(1) Explain the following types of matrixes with the help of examples :

(a) Singular Matrix:

A singular matrix (m=n) that is not invertible is called singular A square matrix is singular if and only if its determinant is 0.

If the determinant of a singular matrix is equal to 0 that matrix does not exist inverse.

Properties:

Some of the important properties of a singular matrix are listed below :

(1) The determinant of a matrix is zero.

(2) A non-zero matrix is defined as singular matrix when the determinant of a matrix is zero, we cannot find its inverse.

(3) Singular matrix is defined only for square matrix..

(4) There will be no multicative inverse for this matrix.

Let's consider with an example :

$$A = \begin{bmatrix} 2 & 4 & 6 \\ 2 & 0 & 2 \\ 6 & 8 & 14 \end{bmatrix}$$

The determinant of A is $\det(A) = 2(0-16)-4(28-12)+6(16-0)$

$$= -32-64+96=0$$

As the determinant is equal to 0, hence it is a singular matrix and so this matrix does not have inverse.

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The inverse of a matrix 'A' is given as

$$A' = \frac{adj(A)}{det|A|}$$

As $det(A) = 0$; $A' = \frac{adj(A)}{0}$

That is not defined.

Therefore, the inverse of a singular matrix does not exist.

(b) Non-singular Matrix:

A non-singular matrix is a square matrix whose determinant is not equal to zero. The non singular matrix is an invertible matrix, and its inverse can be computed as it has a determinant value.

For a square matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, as a ~~singular~~

As it is a non-singular matrix, the determinant of this matrix A is a non-zero value.

$$|A| = |ad - bc| \neq 0.$$

properties of a non-singular Matrix:

The following are some of the important properties of a non singular matrix.

① The determinant of a non-singular matrix is a non-zero value.

② The non-singular matrix is also called an invertible matrix because its determinant can be calculated.

③ The non-singular matrix is a square matrix because its determinants can be calculated only for non-singular matrixes.

④ The product of two non-singular matrices is a non-singular matrix.

⑤ If A is a non singular matrix, K is a constant, then KA is also a non-singular matrix.

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let's consider an example $A = \begin{bmatrix} 1 & -4 \\ 3 & 5 \end{bmatrix}$

$\det(A) = 5+12 = 17$

Therefore, the determinant of the matrix $|A| = 17$, and it is a non-singular matrix as ~~it is a~~ A is a non zero value.

So, it has an inverse.

The inverse of matrix A is denoted as A^{-1}:

$$A^{-1} = \frac{1}{\det(A)} \operatorname{Adj} A$$

$$= \frac{1}{17} \begin{bmatrix} 5 & -3 \\ 4 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{5}{17} & -\frac{3}{17} \\ \frac{4}{17} & \frac{1}{17} \end{bmatrix}$$

(c) Triangular Matrix

A triangular matrix is a square matrix in which elements below ~~and~~ or above the diagonal are all zeros.

There are two types of triangular matrixes. They are:

① **Lower Triangular Matrix:**

A square matrix whose all elements above the main diagonal are zero is called a lower triangular matrix.

An $n \times n$ square matrix $A = [a_{ij}]$ is said to be a lower triangular matrix if and only if $a_{ij} = 0$, for all $i < j$. This implies that all elements above the main diagonal of a square matrix are zero in a lower triangular matrix.

An example of a lower triangular matrix is given below;

$$L = \begin{bmatrix} -1 & 0 & 0 \\ 2 & -8 & 0 \\ 3 & -4 & 2 \end{bmatrix}$$

Let's consider ~~Aes~~ matrix $A = \begin{bmatrix} 2 & 1 & 3 \\ 7 & 5 & 2 \\ 3 & 2 & 1 \end{bmatrix}$ convert into a lower triangular matrix.

$$A = \begin{bmatrix} 2 & 1 & 3 \\ 7 & 5 & 2 \\ 3 & 2 & 1 \end{bmatrix}$$

$$\Rightarrow A = \begin{bmatrix} -7 & -5 & 0 \\ 1 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix} \quad [R_1 \rightarrow R_1 - 3R_3]$$

$$\Rightarrow A = \begin{bmatrix} -2 & 0 & 0 \\ 1 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix} \quad [R_1 \rightarrow R_1 + 5R_2]$$

Thus, the matrix $A = \begin{bmatrix} -2 & 0 & 0 \\ 1 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix}$ is now a lower triangular matrix.

(2) Upper Triangular Matrix:

A square matrix whose all elements below the main diagonal are zero is called an upper triangular matrix.

A ~~matrix~~ $n \times n$ square matrix $A = [a_{ij}]$ is said to be an upper triangular matrix if and only if $a_{ij} = 0$, for all $i > j$. This implies that all elements below the main diagonal of a square matrix are zero in an upper triangular matrix.

An example of an upper triangular matrix is given below;

$$U = \begin{bmatrix} 6 & 0 & 8 \\ 0 & 10 & -12 \\ 0 & 0 & 2 \end{bmatrix}$$

Let's consider a matrix $A = \begin{bmatrix} 1 & -1 & 2 \\ 2 & 1 & 3 \\ 3 & 2 & 4 \end{bmatrix}$ convert into a upper triangular matrix.

S. $A = \begin{bmatrix} 1 & -1 & 2 \\ 2 & 1 & 3 \\ 3 & 2 & 4 \end{bmatrix}$

$\Rightarrow A = \begin{bmatrix} 1 & -1 & 2 \\ 0 & 3 & 1 \\ 0 & 5 & -2 \end{bmatrix}$ $\begin{bmatrix} R_3 \rightarrow R_3 - 3R_1 \\ R_2 \rightarrow R_2 - 2R_1 \end{bmatrix}$

$\Rightarrow A = \begin{bmatrix} 1 & -1 & 2 \\ 0 & 3 & 1 \\ 0 & 0 & -\frac{1}{3} \end{bmatrix}$ $\left[R_3 \rightarrow R_3 - \frac{5}{3}R_2 \right]$

The matrix $A = \begin{bmatrix} 1 & -1 & 2 \\ 0 & 3 & 1 \\ 0 & 0 & \frac{1}{3} \end{bmatrix}$ is now a upper triangular matrix.

(d) Symmetric Matrix:

A symmetric matrix is a square matrix that is equal to its transpose matrix. ~~The transpose~~

If A is a ~~matrix~~ symmetric matrix, then $A = A^T$.

A square matrix 'A' which of size $n \times n$ is considered to be symmetric if and only if ~~$B^T = B$~~. $A^T = A$.

This can be represented as: If $A = [a_{ij}]_{n \times n}$ is the symmetric matrix, then $a_{ij} = b_{ji}$ for all i and j or $1 \le i \le n$, and $1 \le j \le n$. Here,

- n is any natural number.
- b_{ij} is an element position (i, j) which is ith row and jth column in matrix ~~B~~ A and
- ~~B_{ji}~~ b_{ji} is an element at position (j, i) which is jth row and ith column in matrix A.

Example:

Let's consider an ~~exple~~ example of a matrix A, symmetric matrix:

$$A = \begin{bmatrix} 2 & 3 & 6 \\ 3 & 4 & 5 \\ 6 & 5 & 9 \end{bmatrix}$$

$$A^T = \begin{bmatrix} 2 & 3 & 6 \\ 3 & 4 & 5 \\ 6 & 5 & 9 \end{bmatrix}$$

As $A = A^T$ then, A is a symmetric matrix.

(e) ~~Asymmet~~ Asymmetric Matrix:

An asymmetric Matrix is one that is not equal to its transpose matrix

~~A~~ If A be a matrix then $A \neq A^T$

Ex:

Let's considere be an example of A matrix:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

$$A^T = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix}$$

$\therefore A \neq A^T$

So, A is an ~~asso~~ asymmetric matrix.

(f) Skew-symmetric Matrix:

A skew-symmetric matrix is a square matrix that is equal to the negative of its transpose matrix. Here we considered a matrix A.

Now, ~~the~~ ~~so~~ a skew symmetric Matrix as follows of matrix A is given below —

$$A = -A^T$$

A square matrix A which of size $n \times n$ is considered to be skew-symmetric matrix if and only if $A^T = -A$.

If $B = [b_{ij}]_{n \times n}$ is the skew-symmetric matrix,

Then, $b_{ij} = -b_{ji}$ for all i and j or $1 \leq i \leq n$ and $1 \leq j \leq n$

Example:

Let's consider an example of a matrix A,

$$A = \begin{bmatrix} 0 & 3 \\ -3 & 0 \end{bmatrix}$$

$$A^T = \begin{bmatrix} 0 & -3 \\ 3 & 0 \end{bmatrix}$$

$$-A^T = -\begin{bmatrix} 0 & 3 \\ -3 & 0 \end{bmatrix}$$

$$\therefore A = -A^T$$

Thus, B is a skew-symmetric matrix.

(g) Identity Matrix:

Indentity matrix is the matrix which is $n \times n$ square matrix where the diagonal consist of ones and the other elements are all zeros. It is represented as ~~I or either~~ I_n ; where n represents the size of the square matrix.

Properties of an Identiy Matrix:

① It's always a square matrix

② By multiplying any matrix by the unit matrix, given the matrix itself.

③ We always get an identity after multiplying two inverse matricies.

Example:

2X2 Identity Matrix:

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

3X3 Identity Matrix:

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

nxn Identity Matrix:

$$I_n = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 1 \end{bmatrix}$$

(h) Transpose of a matrix:

The transpose of a matrix is found by interchanging its rows and columns or columns into rows. The

For example, if "A" is the given matrix, then the transpose of the matrix is represented by A' or A^T.

The following statement generalizes the matrix transpose:

If $A = [a_{ij}]_{m \times n}$, then $A^T = [a_{ji}]_{n \times m}$.

Let's consider an example of 3x3 matrix. $A = \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}_{3 \times 3}$

Transpose matrix, $A^T = \begin{bmatrix} a & d \\ b & e \\ c & f \end{bmatrix}_{3 \times 2}$

Now, now matrix becomes 3x2, as it has 3 rows and 2 columns.

(I) Inverse of a Matrix:

If A is a non-singular matrix, there is an existence of $n \times n$ matrix A^{-1}, which is called the inverse matrix.

For a square matrix A, its inverse is A^{-1} and $A \cdot A^{-1} = I$, where I is the identity matrix. The matrix whose determinant is non-zero and for which the inverse matrix can be ~~cabult~~ calculated is called an invertible matrix.

Inverse Matrix Method:

Method 01:

Let's consider a matrix of 2×2, $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = A^{-1} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1}$

$$A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Method 02:

$$A^{-1} = adj(A)/det(A)$$

adj(A) = Let's considere a matrix 3×3

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Here,

The adjugate of this matrix is given by:

$$adj A = \underbrace{\begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}}_{\text{cofactors of } A}^{T} = \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix}$$

Example:

Let's consider a 2x2 matrix $A = \begin{bmatrix} 2 & 1 \\ 7 & 4 \end{bmatrix}$; Find its inverse matrix.

$$A^{-1} = \Big[\det(A) = 8 - 7 = 1$$

$$A^{-1} = \frac{1}{\det(A)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$= \frac{1}{1} \begin{bmatrix} 4 & -1 \\ -7 & 2 \end{bmatrix} = \begin{bmatrix} 4 & -1 \\ -7 & 2 \end{bmatrix}$$

Example:

Now, Again, Let's consider a 3X3 matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 2 & 9 \end{bmatrix}$. Find it's inverse matrix.

$$\det(A) = 1(45-12) - 2(36-42) + 3(8-35)$$
$$= -36$$

$$M_{11} = \det \begin{bmatrix} 5 & 6 \\ 2 & 9 \end{bmatrix} = 33$$

$$M_{12} = \det \begin{bmatrix} 4 & 6 \\ 7 & 9 \end{bmatrix} = -6$$

$$M_{13} = \det \begin{bmatrix} 4 & 5 \\ 7 & 2 \end{bmatrix} = -27$$

$$M_{21} = \det \begin{bmatrix} 2 & 3 \\ 2 & 9 \end{bmatrix} = 12$$

$$M_{22} = \det \begin{bmatrix} 1 & 3 \\ 7 & 9 \end{bmatrix} = -12$$

$$M_{23} = \det \begin{bmatrix} 1 & 2 \\ 7 & 2 \end{bmatrix} = -12$$

$$M_{31} = \det \begin{bmatrix} 2 & 3 \\ 5 & 6 \end{bmatrix} = -3$$

$$M_{32} = \det \begin{bmatrix} 1 & 3 \\ 4 & 6 \end{bmatrix} = -6$$

$$M_{33} = \det\begin{bmatrix} 1 & 2 \\ 4 & 5 \end{bmatrix} = -3$$

cofactors : $\begin{bmatrix} 33 & 6 & -27 \\ -12 & -12 & 12 \\ -3 & 6 & -3 \end{bmatrix}$

Find the transpose of it.

$$\text{Adj}A = \begin{bmatrix} 33 & 6 & -27 \\ -12 & -12 & 12 \\ -3 & 6 & -3 \end{bmatrix}^T = \begin{bmatrix} 33 & -12 & -3 \\ 6 & -12 & 6 \\ -27 & 12 & -3 \end{bmatrix}$$

$$A^{-1} = \frac{1}{A} \text{Adj}A$$

$$= \frac{1}{-36}\begin{bmatrix} 33 & -12 & -3 \\ 6 & -12 & 6 \\ -27 & 12 & -3 \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{11}{12} & \frac{1}{3} & \frac{1}{12} \\ -\frac{1}{6} & \frac{1}{3} & -\frac{1}{6} \\ \frac{3}{4} & -\frac{1}{3} & \frac{1}{12} \end{bmatrix}$$

(2) What are the minors, co-factors, ranks, determinants of the following matrix.

$$A = \begin{bmatrix} 1 & 6 & 7 \\ 2 & 5 & 8 \\ 3 & 4 & 9 \end{bmatrix}$$

Minors of the matrix A;

$$M_{11} = \begin{vmatrix} 5 & 8 \\ 4 & 9 \end{vmatrix} = 45 - 32 = 13$$

$$M_{12} = \begin{vmatrix} 2 & 8 \\ 3 & 9 \end{vmatrix} = 18 - 24 = -6$$

$$M_{13} = \begin{vmatrix} 2 & 5 \\ 3 & 4 \end{vmatrix} = 8 - 15 = -7$$

$$M_{21} = \begin{vmatrix} 6 & 7 \\ 4 & 9 \end{vmatrix} = 54 - \cancel{56}\,28 = 26$$

$$M_{22} = \begin{vmatrix} 1 & 7 \\ 3 & 9 \end{vmatrix} = 9 - 21 = -12$$

$$M_{23} = \begin{vmatrix} 1 & 6 \\ 3 & 4 \end{vmatrix} = 4 - 18 = -14$$

$$M_{31} = \begin{vmatrix} 6 & 7 \\ 5 & 8 \end{vmatrix} = \cancel{0}\, 48 - 35 = 13$$

$$M_{32} = \begin{vmatrix} 1 & 7 \\ 2 & 8 \end{vmatrix} = -6$$

$$M_{33} = \begin{vmatrix} 1 & 6 \\ 2 & 5 \end{vmatrix} = -7$$

cofactors ~~by the form of~~ of Matrix A;

$A_{11} = (-1)^{1+1} \cdot 13 = 13$

$A_{12} = (-1)^{1+2} \cdot (-6) = 6$

$A_{13} = (-1)^{1+3} \cdot (-7) = -7$

$A_{21} = (-1)^{2+1} \cdot (26) = -26$

$A_{22} = (-1)^{2+2} \cdot (-12) = -12$

$A_{23} = (-1)^{2+3} (-14) = 14$

$A_{31} = (-1)^{3+1} \cdot 13 = 13$

$A_{32} = (-1)^{3+2} \cdot (-6) = 6$

$A_{33} = (-1)^{3+3} \cdot (-7) = -7$

cofactors of matrix A; C_A:

$$\begin{bmatrix} 13 & 6 & -7 \\ -26 & -12 & 14 \\ 13 & 6 & -7 \end{bmatrix}$$

Q Given here, $A = \begin{bmatrix} 1 & 6 & 7 \\ 2 & 5 & 8 \\ 3 & 4 & 9 \end{bmatrix}$

$\det(A) = 1\,(45-32) + 6\,(18-24) + 7\,(8-15)$

$= 13 + 6 - 49$

$= -30$

Given, $A = \begin{bmatrix} 1 & 6 & 7 \\ 2 & 5 & 8 \\ 3 & 4 & 9 \end{bmatrix}$

For Finding rank of this matrix, we have to ~~find the~~ ~~RE~~ apply the REF Formula:

$$\begin{bmatrix} 1 & 6 & 7 \\ 2 & 5 & 8 \\ 3 & 4 & 9 \end{bmatrix} \xrightarrow[R_3' \to R_3 - 3R_1]{R_2' \to R_2 - 2R_1} \begin{bmatrix} 1 & 6 & 7 \\ 0 & -7 & -6 \\ 0 & -14 & -12 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 6 & 7 \\ 0 & -7 & -6 \\ 0 & 0 & 0 \end{bmatrix} \quad [R_3' \to R_3 - 2R_2]$$

As there are two non-zero rows.

So, the rank of this matrix is 2.

(5) compare Gauss - Jordan Elimination and Cramer's rule
Q. for ~~two differe~~ solving the linear equations.

Sol^n:

Gauss–Jordan Elimination Method:

1. It is an algorithmic method that uses row operations to transform a ~~linee~~ system of linear equations into reduced row echelon form.

2. The goal is to obtain a triangular or diagonal form for the augmented matrix representing the systems.

Cramer's Rule Method:

1. Cramer's rule is a formula for solving a system of linear equations using determinants.

2. It expresses each variable as the ratio of the determinant of a matrix obtained by replacing one column of the coefficient matrix with the column of constants.

~~Compari~~. Let's consider an example ~~of~~ for solving by the both method of gauss -Jordan elimination and cramer's rule for comparison:

As $2x + 5y = 21$ ——— (I)
$x + 2y = 8$ ——— (II)

converted the equations augmented matrix form: $Ax = B$

$$\left[\begin{array}{cc|c} 2 & 5 & 21 \\ 1 & 2 & 8 \end{array}\right]$$

$$= \left[\begin{array}{cc|c} 1 & \frac{5}{2} & \frac{21}{2} \\ 1 & 2 & 8 \end{array}\right] \quad R_1 \to R_1 \div 2$$

$$= \left[\begin{array}{cc|c} 1 & \frac{5}{2} & \frac{21}{2} \\ 0 & -\frac{1}{2} & -\frac{5}{2} \end{array}\right] \quad R_2 \to R_2 - R_1$$

$$= \left[\begin{array}{cc|c} 1 & \frac{5}{2} & \frac{21}{2} \\ 0 & 1 & 5 \end{array}\right] \quad R_2 \to R_2 \times -2$$

$$= \left[\begin{array}{cc|c} 1 & 0 & -2 \\ 0 & 1 & 5 \end{array}\right] \quad R_1 \to R_1 - \frac{5}{2} \times R_2$$

$x = -2$
$y = 5$

By Cramer's Rule:

$2x + 5y = 21$
$x + 2y = 8$

$$D = \begin{vmatrix} 2 & 5 \\ 1 & 2 \end{vmatrix} = |4 - 5| = -1$$

$$|D_x| = \begin{vmatrix} -21 & 5 \\ -8 & 2 \end{vmatrix}$$

$= \cancel{-16+2} \; 2$

$$|D_y| = \begin{vmatrix} 2 & -21 \\ 1 & -8 \end{vmatrix}$$

$= 5$

$$x = \frac{|D_x|}{|D|} = \frac{2}{-1} = -2$$

$$y = \frac{|D_y|}{|D|} = \frac{-5}{-1} = 5$$

Comparison between Gauss - Jordan Elimination and Cramer's rule:

	Gauss-Jordan Elimination	Cramer's Rule
① Efficiency	① It is generally more efficient for larger systems as it involves systematic elimination steps.	① It becomes computationally expensive as the size of the system increases due to the need to calculate determinants.
② Stability	② As per the calculation of the example we notice that, it is more ~~more~~ stable for systems with zero determinants	② It is unstable due to the division by determinants.
③ Applicability	③ Applicable ~~for~~ to any square or rectangular system.	③ More practical for small systems, especially when the coefficient matrix is small.
④ Solutions	④ Directly provides the solution in a reduced row-echelon form	④ Express each variable as a ratio of determinant providing explicit forms for the solution.
⑤ Calculation	⑤ Requires fewer calculation	⑤ Requires more complex calculation for linear larger systems.

(6) What is RREF? obtain the RREF of the following matrix.

$$A = \begin{bmatrix} 1 & 6 & 7 \\ 2 & 5 & 8 \\ 3 & 4 & 9 \end{bmatrix}$$

Sol^n:

RREF stands for Reduced Row Echelon form. It is a specific form that a matrix can be transformed into using row operations.

properties of RREF:

1. All zero rows are at the bottom.
2. The leading entry of each nonzero row is 1.
3. The leading 1 in each non zero row is to the right of the leading 1 in previous row.
4. All entries in a column below a leading entry are zeros.
5. Each leading 1 is the only nonzero entry in its column.

Given, $A = \begin{bmatrix} 1 & 6 & 7 \\ 2 & 5 & 8 \\ 3 & 4 & 9 \end{bmatrix}$

$$= \begin{bmatrix} 1 & 6 & 7 \\ 0 & -7 & -6 \\ 0 & -14 & -12 \end{bmatrix} \quad \begin{matrix} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - 3R_1 \end{matrix}$$

$$= \begin{bmatrix} 1 & 0 & 13/7 \\ 0 & 1 & 6/7 \\ 0 & 0 & 0 \end{bmatrix} \quad \begin{matrix} R_1 \to R_1 - 6R_1 \\ R_3 \to R_3 + 14R_2 \end{matrix}$$

$$RREF(A) = \begin{bmatrix} 1 & 0 & 13/7 \\ 0 & 1 & 6/7 \\ 0 & 0 & 0 \end{bmatrix}$$

(7)

Explain the cramer's rule with example. what are the best usages of it?

~~cate~~ cramer's Rule:

It is a method for solving a system of linear equations using determinants.

It is applicable to the systems when that are square and coefficient matrix is non-singular.

Example:

Let's consider an example. ~~of A =~~

$2x_1 + 5x_2 = 21$ ——— (I)
$x_1 + 2x_2 = 8$ ——— (II)

Now,

$A = \begin{bmatrix} 2 & 5 \\ 1 & 2 \end{bmatrix} \Rightarrow \det(A) = 4 - 5 = -1$

$Ax_1 = \begin{bmatrix} 21 & 5 \\ 8 & 2 \end{bmatrix} \Rightarrow \det(Ax_1) = 42 - 40 = 2$

$Ax_2 = \begin{bmatrix} 2 & 21 \\ 1 & 8 \end{bmatrix} \Rightarrow \det(Ax_2) = 16 - 21 = -5$

$x_1 = \frac{\det(Ax_1)}{\det(A)} = \frac{2}{-1} = -2$

$x_2 = \frac{\det(Ax_2)}{\det(A)} = \frac{-5}{-1} = 5$

$(x_1, x_2) = (-2, 5)$

Best Usage:

1. Suitable for small systems of linear equations, particularly when the number of variables is relatively small (2 or 3).
2. solving problem in differential geometry.
3. computationally straightforward and provides explicit formulas for each variable.
4. proving a theorem in integer programming.

(4) Discuss the properties of the determinant with example.

(i) The determinant of the identity matrix is 1.

Example:

$$\det\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} = 1, \quad \text{and} \quad \det\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix} = 1$$

(ii) The determinant changes sign when two rows are exchanged.

Ex:

$$\det\begin{vmatrix}c & d\\ a & b\end{vmatrix} = bc - ad$$

row exchange: $\det\begin{vmatrix}a & b\\ c & d\end{vmatrix} = ad - bc = -(bc-ad)$

(iii) The determinant depends linearly on the first row.

Suppose, A, B and C are the same form the second row down and row 1 of A is a linear combination of rows B and C. Then the rule says; det A is the same combination of det B and det C.

Linear combination involve two operation; adding vectors and multiplying by scalers.

Therefore, this rule can be split into two parts.

Add vectors in row 1 $\begin{vmatrix}a+a' & b+b'\\ c & d\end{vmatrix} = \begin{vmatrix}a & b\\ c & d\end{vmatrix} + \begin{vmatrix}a' & b'\\ c & d\end{vmatrix}$

Multiply t in row 1 $\begin{vmatrix} ta & tb \\ c & d \end{vmatrix} = t\begin{vmatrix} a & b \\ c & d \end{vmatrix}$

(IV) If two rows of A are equal, then $\det A = 0$

Ex:

$$A = \begin{vmatrix} a & b \\ a & b \end{vmatrix}$$

$\det(A) = ab - ab = 0$

(V) Subtracting a multiple of one row from another row leaves the same determinant.

Ex: $\begin{vmatrix} a-lc & b-ld \\ c & d \end{vmatrix}$

$$= \begin{vmatrix} a & b \\ c & d \end{vmatrix} - l\begin{vmatrix} c & d \\ c & d \end{vmatrix}$$

$$= \begin{vmatrix} a & b \\ c & d \end{vmatrix} - 0 = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

(VI) If A has a row zeros, then $\det A = 0$

Ex:

$$A = \begin{vmatrix} 0 & 0 \\ c & d \end{vmatrix}$$

$\det A = 0$

(VII) If A is triangulare matrix then $\det A$ is the product $a_{11} a_{22} \cdots a_{nn}$ of the diagonal entries. If the triangulare A has 1s along the diagonal, then $\det A = 1$

Ex: $\begin{vmatrix} a & b \\ 0 & d \end{vmatrix} = ad \quad \begin{vmatrix} a & 0 \\ c & d \end{vmatrix} = ad$

$\det A^T = 4 - 6 = -2$

$\det A = 4 - 6 = -2$ so, $\det A^T = \det A$

(VIII) If A is singular, then $\det A = 0$. If A is invertible, then $\det A \neq 0$

Ex:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$ is not invertible if and only if $ad - bc = 0$

(IX) The determinant of AB is the product of $\det A$ ~~or~~ times $\det B$.

Ex:

$$|A|\,|B| = |AB|$$

Here, $$\begin{vmatrix} a & b \\ c & d \end{vmatrix} \begin{vmatrix} e & f \\ g & h \end{vmatrix} = \begin{vmatrix} ae+bg & af+bh \\ ce+dg & cf+dh \end{vmatrix}$$

Let, $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ $B = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$

$$AB = \begin{bmatrix} 3 & 2 \\ 2 & 4 \end{bmatrix}$$

$\det(A) = -2$

$\det(B) = 1$

$\det(AB) = \cancel{\det(A)}\ -2$

L.H.S. = R.H.S.

(X) The transpose of a A has the same determinant as itself. $\det A^T = \det A$

Ex:

$$\cancel{|A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = \begin{vmatrix} a & c \\ b & d \end{vmatrix} = A^T}$$

Let:

$$A = \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = \begin{vmatrix} 1 & 3 \\ 2 & 4 \end{vmatrix}$$

$\det A^T = 4-6 = -2$

$\det A = 4-6 = -2$ so, $\det A^T = \det A$