Linear Algebra HW3

1

(a)

$$A = LDU$$

$$A^T = U^T D^T L^T$$

$$D \quad and \quad D^T \quad have \quad same \quad pivots. - < ans >$$

(b)

$$A \quad is \quad symmetric \rightarrow A^T = A$$

$$(AA^{-1})^T = (A^{-1})^T A^T = I$$

$$\Rightarrow (A^{-1})^T = (A^T)^{-1} = A^{-1} - < ans >$$

(c)

$$\because A \quad is \quad symmetric \rightarrow A = A^T$$

$$(R^T AR)^T = R^T A^T R = R^T AR$$

$$\Rightarrow R^T AR \quad is \quad symmetric. - < ans >$$

$$dim((R^T)_{nm} A_{mm} R_{mn}) = n * n - < ans >$$

2

(a) True

$$for \quad any \quad triangular \quad matrix \quad A, \quad A \quad can \quad be \quad factorized$$

$$by \quad LD \quad or \quad DU.$$

$$where \quad L = L_1 L_2 \ldots L_n, \quad U = U_1 U_2 \ldots U_n \quad can \quad be \quad solved$$

$$by \quad Gauss \quad Elimination.$$

$$inverse \quad of \quad L_1, \ldots, L_n, U_1, \ldots, U_n \quad are \quad also \quad triangular \quad matrices$$

$$Hence, A^{-1} = D^{-1} L^{-1} \quad or \quad D^{-1} L^{-1} \quad is \quad also \quad a$$

$$triangular \quad matrix. - < ans >$$

(b) True

$$From \quad the \quad result \quad in \quad 1(b), \quad we \quad can \quad get$$

$$(A^{-1})^T = A^{-1} - < ans >$$

(c) False

$$proof \quad by \quad contradiction:$$

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 3 & 4 & 5 \\ 0 & 6 & 7 \end{bmatrix}, \quad A^{-1} = \begin{bmatrix} \frac{1}{22} & \frac{7}{22} & \frac{-5}{22} \\ \frac{21}{44} & \frac{-7}{44} & \frac{5}{44} \\ \frac{-9}{22} & \frac{3}{22} & \frac{1}{22} \end{bmatrix}$$

(d) False

$$When \quad calculating \quad A^{-1} \quad by \quad Gauss - Jordan \quad method, \quad all$$
$$entries \quad might \quad not \quad be \quad whole \quad numbers \quad after \quad calculation.$$

(e) True

$$\because fractions \in \mathbb{Q}, \quad and \quad \mathbb{Q} \quad is \quad a \quad field.$$

$$\therefore fractions \quad have \quad closure \quad with \quad operations(+, -, *, /).$$

3

$$D_{j+1} - 2D_j + D_{j-1} = h^2 f(jh) \quad j = 1, \dots, 5$$

$$\begin{bmatrix} 1 & -2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & -2 & 1 \end{bmatrix} \begin{bmatrix} D_0 \\ D_1 \\ D_2 \\ D_3 \\ D_4 \\ D_5 \\ D_6 \end{bmatrix} = \begin{bmatrix} h^2 f(h) \\ h^2 f(2h) \\ h^2 f(3h) \\ h^2 f(4h) \\ h^2 f(5h) \end{bmatrix}$$

$$because \quad of \quad initail \quad conditions, D_0 = D_1 = 0, \quad the \quad equations \quad can \quad be \quad reduced \quad to$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 & 0 \\ 1 & -2 & 1 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 1 & -2 & 1 \end{bmatrix} \begin{bmatrix} D_2 \\ D_3 \\ D_4 \\ D_5 \\ D_6 \end{bmatrix} = \begin{bmatrix} h^2 f(h) \\ h^2 f(2h) \\ h^2 f(3h) \\ h^2 f(4h) \\ h^2 f(5h) \end{bmatrix}$$

$$Since \quad final \quad speed \quad equals \quad to \quad 0, \quad D_5 = D_6.$$

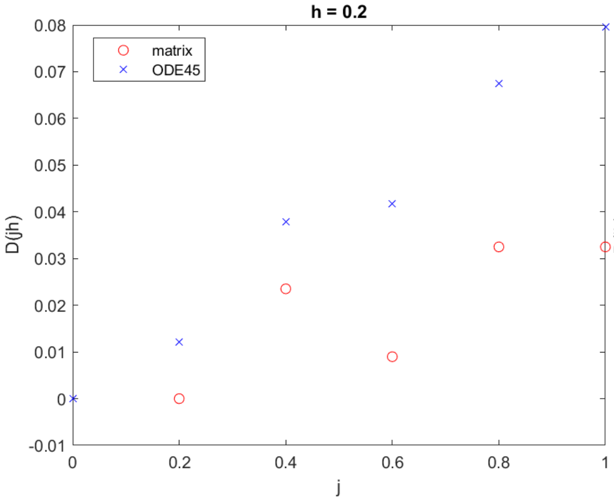
$$the \quad equation \quad can \quad be \quad represented \quad as \quad below:$$

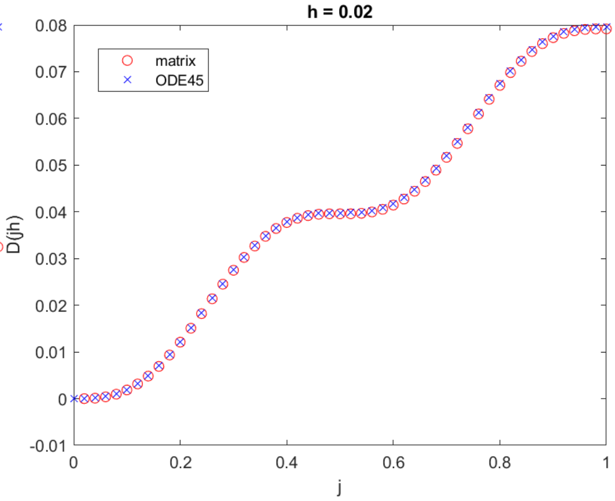
$$\begin{bmatrix} -2 & 1 & 0 & 0 & 0 \\ 1 & -2 & 1 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} D_1 \\ D_2 \\ D_3 \\ D_4 \\ D_5 \end{bmatrix} = \begin{bmatrix} h^2 f(h) \\ h^2 f(2h) \\ h^2 f(3h) \\ h^2 f(4h) \\ h^2 f(5h) \end{bmatrix}$$

$$Let \quad h = 0.2$$

$$\begin{bmatrix} D_1 \\ D_2 \\ D_3 \\ D_4 \\ D_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0.0235 \\ 0.090 \\ 0.0325 \\ 0.0325 \end{bmatrix} - < ans >$$

$$the \quad plot \quad D(jh) \quad vs. \quad j:$$





4

$$without \quad patial \quad pivoting:$$

$$A = \begin{bmatrix} .0001 & 0 \\ 1 & 10000 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 10000 & 1 \end{bmatrix} \begin{bmatrix} .0001 & 0 \\ 0 & 10000 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$with \quad partial \quad pivoting:$$

$$A' = \begin{bmatrix} 1 & 10000 \\ .0001 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ .0001 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 10000 \\ 0 & 1 \end{bmatrix}$$

$$after \quad scaling \quad by \quad multiply \quad 10000$$

$$without \quad patial \quad pivoting:$$

$$A = \begin{bmatrix} 1 & 0 \\ 10000 & 10^8 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 10000 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 10^8 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$with \quad patial \quad pivoting:$$

$$A = \begin{bmatrix} 10000 & 10^8 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ .0001 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -10000 \end{bmatrix} \begin{bmatrix} 1 & 10^8 \\ 0 & 1 \end{bmatrix}$$

$$after \quad pivoting, \quad we \quad can \quad get \quad more \quad stable \quad pivots.$$

$$after \quad scaling, \quad we \quad can \quad avoid \quad gettig \quad small \quad pivots.$$

5

(a) True

$$b_1 = 0, \quad \vec{B} = (0, b_2, b_3), \quad where \quad b_2, b_3 \in \mathbb{R}$$

$$For \quad any \quad two \quad vectors \quad \vec{B}_1, \vec{B}_2 \quad in \quad \vec{B}, \quad \vec{B}_1 + \vec{B}_2 \quad is \quad still \quad in \quad \vec{B}.$$

$$\forall c \in \mathbb{R}, \quad and \quad \forall \vec{B}_1 \in \vec{B}, \quad c\vec{B}_1 \in \vec{B}$$

(b) False

$$\because for \quad scaling, \quad b_1 \quad might \quad not \quad be \quad 1, \quad as \quad well \quad as \quad scaling.$$

(c) False

$$Proof \quad by \quad contradiction,$$

$$let \quad \vec{b}_1 = (0,0,1), \vec{b}_2 = (0,1,0), \vec{b}_1 + \vec{b}_2 = (0,1,1)$$

(d) True

$$All \quad combinations \quad means \quad that \quad for \quad addition \quad and \quad scaling \quad still \quad in$$

$$the \quad span \quad of \quad given \quad vectors.$$

(e) True

$$For \quad addition \quad of \quad any \quad two \quad vectors \quad which \quad are \quad lying \quad in \quad plane$$

$$is \quad still \quad lying \quad in \quad plane, as \quad well \quad as \quad scaling.$$

6

(f) False

$$Proof \quad by \quad contradictions:$$

$$Let \quad A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, b_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, b_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

$$b_1 + b_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, in \quad the \quad column \quad space \quad of \quad A.$$

(g) True

$$\because the \quad column \quad space \quad of \quad A \quad contains \quad only \quad zero \quad vector,$$

$$\therefore All \quad entries \quad of \quad A \quad is \quad 0.$$

(h) True

$$Scaling \quad would \quad not \quad change \quad column \quad space.$$

(i) False

$$Proof \quad by \quad contradiction:$$

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad A - I = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

(j) True

$$\because \; Dimension \quad of \quad \mathbb{R}^2 \quad and \quad \mathbb{R}^3 \quad is \quad different.$$

$$Although \quad \mathbb{R}^2 \quad is \quad isomorphic \quad to \quad the \quad subset(a,b,0) \quad of \quad R^3,$$

$$but \quad it's \quad also \quad isomorphic \quad to \quad infinitely \quad many \quad other \quad subspaces$$

$$of \quad \mathbb{R}^3.$$

7

(a)

$$\begin{aligned}[A \quad B] &= \begin{bmatrix} 1 & 2 & 0 & 3 & b_1 \\ 4 & 5 & 1 & 8 & b_2 \\ 2 & 1 & 1 & 2 & b_3 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 2 & 0 & 3 & b_1 \\ 0 & -3 & 1 & -4 & b_2 - 4b_1 \\ 0 & -3 & 1 & -4 & b_3 - 2b_1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 2 & 0 & 3 & b_1 \\ 0 & -3 & 1 & -4 & b_2 - 4b_1 \\ 0 & 0 & 0 & 0 & 2b_1 - b_2 + b_3 \end{bmatrix}\end{aligned}$$

(b)

$$b \quad have \quad to \quad lie \quad in \quad the \quad column \quad space \quad of \quad A.$$

$$That \quad is, \quad 2b_1 - b_2 + b_3 = 0 - < ans >$$

(c)

$$U = \begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & -3 & 1 & -4 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 1 & \frac{-1}{3} & \frac{4}{3} \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$Null \quad space \quad can \quad be \quad ontained \quad from \quad solving \quad Ux = 0$$

$$Null \quad space \quad of \quad U \quad is: \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \\ -1 \\ -2 \end{bmatrix} - < ans >$$