MA204: Mathematics IV

Complex Analysis: Complex valued functions

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Complex valued functions

Let $S \subseteq \mathbb{C}$. A complex valued function $f : S \to \mathbb{C}$ is a rule which assigns every element of $z \in S$ to a complex number $z \in \mathbb{C}$. In this case, we write $w = f(z)$[1].

The set S is called domain of the function and the set $\{w \in \mathbb{C} : w = f(z) \text{ for some } z \in S\}$ is called range of the function.

For the function $f(z) = \frac{z}{(z-1)(z-i)}$, the domain is $\mathbb{C} - \{1, i\}$.

If $z = x + iy$, the we can write $w = f(z)$ as

$$f(x + iy) = f(x, y) = u(x, y) + iv(x, y),$$

where $u(x, y), v(x, y) : \mathbb{R}^2 \to \mathbb{R}$ are real valued functions[2].

The function also can have polar representation. If $z = r\cos\theta + i\sin\theta$, then we have[3]

$$f(re^{i\theta}) = f(r, \theta) = u(r, \theta) + iv(r, \theta).$$

Limit of a function

Limit of a function: Let f be a complex valued function defined at all points z in some deleted neighborhood of z_0. We say that f has a limit a as $z \to z_0$ if for every $\epsilon > 0$, there is a $\delta > 0$ such that

$$|f(z) - a| < \epsilon \text{ whenever } 0 < |z - z_0| < \delta.$$

In this case, we write $\lim_{z \to z_0} f(z) = a$.

Theorem

If limit of a function at a point exists, then it is unique.

Limit of a function

If $\lim_{z \to z_0} f(z)$ exists, then $f(z)$ must approach to a unique limit, no matter how or in which direction z approaches z_0. Thus the limit is independent of the path taken by z along z_0.

If the value of $\lim_{z \to z_0} f(z)$ is different along atleast any two paths approaching z to z_0, then we say that the limit does not exists.

Theorem

If $f(z) = u(x, y) + iv(x, y)$ and $z_0 = x_0 + iy_0$ then, $\lim_{z \to z_0} f(z) = u_0 + iv_0$ if and only if $\lim_{(x,y) \to (x_0, y_0)} u(x, y) = u_0$ and $\lim_{(x,y) \to (x_0, y_0)} v(x, y) = v_0$.

Limit of a function

Properties: Let $f(z)$ and $g(z)$ be two complex valued function with $\lim_{z\to z_0} = a$ and $\lim_{z\to z_0} = b$. Then

(a) $\lim_{z\to z_0}(f(z) \pm g(z)) = \lim_{z\to z_0} f(z) \pm \lim_{z\to z_0} g(z) = a + b$.

(b) $\lim_{z\to z_0}(f(z) \cdot g(z)) = \lim_{z\to z_0} f(z) \cdot \lim_{z\to z_0} g(z) = ab$.

(c) $\lim_{z\to z_0} \left(\frac{f(z)}{g(z)}\right) = \frac{\lim_{z\to z_0} f(z)}{\lim_{z\to z_0} g(z)} = \frac{a}{b}$ provide $b \neq 0$.

(d) $\lim_{z\to z_0} kf(z) = k \lim_{z\to z_0} f(z) = ka$ for every $k \in \mathbb{R}$.

Problem: Find limit of the following functions, if exist. If exists, then verify.

(a) $f(z) = \frac{z}{\bar{z}}$ at $z = 0$.

(b) $f(z) = \frac{z+1}{iz+3}$ at $z = -1$.

(c) $f(z) = \frac{2+iz}{1+z}$ at $z = 0$.

Continuity of a function

Continuity at a point: A function $f : D \to \mathbb{C}$ is continuous at a point $z_0 \in D$ if for for every $\epsilon > 0$, there is a $\delta > 0$ such that

$$|f(z) - f(z_0)| < \epsilon \text{ whenever } |z - z_0| < \delta.$$

In other words, f is continuous at a point z_0 if

$\lim_{z \to z_0} f(z)$ exists
$f(z_0)$ exists
$\lim_{z \to z_0} = f(z_0)$.

A function f is continuous on D if it is continuous at each and every point in D.

Property: If $f(z)$ and $g(z)$ are two complex valued continuous functions at $z = z_0$, then $(f \pm g)(z)$, $(fg)(z)$, $\left(\frac{f}{g}\right)(z)$, $(kf)(z)$, and $(f \circ g)(z)$ are also continuous at $z = z_0$.

Continuity of a function

Theorem

If a function $f(z)$ is continuous and nonzero at a point z_0, then there is an open ball $B_r(z_0)$ such that $f(z) \neq 0$ for all $z \in B_r(z_0)$.

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Theorem

If $f(z) = u(x, y) + iv(x, y)$ and $z_0 = x_0 + iy_0$, then $f(z)$ is continuous at $z = z_0$ if and only if $u(x, y)$ and $v(x, y)$ are continuous at (x_0, y_0).

Problem

Problem: Check continuity of the following functions:

(a) $f(z) = \frac{z+i}{2z-3}$ at $z = i$.

(b) $f(z) = \frac{z}{|z|}$ at $z = 0$.

Derivative of a function

Let S be a nonempty open subset of $\mathbb{C}$ and $z_0 \in S$. The function $f : S \to \mathbb{C}$ is differentiable at z_0 if the limit

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} \text{ or } \lim_{h \to 0} \frac{f(z_0 + h) - f(z_0)}{h}$$

exists. The value of the limit is denoted by $f'(z_0)$, and is called the derivative of f at the point z_0.

Problem: Find derivatives of $f(z) = z^2$, $g(z) = \bar{z}$, $h(z) = |z|^2$, and $k(z) = \frac{z-1}{2z+1}$ at any point z_0, if exists.

Theorem

If f is differentiable at z_0 then f is continuous at z_0. The converse is not necessarily true.

Properties

Properties: Suppose f, g are two differentiable complex valued functions at z_0 and $\alpha, \beta \in \mathbb{C}$. Then

(a) $(\alpha f + \beta g)'(z_0) = \alpha f'(z_0) + \beta g'(z_0)$.

(b) If $h(z) = f(z)g(z)$, then $h'(z_0) = f'(z_0)g(z) + g'(z_0)f(z0)$.

(c) If $h(z) = \frac{f(z)}{g(z)}$ and $h(z_0) \neq= 0$, then $h'(z_0) = \frac{f'(z_0)g(z_0) - f(z_0)g'(z_0)}{g(z_0)^2}$.

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Theorem (Chain Rule)

$(f \circ g)'(z_0) = \frac{d}{dz}\{f(g(z))\} = f'(g(z_0))g'(z_0)$ *whenever all the terms make sense.*

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Question: Is there any difference between the differentiability in $\mathbb{R}^2$ and in $\mathbb{C}$?[4]

If the real and imaginary parts of a complex function are differentiable at a point, it is not necessary that the function is differentiable at that point.

Cauchy-Riemann Equations

Theorem (C-R Equations)

Suppose that $f(z) = f(x + iy) = u(x, y) + iv(x, y)$ is differentiable at $z_0 = x_0 + iy_0$. Then the partial derivatives of $u(x, y)$ and $v(x, y)$ exist at the point $z_0 = (x_0, y_0)$ and

$$f'(z_0) = u_x(x_0, y_0) + iv_x(x_0, y_0) = v_y(x_0, y_0) - iu_y(x_0, y_0).$$

Thus equating the real and imaginary parts we get

$$u_x = v_y, u_y = -v_x, \text{ at } z_0 = x_0 + iy_0.$$

Cauchy-Riemann Equations

The C-R equations are necessary conditions for differentiability of a complex valued function at a point.

Note:

(a) If the function $f(z) = u(x, y) + iv(x, y)$ is differentiable at $z_0 = x_0 + iy_0$, then $u_x(x_0, y_)) = v_y(x_0, y_0)$ and $v_x(x_0, y_0) = -u_y(x_0, y_0)$. For example, $f(z) = z^2$ is differentiable everywhere and hence satisfies the C-R equations everywhere.

(b) If $u_x(x_0, y_)) \neq v_y(x_0, y_0)$ or $v_x(x_0, y_0) \neq -u_y(x_0, y_0)$, then $f(z) = u(x, y) + iv(x, y)$ is not differentiable at $z_0 = x_0 + iy_0$. For example, $f(x) = \bar{z}$ does not satisfy the C-R equations at any point, and hence nowhere differentiable.

(c) If $u_x(x_0, y_)) = v_y(x_0, y_0)$ and $v_x(x_0, y_0) \neq -u_y(x_0, y_0)$, then it is not necessary that $f(z) = u(x, y) + iv(x, y)$ is differentiable at $z_0 = x_0 + iy_0$. For example, $f(z) = \frac{\bar{z}^2}{z}$ if $z \neq 0$ and $f(0) = 0$ satisfies the C-R Equations at 0, but not differentiable at 0.

Cauchy-Riemann Equations

Theorem (Sufficient condition for differentiability)

Let $z_0 = x_0 + iy_0 \in \mathbb{C}$ and the function $f(z) = u(x, y) + iv(x, y)$ be defined on $B_r(z_0)$ for some r. If u_x, u_y, v_x, v_y exist on $B_r(z_0)$ and are continuous at z_0. If u and v satisfies C-R equations at z_0, then $f'(z_0)$ exist and

$$f'(z_0) = u_x(x_0, y_0) + iv_x(x_0, y_0).$$

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Problem: Check differentiability of the following functions:

(a) $f(z) = z^3$.

(b) $f(z) = \cos z$.

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Result: Let D be a domain in $\mathbb{C}$. If $f : D \to \mathbb{C}$ is such that $f'(z) = 0$ for all $z \in D$, then f is a constant function.

Cauchy-Riemann Equations

C-R Equations in polar form: Let $f(z) = f(re^{i}) = u(r, \theta) + iv(r, \theta)$. The polar form of Cauchy-Riemann equations are

$$u_r = \frac{1}{r} v_\theta \text{ and } v_r = -\frac{1}{r} u_\theta .$$

Thank You

Any Question!!!