Linear Algebra and its Applications

HW#07

1. Show **"geometrically"** that the following transformation are linear transformations:
   (a) Rotation of a vector in $\mathbf{R}^2$ through an angle θ.
   (b) Reflection of a vector in $\mathbf{R}^2$ through the mirror θ-line.
   (c) Projection of a vector in $\mathbf{R}^2$ onto the θ-line.

2. The input is a 1 by 2 matrix $v=(v_1, v_2)$. Which of these transformations is linear? If the transformation is linear, find its *Range* (column space) and *Kernel* (null space).
   (a) $T(v)=(v_2, v_1)$ (b) $T(v)=(v_1, v_1)$ (c) $T(v)=(0, v_1)$ (d) $T(v)=(0, 1)$

3. On the 4-dimensional space of cubic (degree 3) polynomials, let the basis consists of four terms: $1, t, t^2, t^3$ (i.e., any cubic polynomial is a linear combination of the four basis terms).
   (a) Show that the second derivative of the cubic polynomials is a linear transformation.
   (b) Take the second derivative of each basis term. Suppose there is a cubic polynomial $a_0+a_1t+a_2t^2+a_3t^3$. Express the second derivative of this cubic polynomial as the linear combination of the second derivatives of the four basis terms.
   (c) Find a 4 by 4 matrix that represents taking the second derivative of any cubic polynomial. (a square polynomial is a special case of cubic polynomial with the cubic coefficient equal to zero)
   (d) Find a 2 by 4 matrix A that represents taking the second derivative of any cubic polynomial in the four-dimensional space with basis $1, t, t^2/2, t^3/6$ to become a degree-1 polynomial in the two-dimensional space with $1, t$ as its basis. Find the second derivative of $4+3t+2t^2+t^3$ by expressing the polynomial by a vector x and take the second derivative by Ax.

4. Show that the following matrices are reflections:
$$\begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix} \quad \begin{bmatrix} \cos 2\alpha & \sin 2\alpha \\ \sin 2\alpha & -\cos 2\alpha \end{bmatrix}$$
   Show that the product of two reflections is a rotation. Multiply the above reflection matrices to find the rotation angle.

5. Find a vector x orthogonal to the row space of $\boldsymbol{A}$, and a vector y orthogonal to the column space, and a vector z orthogonal to the nullspace:

$$A=\begin{bmatrix}1 & 2 & 1\\ 2 & 4 & 3\\ 3 & 6 & 4\end{bmatrix}$$

6. Show that $x-y$ is orthogonal to $x+y$ if and only if $\| x \| = \| y \|$

7. Find the orthogonal complement of the plane spanned by the vectors (1,1,2) and (1,2,3), by taking these to be the rows of A and solving $Ax = 0$.

8. Draw figure of 4-subspaces of $\boldsymbol{A}$ on page 5 of class note "projection and least square") to show each subspace for
$A=\begin{bmatrix}1 & 2\\ 3 & 6\end{bmatrix}$ and $B=\begin{bmatrix}1 & 0\\ 3 & 0\end{bmatrix}$.