數位控制系統 Project #1

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A. Develop discrete-time models of two motor subsystems:

1. Use Method#1 to find both k-domain difference equations and z-domain transfer function for the position motion system in (2).

Step 1. Develop a physical system model.

馬達的物理系統可以透過電路學、電磁學和經典力學中的定理獲得 ODE。

Step 2. Prepare differential equation (ODE) description.

當輸入為電流、輸出為位置時，其系統之 ODE 可以表示為：

$$J_m \frac{\mathrm{d}\theta^2(\mathrm{t})}{\mathrm{dt}^2} = K_T i_a(\mathrm{t}) \tag{A.1.1}$$

Step 3. Form the Laplace transform of the differential equations.

將(A.1.1)取 Laplace transform：

$$\mathscr{L}.\mathscr{T}. \Rightarrow \mathscr{L}\left[J_m \frac{\mathrm{d}\theta^2(\mathrm{t})}{\mathrm{dt}^2}\right] = \mathscr{L}[K_T i_a(\mathrm{t})]$$

$$\Rightarrow J_m(s^2\Theta(s) - s\theta(0) - \theta'(0)) = K_T \cdot I_a(s) \tag{A.1.2}$$

Step 4. Cross-solve the Laplace transformed differential equations for each state as needed.

令$\theta'(0) = \omega(0)$，將(A.1.2)整理過後可得：

$$\Theta(s) = \frac{\theta(0)}{s} + \frac{\omega_m(0)}{s^2} + \frac{K_T}{J_m} \cdot \frac{I_a(s)}{s^2} \tag{A.1.3}$$

Step 5. Substitute the zero order hold (step input) model for the manipulated input.

令輸入電流$I_a(s)$為步階輸入：

$$I_a(s)\Big|_{t=0\sim T} = \frac{i_a(0)}{s} \tag{A.1.4}$$

Step 6. Find the continuous time step response solution (cross-coupled initial conditions).

將(A.1.4)代入(A.1.3)可得：

$$\Theta(s)=\frac{\theta(0)}{s}+\frac{\omega_m(0)}{s^2}+\frac{K_T}{J_m}\cdot\frac{i_a(0)}{s^3} \tag{A.1.5}$$

將(A.1.5)取反拉式轉換可得：

$$\text{inverse } \mathscr{L}.\mathscr{T}. \Rightarrow \mathscr{L}^{-1}[\Theta(s)]=\mathscr{L}^{-1}\left[\frac{\theta(0)}{s}+\frac{\omega_m(0)}{s^2}+\frac{K_T}{J_m}\cdot\frac{i_a(0)}{s^3}\right]$$

$$\Rightarrow \theta(t)=\theta(0)+\omega_m(0)\cdot t+\frac{K_T}{2J_m}\cdot i_a(0)\cdot t^2 \tag{A.1.6}$$

Step 7. Find the cross-coupled response at the next sample instant.

分別對每一個週期 T 取方程式：

$$\begin{aligned}
0 \sim T:\ &\theta(T)=\theta(0)+\omega_m(0)\cdot T+\frac{K_T T^2}{2J_m}\cdot i_a(0)\\
T \sim 2T:\ &\theta(2T)=\theta(T)+\omega_m(T)\cdot T+\frac{K_T T^2}{2J_m}\cdot i_a(T)\\
2T \sim 3T:\ &\theta(3T)=\theta(2T)+\omega_m(2T)\cdot T+\frac{K_T T^2}{2J_m}\cdot i_a(2T)\\
&\vdots \qquad\qquad\qquad \vdots
\end{aligned}$$

Step 8. Find the cross-coupled difference equation models in sampled time domain.

透過前一個步驟可得系統經取樣後在 kT-domain 的方程式為：

$$\theta(kT)=\theta((k-1)T)+\omega_m((k-1)T)\cdot T+\frac{K_T T^2}{2J_m}\cdot i_a((k-1)T) \tag{A.1.7}$$

將(A.1.7)改寫為 index form：

$$\theta(k)=\theta(k-1)+\omega_m(k-1)\cdot T+\frac{K_T T^2}{2J_m}\cdot i_a(k-1) \tag{A.1.8}$$

Step 9. Find the Z-domain model of the cross-coupled difference equations.

將(A.1.8)取 Z 轉換可得：

$$\mathscr{Z}.\mathscr{T}. \Rightarrow \mathscr{Z}[\theta(k)]=\mathscr{Z}\left[\theta((k-1)T)+\omega_m((k-1)T)\cdot T+\frac{K_T T^2}{2J_m}\cdot i_a((k-1)T)\right]$$

$$\Rightarrow \Theta(z)=\Theta(z)\cdot z^{-1}+\Omega_m(z)\cdot z^{-1}\cdot T+\frac{K_T T^2}{2J_m}\cdot I_a(z)\cdot z^{-1} \tag{A.1.9}$$

注意：其中$\Omega_m(z)$轉移函數仍未知。

NOTE：

可以注意到(A.1.9)中，$\Omega(z)$ 轉移函數仍未知。透過速度與電流的 ODE：

$$J_m \frac{d\omega_m(t)}{dt} = K_T \cdot i_a(t) \tag{A.1.10}$$

透過方法一可得：

$$\begin{aligned} \mathscr{L}.\mathscr{T}. &\Rightarrow J_m(s\Omega_m(s) - \omega(0)) = K_T \cdot I_a(s) \\ &\Rightarrow \Omega_m(s) = \frac{\omega(0)}{s} + \frac{K_T}{J_m} \cdot \frac{I_a(s)}{s} \end{aligned} \tag{A.1.11}$$

將(A.1.4)輸入電流 $I_a(s)$ 代入(A.1.11)可得：

$$\Omega(s) = \frac{\omega(0)}{s} + \frac{K_T}{J_m} \cdot \frac{i_a(0)}{s^2} \tag{A.1.12}$$

(A.1.12)取反拉氏轉換並取樣可得：

$$\text{inverse } \mathscr{L}.\mathscr{T}. \Rightarrow \omega_m(t) = \omega_m(0) + \frac{K_T \cdot i_a(0)}{J_m} \cdot t$$

$$\text{Sampling} \Rightarrow \omega_m(k) = \omega_m(k-1) + \frac{K_T \cdot i_a(k-1)}{J_m} \cdot T \tag{A.1.13}$$

將(A.1.13)取 Z 轉換可得當輸入為電流、輸出為角速度的轉移函數為：

$$\mathscr{Z}.\mathscr{T}. \Rightarrow \Omega_m(z)[1 - z^{-1}] = \frac{K_T \cdot T}{J_m} I_a(z) \cdot z^{-1}$$

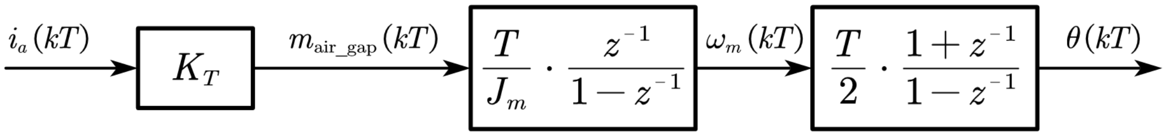
$$\frac{\Omega_m(z)}{I_a(z)} = \frac{K_T \cdot T}{J_m} \cdot \frac{z^{-1}}{1 - z^{-1}} \tag{A.1.14}$$

將(A.1.14)代入(A.1.9)可得當輸入為電流、輸出為角位置的轉移函數為：

$$\Theta(z) = \Theta(z) \cdot z^{-1} + \frac{K_T T^2}{J_m} \cdot \frac{z^{-2}}{1 - z^{-1}} + \frac{K_T T^2}{2J_m} \cdot z^{-1} I_a(z)$$

$$\frac{\Theta(z)}{I_a(z)} = \frac{K_T T^2}{2J_m} \cdot \frac{z^{-1} \cdot (1 + z^{-1})}{(1 - z^{-1})} I_a(z) \tag{A.1.15}$$

最後，可將該系統的方塊圖表示為：



2. Use Method#2 and Z-table to find difference equation and transfer function for the electric system in (1).

Step 1. Develop a physical system model by differential equation.

當輸入為電壓、輸出為電流時，其系統之 ODE 可以表示為：

$$e_a(t) = R_p i_a(t) + L_p \frac{di_a(t)}{dt} + K_e \omega_m(t) \tag{A.2.1}$$

Step 2. Form the Laplace transform of the differential equations.

將(A.2.1)取 Laplace transform 且初始條件為 0：

$$\begin{aligned} \mathscr{L}.\mathscr{T}. \Rightarrow E_a(s) &= R_p I_a(s) + L_p\left(sI_a(s) - \cancel{i_a(0)}\right) + K_e \Omega_m(s) \\ \Rightarrow E_a(s) &= I_a(s)(R_p + L_p s) + E_b(s) \quad (\text{note: } E_b(s) = K_e \Omega_m(s)\ ) \end{aligned}$$

可得系統的轉移函數為：

$$\frac{I_a(s)}{E_a(s) - E_b(s)} = \frac{1}{R_p + L_p s} \tag{A.2.2}$$

Step 3. Consider ZOH effect in the original transfer function.

對原系統加入零階保持效應：

$$\begin{aligned} NDS(s)\big|_{b.s.} &= ZOH(s) \cdot \frac{I_a(s)}{E_a(s) - E_b(s)} \\ &= \frac{1 - e^{-Ts}}{s} \cdot \frac{1}{R_p + L_p s} \end{aligned} \tag{A.2.3}$$

Step 4. Transfer the continuous-time system to discrete-time using Z.T. table.

透過 Z 轉換表，將(A.2.3)轉為離散域的轉移函數：

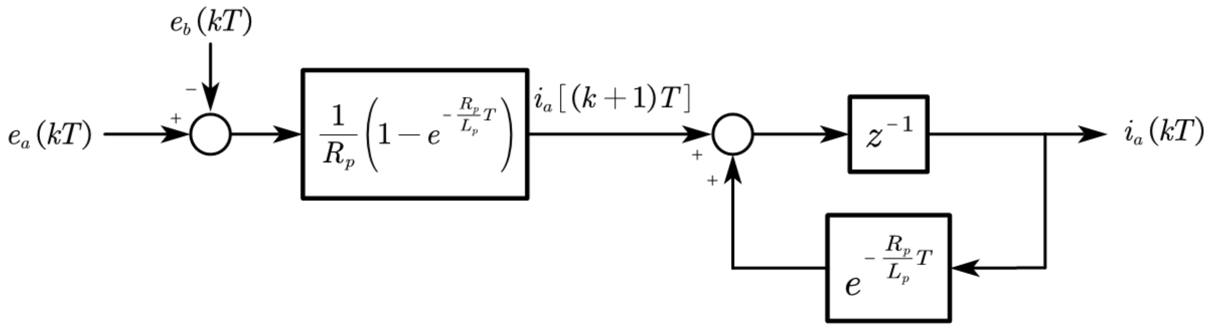
$$\begin{aligned} NDS(z)\big|_{a.s.} &= \mathscr{Z}\left[\frac{1 - e^{-Ts}}{s(R_p + L_p s)}\right] \\ &= \mathscr{Z}[1 - e^{-Ts}] \cdot \mathscr{Z}\left[\frac{1}{s(R_p + L_p s)}\right] \\ &= \cancel{(1 - z^{-1})} \cdot \frac{1}{R_p} \cdot \frac{z^{-1}\left(1 - e^{-\frac{R_p}{L_p}T}\right)}{\cancel{(1 - z^{-1})}\left(1 - z^{-1} e^{-\frac{R_p}{L_p}T}\right)} \end{aligned}$$

$$\text{note: } \mathscr{Z}\left[\frac{a}{s(s+a)}\right] = \frac{z^{-1}(1 - e^{-aT})}{(1 - z^{-1})(1 - z^{-1}e^{-aT})}$$

可得離散的系統轉移函數為：

$$\frac{I_a(z)}{E_a(z)-E_b(z)}=\frac{1}{R_p}\cdot\frac{z^{-1}\left(1-e^{-\frac{R_p}{L_p}T}\right)}{\left(1-z^{-1}e^{-\frac{R_p}{L_p}T}\right)} \tag{A.2.4}$$

最後，可將該系統的方塊圖表示為：



B. Develop the discrete-time model of full motor drive:

1. At part B, you are asked to analyze an actual motor drive where the computer input is voltage ea(t=kT), and output is the sampled speed ωm(t=kT). Analyze difference equations of (4) and (5). Show parameters of C1T, C2T and C3T.

Step 1. Develop a physical system model.

馬達的物理系統可以透過電路學、電磁學和經典力學中的定理獲得 ODE。

Step 2. Prepare differential equation (ODE) description.

系統之**電路系統**與**動力系統之** ODE 可以分別表示為：

$$\begin{cases} e_a(t)=R_p i_a(t)+L_p\dfrac{di_a(t)}{dt}+K_e\omega_m(t) \\ J_m\dfrac{\mathrm{d}\omega_m(\mathrm{t})}{\mathrm{dt}}=K_T i_a(\mathrm{t}) \end{cases} \tag{B.1.1}$$

Step 3. Form the Laplace transform of the differential equations.

對(B.1.1)分別取拉氏轉換可得：

$$\mathscr{L}.\mathscr{T}.\Rightarrow\begin{cases} E_a(s)=R_p I_a(s)+L_p(sI_a(s)-i_a(0))+K_e\Omega_m(s) \\ J_m(s\Omega_m(s)-\omega_m(0))=K_T\cdot I_a(s) \end{cases} \tag{B.1.2}$$

Step 4. Cross-solve the Laplace transformed differential equations for each state as needed.

將(B.1.2)兩式互相代入可分別解出$I_a(s)$和$\Omega_m(s)$為：

$$\begin{cases} I_a(s) = \dfrac{\frac{1}{L_p}E_a(s)\cdot s - \frac{K_e}{L_p}\omega_m(0) + i_a(0)s}{s^2 + \frac{R_p}{L_p}s + \frac{K_eK_T}{L_pJ_m}} \\ \\ \Omega_m(s) = \dfrac{\frac{K_T}{L_pJ_m}E_a(s) + \frac{K_T}{J_m}i_a(0) + \omega_m(0)\cdot\left(\frac{R_p}{L_p}+s\right)}{s^2 + \frac{R_p}{L_p}s + \frac{K_eK_T}{L_pJ_m}} \end{cases} \tag{B.1.3}$$

分別令$\omega_n = \sqrt{\dfrac{K_eK_T}{L_pJ_m}}$，$\zeta = \dfrac{R_p}{2\omega_nL_p} = \dfrac{R_p}{2}\sqrt{\dfrac{J_m}{K_eK_TL_p}}$，$\varphi = \tan^{-1}\dfrac{\sqrt{1-\zeta^2}}{\zeta}$化簡(B.1.3)：

$$\begin{cases} I_a(s) = \dfrac{\frac{1}{L_p}E_a(s)\cdot s - \frac{K_e}{L_p}\omega_m(0) + i_a(0)s}{s^2 + 2\zeta\omega_ns + {\omega_n}^2} \\ \\ \Omega_m(s) = \dfrac{\frac{K_T}{L_pJ_m}E_a(s) + \frac{K_T}{J_m}i_a(0) + \omega_m(0)\cdot\left(\frac{R_p}{L_p}+s\right)}{s^2 + 2\zeta\omega_ns + {\omega_n}^2} \end{cases} \tag{B.1.4}$$

Step 5. Substitute the zero order hold (step input) model for the manipulated input.

令輸入電壓$E_a(s)$為步階輸入：

$$E_a(s)\Big|_{t=0\sim T} = \frac{e_a(0)}{s} \tag{B.1.5}$$

Step 6. Find the continuous time step response solution (cross-coupled initial conditions).

將(B.1.5)代入(B.1.4)可得：

$$\begin{cases} I_a(s) = \dfrac{\frac{1}{L_p}e_a(0) - \frac{K_e}{L_p}\omega_m(0) + i_a(0)s}{s^2 + 2\zeta\omega_ns + {\omega_n}^2} \\ \\ \Omega_m(s) = \dfrac{\frac{K_T}{L_pJ_m}e_a(0)\cdot\frac{1}{s} + \frac{K_T}{J_m}i_a(0) + \omega_m(0)\cdot\left(\frac{R_p}{L_p}+s\right)}{s^2 + 2\zeta\omega_ns + {\omega_n}^2} \end{cases} \tag{B.1.6}$$

將(B.1.6)取反拉式轉換可得：

$$\text{inverse } \mathscr{L}.\mathscr{T}. \Rightarrow \begin{cases} i_a(t) = \dfrac{\dfrac{1}{L_p}e_a(0) - \dfrac{K_e}{L_p}\omega_m(0)}{\omega_n^2}C_{1t} + \dfrac{i_a(0)}{\omega_n^2}C_{2t} \\ \omega_m(t) = \left[\dfrac{\dfrac{K_T}{L_pJ_m}e_a(0)}{\omega_n{}^2} - \omega_m(0)\right]C_{3t} + \dfrac{\dfrac{K_T}{J_m}i_a(0)}{\omega_n{}^2}C_{1t} + \omega_m(0) \end{cases} \tag{B.1.7}$$

其中C_{1t}、C_{2t}及C_{3t}分別為：

$$\begin{aligned} C_{1t} &= L^{-1}\left\{\frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n{}^2}\right\} = \frac{\omega_n}{\sqrt{1-\zeta^2}}e^{-\zeta\omega_n t}\sin\left(\sqrt{1-\zeta^2}\,\omega_n t\right) \\ C_{2t} &= L^{-1}\left\{\frac{\mathrm{s}\omega_{\mathrm{n}}^2}{s^2+2\zeta\omega_n s+\omega_n{}^2}\right\} = -\frac{\omega_n{}^2}{\sqrt{1-\zeta^2}}e^{-\zeta\omega_n t}\sin\left(\sqrt{1-\zeta^2}\,\omega_n t-\varphi\right) \\ C_{3t} &= L^{-1}\left\{\frac{\omega_{\mathrm{n}}^2}{s(s^2+2\zeta\omega_n s+\omega_n{}^2)}\right\} = 1-\frac{1}{\sqrt{1-\zeta^2}}e^{-\zeta\omega_n t}\sin\left(\sqrt{1-\zeta^2}\,\omega_n t+\varphi\right) \end{aligned} \tag{B.1.8}$$

Note: 提供的 Laplace transform table 是錯的!!

Step 7. Find the cross-coupled response at the next sample instant.

分別對每一個週期 T 取方程式(方程式過於複雜，在此略過)。

Step 8. Find the cross-coupled difference equation models in sampled time domain.

透過前一個步驟可得系統經取樣後在 kT-domain 的方程式為：

$$\begin{cases} i_a(k) = \dfrac{\dfrac{1}{L_p}e_a(k-1) - \dfrac{K_e}{L_p}\omega_m(k-1)}{\omega_n^2}C_{1T} + \dfrac{i_a(k-1)}{\omega_n^2}C_{2T} \\ \omega_m(k) = \left[\dfrac{\dfrac{K_T}{L_pJ_m}e_a(k-1)}{\omega_n{}^2} - \omega_m(k-1)\right]C_{3T} + \dfrac{\dfrac{K_T}{J_m}i_a(k-1)}{\omega_n{}^2}C_{1T} + \omega_m(k-1) \end{cases} \tag{B.1.9}$$

最後，可得C_{1T}、C_{2T}及C_{3T}分別為：

$$\begin{aligned} C_{1T} &= \frac{\omega_n}{\sqrt{1-\zeta^2}}e^{-\zeta\omega_n T}\sin\left(\sqrt{1-\zeta^2}\,\omega_n T\right) \\ C_{2T} &= -\frac{\omega_n{}^2}{\sqrt{1-\zeta^2}}e^{-\zeta\omega_n T}\sin\left(\sqrt{1-\zeta^2}\,\omega_n T-\varphi\right) \\ C_{3T} &= 1-\frac{1}{\sqrt{1-\zeta^2}}e^{-\zeta\omega_n T}\sin\left(\sqrt{1-\zeta^2}\,\omega_n T+\varphi\right) \end{aligned} \tag{B.1.10}$$

2. Replot the block diagram of Fig. 4 based on your derived transfer function. Indicate A_1, A_{ie1}, A_{ie2}, B_1, $B_{\omega e1}$, $B_{\omega e2}$ and B_{ie}.

Transfer function of 1st equation:

將(B.1.9)第一式做 z 轉換可得：

$$I_a(z)=\frac{\frac{1}{L_p}E_a(z)\cdot z^{-1}-\frac{K_e}{L_p}\Omega_m(z)\cdot z^{-1}}{\omega_n^2}C_{1T}+\frac{I_a(z)\cdot z^{-1}}{\omega_n^2}C_{2T} \quad \text{(B.2.1)}$$

整理後可得輸入電壓$E_a(z)-Ke\cdot\Omega_m(z)$與輸出電流$I_a(z)$的轉移函數為：

$$\frac{I_a(z)}{E_a(z)-Ke\cdot\Omega_m(z)}=\frac{B_1\cdot z^{-1}}{1-A_1\cdot z^{-1}} \quad \text{(B.2.2)}$$

其中係數A_1和B_1為：

$$\begin{cases} A_1=\dfrac{C_{2T}}{\omega_n^2} \\ B_1=\dfrac{C_{1T}}{L_p\omega_n^2} \end{cases} \quad \text{(B.2.3)}$$

Transfer function of 2nd equation:

將(B.1.9)第二式做 z 轉換可得：

$$\Omega_m(z)=\left[\frac{\frac{K_T}{L_pJ_m}E_a(z)\cdot z^{-1}}{{\omega_n}^2}-\Omega_m(z)\cdot z^{-1}\right]C_{3T}+\frac{\frac{K_T}{J_m}I_a(z)\cdot z^{-1}}{{\omega_n}^2}C_{1T}+\Omega_m(z)\cdot z^{-1} \quad \text{(B.2.4)}$$

將$I_a(z)=\dfrac{B_1\cdot z^{-1}}{1-A_1\cdot z^{-1}}[E_a(z)-Ke\cdot\Omega_m(z)]$代入(B.2.4)整理，可得當輸入電壓

$E_a(z)$與輸出角速度$\Omega_m(z)$的轉移函數為：

$$\frac{\Omega_m(z)}{E_a(z)}=\frac{\frac{K_T}{Jm\cdot\omega_n^2}\left(\frac{C_{3T}}{L_p}+C_{1T}\cdot\frac{B_1\cdot z^{-1}}{1-A_1\cdot z^{-1}}\right)\cdot z^{-1}}{1+\left(C_{3T}-1+\frac{K_TK_eC_{1T}}{J_m{\omega_n}^2}\cdot\frac{B_1\cdot z^{-1}}{1-A_1\cdot z^{-1}}\right)\cdot z^{-1}} \quad \text{(B.2.5)}$$

將(B.2.5)化簡為負回授的形式可得：

$$\frac{\Omega_m(z)}{E_a(z)}=\frac{\left(\frac{B_1\cdot z^{-1}}{1-A_1\cdot z^{-1}}\right)\cdot K_T\cdot V(z)}{1+\left(\frac{B_1\cdot z^{-1}}{1-A_1\cdot z^{-1}}\right)\cdot K_T\cdot V(z)\cdot K_e} \quad \text{(B.2.6)}$$

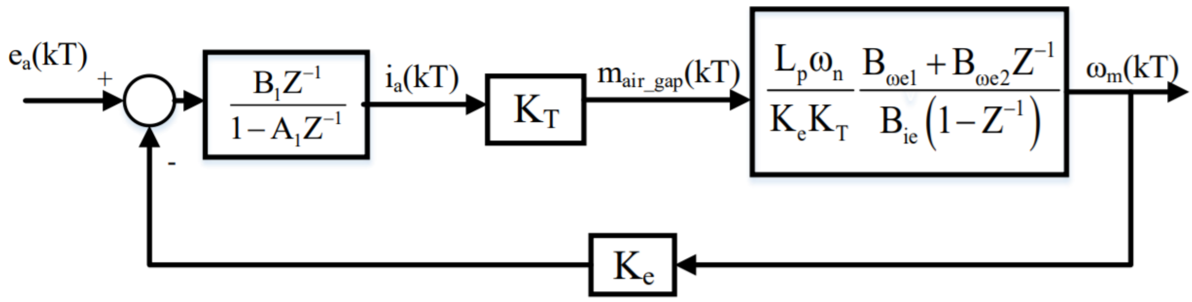
其中$V(z)$為：

$$V(z)=\frac{L_p\omega_n}{K_eK_T}\frac{B_{\omega e1}+B_{\omega e2}\cdot z^{-1}}{B_{ie}(1-z^{-1})} \tag{B.2.7}$$

其中係數$B_{\omega e1}$、$B_{\omega e2}$和B_{ie}為：

$$\begin{cases} B_{\omega e1}= C_{3T} \\ B_{\omega e2}=\dfrac{-C_{2T}C_{3T}+{C_{1T}}^2}{\omega_n^2} \\ B_{ie}=\dfrac{C_{1T}}{\omega_n} \end{cases} \tag{B.2.8}$$

最後，可將該系統的方塊圖表示為：



其中的參數為(B.2.3)及(B.2.8)所得：

$$\begin{cases} A_1=\dfrac{C_{2T}}{\omega_n^2} \\ B_1=\dfrac{C_{1T}}{L_p\omega_n^2} \\ B_{\omega e1}= C_{3T} \\ B_{\omega e2}=\dfrac{-C_{2T}C_{3T}+{C_{1T}}^2}{\omega_n^2} \\ B_{ie}=\dfrac{C_{1T}}{\omega_n} \end{cases} \tag{B.2.9}$$

3. Explain the difference: why the model in part B is different from the combination of model in Part A. There are two issues to cause this difference.

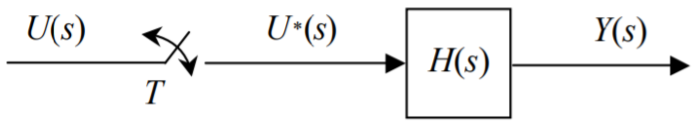
為什麼 B 部分中的離散域模型與 A 部分中分別推導的離散域模型的組合不同？我們所想到可能的兩個分別為：

1. **零階保持後會對系統造成的影響(Average process)。**
2. **負回授對系統造成的影響。**

以下將對兩個原因分別做說明。

原因一： 系統加入零階保持後會對整個系統造成影響(Average process)。直接相連的子系統在取樣後不能直接分離。然而，若是相鄰的轉移函數透過取樣器(samplers)分離，則每個子系統都具有個別取樣後的輸出和輸入以及 z 域傳遞函數。

首先，我們要先補充離散域 LIT 系統的響應。考慮一 LIT 系統$H(s)$當系統方塊圖為以下所示時：



系統的響應可以表示為：

$$
\begin{aligned}
y(t) &= \int_0^t h(t-\tau)u^*(\tau)\,d\tau \\
&= \int_0^t h(t-\tau)\left[\sum_{i=0}^{\infty} u(iT)\delta(\lambda - iT)\right]d\tau
\end{aligned}
\tag{B.2.10}
$$

改變求和與積分順序可得：

$$
\begin{aligned}
y(t) &= \sum_{i=0}^{\infty} u(iT)\int_0^t h(t-\tau)\delta(\lambda - iT)\,d\tau \\
&= \sum_{i=0}^{\infty} u(iT)h(t-iT)
\end{aligned}
\tag{B.2.11}
$$

經過取樣後可得其離散的響應為：

$$y(kT)=\sum_{i=0}^{\infty}u(iT)h(kT-iT),\quad k=0,1,2,3,\ldots \tag{B.2.12}$$

改寫為 index form 可得：

$$y(k)=\sum_{i=0}^{\infty}u(i)h(k-i),\quad k=0,1,2,3,\ldots \tag{B.2.13}$$

根據離散域摺積的 z 轉換可得：

$$\begin{aligned} Y(z)=\mathscr{Z}[y(k)]&=\sum_{k=0}^{\infty}y(k)z^{-k} \\ &=\sum_{k=0}^{\infty}[u(k)*h(k)]z^{-k} \\ &=\sum_{k=0}^{\infty}\left[\sum_{i=0}^{\infty}u(i)h(k-i)\right]z^{-k} \end{aligned} \tag{B.2.14}$$

令 $j=k-i$ 可得：

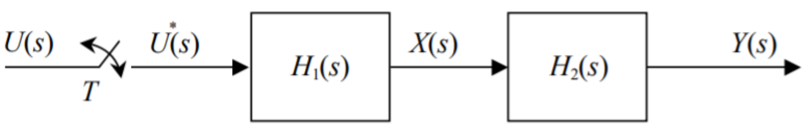
$$\begin{aligned} Y(z)&=\sum_{i=0}^{\infty}\sum_{j=-i}^{\infty}u(i)h(j)z^{-(i+j)} \\ &=\left[\sum_{i=0}^{\infty}u(i)z^{-i}\right]\left[\sum_{j=0}^{\infty}h(j)z^{-j}\right] \end{aligned} \tag{B.2.15}$$

因此系統最終的響應為：

$$Y(z)=H(z)U(z) \tag{B.2.16}$$

- **考慮子系統為直接相連的形式：**

考慮方塊圖如下所示：



透過拉式轉換可得整個系統的轉移函數為：

$$Y(s)=H_2(s)X(s)=H_2(s)H_1(s)U^*(s) \tag{B.2.17}$$

取拉式反轉換後可得：

$$\begin{aligned} y(t) &= \int_0^t h_2(t-\tau)x(\tau)\,d\tau \\ &= \int_0^t h_2(t-\tau)\left[\int_0^\tau h_1(\tau-\lambda)u^*(\lambda)\,d\lambda\right]d\tau \end{aligned} \tag{B.2.18}$$

經由改變積分順序與變數可得：

$$\begin{aligned} y(t) &= \int_0^t u^*(t-\tau)\left[\int_0^\tau h_1(\tau-\lambda)h_2(\lambda)\,d\lambda\right]d\tau \\ &= \int_0^t u^*(t-\tau)h_{eq}(\tau)\,d\tau = \int_0^t h_{eq}(t-\tau)u^*(\tau)\,d\tau \end{aligned} \tag{B.2.19}$$

其中 $h_{eq}(t) = \int_0^t h_1(t-\tau)h_2(\lambda)\,d\tau$ 。

將 $u^*(\tau)$ 帶入可得系統最終的響應為：

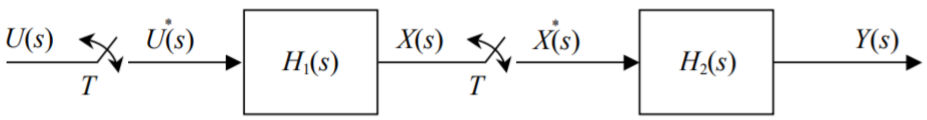
$$y(t) = \int_0^t h_{eq}(t-\tau)u^*(\tau)\,d\tau \tag{B.2.20}$$

根據(B.2.16)的推倒可知系統最終的響應為：

$$\begin{aligned} Y(z) &= H_{eq}(z)U(z) \\ &= \overline{H_1H_2}(z)U(z) \end{aligned} \tag{B.2.21}$$

- **考慮子系統間，透過取樣器(samplers)分離：**

考慮方塊圖如下所示：



系統的響應可以表示為：

$$y(t) = \int_0^t h_2(t-\tau)x^*(\tau)\,d\tau \tag{B.2.22}$$

根據(B.2.16)的推倒可知系統最終的響應為：

$$Y(z) = H_2(z)X(z) \tag{B.2.23}$$

而其中，前一個子系統的響應 $x(t)$ 可以表示為：

$$x(t) = \int_0^t h_1(t-\tau)u^*(\tau)\,d\tau \tag{B.2.24}$$

根據(B.2.16)的推倒可得$X(z)$為：

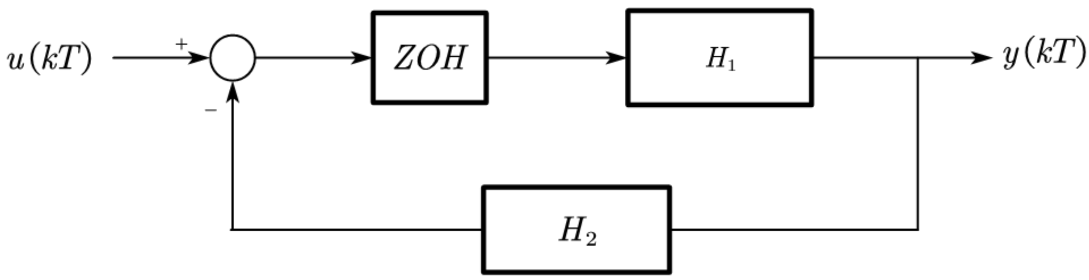
$$X(z) = H_1(z)U(z) \tag{B.2.25}$$

將(B.2.25)帶入(B.2.23)可得最後的系統響應為：

$$Y(z) = H_1(z)H_2(z)U(z) \tag{B.2.26}$$

原因二： 負回授對系統造成的影響。

由於現在的系統是包含反向電動勢(EMF)的負回授系統，系統輸入為參考輸入與系統的輸出相減後再取樣。也就是說，系統的輸入需要得到系統前一個step 的輸出，因此會造成系統產生偶合的情況。從負回授系統的方塊圖中可以發現，除了原本的輸入需要考慮 ZOH，負回授的訊號同樣也需要考慮。



結論：

回到最一開始的問題:「為什麼整個系統的轉移函數，不是電路系統與動力系統單純的結合?」

比較(B.2.21)與(B.2.26)的結果。根據第一個原因可以得知，直接相連的子系統在取樣後不能個別做 z 轉換後再相成，需要考慮 ZOH 所造成的影響。

而即便在對參考輸入考慮 ZOH 所造成的影響。由於反向電動勢的關係，此系統為一負回授。因此，輸出回授同樣也需要考慮 ZOH 所造成的影響。根據第二個原因，導致系統產生偶合的情況，這點同樣可以由前一題的推倒可發現。因此最後才會在動力系統中看到電感L_p在其中。

C. Real-time simulation:

1.Use Matlab to compare both the “actual computer controlled system in Fig. 5” and “fully discrete-time system in Z-domain in Fig. 4”. Both the sampled current ia(t=kT) and sampled speed ωm(t=kT) should be illustrated based on

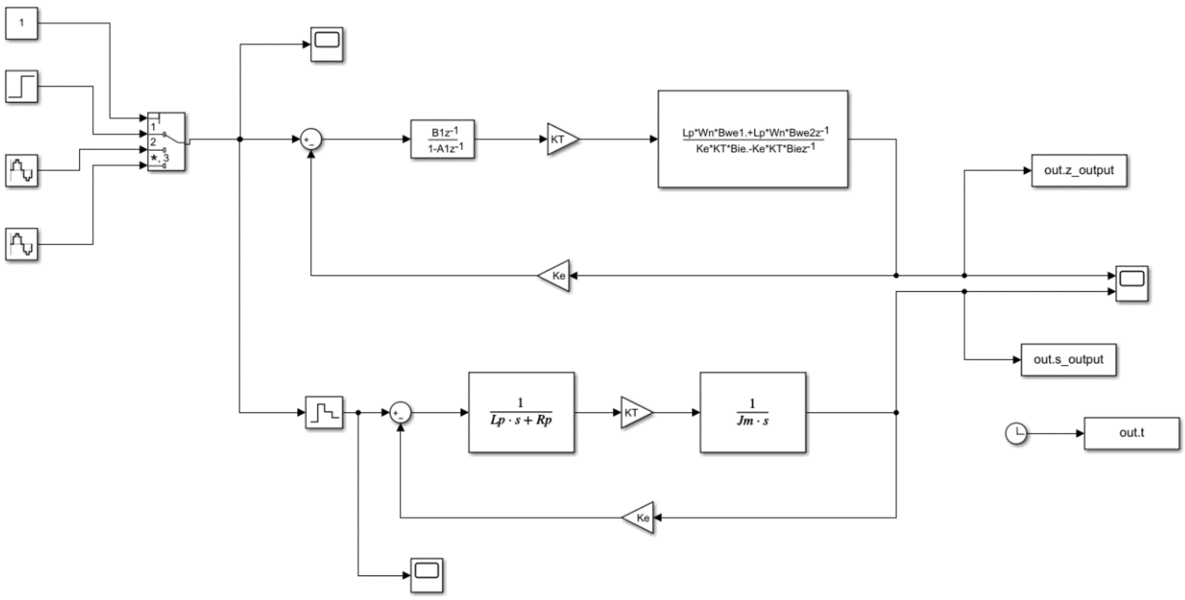
系統的參數為：

		Unit	Value
$\boldsymbol{J_p}$	inertia	Kg-m^2	0.015x10^-3
$\boldsymbol{K_t}$	torque constant	Nm/Amp	0.14
$\boldsymbol{K_e}$	back emf constant	volts/rad/sec	0.14
$\boldsymbol{R_p}$	motor resistance	ohm	2.6
$\boldsymbol{L_p}$	motor inductance	mH	4.3
$\boldsymbol{T}$	sample period (frequency)	sec (5-kHz)	0.0002

轉移函數其中的係數為：

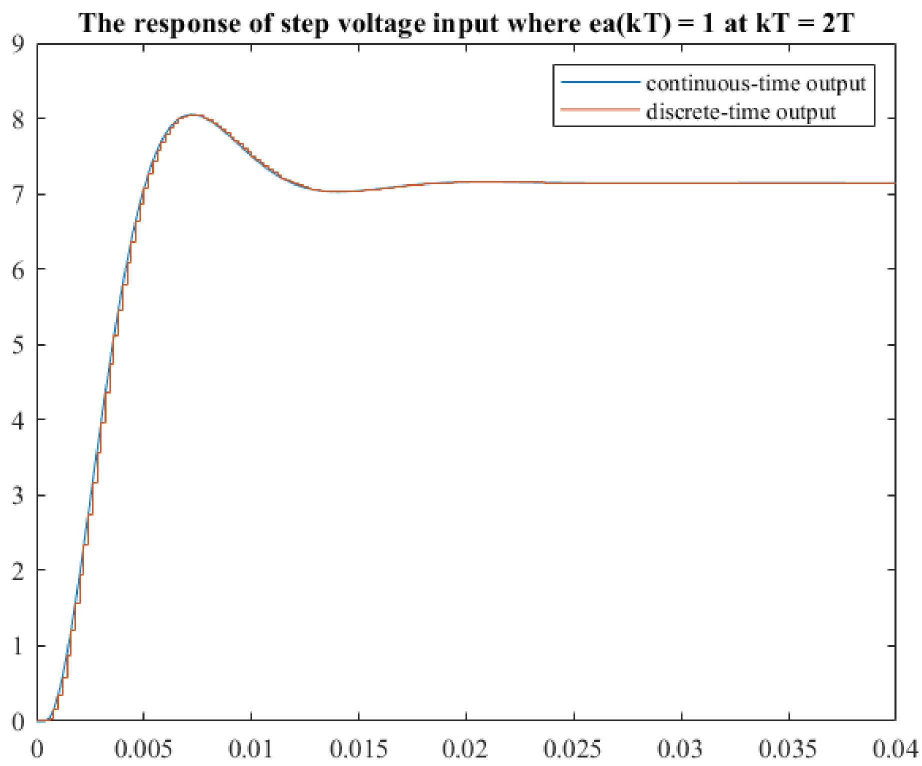
	Value
C_{1T}	57.1283
C_{2T}	2.6756e+5
C_{3T}	0.0058
A_1	0.8805
B_1	0.0437
$B_{\omega e1}$	0.0058
$B_{\omega e2}$	0.0056
B_{ie}	0.1036

透過 Simulink 建構以下方塊圖，用以分別模擬當系統為離散域及連續域時候的響應：



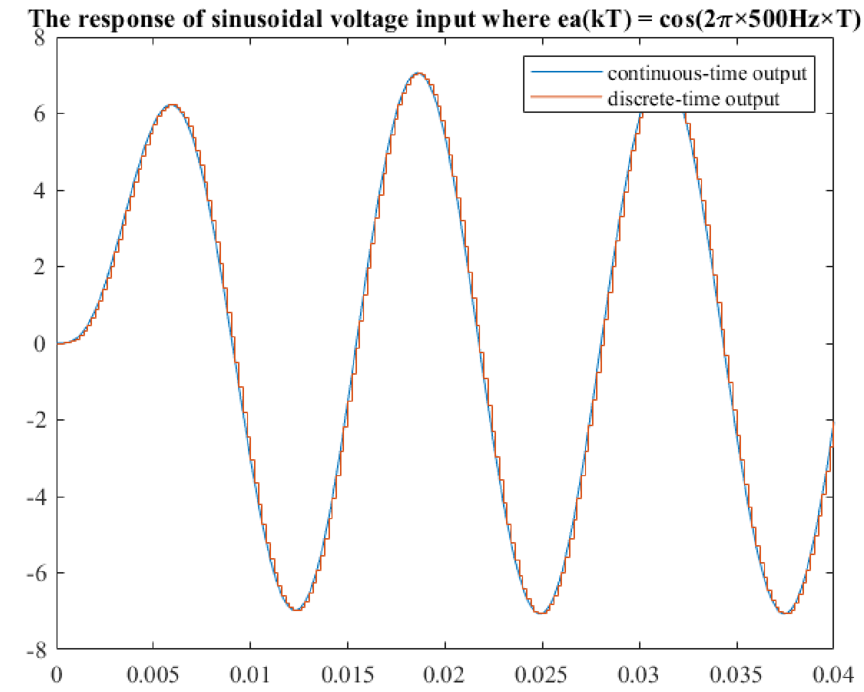
(a) a step voltage input where ea(kT) =1 at kT = 2T

- 當系統輸入 ea(kT)為一在 2T 時躍起的單位步階函數時，離散域及連續域的響應分別為：



(b) a sinusoidal voltage input where ea(kT) = cos(2π×500Hz×T) and ea(kT) = cos(2π×1.25kHz×T)

- 當系統輸入 ea(kT)為 cos(2π×500Hz×T)的弦波函數時，離散域及連續域的響應分別為：



- 當系統輸入 ea(kT)為 cos(2π×1.25kHz×T)的弦波函數時，離散域及連續域的響應分別為：