國立臺灣大學 期中/學期 考試答案卷

National Taiwan University Midterm/Final Examination Answer Sheet

記分 Score	教師簽名或蓋章 Lecturer's signature
75.3	

課程編號 Course no. ______

科目 Course title 線性代數與應用 ______ College 學院 ______ Department 學系 ______ Division 組 ______ Year 年級

考試日期 Date ___ 年 Y ___ 月 M ___ 日 D ___ 學號 Student ID no. R08522830 姓名 Name 陳皓澤

從此處開始寫起。試卷用紙務須節用。非經主試認可不得續用其他紙張作答。
Please write from here.

記分欄

1-0, 5-0.5, 2-4, 3-5, 4-9, 6-6.2

1. $\dfrac{du}{dx} = \dfrac{u(x+h)-u(x)}{h}$, $\dfrac{d^2u}{dx^2} = \dfrac{\frac{u(x+h)-u(x)}{h} - \frac{u(x)-u(x-h)}{h}}{h} \approx \dfrac{u(x+h)-2u(x)+u(x-h)}{h^2} \Rightarrow \dfrac{u_{j+1}-2u_j+u_{j-1}}{h^2}$

$$-\frac{d^2u}{dx^2}+u=8x \Rightarrow -\left[\frac{u_{j+1}-2u_j+u_{j-1}}{h^2}\right]+u_j = 8(jh) \qquad j=1,2,3$$

$$u(0)=1 \Rightarrow u_0=1 \ , \ u(1)=1 \Rightarrow u_4=1$$

$$j=1,\ -\frac{1}{0.25^2}[u_2-2u_1+1]+u_1 = 8(0.25)=2 \Rightarrow 33u_1-16u_2=18$$

$$j=2,\ -\frac{1}{0.25^2}[u_3-2u_2+u_1]+u_2 = 8(2\times 0.25)=4 \Rightarrow -16u_1+33u_2-16u_3=4$$

$$j=3,\ -\frac{1}{0.25^2}[1-2u_3+u_2]+u_3 = 8(3\times 0.25)=6 \Rightarrow -16u_2+33u_3=22$$

$$\begin{bmatrix}33&-16&0\\-16&33&-16\\0&-16&33\end{bmatrix}\begin{bmatrix}u_1\\u_2\\u_3\end{bmatrix}=\begin{bmatrix}18\\4\\22\end{bmatrix} \ , \ \begin{bmatrix}u_1\\u_2\\u_3\end{bmatrix}=\begin{bmatrix}1.1942\\1.33795\\1.31537\end{bmatrix} \text{※}$$

2.

(a) $Ax=(LU)x=b,\ Ux=L^{-1}b=c \Rightarrow Lc=b$

$$\begin{bmatrix}1&0&0\\1&1&0\\0&1&1\end{bmatrix}\begin{bmatrix}c_1\\c_2\\c_3\end{bmatrix}=\begin{bmatrix}b_1\\b_2\\b_3\end{bmatrix} \Rightarrow \begin{bmatrix}c_1\\c_2\\c_3\end{bmatrix}=\begin{bmatrix}b_1\\b_2-b_1\\b_3-b_2+b_1\end{bmatrix}$$, 若 $c_3=0$, 則 solvable $\Rightarrow b_3-b_2+b_1=0$

$$\begin{bmatrix}0&1&2&3&4\\0&0&0&1&2\\0&0&0&0&0\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\\x_4\\x_5\end{bmatrix}=\begin{bmatrix}b_1\\b_2-b_1\\0\end{bmatrix} \Rightarrow \begin{matrix}x_2+2x_3+3x_4+4x_5=b_1\\x_4+2x_5=b_2-b_1\end{matrix}$$

令 $x_1=\alpha,\ x_3=\beta,\ x_5=\gamma$.

$x=?$

$$\Rightarrow x_4=b_2-b_1-2\gamma$$

$$x_2=b_1-2\beta-3(b_2-b_1-2\gamma)-4\gamma=4b_1-3b_2-2\beta+2\gamma$$

故 $$\begin{bmatrix}x_1\\x_2\\x_3\\x_4\\x_5\end{bmatrix}=\underbrace{b_1\begin{bmatrix}0\\4\\0\\-1\\0\end{bmatrix}+b_2\begin{bmatrix}0\\-3\\0\\1\\0\end{bmatrix}}_{x_r}+\underbrace{\alpha\begin{bmatrix}1\\0\\0\\0\\0\end{bmatrix}+\beta\begin{bmatrix}0\\-2\\1\\0\\0\end{bmatrix}+\gamma\begin{bmatrix}0\\2\\0\\-2\\1\end{bmatrix}}_{x_n}$$ (其中 α,β,γ 為 constant). ※

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(b)
$$A=\begin{bmatrix}0&1&2&3&4\\0&1&2&4&6\\0&0&0&1&2\end{bmatrix}$$

column space $R(A)=c_1\begin{bmatrix}1\\1\\0\end{bmatrix}+c_2\begin{bmatrix}3\\4\\1\end{bmatrix}$

row space $R(A^T)=c_1\begin{bmatrix}0&1&2&3&4\end{bmatrix}+c_2\begin{bmatrix}0&0&0&1&2\end{bmatrix}$

nullspace $N(A)=c_1\begin{bmatrix}1\\0\\0\\0\\0\end{bmatrix}^T+c_2\begin{bmatrix}0\\-2\\1\\0\\0\end{bmatrix}^T+c_3\begin{bmatrix}0\\2\\0\\-2\\1\end{bmatrix}^T$

left-nullspace $N(A^T)=c_1\begin{bmatrix}1\\-1\\1\end{bmatrix}$ ※

(c) $x_r=7\begin{bmatrix}0\\4\\0\\-1\\0\end{bmatrix}+9\begin{bmatrix}0\\-3\\0\\1\\0\end{bmatrix}=\begin{bmatrix}0\\1\\0\\2\\0\end{bmatrix}$ ※ -4

3. ① $(1+t)\cdot 0=0$ -2

② $[a_0+a_1t+a_2t^2+\dots]\cdot(1+t)=a_0(1+t)+a_1(1+t)t+a_2(1+t)t^2+\dots$? $(1+t)[\alpha P(t)]=\alpha(1+t)P(t)$

③ $[a_0+a_1t+a_2t^2+\dots](1+t)+[b_0+b_1t+b_2t^2+\dots](1+t)=[(a_0+b_0)+(a_1+b_1)t+(a_2+b_2)t^2+\dots](1+t)$

由①②③得，multiplication of polynomials by $(1+t)$ is a linear transformation ※

$a_0+a_1t+a_2t^2 \xrightarrow{T_2} (a_0+a_1t+a_2t^2)(1+t)=a_0+(a_0+a_1)t+(a_1+a_2)t^2+a_2t^3$

$$\begin{bmatrix}1&0&0\\1&1&0\\0&1&1\\0&0&1\end{bmatrix}\begin{bmatrix}a_0\\a_1\\a_2\end{bmatrix}=\begin{bmatrix}a_0\\a_0+a_1\\a_1+a_2\\a_2\end{bmatrix}，故 T_2=\begin{bmatrix}1&0&0\\1&1&0\\0&1&1\\0&0&1\end{bmatrix}$$ ※ -3

$a_0+a_1t+a_2t^2+a_3t^3 \xrightarrow{T_3} (a_0+a_1t+a_2t^2+a_3t^3)(1+t)=a_0+(a_0+a_1)t+(a_1+a_2)t^2+(a_2+a_3)t^3+a_3t^4$

$$\begin{bmatrix}0&0&0&0\\1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}\begin{bmatrix}a_0\\a_1\\a_2\\a_3\end{bmatrix}=\begin{bmatrix}0\\a_0\\a_1\\a_2\\a_3\end{bmatrix}，故 T_3=\begin{bmatrix}0&0&0&0\\1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}$$ ※

4. (a) ~~$\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=x_1\begin{bmatrix}1\\0\\1\end{bmatrix}+x_2\begin{bmatrix}0\\1\\0\end{bmatrix}$ (x_1, x_2為任意數)~~ -4

(b) basis 為 $\begin{bmatrix}1\\0\\1\end{bmatrix},\begin{bmatrix}0\\1\\0\end{bmatrix}$

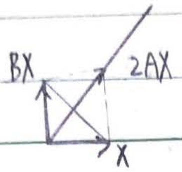
(c) 令 $S=\begin{bmatrix}1&0\\0&1\\1&0\end{bmatrix}$

$$A=S(S^TS)^{-1}S^T=\begin{bmatrix}1&0\\0&1\\1&0\end{bmatrix}\left(\begin{bmatrix}1&0&1\\0&1&0\end{bmatrix}\begin{bmatrix}1&0\\0&1\\1&0\end{bmatrix}\right)^{-1}\begin{bmatrix}1&0&1\\0&1&0\end{bmatrix}=\begin{bmatrix}1&0\\0&1\\1&0\end{bmatrix}\begin{bmatrix}2&0\\0&1\end{bmatrix}^{-1}\begin{bmatrix}1&0&1\\0&1&0\end{bmatrix}$$

記分欄 轉頁從此開始寫起。

$$=\frac{1}{2}\begin{bmatrix}1&0\\0&1\\1&0\end{bmatrix}\begin{bmatrix}1&0\\0&2\end{bmatrix}\begin{bmatrix}1&0&1\\0&1&0\end{bmatrix}=\begin{bmatrix}\frac{1}{2}&0&\frac{1}{2}\\0&1&0\\\frac{1}{2}&0&\frac{1}{2}\end{bmatrix} ※$$

(d)



$$\Rightarrow BX+X=2AX \Rightarrow BX=(2A-I)X$$

$$B=\begin{bmatrix}0&0&1\\0&1&0\\1&0&0\end{bmatrix} ※$$

(e)

-5

5. (a) ① $C+DX+EX^2=1-e^X$

$E^2=(C+DX+EX^2-1+e^X)^2$

$\frac{dE^2}{dC}=0$, $\frac{dE^2}{dD}=0$, $\frac{dE^2}{dE}=0$

② $\begin{bmatrix}1&0&0\\1&1&1\\1&2&4\end{bmatrix}\begin{bmatrix}C\\D\\E\end{bmatrix}=\begin{bmatrix}0\\1-e\\1-e^2\end{bmatrix} \Rightarrow A\hat{X}=b$, $\hat{X}=(A^TA)^{-1}A^Tb$ ※

(b)

$q_1=1$

$$q_2=X-\frac{(q_1,X)}{(q_1,q_1)}q_1=X-\frac{\int_0^4 X\,dX}{\int_0^4 dX}=X-\frac{8}{4}=X-2$$

$$q_3=X^2-\frac{(q_1,X^2)}{(q_1,q_1)}q_1-\frac{(q_2,X^2)}{(q_2,q_2)}q_2=X^2-\frac{\int_0^4 X^2dX}{\int_0^4 dX}-\frac{\int_0^4 X^2(X-2)dX}{\int_0^4(X-2)^2dX}(X-2)$$

$$=X^2-\frac{\frac{64}{3}}{4}-\frac{\frac{64}{3}}{\frac{16}{3}}(X-2)=X^2-4X-\frac{4}{3} ※$$

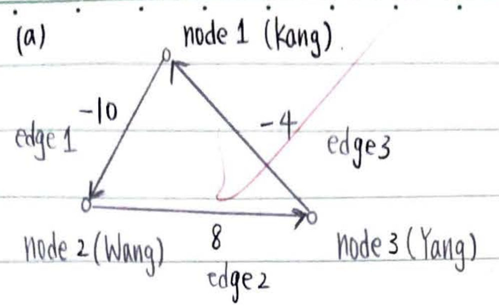
-0.5 (c)

$$\begin{bmatrix}(q_1,q_1)&(q_1,q_2)&(q_1,q_3)\\(q_2,q_1)&(q_2,q_2)&(q_2,q_3)\\(q_3,q_1)&(q_3,q_2)&(q_3,q_3)\end{bmatrix}\begin{bmatrix}C'\\D'\\E'\end{bmatrix}=\begin{bmatrix}(q_1,1-e^X)\\(q_2,1-e^X)\\(q_3,1-e^X)\end{bmatrix}=\begin{bmatrix}\int_0^4 1-e^X dX\\\int_0^4(X-2)(1-e^X)dX\\\int_0^4(X^2-4X-\frac{4}{3})(1-e^X)dX\end{bmatrix}=\begin{bmatrix}-49.598\\-57.598\\170.66\end{bmatrix}$$

$$(q_3,q_3)=\int_0^4\left(X^2-4X-\frac{4}{3}\right)^2dX=\frac{3136}{45}$$

$$\begin{bmatrix}4&0&0\\0&\frac{16}{3}&0\\0&0&\frac{3136}{45}\end{bmatrix}\begin{bmatrix}C'\\D'\\E'\end{bmatrix}=\begin{bmatrix}-49.598\\-57.598\\170.66\end{bmatrix}\Rightarrow\begin{bmatrix}C'\\D'\\E'\end{bmatrix}=\begin{bmatrix}-12.3995\\-10.7996\\2.44\cancel{89}\end{bmatrix} ※$$

6. (a)



$$\begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 1 & 0 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -10 \\ 8 \\ -4 \end{bmatrix}$$

-3.2

(b)
$$\begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 1 & 0 & -1 \end{bmatrix} \to \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 1 & -1 \end{bmatrix} \to \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

nullspace $= c\,[1\ \ 1\ \ 1]$

left-nullspace $= c\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ (所有人英文程度的位能基準)

row-space $= c_1[-1\ \ 1\ \ 0] + c_2[0\ \ -1\ \ 1]$ (判斷位能差有效的場數)

(c) ① 由高斯消去法知

$$\begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -10 \\ 8 \\ -22 \end{bmatrix}$$ ，故 $AX=b$ is not solvable

② 保留 1,2 的 row，$A = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$，則 $A^TAb = A^Tb$ is solvable ※

-2

(d)
$$\begin{bmatrix} -1 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -10 \\ 75 \\ 86 \end{bmatrix}$$

令 $a = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$，$b = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

$$q_1 = \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$b' = b - (q_1^Tb)q_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} - \left(-\frac{1}{\sqrt{2}}\right)\begin{bmatrix} -\frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ 1 \\ \frac{1}{2} \end{bmatrix}, \quad q_2 = \frac{b'}{\|b'\|} = \sqrt{\frac{2}{3}}\begin{bmatrix} \frac{1}{2} \\ 1 \\ \frac{1}{2} \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{6}} \\ \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \end{bmatrix}$$

$$B = QR = \begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ 0 & \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \end{bmatrix}\begin{bmatrix} \sqrt{2} & -\frac{1}{\sqrt{2}} \\ 0 & \frac{3}{\sqrt{6}} \end{bmatrix}$$

-1

(e) $QRX=b \Rightarrow (QR)^T(QR)X = (QR)^Tb \Rightarrow RX = Q^Tb$

$$\begin{bmatrix} \sqrt{2} & -\frac{1}{\sqrt{2}} \\ 0 & \frac{3}{\sqrt{6}} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \Big[$$