# 人工智能

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# Knowledge 3

# First-order Logic:
# syntax and semantics

# First-order logic

- Whereas propositional logic assumes world contains facts, first-order logic(like natural language) assumes the world contains
- **Objects:** people, house, numbers, theories, Ronald McDonald, colors, baseball games, wars, centuries…..
- **Relations:** red, round, bogus, prime, multistoried…,
  brother of, bigger than, inside, part of, has color, occurred after, owns, comes between,…
- **Functions:** father of, best friend, third inning of, one more than, end of…

# Syntax of FOL: Basic elements

| | |
|---|---|
| **Constants** | ***KingJohn, 2, UCB,…*** |
| **Predicates** | **Brother, >,…** |
| **Functions** | ***Sqrt, LeftLegOf,…*** |
| **Variables** | **x, y, a, b,…** |
| **Connectives** | $\wedge \quad \vee \quad \neg \quad \Rightarrow \quad \Leftrightarrow$ |
| **Equality** | **=** |
| **Quantifiers** | $\forall \quad \exists$ |

# Atomic sentences

Atomic sentence = $predicate(term_1, \dots, term_n)$
or $term_1 = term_2$

Term = $function(term_1, \dots, term_n)$
or *constant* or *variable*

E.g., $Brother(KingJohn, RichardTheLionheart)$
$> (Length(LeftLegOf(Richard)), Length(LeftLegOf(KingJohn)))$

# Complex sentences

Complex sentences are made from atomic sentences using connectives

$$\neg \boldsymbol{S},\ \ \boldsymbol{S_1} \wedge \boldsymbol{S_2},\ \ \boldsymbol{S_1} \vee \boldsymbol{S_2},\ \ \boldsymbol{S_1} \Rightarrow \boldsymbol{S_2},\ \ \boldsymbol{S_1} \Leftrightarrow \boldsymbol{S_2}$$

**E.g.,** $Slibing(KingJohn, Richard) \Rightarrow Slibing(Richard, KingJohn)$

$> (1,2) \vee \leq (1,2)$

$> (1,2) \wedge \neg > (1,2)$

# Truth in first-order logic

- Sentences are true with respect to a model and an interpretation
- Model contains ≥ 1 objects( domain elements) and relations among them

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- Interpretation specifies referents for

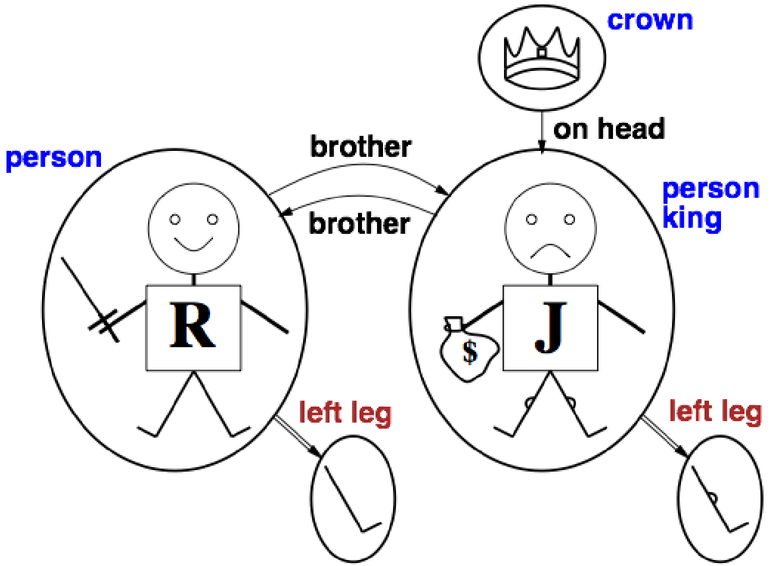
  constant symbols ⟶ objects

  predicate symbols ⟶ relations

  function symbols ⟶ function

- An atomic sentence $predicate(term_1, \ldots, term_n)$ is true iff the objects referred to by $term_1, \ldots, term_n$ are in the relation referred to by $predicate$

# Models for FOL: Example




- Consider the interpretation in which
- $Richard \longrightarrow$ Richard the Lionheart
- $John \longrightarrow$ the evil King John
- $Brother \longrightarrow$ the brotherhood relation

- Under this interpretation, $Brother(Richard, John)$ is true just in case Richard the Lionheart and the evil King John are in the brotherhood relation in the model

# Models for FOL: Lots!

- Entailment in propositional logic can be computed by enumerating models
- We can enumerate the FOL models for a given KB vocabulary:
- For each number of domain elements $n$ from $1$ to $\infty$

    For each $k$-ary predicate $P_k$ in the vocabulary

        For each possible $k$-ary relation on $n$ objects

            For each constant symbol $C$ in the vocabulary

                For each choice of referent for $C$ from $n$ objects …

- Computing entailment by enumerating FOL models is not easy!

# Universal quantification

- $\forall < variables >< sentence >$
- Everyone at Berkeley is smart:

  $\forall x \ \ At(x, Berkeley) \Rightarrow Smart(x)$
- $\forall x \ \ P$ is true in a model *m* iff *P* is true with *x* being each possible object in the model
- **Roughly** speaking, equivalent to the conjunction of instantiations of P

$$\begin{aligned} & \left(At(KingJohn, Berkeley) \Rightarrow Smart(KingJohn)\right) \\ \wedge & \left(At(Richard, Berkeley) \Rightarrow Smart(Richard)\right) \\ \wedge & \left(At(Berkeley, Berkeley) \Rightarrow Smart(Berkeley)\right) \\ \wedge & \ldots\ldots \end{aligned}$$

# A common mistake to avoid

- Typically, $\Rightarrow$ is the main connective with $\forall$
- Common mistake: using $\wedge$ as the main connective with $\forall$

$$\forall x \quad At(x, Berkeley) \wedge Smart(x)$$

- Means “Everyone is at Berkeley and everyone is smart”

# Existential quantification

- $\exists < variables >< sentence >$
- Someone at Stanford is smart:

  $\exists x \;\; At(x, Stanford) \land Smart(x)$
- $\exists x \;\; P$ is true in a model *m* iff *P* is true with *x* being some possible object in the model
- **Roughly** speaking, equivalent to the disjunction of instantiations of P

$$
\begin{aligned}
&\boldsymbol{\big(At(KingJohn, Stanford) \land Smart(KingJohn)\big)} \\
\lor\ &\boldsymbol{\big(At(Richard, Stanford) \land Smart(Richard)\big)} \\
\lor\ &\boldsymbol{(At(Berkeley, Stanford) \land Smart(Berkeley))} \\
\lor\ &\boldsymbol{\ldots..}
\end{aligned}
$$

# Another common mistake to avoid

- Typically, $\wedge$ is the main connective with $\exists$
- Common mistake: using $\Rightarrow$ as the main connective with $\exists$:

$$\exists x \;\; At(x, Stanford) \Rightarrow Smart(x)$$

- Is true if there is anyone who is not at Stanford!

# Properties of quantifiers

- $\forall x\ \forall y$ is the same as $\forall y\ \forall x$ (why??)
- $\exists x\ \exists y$ is the same as $\exists y\ \exists x$ (why??)
- $\exists x\ \forall y$ is not the same as $\forall y\ \exists x$

- $\exists x\ \forall y\ Loves(x, y)$
  "There is a person who loves everyone in the world"

- $\forall y\ \exists x\ Loves(x, y)$
  "Everyone in the world is loved by at least one person"

- Quantifier duality: each can be expressed using the other

$$\forall x\ Likes(x, IceCream) \qquad \neg\exists x\ \neg Likes(x, IceCream)$$

$$\exists x\ Likes(x, Broccoli) \qquad \neg\forall x\ \neg Likes(x, Broccoli)$$

# Fun with sentences

- Brothers are siblings
- $\forall x, y \;\; Brother(x, y) \;\Rightarrow Sibling(x, y)$.
- “Sibling” is symmetric
- $\forall x, y \;\; Sibling(x, y) \;\Leftrightarrow Sibling(y, x)$.
- One’s mother is one’s female parent
- $\forall x, y \;\; Mother(x, y) \;\Leftrightarrow \big(Female(x) \wedge Parent(x, y)\big)$.
- A first cousin is a child of a parent’s sibling
- $\forall x, y \;\; Firstcousin(x, y) \;\Leftrightarrow \exists p, ps\; parent(p, x) \wedge Sibling(ps, p) \wedge Parent(ps, y)$.

# Fun with sentences

- 不到长城非好汉。

- 到了长城就是好汉。

- 理发师只给那些不给自己理发的人理发
  - 理发师悖论

# Equality

- $term_1 = term_2$ is true under a given interpretation if and only if $term_1$ and $term_2$ refer to the same object
- E.g., $1 = 2$ and $\forall x \;\times\left(Sqrt(x), Sqrt(x)\right) = x$ are satisfiable

  $2 = 2$ is valid
- E.g., definition of (full) $Sibling$ in terms of $Parent$:

$$\forall x, y \;\; Sibling(x, y) \Leftrightarrow [\neg(x = y) \wedge \exists m, f \; \neg(m = f) \wedge Parent(m, x) \wedge Parent(f, x) \wedge Parent(m, y) \wedge Parent(f, y)]$$

# Back to the wumpus world again

- Define adjacency:

$$\forall x, y, a, b\ Adjacent([x, y], [a, b]) \Leftrightarrow$$

$$\big(x = a \land (y = b - 1 \lor y = b + 1)\big) \lor \big(y = b \land (x = a - 1 \lor x = a + 1)\big).$$

- Location predicator, x is at square s at time t:

$$\forall t\ At(WUMPUS, [2,2], t).$$

$$\forall x, s_1, s_2, t\ \ At(x, s_1, t) \land At(x, s_2, t) \Rightarrow s_1 = s_2$$

- Define property for squares:

$$\forall s, t\ At(AGENT, s, t) \land Breeze(t) \Rightarrow Breezy(s).$$

$$\forall s, t\ At(PIT, s, t) \Rightarrow Pit(s).$$

$$\forall s, t\ At(WUMPUS, s, t) \Rightarrow Wumpus(s).$$

- Rules of the Wumpus world can be defined.

$$\forall s\ Breezy(s) \Leftrightarrow \exists r\ \ Adjacent(r, s) \land Pit(r)$$

$$\forall t\ HaveArrow(t + 1) \Leftrightarrow \big(HaveArrow(t) \land \neg Action(shoot, t)\big).$$

# Short Summary

- First-order logic:
  - Objects and relations are semantic primitives
  - Syntax: constants, functions, predicates, equality, quantifiers

- Increased expressive power: sufficient to define Wumpus world

# 研究形式逻辑的目的是什么？

- 知识表示
  - 将一组知识形式化为符号
- 知识推理
  - 通过形式推演，自动推出结论
    - 可靠
    - 完备

# Homework

**8.15** Explain what is wrong with the following proposed definition of the set membership predicate $\in$ :

$$\forall x, s \;\; x \in \{x|s\}$$
$$\forall x, s \;\; x \in s \Rightarrow \forall y \;\; x \in \{y|s\} \; .$$

**8.20** Arithmetic assertions can be written in first-order logic with the predicate symbol $<$, the function symbols $+$ and $\times$, and the constant symbols 0 and 1. Additional predicates can also be defined with biconditionals.

**a**. Represent the property "$x$ is an even number."

**b**. Represent the property "$x$ is prime."

**c**. Goldbach's conjecture is the conjecture (unproven as yet) that every even number is equal to the sum of two primes. Represent this conjecture as a logical sentence.