# Linear Algebra HW13

## 1

$$from\ \ det(A-\lambda I)=0 \Rightarrow \lambda^3+25\lambda=0 \Rightarrow \lambda=0,\ \ \pm 5i - < ans >$$
$$for\ \ \lambda=0:(A-\lambda I)x_1=0 \Rightarrow x_1=(4,0,3)^T - < ans >$$
$$for\ \ \lambda=-5i:(A-\lambda I)x_2=0 \Rightarrow x_2=(-3,5i,4)^T - < ans >$$
$$for\ \ \lambda=5i:(A-\lambda I)x_3=0 \Rightarrow x_3=(-3,-5i,4)^T - < ans >$$
$$\because A\ \ is\ \ skew\ \ symmetric \Rightarrow A=-A^T$$
$$\therefore (e^{At})^{-1}=e^{-At}=e^{A^Tt}=(e^{At})^T \Rightarrow e^{At}\ \ is\ \ an\ \ orthogonal\ \ matrix. - < ans >$$
$$\because Real\ \ part\ \ of\ \ eigenvalues\ \ is\ \ zero \Rightarrow it\ \ won't\ \ change\ \ magnitude\ \ of\ \ u(t). - < ans >$$

## 2

$$(a)$$
$$V_1=(1,1)=av_1+bv_2 \Rightarrow a=1,\ b=-1$$
$$V_2=(1,4)=cv_1+dv_2 \Rightarrow c=0,\ d=1$$
$$M_{V\to v}=\begin{bmatrix} a & c \\ b & d \end{bmatrix}=\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} - < ans >$$
$$(b)$$
$$(3,9)=c_1V_1+c_2V_2 \Rightarrow c_1=1,\ c_2=2$$
$$=d_1v_1+d_2v_2 \Rightarrow d_1=1,\ d_2=1$$
$$\Rightarrow Mc=d \Rightarrow \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix}=\begin{bmatrix} 1 \\ 1 \end{bmatrix} - < ans >$$

## 3

$$After\ \ reflection\ \ v_1\ \ becomes\ \ (0,1) = 0v_1 + 1v_2$$
$$After\ \ reflection\ \ v_2\ \ becomes\ \ (1,0) = 1v_1 + 0v_2$$
$$\Rightarrow T_v = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$
$$After\ \ reflection\ \ V_1\ \ becomes\ \ (1,1) = 1V_1 + 0V_2$$
$$After\ \ reflection\ \ V_2\ \ becomes\ \ (-1,1) = 0V_1 - V_2$$
$$\Rightarrow T_V = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$
$$V_1 = (1,1) = av_1 + bv_2 \Rightarrow a = 1,\ b = 1$$
$$V_2 = (1,-1) = cv_1 + dv_2 \Rightarrow c = 1,\ d = -1$$
$$M_{V\to v} = \begin{bmatrix} a & c \\ b & d \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$
$$T_v = M_{V\to v} T_V M_{v\to V}$$
$$\Rightarrow \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}^{-1} - < ans >$$

## 4

$$A^H = \begin{bmatrix} 1 & -i \\ -i & 0 \\ 0 & 1 \end{bmatrix} - < ans >$$
$$C = A^H A = \begin{bmatrix} 2 & i & -i \\ -i & 1 & 0 \\ i & 0 & 1 \end{bmatrix} - < ans >$$
$$C = C^H - < ans >$$
$$C^H = (A^H A)^H = A^H A = C \Rightarrow The\ \ relationship\ \ holds. - < ans >$$

## 5

$$det(P-\lambda I)=0 \Rightarrow \lambda=0,\ 1$$
$$for\ \ \lambda=0,\ (P-\lambda I)x_1=0 \Rightarrow x_1=c_1(1,1)^T$$
$$for\ \ \lambda=1,\ (P-\lambda I)x_2=0 \Rightarrow x_1=c_2(1,-1)^T$$
$$Let\ \ c_1=||x_1||=\frac{\sqrt{2}}{2},\ and\ \ c_2=||x_2||=\frac{\sqrt{2}}{2}$$
$$P=\lambda_1 x_1 x_1^H+\lambda_2 x_2 x_2^H$$
$$=\begin{bmatrix}\frac{\sqrt{2}}{2}\\ \frac{\sqrt{2}}{2}\end{bmatrix}\begin{bmatrix}\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2}\end{bmatrix}=\begin{bmatrix}\frac{1}{2} & \frac{1}{2}\\ \frac{1}{2} & \frac{1}{2}\end{bmatrix}-<ans>$$
$$det(Q-\lambda I)=0 \Rightarrow \lambda=-1,\ 1$$
$$for\ \ \lambda=1,\ (Q-\lambda I)x_1=0 \Rightarrow x_1=c_1(1,1)^T$$
$$for\ \ \lambda=-1,\ (Q-\lambda I)x_2=0 \Rightarrow x_1=c_2(1,-1)^T$$
$$Let\ \ c_1=||x_1||=\frac{\sqrt{2}}{2},\ and\ \ c_2=||x_2||=\frac{\sqrt{2}}{2}$$
$$Q=\lambda_1 x_1 x_1^H+\lambda_2 x_2 x_2^H$$
$$=\begin{bmatrix}\frac{\sqrt{2}}{2}\\ \frac{\sqrt{2}}{2}\end{bmatrix}\begin{bmatrix}\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2}\end{bmatrix}-\begin{bmatrix}\frac{\sqrt{2}}{2}\\ -\frac{\sqrt{2}}{2}\end{bmatrix}\begin{bmatrix}\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2}\end{bmatrix}=\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}-<ans>$$
$$det(R-\lambda I)=0 \Rightarrow \lambda=-5,\ 5$$
$$for\ \ \lambda=1,\ (R-\lambda I)x_1=0 \Rightarrow x_1=c_1(2,1)^T$$
$$for\ \ \lambda=-1,\ (R-\lambda I)x_2=0 \Rightarrow x_1=c_2(1,-2)^T$$
$$Let\ \ c_1=||x_1||=\frac{\sqrt{5}}{5},\ and\ \ c_2=||x_2||=\frac{\sqrt{5}}{5}$$
$$Q=\lambda_1 x_1 x_1^H+\lambda_2 x_2 x_2^H$$
$$=5\begin{bmatrix}\frac{2\sqrt{5}}{5}\\ \frac{\sqrt{5}}{5}\end{bmatrix}\begin{bmatrix}\frac{2\sqrt{5}}{5} & \frac{\sqrt{5}}{5}\end{bmatrix}-5\begin{bmatrix}\frac{\sqrt{5}}{5}\\ -\frac{2\sqrt{5}}{5}\end{bmatrix}\begin{bmatrix}\frac{\sqrt{5}}{5} & -\frac{2\sqrt{5}}{5}\end{bmatrix}=\begin{bmatrix}3 & 4\\ 4 & -3\end{bmatrix}-<ans>$$

## 6

$(a)\ All\ \ eigenvalues\ \ of\ \ real\ \ symmetric\ \ matrix\ \ are\ \ real.$

$(b)\ Real\ \ part\ \ of\ \ all\ \ eigenvalues\ \ are\ \ less\ \ than\ \ 1.$

$(c)\ Magnitude\ \ of\ \ all\ \ eigenvalues\ \ are\ \ equal\ \ to\ \ 1.$

$\Rightarrow ||Ax||=||\lambda||\ ||x||$

$(d)\ Magnitude\ \ of\ \ all\ \ eigenvalues\ \ are\ \ less\ \ than\ \ 1.$

$(e)\ If\ \ A_{n\times n}\ \ is\ \ nondiagonalizable,\ \ then\ \ A\ \ must\ \ have\ \ fewer\ \ than\ \ n\ \ engenvalues.$

$(f)\ det(A)=0$

## 7

$1.\ For\ any\ x,\ x^HKx\ is\ pure\ imaginary.$

$(x^HKx)^H = x^HK^Hx = -x^HKx$

$\Rightarrow x^HKx\ is\ pure\ imaginary.$

$2.\ Every\ eigenvalue\ of\ K\ is\ pure\ imaginary.$

$Kx = \lambda x \Rightarrow x^HKx = \lambda x^Hx$

$From\ 1.\ x^HKx\ is\ pure\ imaginary,\ and\ x^Hx \in \mathbb{R}$

$\Rightarrow \lambda\ is\ pure\ imaginary.$

$3.\ Eigenvectors\ corresponding\ to\ different\ eigenvalues\ are\ orthogonal.$

$If\ K\ has\ eighevalues\ \lambda\ and\ \mu\ with\ corresponding\ eigenvectors\ x\ and\ y.$

$(\lambda x)^Hy = (Kx)^Hy = x^H(-K)y = x^H(-\mu y)$

$\Rightarrow \bar{\lambda}x^Hy = -\mu x^Hy \Rightarrow (\bar{\lambda} + \mu)x^Hy = 0$

$\because \lambda\ and\ \mu\ are\ pure\ imaginary,\ and\ \lambda \neq \mu.$

$\therefore x^Hy = 0$