5. 哈尔变换可以用矩阵的形式表示为：

$$T = HFH^T$$

其中，F是一个$N \times N$的图像矩阵，H是$N \times N$变换矩阵，T是$N \times N$变换结果。对于哈尔变换，变换矩阵H包含基函数$h_k(z)$，它们定义在连续闭区间$z \in [0,1], k = 0,1,2 \cdots N-1$，其中$N = 2^n$。为了生成矩阵，定义整数k，即$k = 2^p + q - 1$（这里$0 \leq p \leq n-1$，当 p=0 时 q=0，或 1；当$p \neq 0$时，$1 \leq q \leq 2^p$）。可得哈尔基函数为：

$$h_0(z) = h_{00}(z) = \frac{1}{\sqrt{N}}, z \in [0,1]$$

且

$$h_k(z) = h_{pq}(z) = \frac{1}{\sqrt{N}} \begin{cases} 2^{\frac{p}{2}}, & (q-1)/2^p \leq z < (q-0.5)/2^p \\ -2^{\frac{p}{2}}, & (q-0.5)/2^p \leq z < q/2^p \\ 0, & 其它，z \in [0,1] \end{cases}$$

N×N 哈尔变换矩阵的第i行包含了元素$h_i(z)$，其中$z = \frac{0}{N}, \frac{1}{N}, \cdots (N-1)/N$。计算当$N = 16$时的$H_{16}$矩阵。

得到表格：

k	p	q	$(q-1)/2^p$	$(q-0.5)/2^p$	$q/2^p$
0	0	0	−1	−0.5	0
1	0	1	0	0.5	1
2	1	1	0	0.5/2	1/2
3	1	2	1/2	1.5/2	2/2
4	2	1	0	0.5/4	1/4
5	2	2	1/4	1.5/4	2/4
6	2	3	2/4	2.5/4	3/4
7	2	4	3/4	3.5/4	1
8	3	1	0	0.5/8	1/8
9	3	2	1/8	1.5/8	2/8
10	3	3	2/8	2.5/8	3/8
11	3	4	3/8	3.5/8	4/8
12	3	5	4/8	4.5/8	5/8
13	3	6	5/8	5.5/8	6/8
14	3	7	6/8	6.5/8	7/8
15	3	8	7/8	7.5/8	1

第 k 行元素为 $h_k(z)$：

$$h_k(z) = h_{pq}(z) = \frac{1}{\sqrt{N}}\begin{cases} 2^{p/2}, & \dfrac{q-1}{2^p} \le z < \dfrac{q-0.5}{2^p} \\ -2^{p/2}, & \dfrac{q-0.5}{2^p} \le z < \dfrac{q}{2^p} \\ 0, & otherwise \end{cases}$$

其中在本题中：

$$z = list\left(range\left(0,1,\frac{1}{16}\right)\right)$$

得到 16×16 的变换矩阵 $\boldsymbol{H_{16}}$ 为：（矩阵应用 LaTex 书写截图所得）

$$\boldsymbol{H_{16}} = \frac{\mathbf{1}}{\sqrt{\mathbf{16}}}\begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \\
\sqrt{2} & \sqrt{2} & \sqrt{2} & \sqrt{2} & -\sqrt{2} & -\sqrt{2} & -\sqrt{2} & -\sqrt{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \sqrt{2} & \sqrt{2} & \sqrt{2} & \sqrt{2} & -\sqrt{2} & -\sqrt{2} & -\sqrt{2} & -\sqrt{2} \\
2 & 2 & -2 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 2 & 2 & -2 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 2 & -2 & -2 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 2 & -2 & -2 \\
2\sqrt{2} & -2\sqrt{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 2\sqrt{2} & -2\sqrt{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 2\sqrt{2} & -2\sqrt{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 2\sqrt{2} & -2\sqrt{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2\sqrt{2} & -2\sqrt{2} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2\sqrt{2} & -2\sqrt{2} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2\sqrt{2} & -2\sqrt{2} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2\sqrt{2} & -2\sqrt{2}
\end{bmatrix}$$

矩阵源码：

```
\begin{equation*}
	\setlength{\arraycolsep}{1.5pt}
	\addtocounter{MaxMatrixCols}{16}
	\begin{bmatrix}
	 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1\\
	 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1\\
	 \sqrt{2} & \sqrt{2} & \sqrt{2} & \sqrt{2} & -\sqrt{2} & -\sqrt{2} & -\sqrt{2} & -\sqrt{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
	 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \sqrt{2} & \sqrt{2} & \sqrt{2} & \sqrt{2} & -\sqrt{2} & -\sqrt{2} & -\sqrt{2} & -\sqrt{2}\\
	 2 & 2 & -2 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
	 0 & 0 & 0 & 0 & 2 & 2 & -2 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
	 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 2 & -2 & -2 & 0 & 0 & 0 & 0\\
	 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 2 & -2 & -2 \\
	 2\sqrt{2} & -2\sqrt{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
	 0 & 0 & 2\sqrt{2} & -2\sqrt{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
	 0 & 0 & 0 & 0 & 2\sqrt{2} & -2\sqrt{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
	 0 & 0 & 0 & 0 & 0 & 0 & 2\sqrt{2} & -2\sqrt{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
	 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2\sqrt{2} & -2\sqrt{2} & 0 & 0 & 0 & 0 & 0 & 0\\
	 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2\sqrt{2} & -2\sqrt{2} & 0 & 0 & 0 & 0\\
	 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2\sqrt{2} & -2\sqrt{2} & 0 & 0\\
	   0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2\sqrt{2} & -2\sqrt{2}\\
	 \end{bmatrix}
\end{equation*}
```