2. ELEMENTERY INTEGRATION

The following integrals are directly obtained from the derivatives of standard functions.

(a) $\int 0.dx = c$

(b) $\int 1.dx = x + c$

(c) $\int k.dx = kx + c(k \in R)$

(d) $\int x^n dx = \frac{x^{n+1}}{n+1} + c(n \neq -1)$

(e) $\int \frac{1}{x} dx = \log_e x + c$

(f) $\int e^x dx = e^x + c$

(g) $\int a^x dx = \frac{a^x}{\log_e a} + c = a^x \log_a e + c$

(h) $\int \sin x\, dx = -\cos x + c$

(i) $\int \cos x\, dx = \sin x + c$

3. BASIC THEOREMS OF INTEGRATION

If f(x), g(x) are two functions of a variable x and k is a constant, then

(a) $\int k\, f(x) dx = k \int f(x) dx$

(b) $\int \left[f(x) \pm g(x) \right] dx = \int f(x) dx \pm \int g(x) dx$

(c) $\frac{d}{dx}\left(\int f(x) dx \right) = f(x)$

(d) $\int \left(\frac{d}{dx} f(x) \right) dx = f(x) + c$

4. METHODS OF INTEGRATION

When the integrand can't be reduced into some standard form then integration is performed using following methods

4.1 Integration by Substitution

4.1.1 Integrand is a Function of Another Function

If the integral is of the form $\int f\left[\phi(x)\right]\phi'(x)dx$, then we put $\phi(x) = t$ so that $\phi'(x)\,dx = dt$. Now integral is reduced $\int f(t)\,dt$.

4.1.2 Integrand is the Product of Function and its Derivative

If the integral is of the form I= $\int f^{'}(x)\ f(x)\,dx$ we put f(x) = t and convert it into a standard integral.

Illustration 6: Evaluate: $\int \tan x \sec^2 x\,dx$ **(JEE MAIN)**

Sol: Here $\sec^2 x$ is a derivatives of $\tan x$ hence we can put tan x = t and $\sec^2 x.dx = dt$ thereafter we can solve the given problem.

Let tan x =t $\Rightarrow$ $\sec^2 x.dx = dt$

$$\therefore\ I = \int \tan x \sec^2 x\,dx = \int t\,dt = \frac{t^2}{2} + c = \frac{\tan^2 x}{2} + c$$

4.1.3 Integrand is a Function of the Form f(ax+b)

Here we put ax+b =t and convert it into a standard integral. Now if,

$$\int f(x)dx = \phi(x), \text{ then } \int f(ax+b)dx = \frac{1}{a}\phi(ax+b)$$

4.1.4 Integral of the Form

$\int \frac{dx}{a\sin x + b\cos x}$ then substitute a=r cos θ and b=r sin θ, $\tan\theta = \frac{b}{a} \Rightarrow \theta = \tan^{-1}\left(\frac{b}{a}\right)$,we get

$$I = \int \frac{dx}{r\sin(x+\theta)} = \frac{1}{r}\int \operatorname{cosec}(x+\theta)dx = \frac{1}{r}\log\tan\left(\frac{x+\theta}{2}\right) + c = \frac{\log\tan\left((x/2)+(1/2)\tan^{-1}(b/a)\right)}{\sqrt{a^2+b^2}} + c$$

$\int \sin^m x \cos^n x \quad dx$, where $m, n \in N$

$\Rightarrow$ If m is odd put $\cos x = t$

If n is odd put $\sin x = t$

If both m and n are odd, put $\sin x = t$ if $m \geq n$ and $\cos x = t$ otherwise.

If both m and n are even, use power reducing formulae

$$\sin^2 x = \frac{1 - \cos 2x}{2} \text{ or } \quad \cos^2 x = \frac{1 + \cos 2x}{2}$$

If $m+n$ is a negative even integer, put $\tan x = t$

4.1.5 Standard Substitutions

The following standard substitutions will be useful

Integrand form	Substitutions
$\sqrt{a^2 - x^2}$ or $\frac{1}{\sqrt{a^2 - x^2}}$	$x = a \sin\theta$ or $x = a \cos\theta$
$\sqrt{x^2 + a^2}$ or $\frac{1}{\sqrt{x^2 + a^2}}$	$x = a \tan\theta$ or $x = a \cot\theta$ or $x = a \sinh\theta$
$\sqrt{x^2 - a^2}$ or $\frac{1}{\sqrt{x^2 - a^2}}$	$x = a \sec\theta$ or $x = a \operatorname{cosec}\theta$
$\sqrt{\frac{x}{a+x}}$ or $\sqrt{\frac{a+x}{x}}$ or $\sqrt{x(a+x)}$ or $\frac{1}{\sqrt{x(a+x)}}$	$x = a \tan^2\theta$
$\sqrt{\frac{x}{a-x}}$ or $\sqrt{\frac{a-x}{x}}$ or $\sqrt{x(a-x)}$ or $\frac{1}{\sqrt{x(a-x)}}$	$x = a \sin^2\theta$ or $x = a \cos^2\theta$

$\sqrt{\frac{x}{x-a}}$ or $\sqrt{\frac{x-a}{x}}$ or $\sqrt{x(x-a)}$ or $\frac{1}{\sqrt{x(x-a)}}$	$x=a\ \sec^2\theta$ or $x=a\ \text{cosec}^2\theta$
$\sqrt{\frac{a-x}{a+x}}$ or $\sqrt{\frac{a+x}{a-x}}$	$x = a\cos 2\theta$
$\sqrt{\frac{x-\alpha}{\beta-x}}$ or $\sqrt{(x-\alpha)(\beta-x)}$ $(\beta>\alpha)$	$x = \alpha\cos^2\theta+\beta\sin^2\theta$

Some Standard Integrals

(a) $\int \tan x dx = \log\sec x + c = -\log\cos x + c$

(b) $\int \cot x dx = \log\sin x + c = -\log \text{cosec}\ x + c$

(c) $\int \sec x dx = \log(\sec x + \tan x) + c = -\log(\sec x - \tan x) + c = \log\tan\left(\frac{\pi}{4}+\frac{x}{2}\right)+c$

(d) $\int \text{cosec} x dx = -\log(\text{cosec} x + \cot x) + c = \log(\text{cosec} x - \cot x) + c = \log\ \tan\left(\frac{x}{2}\right)+c$

(e) $\int \sec x \tan x dx = \sec x + c$

(f) $\int \text{cosec}\ x \cot x\ dx = -\text{cosec}\ x + c$

(g) $\int \sec^2 x dx = \tan x + c$

(h) $\int \text{cosec}^2 x dx = -\cot x + c$

(i) $\int \log x dx = x\log\left(\frac{x}{e}\right)+c = x\left(\log x + 1\right)+c$

4.2 Integration by Parts

If u and v are two functions of x, then

$$\int (u.v)dx = u\left(\int v\ dx\right) - \int \left(\frac{du}{dx}\right)\left(\int v\ dx\right)dx$$

This is also known as uv rule of integration. This method of integrating is called integration by parts.

4.2 Integration by Parts

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- From the first letter of the words inverse circular, logarithmic, Algebraic, Trigonometric, Exponential functions, we get a word ILATE. Therefore the preference of selecting the u function will be according to the order ILATE.
- In some problems we have to give preference to logarithmic function over inverse trigonometric functions. Hence sometimes the word LIATE is used for reference.
- For the integration of Logarithmic or Inverse trigonometric functions alone, take unity (1) as the v function.

4.2.2 Integration of the Form:

If the integral is of the form $\int e^x\left[f(x)+f^{'}(x)\right]dx$, then use the formula;

$$\int e^x\left[f(x)+f'(x)\right]dx = e^x f(x)+c$$

Illustration 20: Evaluate: $\int e^x(\log x + 1/x)dx$ **(JEE MAIN)**

Sol: Solution of this problem is based on the method mentioned above, here f(x) = log x and f′(x) = 1 / x. $=\int e^x\left[\log x+\frac{1}{x}\right]dx$

$$I = \int\left(e^x\log x+\frac{e^x}{x}\right)dx = e^x\log x + c \quad \begin{bmatrix}\text{Here, } f(x)=\log x \\ \& f^{'}(x)=1/x\end{bmatrix}$$

If the integral is of the form $\int\left[xf^{'}(x)+f(x)\right]dx$ then use the formula; $\int\left[xf^{'}(x)+f(x)\right]dx = xf(x)+c$

4.3 Integration of Rational Functions

4.3.1 When the Denominator can be Factorized (Using Partial Fraction)

Let the integrand be of the form $\frac{f(x)}{g(x)}$, where both f(x) and g(x) are polynomials. If degree of f(x) is greater than degree of g(x), then first divide f(x) by g(x) till the degree of the remainder becomes less than the degree of g(x). Let Q(x) be the quotient and R(x), the remainder then

$$\frac{f(x)}{g(x)} = Q(x) + \frac{R(x)}{g(x)}$$

Now in R(x)/g(x), factorize g(x) and then write partial fractions in the following manner:

(a) For every non-repeated linear factor in the denominator. Write

$$\frac{1}{(x-a)(x-b)} = \frac{A}{x-a} + \frac{B}{x-b}$$

(b) For every repeated linear factor in the denominator. Write

$$\frac{1}{(x-a)^3(x-b)} = \frac{A}{(x-a)} + \frac{B}{(x-a)^2} + \frac{C}{(x-a)^3} + \frac{D}{(x-b)}$$

(c) For every non-repeated quadratic factor in the denominator. Write

$$\frac{1}{(ax^2+bx+c)(x-d)} = \frac{Ax+B}{ax^2+bx+c} + \frac{C}{x-d}$$

(d) For every repeated quadratic factor in the denominator. Write

$$\frac{1}{(ax^2+bx+c)^2(x-d)} = \frac{Ax+B}{(ax^2+bx+c)^2} + \frac{Cx+D}{ax^2+bx+c} + \frac{E}{x-d}$$

Consider f(x) as the function we need to factorize

1. For non- repeated linear factor in the denominator.

Let $f(x)=\frac{1}{(x-a)(x-b)}=\frac{A}{(x-a)}+\frac{B}{(x-b)}$

To obtain the value of A remove (x–a) from f(x) and find f(a).

Similarly, to obtain value of B, remove (x-b) from f(x) and find f(b).

2. For repeated linear factor in the denominator.

Let $f(x) = \frac{1}{(x-a)^3(x-b)}=\frac{A}{(x-a)}+\frac{B}{(x-a)^2}+\frac{C}{(x-a)^3}+\frac{D}{(x-b)}$

To obtain value of D remove (x–b) from f(x) and find f(b).

To obtain value of c remove $(x-a)^3$ from f(x) and find f(a).

Now that we have reduced the number of unknowns from 4 to 2, we can find A and B easily by equating.

Now let's try this method for $\frac{x^4+x^3+2x^2-x+4}{x(x^2+2)(x^2+1)^3}$

Partial fraction will be of the form

$$\frac{x^4+x^3+2x^2-x+4}{x(x^2+2)(x^2+1)^3}=\frac{A}{x}+\frac{Bx+C}{(x^2+2)}+\frac{Dx+E}{(x^2+1)}+\frac{Fx+G}{(x^2+1)^2}+\frac{Hx+I}{(x^2+1)^3}$$

Now remove x and put x=0, we get A=2

Now remove $(x^2+1)^3$ and put x^2 =-1 i.e. x = i (you can also substitute x = –i).

We get Hi+I = –3i –2. Hence H = –3 and I = –2.

Now remove (x^2+2) and put x= $\sqrt{2}i$. We get B ($\sqrt{2}i$) +C=2 $\sqrt{2}i$ +3. Hence B = 2 and C = 3

Now the number of unknowns have reduced from 9 to 4 and the remaining unknowns can be solved easily.

This method very useful instead of solving for all the unknowns at the same time.

Also remember that substituting an imaginary number for x is not discussed anywhere in NCERT. So, use this method only for competitive exams.

4.3.2 When the Denominator cannot be Factorized

In this case the integral may be in the form

(i) $\int \frac{dx}{ax^2 + bx + c}$ (ii) $\int \frac{(px+q)}{ax^2 + bx + c}dx$

Method:

(i) Here taking the coefficient of x^2 common from the denominator , write

$$x^2 + (b/a)x + c/a = (x + b/2a)^2 - \frac{b^2 - 4ac}{4a^2}$$

Now the integrand obtained can be evaluated easily by using standard formulae.

(ii) Here suppose that $px + q = A\left[\frac{d}{dx}(ax^2 + bx + c)\right] + B = A(2ax+b)+B$

Now comparing coefficient of x and constant terms.

We get A=p/2a, B=q-(pb/2a)

$$\therefore I = \frac{p}{2a}\int \frac{2ax+b}{ax^2bx+c}dx + \left(q - \frac{pb}{2a}\right)\int \frac{dx}{ax^2 + bx + c}$$

Now we can integrate it easily.

4.4 Integration of Irrational Functions

If any one term in numerator or denominator is irrational then it is made rational by a suitable substitution. Also if the integral is of the form

$$\int \frac{dx}{\sqrt{ax^2+bx+c}} \text{ or } \int \sqrt{ax^2+bx+c}\,dx$$

Then we integrate it by expressing $ax^2+bx+c=(x+\alpha)^2+\beta^2$

Also for integrals of the form $\int \frac{px+q}{\sqrt{ax^2+bx+c}}dx$ or $\int (px+q)\sqrt{ax^2bx+c}\,dx$

First we express px+q in the form

px+q = A [$\frac{d}{dx}(ax^2bx+c)$]+B and then proceed as usual with standard form.

Illustration 26: Evaluate : $\int \frac{e^x}{\sqrt{5-4e^x-e^{2x}}}dx$ **(JEE MAIN)**

Sol: Simply by putting $e^x = t$, then $e^x dx = dt$, we can solve the given problem.

Put $e^x = t$, then $e^x dx = dt$

$$\therefore \int \frac{e^x}{\sqrt{5-4e^x-e^{2x}}}dx = \int \frac{dt}{\sqrt{5-4t-t^2}} = \int \frac{dt}{\sqrt{5-(t^2+4t)}} = \int \frac{dt}{\sqrt{5-(t^2+4t+4)+4}} = \int \frac{dt}{\sqrt{9-(t+2)^2}}$$

$$= \int \frac{dt}{\sqrt{(3)^2-(t+2)^2}} = \sin^{-1}\left(\frac{t+2}{3}\right)+C = \sin^{-1}\left(\frac{e^x+2}{3}\right)+C$$

5. STANDARD INTEGRALS

(a) $\int \frac{1}{x^2+a^2}dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)+c$

(b) $\int \frac{1}{x^2-a^2}dx = \frac{1}{2a}\log\left|\frac{x-a}{x+a}\right|+c$

(c) $\int \frac{1}{a^2-x^2}dx = \frac{1}{2a}\log\left|\frac{a+x}{a-x}\right|+c$

(d) $\int \frac{1}{\sqrt{a^2-x^2}}dx = \sin^{-1}\left(\frac{x}{a}\right)+c = -\cos^{-1}\left(\frac{x}{a}\right)+c$

(e) $\int \frac{1}{\sqrt{x^2+a^2}}dx = \sinh^{-1}\left(\frac{x}{a}\right)+c = \log\left(x+\sqrt{x^2+a^2}\right)+c$

(f) $\int \frac{1}{\sqrt{x^2-a^2}}dx = \cosh^{-1}\left(\frac{x}{a}\right)+c = \log\left(x+\sqrt{x^2-a^2}\right)+c$

(g) $\int \sqrt{a^2-x^2}dx = \frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}\frac{x}{a}+c$ (Substitute $x = a\cos\theta$ or $x = a\sin\theta$ and proceed)

(h) $\int \sqrt{x^2+a^2}dx = \frac{x}{2}\sqrt{x^2+a^2}+\frac{a^2}{2}\log\left|x+\sqrt{x^2+a^2}\right|+c$ (Substitute $x = a\tan\theta$ or $x = a\cot\theta$ and proceed)

(i) $\int \sqrt{x^2-a^2}dx = \frac{x}{2}\sqrt{x^2-a^2}-\frac{a^2}{2}\log\left|x+\sqrt{x^2-a^2}\right|+c$ (Substitute $x = a\sec\theta$ or $x = a\operatorname{cosec}\theta$ and proceed)

(j) $\int \frac{1}{x\sqrt{x^2-a^2}}dx = \frac{1}{a}\sec^{-1}\frac{x}{a}+c$ (Valid for $x > a > 0$)

(k) $\int e^{ax}\sin bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx)+c = \frac{e^{ax}}{\sqrt{a^2+b^2}}\sin\left\{bx-\tan^{-1}\left(\frac{b}{a}\right)\right\}+c$

(I) $\int e^{ax}\cos bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\cos bx + b\sin bx) + c = \frac{e^{ax}}{\sqrt{a^2+b^2}}\cos\left\{bx - \tan^{-1}\frac{b}{a}\right\} + c$

Integration of irrational algebraic functions:

Type 1: (a) $\int \frac{dx}{(x-\alpha)\sqrt{(x-\alpha)(\beta-x)}}$ (Put : $x = a\cos^2 q + b\sin^2 q$)

(b) $\int \frac{dx}{(x-\alpha)\sqrt{(x-\beta)}}$ (Put : $x = a\sec^2 q - b\tan^2 q$)

Type 2: $\int \frac{dx}{(ax+b)\sqrt{px+q}}$ (Put: $px+q=t^2$)

Type 3: $\int \frac{dx}{(ax+b)\sqrt{px^2+qx+r}}$ (Put: $ax+b=\frac{1}{t}$)

Type 4: $\int \frac{dx}{(ax^2bx+c)\sqrt{px+q}}$ (Put: $px+q=t^2$)

Type 5: $\int \frac{dx}{(ax^2+bx+c)\sqrt{px^2+qx+r}}$

Case I: When (ax^2bx+c) breaks up into two linear factors, e.g.

$I = \int \frac{dx}{\left(x^2-x-2\right)\sqrt{x^2+x+1}}dx$ then $= \int\left(\frac{A}{x-2}+\frac{B}{x+1}\right)\frac{1}{\sqrt{x^2+x+1}}dx = A\int \underbrace{\frac{dx}{(x-2)\sqrt{x^2+x+1}}} + B\int \underbrace{\frac{dx}{(x+1)\sqrt{x^2+x+1}}}$

Put $x-2=\frac{1}{t}$ Put $x+1=\frac{1}{t}$

Case II: If ax^2+bx+c is a perfect square say $(lx+m)^2$, then put $lx+m=\frac{1}{t}$

Case III: If b = 2, q = 0

e.g. $\int \frac{dx}{(ax^2+b)\sqrt{px^2+r}}$ then, put $x=\frac{1}{t}$ or trigonometric substitutions are also helpful.

Integral of the form $\int \frac{dx}{P\sqrt{Q}}$, where P, Q are linear or quadratic functions of x.

Integral	**Substitutions**
$\int \frac{1}{(ax+b)\sqrt{cx+d}}dx$	$cx+d=z^2$
$\int \frac{dx}{\left(ax^2+bx+c\right)\sqrt{px+q}}$	$px+q=z^2$
$\int \frac{dx}{(px+q)\sqrt{ax^2+bx+c}}$	$px+q=\frac{1}{z}$
$\int \frac{dx}{\left(ax^2+b\right)\sqrt{cx^2+d}}$	$x=\frac{1}{z}$