Linear Algebra and its Applications

HW#4

1. Let

$$A=\begin{bmatrix}1 & 2 & 0 & 3\\ 4 & 5 & 1 & 8\\ 2 & 1 & 1 & 2\end{bmatrix},\quad b=\begin{bmatrix}b_1\\ b_2\\ b_3\end{bmatrix}$$

   (a) Express the null space of A by the linear combination of free variables.
   (b) Let b=[0, -1, -1]^T. Please find the complete solution.

2. Find the value of c that makes it possible to solve $Ax = b$, and solve it:

$$\begin{aligned} u + v + 2w &= 2\\ 2u + 3v - w &= 5\\ 3u + 4v + w &= c. \end{aligned}$$

3. Write the complete solutions $x=x_p+x_n$ to the following systems:

$$\begin{bmatrix}1 & 2 & 3\\ 2 & 4 & 5\end{bmatrix}\begin{bmatrix}u\\ v\\ w\end{bmatrix}=\begin{bmatrix}1\\ 4\end{bmatrix}\quad \begin{bmatrix}1 & 2 & 3\\ 2 & 4 & 6\end{bmatrix}\begin{bmatrix}u\\ v\\ w\end{bmatrix}=\begin{bmatrix}1\\ 4\end{bmatrix}.$$

4. Write all known relations between r and m and n if $Ax = b$ has
   (a) no solution for some b.
   (b) infinitely many solutions for every b.
   (c) exactly one solution for some b, no solution for other b.
   (d) exactly one solution for every b.

5. Explain why all these statements are false:
   (a) The complete solution is any linear combination of x_p and x_n.
   (b) A system $Ax=b$ has at most one particular solution.
   (c) If A is invertible there is no solution x_n in the nullsplace.

6. Choose three independent columns of U. Then make two other choices. Do the same for A. You have found bases for which spaces?

$$U=\begin{bmatrix}2 & 3 & 4 & 1\\ 0 & 6 & 7 & 0\\ 0 & 0 & 0 & 9\\ 0 & 0 & 0 & 0\end{bmatrix}\quad \text{and}\quad A=\begin{bmatrix}2 & 3 & 4 & 1\\ 0 & 6 & 7 & 0\\ 0 & 0 & 0 & 9\\ 4 & 6 & 8 & 2\end{bmatrix}.$$

7. Every column of AB is a combination of the columns of A. Prove rank(AB)$\leqq$rank(A) by the dimensions of the column space. Prove also that rank(AB)$\leqq$rank(B).

8. Find a basis for each of these subspaces of $\mathbf{R}^4$:

(a) All vectors whose components are equal.

(b) All vectors whose components add to zero.

(c) All vectors that are perpendicular to (1,1,0,0) and (1,0,1,1).

(d) The column space (in $\mathbf{R}^2$) and nullspace (in $\mathbf{R}^5$) of

$$U = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \end{bmatrix}.$$

9. The nullspace of a 4 by 3 matrix A is the line through $(2,3,0)^T$.
   (a) What is the rank of A and the complete solution to $Ax=0$?
   (b) What is the exact row reduced echelon form U of A?

10. Write all known relations between r and m and n if $Ax=b$ has

(a) no solution for some b.

(b) infinitely many solutions for every b.

(c) exactly one solution for some b, no solution for other b.

(d) exactly one solution for every b.

11. Prove that if either d=0 or f=0 (2 cases), the columns of U are dependent:

$$U = \begin{bmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{bmatrix}$$

12. Find a basis for each of these following subspaces of R^4

(a) All vectors whose components are equal;

(b) All vectors whose components add to zero;

(c) All vectors that are perpendicular to $(1, 1, 0, 0)^T$ and $(1, 0, 1, 1)^T$;

13. By performing the elimination to A and b so that A is reduced to a echelon form:

$$\begin{bmatrix} A & b \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 & b_1 \\ 4 & 5 & 6 & b_2 \\ 7 & 8 & 9 & b_3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 3 & b_1 \\ 0 & -3 & -6 & b_2 - 4b_1 \\ 0 & 0 & 0 & b_3 - 2b_2 + b_1 \end{bmatrix}$$

For b to be in the column space, what condition does b has to meet?