Radial Basis Function (RBF) Kernel

$$K(x, x') = \exp\left(-\frac{\|x - x'\|^2}{\sigma^2}\right)$$ is between 0 & 1

$\|x - x'\|^2 \rightarrow$ square of the Euclidian distance between x and x'

$\sigma^2 \rightarrow$ bandwidth

it is a free parameter

i.e, it cannot be predicted precisely, therefore it must be estimated experimentally or theoretically.

Kernels for Comparing Documents

Cosine Similarity for Bag of Words

$$K(x_i, x_{i'}) = \frac{x_i^T \cdot x_{i'}}{\|x_i\|_2 \cdot \|x_{i'}\|_2}$$

Is this function symmetric?

Assume there are V different words

x_i : $V \times 1$ $\qquad$ $\underbrace{x_i^T \cdot x_{i'}}_{\text{scalar}}$: $(1 \times V) \cdot (V \times 1)$: 1×1

$x_{i'}$: $V \times 1$

$$K(x_{i'}, x_i) = \frac{(x_{i'})^T \cdot (x_i)}{\|x_{i'}\|_2 \cdot \|x_i\|_2} = \frac{\left[(x_i)^T \cdot (x_{i'})\right]^T}{\|x_{i'}\|_2 \cdot \|x_i\|_2} = \frac{(x_i)^T \cdot (x_{i'})}{\|x_{i'}\|_2 \cdot \|x_i\|_2}$$

$$= K(x_i, x_{i'})$$

This function is symmetric

(*) A positive definite matrix is a symmetric matrix with all positive eigenvalues.

X : $N \times N$

$X \cdot v = \lambda \cdot v$ where v is non-zero

v: $N \times 1$ → eigenvector

λ : 1×1 (scalar) → eigenvalue

$X \cdot v = \lambda \cdot v$

$X \cdot v - \lambda \cdot v = 0$

$X \cdot v - \lambda \cdot I_N \cdot v = 0$

$(X - \lambda I_N) \cdot \underset{\substack{\uparrow \\ \text{non-zero}}}{v} = 0$

$|X - \lambda I_N| = 0$ equality will give us the eigenvalues.

Example :

$$X = \begin{bmatrix} 0 & -1 \\ 2 & 3 \end{bmatrix}$$

$$|X - \lambda I| = \det\left(\begin{bmatrix} 0 & -1 \\ 2 & 3 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}\right) = \det\left(\begin{bmatrix} -\lambda & -1 \\ 2 & 3-\lambda \end{bmatrix}\right) = 0$$

$$\Rightarrow -3\lambda + \lambda^2 + 2 = 0$$

$$(\lambda - 1)(\lambda - 2) = 0$$

$\lambda_1 = 1 \quad \lambda_2 = 2$ ← eigenvalues of X

$\lambda_1 = 1 \quad v_1 = ?$

$$\begin{bmatrix} -1 & -1 \\ 2 & 2 \end{bmatrix} \cdot \begin{bmatrix} v_{11} \\ v_{12} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad v_1 = k_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$-v_{11} - v_{12} = 0 \qquad v_{11} = -v_{12}$

$2v_{11} + 2v_{12} = 0$

$\lambda_2 = 2 \qquad v_2 = ?$

$$\begin{bmatrix} -2 & -1 \\ 2 & 1 \end{bmatrix} \cdot \begin{bmatrix} v_{21} \\ v_{22} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$-2 v_{21} - v_{22} = 0$$

$$2 v_{21} + v_{22} = 0$$

$$v_2 = k_2 \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$