Linear Algebra HW16

1

$$Definition\ \ of\ \ norm:\ ||A|| = max_{x\neq 0}\frac{||Ax||}{||x||} - < ans >$$
$$Definition\ \ of\ \ condition\ \ number: c = ||A||\ ||A^{-1}|| - < ans >$$
$$||A^{-1}|| \geq \frac{||A^{-1}(\delta b)||}{||\delta b||} \Rightarrow ||\delta x|| \leq ||A^{-1}||\ ||\delta b||$$
$$multiply\ \ ||A|| \Rightarrow ||A||\ ||\delta x|| \leq ||A||\ ||A^{-1}||\ ||\delta b||$$
$$\Rightarrow \frac{||Ax||}{||x||}||\delta x|| \leq c||\delta b|| \Rightarrow \frac{||b||}{||x||}||\delta x|| \leq c||\delta b||$$
$$\Rightarrow \frac{||\delta x||}{||x||} \leq c\frac{||\delta b||}{||b||}$$

2

$$A^TAv_i = \lambda_i v_i \Rightarrow v_i\ \ in\ \ the\ \ column\ \ space\ \ of\ \ A^TA$$
$$\because A^TA\ \ is\ \ a\ \ projection\ \ matrix\ \ to\ \ row\ \ space\ \ of\ \ A$$
$$\therefore v_i\ \ is\ \ in\ \ row\ \ space\ \ of\ \ A.$$

3

$$A = Q_1\Sigma Q_2^T$$
$$AA^T = [4],\ \lambda = 4,\ Q_1 = [1],\ \Sigma = [2\ \ 0\ \ 0\ \ 0]$$
$$A^TA = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix},\ det(A - \lambda I) = 0 \Rightarrow \lambda(A) = 4, 0, 0, 0$$
$$for\ \ \lambda = 4,\ (A^TA - \lambda I)x_1 = 0 \Rightarrow x_1 = \frac{1}{2}(1, 1, 1, 1)$$
$$for\ \ \lambda = 0,\ (A^TA - \lambda I)x_2 = 0 \Rightarrow x_2 = \frac{1}{\sqrt{2}}(1, -1, 0, 0)$$
$$\Rightarrow x_3 = \frac{1}{\sqrt{2}}(1, 0, -1, 0)$$
$$\Rightarrow x_4 = \frac{1}{\sqrt{2}}(1, 0, 0, -1)$$
$$Q_2 = [v_1 \quad v_2 \quad v_3 \quad v_4],\ but\ \ Q_2\ \ isn't\ \ an\ \ orthogonal\ \ matrix.$$
$$by\ \ Gram - Schmidt\ \ process: Q_2 = \begin{bmatrix} \frac{1}{2} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{2\sqrt{3}} \\ \frac{1}{2} & \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{2\sqrt{3}} \\ \frac{1}{2} & 0 & \frac{\sqrt{2}}{\sqrt{3}} & \frac{1}{2\sqrt{3}} \\ \frac{1}{2} & 0 & 0 & \frac{-\sqrt{3}}{2} \end{bmatrix}$$
$$A = Q_1\Sigma Q_2^T = [1]\,[2 \quad 0 \quad 0 \quad 0] \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & 0 & 0 \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{\sqrt{2}}{\sqrt{3}} & 0 \\ \end{bmatrix}$$

$$\begin{bmatrix} \frac{1}{2\sqrt{3}} & \frac{1}{2\sqrt{3}} & \frac{1}{2\sqrt{3}} & \frac{-\sqrt{3}}{2} \end{bmatrix}$$

$$A^{+} = Q_2\Sigma^{+}Q_1^T = \begin{bmatrix} \frac{1}{4} \\ \frac{1}{4} \\ \frac{1}{4} \\ \frac{1}{4} \end{bmatrix} - < ans >$$

$$B = Q_1\Sigma Q_2^T$$

$$BB^T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},\ det(BB^T - \lambda I) = 0,\ \lambda = 1, 1$$

$$\Sigma = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix},\ Q_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$B^TB = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix},\ \lambda = 1, 1, 0,\ Q_2 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$B = Q_1\Sigma Q_2^T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$B^{+} = Q_2\Sigma^{+}Q_1^T = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 0 \end{bmatrix} - < ans >$$

$$C = Q_1\Sigma Q_2^T$$

$$CC^T = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} \Rightarrow det(CC^T - \lambda I) \Rightarrow \lambda = 2, 0$$

$$\Rightarrow \Sigma = \begin{bmatrix} \sqrt{2} & 0 \\ 0 & 0 \end{bmatrix},\ Q_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$C^TC = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix},\ \lambda = 2, 0 \Rightarrow \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \sqrt{2} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{bmatrix}$$

$$C^{+} = Q_2\Sigma^{+}Q_1 = \begin{bmatrix} \frac{1}{2} & 0 \\ \frac{1}{2} & 0 \end{bmatrix} - < ans >$$

4

$$A = Q_1 \Sigma Q_2 T = (Q_1 Q_2^T)(Q_2 \Sigma Q_2^T) \Rightarrow Let\ Q = Q_1 Q_2^T,\ S = Q_2 \Sigma Q_2^T$$

$$A^T A = \begin{bmatrix} 10 & 6 \\ 6 & 10 \end{bmatrix},\ det(A^T A - \lambda I) = 0 \Rightarrow \lambda = 4, 16$$

$$for\ \ \lambda = 16,\ (A - \lambda I)x_1 = 0 \Rightarrow x_1 = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})^T$$

$$for\ \ \lambda = 4,\ (A - \lambda I)x_2 = 0 \Rightarrow x_2 = (\frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}})^T$$

$$then\ \ Q_2 = [x_1\ \ x_2]$$

$$Q_1 = A Q_2 \Sigma^{-1} = \frac{1}{\sqrt{10}} \begin{bmatrix} 10 & 6 \\ 0 & 8 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \frac{1}{4} & 0 \\ 0 & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \frac{-2}{\sqrt{5}} \end{bmatrix}$$

$$Q = Q_1 Q_2^T = \begin{bmatrix} \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \frac{-2}{\sqrt{5}} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} \frac{3}{\sqrt{10}} & \frac{1}{\sqrt{10}} \\ \frac{-1}{\sqrt{10}} & \frac{3}{\sqrt{10}} \end{bmatrix} - < ans >$$

$$S = Q_2 \Sigma Q_2^T = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} - < ans >$$

$$S' = Q_1 \Sigma Q_1^T = \begin{bmatrix} \frac{18}{5} & \frac{4}{5} \\ \frac{4}{5} & \frac{12}{5} \end{bmatrix} - < ans >$$

5

$$A = Q_1 \Sigma Q_2^T,\ AA^T\ \ is\ \ positive\ \ definite. \Rightarrow AA^T\ \ is\ \ invertible.$$

$$\begin{aligned} A^+ : A^T(AA^T)^{-1} &= (Q_2 \Sigma^T Q_1^T)(Q_1 \Sigma \Sigma^T Q_1^T)^{-1} \\ &= Q_2 \Sigma^T Q_1^T Q_1 (\Sigma^T)^{-1} \Sigma^{-1} Q_1^{-1} \\ &= Q_2 \Sigma^{-1} Q_1^{-1} = Q_2 \Sigma^+ Q_1^T = A^+ - < ans > \end{aligned}$$