Linear Algebra HW5

1

$$After \quad Guassian \quad elimination, \quad A \quad becomes \quad U$$

$$For \quad matrix \quad A:$$

$$dimension \quad of \quad column \quad space: 2$$

$$basis \quad of \quad column \quad space: [1,0,1]^T, [2,1,2]^T (from \quad pivots \quad in \quad U)$$

$$dimension \quad of \quad row \quad space: 2$$

$$basis \quad of \quad row \quad space: [1,2,0,1], [0,1,1,0]$$

$$Find \quad null \quad space \quad by \quad solving \quad UX = 0, \quad there \quad are \quad two \quad free \quad variables.$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 1 \\ 0 \end{bmatrix} x_3 + \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix} x_4$$

$$dimension \quad of \quad null \quad space: 4-2=2$$

$$basis \quad of \quad null \quad space: [2,-1,1,0]^T, [-1,0,0,1]^T$$

$$Find \quad left \quad null \quad space \quad by \quad solving \quad A^T y = 0, \quad there \quad is \quad a \quad free \quad variable.$$

$$\begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 2 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} y_3$$

$$dimension \quad of \quad left \quad null \quad space: 1$$

$$basis \quad of \quad \quad leftnull \quad space: [-1,0,1]^T$$

$$For \quad matrix \quad U:$$

$$dimension \quad of \quad column \quad space: 2$$

$$basis \quad of \quad column \quad space: [1,0,0]^T, [2,1,0]^T$$

$$dimension \quad of \quad row \quad space: 2$$

$$basis \quad of \quad row \quad space: [1,2,0,1], [0,1,1,0]$$

$$Find \quad null \quad space \quad by \quad solving \quad UX = 0, \quad there \quad are \quad two \quad free \quad variables.$$

$$dimension \quad of \quad null \quad space: 4-2=2$$

$$basis \quad of \quad null \quad space: [2,-1,1,0]^T, [-1,0,0,1]^T$$

$$Find \quad left \quad null \quad space \quad by \quad solving \quad U^Ty=0, \quad there \quad is \quad a \quad free \quad variable.$$

$$\begin{bmatrix}1&0&0\\2&1&0\\0&1&0\\1&0&0\end{bmatrix}\begin{bmatrix}y_1\\y_2\\y_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\\0\end{bmatrix}\Rightarrow\begin{bmatrix}1&0&0\\0&1&0\\0&0&0\\0&0&0\end{bmatrix}\begin{bmatrix}y_1\\y_2\\y_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\\0\end{bmatrix}$$
$$\begin{bmatrix}y_1\\y_2\\y_3\end{bmatrix}=\begin{bmatrix}0\\0\\1\end{bmatrix}y_3$$

$$dimension \quad of \quad left \quad null \quad space: 1$$

$$basis \quad of \quad\quad leftnull \quad space: [0,0,1]^T$$

2

(a)

$$Find \quad null \quad space \quad by \quad solving \quad UX=0, \quad there \quad is \quad a \quad free \quad variable.$$

$$\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}1\\-2\\1\end{bmatrix}x_3, \quad the \quad basis \quad is \begin{bmatrix}1\\-2\\1\end{bmatrix}-<ans>$$

(b)

$$Find \quad left \quad null \quad space \quad by \quad solving \quad A^Ty=0, \quad there \quad is \quad a \quad free \quad variable.$$

$$\begin{bmatrix}1&4&7\\2&5&8\\3&6&9\end{bmatrix}\begin{bmatrix}y_1\\y_2\\y_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}\Rightarrow\begin{bmatrix}1&4&7\\0&-3&-6\\0&0&0\end{bmatrix}\begin{bmatrix}y_1\\y_2\\y_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$
$$\begin{bmatrix}y_1\\y_2\\y_3\end{bmatrix}=y_3\begin{bmatrix}1\\-2\\1\end{bmatrix}, the \quad basis \quad is \begin{bmatrix}1\\-2\\1\end{bmatrix}-<ans>$$

(c)

$$basis \quad are \quad [1,2,3],[0,-3,-6]$$

(d)

$$basis \quad are \quad [1,4,7]^T,[2,5,8]^T$$

(e)

$$The \quad column \quad space \quad is \quad the \quad set \quad of \quad Ax \quad for \quad every \quad vector \quad x,$$
$$the \quad left \quad null \quad space \quad is \quad the \quad set \quad of \quad vector \quad y \quad s.t. \quad y^TA=0.$$
$$The \quad inner \quad product \quad is \quad y^T(Ax)=x^T(A^Ty)=x^T0=0.$$

3

$$x_1 + 3x_3 = 0 \Rightarrow \begin{bmatrix} 1 & 0 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$hence, \quad matrice \quad are \quad \begin{bmatrix} 1 & 0 & 3 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

4

(a)

$$r \le m, n, \quad no \quad solution \Rightarrow r < m$$

(b)

$$m - r > 0$$

5

$$After \quad Guassian \quad elimination:$$

$$U = \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 2 & -2 \\ 0 & 0 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 6 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 3 \end{bmatrix}, \quad Rank(A) = 1$$

$$A = \begin{bmatrix} 2 & -2 \\ 6 & -6 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix} \begin{bmatrix} 2 & -2 \end{bmatrix}, \quad Rank(A) = 1$$

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$$Rank(A) \le 2, \quad and \quad Rank(CA) = Rank(I) = 3, \quad hence \quad there \quad exist \quad no \quad C \quad s.t. \quad CA$$

7

$$A \quad is \quad a \quad 2 \quad by \quad 3 \quad matrix, \quad then \quad find \quad a \quad right \quad inverse \quad A^T(AA^T)^{-1}$$

$$\begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix} \left(\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix} \right)^{-1} = \begin{bmatrix} \frac{2}{3} & -\frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} \end{bmatrix} - < ans >$$

$$M \quad is \quad a \quad 3 \quad by \quad 2 \quad matrix, \quad then \quad find \quad a \quad left \quad inverse \quad (M^T M)^{-1} M^T$$

$$\left(\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix} \right)^{-1} \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{2}{3} & \frac{1}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{1}{3} & \frac{2}{3} \end{bmatrix} - < ans >$$

$$T \quad is \quad a \quad 2 \quad by \quad 2 \quad matrix, \quad then \quad exists \quad both \quad right \quad inverse \quad and \quad left \quad inverse$$

$$(\begin{bmatrix} a & 0 \\ b & a \end{bmatrix}\begin{bmatrix} a & b \\ 0 & a \end{bmatrix})^{-1}\begin{bmatrix} a & 0 \\ b & a \end{bmatrix} = \begin{bmatrix} \frac{1}{a} & \frac{-b}{a^2} \\ 0 & \frac{1}{a} \end{bmatrix} - < ans >$$