Linear Algebra HW2

1

first three pivot are 2,7,6. 因為在做高斯消去時並不會影響到原來這3個pivots

```
syms a1 a2 a3 a4 a5 a6 a7 a8 a9 a10

A = [2  a1 a2 a3;...
     0  7  a4 a5;...
     0  0  6  a6;...
     a7 a8 a9 a10];

[n,~] = size(A);

for i=1:n-1
    m = A(i+1:n,i)/A(i,i);
    A(i+1:n,:) = A(i+1:n,:) - m*A(i,:);
end
```

after calculation

```
A =

[2, a1, a2,                                                                                  a3]
[0,  7, a4,                                                                                  a5]
[0,  0,  6,                                                                                  a6]
[0,  0,  0, a10 - a5*(a8/7 - (a1*a7)/14) + a6*((a4*(a8/7 - (a1*a7)/14))/6 - a9/6 + (a2*a7)/12) - (a3*a7)/2]
```

只要A矩陣經計算後(如上圖)，右下角為9就可以確定第4個pivot為9

2

(a) **True**, Each column of AB is the product of A and a column of B

(b) **False**, Each column of AB is a combination of the columns of A, 所以還是要看A的係數

(c) **False**, $(AB)^2 = ABAB$

(d) **True**, if A is a lower triangle matrix, then $A_{ij} = 0 \, \forall i < j$. By the matrix operation, if both A and B are lower triangle matrices, then

$(AB)_{ij} = 0 \, \forall i < j$, hence AB is a lower triangle matrix.

3

For A_1 , 會將B的第一列乘4，第二列乘3，第三列乘2，第四列乘1，再將第一列到第四列顛倒。

$$(A_1)^{-1} = \begin{bmatrix} 0 & 0 & 0 & \frac{1}{4} \\ 0 & 0 & \frac{1}{3} & 0 \\ 0 & \frac{1}{4} & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

For A_2, 會將B的第一列乘$-\frac{1}{2}$加到第二列，乘4加到第四列，將第二列乘$-\frac{2}{3}$加到第三列，將第三列乘$-\frac{3}{4}$加到第四列

$$(A_2)^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ \frac{1}{2} & 1 & 0 & 0 \\ \frac{1}{3} & \frac{2}{3} & 0 & 0 \\ -\frac{15}{4} & \frac{1}{2} & \frac{3}{4} & 1 \end{bmatrix}$$

4

以下是用高斯消去法的步驟

$$A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 2 & 3 & 4 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 2 & 3 & 4 \end{bmatrix}, P_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A' = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 2 & 3 & 4 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 3 & 2 \end{bmatrix}, L_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$$

$$A' = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 3 & 2 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{bmatrix}, L_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -3 & 1 \end{bmatrix}$$

(a) L和U分別為

$$L = (L_2 * L_1)^{-1}$$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 3 & 1 \end{bmatrix}$$

$$U = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{bmatrix}$$

(b)

$$L^{-1} = L_2 L_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & -3 & 1 \end{bmatrix}$$

5

(1)

$$\begin{aligned} A = & \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 4 & 1 \end{bmatrix} \begin{bmatrix} 5 & 7 & 8 \\ 0 & 2 & 3 \\ 0 & 0 & 6 \end{bmatrix} \\ = & \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 4 & 1 \end{bmatrix} \begin{bmatrix} 5 & 7 & 8 \\ 0 & 2 & 3 \\ 0 & 0 & 6 \end{bmatrix} \\ = & L * U \end{aligned}$$

將第2列乘4加到第3列，將第一列乘2加到第二列，乘1加到第三列

(2) **no row exchange required**

(3)

$$Lc = b \Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 4 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$$c_1 = 1, c_2 = 0, c_3 = 2$$

$$Ux = c \Rightarrow \begin{bmatrix} 5 & 7 & 8 \\ 0 & 2 & 3 \\ 0 & 0 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$$

$$ANS : x = \begin{bmatrix} 1/30 \\ -1/2 \\ 11/30 \end{bmatrix}$$

6

$$P = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$PA = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 2 & 3 & 4 \end{bmatrix}$$

$$from \quad the \quad result \quad in \quad (4)$$

$$LDU = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

7

(1)Gaussian elimination and back substitution

left side: $(n^2 + ... + 1^2) - (n + .. + 1) = \frac{n(n+1)(2n+1)}{6} = \frac{n^3 - n}{3}$

right side : $(1 + 2 + ... + n) + [(n-1) + (n-2) + ... 1] \approx n^2$

back substitution $\approx \frac{n^2}{2}$

total$\approx \frac{2n^3 + 3n^2}{6}$

(2)Gauss-Jordan Method (algorithm)

Reference ：[gauss-complexity.pdf (ryerson.ca)]()

For Each row i (R_i) from 1 to n

If any row j below row i has non zero entries to the right of the first non zero entry in row i

$R_i \leftrightarrow R_j$

$R_i \to \frac{1}{c} R_j$ where c = the first non-zero entry of row i

For each row j < i

$R_i \leftrightarrow R_j - dR_i$ where d = the entry in row j which is directly below the pivot in row i

If any 0 rows have appeared exchange them to the bottom if the matrix.

next i

For each non zero row i (R_i) from n to 1

For each j < i

$R_i \leftrightarrow R_j - dR_i$ where b = the value in row j directly above the pivot in row i.

the complexity is :

$$
\begin{aligned}
&\sum_{i=1}^{n}[(n+1)+(n-i)(n+1)]+\sum_{i=1}^{n}[(n-i)(n+1)] \\
&=\sum_{i=1}^{n}\sum_{i=1}^{n}(2n-2i+1)(n+1) \\
&=\sum_{i=1}^{n} 2n^2+3n^2+1-2(n+1)i \\
&=2n^3+3n^2+n+n(n+1)^2 \\
&=3n^3+5n^2+2n
\end{aligned}
$$

結論

兩者比較起來Gaussian elimination and back substitution會比Gauss-Jordan Method好一些，但資料量很大時複雜度都是$O(n^3)$

8

$P^3 = I$

將identity matrix做兩次permutation

$$P = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$P^4 \neq I$

將identity matrix做三次permutation

$$P = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

9

$$PA = LDU$$
$$(PA)^{-1} = (LDU)^{-1}$$
$$A^{-1}P^{-1} = U^{-1}D^{-1}L^{-1}$$
$$A^{-1} = U^{-1}D^{-1}L^{-1}P$$