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1、复习理解课本中最佳陷波滤波器进行图像恢复的过程，请推导出 w(x,y)最优解的计算过程，即从公式

$$\frac{\partial\sigma^2(x,y)}{\partial\omega(x,y)}=0$$

到

$$\omega(x,y)=\frac{\overline{\eta(x,y)g(x,y)}-\overline{g}(x,y)\overline{\eta}(x,y)}{\overline{\eta^2}(x,y)-\overline{\eta}^2(x,y)}$$

的推导过程。

加上干扰后的图像为 $g(x)$，噪声为 $\eta(x,y)$，调制函数为 $w(x,y)$，因此，得到原图的估计为：

$$\hat{f}(x,y)=g(x,y)-w(x,y)\eta(x,y)$$

优化方法为，估计值 $\hat{f}(x,y)$ 在每一点 (x,y) 的指定邻域内的方差最小。

设定邻域尺寸为 $(2a+1)\times(2b+1)$，领域内均值为 $\bar{\hat{f}}(x,y)$，局部方差估计为：

$$\sigma^2(x,y)=\frac{1}{(2a+1)(2b+1)}\sum_{s=-a}^{a}\sum_{t=-b}^{b}\left[\hat{f}(x+s,y+t)-\bar{\hat{f}}(x,y)\right]^2$$

带入估计式为：

$$\sigma^2(x,y)=\frac{1}{(2a+1)(2b+1)}\sum_{s=-a}^{a}\sum_{t=-b}^{b}\left[\hat{f}(x+s,y+t)-\bar{\hat{f}}(x,y)\right]^2$$

$$=\frac{1}{(2a+1)(2b+1)}\sum_{s=-a}^{a}\sum_{t=-b}^{b}\Big[g(x+s,y+t)$$

$$-w(x+s,y+t)\eta(x+s,y+t)-\left(\bar{g}(x,y)-\overline{w(x,y)\eta(x,y)}\right)\Big]^2$$

假设调制函数在邻域内保持不变，则有近似式：

$$w(x+s,y+t)=w(x,y)$$

$$\overline{w(x,y)\eta(x,y)}=w(x,y)\,\bar{\eta}(x,y)$$

则估计式变为：

$$\sigma^2(x,y)=\frac{1}{(2a+1)(2b+1)}\sum_{s=-a}^{a}\sum_{t=-b}^{b}\Big[g(x+s,y+t)-w(x,y)\eta(x+s,y+t)$$

$$-\left(\bar{g}(x,y)-w(x,y)\,\bar{\eta}(x,y)\right)\Big]^2$$

$$=\frac{1}{(2a+1)(2b+1)}\sum_{s=-a}^{a}\sum_{t=-b}^{b}[g(x+s,y+t)-\bar{g}(x,y)$$

$$-w(x,y)\eta(x+s,y+t)+w(x,y)\,\bar{\eta}(x,y)]^2$$

求其最小化，因此对 $w(x,y)$ 求偏导数：

$$\frac{\partial\sigma^2(x,y)}{\partial w(x,y)}=\frac{1}{(2a+1)(2b+1)}\sum_{s=-a}^{a}\sum_{t=-b}^{b}2[g(x+s,y+t)-\bar{g}(x,y)$$

$$-w(x,y)\eta(x+s,y+t)+w(x,y)\,\bar{\eta}(x,y)]\cdot[\bar{\eta}(x,y)-\eta(x+s,y+t)]$$

$$=\frac{2}{(2a+1)(2b+1)}\left\{\sum_{s=-a}^{a}\sum_{t=-b}^{b}[g(x+s,y+t)-\bar{g}(x,y)][\bar{\eta}(x,y)\right.$$

$$\left.-\eta(x+s,y+t)]+\sum_{s=-a}^{a}\sum_{t=-b}^{b}w(x,y)[\bar{\eta}(x,y)-\eta(x+s,y+t)]^2\right\}$$

$$=\frac{2}{(2a+1)(2b+1)}\left\{\sum_{s=-a}^{a}\sum_{t=-b}^{b}[g(x+s,y+t)-\bar{g}(x,y)][\bar{\eta}(x,y)\right.$$

$$\left.-\eta(x+s,y+t)]+w(x,y)\sum_{s=-a}^{a}\sum_{t=-b}^{b}[\bar{\eta}(x,y)-\eta(x+s,y+t)]^2\right\}$$

令上式等于 0，得到：

$$w(x,y)=\frac{\sum_{s=-a}^{a}\ \sum_{t=-b}^{b}[\bar{g}(x,y)-g(x+s,y+t)][\bar{\eta}(x,y)-\eta(x+s,y+t)]}{\sum_{s=-a}^{a}\ \sum_{t=-b}^{b}[\bar{\eta}(x,y)-\eta(x+s,y+t)]^2}$$

$$=\frac{\sum\sum(\bar{g}\bar{\eta}+g\eta-g\bar{\eta}-\eta\bar{g})}{\sum\sum\bar{\eta}^2+\sum\sum\eta^2-2\sum\sum\eta\bar{\eta}}$$

易知：

$$\sum_{s=-a}^{a}\sum_{t=-b}^{b}g(x+s,y+t)=\sum_{s=-a}^{a}\sum_{t=-b}^{b}\bar{g}(x,y)=$$

因此分子：

$$\begin{aligned}&\sum\sum(\bar{g}\bar{\eta}+g\eta-g\bar{\eta}-\eta\bar{g})\\&=\sum\sum\bar{\eta}\cdot\left(\sum\sum(\bar{g}-g)\right)+\sum\sum(g\eta-\eta\bar{g})\\&=\sum\sum(g\eta-\eta\bar{g})\\&=(2a+1)(2b+1)(\,\overline{g\eta}-\bar{g}\bar{\eta})\end{aligned}$$

而分母：

$$\begin{aligned}&\sum\sum\bar{\eta}^2+\sum\sum\eta^2-2\sum\sum\eta\bar{\eta}\\&=\sum\sum\eta^2+(\sum\sum\bar{\eta}^2-2\sum\sum\eta\bar{\eta})\\&=\sum\sum\eta^2+\sum\sum(\bar{\eta}^2-2\eta\bar{\eta})\\&=(2a+1)(2b+1)\left(\overline{\eta^2}-\bar{\eta}^2\right)\end{aligned}$$

因此得到：

$$\begin{aligned}w(x,y)&=\frac{\sum\sum(\bar{g}\bar{\eta}+g\eta-g\bar{\eta}-\eta\bar{g})}{\sum\sum\bar{\eta}^2+\sum\sum\eta^2-2\sum\sum\eta\bar{\eta}}\\&=\frac{(2a+1)(2b+1)(\,\overline{g\eta}-\bar{g}\bar{\eta})}{(2a+1)(2b+1)\left(\overline{\eta^2}-\bar{\eta}^2\right)}\\&=\frac{(\,\overline{g\eta}-\bar{g}\bar{\eta})}{\left(\overline{\eta^2}-\bar{\eta}^2\right)}\end{aligned}$$