riemann sum is a "good" approximation of the volume above R and below $z = f(x, y)$ when the ΔA's are small.

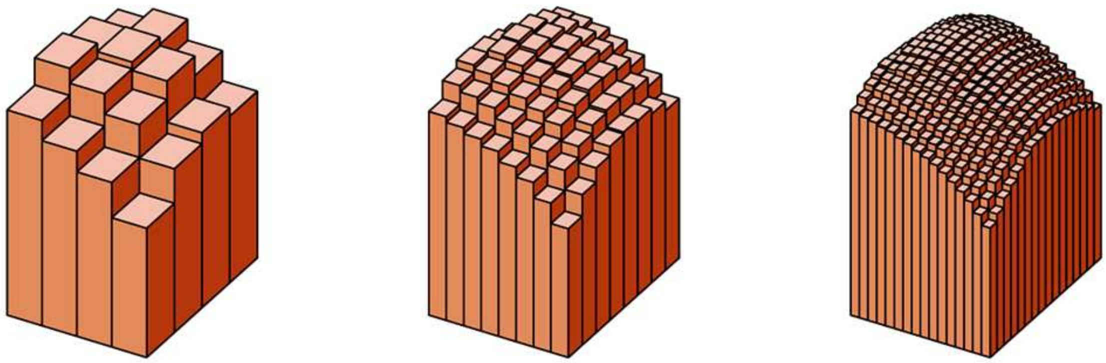


Figure 2

Definition. The *norm* of a partition P, denoted $\|P\|$, is the largest width or height of any rectangle in the partition:

$$\|P\| = \max_{1 \leq k \leq n} \{\Delta x_k, \Delta y_k\}.$$

If the limit

$$\lim_{\|P\| \to 0} \sum_{k=1}^{n} f(x_k, y_k) \Delta A_k$$

exists and is the same regardless of how the partition and (x_k, y_k) are chosen, then f is *integrable* over R and the value of the limit is the *double integral* of f over R, denoted:

$$\iint_R f(x, y)\, dA = \iint_R f(x, y)\, dx\, dy.$$

Theorem. If $f(x, y)$ is continuous on rectangular region R, then f is integrable over R.

Note. When $f(x, y)$ is a nonnegative function over a rectangular region R in the xy-plane, we may interpret the double integral of f over R as the volume of the 3-dimensional solid region over the xy-plane bounded below by R and above by the surface $z = f(x, y)$. In fact, we take this as the definition of such a volume.

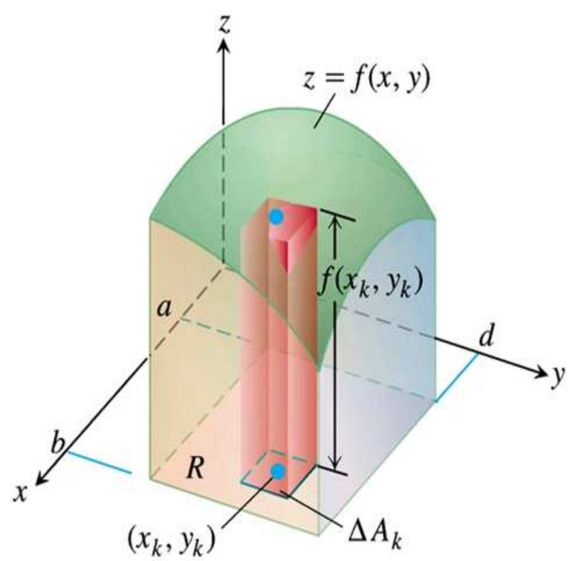


Figure 3

Theorem 1. Fubini's Theorem (First Form)

If $f(x, y)$ is continuous throughout the rectangular region $R = \{(x, y) \mid x \in [a, b], y \in [c, d]\}$, then

$$\iint_R f(x, y)\, dA = \int_c^d \int_a^b f(x, y)\, dx\, dy = \int_a^b \int_c^d f(x, y)\, dy\, dx.$$

The second two integrals are called *iterated integrals*.

Note. Fubini's Theorem allows us to evaluate double integrals by integrating with respect to one variable at a time. This means that when we calculate a volume by "slices" (slices are really differentials), we may start with either dx-slices or dy-slices.

Multiple Integrals

Double Integrals over General Regions

Note. Let R be a non-rectangular region in the plane. A partition of R is formed in a manner similar to rectangular regions, but we now only take rectangles which lie entirely inside region R (see Figure 15.8 below). As before, we number the rectangles and let ΔA_k be the area of the kth rectangle. Choose a point (x_k, y_k) in the kth rectangle and compute a Riemann sum as

$$S_n = \sum_{k=1}^{n} f(x_k, y_k)\Delta A_k.$$

Again, we define the *double integral* of $f(x, y)$ over R as

$$\lim_{\|P\|\to 0} \sum_{k=1}^{n} f(x_k, y_k)\Delta A_k = \iint_R f(x, y)\, dA.$$

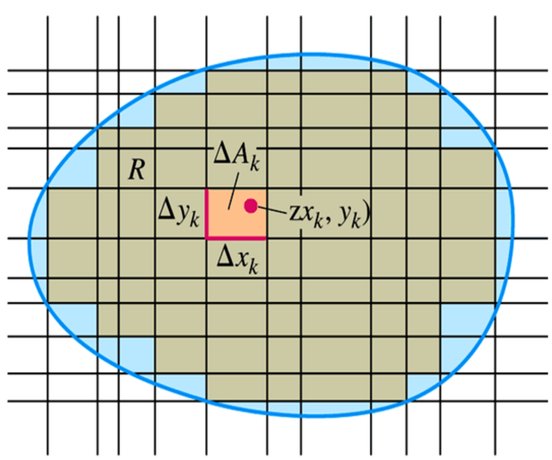


Figure 4

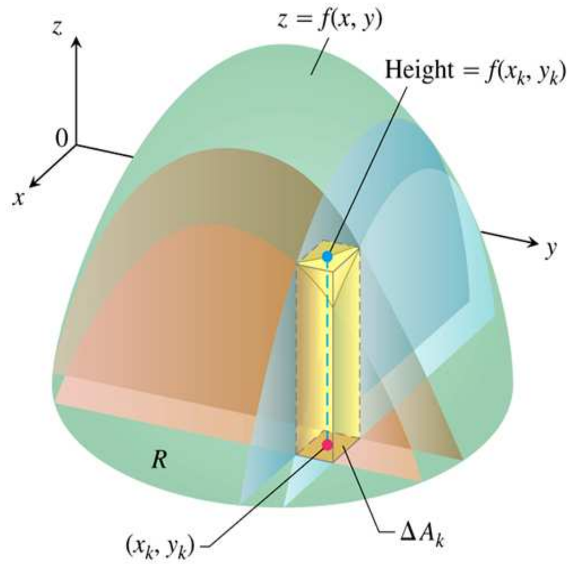


Figure 5

Definition. When $f(x, y)$ is a positive function over a region R in the xy-plane, we define the *volume* bounded below by R and above by the surface $z = f(x, y)$ to be the double integral of f over R.

Theorem 2. Fubini's Theorem (Stronger Form)

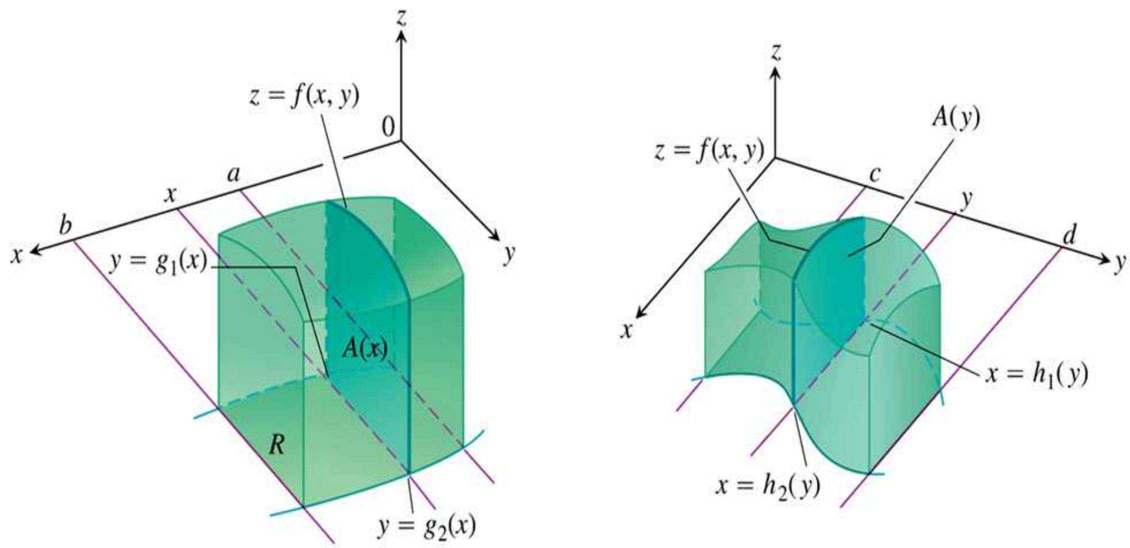
Let $f(x, y)$ be continuous on a region R.

1. If R is defined by $x \in [a, b]$, $g_1(x) \leq y \leq g_2(x)$, with g_1 and g_2 continuous on $[a, b]$, then

$$\iint_R f(x, y)\, dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y)\, dy\, dx.$$

2. If R is defined by $y \in [c, d]$, $h_1(y) \le x \le h_2(y)$, with h_1 and h_2 continuous on $[c, d]$, then

$$\iint_R f(x, y)\, dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y)\, dx\, dy.$$



Figures 6

Note. Using Vertical Cross-Sections.

When faced with evaluating $\iint_R f(x, y)\, dA$, integrating first with respect to y and then with respect to x, do the following three steps:

1. *Sketch.* Sketch the region of integration and label the bounding curves.

2. *Find the y-limits of integration.* Imagine a vertical line L cutting through R in the direction of increasing y. Mark the y-values where L enters and leaves. These are the y-limits of integration and are usually functions of x (instead of constants).

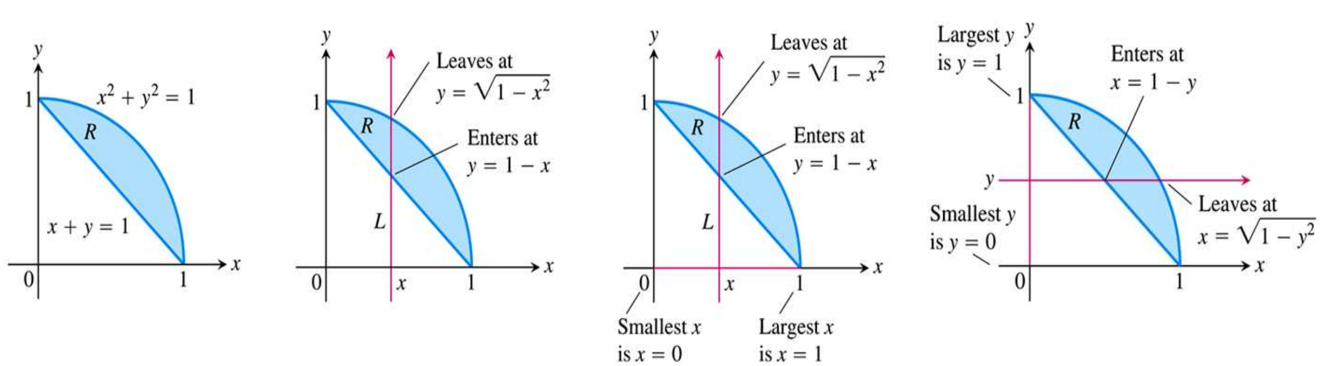
3. *Find the x-limits of integration.* Choose x-limits that include all the vertical lines through R. The integral shown below is

$$\iint_R f(x,y)\,dA = \int_{x=0}^{x=1} \int_{y=1-x}^{y\sqrt{1-x^2}} f(x,y)\,dy\,dx.$$

Using Horizontal Cross-Sections.

To evaluate the same double integral as an iterated integral with the order of integration reversed, use horizontal lines instead of vertical lines in Steps 2 and 3. The integral below is

$$\iint_R f(x,y)\,dA = \int_{y=0}^{y=1} \int_{x=1-y}^{x=\sqrt{1-y^2}} f(x,y)\,dx\,dy.$$



Figures 7

Theorem. Properties of Double Integrals.

If $f(x,y)$ and $g(x,y)$ are continuous on the bounded region R, then the following properties hold.

1. *C onstant Multiple:* $\iint_R cf(x,y)\,dA = c\iint_R f(x,y)\,dA$ for any constant c

2. *Sum and Difference:* $\iint_R (f(x,y) \pm g(x,y))\,dA = \iint_A f(x,y)\,dA \pm \iint_R g(x,y)\,dA$

3. *Domination:*

(a) $\iint_R f(x,y)\,dA \geq 0$ if $f(x,y) \geq 0$ on R

(b) $\iint_R f(x,y)\,dA \geq \iint_R g(x,y)\,dA$ if $f(x,y) \geq g(x,y)$ on R

4. *Additivity:* $\iint_R f(x,y)\,dA = \iint_{R_1} f(x,y)\,dA + \iint_{R_2} f(x,y)\,dA$ if R is the union of two non-overlapping regions R_1 and R_2

Multiple Integrals

Area by Double Integration

Note. If we take $f(x, y) = 1$ in the definition of the double integral over a region R in the preceding section, the Riemann sums reduce to

$$S_n = \sum_{k=1}^{n} f(x_k, y_k)\Delta A_k = \sum_{k=1}^{n} \Delta A_k.$$

This is simply the sum of the areas of the small rectangles in the partition of R, and approximates what we would like to call the area of R.

Definition. The *area* of a closed, bounded plane region R is

$$A = \iint_R dA.$$

Definition. The *average value* of $f(x, y)$ over region R is

$$\frac{1}{\text{area of } R} \iint_R f\, dA.$$

Multiple Integrals

Double Integrals in Polar Form

Note. Suppose that a function $f(r, \theta)$ is defined over a region R that is bounded by the rays $\theta = \alpha$ and $\theta = \beta$ and the continuous curves $r = g_1(\theta)$ and $r = g_2(\theta)$. Suppose also that $0 \leq g_1(\theta) \leq g_2(\theta) \leq a$ for every value of θ between α and β. Then R lies in a fanshaped region Q defined by $\{(r, \theta) \mid r \in [0, a], \theta \in [\alpha, \beta]\}$.

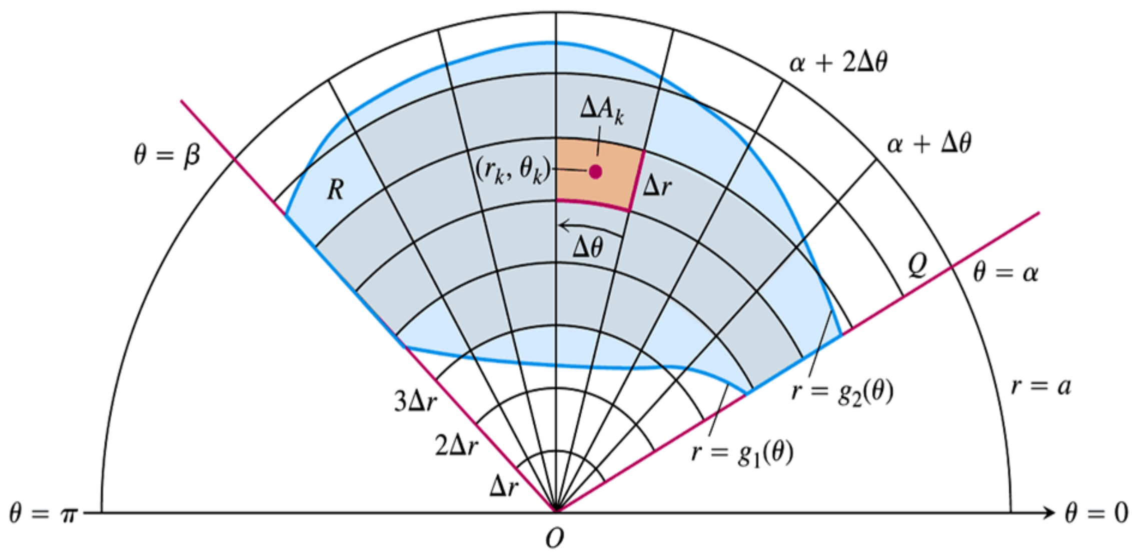


Figure 8

Note. We cover Q by a grid of a circular arcs and rays. The arcs are cut from circles centered at the origin, with radii Δr, $2\Delta r$, $\ldots$, $m\Delta r$, where $\Delta r = a/m$. The rays are given by:

$$\theta = \alpha, \theta = \alpha + \Delta\theta, \theta = \alpha + 2\Delta\theta, \ldots, \theta = \alpha + m'\Delta\theta = \beta$$

where $\Delta\theta = (\beta - \alpha)/m'$. The arcs and rays partition Q into small patches called "polar rectangles." We number the polar rectangles that lie inside R, calling their areas $\Delta A_1, \Delta A_2, \ldots, \Delta A_n$. We let (r_k, θ_k) be any pont in the polar rectangle whose area is ΔA_k. We then form the sum $S_n = \displaystyle\sum_{k=1}^{n} f(r_k, \theta_k)\,\Delta A_k$. If f is continuous throughout R, this sum will approach a limit as we refine the grid to make Δr and $\Delta\theta$ for to zero. The limit is called the double integral of f over R. We define the *norm* $\|P\|$ of this partition of the region as $\|P\| = \max_{1\leq k\leq n}\{\Delta r_k, \Delta\theta_k\}$. In symbols,

$$\iint_R f(r,\theta)\,dA = \lim_{\|P\|\to 0} \sum_{k=1}^{n} f(r_k,\theta_k)\,\Delta A_k.$$

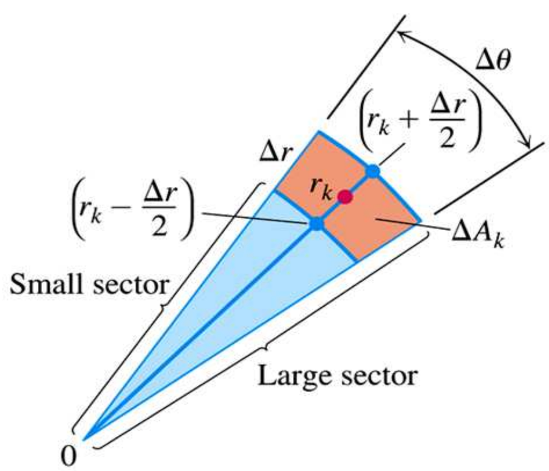


Figure 9

Note. To evaluate the limit above, we need to evaluate ΔA_k in terms of Δr and $\Delta\theta$. We choose r_k to be the average of the radii of the inner and outer arcs bounding the kth polar rectangle ΔA_k. The radius of the inner arc bounding ΔA_k is then $r_k - (\Delta r/2)$. The radius of the outer arc is $r_k + (\Delta r/2)$. The area of a wedge-shaped sector of a circle having radius r and angle θ is $A = \dfrac{1}{2}\theta r^2$, as can be seen by multiplying πr^2, the area of the circle, by $\theta/2\pi$, the fraction of the circle's area contained in the wedge. So the areas of the circular sectors subtended by these arcs at the origin are

$$\text{Inner radius: } \frac{1}{2}\left(r_k - \frac{\Delta r_k}{2}\right)^2 \Delta\theta$$

$$\text{Outer radius: } \frac{1}{2}\left(r_k + \frac{\Delta r_k}{2}\right)^2 \Delta\theta.$$

Therefore,

$$\begin{aligned}
\Delta A_k &= \text{ area of large sector } - \text{ area of small sector} \\
&= \frac{\Delta\theta}{2}\left[\left(r_k + \frac{\Delta r}{2}\right)^2 - \left(r_k - \frac{\Delta r}{2}\right)^2\right] \\
&= \frac{\Delta\theta}{2}(2r_k\Delta r) = r_k\,\Delta r\,\Delta\theta.
\end{aligned}$$

Combining this result with the sum defining S_n gives

$$S_n = \sum_{k=1}^{n} f(r_k, \theta_k)\, r_k\,\Delta r\,\Delta\theta.$$

As $\|P\| \to 0$, these sums converge to the double integral

$$\iint_R f(r, \theta)\, r\, dr\, d\theta.$$

Note. The procedure for finding limits of integration in rectangular coordinates also works for polar coordinates. To evaluate $\iint_R f(r, \theta)\, dA$ over a region R in polar coordinates, integrating first to r and then with respect to θ take the following steps.

1. *Sketch.* Sketch the region and label the bounding curves.

2. *Find the r-limits of integration.* Imagine a ray L from the origin cutting through R in the direction of increasing r. Mark the r-values where L enters and leaves R. These are the r-limits of integration. They usually depend on the angle θ that L makes with the positive x-axis.

3. *Find the θ-limits of integration.* Find the smallest and largest θ-values that bound R. These are the θ-limits of integration.

Definition. The *area* of a closed and bounded region R in the polar coordinate plane is

$$A = \iint_R r\,dr\,d\theta.$$

Note. The procedure for changing a Cartesian integral $\iint_R f(x,y)\,dx\,dy$ into a polar integral has two steps. First substitute $x = r\cos\theta$ and $y = r\sin\theta$, and replace $dx\,dy$ by $r\,dr\,d\theta$ in the Cartesian integral. Then supply the polar limits of integration for the boundary R. The Cartesian integral then becomes

$$\iint_R f(x,y)\,dx\,dy = \iint_G f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta.$$

Multiple Integrals

Triple Integrals in Rectangular Form

Note. If $F(x, y, z)$ is a function defined on a closed, bounded region D in space, then the integral of F over D may be defined in the following way. We partition a rectangular boxlike region containing D into rectangular cells by planes parallel to the coordinate axes. We number the cells that lie completely inside D from 1 to n in some order, the kth cell having dimensions Δx_k by Δy_k by Δz_k and volume $\Delta V_k = \Delta x_k \Delta y_k \Delta z_k$. We choose a point (x_k, y_k, z_k) in each cell and form the sum $S_n = \sum_{k=1}^{n} F(x_k, y_k, z_k)\, \Delta V_k$. We are interested in what happens as D is partitioned by smaller and smaller cells, so that Δx_k, Δy_k, Δz_k and the norm of the partition $\|P\| = \max\{\Delta x_k, \Delta y_k, \Delta z_k\}$ approaches zero. When a single limiting value is attained, no matter how the partitions and points (x_k, y_k, z_k) are chosen, we say that F is *integrable* over D. If F is continuous on D and D is formed from finitely many smooth surfaces joined together along finitely many smooth curves, then F is integrable. As $\|P\| \to 0$, if the sums S_n approach a limits, then the limit is the *triple integral of* F *over* D, denoted

$$\lim_{\|P\| \to 0} S_n = \int\int\int_D F(x, y, z)\, dx\, dy\, dz.$$

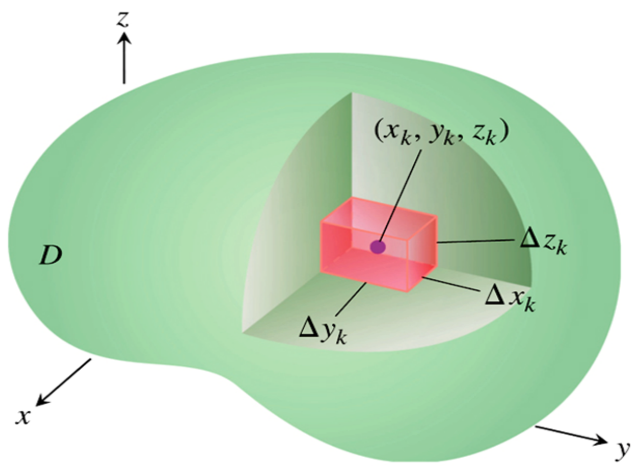


Figure 10

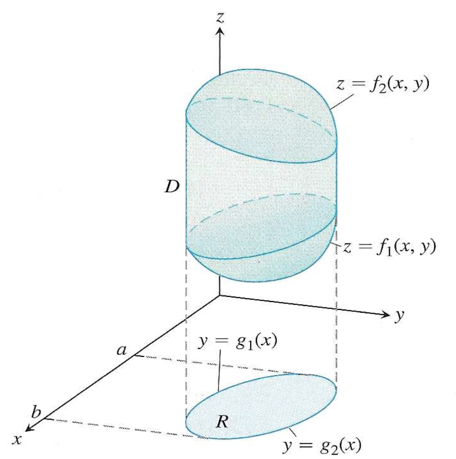
Definition. The *volume* of a closed and bounded region D in space is the triple integral of the function $F(x,y,z)=1$ over D:

$$V=\int\int\int_D dV.$$

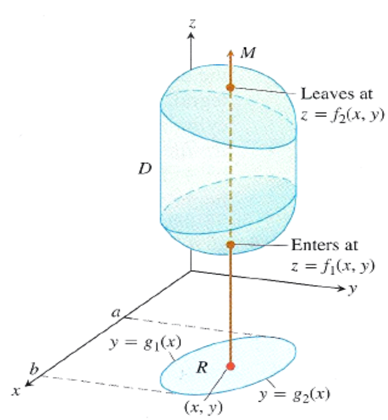
Note. Finding Limits of Integration in the Order $dz\ dy\ dx$

To evaluate $\displaystyle\int\int\int_D F(x,y,z)\,dV$, we illustrate how to find bounds for integrating first with repsect to z, then y, and then x.

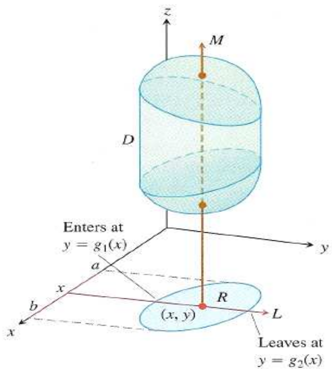
1. *Sketch.* Sketch the region D along with its 'shadow' R (vertical projection) in the xy-plane. Label the upper and lower bounding surfaces of D and the upper and lower bounding curves of R.



2. *Find the z-limits of integration.* Draw a line M passing through a typical point (x, y) in R parallel to the z-axis. As z increases, M enters D at $z = f_1(x, y)$ and leaves at $z = f_2(x, y)$. These are the z-limits of integration.



3. *Find the y-limits of integration.* Draw a line L through (x, y) parallel to the y-axis. As y increases, L enters R at $y = g_1(x)$ and leaves at $y = g_2(x)$. These are the y-limits of integration.



4. *Find the x-limits of integration.* Choose x-limits that include all lines through R parallel to the y-axis. These are the x-limits of integration.

In conclusion, the integral is

$$\int_{x=a}^{x=b} \int_{y=g_1(x)}^{y=g_2(x)} \int_{z=f_1(x,y)}^{z=f_2(x,y)} F(x, y, z)\, dz\, dy\, dx.$$

Of course, we can modify the order of integration by interchanging the variables.

Example. (set up the integral).

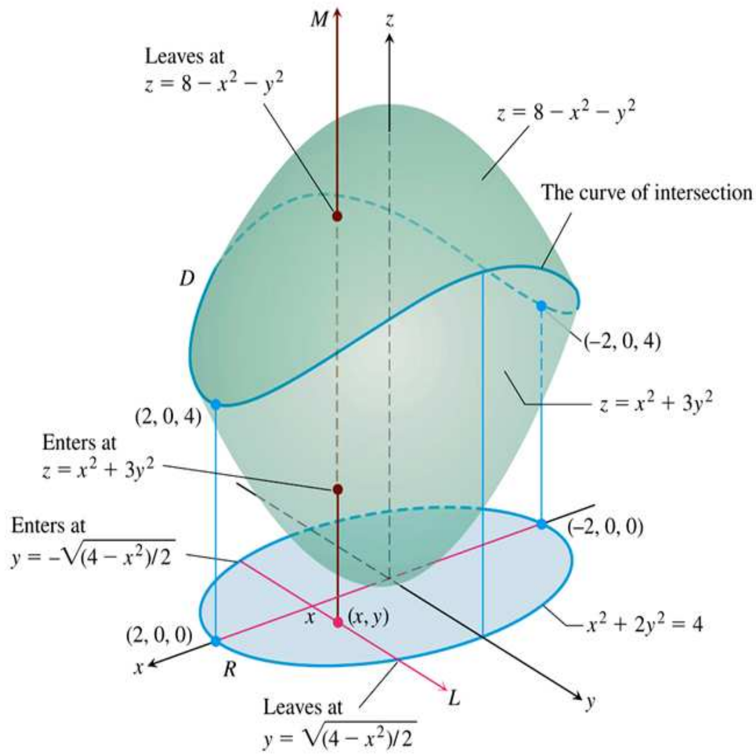


Figure 14

Definition. The average value of a function F over a region D in space is

$$\text{Average value } = \frac{1}{\text{volume of } D} \int\int\int_D F\, dV.$$

Multiple Integrals

Triple Integrals in Cylindrical and Spherical Coordinates

Definition. *Cylindrical coordinates* represent a point P in space by ordered triples (r, θ, z) in which

1. r and θ are polar coordinates for the vertical projection of P on the xy-plane

2. z is the rectangular vertical coordinate.

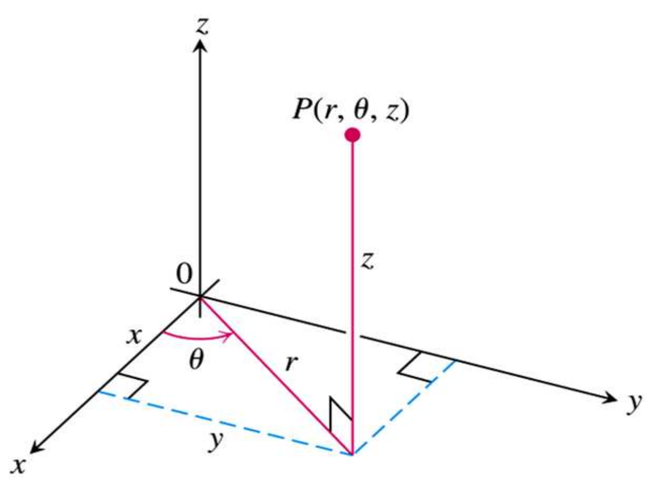


Figure 17

Note. The equations relating rectangular (x, y, z) and cylindrical (t, θ, z) coordinates are

$$x = r\cos\theta,\ \ y = r\sin\theta,\ \ z = z$$

$$r^2 = x^2 + y^2,\ \tan\theta = y/x.$$

Note. In cylindrical coordinates, the equation $r = a$ describes not just a circle in the xy-plane but an entire cylinder about the z-axis. The z-axis is given by $r = 0$. The equation $\theta = \theta_0$ describes the plane that contains the z-axis and makes an angle θ_0 with the positive x-axis. And, just as in rectangular coordinates, the equation $z = z_0$ describes a plane perpendicular to the z-axis.

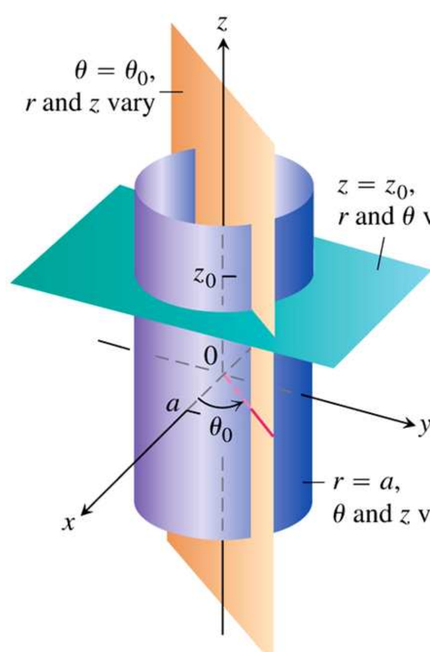


Figure 18

Note. When computing triple integrals over a region D in cylindrical coordinates, we partition the region into n small cylindrical wedges, rather than into rectangular boxes. In the kth cylindrical wedge, r, θ and z change by Δr_k, $\Delta\theta_k$, and Δz_k, and the largest of these numbers among all the cylindrical wedges is called the *norm* of the partition. We define the triple integral as a limit of Riemann sums using these wedges. Thee volume of such a cylindrical wedge ΔV_k is obtained by taking the area ΔA_k of its base in the $r\theta$-plane and multiplying by the height Δz. For a point (r_k, θ_k, z_k) in the center of the kth wedge, we calculated in polar coordinates that $\Delta A_k = r_k \Delta r_k \Delta\theta_k$. So $\Delta V_k = \Delta z_k r_k \Delta r_k \Delta\theta_k$ and a Riemann sum for f over D has the form

$$S_n = \sum_{k=1}^{n} f(r_k, \theta_k, z_k)\, \Delta z_k\, r_k\, \Delta r_k\, \Delta\theta_k.$$

The triple integral of a function f over D is obtained by taking a limit of such Riemann sums with partitions whose norms approach zero:

$$\lim_{\|P\|\to 0} S_n = \int\int\int_D f\, dV = \int\int\int_D f\, dz\, r\, dr\, d\theta.$$

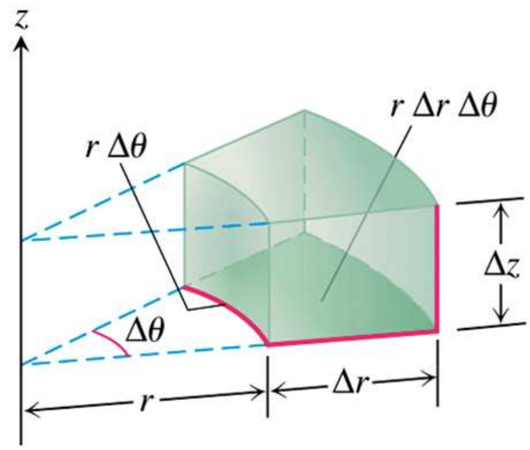


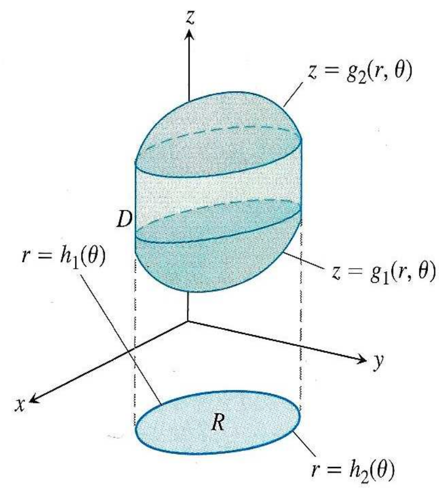
Figure 19

How to Integrate in Cylindrical Coordinates

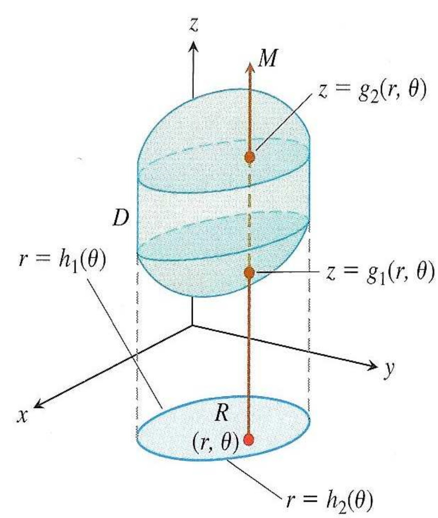
To evaluate $\int\int\int_D f(r,\theta,z)\,dV$ over a region D in space in cylindrical coordinates, integrating first with respect to z, then with respect to r, and finally with respect to θ, take the following steps.

1. *Sketch.* Sketch the region D along with its projection R on the xy-

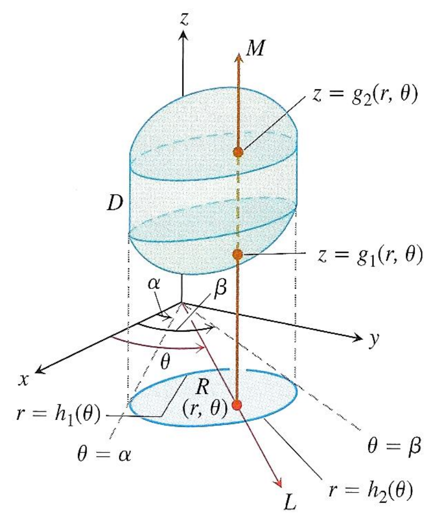
plane. Label the surfaces and curves that bound D and R.



2. *Find the z-limits of integration.* Draw a line M passing through a typical point (r, θ) of R parallel to the z-axis. As z increases, M enters D at $z = g_1(r, \theta)$ and leaves at $z = g_2(r, \theta)$. These are the z-limits of integration.



3. *Find the r-limits of integration.* Draw a ray L through (r, θ) from the origin. The ray enters R at $r = h_1(\theta)$ and leaves at $r = h_2(\theta)$. These are the r-limits of integration.



4.*F ind the θ-limits of integration.* As L sweeps across R, the angle θ it makes with the positive x-axis runs from $\theta = \alpha$ to $\theta = \beta$. These are the θ-limits if integration. The integral is

$$\int\int\int_D f(r, \theta, z)\, dV = \int_{\theta=\alpha}^{\theta=\beta} \int_{r=h_1(\theta)}^{r=h_2(\theta)} \int_{z=g_1(r,\theta)}^{z=g_2(r,\theta)} f(r, \theta, z)\, dz\, r\, dr\, d\theta.$$

Definition. *Spherical coordinates* represent a point P in space by ordered triples (ρ, ϕ, θ) in which

1. ρ is the distance from P to the origin (notice that $\rho > 0$).

2. ϕ is the angle $\vec{OP}$ makes with the positive z-axis ($\phi \in [0, \pi]$).

3. θ is the angle from cylindrical coordinate ($\theta \in [0, 2\pi]$).

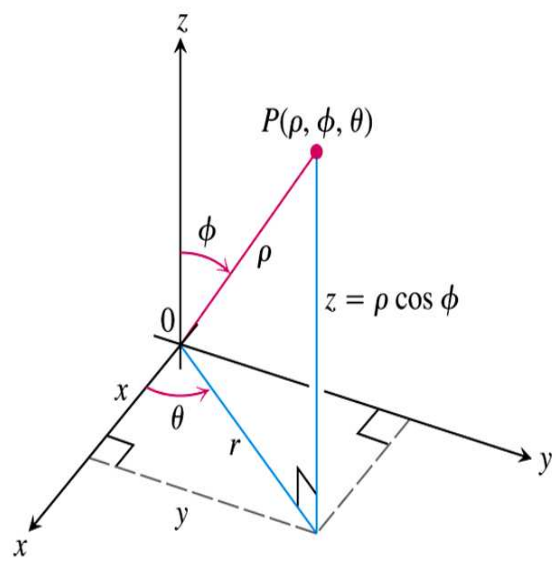


Figure 22

Note. The equation $\rho = a$ describes the sphere of radius a centered at the origin. The equation $\phi = \phi_0$ describes a single cone whose vertex lies at the origin and whose axis lies along the z-axis.

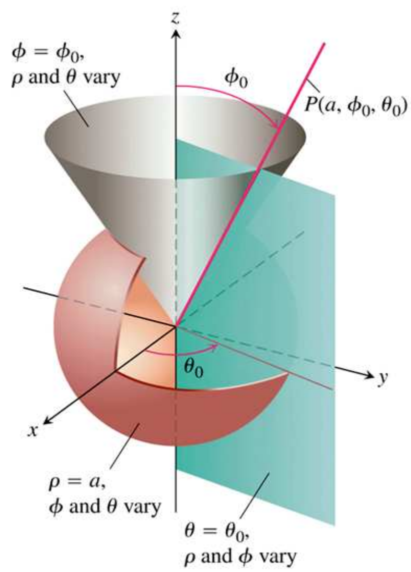


Figure 23

Note. The equations relating spherical coordinates to Cartesian coordinates and cylindrical coordinates are

$$r = \rho \sin \theta, \ x = r \cos \theta = \rho \sin \phi cos\theta,$$

$$z = \rho \cos \phi, \ y = r \sin \theta = \rho \sin \phi \sin \theta,$$

$$\rho = \sqrt{x^2 + y^2 + z^2} = \sqrt{r^2 + z^2}.$$

Note. When computing triple integrals over a region D in spherical coordinates, we partition the region into n spherical wedges. The size of the kth spherical wedge, which contains a point $(\rho_k, \phi_k, \theta_k)$, is given be the changes $\Delta\rho_k$, $\Delta\theta_k$, and $\Delta\phi_k$ in ρ, θ, and ϕ. Such a spherical wedge has one edge a circular arc of length $\rho_k \Delta\phi_k$, another edge a circular arc of length $\rho_k \sin\phi_k \Delta\theta_k$, and thickness $\Delta\rho_k$. The spherical wedge closely approximates a cube of these dimensions when $\Delta\rho_k$, $\Delta\theta_k$, and $\Delta\phi_k$ are all small. It can be shown that the volume of this spherical wedge ΔV_k is $\Delta V_k = \rho_k^2 \sin\phi_k \Delta\rho_k \Delta\phi_k \Delta\theta_k$ for $(\rho_k, \phi_k, \theta_k)$ a point chosen inside the wedge. The corresponding Riemann sum for a function $f(\rho, \phi, \theta)$ is

$$S_n = \sum_{k=1}^{n} f(\rho_k, \phi_k, \theta_k) \rho_k^2 \sin\phi_k \, \Delta\rho_k \, \Delta\phi_k \, \Delta\theta_k.$$

As the norm of a partition approaches zero, and the spherical wedges get smaller, the Riemann sums have a limit when f is continuous:

$$\lim_{\|P\|\to 0} S_n = \int\int\int_D f(\rho, \phi, \theta) \, dV = \int\int\int_D f(\rho, \phi, \theta) \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta.$$

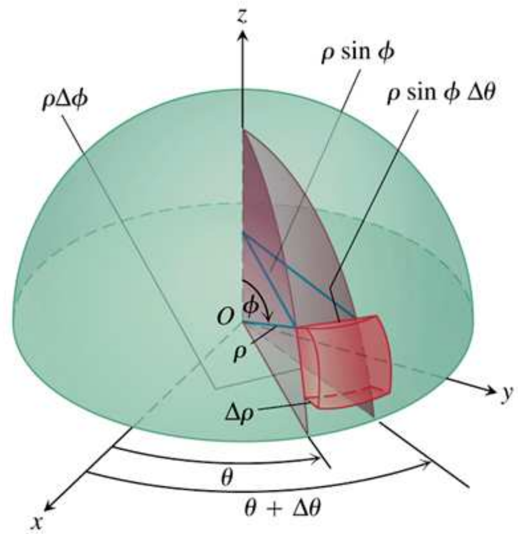


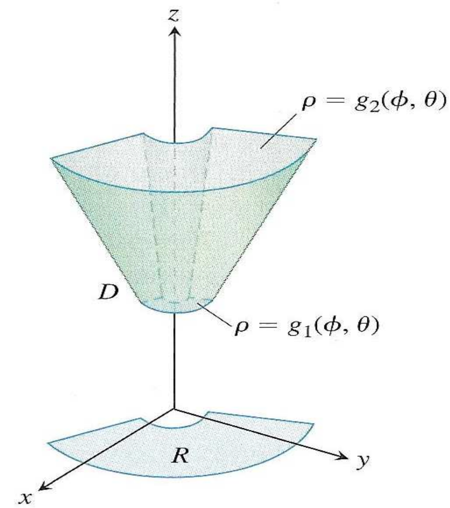
Figure 24

How to Integrate in Spherical Coordinates

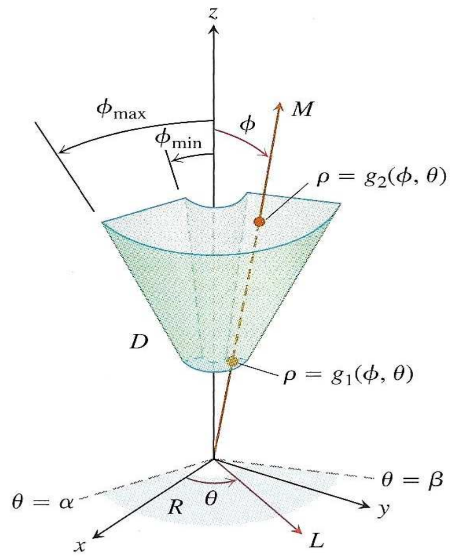
To evaluate $\displaystyle\int\int\int_D f(\rho, \phi, \theta)\, dV$ over a region D in space in spherical coordinates, integrating first with respect to ρ, then with respect to ϕ, and finally with respect to θ, take the following steps.

1. *Sketch.* Sketch the region D along with its projection R on the xy-

plane. Label the surfaces and curves that bound D and R.



2. *Find the ρ-limits of integration.* Draw a ray M from the origin through D making an angle ϕ with the positive z-axis. Also draw the projection of M on the xy-plane (call the projection L). The ray L makes an angle θ with the positive x-axis. As ρ increases, M enters D at $\rho = g_1(\phi, \theta)$ and leaves at $\rho = g_2(\phi, \theta)$. These are the ρ-limits of integration.



3. *Find the ϕ-limits of integration.* For any given θ, the angle ϕ that M makes with the z-axis runs from $\phi = \phi_{\text{min}}$ to $\phi = \phi_{\text{max}}$. These are the ϕ-limits of integration.

4. *Find the θ-limits of integration.* The ray L sweeps over R as θ runs from α to β. These are the θ-limits of integration. The integral is

$$\int\int\int_D f(\rho,\phi,\theta)\,dV = \int_{\theta=\alpha}^{\theta=\beta}\int_{\phi=\phi_{\text{min}}}^{\phi=\phi_{\text{max}}}\int_{\rho=g_1(\phi,\theta)}^{\rho=g_2(\phi,\theta)} f(\rho,\phi,\theta)\rho^2\sin\phi\,d\rho\,d\phi\;d\theta.$$

Note. In summary, we have the following relationships.

Cylindrical to Rectangular	**Spherical to Rectangular**	**Spherical to Cylindrical**
$x = r\cos\theta$	$x = \rho\sin\phi\cos\theta$	$r = \rho\sin\phi$
$y = r\sin\theta$	$y = \rho\sin\phi\sin\theta$	$z = \rho\cos\phi$
$z = z$	$z = \rho\cos\theta$	$\theta = \theta$

In terms of the differential of volume, we have

$$dV = dx\,dy\,dz = dz\,r\,dr\,d\theta = \rho^2\sin\phi\,d\rho\,d\phi\,d\theta.$$

Multiple Integrals

Substitutions in Multiple Integrals

Note. Suppose that a region G in the uv-plane is transformed one-to-one into the region R n the xy-plane by equations of the form

$$x = g(u, v),\ y = h(u, v).$$

We call R the *image* of G under the transformation, and G the *preimage* of R. Any function $f(x, y)$ defined on R can be thought of as a function $f(g(u, v), h(u, v))$ defined on G as well. How is hte integral of $f(x,)$ over R related to the integral of $f(g(u, v), h(u, v))$ over G? The answer is: If g, h, and f have continuous partials derivatives and $J(u, v)$ is zero only at isolated points, then

$$\int\int_R f(x, y)\, dx\, dy = \int\int_G f(g(u, v), h(u, v))|J(u, v)|\, du\, dv.$$

The factor $J(u, v)$, whose absolute value appears above, is the *Jacobian* of the coordinate transformation. It measures how much the transformation is expanding or contracting the area around a point in G as G is transformed into R.

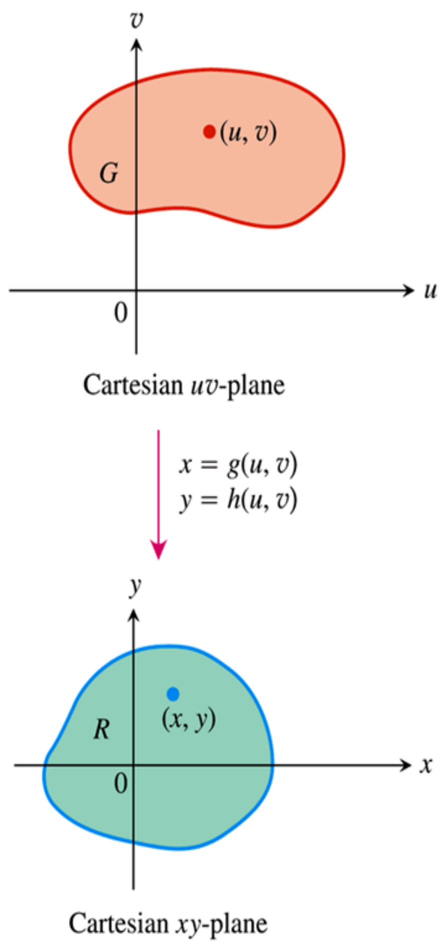


Figure 27

Definition. The *Jacobian determinant* or *Jacobian* of the coordinate transformation $x = g(u,v)$, $y = h(u,v)$ is

$$J(u,v) = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial y}{\partial u}\frac{\partial x}{\partial v}.$$

Note. The proof of the above is “intricate and properly belongs to a course in advances calculus. We do not give the derivation here.”

Example. Evaluate $\displaystyle\int_0^1 \int_0^{1-x} \sqrt{x+y}(y-2x)^2\, dy\, dx$.

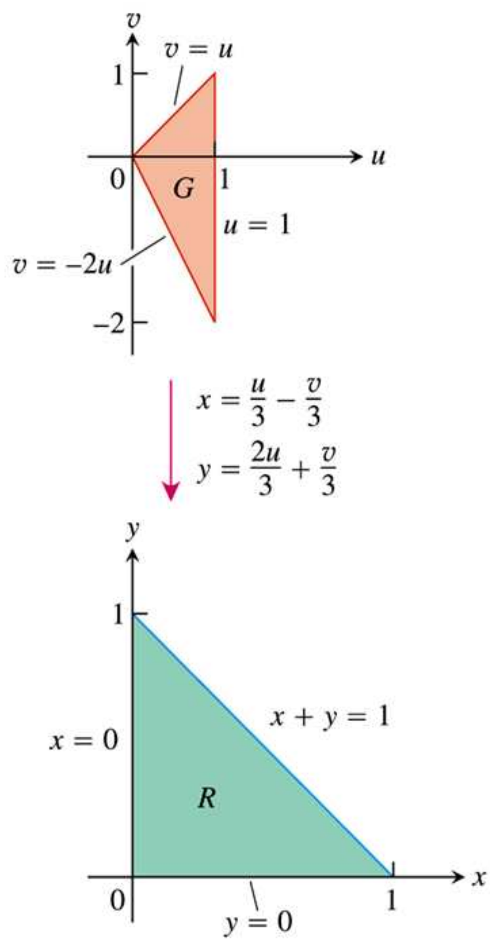


Figure 28

Note. Suppose that a region G in uvw-space is transformed one-to-one into the region D in xyz-space by differentiable equations of the form

$$x = g(u, v, w),\ \ y = h(u, v, w),\ \ z = k(u, v, w).$$

Then any function $F(x, y, z)$ defined on D can be thought of as a function

$$F(g(u, v, w), h(u, v, w), k(u, v, w)) = H(u, v, w)$$

defined on G. If g, h, and k have continuous first partial derivatives, then the integral of $F(x, y, z)$ over D is related o the integral of $H(u, v, w)$ over G by the equation

$$\int\int\int_D F(x, y, z)\, dx\, dy\, dz = \int\int\int_G H(u, v, w)|J(u, v, w)|\, du\, dv\, dw.$$

The factor $J(u, v, w)$ whose absolute value appears in this equation, is the *Jacobian determinant*

$$J(u, v, w) = \frac{\partial(x, y, z)}{\partial(u, v, w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}.$$

This determinant measures how much the volume near a point in G is being expanded or contracted by the transformation from (u, v, w) to (x, y, z) coordinates.