MA204: Mathematics IV

Partial Differential Equation (Fourier Series Convergence and Fourier Integral)

Fourier Series Convergence

Recall that Fourier Series of the function $f(x) = x$ for $-L \leq x \leq L$ is given by

$$\frac{2L}{\pi}\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n}\sin\frac{n\pi x}{L}.$$

Observe that value of the Fourier series of $f(x)$ at $x = \pm L$ is zero. But, value of $f(\pm L)$ is $\pm L$.

Thus, the Fourier series value of $f(x)$ is not equal to $f(x)$ at $x = \pm L$.

Theorem

Let $f \in C^2[-L, L]$ with $f(-L) = f(L)$ and $f'(-L) = f'(L)$. If $a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}dx$, $n = 0, 1, 2, \ldots$ and $b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}dx$, $n = 1, 2, \ldots$, then the Fourier series

$$\frac{a_0}{2} + \sum_{n=1}^{\infty}\{a_n\cos\frac{n\pi x}{L} + b_n\sin\frac{n\pi x}{L}$$

converges to $f(x)$.

Fourier Series Convergence

Theorem

Let $f : [-L, L] \to \mathbb{R}$ be a piecewise C^1 function such that $f(-L) = f(L)$. Then the Fourier series of $f(x)$ converges to $f(x)$ on $[-L, L]$.

Fourier Integral

Recall that the Fourier series expansion for a periodic function $f(x)$ in $[-L, L]$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\{a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\},$$

where $a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}dx$ and $b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}dx$.

The Fourier series representation can be extended to some non-periodic functions, provided $\int_{-\infty}^{\infty}|f(x)|dx$ is finite.

Substituting the values of a_n and b_n in the Fourier series expansion of $f(x)$, we have

$$f(x) = \frac{1}{2L}\int_{-L}^{L} f(t)dt + \sum_{n=1}^{\infty}\{\frac{1}{L}\int_{-L}^{L} f(t)\cos\frac{n\pi t}{L}\cos\frac{n\pi x}{L}dt + \frac{1}{L}\int_{-L}^{L} f(t)\sin\frac{n\pi t}{L}\sin\frac{n\pi x}{L}dt\}$$

Fourier Integral

Using $\cos(A-B)=\cos A\cos B+\sin A\sin B$, and then interchanging the sum and integration, we obtain

$$f(x)=\frac{1}{2L}\int_{-L}^{L}f(t)dt+\frac{1}{L}\int_{-L}^{L}f(t)\sum_{n=1}^{\infty}\cos\frac{n\pi(t-x)}{L}dt.$$

Noting $\int_{-\infty}^{\infty}|f(t)|dt$ is finite, we must have

$$\lim_{L\to\infty}\frac{1}{2L}\int_{-L}^{L}f(t)dt=0.$$

For the remaining, we set $\Delta s=\frac{\pi}{L}$, and hence

$$\begin{aligned}
f(x)=&\lim_{\Delta s\to 0}\frac{1}{\pi}\int_{-\frac{\pi}{\Delta s}}^{\frac{\Delta s}{L}}f(t)\sum_{n=1}^{\infty}\cos\{n\Delta(t-s)\}dt\\
=&\frac{1}{\pi}\int_{-\infty}^{\infty}f(t)\int_{0}^{\infty}\cos\{s(t-x)\}dsdt\\
=&\frac{1}{\pi}\int_{0}^{\infty}\int_{-\infty}^{\infty}f(t)\cos\{s(t-x)\}dtds.
\end{aligned}$$

Fourier Integral

Thus the expression

$$f(x) = \frac{1}{\pi} \int_0^{\infty} \int_{-\infty}^{\infty} f(t) \cos\{s(t-x)\} dtds$$

is called the Fourier integral representation of $f(x)$.

Fourier Transform

Noting that $\cos\frac{n\pi x}{L} = \frac{1}{2}(e^{\frac{in\pi x}{L}} + e^{-\frac{in\pi x}{L}})$ and $\sin nx = \frac{1}{2i}(e^{\frac{in\pi x}{L}} + e^{-\frac{in\pi x}{L}})$, we can write the Fourier series expansion of a function $f(x)$ as

$$f(x) \equiv \sum_{-\infty}^{\infty} c_n e^{\frac{in\pi x}{L}},$$

where $c_n = \frac{1}{2}(a_n - ib_n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-\frac{in\pi x}{L}}dx$ and
$c_{-n} = \bar{c}_n = \frac{1}{2}(a_n + ib_n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{\frac{in\pi x}{L}}dx$.

This expression is called **complex form of the Fourier series** expansion of $f(x)$.

In terms of this expansion, we can define the Fourier integral of $f(x)$ as

$$f(t) = \int_{-\infty}^{\infty}\left\{\frac{1}{2\pi}e^{-ist}\int_{-\infty}^{\infty} f(t)e^{ist}dt\right\}ds.$$

Fourier Transform

Definition

Let $f(t)$ be a function defined for all $x \in \mathbb{R}$ with values in $\mathbb{C}$. The Fourier transform of $f(t)$, denoted by $\mathcal{F}\{f(t)\}$, is a mapping from real numbers to complex numbers defined as

$$\mathcal{F}\{f(t)\} = F(\alpha) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{i\alpha t} dt.$$

On the other hand, the inverse Fourier transform of $F(\alpha)$, denoted by $f(t) = \mathcal{F}^{-1}(t)$, is defined as

$$\mathcal{F}^{-1}\{F(\alpha)\} = f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(\alpha) e^{-i\alpha t} d\alpha.$$

Here $F(\alpha)$ is in the frequency domain and $f(t)$ in the time domain.

Fourier Transform

We can define the Fourier cosine transform of $f(t)$ as

$$\mathcal{F}_c\{f(t)\} = F_c(\alpha) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(t) \cos \alpha t dt.$$

with inverse transform

$$\mathcal{F}^{-1}\{F_c(\alpha)\} = f(t) = \sqrt{\frac{2}{\pi}} \int_0^\infty F_c(\alpha) \cos \alpha t d\alpha.$$

In the similar way, the Fourier sine transform of $f(t)$ is defined as

$$\mathcal{F}_c\{f(t)\} = F_c(\alpha) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(t) \sin \alpha t dt.$$

with inverse transform

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Fourier Transform

Problem: Find Fourier transform of the following functions:

(a) $f(t) = \begin{cases} e^{at}, & \text{if } t < 0; \\ e^{-at}, & \text{if } t > 0; \end{cases}$ with $a > 0$.

(b) $f(x) = e^{x^2}$

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Problem: Find the value of the integral $\int_0^\infty \frac{\cos \alpha x}{\alpha^2 + b^2} d\alpha$.

Fourier Transform

Some basic properties of Fourier transform are stated below:

(1) **Linear Property:** If $\mathcal{F}\{f_1(t)\} = F_1(\alpha)$, $\mathcal{F}\{f_2(t)\} = F_2(\alpha)$, then

$$\mathcal{F}\{c_1 f_1(t) \pm c_2 f_2(t)\} = c_1 \mathcal{F}\{f_1(t)\} \pm c_2 \mathcal{F}\{f_2(t)\} = c_1 F_1(\alpha) \pm c_2 F_2(\alpha),$$

where c_1 and c_2 are constants.

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(4) **Translation property:** If $\mathcal{F}\{f(t)\} = F(\alpha)$, then

$$\mathcal{F}\{e^{iat} f(t)\} = F(\alpha - a).$$

Fourier Transform

(5) Let $\mathcal{F}\{f(t)\} = F(\alpha)$. If $f(t)$ is continuously differentiable and $f(t) \to 0$ as $|t| \to \infty$, then

$$\mathcal{F}\{f'(t)\} = i\alpha.$$

More generally, if $f(t)$ is continuously n-times differentiable and $f^{(k)}(t) \to 0$ as $|t| \to \infty$ for $k = 1, 2, \ldots, (n-1)$, then the Fourier transform of the n-th derivative of $f(t)$ is given by

$$\mathcal{F}\{f^{(n)}(t)\} = (i\alpha)^n \mathcal{F}\{f(t)\} = (i\alpha)^n F(\alpha).$$

Thank you

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