MA 102
Calculus
Tutorial–2

Note: $f_{xy} = \frac{\partial^2 f}{\partial y \partial x}, f_{xyz} = \frac{\partial^3 f}{\partial z \partial y \partial x}$

(1) Sketch the domain and graph for the functions $f(x, y) = ln(x^2 + y^2 - 4), f(x, y) = e^{-(x^2+y^2)}$.

(2) Label level curve with its function value of the function $f(x, y) = \sqrt{x^2 + y^2}$.

(3) At what points (x, y) in the plane are the functions $f(x, y) = \frac{x+y}{x-y}$, $f(x, y) = \frac{x^2}{x^2-y}$, $f(x, y, z) = \sqrt{4 - x^2 - y^2 - z^2}$ continuous.

(4) Find $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$.
  (a) $f(x, y) = x^2 - xy + y^2$
  (b) $f(x, y) = (2x - 3y)^3$
  (c) $f(x, y) = e^{xy} ln(y)$

(5) Find $f_x, f_y, f_z, f_{xy}, f_{yz}, f_{zx}, f_{xyz}$.
  (a) $f(x, y, z) = 1 - xy^2z + 2z^2$
  (b) $f(x, y, z) = ln(xy + 2yz + 3xyz)$

(6) Find $\frac{dw}{dt}$ where $w = ln(x^2 + y^2 + z^2), x = cos(t), y = sin(t), z = 4\sqrt{t}$ (using the Chain Rule).

(7) Find ∇f at $(1, 1, 1)$ where $f(x, y, z) = x^2 + y^2 - 2z^2 + z\ lnx$.

(8) Find the derivative of the function $f(x, y, z) = xy+yz+zx$ at $(1, -1, 2)$ in the direction $u = 3i + 6j - 2k$.

(9) Find the directions of $f(x, y, z) = ln\ xy + ln\ yz + ln\ xz$ in which the functions increase and decrease most rapidly

(10) In what direction is the derivative of $f(x, y) = xy + y^2$ at $(3, 2)$ equal to zero? Ans: $\nabla f(3, 2) = 2i + 7j, v = 7i - 2j, u = v/|v|$, $-u$ is the directions where the derivative is zero.

(11) Find equations of the tangent plane and normal line of $x^2 + 2xy - y^2 + z^2 = 7$ at $(1, -1, 3)$. Ans: $\nabla f(1, -1, 3) = 4j + 6k$. Tangent plane : $4(y + 1) + 6(z - 3) = 0$ and normal line: $x = 1, y = -1 + 4t, z = 3 + 6t$.

(12) By about how much will $f(x, y, z) = e^x cos(yz)$ change as the point (x, y, z) moves from the origin a distance of $ds = 0.1$ unit in the direction of $2i + 2j - 2k$. Ans: $\nabla f(0, 0, 0) = i, u = v/|v| = \frac{1}{\sqrt{3}}i + \frac{1}{\sqrt{3}}j - \frac{1}{\sqrt{3}}k$. $\nabla f(0, 0, 0) \cdot u = \frac{1}{\sqrt{3}}, df = (\nabla f(0, 0, 0) \cdot u)ds$.

(13) Find the linearization $L(x, y)$ of the function $f(x, y) = e^{2y-x}$ at $(0, 0)$. Ans: $f(0, 0) = 1, f_x(0, 0) = -1, f_y(0, 0) = 2, L(x, y) = f(0, 0) + 1(x - 0) + 2(y - 0)$.

(14) Find all the local maxima, local minima, and saddle points of the $f(x, y) = x^2 - 4xy + y^2 + 6y + 2$. Ans: Critical point : $(2, 1)$, $f_{xx}f_{yy} - f_{xy}^2 < 0$, the point is saddle point.

(15) Among all closed rectangular boxes of volume 27 cm^3, what is the smallest surface area? Ans: $f(x, y, z) = 2xy+2yz+2zx, xyz = 27$. Hence, $f(x, y) = 2xy+(2y+2x)\frac{27}{xy}$. Critical point: $(3, 3, 3)$. Now, $f_{xx}f_{yy} - f_{xy}^2 = 12 > 0, f_x(3, 3, 3) > 0$, so, $(3, 3, 3)$ is a minima. So, the smallest surface area $= 54$.

(16) Suppose that the Celsius temperature at the point (x, y, z) on the sphere $x^2+y^2+z^2 = 1$ is $T = 6xyz$. Locate the highest and lowest temperatures on the sphere. Ans: The dimensions of the box are $\frac{2}{\sqrt{3}}, \frac{2}{\sqrt{3}}, \frac{2}{\sqrt{3}}$ by for maximum volume. Note that there is no minimum volume since the box could be made arbitrarily thin.

(17) Use Taylor's formula for $f(x, y)$ at the origin to find quadratic and cubic approximations of $f(x, y) = sin(x)cos(y)$ near the origin. Ans: $L(x, y) = f(0, 0) + (x-0)f_x(0, 0) + (y-0)f_y(0, 0) + \frac{1}{2}[x^2 f_{xx} + 2xy f_{xy} + y^2 f_{yy}(0, 0)]$.