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An Introduction to Instantons in Higher Dimensions

Tathagata Ghosh

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University of Leeds

Outline

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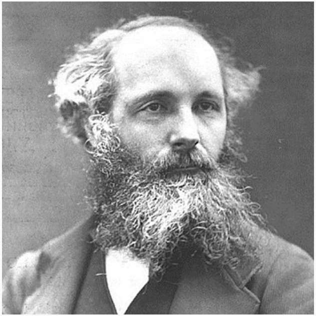

James Clerk Maxwell

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Since then, the Yang–Mills theory in 4-dimensions have been studied extensively by physicists and mathematicians alike.

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In two dimensions $\mathbb{R}^2$, you have integrated a function $f(x, y)$ over some region $R \subset \mathbb{R}^2$:

$$\iint\limits_R f(x, y)\ dxdy$$

We called $dxdy$ the 'area element'.

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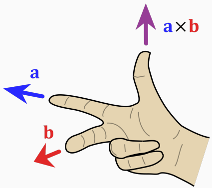

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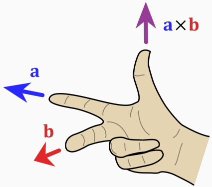

Now, if we take dy first then dx, the area element $dydx$ has a normal vector ($dy \times dx$) is downwards, opposite to the previous one.

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Similarly, we have the volume element $dx \wedge dy \wedge dz$.

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Let we are still in $\mathbb{R}^3$, we call x, y, z or any real valued function a 0-form. We call dx, dy, dz 1-forms, $dx \wedge dy, dy \wedge dz, dz \wedge dx$ 2-forms, and $dx \wedge dy \wedge dz$ a 3-form. Clearly you can generalize this to any dimensions, for $\mathbb{R}^n$.

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Note that in dimensions 3, any 4-form is zero (just like the volume of a plane object is zero).

Differential Forms

Any 1-form on $\mathbb{R}^3$ can be written as $fdx + gdy + hdz$, for real valued functions $f, g, h \in \Omega^0$. any 2 form can be written as $fdx \wedge dy + gdy \wedge dz + hdz \wedge dx$ etc.

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Important note: $d^2 = 0$.

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Hodge Star Operator

We are still in $\mathbb{R}^3$. Recall the volume form $dx \wedge dy \wedge dz$. Hodge star of a form is the rest of the terms in the volume form. So,

$$*dx = dy \wedge dz$$

Since, $dx \wedge dy \wedge dz = -dy \wedge dx \wedge dz$,

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Using the Hodge star, we define the 'codifferential' d^* which decreases forms. Define

$$d^* = *d*$$

Hodge Star Operator

Then, in 3-dimensions, for a 2-form F,

$$
\begin{aligned}
F &\in \Omega^2 \\
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Historically, Hodge introduced the star operator precisely to write down the Maxwell's equations in easier manner. So, let's do that now.

Maxwell's Equations

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In $\mathbb{R}^3$, the electric field $\mathbf{E} \in \Omega^1(\mathbb{R}^3)$ is a 1-form and the magnetic field $\mathbf{B} \in \Omega^2(\mathbb{R}^3)$ is a 2-form. Let dx^1, dx^2, dx^3 spans $\Omega^1(\mathbb{R}^3)$. Introduce the 4th coordinate $t = x^0$, and so, $dt(= dx^0), dx^1, dx^2, dx^3$ spans $\Omega^1(\mathbb{R}^4)$.

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The Electromagnetic field (strength) is a 2-form on space-time $\mathbb{R}^4$,

$$\mathbf{F} = \mathbf{B} + dt \wedge \mathbf{E} \in \Omega^2(\mathbb{R}^4)$$

We can write

$$\mathbf{F} = \sum_{0 \leqslant \mu, \nu \leqslant 3} F_{\mu\nu} dx^\mu \wedge dx^\nu.$$

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Now, we have

$$d\mathbf{F} = 0$$

which is Bianchi identity, it always holds.

Maxwell's Equations

Since $d^2 = 0$, we have an 1-form $\mathbf{A} \in \Omega^1(\mathbb{R}^4)$ such that

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The Maxwell's equations (in vacuum) in space-time are

$$\begin{aligned} d\mathbf{F} &= 0 \\ d^*\mathbf{F} &= 0. \end{aligned}$$

Yang–Mills Equations

The connection 1-form (or, electromagnetic potential) is real valued (we can think of it as a function $A : \mathbb{R}^4 \to \mathbb{R}$). So, for $x, y \in \mathbb{R}^4$,

$$[A, A](x, y) := A(x)A(y) - A(y)A(x) = 0$$

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We can think of Yang–Mills equations as generalisations of Maxwell's equations where we consider the potential to 2×2 trace-less skew-Hermitian matrix valued function (rather than real i.e., 1×1 matrix-valued). Now, because of non-commutativity of matrices, in general,

$$[A, A](x, y) := A(x)A(y) - A(y)A(x) \neq 0$$

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Fianlly, we have the equations

$$d_A F_A = 0 \quad \text{(Bianchi identity)}$$
$$d_A^* F_A = 0 \quad \text{(Yang–Mills equation)}$$

Finally, Instantons (in 4-dimensions)

Instantons in 4-dimensions

We are in 4-diemensional Euclidean space. Let's look at two particular equations

$$*F_A = F_A \quad \text{(Self-dual Instanton equation)}$$
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The solutions for these equations are the connections for which the curvature satisfies these equations. These connections are called Instantons.

Instantons in 4-dimensions

Now,

$$\begin{aligned} d_A^* F_A &= *d_A * f_A \\ &= d_A(\pm F_A) \quad \text{(using instanton equations)} \\ &= \pm * d_A F_A \\ &= 0 \quad \text{(using Bianchi identity)} \end{aligned}$$

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So, we see that instantons are indeed solutions of the Yang–Mills equations.

Instantons in higher Dimensions

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Now, we want to do the same thing on $\mathbb{R}^7$. Here $F_A \in \Omega^2(\mathbb{R}^7)$. But then, $F_A \in \Omega^2 \Rightarrow *F_A \in \Omega^{7-2} = \Omega^5 \not\ni \pm F_A$.

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Similarly, in dimensions 8, we use a 4-form $\Phi \in \Omega^2(\mathbb{R}^8)$.

Why would we care?

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Although I described the Yang–Mills equations and Instantons on Euclidean space $\mathbb{R}^n$, things become more interesting when we consider 'smooth manifolds'. Euclidean spaces are also manifolds, so are the Earth, a football, a rugby ball, a doughnut etc. And then, there are these complicated manifolds:

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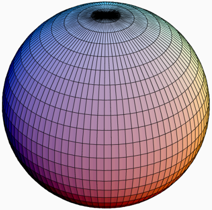
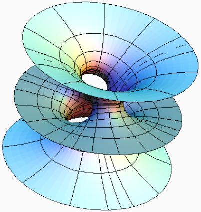
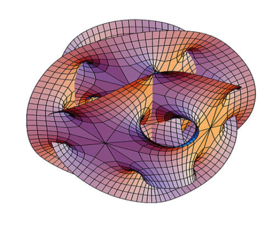

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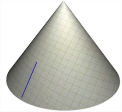

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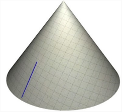

Manifolds are also ‘locally Euclidean’, just like the football field is flat, but the whole Earth isn’t.

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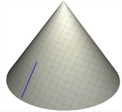
A manifold is smooth means it dose not have a pointy part:



Manifolds are also ‘locally Euclidean’, just like the football field is flat, but the whole Earth isn’t.
When we define Yang–Mills / Instanton equations on these manifolds, the space of all solutions of these equations (we call them moduli spaces) contain a lot of topological / geometric information about the manifolds, and hence extremely lucrative to mathematicians. For similar reasons, the higher dimensional theory is extensively studied by physicists and mathematicians working on string theory and Quantum field theory.

Thank you!