$$= \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} E\left[b_n b_m \right] p(t - nT_b) \; p(t - mT_b - \tau)$$

$\tau \cdot \frac{1}{2}$

$\tau \cdot \frac{1}{2} \, d\tau$

$\int_{-\infty}^{\infty} \frac{\tau^2}{2}$

$\frac{1}{4} \quad \tau^2$

$\underbrace{b_n = b_m}_{\substack{\text{Probability} \\ = 1/4}}$, $b_n \neq b_m$ for $m \neq n$

NRZ $E\{ b_n b_m \} = 0 \quad = \frac{1}{4} A^2 + \frac{1}{4} A^2 + \frac{1}{4}(-A^2) + \frac{1}{4}(-A^2)$.

for $m = n$,

Tutorial 10/08/23

Q. Consider a sinusoidal process $X(t) = A \cos(2\pi f_c t)$ where the frequency f_c is constant and the amplitude A is uniformly distributed

$$f_A(a) = \begin{cases} 1 & 0 \le a \le 1 \\ 0 & \text{o.w.} \end{cases}$$

Determine whether or not this process is strictly stationary.

Sol^n:-

$E_A[X(t)] = \cos(2\pi f_c t) \; E_A[A]$

$= \cos(2\pi f_c t) \cdot \int_0^1 a \, f_A(a) \, da$

$= \cos(2\pi f_c t) \int_0^1 a \, da = \frac{1}{2} \cos(2\pi f_c t)$

Expectation is determined on the random variable.

1st order stationarity	$\longrightarrow$	mean should be constant.
2nd order "	$\longrightarrow$	Auto-correlation.

$t = \frac{1}{f_c}$; $\mu_X(t) = \frac{1}{2} \times 1 = \frac{1}{2}$

$t = \frac{1}{2f_c}$; $\mu_X(t) = \frac{1}{2} \times -1 = -\frac{1}{2}$

depends on t_1. not t_1, $t_1 + \tau$.

Method. 2

$0 \leq a \leq 1$

$$f_{x(t_1)}(x_1) = \begin{cases} \dfrac{1}{\cos 2\pi f_c t_1} & 0 \leq x(t) \leq \cos 2\pi f t_1 \\ 0 & \text{o.w} \end{cases}$$

Q. Consider a random process $X(t)$ defined by $x(t) = \sin(2\pi f_c t)$ in which the frequency f_c is random variable uniformly distributed over the interval $[0, \omega]$. Show that $X(t)$ is non-stationary?

Hint: Examine specific sample function for the freq.

$f = \frac{\omega}{4}, \frac{\omega}{2}, \omega.$

$f = 4/\omega, 2/\omega, 1/\omega$

Soln:-

$$E_{f_c}(X(t)) = E[x(t)]$$

$$= E\{\sin(2\pi f_c t)\}.$$

$$= \int_0^{\omega} f_\omega(\omega) \cdot \sin(2\pi f_c t)\, df_c$$

$$\int_0^{\omega} k\, df = 1$$

$$k\{\omega\}_0^{\omega} = 1$$

$$k = 1/\omega.$$

$\sin(2\pi r$

$$= \frac{1}{\omega \cdot 2\pi t}(1 - \cos 2\pi \omega t).$$

1.3

Q. A random process $X(t)$ is defined by $X(t) = A\cos 2\pi f_c t$ where A is gaussian r.v with $A \sim \{0, \frac{\sigma^2}{4}\}$. This random process is applied to ideal integrator

$$Y(t) = \int_0^t X(\tau)\, d\tau.$$

(a) Determine the PDF of the output random process $y(t)$ $f_{Y(t_k)}(y_k)$.

Soln :- $Y(t_k)$ is a Gaussian R.V.

R.V because we are taking Sampling.

$$Y(t_k) = \int_0^{t_k} x(\tau)\, d\tau)$$

$$E_A[Y(t_k)] = E[A] \cos 2\pi f_c t_k = 0$$

$$Var[Y(t_k)] = E[(A-0)^2] \cos 2\pi f_c t_k$$

$$= E[A^2] \cos^2 2\pi f_c t_c$$

$$= \sigma_A^2 \cos^2 2\pi f_c t_k.$$

$$y(t_k) \sim N(0, \cos^2 2\pi f_c t_k \; \sigma_A^2)$$

y_{t_k} is not Stationary

not erogidicity

16/08/23
class

Line code
NRZ

$$R_X'(\tau) = \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} E[b_n b_m] \; p(t - nT_b) \; p(t - mT_b - \tau).$$

$b_n = b_m$, $b_n \neq b_m$

$$E[b_n b_m] = 0 = \tfrac{1}{4} A^2 + \tfrac{1}{4} A^2 + \tfrac{1}{4}(-A^2) + \tfrac{1}{4}(-A^2)$$

for, $m = n$

$$E[b_n b_m] = \tfrac{1}{2} \cdot A \cdot A + \tfrac{1}{2}(-A)(-A)$$

$$= A^2.$$

$P_r(0,0) = P_r(0) = 1/2$

$P_r(0,1) = 0$

$P_r(1,0) = 0.$

Comment :-

$R'(\tau)$ is a cyclostationary process.

H.W

$$R_X'(\tau) = R_X'(\tau \pm kT).$$

Prove

$R_X'(\tau)$ is periodic Signal.

We never define energy of periodic Signal.

$$R_x(\tau) = \frac{1}{T_b}\underbrace{\int_0^{T_b} R_x'(t,\, t-\tau)\, dt}$$

$\Downarrow$ Average over one time period.

$$R_x(\tau) = \frac{1}{T_b}\int_0^{T_b} \sum_n \sum_m E[b_n b_m]\ p(t-nT)\ p(t-mT-\tau)\ dt$$

$$= \frac{1}{T_b}\sum_n \sum_m E[b_n b_m] \underbrace{\int_0^{T_b} p(t-nT)\ p(t-mT-\tau)\ dt}_{A}$$

$$A = \int_0^{T_b} p(t-nT_b)\ p(t-mT_b+\tau)\, dt$$

$t - mT = \lambda$

$dt = d\lambda$

$$A = \int_{-mT_b}^{-T_b(m-1)} p(\lambda + (m-n)T_b)\ p(\lambda+\tau)\ d\lambda$$

Now,

$$R_x(\tau) = \frac{1}{T_b}\sum_n \left[\sum_{m=-\infty}^{\infty} E[b_n b_m] \int_{-mT_b}^{-(m-1)T_b} p(\lambda + (m-n)T_b)\ p(\lambda+\tau)\ d\lambda\right.$$

let $m-n = K$

$$= \frac{1}{T_b}\sum_{K=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} \underbrace{E[b_m b_{m+K}]}_{R_b(K)} \int_{-mT_b}^{-(m-1)T_b} p(\lambda - KT_b)\ p(\lambda+\tau)\, d\lambda$$

$$= \frac{1}{T_b}\sum_{K=-\infty}^{\infty} R_b(K) \underbrace{\sum_{m=-\infty}^{\infty}\int_{-mT_b}^{-(m-1)T_b}} p(\lambda - KT_b)\ p(\lambda+\tau)\ d\lambda$$

$$= \frac{1}{T_b}\sum_{K=-\infty}^{\infty} R_b(K) \int_{-\infty}^{\infty} p(\lambda - KT_b)\ p(\lambda+\tau)\ d\tau$$

$S_x(f) = F.T\,[R_x(\tau)]$

$$S_x(f) = \frac{1}{T_b} \sum_{k=-\infty}^{\infty} R_b(k)\; F.T \left[\int_{-\infty}^{\infty} p(\lambda - kT_b)\, p(\lambda + \tau)\, d\lambda \right]$$

$\Downarrow$

Proof $\rightarrow$ $|p(t)|^2$

Hint :- you need to apply the relationship b/w Correlation and convolution

$\Downarrow$

Wiener - Kinchin theorem

$$\boxed{\int_{-\infty}^{\infty} p(\lambda + \tau)\, p(\lambda)\, d\lambda = |P(f)|^2}$$ — H.W

$p(-(-\lambda)+\tau)\; p(\lambda)\, d\lambda.$

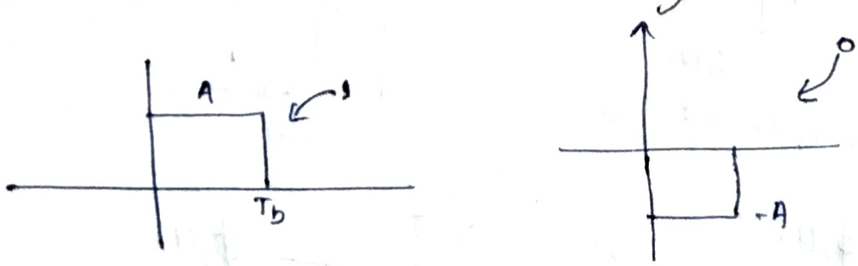
Rem.

$$\boxed{S_x(f) = \frac{1}{T_b} \sum_{k=-\infty}^{\infty} R_b(k)\; |P(f)|^2\, e^{-j2\pi f k T_b}.}$$

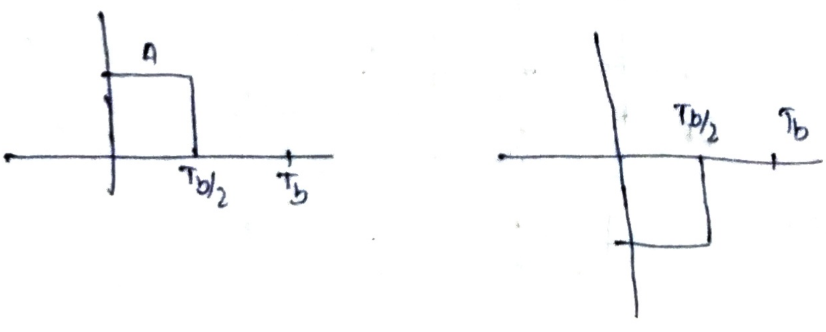
Q. Assignment

Q. PSD of NRZ, RZ, ON-OFF; } — Proof and also plot in Matlab.

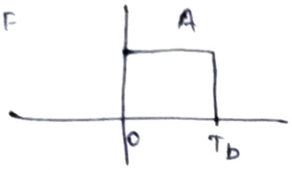
NRZ



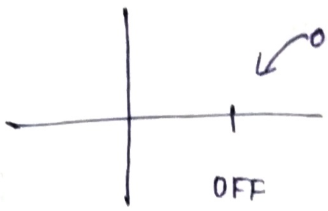
RZ



ON-OFF

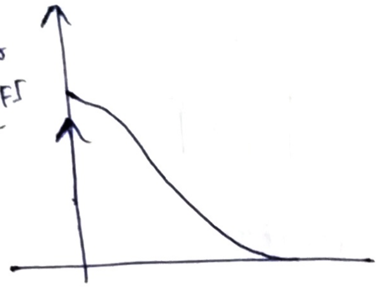


ON



OFF

PSD of ON-OFF



Q. Derive PSD for all the cases given above and plot PSD vs. freq.

Vector Representation of Signals

Vectors :-

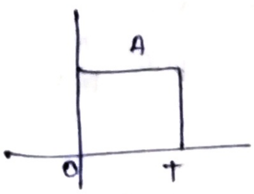
$|i| = 1$
$|j| = 1$
$\langle i \cdot j \rangle = 0$

— Basis fn

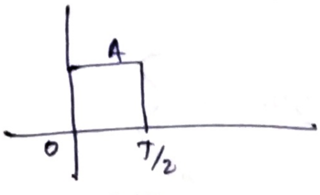
$\frac{1}{\sqrt{6}} \begin{Bmatrix} 1 \\ -2 \\ 1 \end{Bmatrix}$; $\frac{1}{\sqrt{3}} \begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix}$.

→ Now these two are basis function } ⇒ Qualified Basis fn.

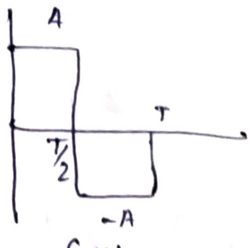
we take orthogonal because we don't have any common b/w two axis. i.e projection.



$S_1(t)$



$S_2(t)$

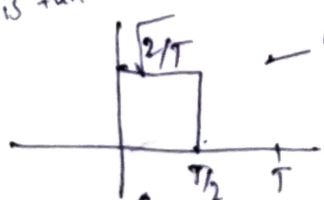


$S_3(t)$

← Minimum no. of basis function

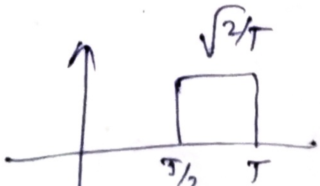
Energy should be 1.

Basis function —



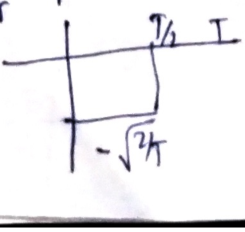
— $\phi_1(t)$

energy = $\int (\ )^2 dt = 1$



— $\phi_2(t)$

or

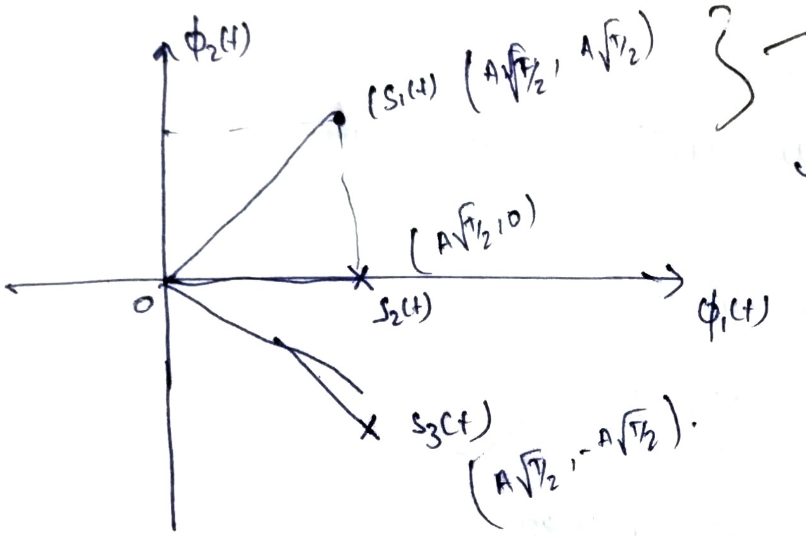


$s_1(t) = A\sqrt{T/2}\,\phi_1(t) + A\sqrt{T/2}\,\phi_2(t)$ | Reference, $\phi_1(t)$, $\phi_2(t)$

$s_2(t) = A\sqrt{T/2}\,\phi_1(t) + 0\,\phi_2(t)$.

$s_3(t) = A\sqrt{T/2}\,\phi_1(t) - A\sqrt{T/2}\,\phi_2(t)$

constellation of s_1, s_2 & s_3.



Vector representation of signal
⇓
not unique
⇓
Because basis for is not unique.

$d_{s_1(t)} = A\sqrt{T}$

$d_{s_2(t)} = A\sqrt{T/2}$

$d_{s_3(t)} = A\sqrt{T}$.

squaring

$d^2_{s_1(t)} = A^2T$

$d^2_{s_2(t)} = A^2T/2$

$d^2_{s_3(t)} = A^2T$.

→ Square of distance ⇓ Energy of Signal.

$d_{12} = d_{21} = A\sqrt{T/2}$

$d_{23} = d_{32} = A\sqrt{T/2}$

Dot product of two signals

$s_1(t)$, $s_2(t)$ — projection of s_1 on s_2.

$$\langle s_1(t), s_2(t)\rangle = \int_{-\infty}^{\infty} s_1(t)\, s_2(t)\, dt$$

$$= \langle \underline{s_1}, \underline{s_2} \rangle \text{ ↳ vector}$$

Similarly we can visualize correlation & convolution as projection.

$\underline{s_1} = \begin{bmatrix} s_{11} \\ s_{12} \end{bmatrix}$

$\underline{s_2} = \begin{bmatrix} s_{21} \\ s_{22} \end{bmatrix}$

H.W

$$\langle s_1(t), s_2(t)\rangle = \langle \underline{s_1}, \underline{s_2}\rangle = s_{11}s_{21} + s_{12}s_{22}$$

Energy of $S_1(t)$

$$E_{S_1(t)} = \int_{-\infty}^{\infty} S_1^2(t)\, dt = \langle S_1(t), S_1(t) \rangle$$

$$= S_{11}^2 + S_{12}^2$$

$$= S_1^T S_1$$

Distance of Signal $S_1(t)$ and $S_2(t)$

$$d_{12} = d_{21} = \int_{-\infty}^{\infty} \{S_1(t) - S_2(t)\}^2\, dt \quad \text{— Prove}$$

$$= \sqrt{(S_{11} - S_{21})^2 + (S_{12} - S_{22})^2}$$

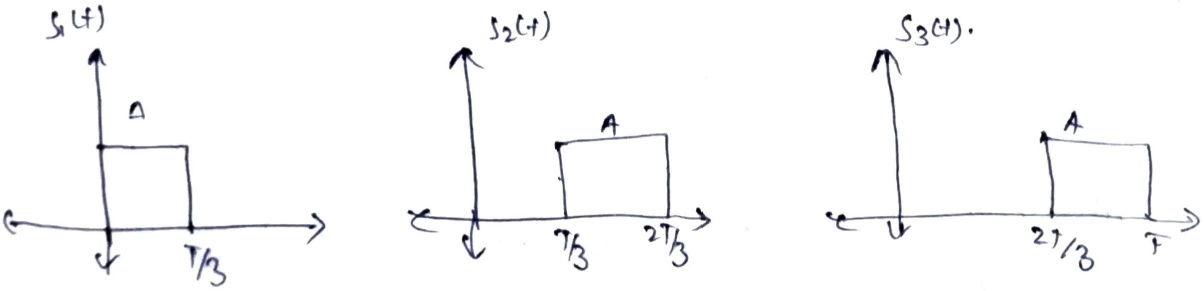
$$\langle S_1(t), S_2(t) \rangle = |S_1(t)|\, |S_2(t)| \cos\theta.$$

The minimum Number of basis function

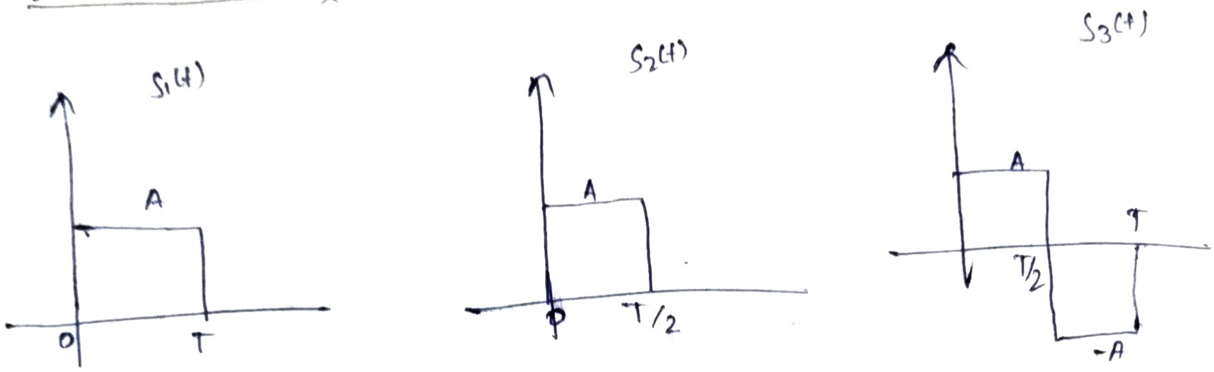
Max^m no. of minimum basis function = M.

For an M number arbitary signals, we require maximum M no. of minimum basis function.

$$\boxed{\text{The no. of Basis function required } \leq M.}$$

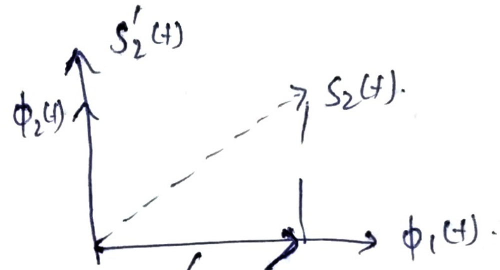


Gram Schmitt' Orthogonalization



Based on $s_1(t)$

$$\phi_1(t) = \frac{s_1(t)}{|s_1(t)|}$$



$\langle s_2(t), \phi_1(t) \rangle \phi_1(t)$

projection of $s_2(t)$ on $\phi_1(t)$.

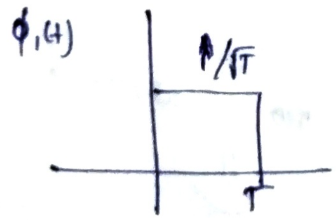
$$s_2(t) = s_2'(t) + \langle s_2(t), \phi_1(t) \rangle \phi_1(t)$$

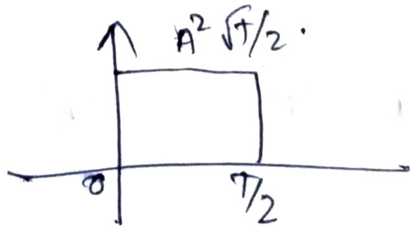
$$\boxed{s_2'(t) = s_2(t) - \langle s_2(t), \phi_1(t) \rangle \phi_1(t).}$$

$$\phi_2(t) = \frac{s_2'(t)}{|s_2'(t)|}$$

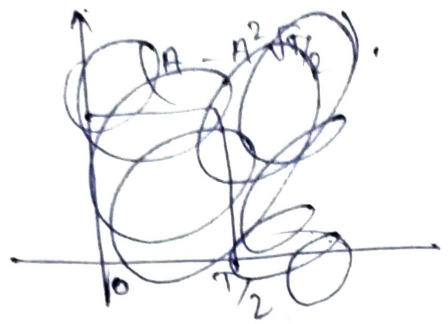
$$s_3'(t) = s_3(t) - \langle s_3(t), \phi_1(t) \rangle \phi_1(t) - \langle s_3(t), \phi_2(t) \rangle \phi_2(t).$$

$$\phi_3(t) = \frac{s_3'(t)}{|s_3'(t)|}.$$





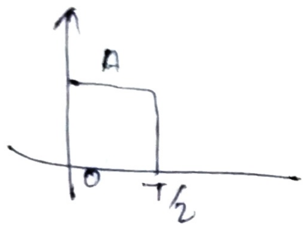
$$\langle s_2(t), \phi_1(t) \rangle \phi_1(t) = \frac{A^2}{\sqrt{T}} \cdot \frac{T}{2} = \begin{cases} \frac{A^2\sqrt{T}}{2}. & 0 \le t \le T/2 \\ 0 & \text{o.w.} \end{cases}$$



$$\left(A - A^2\sqrt{T/2}\right) x \times T/2 = 1$$

$$x = \frac{2}{T\left(A - A^2\sqrt{T/2}\right)}$$

$$\times\ A\sqrt{T/2} = A/2.$$



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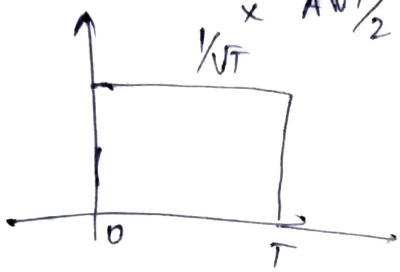
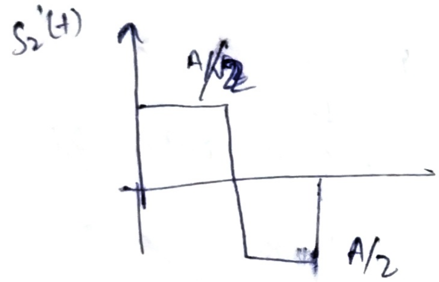
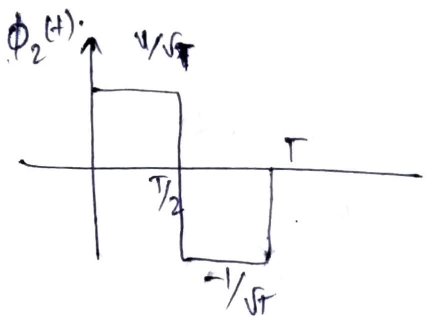






$$s_3'(t) = s_3(t) - \langle s_3(t), \phi_1(t)\rangle\, \phi_1(t) - \langle s_3(t), \phi_2(t)\rangle\, \phi_2(t).$$

23/08/23

Tutorial book

Q. 1.3

$$y(t) = \frac{A}{2\pi f_c}\left(\sin(2\pi f_c t)\right).$$

$$\mathrm{Var}(y(t_k)) = \sigma_A^2 \frac{\sin^2 2\pi f_c t}{(2\pi f_c)^2}$$ — Variance will depend on t_k.

$\Downarrow$

no Stationary

$\Downarrow$

non ergodic.

Exam 1.4 $z(t) = \underbrace{X}_{N.S} \cos(2\pi t) + \underbrace{Y}_{N.S} \sin(2\pi t)$

joint PDF $\longrightarrow$ depend on time difference.

$z(t_1) = X\cos(2\pi t_1) + Y\sin(2\pi t_1)$

$E[z(t)] = \cancel{E[X]}^{0} \cdot \cos 2\pi t + \cancel{E[Y]}^{0} \sin(2\pi t)$

$= 0.$

$Var(z(t)) = E[X^2] \cos^2 2\pi t_1 + E[Y^2] \sin^2(2\pi t_1) + \cancel{E[XY]}^{0} 2 \sin 2\pi t \cos 2\pi t$

$= 1 \cdot \cos^2 2\pi t_1 + 1 \cdot \sin^2 2\pi$

$E[(z(t_k) - E[z(t_k)])^2] = 1.$

$E[(z(t_k))^2].$

covariance matrix $= \begin{bmatrix} \sigma_x^2 & \rho\,\sigma_x \sigma_y \\ \rho\,\sigma_x \sigma_y & \sigma_y^2 \end{bmatrix}.$

$cov(z(t_1), z(t_2)) = \begin{bmatrix} \sigma^2_{z(t_1)} & \rho\,\sigma_{z(t_1)} \\ & \end{bmatrix}$

$\rho = \cos 2\pi(t_1 - t_2) = \dfrac{cov(z(t_1)\, z(t_2))}{\sigma_{z(t_1)} \cdot \sigma_{z(t_2)}}.$

$f_{z_1,z_2}(z_1, z_2) = \left(\dfrac{1}{\sqrt{2\pi}}\right)^2 \dfrac{1}{\sqrt{det(cov)}} \, Exp\left(\dfrac{-z^T (cov)^{-1} z}{2}\right)$

$$\text{Cov}(\ ) = \begin{bmatrix} 1 & \cos 2\pi(t_1-t_2) \\ \cos 2\pi(t_1-t_2) & 1 \end{bmatrix}.$$

$$= 1 - \cos^2 2\pi(t_1-t_2).$$

$$f_{z_1,z_2}(z_1,z_2) = \frac{1}{2\pi\sqrt{1-\cos^2 2\pi(t_1-t_2)}} \text{Exp}\left\{ \frac{-1}{2\sin^2(2\pi(t_1-t_2))} \left\{ z_1^2 - 2\cos 2\pi(t_1-t_2) z_1 z_2 + z_2^2 \right\} \right\}$$

$$\left\{ z_1^2 - 2\cos 2\pi(t_1-t_2) \cancel{z_1} z_2 + z_2^2 \right\}.$$

$z(t)$ is stationary since $z_1(t)$ is independent of t_1 & t_2.

Q. Consider a random process $X(t) = A \cos(2\pi f_c t + \phi)$, ϕ is uniformly distributed with $(0, 2\pi)$ check whether $X(t)$ is ergodic or not.

Soln:

$$E[x(t)] = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} x(t)\, dt.$$

Ensemble Avg. — time Average.

$$E\{x(t)\, x(t+\tau)\} = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} x(t)\, x(t+\tau)\, dt.$$

$$E[x(t)] = \int_{-\infty}^{\infty} \frac{1}{2\pi} \times A \cos(2\pi f_c t + \phi)\, d\phi$$

$$= \frac{A}{2\pi} \left[\sin(2\pi f_c t + \phi \right]_0^{2\pi}$$

$$= \frac{A}{2\pi} \quad \sin(2\pi f_c t + 2\pi) - \sin(2\pi f_c t)$$

$$= 0.$$

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} x(t)\, dt = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} A \cos(2\pi f_c t + \phi)\, dt.$$

$$= \lim_{T\to\infty} \frac{1}{2T} \quad A \left[\sin(2\pi f_c t + \phi) \right]_{-T}^{T}$$

$$A \{ \sin(2\pi f_c T + \phi) - \sin(2\pi f_c (-T) + \phi) \}$$

$$= 0.$$

$$E[x(t)\, x(t+\tau)] = \int_{-\infty}^{\infty} \frac{1}{2\pi} A \cos(2\pi f_c t + \phi) A \cos(2\pi f_c (t+\tau) + \phi).\ d\phi$$

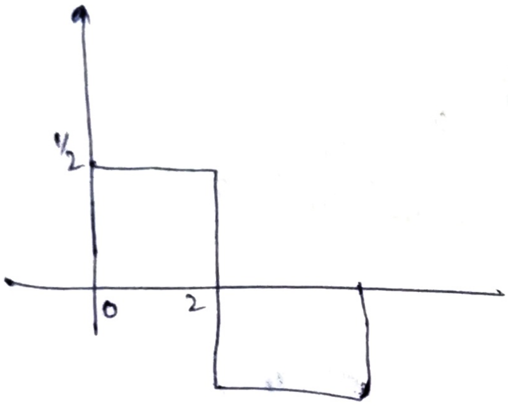
$$= \frac{A^2}{2} \cos(2\pi f_c \tau)$$

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} x(t)\, x(t+\tau)\, dt = \frac{A^2}{2} \cos(2\pi f_c \tau).$$

Q. 5.1

$$x(t) = \begin{cases} -1 & 0 \le t \le 1 \end{cases}$$

$\frac{1/\sqrt{2}+1/\sqrt{T}}{\cancel{2/\sqrt{T}}}$ $\quad A \times \cancel{\sqrt{T}}/2$



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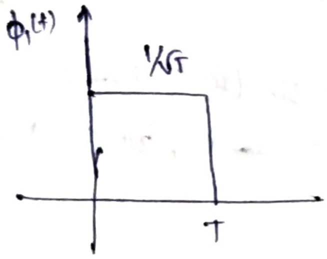
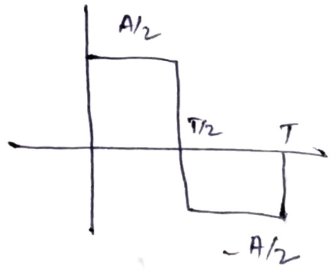
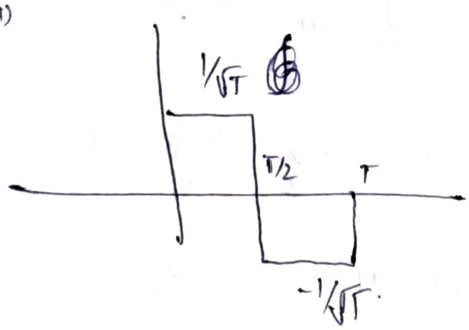
class



$s_2'(t) =$



$\phi_2(t) = \frac{s_2'(t)}{|s_2'(t)|}$

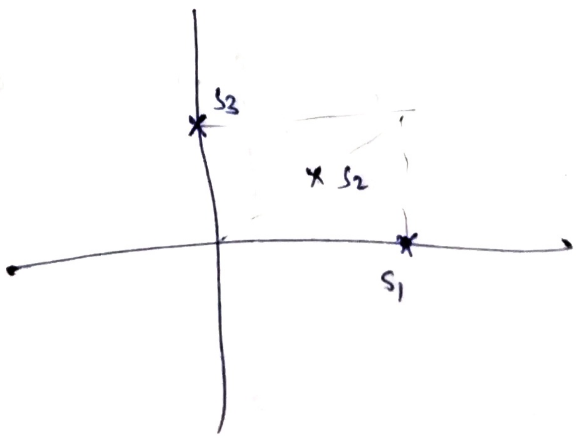


$\phi_3(t) = 0$

$s_1(t) = A\sqrt{T}\,\phi_1(t) + 0\,\phi_2(t)$

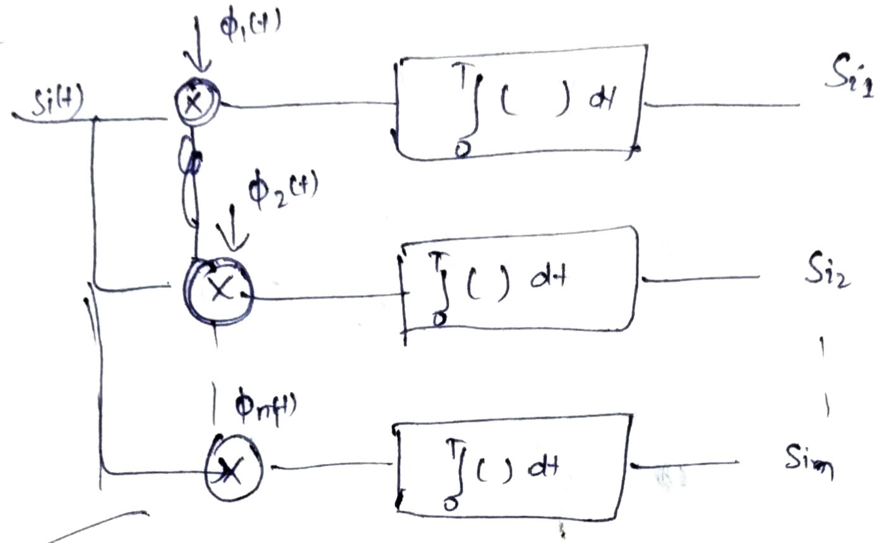
$s_2(t) = \left(\phi_1(t) + \phi_2(t)\right)\frac{A\sqrt{T}}{2}$

$s_3(t) = 0\,\phi_1(t) + A\sqrt{T}\,\phi_2(t)$



~~AWGN Channel Model~~

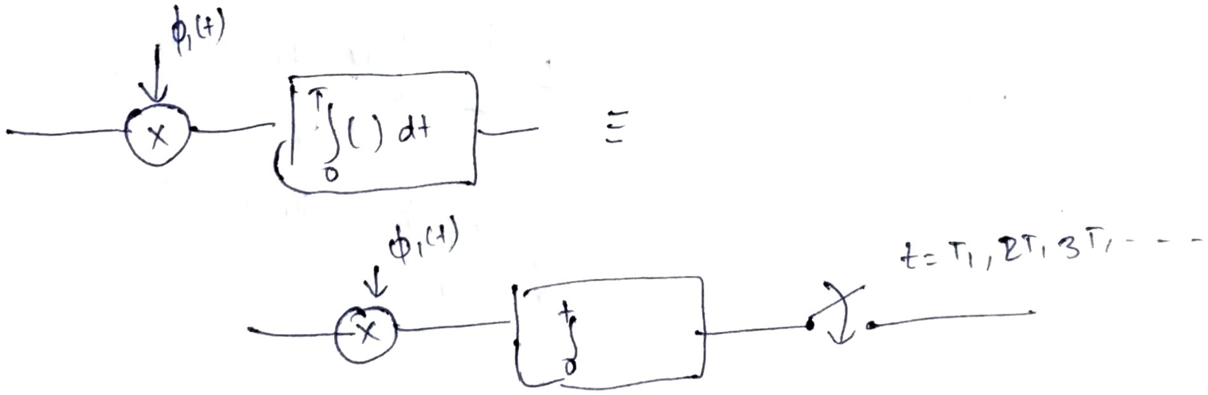
Receiver



projection of s_i on ϕ_1.

$$s_i(t) = s_{i1}\,\phi_1(t) + s_{i2}\,\phi_2(t).$$

$$\int_0^T s_i(t)\,\phi_1(t) = s_{i1}\int_0^T \phi_1(t)\,\phi_1(t)\,dt + s_{i2}\int_0^T \cancelto{0}{\phi_2(t)\,\phi_1(t)}\,dt$$



AWGN Channel Model

$$s_i(t) \longrightarrow \overset{n(t)}{\otimes} \longrightarrow x(t) = s_i(t) + n(t).$$

$n(t)$ = Gaussian noise process with zero mean and PSD $\frac{N_o}{2}$.

$\Downarrow$

due to brownian motion

$$x(t) = \sum_{m=1}^{M} \phi_m(t)\, s_{im} + n(t).$$

$$= \sum_{m=1}^{M} \phi_m(t)\, x_{im}$$

$x_{im} = \langle x(t), \phi_m(t) \rangle$ — projection of $x(t)$ on $\phi_m(t)$.

$$= \left\langle \left(\sum_{n=1}^{M} s_{in} \phi_n(t) + n(t) \right) ; \phi_m(t) \right\rangle.$$

$$= \left\langle \sum_{n=1}^{M} s_{in} \phi_n(t), \phi_m(t) \right\rangle + \langle n(t), \phi_m(t) \rangle$$

$$\boxed{x_{im} = s_{im} + \tilde{n}}$$

All projection expect $m=n$ will be zero.

Constant ⇓ not dependent on time.

Let $\tilde{n} = n_m$.

$$f_{x_{im}/s_{im}}(x_{im})$$

$$E(\tilde{n}) = 0 = E\left[\underbrace{\int_0^T n(t)\,\phi_m(t)\,dt}_{\tilde{n}} \right]$$

$$= \int_0^T E[n(t)]\,\phi_m(t)\,dt$$

$$= 0.$$

$$E\{\tilde{n}^2\} = E\left[\left[\int_0^T n(t)\,\phi_m(t)\,dt \right]^2 \right]$$

$$= E\left[\int_0^T n(t)\,\phi_m(t)\,dt \int_0^T n(u)\,\phi_m(u)\,du \right]$$

$$= E\left[\int_0^T \int_0^T n(t)\,n(u)\,\phi_m(t)\,\phi_m(u)\,dt\,du \right].$$

$$= \int_0^T \int_0^T E[n(t) \cdot n(u)]\,\phi_m(t)\,\phi_m(u)\,dt\,du$$

$$= \int_0^T \int_0^T R_n(t-u)\ \phi_m(t)\ \phi_m(u)\ dt\ du$$

F.T. $S_n(f) = N_0/2$ [∵ Given].

$S_n(f) \xrightleftharpoons{\text{I.F.T.}} \frac{N_0}{2}\ \delta(\tau)$ — Autocorrelation.

$$= \int_0^T \int_0^T \boxed{\frac{N_0}{2}\ \delta(t-u)}\ \phi_m(t)\ \phi_m(u)\ dt\ du$$

$$= \frac{N_0}{2} \int_0^T \int_0^T \delta(t-u)\ \phi_m(t)\ \phi_m(u)\ dt\ du$$

$$\boxed{\int \delta(t-u)\ y(t)\ dt = y(u)}$$

$$= \int_0^T \frac{N_0}{2}\ \phi_m(u)\ \phi_m(u)\ du$$

$$= \frac{N_0}{2} \int_0^T \phi_m^2(u)\ dt \qquad \phi_m^2(u) = 1 \text{ orthonormal}$$

$$= \frac{N_0 T}{2}$$

x_i - mean S_{in}, Variance $\frac{N_0}{2}$.

$x_{ip} = S_{ip} + n_p$, $n_p \sim \mathcal{N}\left(0, \frac{N_0}{2}\right)$.

Derive ⇒ n_p and n_m (i.e $m \neq p$) are independent & identically

iid

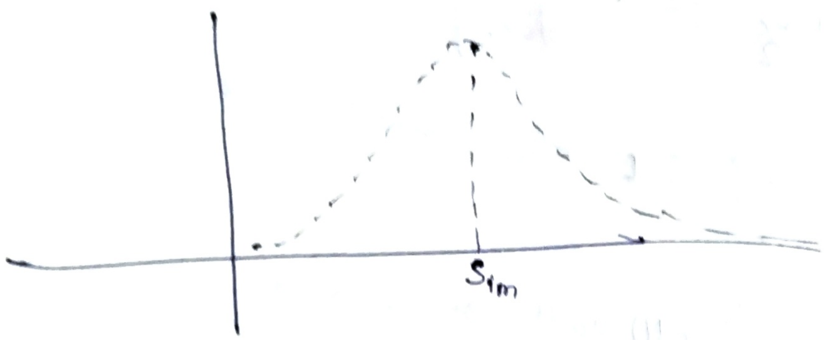
$$X_{im}\big|_{s_{im}} \sim \mathcal{N}\left(s_{im}, \frac{N_0}{2}\right).$$

If X is gaussian with mean = 0 var = σ^2

$Y = a + X$

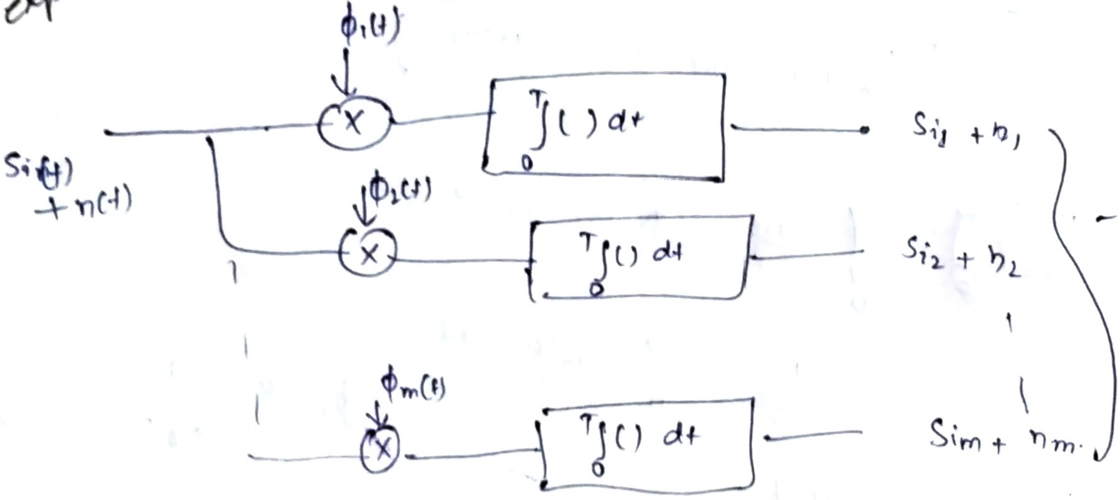
$var(Y) = var(X) = \sigma^2$

$mean(Y) = a$



$$\boxed{n(t) = \sum_{m=-\infty}^{\infty} \phi_m(t)\, n_m} = \sum_{\substack{m=-\infty \\ m \neq 1,2}}^{\infty} \phi_m(t)\, n_m + \sum_{m=1}^{2} \phi_m(t)\, n_m$$

General ways to express noise.

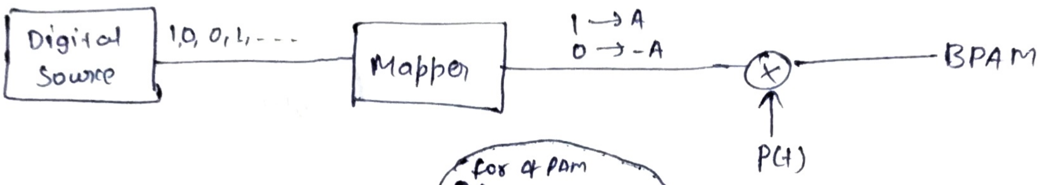


– discretization / time to scalar value

Correlation Receiver

Baseband Digital Modulation

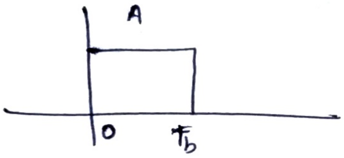
Binary Pulse Amplitude Modulation (BPAM)



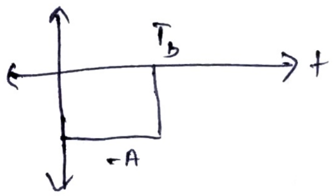
for 4 PAM
00 → -2A
01 → -A
10 → A
11 → 2A

Standard PAM

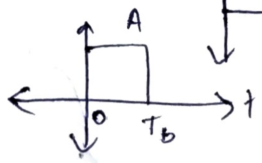
P(t) =



$S_1(t)\big|_0 = -p(t)$:

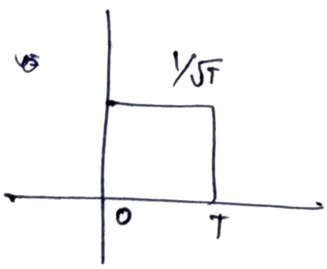


$S_2(t)\big|_1 = p(t)$:



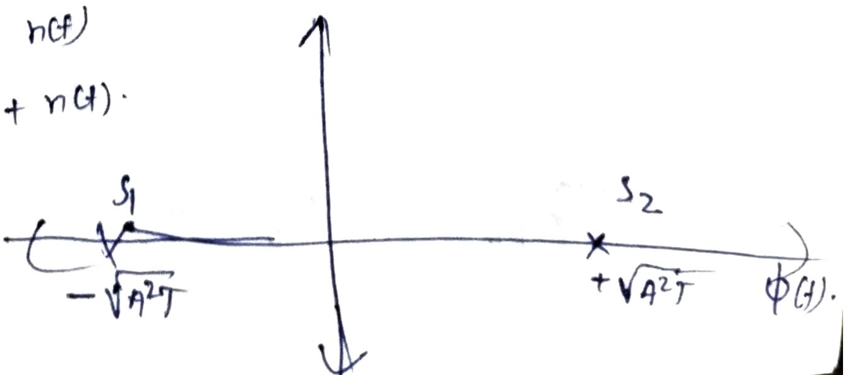
PAM - 1 dimension modulation technique.

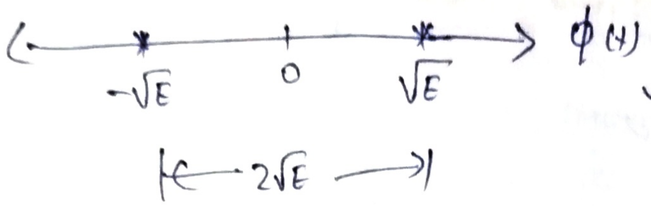
$\phi(t)$ vs



$$X(t)\big|_{S_1(t)} = S_1(t) + n(t)$$
$$= -A\sqrt{T}\,\phi_1(t) + n(t)$$
$$= -\sqrt{A^2T}\,\phi_1(t) + n(t)$$
$$= -\sqrt{E}\,\phi_1(t) + n(t)$$

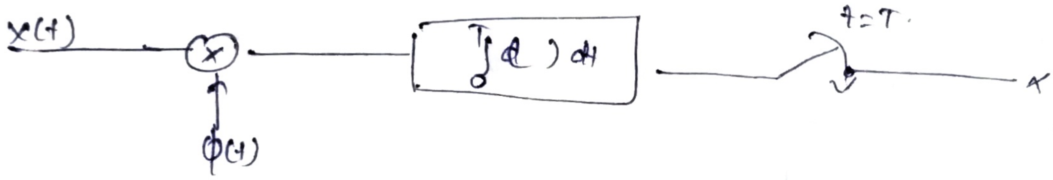
$$x(t)\big|_{S_2(t)} = S_2(t) + n(t)$$
$$= A\sqrt{T}\,\phi_1(t) + n(t)$$
$$= \sqrt{E}\,\phi_1(t) + n(t).$$





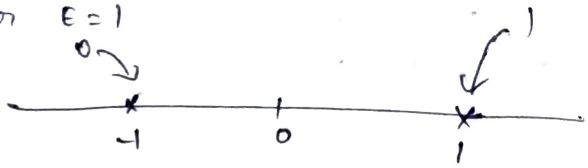
— without Noise

— noise will be highly concentrated at mean.



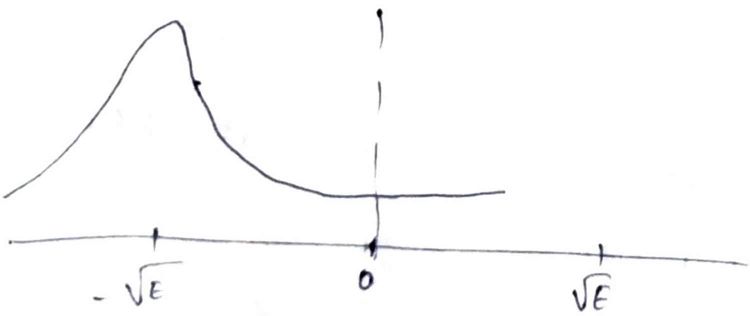
Intituitive Receiver

Consider $E = 1$



E_1 = when you transmit -1 V & receive > 0 V

E_2 = when you transmit 1V & Receive < 0 V.



$E|_0 \Rightarrow x|_0 > 0.$

$P_r[x|_0 > 0] = \int_0^{\infty} f_{x|0}(x|0)\,dx$

$$Pr[x|_0 > 0] = \int_0^\infty f_{x|_0}(x|0)\,dx$$

$$= \int \frac{1}{\sqrt{2\pi \frac{N_0}{2}}} e^{-\frac{(x+\sqrt{E})^2}{2 N_0/2}} dx$$

$$= \boxed{Q\left(\sqrt{\frac{2E}{N_0}}\right)}.$$

$$Q(y) = \int_y^\infty \frac{1}{\sqrt{2\pi}} e^{-u^2/2}\, du.$$

$$\frac{(x+\sqrt{E})}{\sqrt{\frac{N_0}{2}}} = u.$$

$$\frac{u^2}{2} = \frac{(x+\sqrt{E})^2}{2\,N_0/2} = u$$

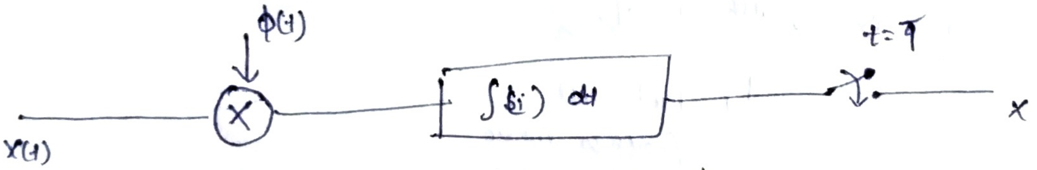
$$\boxed{Pr(x>0] = Q\left(\sqrt{\frac{2E}{N_0}}\right) = Pr(\varepsilon|_0) = Pr(\varepsilon|_1).}$$

ε - Error

28/08/23

Recap.

Bitwise Receiver for Binary PAM



$x = s_i + n$; $x \sim N(s_i, \frac{N_0}{2})$.

$0 \to -1V$

$1 \to 1V$

$$\begin{bmatrix} 0.7 \to 1 \\ -0.2 \to 0 \end{bmatrix}$$

$$P_{e|0} = P_{e|s_i(t)} = Pr[x|_0 \geq 0] = P[x|_{s_i(t)} \geq 0].$$

Probability of error in transmitting 0.

$X|_0 \sim \left(-\sqrt{E_b}, \frac{N_0}{2}\right)$

$$= \int_{x=0}^{\infty} \frac{1}{\sqrt{2\pi \frac{N_0}{2}}} \exp\left[-\frac{1}{2\frac{N_0}{2}} \left(x-(-\sqrt{E_b})\right)^2\right] dx$$

$$= Q\left(\sqrt{\frac{2E_b}{N_0}}\right) ; \qquad \boxed{\frac{x+\sqrt{E_b}}{\sqrt{N_0/2}} = u}$$

$$= \int_{\sqrt{\frac{2E_b}{N_0}}}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{u^2}{2}} du.$$

E_b : bit Energy

$\frac{N_0}{2}$: Noise Energy

$$= Q\left(\sqrt{\frac{E_b}{N_0/2}}\right)$$

$$= Q\left(\sqrt{SNR}\right)$$

$SNR = \frac{\text{Signal power}}{\text{Noise power}}$

SNR $\uparrow$ $\rightarrow$ $Q(\sqrt{SNR}) \downarrow$

↳ tail will further move forward:

Assumption in Intuitive receiver :-

Probability of Generation of 0 and 1 is same

$\Downarrow$

uniform distribution of the source

$P_r(0) = P_r(1) = \frac{1}{2}.$

ML Desion rule

$$\boxed{P_{e|1} = P_{e|s_2(t)} = Q\left(\sqrt{SNR}\right).}$$

$$P_e = Pr(0)\, P_{e|0} = Pr(1) \cdot P_{e|1}$$

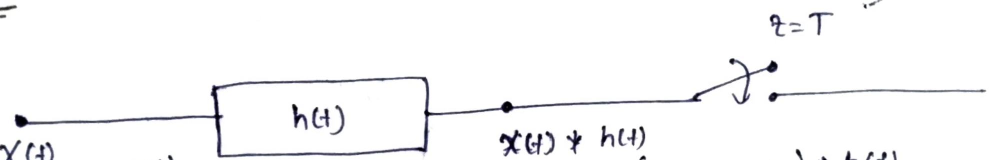
$$= \tfrac{1}{2} Q\left(\sqrt{SNR}\right) + \tfrac{1}{2} Q\left(\sqrt{SNR}\right)$$

$$\boxed{P_e = Q\left(\sqrt{SNR}\right)} = P_{e|0} = P_{e|1}.$$

Probability of error.

Receiver

why sampling at T?



$x(t) = s(t) + n(t)$

$$x(t) * h(t) = \left(s(t) + n(t)\right) * h(t)$$

$$= s(t) * h(t) + n(t) * h(t).$$

$$y(t) = \underbrace{s(t) * h(t)}_{\text{Signal Component}} + \underbrace{n(t) * h(t)}_{\text{noise Component}}.$$

$$n'(t) = n(t) * h(t) = \int_{0}^{t} n(\tau)\, h(t-\tau)\, d\tau$$

$$E\left[n(t) * h(t)\right] = \int_{0}^{t} E\left[n(\tau)\right] h(t-\tau)\, d\tau = 0$$

$$n = n'(t)\Big|_{t=T} = \int_0^T h(\tau)\, h(T-\tau)\, d\tau$$

$$E[n] = 0$$

$$E[n^2] = E\left[\int_0^T n(\tau)\, h(t-\tau)\, d\tau \int_0^T h(u)\, h(t-u)\, du\right]$$

$$= \int_0^T \int_0^T \frac{N_0}{2} \delta(\tau - u)$$

$$= \int_0^{T}\int_0^{T} \frac{N_0}{2}\, \delta(\tau - u)\, h(t-\tau)\, h(t-u)\, d\tau\, du.$$

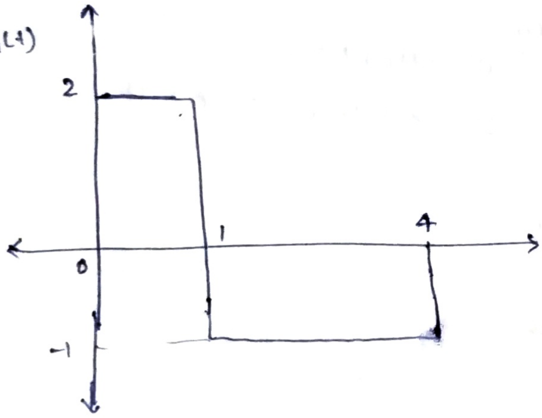
$$\int_0^{T} \delta(t - t_0)\, x(t)\, dt = x(t_0)$$

$$= \int_0^{T} \frac{N_0}{2}\, h(T-u)\, h(T-u)\, du$$

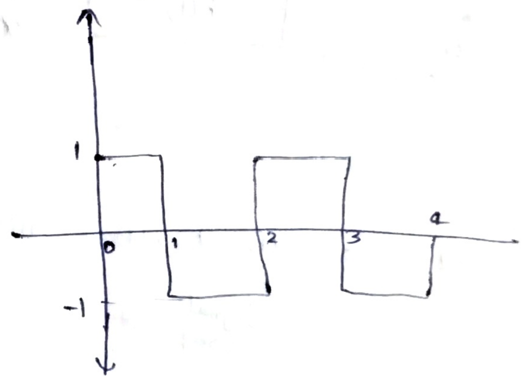
$$= \frac{N_0}{2} E_h$$ ← Energy of filter.

30/08/23
Tutorial

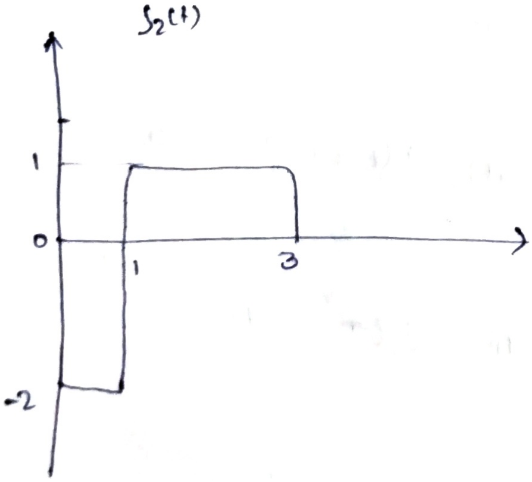
$S_1(t)$



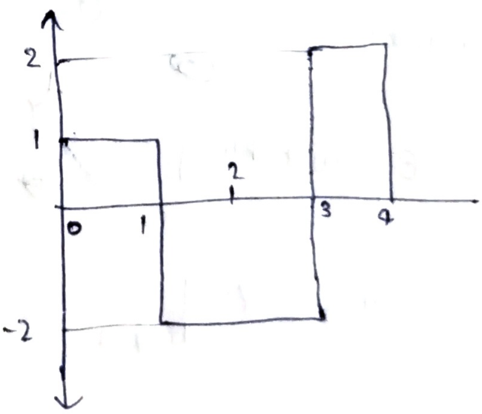
$S_3(t)$



$S_2(t)$



$S_4(t)$



Q. Determine the dimensionality of the waveform and a set of basis function

Q. Use the basis function to represent the four waveform by $S_1\ S_2,\ S_3\ S_4$.

Q3. Determine the minimum distance b/w pair of vectors.

$\phi_1(t) = \dfrac{s_1(t)}{\sqrt{E_1}}$

$\int_0^4 s_1(t)^2 \, dt = \int_0^1 4 + \int_1^4 1$

$= 4 + 1 \times (4-1) = 3.$

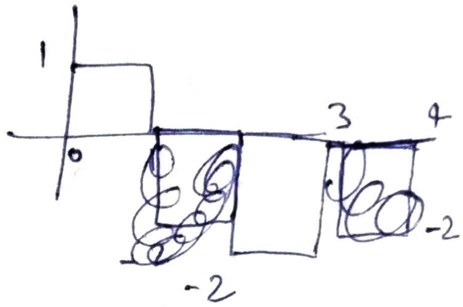
$= 4 + 3$

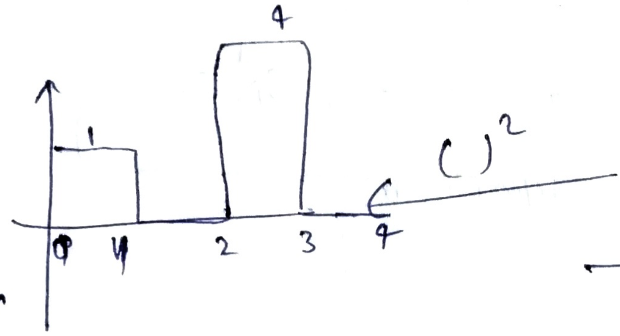
$= 7.$

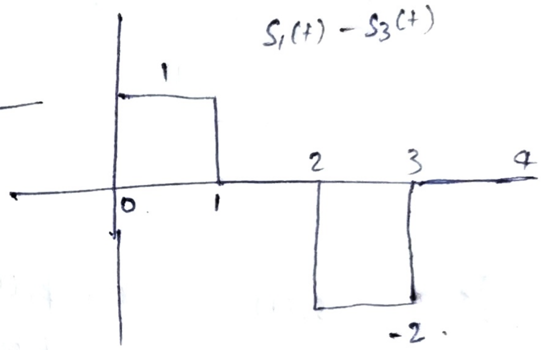
$= \dfrac{s_1(t)}{\sqrt{7}}.$

$$d_{13} = \sqrt{\int_0^T \{s_1(t) - s_3(t)\}^2 dt}$$

distance b/w 1 & 3







Area under curve ;

$A_{0-1} + A_{1-2} + A_{2-3} + A_{3-4}$

$= 1 + 0 + 4 + 0$

$= 5.$

$d_{13} = \sqrt{5}.$

Q. 5.8

p 5.9

$$s_i(t) = \sum_{m=1}^{M} \phi_m(t) \, s_{im}$$

$$s_k(t) = \sum_{m=1}^{M} \phi_m(t) \, s_{km}$$

$$\int s_i(t) \cdot s_k(t) \, dt = \int \left(\sum_{m=1}^{M} \phi_m(t) \, s_{im} \right) \left(\sum_{l=1}^{M} \phi_l(t) \, s_{kl} \right) dt.$$

Orthogonality and orthonormality property.

$$= \sum_{l=1}^{M} \sum_{m=1}^{M} s_{im} \, s_{kl} \int \phi_m(t) \, \phi_l(t) \, dt$$

$m = l \Rightarrow 1$

$m \neq l = 0.$

$$\Rightarrow \sum_{m=1}^{M} \sum_{l=1}^{M} s_{im} s_{kl} \int \phi_m(t) \phi_l(t) \, dt$$

when $m \neq l$

$$\int \phi_m(t) \phi_l(t) \, dt = 0$$

$m = l$ $\quad \int \phi_m(t) \phi_l(t) \, dt = 1.$

$$\Rightarrow \boxed{\sum_{m=1}^{M} s_{im} \underbrace{s_{km}}_{l \text{ replace by 'm'}} = s_i^T s_k.}$$

5.9.

$$s_1(t) = a_{11} \phi_1(t) + a_{12} \phi_2(t)$$
$$s_2(t) = a_{21} \phi_1(t) + a_{22} \phi_2(t)$$

$$\langle v_1 \, v_2 \rangle = |v_1| \, |v_2| \cos\theta$$

Soln.

$$\langle s_1(t), s_2(t) \rangle = \int s_1(t) \, s_2^*(t) \, dt.$$

$$\langle s_1(t), s_2(t) \rangle = |s_1(t)| \, |s_2(t)| \cos\theta.$$

Squaring both side

$$|\langle s_1(t), s_2(t) \rangle|^2 = |s_1(t)|^2 \, |s_2(t)|^2 \cos^2\theta$$

$$-1 \leq \cos\theta \leq 1 \quad \Rightarrow 0 \leq \cos^2\theta \leq 1.$$

$$\langle s_1(t), s_2(t) \rangle^2 \leq |s_1(t)|^2 \, |s_2(t)|^2$$

$$|s_1(t)| = \sqrt{\int s_1^2(t) \, dt}$$

$$|s_2(t)| = \sqrt{\int s_2^2(t) \, dt.}$$

$$\langle s_1(t), s_2(t) \rangle \leq E_1 E_2.$$

$$E_1 = \int s_1^2(t)\, dt$$

$$E_2 = \int s_2^2(t)\, dt.$$

30/8/23

Signal component

$$s(t) * h(t) = \int s(\tau)\, h(t-\tau)\, d\tau = \int h(\tau)\, s(t-\tau)\, d\tau$$

$$\int h(\tau)\, s(t-\tau)\, d\tau = \langle h(\tau), s(t-\tau) \rangle$$ dot product.

$$\left(\int h(\tau)\, s(t-\tau)\, d\tau \right)^2 \leq \int h^2(t)\, \tau \cdot \int s^2(t-\tau)\, d\tau$$

$$(\langle v_1, v_2 \rangle)^2 = (|v_1||v_2| \cos\theta)^2$$

$$= |v_1|^2 |v_2|^2 \cos^2\theta.$$

$-1 \leq \cos\theta \leq 1$

$$\Rightarrow 0 \leq \cos^2\theta \leq 1$$

$$(\langle v_1, v_2 \rangle)^2 \leq |v_1|^2 |v_2|^2.$$

Cauchy - Schwartz Inequality.

Note :-
The above inequality has equality when $v_1 = a v_2$.

$$\left(\int_0^t h(\tau)\, s(t-\tau)\, d\tau \right)^2 \leq E_h \int_0^t s^2(t-\tau)\, d\tau$$

Inequality will become equal
when iff $\boxed{h(\tau) = k\, s(T-\tau).}$; $t = T$

$K = 1$

$h(\tau) = s(T-\tau)$

$h(t) = s(T-t)$.

↑ matched filter response

$E_h = E$

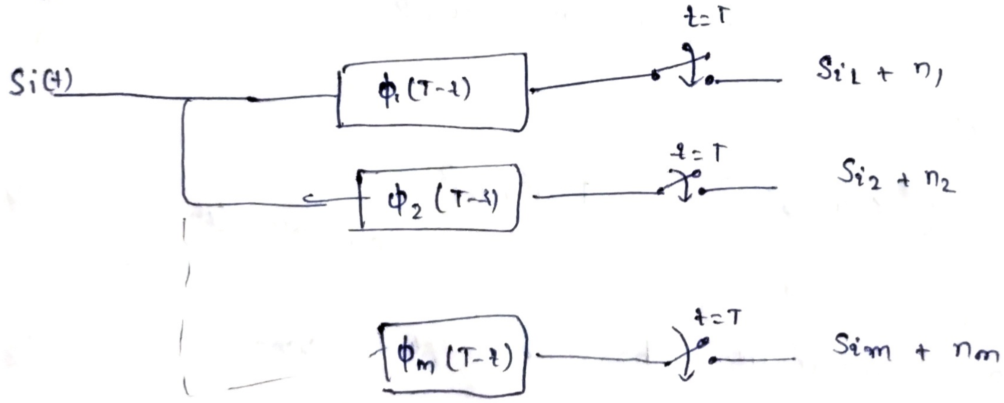
for the matched filter SNR is

$$SNR = \frac{K E_s^2}{K E_s \frac{N_0}{2}}$$

$$\boxed{SNR = \frac{E_s}{N_0/2}}$$ — Same as Intuitive/ receiver.

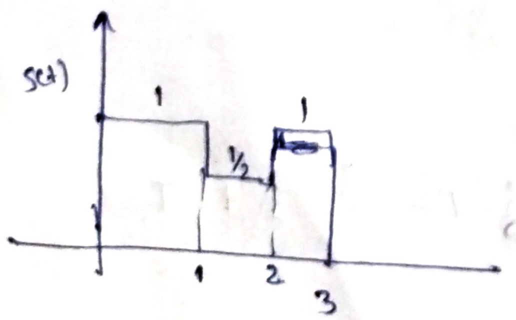
$Y_{corr}(T \pm \Delta t) \neq Y_{matched}(T \pm \Delta t)$

/
output of correlator receiver



Matched filter Based Receiver.

Q.



$h(t) = s(T-t)$.

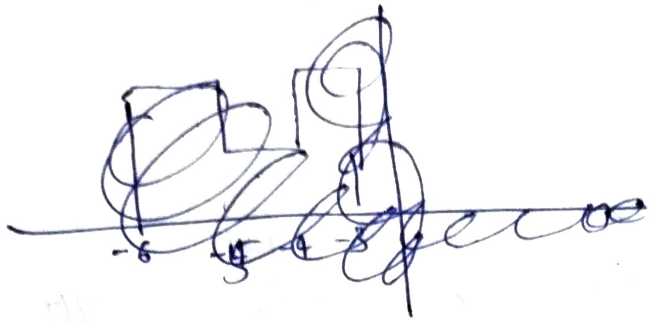
$h(t) = s(3-t) = \underline{s(t)}$.

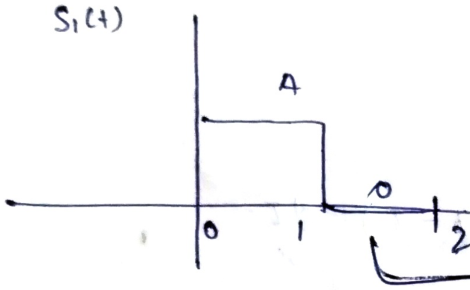
-3 -2 -1 0

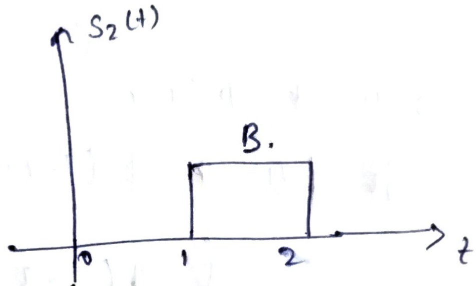
0 1 2 3

1-2 -1

2-2 = 0





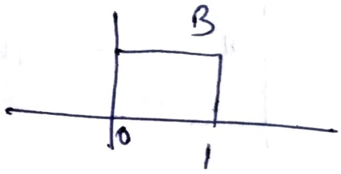


T = 2

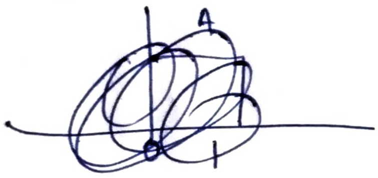
for both

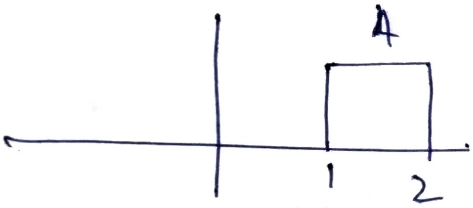
same time sample

$h_2(t) = h(2-t)$



$h(t) = s_1(2-t)$





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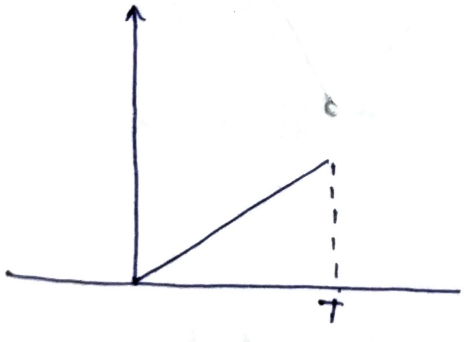

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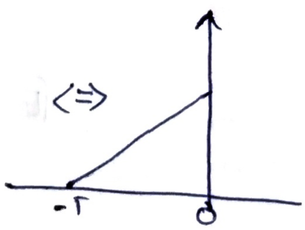

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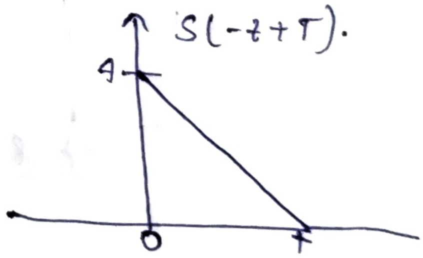


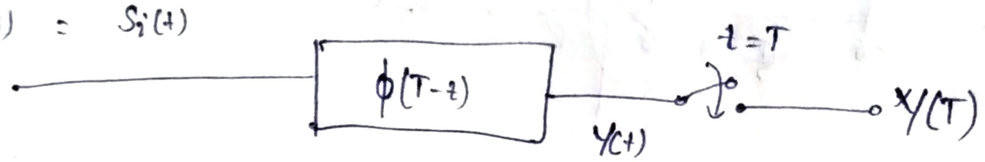
Standard PAM

$0 \rightarrow -\sqrt{E}\ \phi(t) = s_1(t)$

$1 \rightarrow \sqrt{E}\ \phi(t) = s_2(t)$

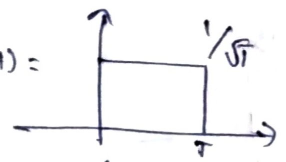
Matched filter based receiver

$x(t) = s_i(t)$



$$y(t) = s_1(t) * \phi(T-t)$$
$$= -\sqrt{E}\ \phi(t) * \phi(T-t)$$
$$= -\sqrt{E}\int \phi(T-\tau)\ \phi(t-\tau)\, d\tau.$$

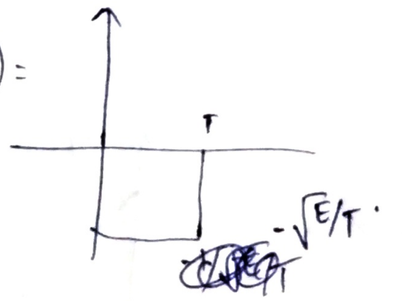
$\phi(t) =$

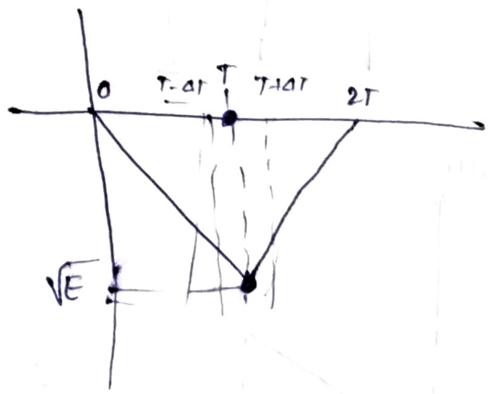


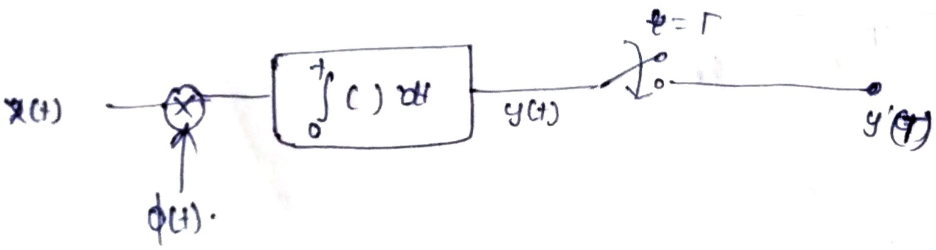
$= -\sqrt{E}\int \phi(t)$

convulation $\phi(t) * \phi(t)$

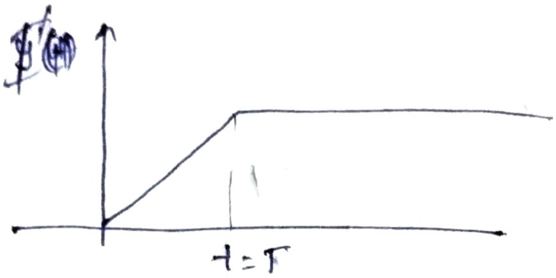
$s_1(t) =$

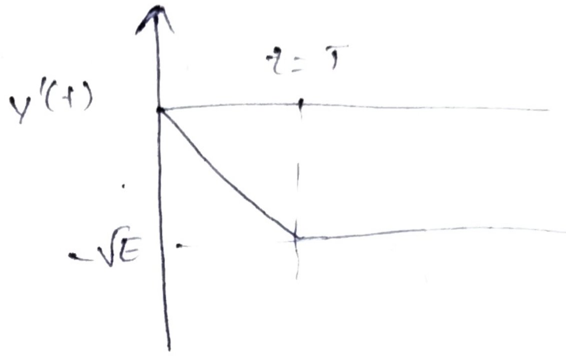






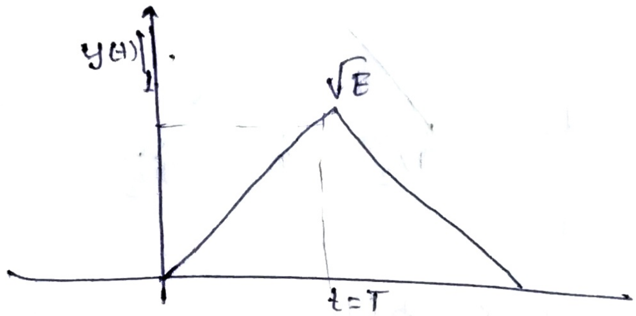
$$y'(t) = -\sqrt{E}\int_0^t \phi(u)\ \phi(v)\ du.$$





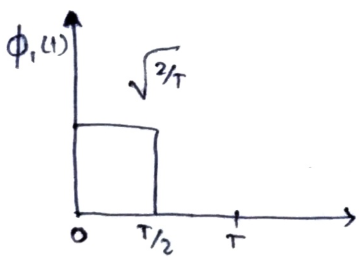
$$\boxed{y(t \pm \Delta t) \neq y'(t \pm \Delta t).}$$

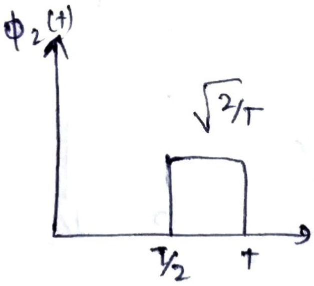
$$y(t)\big|_1 = y(t)\big|_{s_2(t)}$$



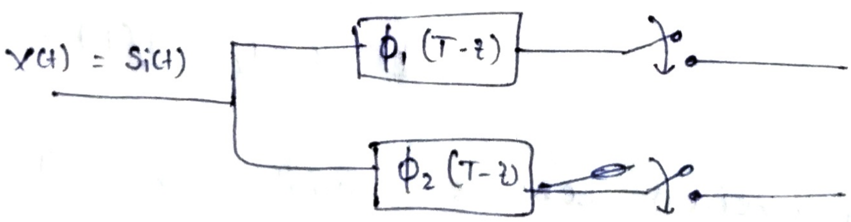
$$y(T)\big|_1 = \sqrt{E} + \underbrace{n}_{\text{noise}}.$$

~~Two dimen~~



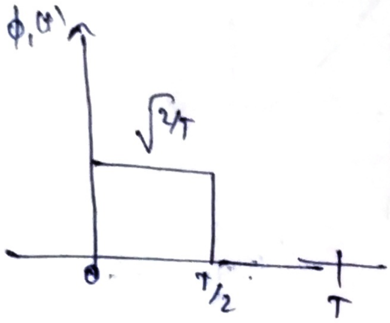


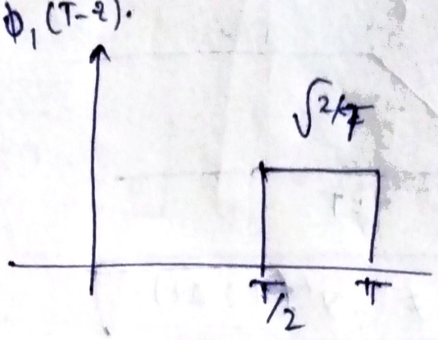
$$\left.\begin{aligned} s_1(t) &= \sqrt{E}\ \phi_1(t) \\ s_2(t) &= \sqrt{E}\ \phi_2(t) \end{aligned}\right\}$$ Two dimension digital filter.



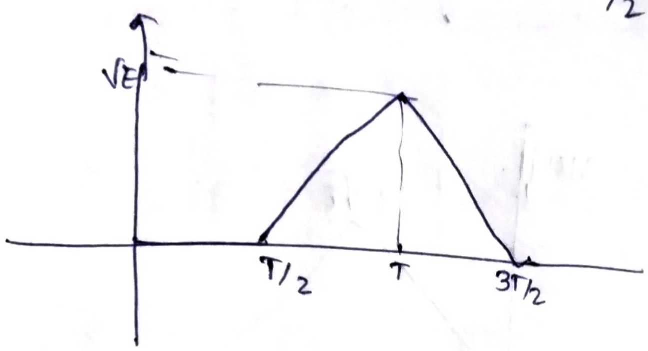
$$y_1(t) = s_1(t) * \phi_1(T-\tau)$$

$$= \sqrt{E}\ \phi_1(t) * \phi_1(T-\tau)$$





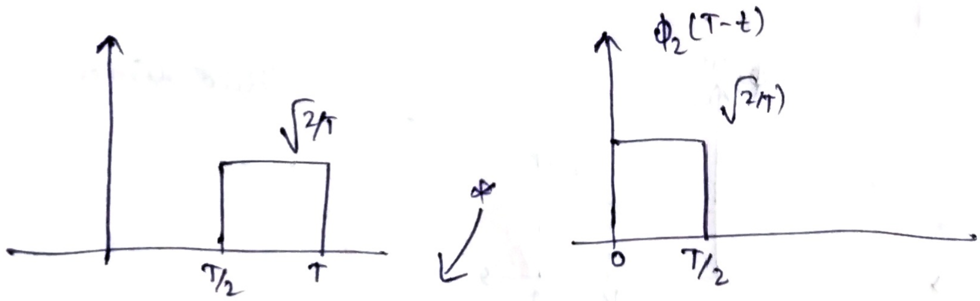
$-T \ \ -T/2$

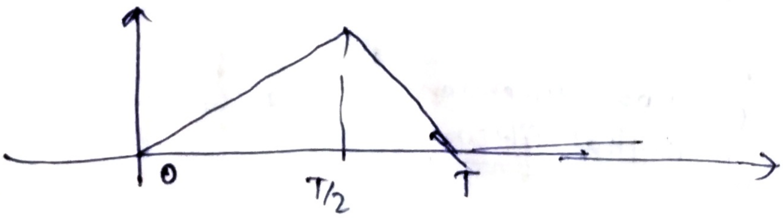


$t = T = \sqrt{E}$

$$y_2(t) = s_2(t) * \phi_2(T-\tau)$$

$$= \sqrt{E}\ \phi_2(t) * \phi_2(T-t)$$





At $t = T$, $y_2(T) = 0$

Note:-

Both types of receivers behaves Similar at $t = T$.

BER Analysis of BPAM

Probability of Error Analysis of BPAM

b_i	a_i	$s_i(t)$	$x(t)$	
0	$-\sqrt{E}$	$= -\sqrt{E}\,\phi(t)$	$= -\sqrt{E}\,\phi(t) + n(t)$	$= -\sqrt{E} + n$
1	$+\sqrt{E}$	$= +\sqrt{E}\,\phi(t)$	$= +\sqrt{E}\,\phi(t) + n(t)$	$= +\sqrt{E} + n.$

Intuitive Receiver

$$x \underset{0}{\overset{1}{\gtrless}} 0 \quad \{BPAM\}.$$

- Decision region
- Decision criteria
- Sufficient statistics.

$P_r(0) \neq P_r(1)$

$P_r(0) = P$

$P_r(1) = 1 - P$

In this case I.R is valid?

$$\boxed{\begin{array}{c} x \overset{1}{\geq} \alpha \\ \hline x \overset{0}{<} \alpha \end{array}}$$

decides '1'

decides '0'.

error region $x > \alpha$

$$P_e = p \cdot P_{e|0} + (1-P)\, P_{e|1}$$

$$\frac{dP_e}{d\alpha} = 0 \Rightarrow \alpha^* \text{ — optimum alpha.}$$

$$P_{e|0} = \int_{\alpha}^{\infty} \frac{1}{\sqrt{2\pi \frac{N_0}{2}}}\, e^{-\frac{(x+\sqrt{E})^2}{2\,N_0/2}}\, dx.$$

$$P_{e|1} = \int_{-\infty}^{\alpha} \frac{1}{\sqrt{2\pi \frac{N_0}{2}}}\, e^{-\frac{(x-\sqrt{E})^2}{2\,N_0/2}}\, dx.$$

$$P_e = \frac{1}{\sqrt{2\pi \frac{N_0}{2}}} \left[p \int_{\alpha}^{\infty} e^{-\frac{(x+\sqrt{E})^2}{N_0}}\, dx + (1-p) \int_{-\infty}^{\alpha} e^{-\frac{(x-\sqrt{E})^2}{N_0}}\, dx \right].$$

$$\frac{dP_e}{d\alpha} = \frac{d}{d\alpha}\left[\quad \cdot \quad - \right]$$

$$\frac{d}{dx}\left[\int_{a(x)}^{b(x)} f(x,t)\,dt\right] = f(b(x),t)\frac{d\,b(x)}{dt} - f(a(x),t)\frac{d\,a(x)}{dt}$$

$$+ \int_{a[x]}^{b[x]} \frac{\partial}{\partial x} f(x,t)\,dt.$$

$$\frac{dP_e}{d\alpha} = p\left[\cancel{f(\infty|s_1(t))\frac{d\infty}{d\alpha}}^{0} - f(\alpha/s_1(t))\frac{d\alpha}{d\alpha}\right] + 0$$

$$+ (1-p)\left[f(\alpha|s_2(t))\frac{d\alpha}{d\alpha} - \cancel{f(-\infty/s_2)\frac{d(-\infty)}{d\alpha}}^{0}\right].$$

$$\boxed{\frac{dP_e}{d\alpha} = p\left(-f(\alpha|s_1(t))\right) + (1-p)\, f(\alpha|s_2(t)).}$$

$$-p\, f(\alpha^*|s_1(t)) + (1-p)\, f(\alpha^*|s_2(t)) = 0$$

$$f(\alpha^*|s_2(t))\,(1-p) = p\, f(\alpha^*|s_1(t))$$

$$\boxed{\frac{f(\alpha^*|s_2(t))}{f(\alpha^*|s_1(t))} = \frac{p}{1-p}}$$

Generalized rule to calculate 'α'.

MAP of Maximum Aposteriori probability

for Gaussian case

$$f(x|s_2(t)) = \frac{1}{\sqrt{2\pi \cdot N_0/2}}\, e^{-\frac{(x-\sqrt{E})^2}{2 N_0/2}}$$

$$f(x|s_1(t)) = \frac{1}{\sqrt{2\pi\, N_0/2}}\, e^{-\frac{(x+\sqrt{E})^2}{2 N_0/2}}.$$

$$e^{\frac{-(x-\sqrt{E})^2}{N_0} + \frac{(x+\sqrt{E})^2}{N_0}}$$

$$-\cancel{x^2} - \cancel{E} + 2x\sqrt{E} + \cancel{x^2} + \cancel{E} + 2x\sqrt{E}$$

$$e^{+\frac{4\alpha^* \sqrt{E}}{N_0}} = \frac{p}{1-p}$$

$$\frac{4\alpha^* \sqrt{E}}{N_0} = \ln p/1-p$$

$$\boxed{\alpha^* = \frac{N_0}{4\sqrt{E}} \ln \frac{p}{1-p}}$$

MAP receiver.

$$\cancel{\alpha^* = \frac{N_0}{4\sqrt{E}} \ln \frac{0.6}{0.4}}$$

for $p = 1/2$

$\alpha^* = 0$ ∴ [Intuitive receiver].

Eg. $p(0) = 0.6$

$p(1) = 0.4$

$$\alpha^* = \frac{N_0}{4\sqrt{E}} \ln 6/4.$$

$$= \frac{0.405\, N_0}{4\sqrt{E}}.$$

~~Pe|o~~

$$x \underset{0}{\overset{1}{\gtrless}} \alpha^*$$

$$P_{e|0} = \int_{\alpha^*}^{\infty} f_{x|s_1(t)}(x)\, dx.$$

$$= \int_{\alpha^*}^{\infty} \frac{1}{\sqrt{2\pi \frac{N_0}{2}}} e^{\frac{-(x+\sqrt{E})^2}{2 \cdot N_0/2}} \cdot dx.$$

$$\frac{x+\sqrt{E}}{\sqrt{N_0/2}} = u \qquad dx = \sqrt{\frac{2}{N_0}}\, du.$$

$$= \int_{\frac{\alpha^* + \sqrt{E}}{\sqrt{N_0/2}}}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-u^2/2}\, du.$$

$$P_{e|0} = Q\left[\frac{\alpha^* + \sqrt{E}}{\sqrt{\frac{N_0}{2}}}\right]$$

$$P_{e|1} = 1 - Q\left[\frac{\alpha^* - \sqrt{E}}{\sqrt{N_0/2}}\right]$$

$$\frac{f(\alpha^* | s_2(t))}{f(\alpha^* | s_1(t))} = \frac{p}{1-p}.$$

$$\boxed{P_r(1)\ f(\alpha^* | s_2(t)) = P_r(0)\ f(\alpha^* | s_1(t))}$$

$$\frac{P_r(1)\ f(\alpha^* | s_2(t))}{P_r(1)\ f(\alpha^* | s_2(t)) + P_r(0)\ f(\alpha^* | s_2(t))} = \frac{P_r(0)\ f(\alpha^* | s_1(t))}{P_r(1)\ f(\alpha^* | s_2(t)) + P_r(0) + f(\alpha^* | s_2(t))}$$

$$\boxed{f(s_2(t) | \alpha^*) = P(s_1(t) | \alpha^*).}$$ — aposterior probability.

So, for any α,

$$\boxed{f(s_2(t) | x) \underset{0}{\overset{1}{\gtrless}} f(s_1(t) | x).}$$ — Greater than it belongs to 1, less than it belongs to 0.

aposterior of $s_2(t)$

aposterior of $s_1(t)$.

If $P_r(0) = P_r(1) = 1/2$

prior probability

$$f(x | s_2(t)) \underset{0}{\overset{1}{\gtrless}} f(x | s_1(t)).$$ — likelihood.

$$\boxed{m^* = \max_{1 \le m \le M} f(r | s_m(t))}$$

maximum likelihood receiver.

$$\boxed{m^* = \max_{1 \le m \le M} f(s_m(t) \mid x)}$$ — MAP detector.

$x < \alpha^*$ $\quad$ $x > \alpha^*$

DR - 0 $\quad$ DR = 1

$$\alpha^* = \frac{4}{N_0\sqrt{E}} \ln \frac{p}{1-p}$$

ML - case

Special case of MAP $p = 1/2$

DR - 0 $\quad$ DR = 1

$-\sqrt{E}$ $\quad$ 0 $\quad$ $\sqrt{E}$

d_1 $\quad$ d_2

distance b/w constellation and point.

$d_1 > d_2$ - 1

$d_2 \ngtr d_1$ - 0.

ML receiver is also known as minimum distance receiver.

$$P_{e|0} = Q\left(\sqrt{\frac{2E}{N_0}}\right)$$

$-\sqrt{E}$ $\quad$ $+\sqrt{E}$

d_{12}

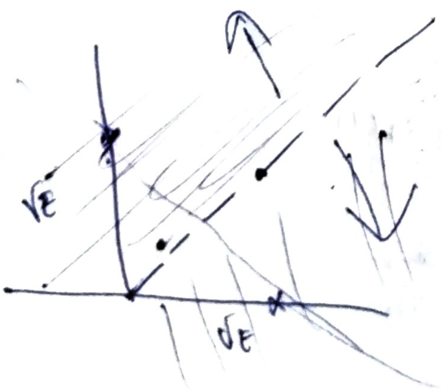
$2\sqrt{E}$

for ML case only.

$$\boxed{P_{e|0} = Q\left(\frac{d_{12}}{\sqrt{2N_0}}\right)}$$

d_{12} = distance b/w two pts.

$$P_e = Q\left(\frac{d_{12}}{\sqrt{2N_0}}\right).$$



$d_{12} = 2E$

$$P_e = Q\left(\frac{2E}{\sqrt{2N_0}}\right)$$

$$= Q\left(\sqrt{\frac{E}{N_0}}\right).$$

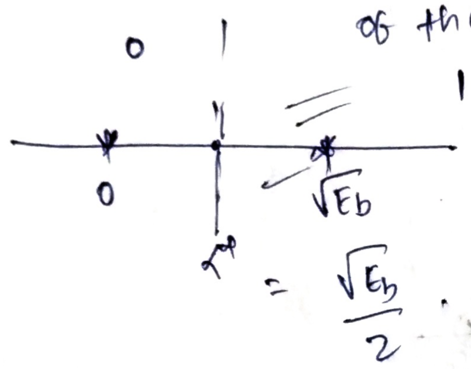
ML detection
↓
In this discussion.

7/09/23

ML detection

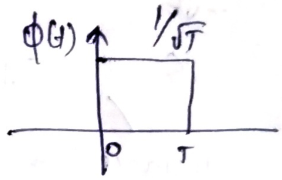
$$\boxed{x \underset{0}{\overset{1}{\gtrless}} 0.}$$ — Antipodal Signalling.

⇑ middle pt. of the constellation.



ON-OFF Signalling.

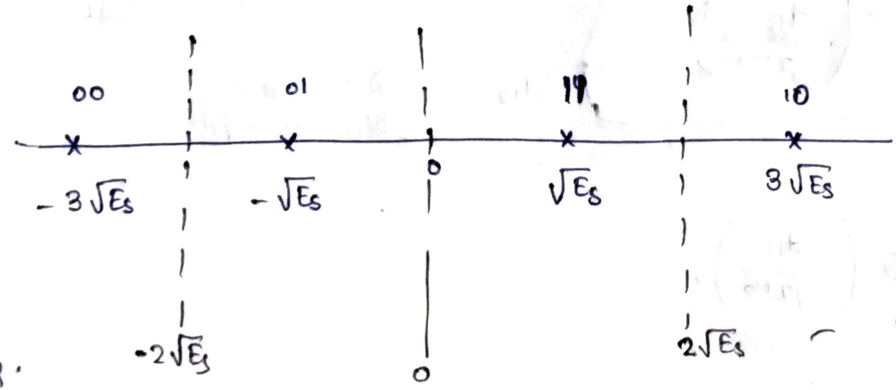
Standard binary PAM.



4-PAM case

4-ary PAM

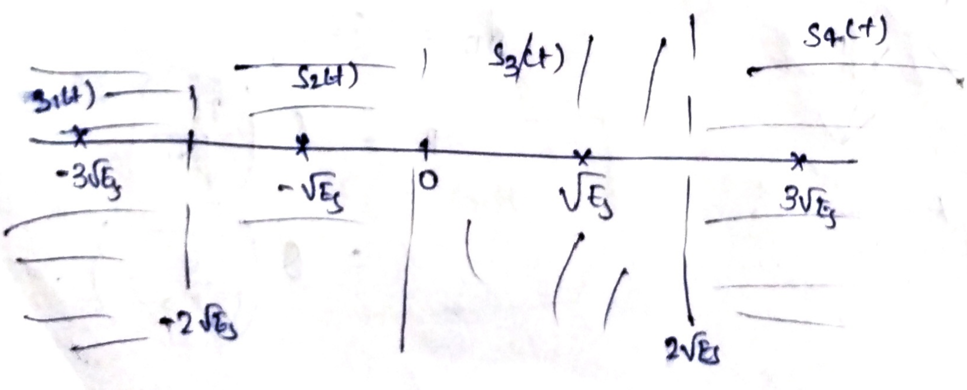
			Symbols		
b_1	→	00	→	$s_1(t)$	$= -3\sqrt{E_s}\,\phi(t)$
b_2	→	01	→	$s_2(t)$	$= -\sqrt{E_s}\,\phi(t)$
b_3	→	10	→	$s_3(t)$	$= \sqrt{E_s}\,\phi(t)$
b_4	→	11	→	$s_4(t)$	$= 3\sqrt{E_s}\,\phi(t)$.



Arbitary bit assignment.

decision region.

Gray coding manner bit assigning.
⇓
one bit changing.



$s_1(t)$ $\quad x < -2\sqrt{E_s}$

$s_2(t)$ $\quad -2\sqrt{E_s} < x \le 0$

$s_3(t)$ $\quad 0 \le x \le 2\sqrt{E_s}$

$s_4(t)$ $\quad x > 2\sqrt{E_s}$.

s_2 & $s_3(t)$ have more probability of confusion.

x

$x < -2\sqrt{E_s}$

$x > 2\sqrt{E_s}$ $\quad 1 - Q(2\sqrt{E_s})$.

$$P_{e|s_1(t)} = \int_{-2\sqrt{E_s}}^{\infty} f_{x|s_1(t)}(x)\,dx.$$

$$\int_{-\infty}^{x} e^{-x^2/2}\,dx.$$

$$P_{e|s_1(t)} = 1 - \int_{-\infty}^{-2\sqrt{E_s}} f_{x|s_1(t)}(x)\,dx.$$

$$\boxed{Q(x) = 1 - Q(-x)}$$

$$f_{x|s_1(t)}(x|s_1(t)) = \frac{1}{\sqrt{2\pi N_0/2}}\, e^{-\frac{(x-(-3\sqrt{E_s}))^2}{2\,N_0/2}}$$

$$= \frac{1}{\sqrt{2\pi N_0/2}}\, e^{-\frac{(x+3\sqrt{E_s})^2}{N_0}}$$

$$P_{e|s_1(t)} = \int_{-2\sqrt{E_s}}^{\infty} \frac{1}{\sqrt{2\pi N_0/2}}\, e^{-\frac{(x+3\sqrt{E_s})^2}{N_0}}\,dx.$$

$$= \int_{\frac{\sqrt{E_s}}{\sqrt{N_0/2}}}^{\infty} \frac{1}{\sqrt{2\pi}}\, e^{-\frac{u^2}{2}}\,du.$$

$$\frac{x+3\sqrt{E_s}}{\sqrt{N_0/2}} = u$$

$$dx = \sqrt{\frac{2}{N_0}}\,du.$$

$$= Q\left(\sqrt{\frac{2E_s}{N_0}}\right).$$

$$\boxed{P_{e|s_4(t)} = P_{e|s_1(t)} = Q\left(\sqrt{\frac{2E_s}{N_0}}\right).}$$

$P_{e|s_2(t)} \rightarrow$

$$f_{x|s_2(t)}(x|s_2(t)) = \frac{1}{\sqrt{2\pi N_0/2}} \, e^{-\frac{(x+\sqrt{E_s})^2}{2 N_0/2}}.$$

$$P_{e|s_2(t)} = \int_{-\infty}^{-2\sqrt{E_s}} \frac{1}{\sqrt{2\pi N_0/2}} \, e^{-\frac{(x+\sqrt{E_s})^2}{2 N_0/2}} \, dx.$$

$$\boxed{P_{e|s_2(t)} = P_{e|s_3(t)} = 2\,Q\left(\sqrt{\frac{2E_s}{N_0}}\right).}$$

$$P_e = \sum p_i \, P_{e|s_i(t)}$$

$$= \frac{1}{4}\left[Q\left(\sqrt{\frac{2E_s}{N_0}}\right) + 2\,Q\left(\sqrt{\frac{2E_s}{N_0}}\right) + 2\,Q\left(\sqrt{\frac{2E_s}{N_0}}\right) + Q\left(\sqrt{\frac{2E_s}{N_0}}\right)\right].$$

$$\boxed{P_e = \frac{3}{2}\,Q\left(\sqrt{\frac{2E_s}{N_0}}\right).} \Rightarrow 1.5\,Q\left(\sqrt{\frac{2E_s}{N_0}}\right).$$

1.5 → Avg. No. of neighbours that encountered by an Symbol in 4-PAM — Immediate Neighbours

Average Energy $= \sum p_i E_i$

$E_1 = (-3\sqrt{E_s})^2 = 9E_s$

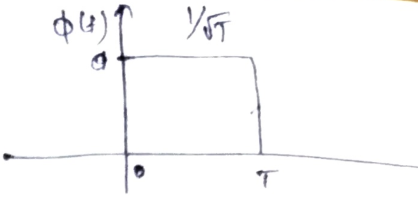
$E_2 = (-\sqrt{E_s})^2 = E_s$

$E_3 = (\sqrt{E_s})^2 = 1E_s$

$E_4 = (3\sqrt{E_s})^2 = 9E_s.$

$$E_{Avg} = \frac{1}{4}\{20E_s\} = 5E_s$$

M-ary - PAM



$$s_i(t) = (2i-1-M)\sqrt{E_s}\,\phi(t) \qquad i = 1 \text{ to } M.$$

$\log_2 M$ — no. of bits.

$M = 2, 4, 8, 16, \ldots$

$M = 4$

$$s_1(t) = (2-1-4)\sqrt{E_s}\,\phi(t) = -3\sqrt{E_s}\,\phi(t)$$

$$s_2(t) = (4-1-4)\sqrt{E_s} = -\sqrt{E_s}\,\phi(t)$$

$$s_3(t) = (6-1-4)\sqrt{E_s} = \sqrt{E_s}\,\phi(t)$$

$$s_4(t) = (8-1-4)\sqrt{E_s} = 3\sqrt{E_s}\,\phi(t).$$

$\underbrace{2-1-8}_{-7}$

for generic 'M'

$$E_{avg} = \sum_{i=1}^{M} \frac{(2i-1-M)^2(\sqrt{E_s})^2}{M}$$

$$= \frac{\sum_{i=1}^{M}(4i^2 - 2iM + m^2)E_s}{M}$$

$$= \frac{(M^2-1)}{3}E_s.$$

$M=4$, $\quad E_{avg} = \frac{(16-1)}{3}E_s = 5E_s.$

$T_s = 2T_b$ — bit rate

symbol rate

$E_s = 2E_b.$

$$P_e = \sum P_i \, P_e|s_i(t)$$

$$= 2Q\left(\sqrt{\frac{2E_s}{N_0}}\right) + (M-2)\times 2\,Q\left(\sqrt{\frac{2E_s}{N_0}}\right).$$

last symbol

$(2M-1-M)\sqrt{E_s}$

$(M-1)\sqrt{E_s}$.

~~l~~ast symbol.

$(2-1-M)\sqrt{E_s}$

$(1-M)\sqrt{E_s}$

$Q\left(\sqrt{\frac{2E_s}{N_0}}\right)$ — edge symbol error

$P_{e|s_0(t)} = Q\left(\sqrt{\frac{2E_s}{N_0}}\right)$.

$$P_e = \frac{1}{M}\left[2\,Q\left(\sqrt{\frac{2E_s}{N_0}}\right) + \underbrace{(M-2)\cdot 2\,Q\left(\sqrt{\frac{2E_s}{N_0}}\right)}\right].$$

↳ probability of middle having two neighbours.

$$P_e = \frac{1}{M} \times (2M-2)\,Q\left(\sqrt{\frac{2E_s}{N_0}}\right).$$

$$\boxed{P_e = \underbrace{\frac{2(M-1)}{M}}\,Q\left(\sqrt{\frac{2E_s}{N_0}}\right).}$$

Average no. of neighbours that each symbol have.

$M=4$,

$$P_e = \frac{2\times 3}{4}\,Q\left(\sqrt{\frac{2E_s}{N_0}}\right) = 3/2\,Q\left(\sqrt{\frac{2E_s}{N_0}}\right).$$

for high M

$$\boxed{P_e = 2\,Q\left(\sqrt{\frac{2E_s}{N_0}}\right)}$$

P_e in terms of d_{min}

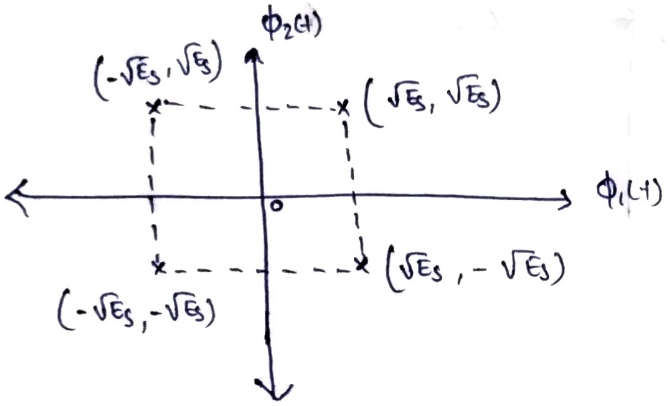
$$\boxed{P_e = \frac{2(M-1)}{M}\,Q\left(\frac{d_{min}}{\sqrt{2N_0}}\right).}$$

⇒ BPSK, BASK is some sort of BPAM.

Two dimensional M-PAM {Baseband}.

→ QPSK, 8 PSK, QAM (pass band)

$$s_i(t) = (2i-1-M)\sqrt{E_s}\,\phi(t) \quad \{\text{one dimensional}\}.$$



$$s_i(t) = a_i\,\phi_1(t) + b_i\,\phi_2(t)$$ } ← combination of $2(\sqrt{M})$ PAM.

$$a_i = \left(2i-1-\sqrt{M}\right)\sqrt{E_s} \; ; \quad b_i = \left(2i-1-\sqrt{M}\right)\sqrt{E_s}$$

$$; \; 1 \le i \le \sqrt{M}.$$

M=4.

$1 \le i \le 2.$

$a_1 = (2-1-2)\sqrt{E_s}$

$= -1\sqrt{E_s}$

$a_2 = (4-1-2)\sqrt{E_s}$

$= \sqrt{E_s}$

$b_1 = (2-1-2)\sqrt{E_s}$

$= -\sqrt{E_s}$

$b_2 = (4-1-2)\sqrt{E_s}$

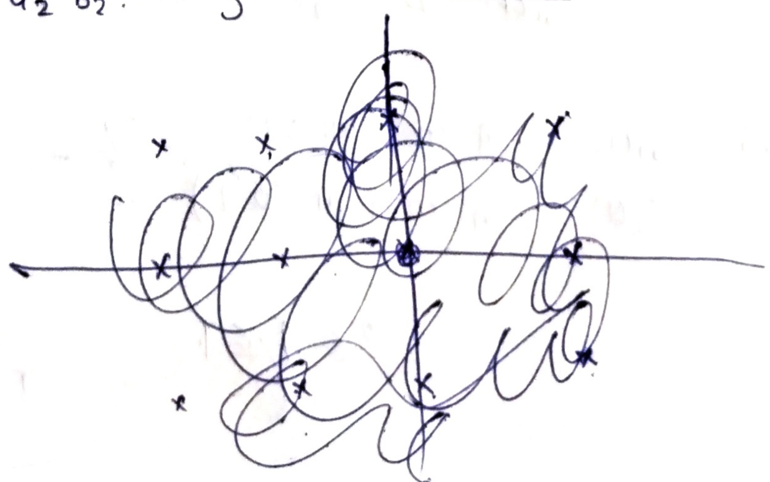
$= \sqrt{E_s}.$

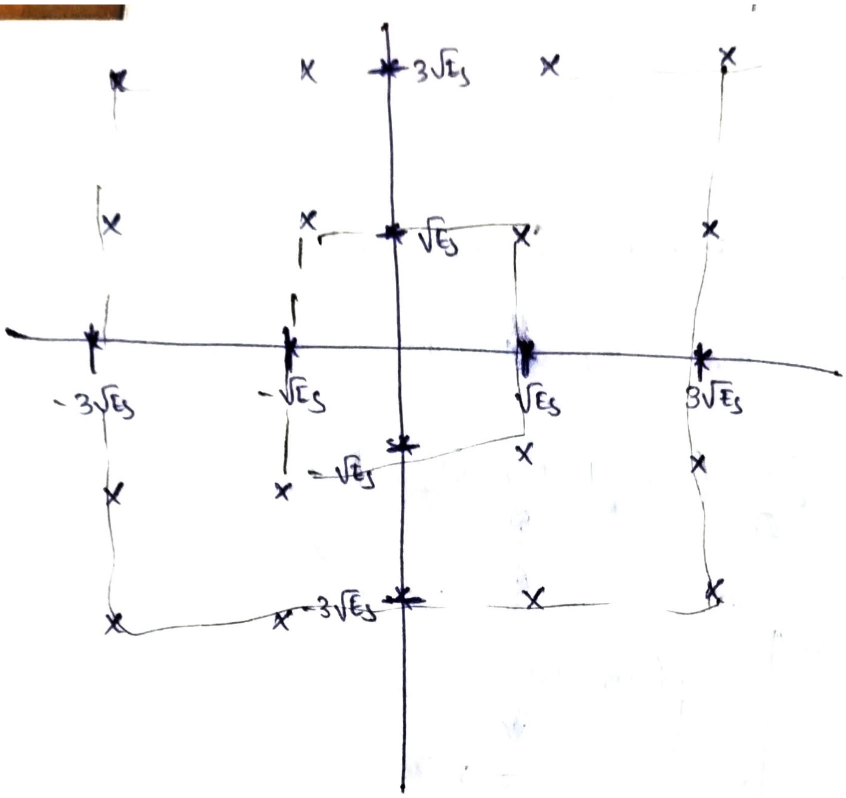
$a_1 b_1, \; a_1 b_2, \; a_2 b_1, \; a_2 b_2.$ } ← combination..

M=16

$a_1 = (2-1-8)\sqrt{E_s}$

=





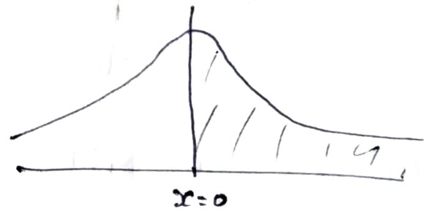
16 PAM in 2 Dimension.

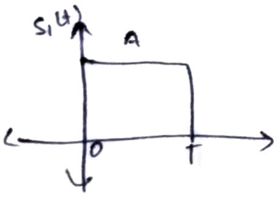
Quiz soln:

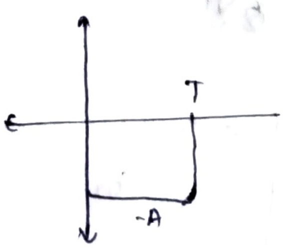
$$P_e = Q(\sqrt{SNR})$$

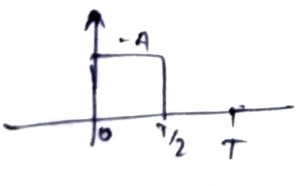
$$= Q(0)$$

$$= \frac{1}{2}$$









$s_1(t)$ & $s_2(t)$ are distinct Signal and $s_3(t)$

Q3.

$$\left.\begin{aligned} s_1(t) &= A\cos(2\pi \cdot 5t) \\ s_2(t) &= A\cos(2\pi \cdot 10 \cdot t) \end{aligned}\right\} - \text{Harmonic Signal}$$

⇓

Orthogonal signal.

$0 \le t \le T$

$T = 1/5$

Energy of $S_1(t)$

$$\frac{A^2 T}{2} = E_2$$

$$\frac{A^2 T}{2} = E_2.$$

$$\int_0^{0.2} S_1(t)\, S_2(t)\, dt = 0.$$

$$|S_1(t)| = \sqrt{A^2 \times \cancel{0.2}\, 0.1 / \cancel{2}}$$

$$= \frac{A}{\sqrt{10}}.$$

$$\phi_1(t) = \sqrt{\frac{2}{T}} \cos 2\pi 5 t \quad ; \; 0 \le t \le T$$

$$\phi_2(t) = \sqrt{2/T} \cos 2\pi \times 20 t. \quad ; \; 0 \le t \le T.$$

Energy Efficient

less Eavg

d). Average energy

$$E_{avg} = \sum_{i=1}^{3} P_i E_i$$

$$= P_1 E_1 + P_2 E_2 + P_3 E_3$$

$$= \frac{1}{3} \left[E_1 + E_2 + E_3 \right].$$

$$= \frac{1}{3} \left\{ \frac{A^2 T}{2} + \frac{A^2 T}{2} + \frac{1}{4} \left[\frac{A^2 T}{2} + \frac{A^2 T}{2} \right] \right\}.$$

$$1 + 1 + 1/2 \}$$

$$E_{avg} = 5/2 \times \frac{A^2 T}{2} \times \frac{1}{3}$$

$$= 5/12\, A^2 T.$$

e

$$2/5 + 2x = 1$$

$$2x = 3/5$$

$$x = 3/10$$

$$P\{S_3(t)\} = 2/5.$$

$$P_r\{S_1(t)\} = P_r\{S_2(t)\} = 3/10$$

$$E_{avg} = \frac{3}{10} E_1 + \frac{3}{10} E_2 + \frac{2}{5} E_3.$$

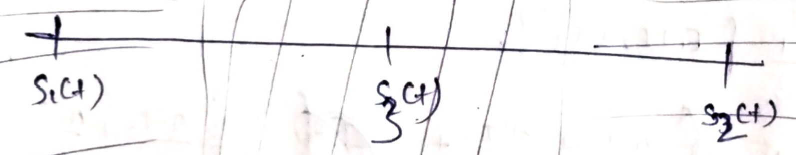
(f) Avg. probability of error;

$$P_{e|1} = Q\left(\frac{d_{13}}{\sqrt{2N_0}}\right)$$

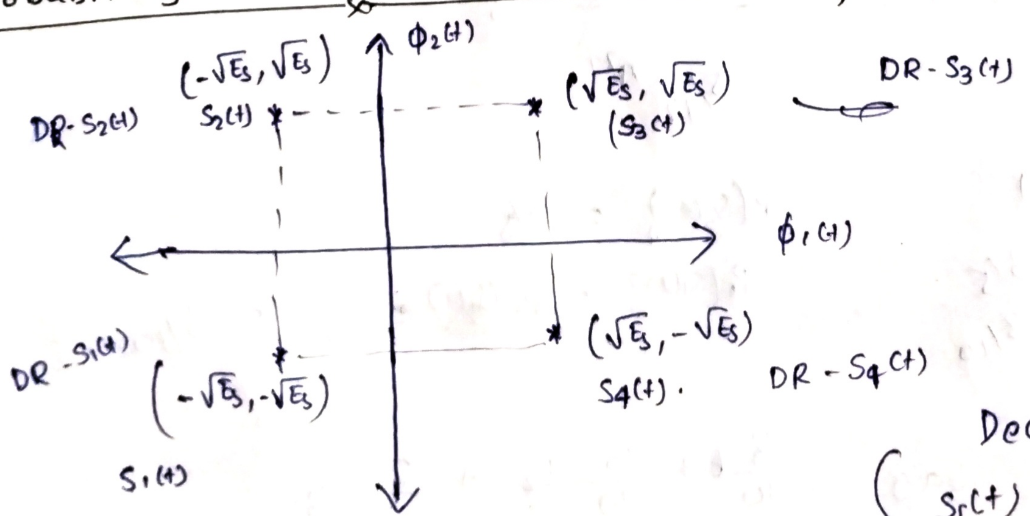
$$P_{e|2} = Q\left(\frac{d_{23}}{\sqrt{2N_0}}\right)$$

$$P_{e|3} = 2\,Q\left(\frac{d_{13}}{\sqrt{2N_0}}\right) = 2Q\left(\frac{d_{23}}{\sqrt{2N_0}}\right).$$

Probability of error doesn't depend on basis fn. } – depend on signal point.



Probability of Error for 2D - 4 PAM



Decision region

ML desion rule:
- $S_1(t)$ = Quadrant 3
- $S_2(t)$ = Quadrant 2
- $S_3(t)$ = Quadrant 1
- $S_4(t)$ = Quadrant 4

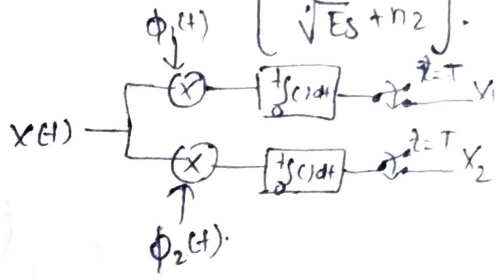
$P_e | s_3(t) = 1 - P_c | s_3(t)$

$$\underline{x}\Big|_{s_3(t)} = \begin{bmatrix} \sqrt{E_s} \\ \sqrt{E_s} \end{bmatrix} + \begin{bmatrix} n_1 \\ n_2 \end{bmatrix} = \begin{bmatrix} \sqrt{E_s} + n_1 \\ \sqrt{E_s} + n_2 \end{bmatrix}.$$

H.W exam

End sem

$E[n_1 n_2] = 0$ — Independent.



$$P_c | s_3(t) = \int_{x_1=0}^{\infty} \int_{x_2=0}^{\infty} f_{x_1 x_2}(x_1, x_2)\, dx_1\, dx_2.$$

x_1 & x_2 are Independent

$$f_{x_1 x_2}(x_1, x_2|_{s_3}) = \underbrace{f_{x_1}(x_1|_{s_3})}_{\mathcal{N}(\sqrt{E_s},\, N_0/2)} \cdot \underbrace{f_{x_2}(x_2|_{s_3})}_{\mathcal{N}(\sqrt{E_s},\, N_0/2)} \Big\} \text{ - Marginal P.D.F.}$$

$$f_{x_1}(x_1|s_3) = \frac{1}{\sqrt{2\pi N_0/2}}\, e^{-\frac{(x - \sqrt{E_s})^2}{2 N_0/2}} = f_{x_2}(x_2|s_3).$$

$$= \int_{x_1=0}^{\infty} \frac{1}{\sqrt{2\pi N_0/2}}\, e^{-\frac{(x_1 - \sqrt{E_s})^2}{2 N_0/2}}\, dx_1 \cdot \int_{x_2=0}^{\infty} \frac{1}{\sqrt{2\pi N_0/2}}\, e^{-\frac{(x_2 - \sqrt{E_s})^2}{2 N_0/2}}\, dx_2.$$

$$\frac{x_1 - \sqrt{E_s}}{\sqrt{N_0/2}} = u$$

$$dx = du.$$

$$\int_{-\frac{\sqrt{E_s}}{\sqrt{N_0/2}}}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{u^2}{2}}\, du \cdot \int_{-\frac{\sqrt{E_s}}{\sqrt{N_0/2}}}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{v^2}{2}}\, dv.$$

$$Q\left(-\sqrt{\frac{2E_s}{N_0}}\right) \quad Q\left(-\sqrt{\frac{2E_s}{N_0}}\right).$$

$$= \left(1 - Q\left(\sqrt{\frac{2E_s}{N_0}}\right)\right)\left(1 - Q\left(\sqrt{\frac{2E_s}{N_0}}\right)\right).$$

Re ~~...~~ l-

$$P_{c|S_3(t)} = \left(1 - Q\left(\sqrt{\frac{2E_s}{N_0}}\right)\right)^2 = P_{c|S_2(t)}$$

$$P_{c|S_1(t)}$$

$$P_{c|S_4(t)}$$

$$\boxed{P_e = \sum P_i \, P_{e|s_i(t)}}.$$

$$P_e = \frac{1}{4} \times 4 \underbrace{\left[1 - \underbrace{\left(1 - Q\left(\sqrt{\frac{2E_s}{N_0}}\right)\right)^2}_{P_{c|s_i(t)}}\right]}_{P_{e|s_i(t)}}$$

$$\boxed{P_e = 2\,Q\left(\sqrt{\frac{2E_s}{N_0}}\right) - Q^2\left(\sqrt{\frac{2E_s}{N_0}}\right)}.$$

Range $\left(0 \text{ to } \frac{1}{2}\right)$.

for high SNR $\qquad \frac{E_s}{N_0/2} \uparrow\uparrow, \quad Q(\cdot) \downarrow\downarrow$

$$\boxed{P_e \simeq 2\,Q\left(\sqrt{\frac{2E_s}{N_0}}\right)}$$

2 → no. of nearest neighbours.

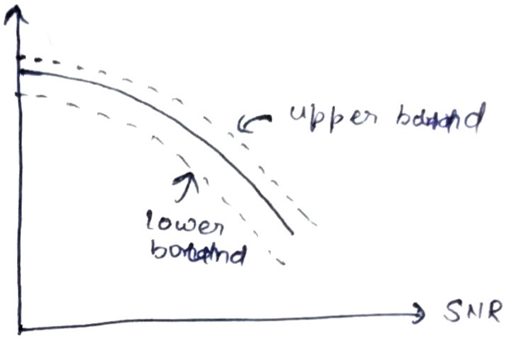
$$\boxed{P_e = 2Q\left(\frac{d_{min}}{\sqrt{2N_0}}\right)}.$$

Symbol error rate.

14/09/23

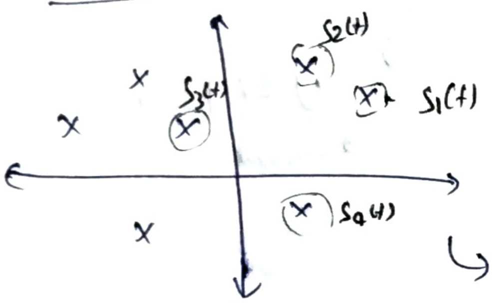
Union Bound on the Probability of Error

$P_e^L \le P_e \le P_e^U$



Bound should never cross Actual graph.

Equiprobable Symbols



↳ difficult to draw decision region } ⇒ difficult to calculate Pe.

$P_e = K\, Q\left(\sqrt{\frac{2E_s}{N_0}}\right)$. — General case

→ Union Bond is an upper bond on probability of error.

$A \cup B = A + B - A \cap B$

$P_r(A \cup B) = P_r(A) + P_r(B) - P_r(A \cap B)$.

$P_r(A \cup B \cup C) = P_r(A) + P_r(B) + P_r(C) - P_r(A \cap B \cap C) + \sum A \cap B + \ldots$

$P_r\left[\bigcup_{i=1}^{M} A_i\right] = \sum P_r(A_i) - \left\{\sum P_r\right\}$.

$$\boxed{P_r\left[\bigcup_{i=1}^{M} A_i\right] \le \sum P_r(A_i)}$$

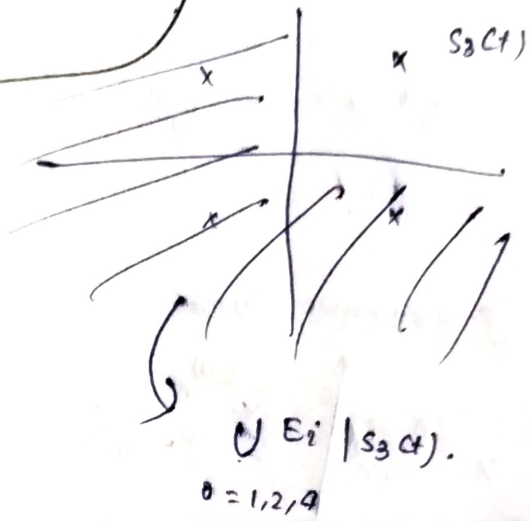
Assumtion :- each symbol causing error to the other symbol.

Error event for $s_i(t)$ is

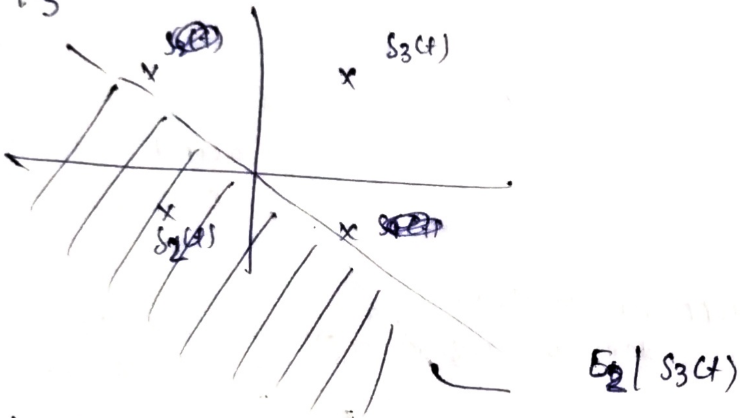
$$\boxed{P_{e|s_i(t)} = P_r\left[\bigcup_{\substack{i=1 \\ i \ne m}}^{M} \varepsilon_i \,\middle|\, s_m(t)\right]}$$

ε_i :- Error event due to ith event.

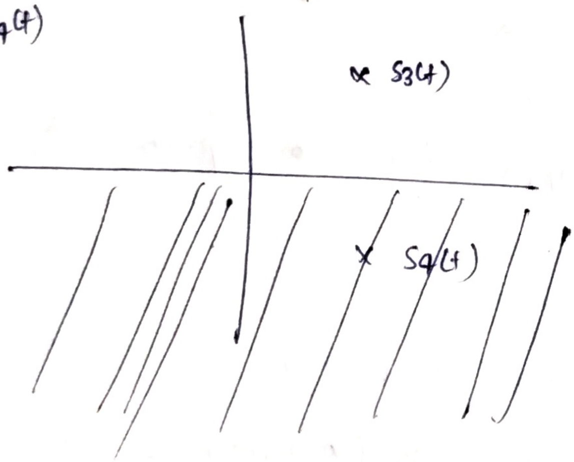
$$\boxed{P_{e|s_m(t)} \le \sum_{\substack{i=1 \\ i\neq m}}^{M} \Pr\left[E_i \mid s_m(t)\right].}$$



$E_1 \mid s_3(t)$



$E_1 \mid s_4(t)$



$\Pr(E_i \mid s_m(t)).$

Pair-wise Error probability — $(M-1)$ pairs.

$$\Pr[E_1 \mid s_3(t)] = Q\left(\frac{d_{13}}{\sqrt{2N_0}}\right).$$

$$\Pr[E_2 \mid s_3(t)] = Q\left(\frac{d_{23}}{\sqrt{2N_0}}\right)$$

$$\Pr[E_4 \mid s_3(t)] = Q\left(\frac{d_{34}}{\sqrt{2N_0}}\right).$$

$$P_{e|s_3(t)} \le Q\left(\frac{d_{13}}{\sqrt{2N_0}}\right) + Q\left(\frac{d_{23}}{\sqrt{2N_0}}\right) + Q\left(\frac{d_{34}}{\sqrt{2N_0}}\right).$$

Union bond.

$$P_e \le 2Q\left(\frac{d_{13}}{\sqrt{2N_0}}\right) + Q\left(\frac{d_{23}}{\sqrt{2N_0}}\right).$$

$d_{13} = d_{34}$

$d_{13} < d_{23}$

$$Q\left(\frac{d_{13}}{\sqrt{2N_0}}\right) > Q\left(\frac{d_{23}}{\sqrt{2N_0}}\right)$$

$$\boxed{P_{e|s_3(t)} \le 3\,Q\left(\frac{d_{min}}{\sqrt{2N_0}}\right).}$$

$$\boxed{P_{e|s_3(t)} \le (M-1)\,Q\left(\frac{d_{min}}{\sqrt{2N_0}}\right)}$$ union bond.

$$\boxed{P_e \le \frac{1}{M}\sum_{i=1}^{M}(M-1)\,Q\left(\frac{d_{min}}{\sqrt{2N_0}}\right)}$$

$$\boxed{P_e \le (M-1)\,Q\left(\frac{d_{min}}{\sqrt{2N_0}}\right)}$$

$$Q(x) \le \frac{1}{2}e^{-x^2/2}$$

$$\boxed{P_e \le \frac{(M-1)}{2}e^{-\frac{d_{min}^2}{4N_0}}}$$ union bond.