# Linear Algebra HW13

## 1

$$from\ \ det(A-\lambda I)=0 \Rightarrow \lambda^3+25\lambda=0 \Rightarrow \lambda=0,\ \ \pm 5i - < ans >$$
$$for\ \ \lambda=0:(A-\lambda I)x_1=0 \Rightarrow x_1=(4,0,3)^T - < ans >$$
$$for\ \ \lambda=-5i:(A-\lambda I)x_2=0 \Rightarrow x_2=(-3,5i,4)^T - < ans >$$
$$for\ \ \lambda=5i:(A-\lambda I)x_3=0 \Rightarrow x_3=(-3,-5i,4)^T - < ans >$$
$$\because A\ \ is\ \ skew\ \ symmetric \Rightarrow A=-A^T$$
$$\therefore (e^{At})^{-1}=e^{-At}=e^{A^Tt}=(e^{At})^T \Rightarrow e^{At}\ \ is\ \ an\ \ orthogonal\ \ matrix. - < ans >$$
$$\because Real\ \ part\ \ of\ \ eigenvalues\ \ is\ \ zero \Rightarrow it\ \ won't\ \ change\ \ magnitude\ \ of\ \ u(t). - < an$$

## 2

$$(a)$$
$$V_1=(1,1)=av_1+bv_2 \Rightarrow a=1,\ b=-1$$
$$V_2=(1,4)=cv_1+dv_2 \Rightarrow c=0,\ d=1$$
$$M_{V\to v}=\begin{bmatrix} a & c \\ b & d \end{bmatrix}=\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} - < ans >$$
$$(b)$$
$$(3,9)=c_1V_1+c_2V_2 \Rightarrow c_1=1,\ c_2=2$$
$$=d_1v_1+d_2v_2 \Rightarrow d_1=1,\ d_2=1$$
$$\Rightarrow Mc=d \Rightarrow \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix}=\begin{bmatrix} 1 \\ 1 \end{bmatrix} - < ans >$$

## 3

$$After\ \ reflection\ \ v_1\ \ becomes\ \ (0,1)=0v_1+1v_2$$
$$After\ \ reflection\ \ v_2\ \ becomes\ \ (1,0)=1v_1+0v_2$$
$$\Rightarrow T_v=\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$
$$After\ \ reflection\ \ V_1\ \ becomes\ \ (1,1)=1V_1+0V_2$$
$$After\ \ reflection\ \ V_2\ \ becomes\ \ (-1,1)=0V_1-V_2$$
$$\Rightarrow T_V=\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$
$$V_1=(1,1)=av_1+bv_2 \Rightarrow a=1,\ b=1$$
$$V_2=(1,-1)=cv_1+dv_2 \Rightarrow c=1,\ d=-1$$
$$M_{V\to v}=\begin{bmatrix} a & c \\ b & d \end{bmatrix}=\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$
$$T_v=M_{V\to v}T_VM_{v\to V}$$
$$\Rightarrow \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}=\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}^{-1} - < ans >$$

## 4

$$A^H = \begin{bmatrix} 1 & -i \\ -i & 0 \\ 0 & 1 \end{bmatrix} - < ans >$$

$$C = A^H A = \begin{bmatrix} 2 & i & -i \\ -i & 1 & 0 \\ i & 0 & 1 \end{bmatrix} - < ans >$$

$$C = C^H - < ans >$$

$$C^H = (A^H A)^H = A^H A = C \Rightarrow The\ \ relationship\ \ holds. - < ans >$$

## 5

$$det(P - \lambda I) = 0 \Rightarrow \lambda = 0,\ 1$$

$$for\ \ \lambda = 0,\ (P - \lambda I)x_1 = 0 \Rightarrow x_1 = c_1(1,1)^T$$

$$for\ \ \lambda = 1,\ (P - \lambda I)x_2 = 0 \Rightarrow x_1 = c_2(1,-1)^T$$

$$Let\ \ c_1 = ||x_1|| = \frac{\sqrt{2}}{2},\ and\ \ c_2 = ||x_2|| = \frac{\sqrt{2}}{2}$$

$$P = \lambda_1 x_1 x_1^H + \lambda_2 x_2 x_2^H$$

$$= \begin{bmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{bmatrix} \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} - < ans >$$

$$det(Q - \lambda I) = 0 \Rightarrow \lambda = -1,\ 1$$

$$for\ \ \lambda = 1,\ (Q - \lambda I)x_1 = 0 \Rightarrow x_1 = c_1(1,1)^T$$

$$for\ \ \lambda = -1,\ (Q - \lambda I)x_2 = 0 \Rightarrow x_1 = c_2(1,-1)^T$$

$$Let\ \ c_1 = ||x_1|| = \frac{\sqrt{2}}{2},\ and\ \ c_2 = ||x_2|| = \frac{\sqrt{2}}{2}$$

$$Q = \lambda_1 x_1 x_1^H + \lambda_2 x_2 x_2^H$$

$$= \begin{bmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{bmatrix} \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix} - \begin{bmatrix} \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} \end{bmatrix} \begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} - < ans >$$

$$det(R - \lambda I) = 0 \Rightarrow \lambda = -5,\ 5$$

$$for\ \ \lambda = 1,\ (R - \lambda I)x_1 = 0 \Rightarrow x_1 = c_1(2,1)^T$$

$$for\ \ \lambda = -1,\ (R - \lambda I)x_2 = 0 \Rightarrow x_1 = c_2(1,-2)^T$$

$$Let\ \ c_1 = ||x_1|| = \frac{\sqrt{5}}{5},\ and\ \ c_2 = ||x_2|| = \frac{\sqrt{5}}{5}$$

$$Q = \lambda_1 x_1 x_1^H + \lambda_2 x_2 x_2^H$$

$$= 5 \begin{bmatrix} \frac{2\sqrt{5}}{5} \\ \frac{\sqrt{5}}{5} \end{bmatrix} \begin{bmatrix} \frac{2\sqrt{5}}{5} & \frac{\sqrt{5}}{5} \end{bmatrix} - 5 \begin{bmatrix} \frac{\sqrt{5}}{5} \\ -\frac{2\sqrt{5}}{5} \end{bmatrix} \begin{bmatrix} \frac{\sqrt{5}}{5} & -\frac{2\sqrt{5}}{5} \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ 4 & -3 \end{bmatrix} - < ans >$$

## 6

$(a)$ *All eigenvalues of real symmetric matrix are real.*
$(b)$ *Real part of all eigenvalues are less than* 1.
$(c)$ *Magnitude of all eigenvalues are equal to* 1.
$\Rightarrow ||Ax|| = ||\lambda|| \, ||x||$
$(d)$ *Magnitude of all eigenvalues are less than* 1.
$(e)$ *If* $A_{n\times n}$ *is nondiagonalizable, then* $A$ *must have fewer than* $n$ *engenvalues.*
$(f)$ $det(A) = 0$

# 7

1. *For any* $x$, $x^H Kx$ *is pure imaginary.*
$(x^H Kx)^H = x^H K^H x = -x^H Kx$
$\Rightarrow x^H Kx$ *is pure imaginary.*
2. *Every eigenvalue of* $K$ *is pure imaginary.*
$Kx = \lambda x \Rightarrow x^H Kx = \lambda x^H x$
*From* 1. $x^H Kx$ *is pure imaginary, and* $x^H x \in \mathbb{R}$
$\Rightarrow \lambda$ *is pure imaginary.*
3. *Eigenvectors corresponding to different eigenvalues are orthogonal.*
*If* $K$ *has eighevalues* $\lambda$ *and* $\mu$ *with corresponding eigenvectors* $x$ *and* $y$.
$(\lambda x)^H y = (Kx)^H y = x^H(-K)y = x^H(-\mu y)$
$\Rightarrow \bar{\lambda} x^H y = -\mu x^H y \Rightarrow (\bar{\lambda} + \mu) x^H y = 0$
$\because \lambda$ *and* $\mu$ *are pure imaginary, and* $\lambda \neq \mu$.
$\therefore x^H y = 0$