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CAT

Permutation, Combination & Probability

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Chapter 1: Permutation & Combination, Probability

1.1 Fundamental Principle of Counting

Multiplication: If there are two jobs such that one of them can be completed in m ways, and when it has been completed in any one of these m ways, second job can be completed in n ways, then the two jobs in succession can be completed in $m \times n$ ways.

Ex1. In a class there are 10 boys and 8 girls. The teacher wants to select a boy and a girl to represent the class in a function. In how many ways can the teacher make the selection?

Sol. Here the teacher has to perform two jobs.
(i) Selecting a boy among 10 boys and
(ii) Selecting a girl among 8 girls.
The first of these can be performed in 10 ways and the second in 8 ways. Therefore by the fundamental principle of multiplication, the required number of ways is $10 \times 8 = 80$.

Addition: If there are two jobs such that they can be performed independently in m and n ways respectively, then either of the two jobs can be performed in $(m + n)$ ways.

Ex.2. In a class there are 10 boys and 8 girls. The teacher wants to select a boy or a girl to represent the class in a function. In how many ways the teacher can make the selection.

Sol. Here the teacher has to perform either of the following two jobs.
(i) Selecting a boy among 10 boys or
(ii) Selecting a girl among 10 girls.
The first of these can be performed in 10 ways and the second in 8 ways. Therefore, by fundamental principle of addition either of the two jobs can be performed in 10 + 8 = 18 ways. Hence the teacher can make the selection of either a boy or a girl in 18 ways.

Note: The above principles of counting can be extended to any finite number of jobs.

1.2 Permutations

Each of the arrangement which can be made by taking some or all of a number of things is called a permutation.

Ex.3 The permutation of three letters A, B, C.

Sol. The permutation of three letters A, B, C taking all at a time are ABC, ACB, BCA, CBA, CAB, BAC

Ex.4 The permutation of three letters A, B, C taken two a time.

Sol. The required permutations are AB, BA, BC, CB, AC, CA.

Permutation of n distinct objects taken 'r' at a time

{Here r and n are positive integers & $1 \le r \le n$} is = $P(n, r)$ = ${}^nP_r = n(n-1)(n-2)\_\_\_\_\_\_(n-r+1)$

Note–1: $P(n, r) = {}^nP_r = \dfrac{n!}{(n-r)!}$

Ex.5 In how many way can three different rings be worn in four fingers with at most one in each finger?

Sol. The total number of ways is same as the number of arrangements of 4 fingers taken 3 at a time. So required number of ways = ${}^4P_3 = \dfrac{4!}{(4-3)!} = \dfrac{4!}{1!} = 4! = 24$.

The above illustration can also be explained by presuming the three rings as R_1, R_2 and R_3

First Ring R_1 can be worn in 4 ways

Now Ring R_2 can be worn in 3 ways

And Ring R_3 can be worn in 2 ways

By the fundamental principle of counting the total number of way in which three different rings can be worn in four fingers = $4 \times 3 \times 2$ ways = 24 ways

Permutation of 'n' distinct objects taken all at a time

(Here n is a positive integer) is $P(n, n) = {}^{n}P_{n} = n!$

Note: $P(n, n) = \frac{n!}{(n-n)!} \Rightarrow n! = \frac{n!}{0!} \Rightarrow 0! = 1.$

Hence zero factorial is 1.

Ex.6 How many words with or without meaning can be formed using all the letters of the word EQUATION, using each letter exactly once.

Sol. There are 8 letters in the word EQUATION. So the total number of words is equal to the number of arrangements of these letters, taken all at a time. The number of such arrangements is ${}^{8}P_{8} = 8!$

<u>Permutation of '*n*' different objects, taken '*r*' at a time, when a particular objects is to be always included in each arrangement is '*r*'.</u> $^{n-1}P_{r-1}$

Ex.7 How many four lettered words, with or without meaning, can be formed using the letters of the word 'MOTHERLY' using each letter exactly once having essentially 'M' as one of the letters.

Sol. (i) Number of four letters words beginning with 'M' $= {}^{8-1}P_{4-1}$

(ii) Number of four letters words having 'M' as 2nd letter = ${}^{8-1}P_{4-1}$

(iii) Number of four letters words having 'M' as 3rd letter = ${}^{8-1}P_{4-1}$

(iv) Number of four letters words having 'M' as last letter = ${}^{8-1}P_{4-1}$

Total number of words

$= {}^{8-1}P_{4-1} + {}^{8-1}P_{4-1} + {}^{8-1}P_{4-1} + {}^{8-1}P_{4-1} = 4.{}^{8-1}P_{4-1}$

<u>Permutation of '*n*' distinct objects taken '*r*' at time when a particular object is never taken in each arrangement</u> is ${}^{n-1}P_r$. Here one particular object out of *n* given objects is never taken. So we have to determine the number of ways in which *r* places can be filled with (*n* – 1) distinct objects. Clearly the number of arrangement is ${}^{n-1}P_r$.

Ex.8 How many four letters words with or without meaning can be formed using the letters of the word EQUATION using each letter exactly once. The words are not to have 'N' as one of the letters.

Sol. Here the total numbers of objects is the numbers of letters of the word EQUATION which is = 8. We can arrange only (8 – 1) objects taken 4 at a time. Required number of ways = ${}^{8-1}P_4 = {}^{7}P_4$.

<u>Permutation of '*n*' different objects taken '*r*' at a time in which two specified objects always occur together</u> is 2! (*r* – 1) ${}^{n-2}P_{r-2}$. Here if leave out two specified objects, then the number of permutations of the remaining (n – 2) objects, taken (r – 2) at a time is ${}^{n-2}P_{r-2}$. Now consider two specified objects temporarily as a single object and add to each of these ${}^{n-2}P_{r-2}$ permutations which can be done is (r – 1) ways. Thus the number of permutations becomes (r – 1) ${}^{n-2}P_{r-2}$. But the two specified things can be put together in 2! ways. Hence the required no. of permutations is 2! (r – 1) ${}^{n-2}P_{r-2}$.

Ex.9 In how many ways can letters of the word PENCIL be arranged so that E and N are always together?

Sol. Let us keep EN together and consider as one letter. Now we have 5 letters which can be arranged in ${}^5P_5$ = 5! = 120 ways. But E and N can be put together in 2! ways (EN or NE). Hence total no. of ways = 2! × 5! = 240 ways.

<u>Permutation of objects not all distinct:</u> till now we have been discussing permutations of distinct objects by taking some or all at a time. Now we will discuss the permutations of a given number of objects when objects are not all different. The number of mutually distinguishable permutations of 'n' things, taken all at a time, of which p are alike of one kind, q are alike of second such that $p + q = n$ is $\frac{n!}{p!q!}$


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Ex.10 How many different words can be formed with the letters ***aaaaiiiippf?***

Sol. There are 11 letters in the given word of which 4 are *a*'s, 4 are *i*'s and 2 are *p*'s. So total number of words is the arrangement of 11 things, of which 4 are alike of one kind, 4 are alike of second kind and 2 are alike of third kind i.e. $\frac{11!}{4!4!2!}$. Hence total number of words = $\frac{11!}{4!4!2!}$ = 34,650.

<u>Permutation when objects can repeat</u> the number of permutations of n different things, taken r at a time, when each may be repeated any number of times in each arrangement is n^r.

The concept can be explained by comparing this permutation with the number of was in which *r* places can be filled in by n different things when each thing can be repeated *r* times.

The first places can be filled in *n* ways by any one of the *n* things. Having filled up the first place n things are again left, therefore the second place can be filled in n ways.

Similarly each of the 3^{rd}, 4^{th}, _ _ _ _ r^{th} place can be filled in n ways. Thus by fundamental principle of counting, the total number of ways of filling 'r' places = $n \times n \times n$ _ _ _ _ _ _ _ to r factors = n^r

Ex.11 In how many ways can 5 letters be posted in 4 letter boxes?

Sol. Since each letter can be posted in any one of the four letter boxes. So a letter can be posted in 4 ways. So total number of ways in which all five letters can be posted = $4 \times 4 \times 4 \times 4 \times 4 = 4^5$ ways.

1.3 Circular Permutations

Permutation of n distinct objects along a circle can be done in $(n - 1)!$ ways

Ex.12 In how many ways can 8 students be seated in a circle

Sol. The number of ways in which 8 students can be seated around a circle = $(8 - 1)! = 7!$ ways.

Note: This concept can be understood by understanding that n linear permutations when considered along a circle give rise to one circular permutation. Thus required circular permutations = $\frac{n!}{n} = (n-1)!$

Permutation along a circle when clockwise and anticlockwise arrangements are considered alike. The number of permutations of n distinct objects when clockwise and anticlockwise arrangements are similar $= \frac{(n-1)!}{2}$.

Ex.13 Find the number of ways in which 10 different flowers can be arranged to form a garland.

Sol. Ten different flowers can be arranged in circular form is $(10 - 1)! = 9!$ ways. Since there is no distinction between the clockwise and anticlockwise arrangements. So, the required number of arrangements $= \frac{9!}{2}$

1.4 Combinations

Each of the different selections made by taking some or all of a number of objects, irrespective of their arrangements is called a combination.

Ex.14 The different combinations formed of three letters A, B, C taken two at a time.

Sol. The different combinations formed of three letters A, B, C are AB, BC, CA.

Difference Between Permutation & Combination

(i) In a combination only selection is made whereas in a permutations not only a selection is made but also an arrangement in a definite order is considered.

(ii) In a combination the ordering of selected objects is immaterial whereas in a permutation, the ordering is essential

(iii) Practically to find the permutations of *n* different items, taken '*r*' at a time, we first select *r* items from *n* items and then arrange them. So usually, the number of permutations exceeds the combinations.

Combination of n different objects, taken r at a time is given by $C(n, r)$ $^{n}C_{r} = \frac{n!}{(n-r)!r!}$

Ex.15 Out of 5 men & 2 women, a committee of 3 is to be formed. In how many ways can it be done if at least one woman is to be included.

Sol. The committee can be formed in the following ways.

(i) By selecting 2 Men and 1 Woman.

(ii) By selecting 1 Man and 2 Women

2 men out of 5 men and 1 woman out of 2 women can be chosen in $^{5}C_{2} \times {}^{2}C_{1}$ ways and 1 men out of 5 men and 2 women out of 2 women can be chose in $^{5}C_{1} \times {}^{2}C_{2}$ ways.

$\therefore$ Total number of ways of forming the committee
$= {}^5C_2 \times {}^2C_1 + {}^5C_1 \times {}^2C_2 + = 20 + 5 = 25.$

Properties of ^nC_r

Prop–I ${}^nC_r = {}^nC_{n-r}$ for $0 \le r \le n$

Prop–II Let n and r be non–negative integers such that $r \le n$. Then ${}^nC_r + {}^nC_{r-1} = {}^{n+1}C_r$

Prop–III Let n and r be non–negative integers such that $1 \le r \le n$. Then ${}^nC_r = \frac{n}{r} \cdot {}^{n-1}C_{r-1}$

<u>Selection of one or more items</u>: The number of ways of selecting one or more items from a group of n distinct items is $2^n - 1$.

In making selection each item can be dealt with in two ways; it is either selected or rejected and corresponding to each way of dealing with one item, any one of the other items can also be dealt with in 2 ways. So the total number of ways of dealing with n items is 2^n. But these 2^n ways also include the case when all the items are rejected. Hence required number of ways = $2^n - 1$

Ex.16 Find the total number of proper factor of 7,875.

Sol. We have $7{,}875 = 3^2 \times 5^3 \times 7^1$. The total number of ways of selecting some or all out of two 3's, three 5's, and one 7, is (2 + 1) (3 + 1) (1 + 1) = 24. But this include the given number itself and one. Therefore the required number of proper factors is 22.

1.5 Division of Items into Groups

Division of items into groups of unequal sizes: Number of ways in which (m + n) items can be divided into two unequal groups containing 'm' and 'n' items is $\frac{(m+n)!}{m!n!}$.

Note: The number of ways in which (m + n) items are divided into two groups containing 'm' and 'n' items is same as the number of combinations of (m + n) things. Thus the required number = ${}^{m+n}C_m$ = $\frac{(m+n)!}{m!n!}$.

Note: The number of ways of dividing (m + n + p) items among 3 groups of size m, n and p respectively is

= (Number of ways to divide) = $\frac{(m+n+p)!}{m!n!p!}$

Note: The number of ways in which mn different items can be divided equally into m groups each containing n objects and the order of group is important is

$$\left\{\frac{(mn)!}{(n!)^m}\times\frac{1}{m!}\right\}m!=\frac{(mn)!}{(n!)^m}.$$

Note: The number of ways in which (mn) different items can be divided equally into m groups each containing n objects and the order of groups is not important is $\left[\frac{(mn)!}{(n!)^m}\right]\frac{1}{m!}$.

1.6 More Solved Examples

Ex.1 Find the LCM of 4!, 5! and 6!.

Sol. We have 5! = 5 × 4! and 6! = 6 × 5 × 4!

$\Rightarrow$ LCM of 4!, 5! and 6! = LCM of 4!, 5 × 4! and 6 × 5 × 4! = (4!) × 5 × 6 = 6! = 720.

Ex.2 If $\frac{n!}{2!(n-2)!}$ and $\frac{n!}{4!(n-4)!}$ are in the ratio of 2 : 1, find the value of n.

Sol. We have $\frac{n!}{2!(n-2)!}$. $\frac{n!}{4!(n-4)!}$ = 2 : 1

$\Rightarrow \frac{n!}{2!(n-2)!}\times\frac{4!(n-4)!}{n!} = \frac{2}{1} \quad \Rightarrow (n-2)(n-3) = 6$

$\Rightarrow n^2 - 5n = 0 \quad \Rightarrow n = 0, 5$. For $n = 0$ $(n-2)!$ and $(n-4)!$ are not meaningful. $\therefore$ $n = 5$

Ex.3 Prove that $(n! + 1)$ is not divisible by any natural number between 2 and n.

Sol. Let m be divisible by k . Let r be any natural number between 1 and k. If $(m + r)$ is divided by k, then we obtain r as the remainder. Since $n! = 1.2.3.4\_\ \_\ \_\ \_\ \_\ (n-1).n$, it follows that $n!$ is divisible by every natural number between 2 and n. So $(n! + 1)$ when divided by any natural number between 2 and n, leaves 1 as the remainder. Hence $(n! + 1)$ is not divisible by any natural number between 2 and n.

Ex.4 Find the exponent of 15 in 100!

Sol. Now E_3 (100!) = 48 and E_5 (100!) = 24.
$\therefore$ Exponent of 15 in 100! = min (24, 48) = 24.

Ex.5 How many four digit numbers can be formed using the digits 0, 1, 2, 3, 4, 5 if
(i) Repetition of digits is not allowed
(ii) Repetition of digits is allowed?

Sol. (i) In a four digit number 0 can't appear in the thousand's place. So thousand's place can be filled in 5 ways (viz 1, 2, 3, 4, 5). Since repetition of digits is not allowed and 0 can be used at hundred's place, so hundred's place can be filled in 5 ways. Now any one of the remaining four digits can be used to fill up ten's place. So ten's place can be filled in 4 ways. One's place be filled from the remaining three digits in 3 ways. Hence the required number of numbers = 5 × 5 × 4 × 3 = 300.
(ii) For a four digit number we have to fill up four places and 0 can not appear in the thousand's

place. So thousand's place can be filled in 5 ways. Since repetition of digits is allowed, so each of the three remaining places viz hundred's, ten's and one's can be filled in 6 ways. Hence required number of numbers = 5 × 6 × 6 × 6 = 1,080.

Ex.6 It is required to seat 5 men and 4 women in a row so that the women occupy the even places. How many such arrangements are possible?

Sol. In all 9 persons are to be seated in a row and in the row of a positions there are exactly four even places viz. second, fourth, sixth and eighth. It is given that these four even places are to be occupied by 4 women. This can be done in ${}^4P_4$ ways. The remaining 5 positions can be filled by the 5 Men in ${}^5P_5$ ways. So by the fundamental principle of counting, the numbers of seating arrangements as required in ${}^4P_4 \times {}^5P_5 = 4! \times 5! = 24 \times 120 = 2,880$.

Ex.7 Find the sum of all the numbers that can be formed with the digits 2, 3, 4, 5 taken all at a time.

Sol. The total number of numbers formed with the digits 2, 3, 4 and 5 taken all at a time = Number of arrangements of 4 digits taken = ${}^4P_4 = 4! = 24$. To find the sum of these 24 numbers we will find the sum of digits at units, tens, hundred's and thousand's place in all these numbers. Consider the digits in the unit's place in all these numbers. Each of the digits 2, 3, 4 and 5 occur in 3! (6) times in the unit's place.

So, total for the digits in the unit's place in all these numbers = $(2 + 3 + 4 + 5) \times 3! = 84$. Since each of the digits 2, 3, 4 and 5 occurs in 3! Times in any one of the remaining places. So, the sum of the digits in the ten's, hundred's and thousand's places in all these numbers = $(2 + 3 + 4 + 5) \times 3! = 84$. Hence the sum of all the numbers = $(10^0 + 10^1 + 10^2 + 10^3) \times 84 = 93{,}324$.

Ex.8 How many words can be formed from the letters of the word 'DAUGHTER' so that the vowels never come together?

Sol. The total number of words formed by using all the eight letters of the word 'DAUGHTER' is $^8P_8 = 8! = 40{,}320$. So, the total number of words in which vowels are never together = Total number of words – Number of words in which vowels are together = 40,320 – 4,320 = 36,000

Ex.9 How many four digit numbers divisible by 4 can be made with the digits 1, 2, 3, 4, 5 if the repetition of digits is not allowed?

Sol. A number is divisible by 4 if the number formed by the last two digits is divisible by 4.

So possible ways of last two places are 12,24,32 and 52.

Now corresponding each such way the remaining three digits at thousand's and hundred's places can be arranged in ${}^3P_2$ ways. Hence the required number of numbers = ${}^3P_2 \times 4 = 3! \times 4 = 24$.

Ex.10 How many words can be formed using the letter *A* thrice, the letter *B* twice and the letter *C* thrice?

Sol. We are given 8 letters viz. *A*, *A*, *A*, *B*, *B*, *C*, *C*, *C*. Clearly there are 8 letters of which three are of one kind, two are of second kind and three are of third kind. So the total number of permutations is = $\frac{8!}{3!2!3!} = 560$. Hence the requisite number of words = 560.

Ex.11 Find the number of ways in which a pack of 52 cards be divided into 4 sets, three of them having 17 cards and the fourth just one card?

Sol. First we divide 52 cards into two groups of 1 cards and 51 cards. This can be done in $\frac{52!}{1!51!}$ ways. Now every group of 51 cards can be divided into 3 groups of 17 cards each is $\frac{51!}{(17!)^3 3!}$. Hence the required number of ways

$$= \frac{52!}{(51!)(1!)} \times \frac{51!}{(17!)^3 3!} = \frac{52!}{(17!)^3 3!}$$

Ex.12 How many words of 4 different letters can be formed out of 7 capital letters, 3 vowels and 5 consonants if each word starts with a capital letter and contains at least one vowel?

Sol. First letter of each word must be a capital letter. Now following cases are possible:

No. of Capital letters	No. of Vowels	No. of Consonants	No. of words	
1	1	2	${}^7C_1 \times {}^3C_1 \times {}^5C_2 \times 3!$	= 1260
1	2	1	${}^7C_1 \times {}^3C_2 \times {}^5C_1 \times 3!$	= 630
1	3	0	${}^7C_1 \times {}^3C_3 \times {}^5C_0 \times 3!$	= 42

Chapter 2: Probability

2.1 Definitions of Probability

A Classical definition of probability is: The probability of an event is the number of cases favorable to that event to the total number of cases, provided that all these are equally likely.

There are two approaches to probability viz.
(i) Classical approach
(ii) Axiomatic approach

In both the approaches we frequently use the term 'experiment' which means an operation which can produce well defined outcome(s). There are two types of experiments:

(i) Deterministic Experiment: Those experiments which when repeated under identical conditions produce the same result or outcome are known as deterministic experiment. When experiments in science or engineering are repeated under identical conditions, we almost get the same result every time.

(ii) Random Experiment: If an experiment when repeated under identical conditions, do not produce the same outcome every time but the outcome in a trial is one of the several possible outcomes then such an experiment is known as probabilistic experiment or a random experiment. For example tossing of a coin, it is not sure that the outcome will be head or tail.

Trial and Elementary Events: Let a random experiment be repeated under identical conditions. Then the experiment is called a trial and the possible outcomes of the experiment are known as elementary events or cases.
The tossing of a coin is trial and getting head or tail is an elementary event.

Compound Event: Events obtained by combing together two or more elementary events are known as the compound events.
In a throw of a dice the event: getting a multiple of 2 is the compound event because this event occurs if any one of the elementary event 2, 4 or 6 occurs.

Exhaustive Number of Cases; The total number of possible outcomes of a random experiment in a trial is known as the exhaustive number of cases. In other words, the total number of elementary events of a random experiment is called the exhaustive number of cases.

In throwing of a dice the exhaustive number of cases is 6 since any one of the six marked with 1, 2, 3, 4, 5, 6 may come uppermost.

Mutually Exclusive Events: Events are said to be mutually exclusive or incompatible if the occurrence of any one of them prevents the occurrence of all the others i.e. if no two or more of them can occur simultaneously in the same trial.

Elementary events related to a random experiment are always mutually exclusive, because elementary events are outcomes of an experiment when it is performed and at a time only one outcome is possible.

Equally Likely Cases: Events are equally likely if there is no reason for an event to occur in preference to any other event. If an unbiased dice is rolled, then each outcome is equally likely to happen. i.e. all elementary events are equally likely.

Favourable Number of Cases: The number of cases favourable to an event in a trial is the number of elementary events such that if any one of them occurs, we say that the event happens.

In other words, the number of cases favourable to an event in a trial is the total number of elementary events such that the occurrence of any of them ensures the happening of that event.

Independent Events: Events are said to be independent if the happening (or non happening) of one event is not effected by the happening (or non–happening) of other.

If two dice are thrown together, then getting an even number on first is independent to getting an odd number on the second.

Classical Definition of Probability

If there are n–elementary events associated with a random experiment and m of them are favorable to an event A then probability of A is denoted by $P(A)$ and is defined as the ratio $\frac{m}{n}$. Thus $P(A) = \frac{m}{n}$, since $0 \le m \le n$ therefore $0 \le \frac{m}{n} \le 1$, therefore $0 \le P(A) \le 1$

The number of cases in which the event A will not happen is $(n - m)$, therefore if $\bar{A}$ denotes not happening of A, then the probability $P(\bar{A})$ of not happening of A is given by $P(\bar{A})$

$= \frac{n-m}{n} = 1 - \frac{m}{n} = 1 - P(A) \Rightarrow P(A) + P(\bar{A}) = 1$. If $P(A) = 1$, A is called certain event and if $P(\bar{A}) = 1$, A is called impossible event.

Mutually Exclusive Events: Let S be the sample space associated with a random experiment and let A_1 and A_2 be two events. Then A_1 and A_2 are mutually exclusive if $A_1 \cap A_2 = \phi$.

Note–1: If A and B are two events associated with a random experiment, then $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

Note–2: If A and B are mutually exclusive events, then $P(A \cap B) = 0$, therefore $P(A \cup B) = P(A) + P(B)$.

Note–3: If A, B, C are three events associated with a random experiment, then $P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(A \cap C) + P(A \cap B \cap C)$.

Note–4: If A & B are two events associated with a random experiment. Then (i) $P(\overline{A} \cap B) = P(B) - P(A \cap B)$ (ii) $P(A \cap \overline{B}) = P(A) - P(A \cap B)$

Conditional Probability

Let A and B be two events associated with a random experiment. Then the probability of occurrence of A under the condition that B has already occurred and $P(B) \neq 0$ is called conditional probability and it is denoted by $P\left(\frac{A}{B}\right)$

Thus $P\left(\frac{A}{B}\right)$ = Probability of occurrence of *A* under the condition that *B* has already occurred.

P $\left(\frac{B}{A}\right)$ = Probability of occurrence of *B* under the condition that *A* has already occurred.

Illustration of Conditional Probability: Suppose a bag contains 5 white and 4 red balls. Two balls are drawn from the bag one after the other without replacement. Consider the following events:

A = Drawing a white ball in the first draw

B = Drawing a red ball in the second draw

Now $P\left(\frac{B}{A}\right)$= probability of drawing a red ball in the second draw given that a white ball has already been drawn in the first draw. Since 8 balls are left after drawing a white ball in the first draw and out of these 8 balls, 4 balls are red, therefore $P\left(\frac{B}{A}\right) = \frac{4}{8} = \frac{1}{2}$.Hence $P\left(\frac{A}{B}\right)$ = Not meaningful in this experiment because *A* cannot occur after the occurrence of *B*.

Note–1: If *A* & *B* are two events associated with a random experiment, then $P(A \cap B) = P(A) \cdot P\left(\frac{B}{A}\right)$ if $P(A) \neq 0$ or $P(A \cap B) = P(B) \cdot P\left(\frac{A}{B}\right)$ if $P(B) \neq 0$

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Note–2: If $A_1, A_2, A_3, \_\_\_\_\_ A_n$ are n events related to a random experiment, then $P(A_1 \cap A_2 \cap A_3 \_\_\_ A_n) = P(A_1)$. $P\left(\frac{A_2}{A_1}\right) . P\left(\frac{A_3}{A_1 \cap A_2}\right) \_\_\_\_ P\left(\frac{A_n}{A_1 \cap A_2 \_\_ A_{n-1}}\right)$

Illustrations:

Ex.1 A bag contains 10 white and 15 black balls. Two balls are drawn in succession without replacement. What is the probability that first is white and second is black?

Sol. Consider the events

A = Getting a white ball in the first draw

B = Getting a Black Ball in the second draw.

Required Probability = Probability of getting a white ball in the first draw and a black ball in the second draw = P(A and B) = P $(A \cap B)$ = P(A). P $\left(\frac{B}{A}\right)$

P(A) = $\frac{^{10}C_1}{^{25}C_1} = \frac{10}{25} = \frac{2}{5}$ and P$\left(\frac{B}{A}\right)$= Probability of getting a black ball in the second draw when a white ball has already been in first draw = $\frac{^{15}C_1}{^{24}C_1} = \frac{15}{24} = \frac{5}{8}$

$\Rightarrow$ Required probability = $P(A \cap B) = P(A) . P\left(\frac{B}{A}\right) =$

$\frac{2}{5} \times \frac{5}{8} = \frac{1}{4}$

Independent Events: Events are said to be independent, if the occurrence or non – occurrence of one does not affect the probability of the occurrence or non – occurrence of the other. Suppose a bag contains 6 white and 3 red balls. Consider the events A = drawing a white ball in the first draw and B = drawing a red ball in the second draw.

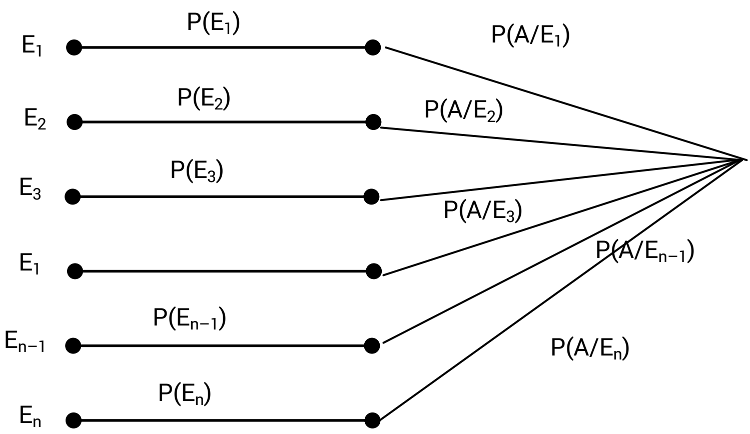
If the ball drawn in the first draw is not replaced back in the bag, then the events A and B are dependent events because $P(B)$ is increased or decreased according as the first draw results as a white or a red ball. If the ball drawn in the first drawn is replaced back in the bag, then A & B are independent events because $P(B)$ remains same whether we get a white ball or a red ball in the first draw i.e. $P(B) = P\left(\frac{B}{A}\right)$ and $P(B) = P\left(\frac{B}{A}\right)$.

From the above, it can be concluded that if A and B are two independent events associated with a random experiment, then $P\left(\frac{A}{B}\right) = P(A)$ and $P\left(\frac{B}{A}\right) = P(B)$.

2.2 Total Probability Law

Let S be the sample space and let E_1, E_2, _ _ _ _ _ E_n be n mutually exclusive and exhaustive events associated with a random experiment. If A is any event which occurs with E_1 or E_2 or _ _ _ _ _ _ _ _ E_n then P(A) = P(E_1) .

$$P\left(\frac{A}{E_1}\right) + P(E_2) \,.\, P\left(\frac{A}{E_2}\right) + \_\_\_\_\_\_ + P(E_n) \,.\, P\left(\frac{A}{E_n}\right) .$$



Ex.2 A bag contains 3 red and 4 black balls. A second bag contains 2 red and 3 black balls. One bag is selected at random and from the selected bag, one ball is drawn. Find the probability that the ball drawn is red.

Sol. A red ball can be drawn in two mutually exclusive ways

(I) Selecting bag I and then drawing a red ball from it.

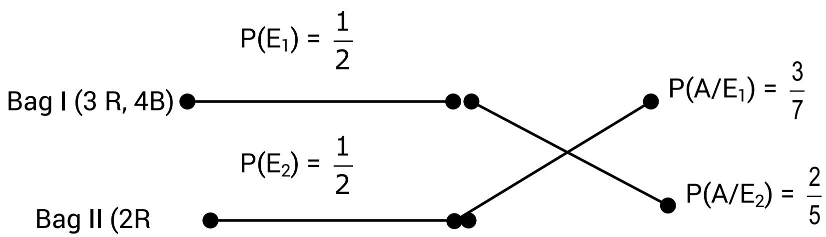
(II) Selecting bag II and then drawing a red ball from it.

Let E_1, E_2, and A denote the events defined as follows

E_1 = Selecting bag I

A = Drawing a red ball

E_2 = Selecting bag II



Since one of the two bags is selected randomly, therefore $P(E_1) = \frac{1}{2}$ and $P(E_2) = \frac{1}{2}$.

Now $P\left(\frac{A}{E_1}\right)$ = Probability of drawing a red ball when the first bag has been chosen = $\frac{3}{7}$ and $P\left(\frac{A}{E_2}\right)$ = Probability of drawing a red ball when the second bag has been selected

$= \frac{2}{5}$ [$\because$ The second bag contains 2 red and 3 black balls]

Using the law of total probability P(red ball) = $P(A)$

$= P(E_1)\ .\ P\left(\frac{A}{E_1}\right) + P(E_2)\ .\ P\left(\frac{A}{E_2}\right) = \frac{1}{2}\times\frac{3}{7}+\frac{1}{2}\times\frac{2}{5}=\frac{29}{70}$.

2.3 Baye's Rule & Miscellaneous Solved

Examples

Let S be the sample space and let E_1, E_2, _ _ _ _ _ _ E_n be n mutually exclusive and exhaustive events associated with a random experiment. If A is any event which occurs with E_1 or E_2, _ _ _ _ _ or E_n.

Then $P\frac{E_2}{A} \quad = \dfrac{P(E_1.P(A|E_1)}{\sum_{i=n}^{n} P(E_1).P(A|E_1)}$

Ex.3 Three bags contains 6 red, 4 black; 4 red, 6 black and 5 red & 5 black balls respectively. One of the bag is selected at random and a ball is drawn from it. If the ball drawn is red, find the probability that it is drawn from the first bag.

Sol. Let E_1, E_2, E_3 and A be the events defined as follows.

E_1 = First bag is chosen

E_2 = Second bag is chosen

E_3 = Third bag is chosen

A = Ball drawn is red

Since there are three bags and one of the bags is chosen at random, so $P(E_1) = P(E_2) = P(E_3) = \frac{1}{3}$.

If E_1 has already occurred, then first bag has been chosen which contains 6 red and 4 black balls. The probability of drawing a red ball from it is $\frac{6}{10}$. So

$P\left(\frac{A}{E_1}\right) = \frac{6}{10}$, similarly

$P\left(\frac{A}{E_2}\right) = \frac{4}{10}$, and $P\left(\frac{A}{E_3}\right) = \frac{5}{10}$. We are required to find

$P\left(\frac{E_1}{A}\right)$ i.e. given that the ball drawn is red, what is the probability that it is drawn from the first bag by Baye's rule $P\,\frac{E_1}{A}$

$$= \frac{P(E_1).P(A|E_1)}{P(E_1).P(A|E_1)+P(E_2).P(A|E_2)+P(E_3).P(A|E_2)}$$

$$= \frac{\frac{1}{3}\times\frac{6}{10}}{\frac{1}{3}\times\frac{6}{10}+\frac{1}{3}\times\frac{4}{10}+\frac{1}{3}\times\frac{5}{10}} = \frac{6}{15} = \frac{2}{5}\quad .$$

Ex.4 From a group of 2 boys and 3 girls, two children are selected. Find the sample space of the experiment.

Sol. Let the two boys be taken as B_1 and B_2 and the three girls be taken as G_1 G_2 G_3 out of 5 children,

two children can be selected in ${}^5C_2$= 10 ways. So the sample space consists of 10 points and is given by
S = {$B_1.B_2$, $B_1.G_1$, $B_1.G_2$, $B_1.G_3$, $B_2.G_1$, $B_2.G_2$, $B_2.G_3$, $G_1.G_2$, $G_1.G_3$, $G_2.G_3$}

Ex.5 Six dice are thrown simultaneously. Find the probability that all of them show the same face.

Sol. The total number of elementary events associated to the random experiment of throwing six dice is 6 × 6 × 6 × 6 × 6 × 6 = 6^6. All the dice show the same face means we are getting same number on all the six dice. The number of ways for which is ${}^6C_1$. Hence required probability = $\frac{{}^6C_1}{6^6} = \frac{1}{6^5}$

Ex.6 What is the probability that four S's come consecutively in the word 'MISSISSIPPI' written in all possible forms?

Sol. The total number of words which can be formed by permuting the 11 letters of the word 'MISSISSIPPI' is $\frac{11!}{4!4!2!}$. Since the sequence of 4 consecutive S's may start either from the first place or from the second place .... or from 8^{th} place. Therefore there are 8 possible ways in which 4 S's can come consecutively and in each case the remaining 7 letters MIIIPPI can be arranged in $\frac{7!}{4!2!}$ ways. Thus, the total number of ways in which 4 S′s can come consecutively is = 8. $\frac{7!}{4!2!}$.

Hence required probability = 8. $\frac{7!}{4!2!} \div \frac{11!}{4!4!2!}$

Ex.7 Two persons each make a single throw with a pair of dice. Show that the probability that both get equal sums is $\frac{73}{648}$.

Sol. The total number of outcomes when each of the two persons throw a pair of dice is $6^2 \times 6^2 = 1296$.

. The sum varies from 2 to 12. The number of ways to get the sum 2, 3, 4, ... 12 in a single throw of a pair of dice are

1–2–3–4–5–6–5–4–3–2–1

Now, if each person can throw the sum in a_i ways, then both of them will throw the sum I in a_i^2 ways. Therefore the number of ways in which the throws of two persons are equal is $2(1^2 + 2^2 + 3^2 + 4^2 + 5^2) + 6^2 = \frac{2\times5\times6\times11}{6} + 36 = 146$.

Hence required probability = $\frac{146}{1296} = \frac{73}{648}$.

Ex.8 An integer is chosen at random from the numbers ranging from 1 to 50. What is the probability that the integer chosen is a multiple of 2 or 3 or 10?

Sol. Out of 50 integers an integer can be chosen in $^{50}C_1$ ways. So exhaustive number of cases = $^{50}C_1$ = 50.

Now let us consider the following events

A = Getting a multiple of 2

B = Getting a multiple of 3

C = Getting a multiple of 10.

$A = \{2, 4, \ldots\ldots\ldots 50\}$

$B = \{3, 6, \ldots\ldots\ldots\ldots 48\}$

$C = \{10, 20, \ldots\ldots\ldots 50\}$

$A \cap B = \{6, 12, \ldots. 48\}$, $B \cap C = \{30\}$

$A \cap C = \{10, 20, \ldots\ldots 50\}$ and $A \cap B \cap C = \{30\}$

$P(A) = \frac{25}{50}$, $P(B) = \frac{16}{50}$, $P(C) = \frac{5}{50}$ $P(A \cap B) = \frac{8}{50}$,

$P(B \cap C) = \frac{1}{50}$, $P(C \cap A) = \frac{5}{50}$ $P(A \cap B \cap C) = \frac{1}{50}$.

Now required probability = $P(A \cup B \cup C)$

$= P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(A \cap C) + P(A \cap B \cap C) =$

$= \frac{25}{50} + \frac{16}{50} + \frac{5}{50} - \frac{8}{50} - \frac{1}{50} - \frac{5}{50} + \frac{1}{50}$

Ex.9 A dice is thrown twice and the sum of the numbers appearing is observed to be 6. What is the conditional probability that the number 4 has appeared at least once?

Sol. Consider the events

A = number 4 appears at least once.

B = the sum of the numbers appearing is 6.

Thus *A* = {(4, 1), (4, 2) (4, 3), (4, 4) (4, 5) (4, 6) (6, 4), (5, 4), (3, 4), (2, 4) (1, 4)} and B = {1, 5) (2, 4), (3, 3), (4, 2) (5, 1)}. The required probability = $\frac{n(A\cap B)}{n(B)}=\frac{2}{5}$.

Ex.10. Two integers are selected at random from integers 1 through 11. If the sum is even, find the probability that both the numbers are odd?

Sol. Out of integers from 1 to 11, 5 are even integers and 6 are odd integers. Consider the following events.

A = Both the numbers chosen are odd

B = The sum of the numbers chosen is even.

Since the sum of two integers is even if either both the integers are even or both are odd, therefore

P(A) = $\frac{^6C_2}{^{11}C_2}$

P(B) = $\frac{^6C_2+^5C_2}{^{11}C_2}$

P(A $\cap$B) = $\frac{^6C_2}{^{11}C_2}$

Now required probability $=P\left(\frac{A}{B}\right)=\frac{P(A\cap B)}{P(B)}$

$$= \frac{{}^{6}C_2/{}^{11}C_{22}}{\dfrac{{}^{6}C_2+{}^{5}C_2}{{}^{11}C_2}} = \frac{{}^{6}C_2}{{}^{6}C_2+{}^{5}C_2}$$

Ex.11 A fair coin is tossed repeatedly. If tail appears on first four tosses, find the probability of head appearing on fifth toss.

Sol. Since the trials are independent, so the probability that head appears on the fifth toss does not depend upon previous results of the toss. Hence required probability = $\frac{1}{2}$

Ex.12. There are four machines and it is known that exactly two of them are faulty. They are tested one by one, in a random order till both the faulty machines are identified. Find the probability that only two tests are required?

Sol. The total number of ways in which two machines can be chosen out of four machines is ${}^{4}C_2$ = 6. If only two tests are required to identify faulty machines, then in first two tests faulty machines are identified. This can be done in one way only.

So, the favorable number of ways = 1. Hence, required probability = $\frac{1}{6}$.

Ex.13 Four numbers are multiplied together. Find the probability that the product will be divisible by 5 or 10.

Sol. The divisibility of the product of four numbers depend upon the value of the last digit of each number. The last digit of a number can be any one of the ten digits 0, 1, 2, ….. 9. So the total number of ways of selecting last digits of four numbers is $10 \times 10 \times 10 \times 10 = 10^4$. If the product of the 4 numbers is not divisible by 5 or 10, then the number of choices for the last digits of each number is 8 (excluding 0 or 5). So favorable number of ways = 8^4.

The probability that the product is not divisible by 5 or 10

$= \frac{8^4}{10^4}$.

Hence required probability = $1 - \frac{8^4}{10^4}$

Ex.14 Find the probability of the first box to contain three balls when 12 balls are distributed among three boxes.

Sol. Since each ball can be put into any one of the three boxes. So the total number of ways in which 12 balls can be put into three boxes is 3^{12}.

Out of 12 boxes, 3 balls can be chosen in ${}^{12}C_3$ ways. Now remaining 9 balls can be put the remaining 2 boxes in 2^9 ways. So the total number of ways in which 3 balls are put in the first box and the remaining in other two boxes is ${}^{12}C_3 \times 2^9$. Hence required probability = $\frac{{}^{12}C_3 \times 2^9}{3^{12}}$

Ex.15 There is a five volume dictionary among 50 books arranged on a shelf in a random order. If the volumes are not necessarily kept side by side, find the probability that the volumes will occur in the increasing order from left to right.

Sol. The total numbers of ways of arranging 50 books in shelf is ${}^{50}P_{50}$ = 50!. Out of 50 places, 5 places for the five–volume dictionary can be chosen in ${}^{50}C_5$ ways. In the remaining 45 places the remaining 45 books can be arranged in ${}^{45}P_{45}$ = 45! ways. In the five places five volumes of dictionary can be arranged in increasing order in one way only. So favorable number of ways = ${}^{50}C_5 \times 45!$.

Hence required probability = $\frac{{}^{50}C_5 \times 45!}{50!} = \frac{1}{120}$.

Ex.16 If ten objects are distributed at random among ten persons. Find the probability that at least one of them will not get anything?

Sol. Since each object can be given to any one of ten persons. So, ten objects can be distributed among 10 persons in 10^{10} ways. Thus the total number of ways = 10^{10} ways.

The number of ways of distribution in which one gets only one thing is 10! So the number of ways of distribution in which at least one of the does not get anything is $10^{10} - 10!$ Hence required probability

$= \frac{10^{10} - 10!}{10^{10}}$