Linear Algebra HW11

1

$\because Any\ \ vectors\ \ lie\ \ on\ \ x_1 = x_3\ \ won't\ \ change\ \ its\ \ direction\ \ and\ \ magnitude$
$after\ \ projection.$
$\therefore eigenvalue\ \ is\ \ 1,\ \ and\ \ eigenvectors\ \ are\ \ (c, a, c)^T,\ \ where\ \ a, c \in \mathbb{R}. - < ans >$
$\because Any\ \ vectors\ \ lie\ \ on\ \ normal\ \ vector\ \ is\ \ zero\ \ after\ \ projection.$
$\therefore eigenvalue\ \ is\ \ 0,\ \ and\ \ eigenvectors\ \ are\ \ (c, 0, -c)^T,\ \ c \in \mathbb{R}. - < ans >$

2

$Av_0 = 0v_0 \Rightarrow v_0\ \ is\ \ null\ \ space\ \ of\ \ A - < ans >$
$Av_1 = 1v_1 \Rightarrow v_1\ \ is\ \ column\ \ space\ \ of\ \ A - < ans >$
$Av_2 = 2v_2 \Rightarrow v_2\ \ is\ \ column\ \ space\ \ of\ \ A - < ans >$
$Ax = v_1 + v_2 = Av_1 + \frac{1}{2}Av_2 \Rightarrow x = v_1 + \frac{1}{2}v_2 - < ans >$
$v_0\ \ is\ \ in\ \ the\ \ null\ \ space\ \ of\ \ A\ \ rather\ \ than\ \ column\ \ space.$
$\therefore Ax = v_0\ \ has\ \ no\ \ solution. - < ans >$

3

$The\ \ eigenvectors\ \ in\ \ the\ \ column\ \ space\ \ has\ \ r'\ \ (rank\ \ of\ \ A -\ \ \lambda I)\ \ linearly$
$independent\ \ vectors\ \ and\ \ the\ \ eigenvectors\ \ in\ \ the\ \ nullspace\ \ has\ \ (n - r')$
$linearly\ \ independent\ \ vectors. - < ans >$
$let\ \ A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix},\ rank(A) = 2,\ \ eigenvalues : 1, 1$
$A\ \ has\ \ only\ \ one\ \ eigenvector\ \ (1, 0)^T - < ans >$
$The\ \ state\ \ is\ \ true\ \ for\ \ projection\ \ matrix.$
$\because If\ \ P\ \ is\ \ projection\ \ matrix\ \ that\ \ project\ \ any\ \ vector\ \ to\ \ A's\ \ column\ \ space,$
$then\ \ P^T\ \ is\ \ projection\ \ matrix\ \ that\ \ project\ \ any\ \ vector\ \ to\ \ A's\ \ row\ \ space.$
$\because\ \ transpose\ \ won't\ \ change\ \ rank.$
$The\ \ eigenvectors\ \ in\ \ the\ \ row\ \ space\ \ has\ \ n - r\ \ linearly\ \ independent\ \ vectors.$
$\Rightarrow The\ \ eigenvectors\ \ in\ \ the\ \ column\ \ space\ \ has\ \ n - r\ \ linearly\ \ independent\ \ vectors. <$

4

$Rank(A) = Rank(A^T),\ \ det(A) = det(A^T)$
$det(A - \lambda I) = det(A - \lambda I)^T = det(A^T - \lambda I)$
$A\ \ and\ \ A^T\ \ has\ \ same\ \ eigenvalues - < ans >$
$Let\ \ A = \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}$
$eigenvectors\ \ of\ \ A : (2, 1)^T, (-1, 1)^T$
$eigenvectors\ \ of\ \ A^T : (-1, 2)^T, (1, 1)^T - < ans >$

5

(a)
$\because \ one \ of \ eigenvalues \ is \ 0.$
$\therefore rank(B) = 2- < ans >$
(b)
$\because det(B^T) \cdot det(B) = det(B^T B), \ B \ and \ B^T \ has \ same \ eigenvalues$
$\therefore det(B^T B) = 0 \cdot 0 = 0$
(c)
$There \ is \ not \ enough \ information \ to \ find \ the \ eigenvalues \ of \ B^T B.$
$Only \ when \ B \ is \ diagonal \ matrix, \ B^T B's \ engenvalues = (B's \ eigenvalues)^2$

6

$$\begin{aligned} det(A - \lambda I) &= (\lambda_1 - \lambda)(\lambda_2 - \lambda)\ldots(\lambda_n - \lambda) \\ &= (-1)^n[\lambda^n + (\lambda_1 + \lambda_2 + \ldots + \lambda_n)\lambda^{n-1} + \ldots. + \lambda_1\lambda_2\ldots\lambda_n] \end{aligned}$$
$det(A) = det(A - \lambda I)|_{\lambda=0} = \lambda_1\lambda_2\ldots\lambda_n - < ans >$

7

$For \ A_1 : eigenvalue = 0, 0 (\because det(A_1) = 0, \ trace = 0)$
$Rank(A_1 - 0I) = 1 \ \ \therefore A_1 \ cannot \ be \ diagonized. - < ans >$
$For \ A_2 : eigenvalue = 2, -2 (\because triangular \ matrix)$
$Rank(A_2 - 2I) = 1 \, , Rank(A_2 - (-2)I) = 1 \ \therefore A_2 \ can \ be \ diagonized. - < ans >$
$For \ A_3 : eigenvalue = 2, 2 (\because triangular \ matrix)$
$Rank(A_3 - 2I) = 1 \ \ \therefore A_3 \ cannot \ be \ diagonized. - < ans >$

8

$det(A) = 0, \ det(A - 3I) = 0, \ trace = 0$
$Eigenvalues : 0, 3, 0 - < ans >$
$Eigenvectors : (1, 0, -1)^T, (1, -1, 0)^T, (1, 1, 1)^T - < ans >$
$A = S_1^{-1}\Lambda_1 S_1 = S_2^{-1}\Lambda_2 S_2$
$$where \ \Lambda_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 3 \end{bmatrix}, \ \Lambda_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$
$$S_1 = \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & 1 \\ -1 & 0 & 1 \end{bmatrix}, \ S_2 = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & -1 \\ -1 & 1 & 0 \end{bmatrix} - < ans >$$