Non-Determined Infinite Games

Descriptive Set Theory

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Historical background

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Descriptive set theory has been historically been motivated by several things: non-analytic functions, non-measurability, and pathological subsets of the real line.

Historically, it has been motivated by the Axiom of Choice, which turns out to propagate **pathologies**. Descriptive set theory attempts to systematically study and classify sets where one can avoid these pathologies–i.e., determine which sets are "nice" enough.

The Axiom of Choice

The Axiom of Choice (AC) is widely accepted by mathematicians, but historically, it has been extremely controversial. It is independent of Zermelo-Fraenkel set theory (ZF), which is the standard axiomatization of set theory.

Definition

Given any indexed family $(S_i)_{i \in I}$ of non-empty sets, there exists an indexed set $(x_i)_{i \in I}$ such that $x_i \in S_i$ for each $i \in I$.

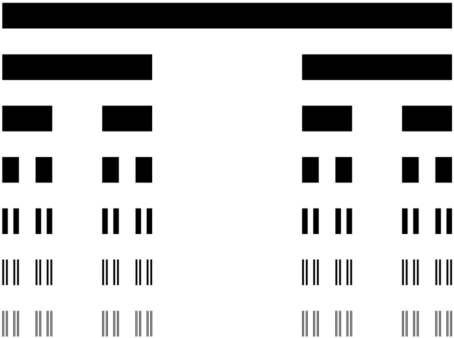
Equivalently: The Cartesian product of a collection of non-empty sets is always non-empty.

While the Axiom of Choice looks fairly non-threatening, it can induce all sorts of strange behaviour.

Infinite games

Spaces of infinite sequences

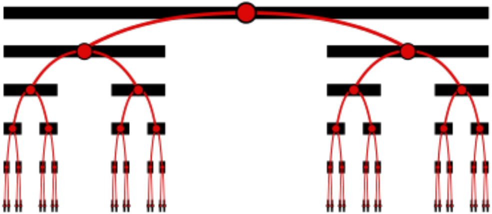
Let's start off with defining infinite sequences. Recall the Cantor set:



One way to think about the Cantor space is as a tree.

Binary trees

From this picture we can see the Cantor set as a binary tree.



From here we define the Cantor space $\mathbb{C}$, which contains all the possible *infinite* sequences of 1 and 0.

So, we define $\mathbb{C} = \{0,1\}^{\mathbb{N}} = 2^{\mathbb{N}}$. Any point in $\mathbb{C}$ is some sort of infinite bitstream or sequence of 0's and 1's. $\{0,1\}^{\mathbb{N}}$ is homeomorphic to the Cantor set.

$A^{\mathbb{N}}$, in general

In general we can define new topological spaces by taking the countably infinite topological product of a set. (Countably infinite means we use $\mathbb{N}$ as the superscript).

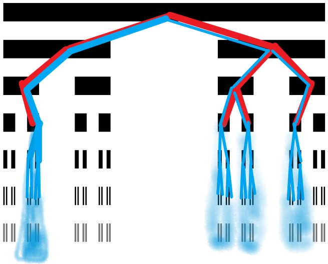
Definition

Consider any set A (possibly infinite). Then $A^{\mathbb{N}}$ is defined as the countably infinite topological product of A, with the standard topology on A.

Basic closed sets

Definition

Given a tree $T \subseteq A^{<\mathbb{N}}$, we denote by $[T]$ the set of infinite branches through T.



As it turns out, for any tree $T \subseteq A^{<\mathbb{N}}$, the set $[T]$ of infinite branches through T is a **closed** subset of $A^{\mathbb{N}}$.

Basically, $[T]$ are our basic closed sets, where A has the discrete topology and $A^{\mathbb{N}}$ is a topological space with the product topology.

Games, games, games

What is a game? As mathematicians we want to abstract this question and formalize it.

We can construct games using any non-empty set A.

Let A be a non-empty set. Consider some subset $D \subseteq A^{\mathbb{N}}$. We can now associate with D the following game:

I	a_0		a_2	
II		a_1		a_3

Together the players will end up making an infinite sequence of moves together: $(a_1, a_2, a_3, ...a_n, ...)$. If the sequence of moves is in D, then Player I wins.

Otherwise Player II wins.

A bit of terminology

Usually, we call D the payoff set.

In general, we call a finite sequence $s \in A^{<\mathbb{N}}$ (up to some point in the game) a *position*.

Each infinite sequence $(a_n)_{n\in\mathbb{N}}$ is called a *run* of the game.

In games like chess and tic-tac-toe, we typically have *strategies*. What does a strategy look like here? And do good strategies always exist?

Strategies as trees

Formally, strategies are *trees* that tell you what move to play.

Definition

A strategy for Player I is a tree σ such that:

1. σ is nonempty;
2. if a position $(a_0, a_1, ..., a_{2n-1}) \in \sigma$, then there exists exactly one $a_{2n} \in A$ such that $(a_0, a_1, ..., a_{2n-1}, a_{2n}) \in \sigma$
   (Player I should always have one **unique** response to Player II's moves);
3. if $(a_0, a_1, ..., a_{2n}) \in \sigma$, then for all $a_{2n+1} \in A, (a_0, a_1, ..., a_{2n}, a_{2n+1}) \in \sigma$.
   (There should be a response for any of the moves that Player II can make.)

All about winning

What's the best kind of strategy? The one that wins.

Definition

We say that a strategy φ is **winning** if for every run of the game that Player I plays φ, the run $(a_n)_{n\in\mathbb{N}} \in D$.

We define a winning strategy for II analogously. Obviously, only one player can have a winning strategy.

Does a winning strategy always exist? We say that a game is *determined* if one of the players has a winning strategy. The question, then, becomes: Are all games determined?

A short proof

A short proof

The Axiom of Choice implies actually not all games are determined. In fact, we can explicitly construct a non-determined game.

The proof makes use of the following lemma, which we use without proof.

Lemma

If A has at least 2 elements and σ is a winning strategy for a game in A with winning set D, then $[\sigma]$ is a non-empty perfect subset of D or D^C.

(Recall: a perfect set is closed and has no isolated points. Equivalently, a perfect set is equal to the set of all its limit points.)

A bad payoff set

Let A be any countable set with at least two elements. (E.g., A could be $\{0, 1\}$.) Let some $D \subseteq A^{\mathbb{N}}$ be the payoff set.

If σ is a winning strategy for our game, then $[\sigma]$ is a nonempty perfect subset of D or D^C by our **Lemma**. So, to construct a game with no winning strategy, it's enough to construct a payoff set $B \subseteq A^{\mathbb{N}}$ such that neither B nor B^C contains a nonempty perfect set.

Constructing a cursed set

Remember we assumed A is countable. So there are at most $2^{\aleph_0} = |\mathbb{R}|$ many subsets. Then there are at most $2^{\aleph_0}$ possible perfect subsets.

The axiom of choice allows us to enumerate all of those nonempty perfect subsets: $(P_\xi)_{\xi < 2^{\aleph_0}}$.

We then follow this algorithm: in each perfect subset P_ξ, choose two distinct points a_ξ, b_ξ that have not been picked before from any of the previous sets $P_\lambda : \lambda < \xi$. (This is always possible.)

Now let $B = \{b_\xi\}_{\xi < 2^{\aleph_0}}$. Clearly, B^C contains $\{a_\xi\}_{\xi < 2^{\aleph_0}}$.

Clearly, there does not exist any perfect subset P_ξ of $A^{\mathbb{N}}$ that belongs either to B or B^C.

So, we're done. Let B be the payoff set. The game is non-determined. There is no winning strategy.

Acknowledgements

Thank you to Ran Tao for being such an excellent mentor.

Many thanks to Anush Tserunyan (and Ruiyuan Chen, other collaborators) for making excellent notes on set theory and descriptive set theory. The short proof shown here is owed to Anush's notes.

Resources

Courses: Analysis 3 and 4 (MATH 454, 455). Mathematical logic (MATH 318). Honours set theory (MATH 488). A bit of discrete math and basic tree theory might be useful, too (MATH 340 or 350). However, don't get overwhelmed with prerequisites, just dive in.

Notes and textbooks: Anush Tserunyan's notes on descriptive set theory and set theory. The textbook Classical Descriptive Set Theory by Alexander S. Kechris. Notes on descriptive set theory by Yiannis N. Moschovakis (available online in PDF form if you Google.)

Historical background (no math required): *Naming Infinity* by Graham Kantor is a very interesting book on the historical development of descriptive set theory, from the "French trio" (Lebesgue, Baire, Borel) to the Russian mathematicians in the early 1900s (Luzin, Egorov). You can borrow it from the McGill library through WorldCat/McLennan-Redpath basement.