Control Systems

Subject Code: EC380

Lecture 10: Root Locus

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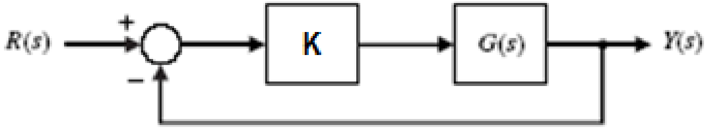
The Root Locus Method

The root locus is a graphical procedure for determining the poles of a closed-loop system given the poles and zeros of a forward-loop system. Graphically, the locus is the set of paths in the complex plane traced by the closed-loop poles as the root locus gain is varied from zero to infinity.

In mathematical terms, given a forward-loop transfer function, $KG(s)$
where K is the root locus gain, and the corresponding closed-loop transfer function

$$\frac{KG(s)}{1 + KG(s)}$$

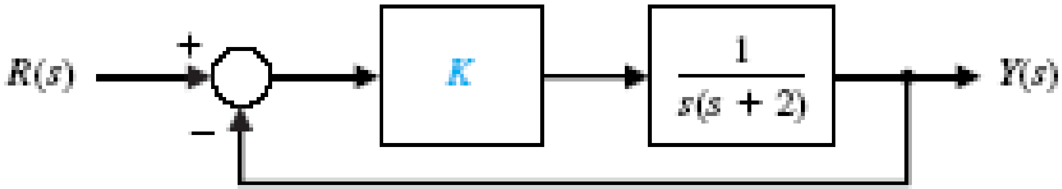


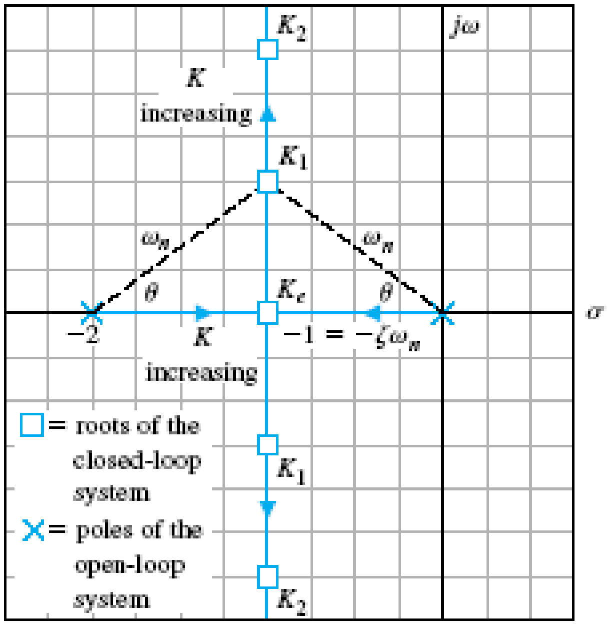
the root locus is the set of paths traced by the roots c..

$$1 + KG(s) = 0$$

as K varies from zero to infinity. As K changes, the solution to this equation changes. This equation is called the characteristic equation. This equation defines where the poles will be located for any value of the root locus gain, K. In other words, it defines the characteristics of the system behavior for various values of controller gain.

The Root Locus Method





Root-Locus Design Procedure

- In Laplace transform domain, when the gain is small the poles start at the poles of the loop transfer function.
- When gain becomes infinity, the poles move to overlap the zeros of the loop transfer function.
- This means that on a root-locus graph, all the poles move towards a zero.

Root-Locus Design Procedure

- Only one pole may move towards one zero and this means that there must be the **same number of poles as zeros**.
- If there are fewer zeros than poles in the transfer function, there are a number of implicit **zeros located at infinity** that the poles will approach.

Root-Locus Design Procedure

- We can start drawing the **root-locus** by first placing the poles of the loop transfer function on the graph with an 'X'.

- Next, we place the zeros of the loop transfer function on the graph, and mark them with an 'O'.

Root-Locus Design Procedure

- Next, we **examine the real-axis**.

- Starting from the right-hand side of the graph and traveling to the left, we draw a root-locus line on the real-axis at every point to the left of an **odd number of poles and zeros** on the real-axis.

Root-Locus Design Procedure

- Now, a **root-locus line starts at every pole**.
- Therefore, any place that two poles appear to be connected by a root locus line on the real-axis, the two poles actually move towards each other, and then they "breakaway", and move off the axis.
- The point where the poles break off the axis is called the **breakaway point**.

Note

- It is important to note that the s-plane is symmetrical about the real axis, so whatever is drawn on the top half of the S-plane, must be drawn in mirror-image on the bottom-half plane.

Root-Locus Design Procedure

- Once a **pole breaks away** from the real axis, they can either travel out towards infinity (to meet an implicit zero) or they can travel to meet an explicit zero, or they can re-join the real-axis to meet a zero that is located on the real-axis.

Root-Locus Design Procedure

- If a pole is traveling towards infinity, it always follows an asymptote.

- The number of asymptotes is equal to the number of implicit zeros at infinity.

Root-Locus Construction Rules

- **Rule 1: Starting Point (*K=0*)**
  - The root locus starts at open loop poles. Or there is one branch of the root-locus for every root of b(s).
- **Rule 2: Terminating Point (*K=infinity*)**
  - The root locus terminates at open loop zeros which include those at infinity.
- **Rule 3: Number of Distinct Root Loci**
  - There will be as many root loci as the highest number of finite open loop poles or zeros.

Root-Locus Construction Rules

- **Rule 4: Symmetry of the Root Loci**
  - The root loci are symmetrical with respect to the real axis and all complex roots are conjugate.
- **Rule 5: Root Locus Location on the Real Axis**
  - The root loci may be found on portions of the real axis to the left of an odd number of open loop poles and zeros.
- **Rule 6: Locus Breakaway Point**
  - The points at which the root locus break away can be calculated by the following:

$$K = -\frac{1}{G(s)H(s)}$$

$$\left.\frac{dK}{ds}\right|_{s=r} = 0$$

Root-Locus Construction Rules

- **Rule 7: Angle of Asymptotes**
  - The root loci are asymptotic to straight lines at large values and the angle of asymptotes is given by $$\Phi_a = \frac{(2q+1)}{n-m} \times 180^o$$
- **Rule 8: Asymptotic Intersection**
  - The asymptotes intersects the real axis at the point given by

$$\frac{\left(\sum poles\ of\ G(s)H(s) - \sum zeros\ of\ G(s)H(s)\right)}{\text{n-m}}$$

Number of asymptotes= n-m

Root-Locus Construction Rules

- **Rule 9: Angle of Departure and Arrival**
  - Find the formula
- **Rule 10: Point of Intersection with the Imaginary Axis**
  - Find the point of intersection using Routh-Hurwitz criterion.
- **Rule 11: Determination of K at any point on root loci**
  - Use K=-1/G(s)H(s)

Root Locus Example

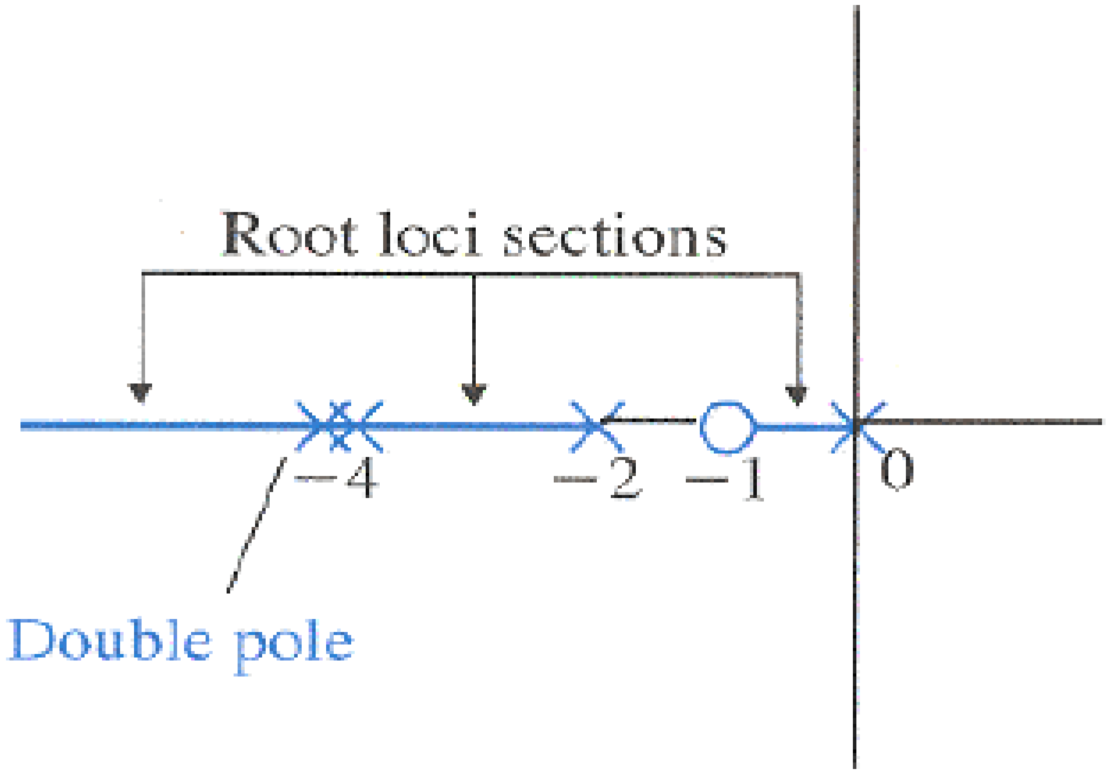
- A single- loop feedback system has a characteristic equation as follows:

$$1 + G(s) = 1 + \frac{K(s+1)}{s(s+2)(s+4)^2} = 0$$

Root Locus Example

- We wish to sketch the root locus in order to determine the effect of the gain K. The poles and the zeros are located in the s-plane as:

Root Locus Example



Root Locus Example

- The root loci on the real axis must be located to the left of an odd number of poles and zeros and are therefore located as shown on the figure above in heavy lines.

Root Locus Example

- The intersection of the asymptotes is:

$$\sigma_a = \frac{(-2) + 2(-4) - (-1)}{4 - 1} = \frac{-9}{3} = -3$$

Root Locus Example

- The angles of the asymptotes are:

$$\Phi_a = +60^\circ \quad \text{for } q = 0$$

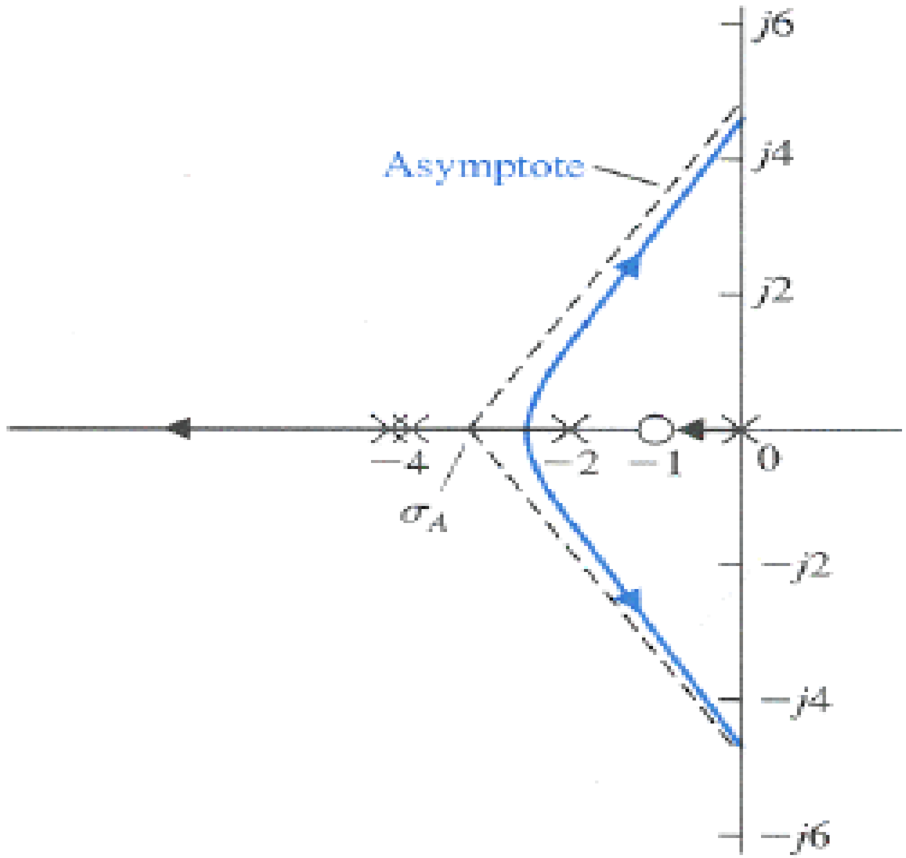
$$\Phi_a = +180^\circ \quad \text{for } q = 1$$

$$\Phi_a = +300^\circ \quad \text{for } q = 2$$

Root Locus Example

- There are three asymptotes, since the number of poles minus the number of zeros, $n – m = 3$.
- Also, we note that the root loci must begin at poles, and therefore two loci must leave the double pole at $s = - 4$. Then, with the asymptotes as sketched below, we may sketch the form of the root locus:

Root Locus Example



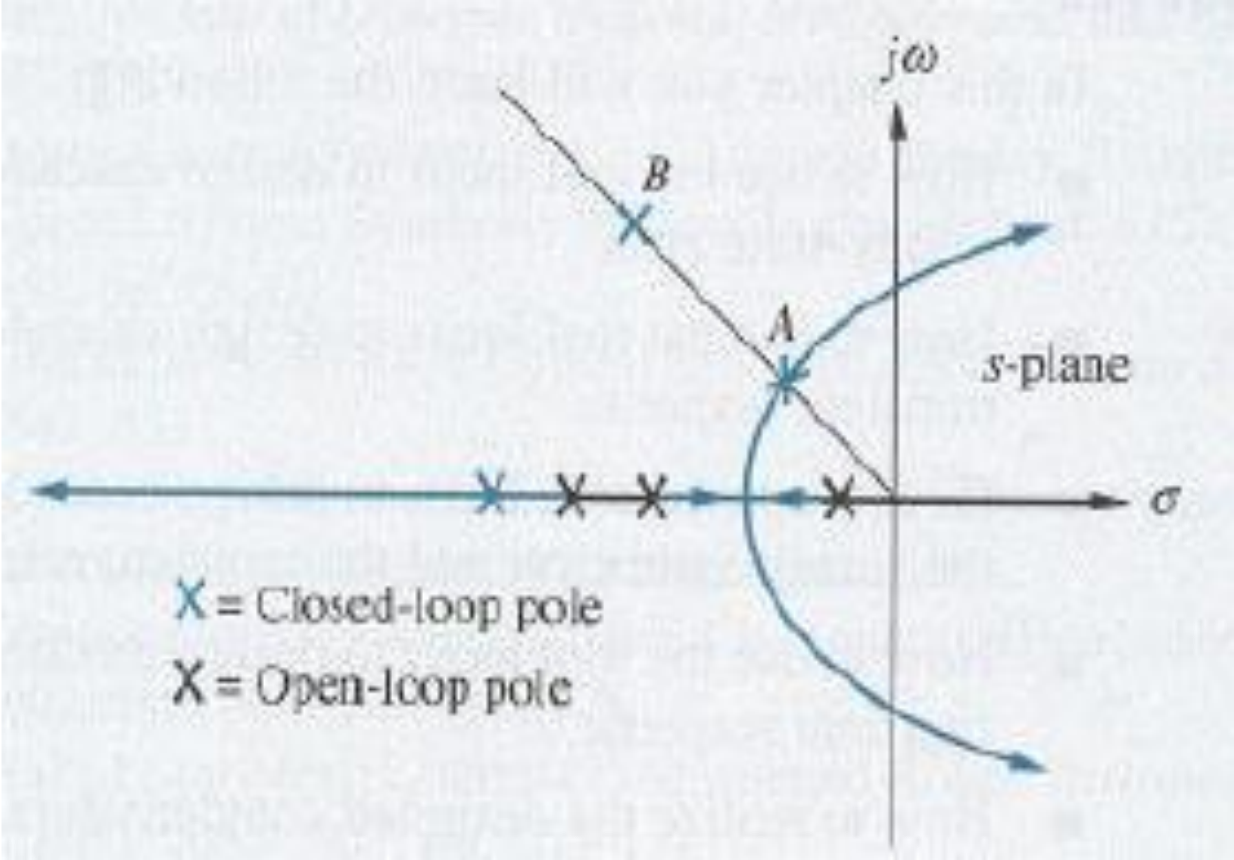
Compensator design using the root locus

- The root locus graphically displays both transient response and stability information.
- The locus can be sketched quickly to get a general idea of the changes in transient response generated by changes in gain.
- Specific points on the locus can also be found accurately to give quantitative design information.

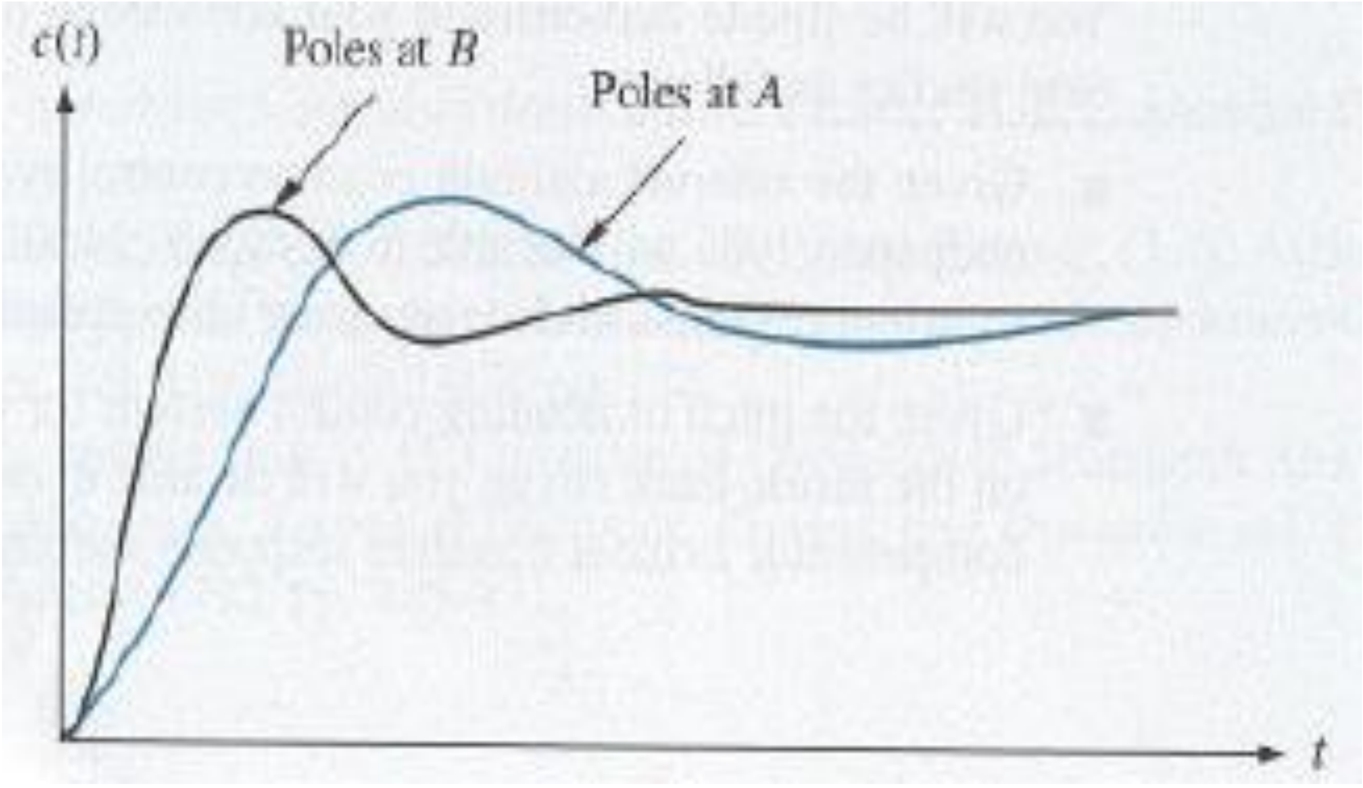
Compensator design using the root locus

- The root locus typically allows us to choose the proper loop gain to meet a transient response specification.
- As the gain is varied, we move through different regions of response.
- Setting the gain at a particular value yields the transient response dictated by the poles at that point on the root locus.
- Thus, we are limited to those responses that exist along the root locus.

Possible Root Locus

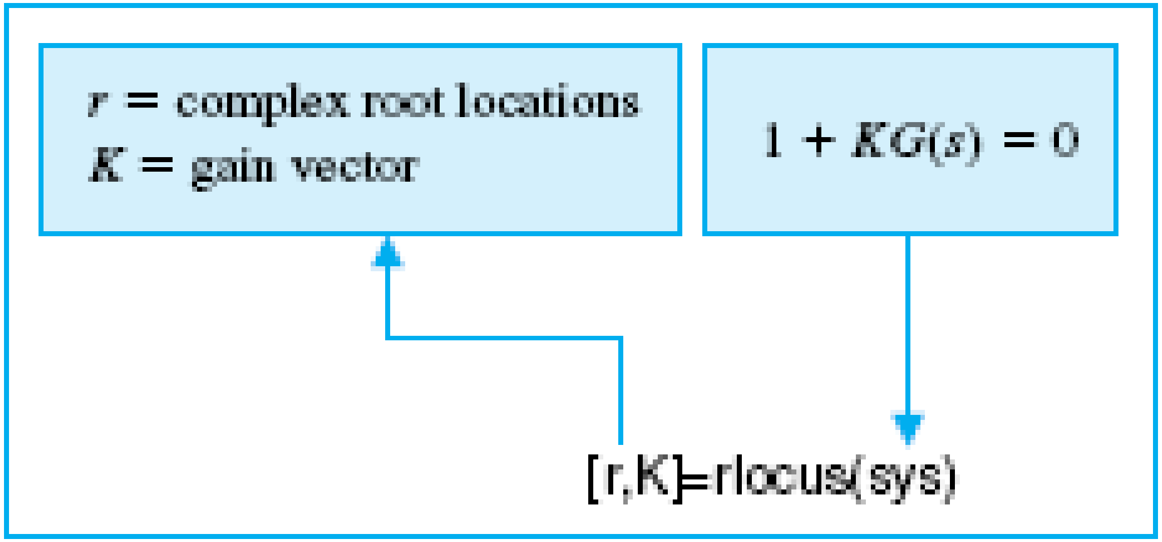


Possible Response Options

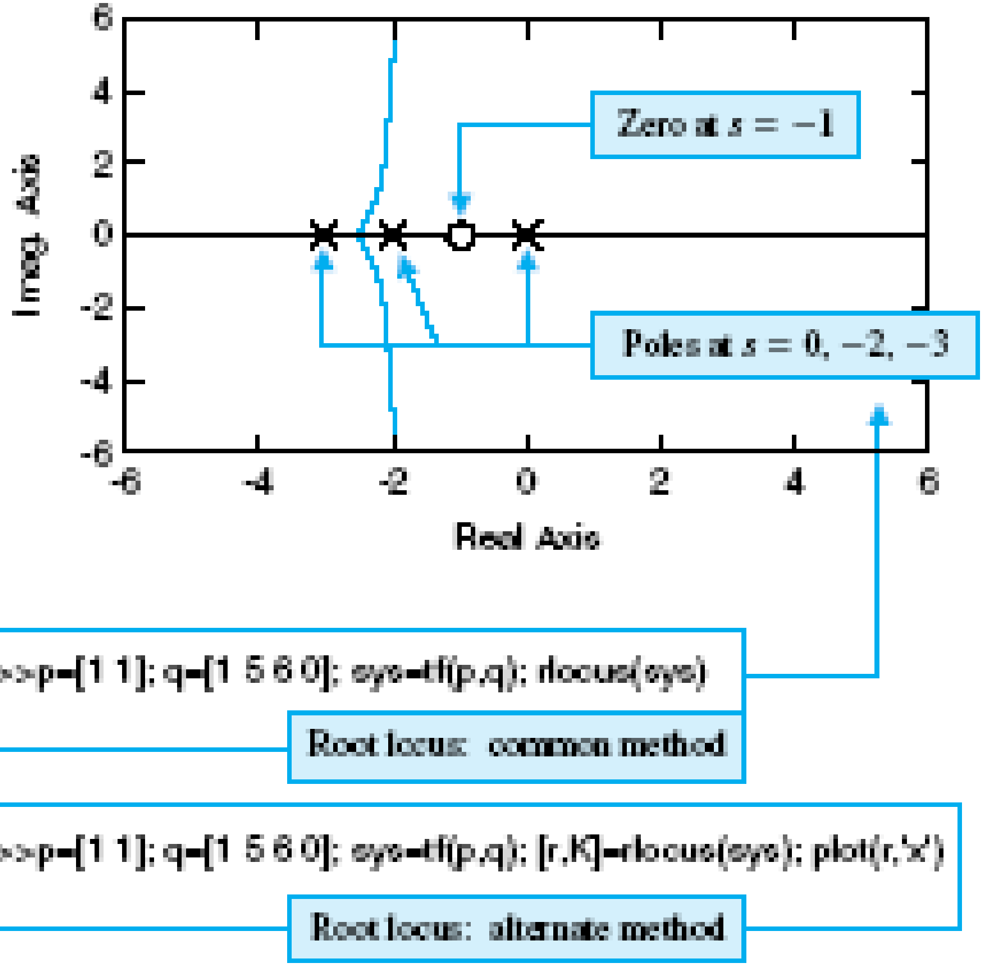


- Root Locus is a very important techniques that can be used for compensation design of various control systems
- Do further research on this topic

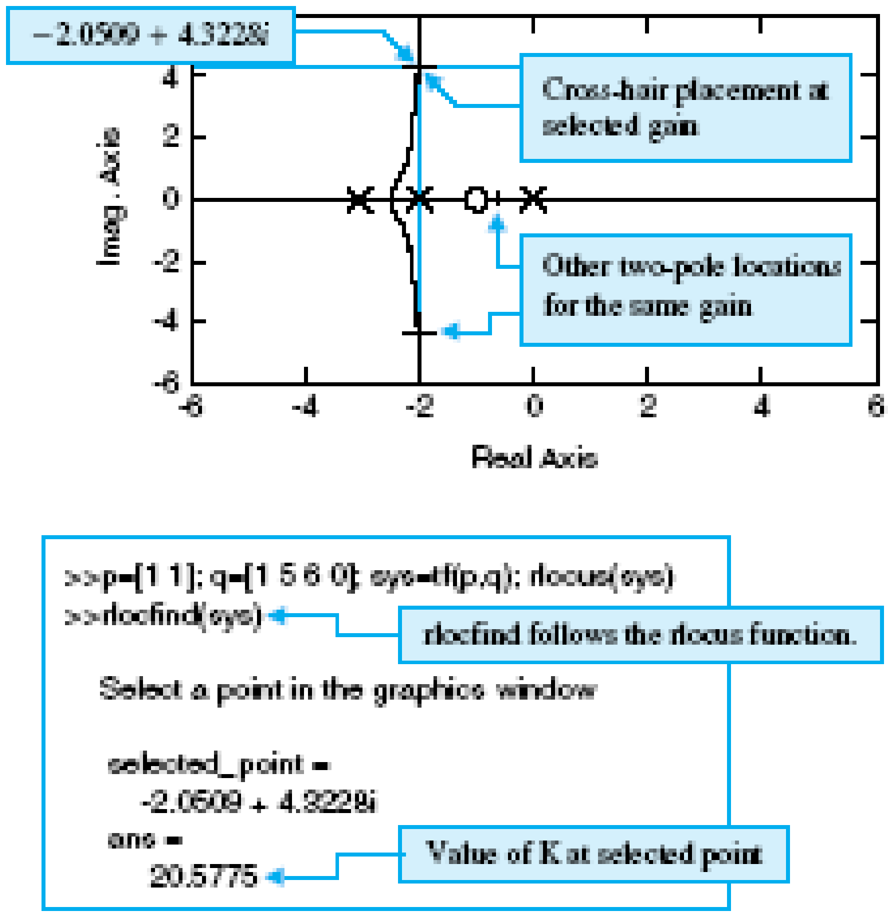
The Root Locus Method



The Root Locus Method



The Root Locus Method



The Root Locus Method

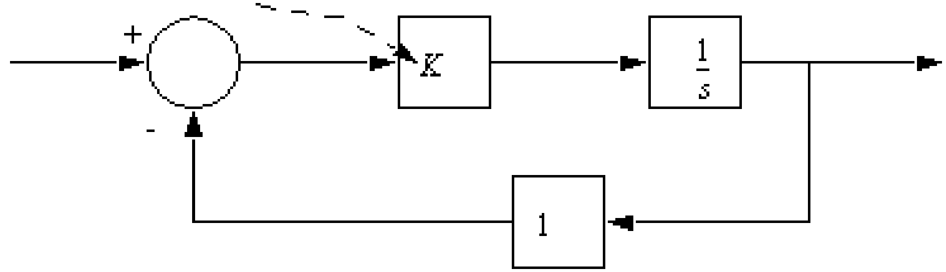
No matter what we pick K to be, the closed-loop system must always have n poles, where n is the number of poles of G(s).
The root locus must have n branches, each branch starts at a pole of G(s) and goes to a zero of G(s).
If G(s) has more poles than zeros (as is often the case), m < n and we say that G(s) has zeros at infinity. In this case, the limit of G(s) as s -> infinity is zero.

The number of zeros at infinity is n-m, the number of poles minus the number of zeros, and is the number of branches of the root locus that go to infinity (asymptotes).

Since the root locus is actually the locations of all possible closed loop poles, from the root locus we can select a gain such that our closed-loop system will perform the way we want. If any of the selected poles are on the right half plane, the closed-loop system will be unstable. The poles that are closest to the imaginary axis have the greatest influence on the closed-loop response, so even though the system has three or four poles, it may still act like a second or even first order system depending on the location(s) of the dominant pole(s).

Example

Note: This controller has adjustable gain. After this design is built we must anticipate that all values of K will be used. It is our responsibility to make sure that none of the possible K values will lead to instability.



$$G(s) = \frac{K}{s} \qquad H(s) = 1$$

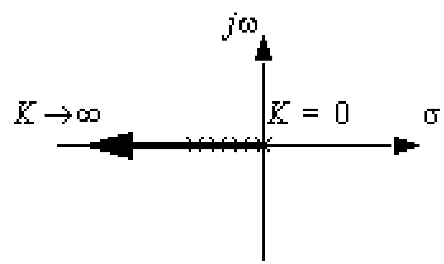
First, we must develop a transfer function for the entire control system.

$$G_S(s) = \frac{G(s)}{1 + G(s)H(s)} = \frac{\left(\frac{K}{s}\right)}{1 + \left(\frac{K}{s}\right)(1)} = \frac{K}{s + K}$$

Next, we use the characteristic equation of the denominator to find the roots as the value of K varies. These can then be plotted on a complex plane. Note: the value of gain 'K' is normally found from 0 to +infinity.

$$s + K = 0$$

K	root
0	
1	
2	
3	
etc..	



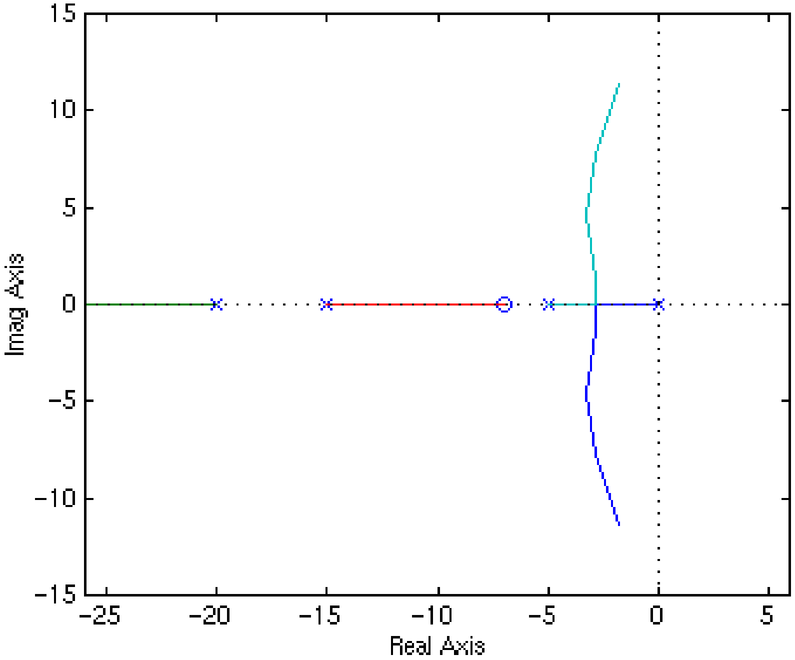
Note: because all of the roots for all values of K are real negative this system will always be stable, and it will always tend to have a damped response. The large the value of K, the more stable the system becomes.

Example

MATLAB Example - Plotting the root locus of a transfer function

Consider an open loop system which has a transfer function of
G(s) = (s+7)/s(s+5)(s+15)(s+20)
How do we design a feedback controller for the system by using the root locus method?
Enter the transfer function, and the command to plot the root locus:

```
num=[1 7];
den=conv(conv([1 0],[1 5]),conv([1 15],[1 20]));
rlocus(num,den)
axis([-22 3 -15 15])
```



Graphical Method

Given the system elements (you should assume negative feedback),

$$G(s) = \frac{K}{s^2+3s+2} \qquad H(s) = 1$$

Step 1: (put equation in standard form)

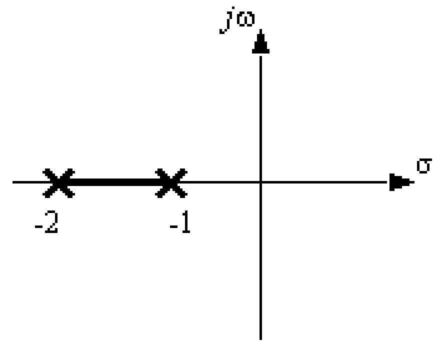
$$1+G(s)H(s) = 1+\left(\frac{K}{s^2+3s+2}\right)(1) = 1+K\frac{1}{(s+1)(s+2)}$$

Step 2: (find loci ending at infinity)

$$m = 0 \qquad n = 2 \qquad \text{(from the poles and zeros of the previous step)}$$

$$n-m = 2 \qquad \text{(loci end at infinity)}$$

Step 3: (plot roots)



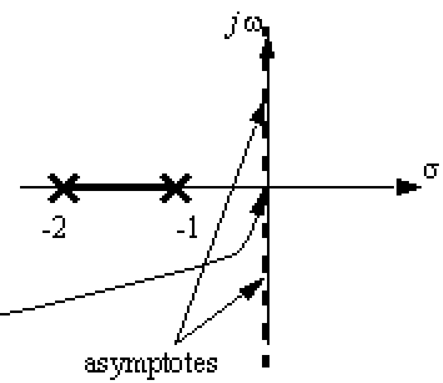
Step 4: (find asymptotes angles and real axis intersection)

$$\beta(k) = \frac{180°(2k+1)}{2} \qquad k \in I[0,1]$$

$$\beta(0) = \frac{180°(2(0)+1)}{2} = 90°$$

$$\beta(1) = \frac{180°(2(1)+1)}{2} = 270°$$

$$\sigma = \frac{(0)(-1-2)}{2} = 0$$



Graphical Method

Step 5: (find the breakout points for the roots)

$$A = 1 \qquad B = s^2 + 3s + 2$$

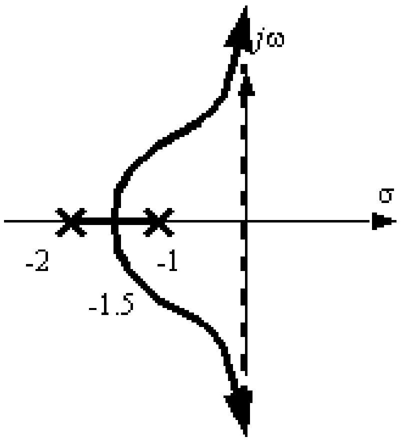
$$\frac{d}{ds}A = 0 \qquad \frac{d}{ds}B = 2s + 3$$

$$A\left(\frac{d}{ds}B\right) - B\left(\frac{d}{ds}A\right) = 0$$

$$1(2s+3) - (s^2 + 3s + 2)(0) = 0$$

$$2s + 3 = 0$$

$$s = -1.5$$



Note: because the loci do not intersect the imaginary axis, we know the system will be stable, so step 6 is not necessary, but we it will be done for illustrative purposes.

Step 6: (find the imaginary intercepts)

$$1 + G(s)H(s) = 0$$

$$1 + K\frac{1}{s^2 + 3s + 2} = 0$$

$$s^2 + 3s + 2 + K = 0$$

$$(j\omega)^2 + 3(j\omega) + 2 + K = 0$$

$$-\omega^2 + 3j\omega + 2 + K = 0$$

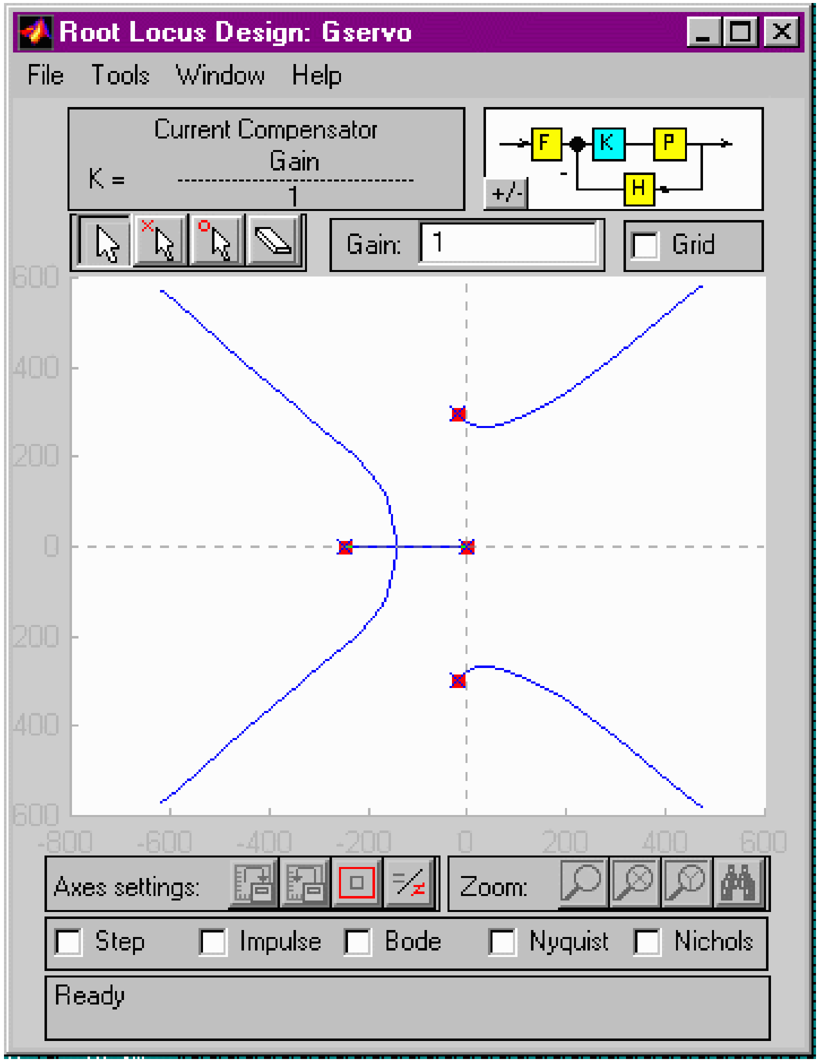
$$\omega^2 + \omega(-3j) + (-2 - K) = 0$$

$$\omega = \frac{3j \pm \sqrt{(-3j)^2 - 4(-2-K)}}{2} = \frac{3j \pm \sqrt{-9 + 8 + 4K}}{2} = \frac{3j \pm \sqrt{4K - 1}}{2}$$

In this case the frequency has an imaginary value. This means that there will be no frequency that will intercept the imaginary axis.

Root Locus Design GUI (rltool)

The Root Locus Design GUI is an interactive graphical tool to design compensators using the root locus method. This GUI plots the locus of the closed-loop poles as a function of the compensator gains. You can use this GUI to add compensator poles and zeros and analyze how their location affects the root locus and various time and frequency domain responses. Click on the various controls on the GUI to see what they do.



Thank You