Linear Algebra HW14

1

$$
\begin{aligned}
Z &= A + K \\
&= \frac{Z+Z^H}{2} + K \Rightarrow K = \frac{Z-Z^H}{2} - < ans > \\
Z &= \begin{bmatrix} 3 & 2 \\ 4 & 4 \end{bmatrix} = \begin{bmatrix} 3 & 3 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} - < ans > \\
Z &= \begin{bmatrix} 3+i & 4+2i \\ 0 & 5 \end{bmatrix} = \begin{bmatrix} 3+i & 2+i \\ 2-i & 5 \end{bmatrix} + \begin{bmatrix} 0 & 2+i \\ -2+i & 0 \end{bmatrix} - < ans > \\
Z &= \begin{bmatrix} i & i \\ -i & i \end{bmatrix} = \begin{bmatrix} i & i \\ -i & i \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} - < ans >
\end{aligned}
$$

2

$$
\begin{aligned}
&A = \begin{bmatrix} 5 & -3 \\ 4 & -2 \end{bmatrix},\ \lambda = 2, 1 \\
&for\ \ \lambda = 2,\ (A - \lambda I)x_1 = 0 \Rightarrow x_1 = (1,1)^T \\
&for\ \ \lambda = 1,\ (A - \lambda I)x_2 = 0 \Rightarrow x_2 = (1,-1)^T \\
&Let\ \ U = [x_1\ \ x_2],\ \ U^{-1}AU = \begin{bmatrix} 2 & 7 \\ 0 & 1 \end{bmatrix} - < ans > \\
&B = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix},\ \lambda = 0, 0, 0 \\
&for\ \ \lambda = 0,\ (B - \lambda I)x_1 = 0 \Rightarrow x_1 = (0,0,1)^T \\
&Select\ \ x_2 = (0,1,0)^T\ \ and\ \ x_3 = (1,0,0)^T,\ and\ \ Let\ \ U_1 = [x_1\, x_2\, x_3] \\
&U_1^{-1}BU = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \\
&Select\ \ two\ \ perpendicular\ \ vectors (0,1)^T, (1,0)^T \\
&U_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix},\ U_2^{-1}(U_1^{-1}BU_1)U_2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} = U^{-1}BU \\
&U = U_1 U2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} - < ans >
\end{aligned}
$$

3

$$
\begin{aligned}
&Let\ \ A \in \mathbb{R}^{n \times n}\ \ be\ \ upper\ \ triangular\ \ matrix.\ \ a_{ij} = 0\ \ \forall i > j \\
&If\ \ A\ \ is\ \ a\ \ normal\ \ matrix,\ \ then\ \ A^H A = AA^H \\
&\Rightarrow (A^H A)\ \ is\ \ a\ \ Hermitian\ \ matrix. \\
&(A^H A)_{ii} = (AA^H)_{ii} \Rightarrow \sum_{m=i}^{m=n} a_{im}\bar{a}_{im} = \sum_{m=1}^{m=i} a_{mi}\bar{a}_{mi} \\
&\Rightarrow a_{ij} = 0\ \forall i \neq j \\
&\Rightarrow A\ \ is\ \ a\ \ diagonal\ \ matrix.
\end{aligned}
$$

4

$$If\ \ P\ \ is\ \ permutation\ \ matrix,\ then\ \ P^{-1} = P^H$$
$$PP^H = PP^{-1} = I = P^{-1}P = P^HP$$
$$\Rightarrow P\ \ is\ \ a\ \ normal\ \ matrix.$$

5

$$(i)$$
$$A = \begin{bmatrix} 5 & 4 \\ 4 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \frac{4}{5} & 1 \end{bmatrix} \begin{bmatrix} 5 & 0 \\ 0 & \frac{9}{5} \end{bmatrix} \begin{bmatrix} 1 & \frac{4}{5} \\ 0 & 1 \end{bmatrix}$$
$$[x \quad y]A \begin{bmatrix} x \\ y \end{bmatrix} = [x \quad y]LDU \begin{bmatrix} x \\ y \end{bmatrix} = 5(x + \frac{4}{5})^2 + \frac{9}{5}y^2 - < ans >$$
$$(ii)$$
$$from\ \ det(A - \lambda I) = 0 \Rightarrow \lambda = 1,\ 9$$
$$for\ \ \lambda = 1,\ (A - \lambda I)x_1 = 0 \Rightarrow x_1 = c_1(1, -1)^T$$
$$for\ \ \lambda = 9,\ (A - \lambda I)x_2 = 0 \Rightarrow x_2 = c_2(1, 1)^T$$
$$Set\ \ c_1 = c_2 = \frac{1}{\sqrt{2}},\ \ and\ \ Q = [x_1\ \ x_2],\ \ then\ \ A = Q\Lambda Q^T$$
$$[x \quad y]Q\Lambda Q^T \begin{bmatrix} x \\ x \end{bmatrix} = (\frac{1}{\sqrt{2}})^2 [x - y \quad x + y] \begin{bmatrix} 1 & 0 \\ 0 & 9 \end{bmatrix} \begin{bmatrix} x - y \\ x + y \end{bmatrix}$$
$$= \frac{1}{2}(x - y)^2 + \frac{9}{2}(x + y)^2 - < ans >$$
$$(iii)$$
$$(i)a > 0,\ ac - b^2 > 0 \Rightarrow positive\ \ definite$$
$$(ii)\lambda(A) = 1, 9 > 0 \Rightarrow positive\ \ definite - < ans >$$

6

$$If\ \ the\ \ columns\ \ of\ \ R\ \ are\ \ linearly\ \ dependent,\ \ then\ \ ||Rx|| \geq 0.$$
$$\because R\ \ has\ \ null\ \ space.$$
$$\Rightarrow X^TR^TRX \geq 0,\ \ R^TR\ \ is\ \ positive\ \ semidefinite. - < ans >$$
$$If\ \ the\ \ columns\ \ of\ \ R\ \ are\ \ linearly\ \ independent,\ \ then\ \ ||Rx|| > 0.$$
$$\Rightarrow X^TR^TRX > 0,\ \ R^TR\ \ is\ \ positive\ \ definite. - < ans >$$

7

$$R=\sqrt{D}L^T,\ \ then\ \ R^TR=L\sqrt{D}^T\sqrt{D}L^T=LDU=A$$
$$A=\begin{bmatrix}1&0\\\frac{4}{5}&1\end{bmatrix}\begin{bmatrix}5&0\\0&\frac{9}{5}\end{bmatrix}\begin{bmatrix}1&\frac{4}{5}\\0&1\end{bmatrix}$$
$$R=\sqrt{D}L^T=\begin{bmatrix}\sqrt{5}&0\\0&\sqrt{\frac{9}{5}}\end{bmatrix}\begin{bmatrix}1&\frac{4}{5}\\0&1\end{bmatrix}=\begin{bmatrix}\sqrt{5}&\frac{4}{\sqrt{5}}\\0&\frac{3}{\sqrt{5}}\end{bmatrix}-<ans>$$
$$R=\sqrt{\Lambda}Q^T,\ \ then\ \ R^TR=Q\sqrt{\Lambda}^T\sqrt{\Lambda}Q^T=Q\Lambda Q^T=A$$
$$R=\sqrt{\Lambda}Q^T=\begin{bmatrix}1&0\\0&\sqrt{9}\end{bmatrix}\begin{bmatrix}\frac{1}{\sqrt{2}}&\frac{-1}{\sqrt{2}}\\\frac{1}{\sqrt{2}}&\frac{1}{\sqrt{2}}\end{bmatrix}=\begin{bmatrix}\frac{1}{\sqrt{2}}&\frac{-1}{\sqrt{2}}\\\frac{3}{\sqrt{2}}&\frac{3}{\sqrt{2}}\end{bmatrix}-<ans>$$
$$R=Q\sqrt{\Lambda}Q^T,\ \ then\ \ R^TR=Q\sqrt{\Lambda}^TQ^TQ\sqrt{\Lambda}Q^T=Q\Lambda Q^T$$
$$R=Q\sqrt{\Lambda}Q^T=\begin{bmatrix}\frac{1}{\sqrt{2}}&\frac{1}{\sqrt{2}}\\\frac{-1}{\sqrt{2}}&\frac{1}{\sqrt{2}}\end{bmatrix}\begin{bmatrix}1&0\\0&\sqrt{9}\end{bmatrix}\begin{bmatrix}\frac{1}{\sqrt{2}}&\frac{-1}{\sqrt{2}}\\\frac{1}{\sqrt{2}}&\frac{1}{\sqrt{2}}\end{bmatrix}=\begin{bmatrix}2&1\\1&2\end{bmatrix}-<ans>$$

8

$$\because A\ \ is\ \ positive\ \ definite.\Rightarrow A=LDU=LDL^T$$
$$Let\ \ C=L\sqrt{D}\Rightarrow CC^T=LDL^T=A$$
$$X^TAX=X^TCC^TX>0$$
$$Hence,\ \ C=L\sqrt{D}\ \ with\ \ positive\ \ diagonal\ \ elements.$$

9

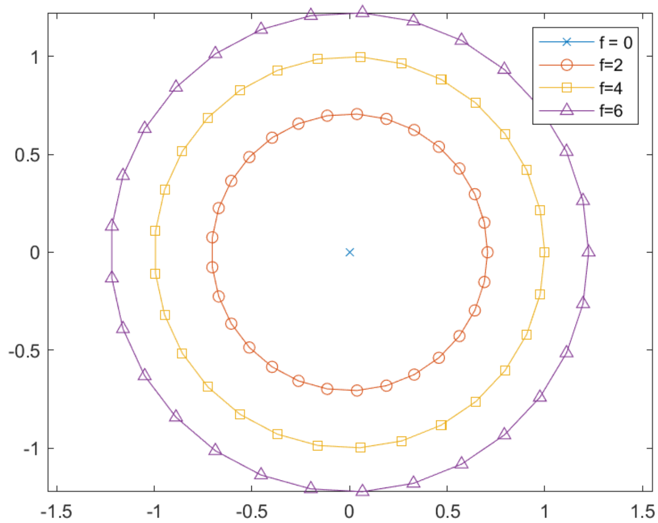
$$(i)$$
$$\frac{\partial^2F}{\partial x^2}|_{x=y=0}=4e^x|_{x=y=0}=4$$
$$\frac{\partial^2F}{\partial y^2}|_{x=y=0}=5xsiny+12|_{x=y=0}=12$$
$$\frac{\partial^2F}{\partial x\partial y}|_{x=y=0}=-5cosy|_{x=y=0}=-5$$
$$\Rightarrow\frac{\partial^2F}{\partial x^2}|_{x=y=0}\frac{\partial^2F}{\partial y^2}|_{x=y=0}=48>(\frac{\partial^2F}{\partial x\partial y}|_{x=y=0})^2=25$$
$$\Rightarrow F\ \ is\ \ positive\ \ definite,\ \ and\ \ (0,0)\ \ is\ \ minimum\ \ point.-<ans>$$
$$(ii)$$
$$\frac{\partial^2F}{\partial x^2}|_{x=1,y=\pi}=2cosy|_{x=1,y=\pi}=-2$$
$$\frac{\partial^2F}{\partial y^2}|_{x=,y=0}=-(x^2-2x)cosy|_{x=1,y=\pi}=-1$$
$$\frac{\partial^2F}{\partial x\partial y}|_{x=1,y=\pi}=-(2x-2)siny|_{x=1,y=\pi}=0$$
$$\Rightarrow\frac{\partial^2F}{\partial x^2}|_{x=1,y=\pi}\frac{\partial^2F}{\partial y^2}|_{x=1,y=\pi}=2>(\frac{\partial^2F}{\partial x\partial y}|_{x=1,y=\pi})^2=0$$
$$\Rightarrow F\ \ is\ \ negative\ \ definite,\ \ and\ \ (1,\pi)\ \ is\ \ maximum\ \ point.-<ans>$$

10

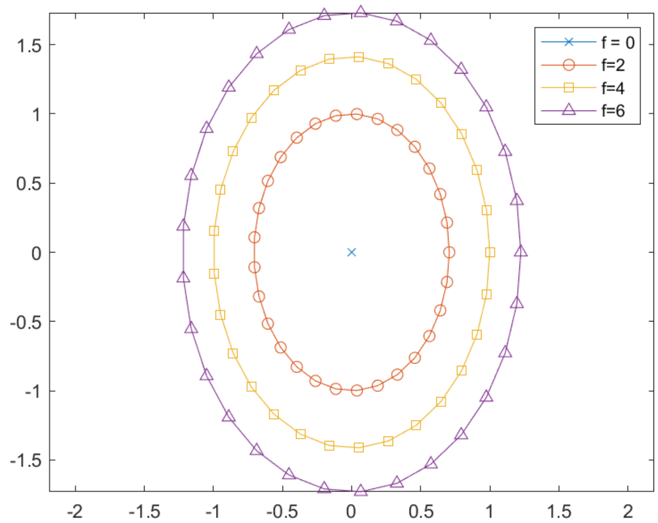
$(i) \because$ *All eigenvalues of positive definite matrix are positive.*
$(ii) \because$ *All projection matrix are singular expect identity matrix* I.
$(iii) \because$ *The eigenvalues of a diagonal matrix are diagonal entries.*

11

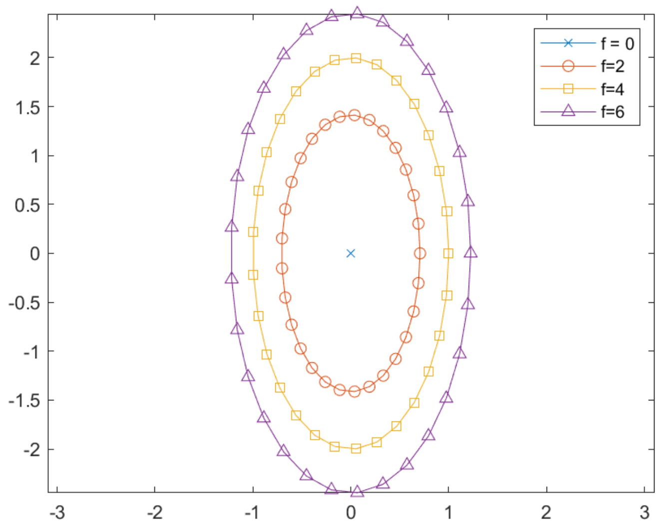
> $\lambda = 4$



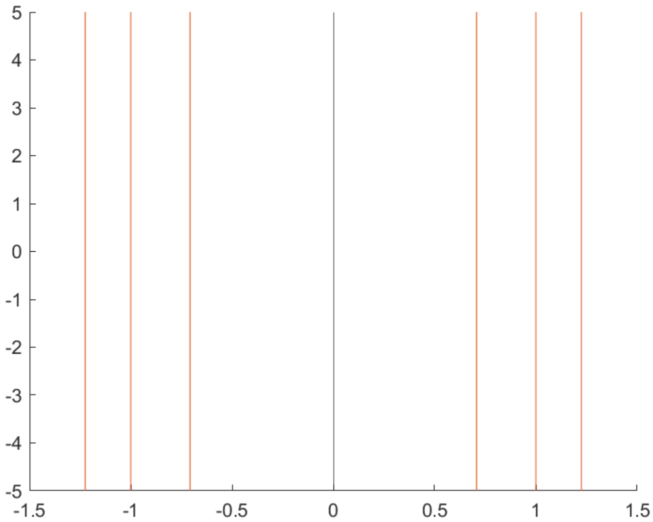
> $\lambda = 2$



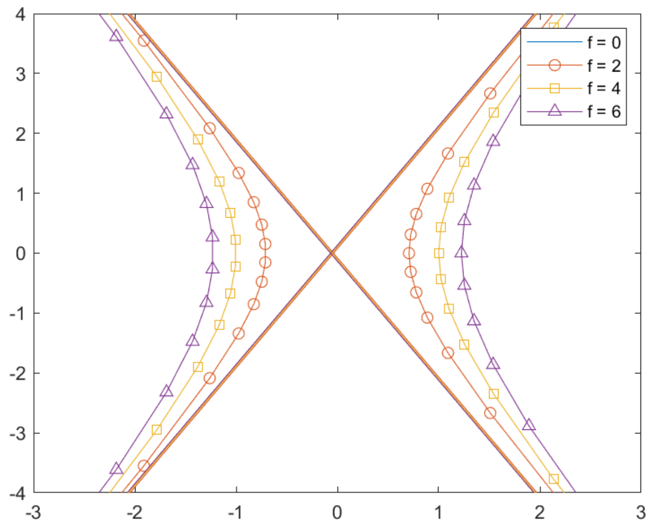
> $\lambda = 1$



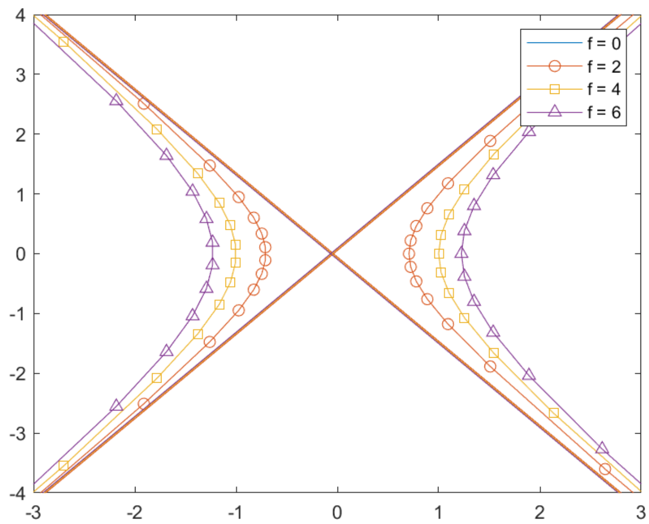
> $\lambda = 0$



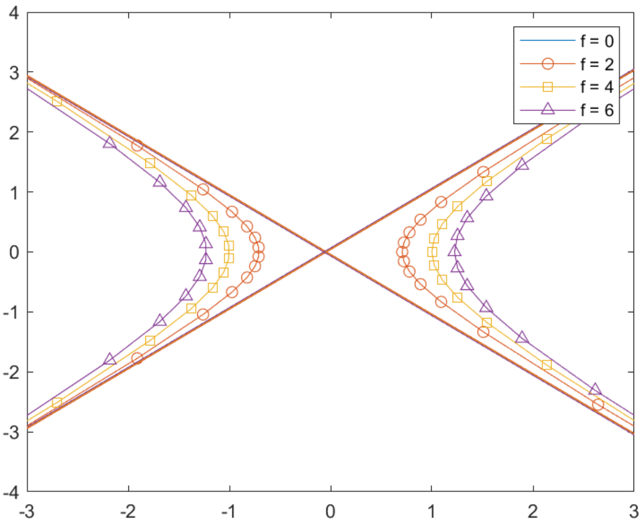
> $\lambda = -1$



> $\lambda = -2$



> $\lambda = -4$



f = 0
f = 2
f = 4
f = 6
4
3
2
1
0
-1
-2
-3
-4
-3
-2
-1
0
1
2
3