$$\frac{d}{dx}\left[p(x)\frac{du}{dx}\right]+[\lambda w(x)-q(x)]u=0 \rightarrow \left(p(x)\frac{d^2u}{dx^2}+\frac{dp(x)}{dx}\frac{du}{dx}\right)+[\lambda w(x)-q(x)]u=0.$$

- $u=u(x) \Rightarrow$ eigenfunction
- $p(x), q(x), w(x)>0$ on $[a,b]$
- $p(x)$ - weight function.
- $w(x)$ - density function

$$-\frac{1}{2}(\cos^4(x))\frac{d^2u}{dx^2}+\frac{\cos^3(x)\cos(2x)}{\sin(x)}\frac{du}{dx}+\left(m^2\frac{\cos^2(x)}{2\sin^2(x)}-\frac{\cos(x)}{\sin(x)}\right)u=\lambda u$$

$$\left(-\frac{1}{2}\cdot\cos^4(x)\frac{d^2u}{dx^2}-\frac{\cos^3(x)\cdot\cos(2x)}{\sin(x)}\cdot\frac{du}{dx}\right)+\left(m^2\frac{\cos^2(x)}{2\cdot\sin^2(x)}-\frac{\cos(x)}{\sin(x)}\right)u-\lambda\cdot u=0$$

$$\left(-\frac{1}{2}\cos^4(x)\cdot\frac{d^2u}{dx^2}-\frac{\cos^3(x)\cdot\cos(2x)}{\sin(x)}\cdot\frac{du}{dx}\right)+\left(\lambda\cdot(-1)-\left(-\frac{m^2\cdot\cos^2(x)}{2\cdot\sin^2(x)}+\frac{\cos(x)}{\sin(x)}\right)\right)u=0$$

multiply both sides by (-1)

$$\left(\underbrace{\frac{1}{2}\cos^4(x)}_{p(x)}\frac{d^2u}{dx^2}+\frac{\cos^3(x)\cdot\cos(2x)}{\sin(x)}\cdot\frac{du}{dx}\right)+\left(\lambda\cdot\underbrace{(1)}_{w(x)}-\underbrace{\left(\frac{m^2\cdot\cos^2(x)}{2\cdot\sin^2(x)}+\frac{\cos(x)}{\sin(x)}\right)}_{q(x)}\right)\cdot u=0.$$

$$\frac{d}{dx}\left[\frac{1}{2}\cos^4(x)\frac{du}{dx}\right]+\left[\lambda-\left(\frac{m^2\cdot\cos^2(x)}{2\cdot\sin^2(x)}-\frac{\cos(x)}{\sin(x)}\right)\right]u=0$$

BV

$u(0)=0$

$u\left(\frac{\pi}{2}\right)=0$

$\left[0,\frac{\pi}{2}\right]$

$a<x<b$

$c_1\cdot u(a)+c_2 u'(a)=0$

$d_1\cdot u(b)+d_2 u'(b)=0$

$c_1, c_2\neq 0 \quad d_1, d_2\neq 0$

$$\boxed{\begin{aligned}&\left(\frac{1}{2}\cos^4(x)\cdot u'\right)'+\left(\lambda-\left(\frac{m^2\cdot\cos^2(x)}{2\cdot\sin^2(x)}-\frac{\cos(x)}{\sin(x)}\right)\right)u=0.\\ &0<x<\frac{\pi}{2} \qquad c_1\cdot u(0)+c_2\cdot u'(0)=0 \qquad (c_1,c_2)\neq(0,0)\\ &u(0)=0 \qquad d_1\cdot u\left(\frac{\pi}{2}\right)+c_2\cdot u'\left(\frac{\pi}{2}\right)=0 \qquad (d_1,d_2)\neq(0,0)\\ &u\left(\frac{\pi}{2}\right)=0\end{aligned}}$$

$\lambda_1<\lambda_2<\dots<\lambda_n\rightarrow\infty$

Shooting method

$y_1=u \qquad y_2=u'=\frac{du}{dx}$

$$\left(\frac{1}{2}\cos^4(x)\cdot y_2'+\frac{\cos^3(x)\cdot\cos(2x)}{\sin(x)}\cdot y_2\right)+\left(\lambda-\left(\frac{m^2\cos^2(x)}{2\cdot\sin^2(x)}+\frac{\cos(x)}{\sin(x)}\right)\right)\cdot u=0$$

$$\frac{1}{2}\cos^4(x)\cdot y_2'+\frac{\cos^3(x)\cdot\cos(2x)}{\sin(x)}\cdot y_2+\left(\lambda-\left(\frac{m^2\cdot\cos^2(x)}{2\cdot\sin^2(x)}+\frac{\cos(x)}{\sin(x)}\right)\right)\cdot u=0$$

$$\boxed{\begin{aligned}&y_2'=\frac{2}{\cos^4(x)}\cdot\left(\left(-\lambda+\frac{m^2\cdot\cos^2(x)}{2\cdot\sin^2(x)}-\frac{\cos(x)}{\sin(x)}\right)u-\frac{\cos^3(x)\cdot\cos(2x)}{\sin(x)}\cdot y_2\right)\\ &y_1(0)=0, \quad y_1\left(\frac{\pi}{2}\right)=0 \qquad \text{guess } y_2(0)=s\end{aligned}}$$

guess $s=1 \Rightarrow y_2(0)=1. \rightarrow \lambda_0=1$

$y_1(x_{i+1})=y_1(x_i)+h y_2(x_i)$

$y_2(x_{i+1})=y_2(x_i)+h\cdot y_2'(x_i, y_1(x_i), y_2(x_i), \lambda)$

$x=0$
step 1: $y_1(0)=0;\ y_2(0)=1$

$x=0.1$
step 2: $y_1(0.1)=y_1(0)+0.1\cdot y_2(0)=0+0.1\cdot 1=0.1$

$$y_2'(0)=\frac{2}{1^4}\left(\lambda+\left(\frac{1}{2x^2}-\frac{1}{2}\right)-\left(\frac{1}{x}+\frac{x}{6}\right)\cdot 0-\frac{(1)^3\cdot 1}{x}\cdot 1\right)$$

$\vdots$

for n number of times.

euler method $\rightarrow y_{n+1}=y_n+h\cdot y_n'$

let $h=0.1$ (step size)

central difference $\Rightarrow y'(x)=\frac{y(x+h)-y(x-h)}{2h}$

let $h=0.1$ (step size)

Taylor expansion $\rightarrow \sum_{n=0}^{\infty}\frac{f^{(n)}(a)}{n!}(x-a)^n$

approximation for vanishing or singular coefficients

Newtons method $\Rightarrow \lambda_{n+1}=\lambda_n-\frac{u(\lambda)}{u'(\lambda)}$