Ex 14.1

Two data points in 1d : $(x_1 = 0, y = -1)$ and $(x_2 = \sqrt{2}, y_2 = 1)$

Feature vector $\phi(x) = \begin{bmatrix} 1 \\ \sqrt{2}\,x \\ x^2 \end{bmatrix}$

The max margin classifier has the form:

$$\min \|w\|^2 \quad \text{s.t.}$$

$$y_1 (w^T \phi(x_1) + w_0) \geq 1$$

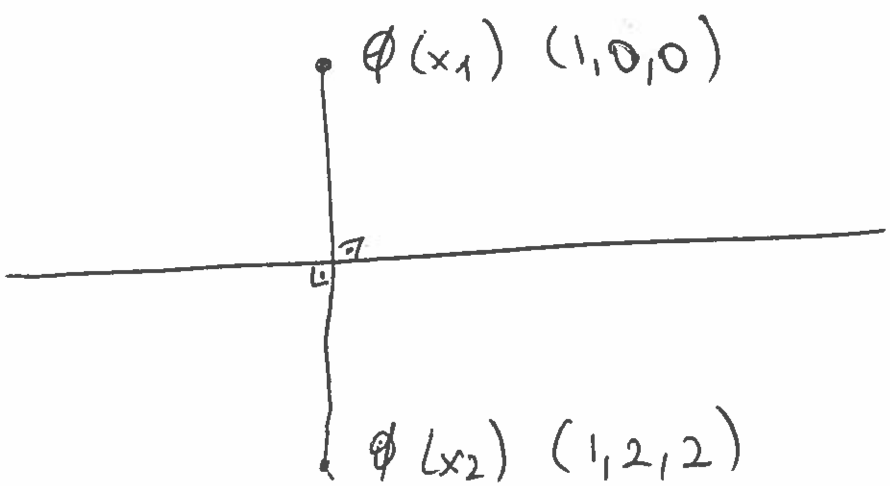
$$y_2 (w^T \phi(x_2) + w_0) \geq 1$$

$$\phi(x_1) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \qquad \phi(x_2) = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$$

$$y_1 = -1 \qquad y_2 = 1$$

14.1.a Write down a vector that is parallel to the optimal vector w. Hint: w is perpendicular to the decision boundary between the two points in the 3d feature space

Sol: The perpendicular to the decision boundary is a line through $\phi(x_1)$ to $\phi(x_2)$, and therefore parallel to $\phi(x_2) - \phi(x_1)$



$$\phi(x_2) - \phi(x_1) = (1,2,2) - (1,0,0)$$
$$= (0,2,2).$$

Any scalar multiple of $(0,2,2)$ is acceptable, e.g. $(0,3,3)$ $(0,1,1)$, $(0,-1,-1)$, etc.

14.1.b what is the value of the margin achieved by this w.

Sol: There are two support vectors (i.e. data points). The decision boundary will be half way between them. The margin will be equal to half the distance between two data points.

$$\text{margin} = \frac{\|\phi(x_2) - \phi(x_1)\|}{2} = \frac{\sqrt{(1-1)^2 + (2-0)^2 + (2-0)^2}}{2} = \frac{\sqrt{8}}{2} = \frac{2\sqrt{2}}{2}$$

$$= \sqrt{2}$$

14.1.c Solve for w, using the fact that the margin is equal to $\frac{1}{\|w\|}$

Sol: $w = \begin{bmatrix} 0 \\ 2k \\ 2k \end{bmatrix}$

$$\|w\| = \sqrt{0 + 4k^2 + 4k^2} = 1/\sqrt{2}$$

$$\sqrt{8k^2} = 1/\sqrt{2}$$

$$8k^2 = 1/2$$

$$k^2 = 1/16$$

$$k = 1/4$$

$$w = \begin{bmatrix} 0 \\ 2k \\ 2k \end{bmatrix} = \begin{bmatrix} 0 \\ 1/2 \\ 1/2 \end{bmatrix}$$

14.1.d Solve for w_0 using w and the equations provided for the max margin classifier. Hint: The points will be on the decision boundary, so the inequalities will be tight.

Sol:

$$w = \begin{bmatrix} 0 \\ 1/2 \\ 1/2 \end{bmatrix}$$

$$y_1 \cdot (w^T \cdot \phi(x_1) + w_0) = 1$$

$$-1 \cdot \left([0 \ \ 1/2 \ \ 1/2] \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + w_0 \right) = 1$$

$$-1 \cdot (0 + w_0) = 1$$

$$w_0 = -1$$

OR

$$y_2 \left(w^T \phi(x_2) + w_0 \right) = 1$$

$$1 \cdot \left(\begin{bmatrix} 0 & 1/2 & 1/2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} + w_0 \right) = 1$$

$$1 \cdot (0 + 1 + 1 + w_0) = 1$$

$$w_0 + 2 = 1$$

$$w_0 = -1$$

14.1.e Write down the form of the discriminant function $f(x) = w_0 + w^T \phi(x)$ as an explicit function of x.

Sol.

$$w_0 + w^T \phi(x) = -1 + \begin{bmatrix} 0 & 1/2 & 1/2 \end{bmatrix} \begin{bmatrix} 1 \\ \sqrt{2}x \\ x^2 \end{bmatrix}$$

$$= -1 + 0 + \frac{\sqrt{2}}{2} x + \frac{1}{2} x^2$$

$$= -1 + \frac{\sqrt{2}}{2} x + \frac{1}{2} x^2$$