MA204: Mathematics IV

Complex Analysis: Analytic Functions

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Analytic function

Definition

A function $f : S \subseteq \mathbb{C} \to \mathbb{C}$ is called **analytic** at a point $z_0 \in \mathbb{C}$ if there exist $r > 0$ such that f is differentiable at every point $z \in B_r(z_0)$. If $f(z)$ is not analytic at z_0, then z_0 is called a **singular** point of $f(z)$.

For example, $f(z) = |z|^2$ is differentiable at $z = 0$, but nowhere differentiable.

Problem: Find the points at which the following functions are analytic and the singular points.

(a) $f(z) = \bar{z}$
(b) $f(z) = e^{iz}$ and $g(z) = e^{-i\bar{z}}$
(c) $f(z) = \frac{1}{z}$
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A function is called analytic in an open set $U \subseteq \mathbb{C}$ if it is analytic at each point $z \in U$.

Analytic function

A function which is analytic on the whole complex plane $\mathbb{C}$ is called an **entire function**.

Note the following:

(a) If f and g are analytic in an open set D. Then $f \pm g$, fg, $\frac{f}{g}$ $(g \neq 0)$, αf $(\alpha \in \mathbb{C})$ are analytic on D.

(b) Composition of two analytic functions is also analytic.

(c) If f is analytic on an open set D, then f satisfies CR-Equation at every point on the open set D.

(d) If $f(x, y) = u(x, y) + iv(x, y)$ satisfies CR-Equation at every point on a open set D and u, v have continuous first order partial derivatives at every point on D, then f is analytic on D.

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Problem: If a function f and its conjugate are both analytic on a domain D, then f is constant.

Harmonic function

Definition

A real valued function $\phi(x, y)$ is said to be harmonic in a domain D if all the partial derivatives up to second order exists and are continuous on D such that they satisfy the Laplace equation $\phi_{xx}(x, y) + \phi_{yy}(x, y) = 0$ at each point of D.

For example, $x^2 - y^2$, $e^x \cos y$, and $\frac{x^2-y^2}{(x^2+y^2)^2}$ $((x, y) \neq (0, 0))$ are harmonic functions. However $x^2 + y^2$ and $x \sin y$ $((x, y) \neq (0, 0))$ are not harmonic.

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Theorem

If $f(z) = u(x, y) + iv(x, y)$ is analytic in a domain D, then the functions $u(x, y)$ and $v(x, y)$ are harmonic in D.

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If $f(z) = u(x, y) + iv(x, y)$ is analytic in a domain D, then the functions $u(x, y)$ and $v(x, y)$ are harmonic in D.

The converse of the above theorem is not true. For example, $u(x, y) = x$ and $v(x, y) = -y$ are harmonic functions everywhere, but $f(z) = x - iy = \bar{z}$ is nowhere analytic .

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(a) If v is harmonic conjugate of u, then is u a harmonic conjugate of v?

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Questions:

(a) If v is harmonic conjugate of u, then is u a harmonic conjugate of v?
(b) Under what condition u and v are harmonic conjugate to each other?

Harmonic conjugate

Problem: Find harmonic conjugate of the following functions, if exists.

(a) $u(x, y) = e^{-y} \cos x$

(b) $u(x, y) = 2x(1 - y)$

(c) $u(x, y) = \log(x^2 + y^2)^{\frac{1}{2}}$

Thank You

Any Question!!!