Linear algebra HW4

1

(a)

$$After \quad Gaussian \quad elimination, \quad Ux = 0 \Rightarrow$$

$$Ux = \begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & -3 & 1 & -4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} u \\ v \\ w \\ y \end{bmatrix} = \begin{bmatrix} \frac{-2w-y}{3} \\ \frac{w-4y}{3} \\ w \\ y \end{bmatrix} = w \begin{bmatrix} \frac{-2}{3} \\ \frac{1}{3} \\ 1 \\ 0 \end{bmatrix} + y \begin{bmatrix} \frac{-1}{3} \\ \frac{-4}{3} \\ 0 \\ 1 \end{bmatrix} - < ans >$$

(b)

$$After \quad Gaussian \quad elimination, \quad Ux = c \Rightarrow$$

$$Ux = \begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & -3 & 1 & -4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} u \\ v \\ w \\ y \end{bmatrix} = \begin{bmatrix} \frac{-2}{3} \\ \frac{1}{3} \\ 0 \\ 0 \end{bmatrix} + w \begin{bmatrix} \frac{-2}{3} \\ \frac{1}{3} \\ 1 \\ 0 \end{bmatrix} + y \begin{bmatrix} \frac{-1}{3} \\ \frac{-4}{3} \\ 0 \\ 1 \end{bmatrix} - < ans >$$

2

$$After \quad Gaussian \quad elimination, \quad Ux = C \Rightarrow$$

$$Ux = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & -5 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ c-7 \end{bmatrix} = C$$

$$if \quad AX = b \quad is \quad possible \quad to \quad solve, \quad then \quad c = 0 - < ans >$$

$$\begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + z \begin{bmatrix} -7 \\ 5 \\ 1 \end{bmatrix} - < ans >$$

3

$$After \quad Gaussian \quad elimination, \quad Ux = C \Rightarrow$$

$$Ux = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$x = v \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} -5 \\ 0 \\ -2 \end{bmatrix} - < ans >$$

$$Ux = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

$$x \quad has \quad no \quad solutions. - < ans >$$

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(a) $r < m$

(b) $r = m \quad and \quad r < n \Rightarrow r = m < n$

(c) $r = n \quad and \quad r < m \Rightarrow r = n < m$

(d) $r = n \quad and \quad r = m \Rightarrow r = n = m$

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(a)

$\because the \quad coefficient \quad of \quad x_p \quad is \quad unique.$

(b)

$Particular \quad solution \quad = x_n \quad + x_p \quad , \quad and \quad there \quad are \quad infinitely$

$many \quad x_n.$

(c)

$x_n \quad is \quad always \quad a \quad zero \quad vector.$

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$choose \quad 3 \quad independent \quad columns \quad of \quad U : (2, 0, 0, 0), (3, 6, 0, 0),$

$(1, 0, 9, 0)$

$Second \quad choice : (2, 0, 0, 0), (4, 7, 0, 0), (1, 0, 9, 0)$

$Third \quad choice : (3,6,0,0),(4,7,0,0),(1,0,9,0)$

$After \quad Gaussian \quad elimination, \quad U = A, \quad therefore \quad 3$

$independent \quad columns \quad of \quad A :$

$Firstchoice : (2,0,0,4),(3,6,0,6),(1,0,9,2)$

$Second \quad choice : (2,0,0,4),(4,7,0,8),(1,0,9,2)$

$Third \quad choice : (3,6,0,6),(4,7,0,8),(1,0,9,2)$

$We \quad have \quad found \quad bases \quad for \quad column \quad space.$

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$\because Col(AB) \subseteq Col(B), \quad and \quad Row(AB) \subseteq Row(B)$

$Hence,\ Rank(AB) \leq Rank(A), \quad and \quad Rank(AB) \leq Rank(B)$

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(a) $\begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}^T$

(b)

$$u + v + w + y = 0, \quad u = -v - w - y$$

$$\begin{bmatrix} u \\ v \\ w \\ y \end{bmatrix} = v\begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix} + w\begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix} + y\begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}, the \quad bases \quad are \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix} - < ans >$$

(c)

$(1,1,0,0)\cdot(a,b,c,d) = 0 \Rightarrow a + b = 0 \Rightarrow -a = b$

$(1,0,1,1)\cdot(a,b,c,d) = 0 \Rightarrow a + c + d = 0 \Rightarrow b = c + d$

$basis : (0,0,1,-1),(-2,2,1,1) - < ans >$

(d)

$Column \quad space \quad basis : (1,0),(0,1) - < ans >$

$$Null \quad space: Ux = 0 \Rightarrow \begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \\ e \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow a + c + e = 0, b + d = 0$$

$$\begin{bmatrix} a \\ b \\ c \\ d \\ e \end{bmatrix} = c \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + d \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + e \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, bases: \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} - < ans >$$

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(a) 4-2 = 2

(b)

$$Ux = \begin{bmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$Ux = \begin{bmatrix} 1 & \frac{-2}{3} & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} - < ans >$$

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$$if \quad d = 0: U = \begin{bmatrix} a & b & c \\ 0 & 0 & e \\ 0 & 0 & f \end{bmatrix}$$

$$\begin{bmatrix} a \\ 0 \\ 0 \end{bmatrix} and \begin{bmatrix} b \\ 0 \\ 0 \end{bmatrix} are \quad depedent.$$

$$if \quad f = 0: U = \begin{bmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & 0 \end{bmatrix}, let \quad v_1 = \begin{bmatrix} a \\ 0 \\ 0 \end{bmatrix}, v_2 = \begin{bmatrix} b \\ d \\ 0 \end{bmatrix}, v_3 = \begin{bmatrix} c \\ e \\ 0 \end{bmatrix}$$

$$\exists c_1, c_2, c_3 \neq 0 \quad s.t \quad c_1v_1 + c_2v_2 + c_3v_3 = 0$$

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$$the \quad bases \quad of \quad column \quad space: \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ 0 \end{bmatrix}$$

$$Hence, b_3 - 2b_2 + b_1 = 0$$