黄磊　计 702　　2022E8013282156

2. 根据书中对傅立叶变换的定义，证明课本 165 页上有关傅立叶变换的平移性质。

平移	$f(x,y)e^{j2\pi(u_0x/M+v_0y/N)} \Leftrightarrow F(u-u_0, v-v_0)$ $f(x-x_0, y-y_0) \Leftrightarrow F(u,v)e^{-j2\pi(ux_0/M+vy_0/N)}$ 当 $x_0 = u_0 = M/2$ 和 $y_0 = v_0 = N/2$ 时， $f(x,y)(-1)^{x+y} \Leftrightarrow F(u-M/2, v-N/2)$ $f(x-M/2, y-N/2) \Leftrightarrow F(u,v)(-1)^{u+v}$

1) **证明$\boldsymbol{f(x,y)e^{j2\pi\left(\frac{u_0x}{M}+\frac{v_0y}{N}\right)} \Leftrightarrow F(u-u_0, v-v_0)}$**

由离散形式的傅立叶反变换：

$$\mathcal{F}^{-1}[F(u,v)] = f(x,y) = \sum_{u=0}^{M-1}\sum_{v=0}^{N-1} F(u,v)e^{j2\pi\left(\frac{ux}{M}+\frac{vy}{M}\right)}$$

有：

$$\mathcal{F}^{-1}[F(u-u_0,v-v_0)] = \sum_{u=0}^{M-1}\sum_{v=0}^{N-1} F(u-u_0,v-v_0)e^{j2\pi\left(\frac{(u-u_0)x+u_0x}{M}+\frac{(v-v_0)y+v_0y}{N}\right)}$$

$$= \sum_{u=0}^{M-1}\sum_{v=0}^{N-1} F(u-u_0,v-v_0)e^{j2\pi\left(\frac{(u-u_0)x}{M}+\frac{(v-v_0)y}{N}\right)}\boxed{e^{j2\pi\left(\frac{u_0x}{M}+\frac{v_0y}{N}\right)}}$$

与 u, v 无关

$$= f(x,y)e^{j2\pi\left(\frac{u_0x}{M}+\frac{v_0y}{N}\right)}$$

因此 $\mathcal{F}\left[f(x,y)e^{j2\pi\left(\frac{u_0x}{M}+\frac{v_0y}{N}\right)}\right] = F(u-u_0, v-v_0)$，证毕

2) **证明$\boldsymbol{f(x-x_0, y-y_0) \Leftrightarrow F(u,v)e^{-j2\pi\left(\frac{ux_0}{M}+\frac{vy_0}{N}\right)}}$**

由离散形式的傅立叶正变换：

$$\mathcal{F}[f(x,y)] = F(u,v) = \frac{1}{MN}\sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f(x,y)e^{-j2\pi\left(\frac{ux}{M}+\frac{vy}{N}\right)}$$

有：

$$\mathcal{F}[f(x-x_0,y-y_0)] = \frac{1}{MN}\sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f(x-x_0,y-y_0)e^{-j2\pi\left(\frac{u(x-x_0)+x_0u}{M}+\frac{v(y-y_0)+vy_0}{N}\right)}$$

$$= \sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f(x-x_0,y-y_0)e^{-j2\pi\left(\frac{u(x-x_0)}{M}+\frac{v(y-y_0))}{N}\right)}\boxed{e^{-j2\pi\left(\frac{ux_0}{M}+\frac{vy_0}{N}\right)}}$$

与 x, y 无关

$$= F(u,v)e^{-j2\pi\left(\frac{ux_0}{M}+\frac{vy_0}{N}\right)}$$

因此有$\mathcal{F}[f(x-x_0,y-y_0)] = F(u,v)e^{-j2\pi\left(\frac{ux_0}{M}+\frac{vy_0}{N}\right)}$，证毕。

3） **当 $\boldsymbol{x_0 = u_0 = \frac{M}{2}, y_0 = v_0 = \frac{N}{2}}$时候，证明：**

$$\boldsymbol{f(x,y)(-1)^{x+y} \Leftrightarrow F\left(u-\frac{M}{2}, v-\frac{N}{2}\right)}$$

由题（1），有：

$$\mathcal{F}\left[f(x,y)e^{j2\pi\left(\frac{u_0 x}{M}+\frac{v_0 y}{N}\right)}\right] = F(u-u_0, v-v_0)$$

代入 $x_0 = u_0 = \frac{M}{2}, y_0 = v_0 = \frac{N}{2}$ 有：

$$\mathcal{F}\left[f(x,y)e^{j2\pi\left(\frac{x}{2}+\frac{y}{2}\right)}\right] = F\left(u-\frac{M}{2}, v-\frac{N}{2}\right)$$

$$\mathcal{F}\left[f(x,y)e^{j\pi(x+y)}\right] = F\left(u-\frac{M}{2}, v-\frac{N}{2}\right)$$

而由相角公式可知：

$$e^{jx} = \cos x + j\sin x$$

因此:

$$e^{j\pi} = \cos\pi + j\sin\pi = -1$$

因此有:

$$\mathcal{F}\left[f(x,y)e^{j\pi(x+y)}\right] = \mathcal{F}\left[f(x,y)(-1)^{x+y}\right] = F\left(u-\frac{M}{2}, v-\frac{N}{2}\right)$$

证毕。

4） **当 $\boldsymbol{x_0 = u_0 = \frac{M}{2}, y_0 = v_0 = \frac{N}{2}}$时候，证明：**

$$\boldsymbol{f\left(x-\frac{M}{2}, y-\frac{N}{2}\right) \Leftrightarrow F(u,v)(-1)^{u+v}}$$

由题（2）有：

$$\mathcal{F}[f(x-x_0, y-y_0)] = F(u,v)e^{-j2\pi\left(\frac{ux_0}{M}+\frac{vy_0}{N}\right)}$$

代入 $x_0 = u_0 = \frac{M}{2}, y_0 = v_0 = \frac{N}{2}$ 有：

$$\mathcal{F}\left[f\left(x-\frac{M}{2}, y-\frac{N}{2}\right)\right] = F(u,v)e^{-j\pi(u+v)}$$

而由相角公式可知：

$$e^{jx} = \cos x + j\sin x$$

因此:

$$e^{-j\pi} = \cos(-\pi) + j\sin(-\pi) = -1$$

因此有：

$$\mathcal{F}\left[f\left(x-\frac{M}{2}, y-\frac{N}{2}\right)\right] = F(u,v)(-1)^{u+v}$$

证毕。