黄磊　2022E8013282156　计 702

1、r,g,b 是 RGB 彩色空间沿 R,G,B 轴的单位向量，定义向量 $\mathrm{u}=\frac{\partial R}{\partial x}r+\frac{\partial G}{\partial x}g+\frac{\partial B}{\partial x}b$ 和 $\mathrm{v}=\frac{\partial R}{\partial y}r+\frac{\partial G}{\partial y}g+\frac{\partial B}{\partial y}b$，$g_{xx}, g_{yy}, g_{xy}$ 定义为这些向量的点乘：

$$g_{xx}=\mathrm{u}\cdot\mathrm{u}=\mathrm{u}^T\mathrm{u}=\left|\frac{\partial R}{\partial x}\right|^2+\left|\frac{\partial G}{\partial x}\right|^2+\left|\frac{\partial B}{\partial x}\right|^2$$

$$g_{yy}=\mathrm{v}\cdot\mathrm{v}=\mathrm{v}^T\mathrm{v}=\left|\frac{\partial R}{\partial y}\right|^2+\left|\frac{\partial G}{\partial y}\right|^2+\left|\frac{\partial B}{\partial y}\right|^2$$

$$g_{xy}=\mathrm{u}\cdot\mathrm{v}=\mathrm{u}^T\mathrm{v}=\frac{\partial R}{\partial x}\frac{\partial R}{\partial y}+\frac{\partial G}{\partial x}\frac{\partial G}{\partial y}+\frac{\partial B}{\partial x}\frac{\partial B}{\partial y}$$

推导出最大变换率方向 θ 和 (x,y) 点在 θ 方向上变化率的值 $\mathrm{F}(\theta)$

梯度即为某点处取得最大方向的导数，假设该点为 $P(x,y)$，图像函数为 $f(x,y)$，则有：

$$\frac{\partial f}{\partial x}=u;\quad \frac{\partial f}{\partial y}=v$$

函数在 $P(x,y)$ 点处的沿 $\vec{l}$ 的方向导数为：

$$grad\frac{\partial f}{\partial \vec{l}}|_{P(x,y)}=f_x'(P)\cos\alpha+f_y'(P)\cos\beta$$

在 $x-y$ 平面上，有 $\alpha+\beta=\frac{\pi}{2}$，因此有：

$$\begin{aligned}grad\frac{\partial f}{\partial \vec{l}}|_{P(x,y)}&=f_x'(P)\cos\alpha+f_y'(P)\cos\beta\\&=f_x'(P)\cos\alpha+f_y'(P)\sin\alpha\\&=u\cdot\cos\alpha+v\cdot\sin\alpha\end{aligned}$$

因此，问题变为：

$$\underset{\alpha}{\operatorname{argmax}}\ |u \cdot \cos\alpha + v \cdot \sin\alpha|^2$$

则有：

$$\begin{aligned} h(\alpha) &= \ |u \cdot \cos\alpha + v \cdot \sin\alpha|^2 \\ &= u^2\cos^2\alpha + v^2\sin^2\alpha + uv\sin 2\alpha \\ &= g_{xx}\frac{1+\cos 2\alpha}{2} + g_{xy}\sin 2\alpha + g_{yy}\frac{1-\cos 2\alpha}{2} \\ &= \frac{1}{2}\left(g_{xx}+g_{yy}\right) + \frac{\cos 2\alpha}{2}\ \left(g_{xx}-g_{yy}\right) + g_{xy}\sin 2\alpha \end{aligned}$$

则有：

$$\underset{\alpha}{\operatorname{argmax}}\, h(\alpha) = \alpha_0|_{h'(\alpha_0)=0}$$

求导取 0:

$$h'(\alpha) = 2g_{xy}\cos 2\alpha - \sin 2\alpha\ \left(g_{xx}-g_{yy}\right)$$

则有：

$$\alpha_0 = \frac{1}{2}\tan^{-1}\frac{2g_{xy}}{g_{xx}-g_{yy}}$$

因此最大变换率方向：

$$\theta = \alpha_0 = \frac{1}{2}\tan^{-1}\frac{2g_{xy}}{g_{xx}-g_{yy}}$$

对应的变化率最值：

$$F(\theta) = h(\alpha_0) = \frac{1}{2}\left(g_{xx}+g_{yy}\right) + \frac{\cos 2\theta}{2}\ \left(g_{xx}-g_{yy}\right) + g_{xy}\sin 2\theta$$