Linear Algebra HW12

1

$$G_{k+2} = \frac{1}{2}(G_{k+1} + G_k)$$
$$G_{k+1} = G_{k+1}$$
$$Let\ \ u_k = \begin{bmatrix} G_{k+1} \\ G_k \end{bmatrix},\ \ the\ \ difference\ \ equation\ \ becomes:$$
$$u_{k+1} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ 1 & 0 \end{bmatrix} u_k,\ \ initial\ \ condition\ \ u_0 = \begin{bmatrix} G_1 \\ G_0 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ 0 \end{bmatrix}$$
$$\Rightarrow u_k = A^k u_0$$
$$det(\lambda I - A) = 0 \Rightarrow \lambda = 1, -\frac{1}{2}$$
$$for\ \ \lambda = 1,\ \ v_1 = (1,1)^T$$
$$for\ \ \lambda = -\frac{1}{2},\ \ v_2 = (1,-2)^T$$
$$let\ \ S = [v_1 \quad v_2],\ \ and\ \ A = S\Lambda S^{-1}$$
$$G_{k+1} = AG_k \Rightarrow G_k = A^k G_0 = S\Lambda^k S^{-1} G_0$$
$$G_\infty = \lim_{k\to\infty} G_k = \begin{bmatrix} 1 & 1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & -\frac{1}{3} \end{bmatrix} \begin{bmatrix} \frac{1}{2} \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} \\ \frac{1}{3} \end{bmatrix} - < ans >$$

2

$$(a)$$
$$\begin{bmatrix} US \\ J \\ E \end{bmatrix}_{k+1} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{2} & 0 \\ \frac{1}{4} & 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} US \\ J \\ E \end{bmatrix}_k - < ans >$$
$$(b)$$
$$det(\lambda I - A) = 0 \Rightarrow \lambda = 0, 1, \frac{1}{2} - < ans >$$
$$for\ \ \lambda = 0,\ \ (\lambda I - A)v_1 = 0 \Rightarrow v_1 = (-2,1,1)^T - < ans >$$
$$for\ \ \lambda = 1,\ \ (\lambda I - A)v_2 = 0 \Rightarrow v_2 = (2,1,1)^T - < ans >$$
$$for\ \ \lambda = \frac{1}{2},\ \ (\lambda I - A)v_3 = 0 \Rightarrow v_3 = (0,1,-1)^T - < ans >$$
$$(c)$$
$$Let\ \ S = [v_1 \quad v_2 \quad v_3],\ \ and\ \ A = S\Lambda S^{-1}$$
$$\Rightarrow A^k = S\Lambda^k S^{-1}$$
$$u_\infty = \lim_{k\to\infty} S\Lambda^k S^{-1} u_0 = \begin{bmatrix} -2 & 2 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} -\frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ 0 & \frac{1}{2} & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} - < ans$$
$$(d)$$
$$u_k = S\Lambda^k S^{-1} u_0 = \begin{bmatrix} -2 & 2 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} 0^k & 0 & 0 \\ 0 & 1^k & 0 \\ 0 & 0 & (\frac{1}{2})^k \end{bmatrix} \begin{bmatrix} -\frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ 0 & \frac{1}{2} & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 - (\frac{1}{2})^k \\ 1 + (\frac{1}{2})^k \end{bmatrix} -$$

3

(a)

$e^{At} = Se^{\Lambda t}S^{-1},\ \ e^{As} = Se^{\Lambda s}S^{-1}$

$e^{At}e^{As} = Se^{\Lambda t}S^{-1}Se^{\Lambda s}S^{-1} = S = e^{\Lambda(s+t)}S^{-1}e^{A(t+s)}$

(b)

$If\ \ either\ \ A\ \ or\ \ B\ \ cannot\ \ be\ \ diagonized,\ \ then\ \ e^{A} \neq Se^{\Lambda s}S^{-1}$

$\therefore e^{A}e^{B} \neq e^{A+B}$

4

(a)

$$e^{At} = I + At + \frac{(At)^2}{2!} + \ldots + \frac{(At)^n}{n!} + \ldots$$

$$\frac{de^{At}}{dt} = A + A^2t + \ldots + \frac{A^n t^{n-1}}{(n-1)!} + \ldots = A(I + At + \ldots + \frac{(At)^{n-1}}{(n-1)!} + \ldots) = Ae^{At}$$

(b)

$$\frac{du}{dt} = \frac{de^{At}u_0}{dt} = Ae^{At}u_0 = Au$$

$u(0) = Ae^{A0}u_0 = u_0$

5

(a)

$Let\ \ u = [v\ \ w]^T$

$$\frac{d}{dt}\begin{bmatrix} v \\ w \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} v \\ w \end{bmatrix} \Rightarrow \frac{du}{dt} = Au,\ \ \lambda(A) = 0, -2$$

$for\ \ \lambda = 0,\ \ (\lambda I - A)v_1 = 0 \Rightarrow v_1 = (1, 1)^T$

$for\ \ \lambda = -2,\ \ (\lambda I - A)v_1 = 0 \Rightarrow v_2 = (-1, 1)^T$

(b)

$Let\ \ S = [v_1\ \ v_2],\ \ then\ \ A = S\Lambda S^{-1}$

$$u(1) = \begin{bmatrix} v \\ w \end{bmatrix} = Se^{\Lambda}S^{-1}u_0 = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & e^{-2} \end{bmatrix}\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix}\begin{bmatrix} 30 \\ 10 \end{bmatrix} = \begin{bmatrix} 21.3534 \\ 18.6466 \end{bmatrix} - < ans >$$

(c)

$$u_\infty = \lim_{t\to\infty} Se^{\Lambda t}S^{-1}u_0 = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix}\begin{bmatrix} 30 \\ 10 \end{bmatrix} = \begin{bmatrix} 20 \\ 20 \end{bmatrix} - < ans >$$

(d)

$$\frac{du}{dt} = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}u = -Au,\ \ \lambda(-A) = 0, 2$$

$for\ \ \lambda = 0,\ \ (\lambda I + A)v_1 = 0 \Rightarrow v_1 = (1, 1)^T$

$for\ \ \lambda = 2,\ \ (\lambda I + A)v_1 = 0 \Rightarrow v_2 = (1, -1)^T$

$Let\ \ S = [v_1\ \ v_2]$

$$u = Se^{\Lambda t}S^{-1}u_0 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & e^{2t} \end{bmatrix}\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix}\begin{bmatrix} 30 \\ 10 \end{bmatrix}$$

$when\ \ t \to \infty,\ \ v \to \infty - < ans >$