Summary of CPSC 320

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1 Review of CPSC 221

1.1 Asymptotic Analysis

- O-notation: $f(n) = O(g(n))$ if there exists $c > 0$ and n_0 such that $f(n) \leq cg(n)$ for all $n \geq n_0$.
- Ω-notation: $f(n) = \Omega(g(n))$ if there exists $c > 0$ and n_0 such that $f(n) \geq cg(n)$ for all $n \geq n_0$.
- Θ-notation: $f(n) = \Theta(g(n))$ if $f(n) = O(g(n))$ and $f(n) = \Omega(g(n))$.
- o-notation: $f(n) = o(g(n))$ if for all $c > 0$, there exists n_0 such that $f(n) < cg(n)$ for all $n \geq n_0$.
- ω-notation: $f(n) = \omega(g(n))$ if for all $c > 0$, there exists n_0 such that $f(n) > cg(n)$ for all $n \geq n_0$.

Some notes:

$$\log n < \sqrt{n} < n < n \log n < n^a < a^n < n!, \forall a > 1$$

1.2 Graph

Definition. *an **articulation point** in an undirected graph is a vertex whose removal increases the number of connected components in the graph.*

Definition. *The **diameter** of an undirected, unweighted graph is the largest possible value of the following quantity:*

- *The smallest number of edges on any path between two nodes.*
- *In other words, it's the largest number of steps required to get between any two nodes in the graph.*

2 Stable Matching

We have exactly n Employers and n Workers. Each Employer has a preference list of the Workers, and each Worker has a preference list of the Employers. We want to find a stable matching between the Employers and Workers.

A matching is a set of pairs (e, w) where e is an Employer and w is a Worker. A matching is stable if there is no pair (e, w) and (e', w') such that e prefers w' over w and w' prefers e over e'.

2.1 Gale-Shapley Algorithm

set all $s \in S$ and $w \in W$ to be free
while there is a free employer e **do**
    $w = e$'s most preferred worker to whom e has not yet proposed
    if w is free **then**
        (e, w) become engaged

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    else
        if w prefers e to her current employer e' then
            e' becomes free
            (e, w) become engaged
        end if
    end if
end while
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Complexity: $O(n^2)$

2.2 Residence Hospital Matching

We have n Residents and m Hospitals, $m < n$. Each hospital has different number of slots s_i. The total number of slots is equal to the number of residents. Each resident has a preference list of the hospitals, and each hospital has a preference list of the residents. We want to find a stable matching between the residents and hospitals.

2.2.1 Reduction to Stable Matching

1. Create new hospital H_{ij} for each slot the hospital H_i has. Copy the preference list of H_i to H_{ij}.
2. Update the preference list of the residents to include the new hospitals, make sure the new hospitals are at the same rank as the original hospital. (expand the preference list without changing the order)
3. Apply the Gale-Shapley algorithm.
4. The matching is stable.
5. Transform the matching back to the original problem by replacing all H_{ij} with H_i.
6. The matching is stable.

3 Reductions

Definition. *A **reduction** from problem A to problem B is a transformation of problem A into problem B in such a way that a solution to problem B can be used to solve problem A.*

3.1 How reductions work

1. Show how to transform an arbitrary instance I_A of problem A into an instance I_B of problem B.
2. Show that the answer to I_A is "yes" if and only if the answer to I_B is "yes".
3. Show how to transform the solution to I_B into a solution to I_A.

The solver is a black box that solves problem B.

4 Greedy Algorithms

Definition. *A **greedy algorithm** is an algorithm that makes a sequence of choices, each of which is the best choice at the time (local maximum), without regard for the future.*

4.1 Greedy stays ahead

Theorem. *If a greedy algorithm always makes a choice that stays ahead (at least as good) of the optimal solution, then the greedy algorithm is optimal.*

- Essentially a proof by induction.
- Compare to an optimal solution, show that the greedy solution is at least as good.

5 Divide and Conquer

Definition. ***Divide and Conquer*** *is a problem-solving strategy that breaks a problem into smaller, simpler subproblems, solves the subproblems, and then combines the solutions to the subproblems to solve the original problem.*

5.1 Recurrence Relations

The running time $T(n)$ of a recursive function can be described using a recurrence relation:

- $T(n)$ is defined in terms of one or more terms of the form $T(m)$ where $m < n$.
- For example: $T(n) = 2T(n/2) + O(n)$ with a base case $T(1) = O(1)$.

5.2 Recurrence Tree

- A tree that represents the recursive calls of a function.
- Each node represents a recursive call. The size of the subproblem is shown at the node.
- Next to each node, we write the cost of the work done at that level besides the recursive calls.
- Compute the total cost of the work done at each level of the tree.
- The total cost of the algorithm is the sum of the costs at each level.

5.3 Master Theorem

Theorem. *Let $a \geq 1$ and $b > 1$ be constants, let $f(n)$ be a function, and let $T(n)$ be defined on the non-negative integers by the recurrence relation:*

$$T(n) = aT(n/b) + f(n)$$

where n/b means either $\lfloor n/b \rfloor$ or $\lceil n/b \rceil$. Then $T(n)$ has the following asymptotic bounds:

1. *If $f(n) = O(n^{\log_b a - \epsilon})$ for some constant $\epsilon > 0$, then $T(n) = \Theta(n^{\log_b a})$.*
2. *If $f(n) = \Theta(n^{\log_b a} \log^k n)$, then $T(n) = \Theta(n^{\log_b a} \log^{k+1} n)$.*
3. *If $f(n) = \Omega(n^{\log_b a + \epsilon})$ for some constant $\epsilon > 0$, and if $af(n/b) \leq kf(n)$ for some constant $k < 1$ and sufficiently large n, then $T(n) = \Theta(f(n))$.*

5.4 Prune and Search

Definition. ***Prune and Search*** *is a problem-solving strategy that divides the search space into two parts, one of which is pruned (discarded) and the other is searched.*

function QUICKSELECT(A[1:n], k) ▷ Returns the element of rank k in an array of n numbers
  if n = 1 **then**
    return A[1]
  end if
  Choose a random pivot element p from A
  Partition A into $L = \{x \in A : x < p\}$, $E = \{x \in A : x = p\}$, $G = \{x \in A : x > p\}$
  if k $\leq |L|$ **then**
    return QuickSelect(L, k)
  else if k $\leq |L| + |E|$ **then**
    return p
  else
    return QuickSelect(G, k - |L| - |E|)
  end if
end function

6 Dynamic Programming

Definition. ***Dynamic Programming*** *is a problem-solving strategy that breaks a problem into smaller, overlapping subproblems, solves the subproblems, and then combines the solutions to the subproblems to solve the original problem.*

Typical DP examples:

$$\text{DP}[i] = \begin{cases} \text{base case} & \text{if } i \text{ is a base case} \\ \text{combine}(\text{DP}[i-1], \text{DP}[i-2], \ldots, \text{DP}[i-k]) & \text{otherwise} \end{cases}$$

Or

$$\text{OPT}(j) = \max\{\text{OPT}(j-1), \text{OPT}(j-2) + v_j\}$$

6.1 Memoization

Definition. ***Memoization*** *is a technique used to store the results of expensive function calls and return the cached result when the same inputs occur again.*

6.2 Tabulation

Definition. ***Tabulation*** *is a technique used to store the results of expensive function calls in a table and return the cached result when the same inputs occur again.*

See the worksheet for LCS (Longest Common Subsequence) for an example of tabulation.

7 NP-Completeness

As long as the algorithm runs in $O(n^k)$ time, it is considered polynomial time. We think it is efficient.

7.1 Decision V.S. Optimization

Definition. *Optimization problem: we want to find the solution s that maximizes or minimizes some objective function $f(s)$.*

Definition. *Decision problem: given a parameter k, we want to know if there is a solution s such that $f(s) \geq k$ (maximization) or $f(s) \leq k$ (minimization).*

These two types of problems are equivalent. If we can solve the decision problem, we can solve the optimization problem by binary search.

7.2 P and NP

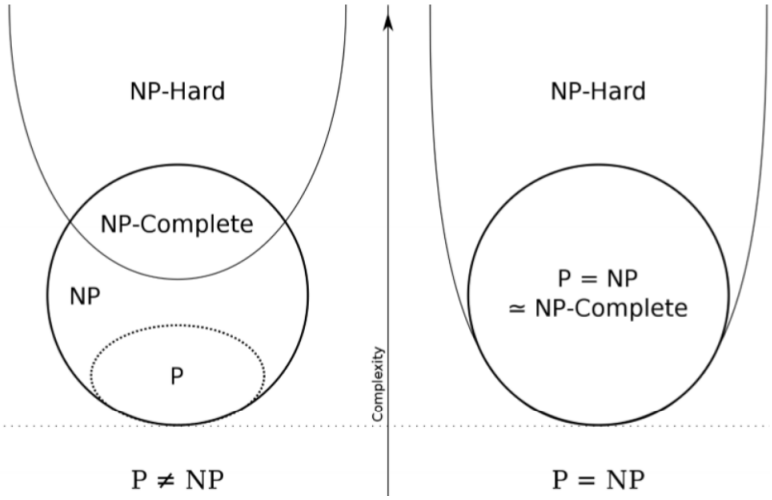
Definition. ***P*** *is the class of decision problems that can be solved in polynomial time. (We have the solver)*

Definition. ***NP*** *is the class of decision problems for which a solution can be verified in polynomial time. (We have the verifier)*

Cook's Theorem:

Theorem. *If SAT can be solved in polynomial time, then every problem in NP can be solved in polynomial time.*

A problem that belongs to NP and is as hard as SAT is called **NP-complete**.



For example, 3-SAT is NP-complete. The 3-SAT problem is: given a Boolean formula in conjunctive normal form (CNF) where each clause has exactly 3 literals, is there an assignment of truth values to the variables that makes the formula true?

7.3 Proof of NP-completeness

1. Show that the problem is in NP.
2. Show that the problem is in NP-hard.

7.3.1 Proof of NP

To prove that P is in NP, we need to **efficiently verify** a **certificate**.

- A certificate is a proof that the solution is correct.
- A verifier is an algorithm that checks the certificate. Needs to run in polynomial time.
- For example, an optimization problem can be: Does there exists X such that Y is true?

- X is the certificate.
- Y is the verifier.

7.3.2 Proof of NP-hard

Definition. *A problem A is **NP-hard** if P is at least as hard as any other problem in NP.*

We prove "at least as hard" via polynomial-time reduction.

- Pick a known NP-complete problem B. Let A be the problem we want to prove is NP-hard.
- Show that B can be reduced to A in polynomial time with the same YES/NO answer.
- Since B is NP-complete, A is NP-hard.
- Intuition: Since we can use a solver for A to solve B, this tells us that A is at least as hard as B (It is powerful enough to handle B)