Circuits and Transforms

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Abstract—**This manual provides a simple introduction to Transforms**

1 Definitions

1. The unit step function is

$$u(t) = \begin{cases} 1 & t > 0 \\ \frac{1}{2} & t = 0 \\ 0 & t < 0 \end{cases} \tag{1.1}$$

2. The Laplace transform of $g(t)$ is defined as

$$G(s) = \int_{-\infty}^{\infty} g(t)e^{-st}\,dt \tag{1.2}$$

2 Laplace Transform

1. In the circuit, the switch S is connected to position P for a long time so that the charge on the capacitor becomes $q_1\,\mu C$. Then S is switched to position Q. After a long time, the charge on the capacitor is $q_2\,\mu C$.

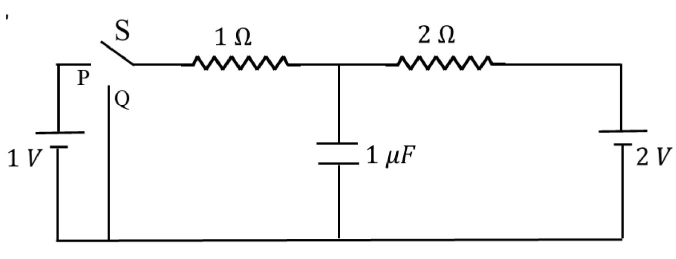
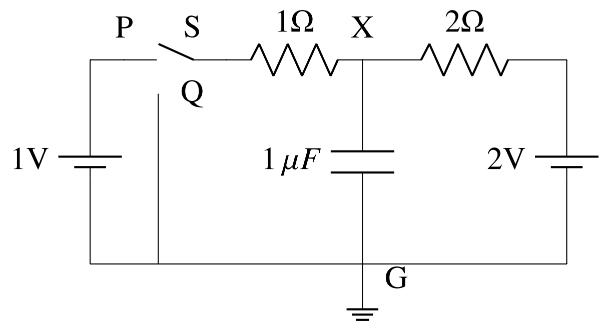
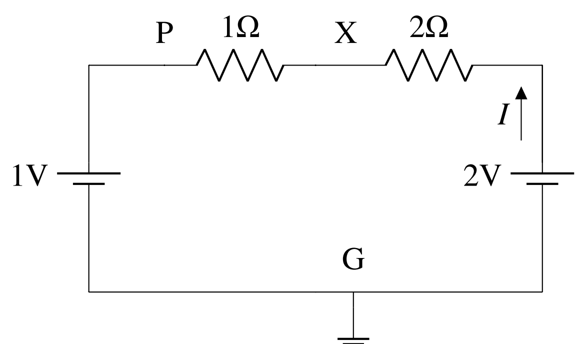


Fig. 2.1

2. Draw the circuit using latex-tikz.
Solution:



3. Find q_1.
Solution:
After a long time in position P, the charge on the capacitor is $q_1\,\mu C$.
Current through the capacitor is 0 at steady state. So, current only flows in the outer loop. Let the total current in the circuit be I.



$$I = \frac{(2-1)V}{(1+2)\Omega} \tag{2.1}$$
$$I = \frac{1}{3}A \tag{2.2}$$

Potential difference across the capacitor (across the points G and X) is

$$V_{C_0} = 2 - 2 \times \frac{1}{3} = 1.33V \tag{2.3}$$
$$q_1 = CV_{C_0} \tag{2.4}$$
$$q_1 = 1\mu F \times 1.33V \tag{2.5}$$
$$q_1 = 1.33\mu C \tag{2.6}$$

4. Show that the Laplace transform of $u(t)$ is $\frac{1}{s}$ and find the ROC.
Solution:

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By definition of Laplace transform of $u(t)$, we have

$$\mathcal{L}\{u(t)\} = \int_{-\infty}^{\infty} u(t)e^{-st}\,dt \tag{2.7}$$

$$= \int_{0}^{\infty} e^{-st}\,dt \tag{2.8}$$

$$= \left(-\frac{e^{-st}}{s}\right)_0^{+\infty} \tag{2.9}$$

$$\mathcal{L}\{u(t)\} = \frac{1}{s} \tag{2.10}$$

$\lim_{s\to\infty} e^{-st} = 0$ only when $Re(s) > 0$.
The ROC is $Re(s) > 0$.

5. Show that

$$e^{-at}u(t) \overset{\mathcal{H}}{\longleftrightarrow} L\frac{1}{s+a}, \quad a > 0 \tag{2.11}$$

and find the ROC.
Solution:

$$\mathcal{L}\{e^{-at}u(t)\} = \int_{-\infty}^{\infty} e^{-at}u(t)e^{-st}\,dt \tag{2.12}$$

$$= \int_{0}^{\infty} e^{-at}e^{-st}\,dt \tag{2.13}$$

$$= \left(-\frac{e^{-at}e^{-st}}{s+a}\right)_0^{+\infty} \tag{2.14}$$

$$\mathcal{L}\{e^{-at}u(t)\} = \frac{1}{s+a} \tag{2.15}$$

$\lim_{s\to\infty} e^{-(s+a)t} = 0$ only when $Re(s) > -a$.
The ROC is $Re(s) > -a$.

6. Now consider the following resistive circuit transformed from Fig. 2.1 where

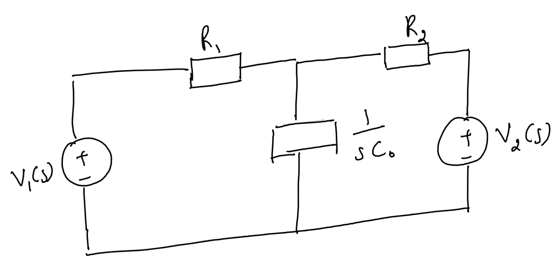


Fig. 2.2

$$u(t) \overset{\mathcal{H}}{\longleftrightarrow} LV_1(s) \tag{2.16}$$

$$2u(t) \overset{\mathcal{H}}{\longleftrightarrow} LV_2(s) \tag{2.17}$$

Find the voltage across the capacitor $V_{C_0}(s)$.

Solution:

$$V_1(s) = \mathcal{L}(u(t)) = \frac{1}{s} \tag{2.18}$$

$$V_2(s) = \mathcal{L}(2u(t)) = \frac{2}{s} \tag{2.19}$$

Potential across the capacitor is $V_{C_0}(s)$ and assume that the bottom is grounded.
Applying Kirchoff's circuital law at the junction, we have

$$\frac{V_{C_0}(s) - V_1(s)}{R_1} + \frac{V_{C_0}(s) - V_2(s)}{R_2} + \frac{V_{C_0}(s)}{\frac{1}{sC_0}} = 0 \tag{2.20}$$

$$V_{C_0}(s)\left(\frac{1}{R_1} + \frac{1}{R_2} + sC_0\right) = \frac{V_1(s)}{R_1} + \frac{V_2(s)}{R_2} \tag{2.21}$$

$$V_{C_0}(s) = \frac{V_1(s)R_2 + V_2(s)R_1}{R_1 + R_2 + sC_0R_1R_2} \tag{2.22}$$

$$V_{C_0}(s) = \frac{R_2 + 2R_1}{s(R_1 + R_2 + sC_0R_1R_2)} \tag{2.23}$$

7. Find $v_{C_0}(t)$. Plot using python.
Solution: Factoring $V_{C_0}(s)$, we have

$$V_{C_0}(s) = \frac{\left(\frac{R_2+2R_1}{R_1+R_2}\right)}{s} - \frac{R_1R_2C_0\left(\frac{R_2+2R_1}{R_1+R_2}\right)}{R_1 + R_2 + sC_0R_1R_2} \tag{2.24}$$

$$= \left(\frac{R_2 + 2R_1}{R_1 + R_2}\right)\left(\frac{1}{s} - \frac{R_1R_2C_0}{R_1 + R_2 + sC_0R_1R_2}\right) \tag{2.25}$$

$$= \left(\frac{R_2 + 2R_1}{R_1 + R_2}\right)\left(\frac{1}{s} - \frac{1}{\frac{1}{R_2C_0} + \frac{1}{R_1C_0} + s}\right) \tag{2.26}$$

Applying inverse Laplace transform on both sides, we obtain

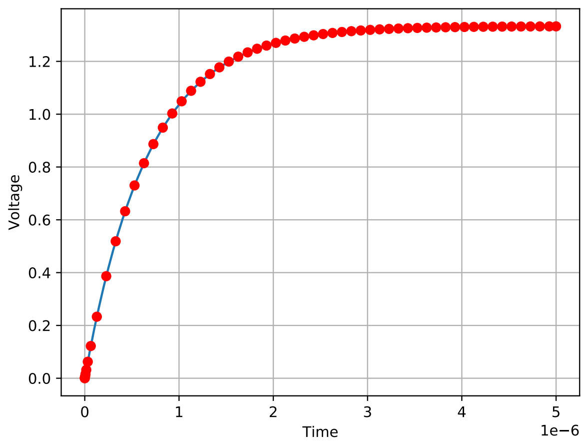
$$v_{C_0}(t) = \frac{R_2 + 2R_1}{R_1 + R_2}\left(u(t) - e^{-\left(\frac{1}{R_1C_0} + \frac{1}{R_2C_0}\right)t}u(t)\right) \tag{2.27}$$

Using the values $R_1 = 1\,\Omega$, $R_2 = 2\,\Omega$, $C_0 = 1\,\mu F$, we have

$$v_{C_0}(t) = \frac{4}{3}u(t)\left(1 - e^{-1.5\times10^6\,t}\right) \tag{2.28}$$

Download the following python code for plot of $v_{C_0}(t)$.

```
wget 2_7.py
```



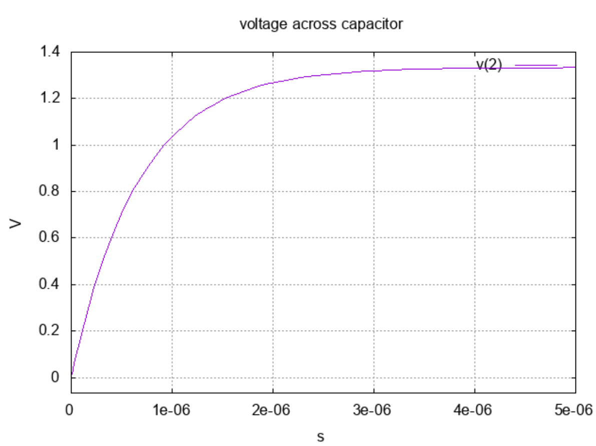
8. Verify your result using ngspice.
Solution: Download the following code for ngspice simulation.

```
wget 2_8.spice
```

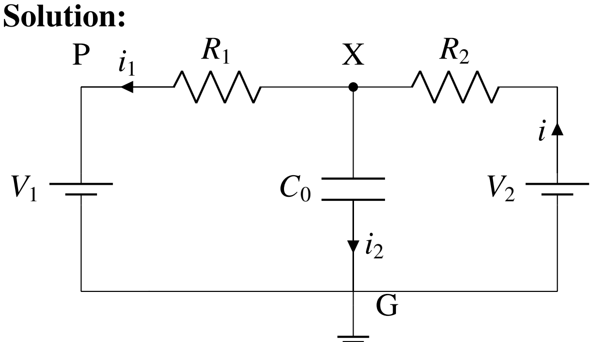
Run using the command

```
ngspice 2_8.spice
```

The plot obtained is show below



9. Obtain Fig. 2.2 using the equivalent differential equation.
Solution:



Applying Kirchoff's loop law, we have

$$V_{C_0} = i_1R_1 + V_1 \tag{2.29}$$

$$V_2 - iR_2 = V_{C_0} \tag{2.30}$$

$$V_2 - (i_1 + i_2)R_2 = V_{C_0} \tag{2.31}$$

$$i_2 = \frac{d\left(C_0V_{C_0}\right)}{dt} \tag{2.32}$$

To find resistance of capacitor in $s-$domain, we calculate the value of $\frac{V_{C_0}}{s}$

$$\mathcal{L}\left(\frac{dV_{C_0}}{dt}\right) = \int_{-\infty}^{\infty}\frac{dV_{C_0}}{dt}e^{-st}dt \tag{2.33}$$

$$= e^{-st}V_{C_0}(t) + \int_{-\infty}^{\infty} se^{-st}V_{C_0}(t)dt \tag{2.34}$$

$$= sV_{C_0}(s) - V_{C_0}(0) \tag{2.35}$$

$$= sV_{C_0}(s) \tag{2.36}$$

To find resistance of capacitor in $s-$domain, we calculate the value of

$$\frac{V_{C_0}(s)}{i_2(s)} = \frac{V_{C_0}(s)}{s\,C_0\,V_{C_0}(s)} \tag{2.37}$$

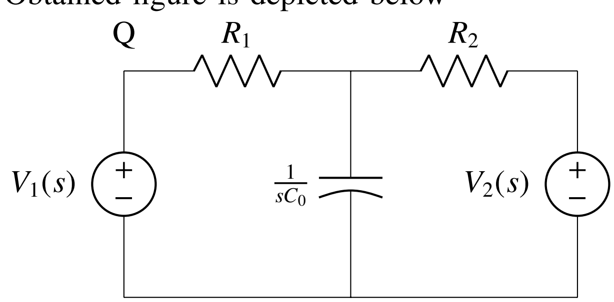
$$= \frac{1}{sC_0} \tag{2.38}$$

The circuit elements will be as follows

$$V_1(t) \xleftrightarrow{\mathcal{L}} \frac{1}{s} \quad V_2(t) \xleftrightarrow{\mathcal{L}} \frac{2}{s} \tag{2.39}$$

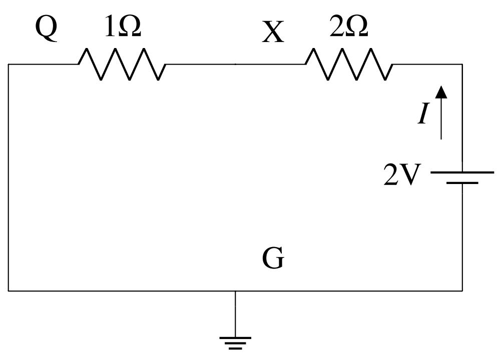
$$C_0 \xleftrightarrow{\mathcal{L}} \frac{1}{sC_0} \tag{2.40}$$

Resistors will remain unchanged in $s-$domain Obtained figure is depicted below



3 Initial Conditions

1. Find q_2 in Fig. 2.1.
Solution: After a long time in position Q, the charge on the capacitor is $q_2\,\mu C$.
Current through the capacitor is 0 in steady state. So, current only flows in the outer loop. Let the total current in the circuit be I.



$$I = \frac{2V}{(1+2)\Omega} = \frac{2}{3}A \tag{3.1}$$

So, the potential difference across the capacitor (across the points G and X) is

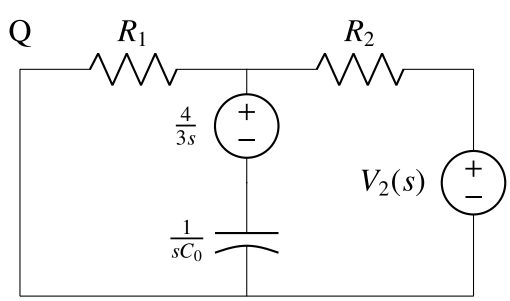
$$V_{C_0} = 2 - \left(\frac{2}{3} \times 2\right) = 0.67V \tag{3.2}$$

$$q_2 = CV_{C_0} \tag{3.3}$$

$$q_2 = 1\,\mu C \times 0.67V \tag{3.4}$$

$$q_2 = 0.67\,\mu C \tag{3.5}$$

2. Draw the equivalent s-domain resistive circuit when S is switched to position Q. Use variables R_1, R_2, C_0 for the passive elements. Use latex-tikz.
Solution:



3. $V_{C_0}(s) = ?$
Solution:

$$V_{C_0}(s)\left(\frac{R_1 + R_2 + sC_0R_1R_2}{R_1R_2}\right) = \frac{6 + 4C_0sR_2}{3sR_2} \tag{3.6}$$

$$V_{C_0}(s) = \frac{\left(\frac{2R_1}{s}\right) + \left(\frac{4C_0}{3}\right)R_1R_2}{(R_1 + R_2 + sC_0R_1R_2)} \tag{3.7}$$

4. $v_{C_0}(t) = ?$ Plot using python.
Solution: Splitting into partial fractions, we have

$$V_{C_0}(s) = \frac{\left(\frac{2R_1}{s}\right) + \left(\frac{4C_0}{3}\right)R_1R_2}{(R_1 + R_2 + sC_0R_1R_2)} \tag{3.8}$$

$$V_{C_0}(s) = \frac{2R_1}{s\,(R_1 + R_2 + sC_0R_1R_2)} + \tag{3.9}$$

$$\frac{4C_0R_1R_2}{3\,(R_1 + R_2 + sC_0R_1R_2)} \tag{3.10}$$

$$= \frac{2R_1}{R_1 + R_2}\left(\frac{1}{s} - \frac{1}{\left(\frac{1}{R_1C_0} + \frac{1}{R_2C_0} + s\right)}\right) + \tag{3.11}$$

$$\frac{4}{3\left(\frac{1}{R_1C_0} + \frac{1}{R_2C_0} + s\right)} \tag{3.12}$$

Applying inverse Laplace transform,

$$v_{C_0}(t) = \frac{2R_1}{R_1 + R_2}\left(1 - e^{\frac{-t}{C_0}\left(\frac{1}{R_1}+\frac{1}{R_2}\right)}\right)u(t) + \frac{4}{3}e^{\frac{-t}{C_0}\left(\frac{1}{R_1}+\frac{1}{R_2}\right)}u(t) \tag{3.13}$$

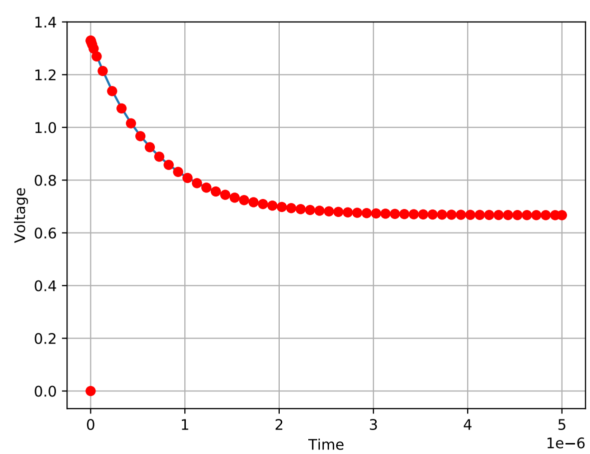
Substituting the values, we have

$$v_{C_0}(t) = \frac{2}{3}\left(1 - e^{-1.5t\times10^6}\right)u(t) + \frac{4}{3}e^{-1.5t\times10^6}u(t) \tag{3.14}$$

$$v_{C_0}(t) = \frac{2}{3}\left(1 + e^{-1.5t\times10^6}\right)u(t) \tag{3.15}$$

Download the following python code for the plot

```
wget 3_4.py
```



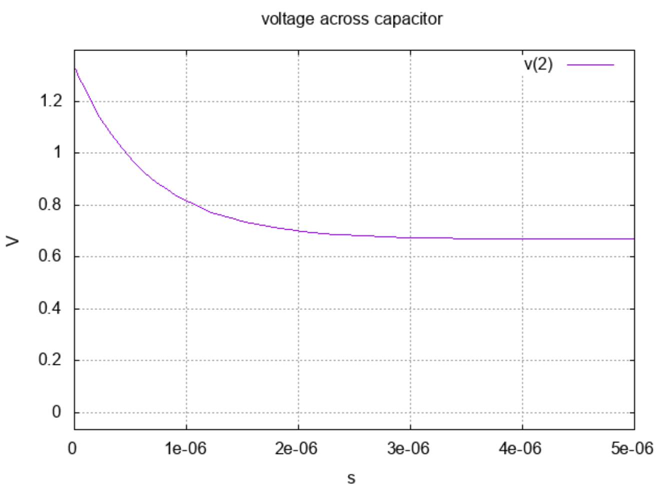
5. Verify your result using ngspice.
Download the following code for ngspice simulation.

```
wget 3_4.spice
```

Run using the command

```
ngspice 3_4.spice
```

The plot obtained is show below



6. Find $v_{C_0}(0-), v_{C_0}(0+)$ and $v_{C_0}(\infty)$.
Solution: At $t = 0^-$, the switch is in position P for a long time

$$v_{C_0}(0-) = \frac{4}{3}V \tag{3.16}$$

At $t = 0^+$, the switch has just shifted to position Q

$$\lim_{t->0^+} v_{C_0}(t) = \lim_{t->0^+} \frac{2}{3}\left(1 + e^{-1.5t\times10^6}\right)u(t) \tag{3.17}$$

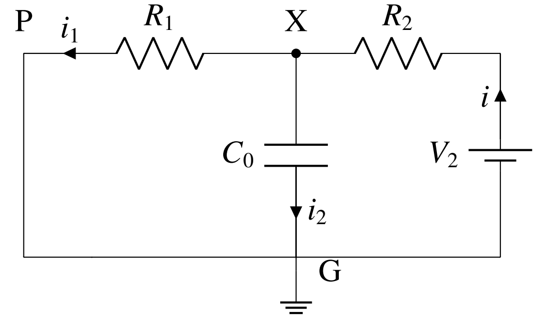
$$v_{C_0}(0+) = \frac{4}{3}V \tag{3.18}$$

At $t = \infty$, the switch is in position Q for a long time

$$lim_{t->\infty} v_{C_0}(t) = \lim_{t->\infty} \frac{2}{3}\left(1 + e^{-1.5t\times10^6}\right)u(t) \tag{3.19}$$

$$v_{C_0}(\infty) = \frac{2}{3}V \tag{3.20}$$

7. Obtain the Fig. in problem 3.2 using the equivalent differential equation.
Solution:



Applying Kirchoff's loop law, we have

$$V_{C_0} = i_1R_1 \tag{3.21}$$

$$V_2 - iR_2 = V_{C_0} \tag{3.22}$$

$$V_2 - (i_1 + i_2)R_2 = V_{C_0} \tag{3.23}$$

$$i_2 = \frac{d\left(C_0V_{C_0}\right)}{dt} \tag{3.24}$$

To find resistance of capacitor in $s-$domain, we calculate the value of $\frac{V_{C_0}}{s}$

$$\mathcal{L}\left(\frac{dV_{C_0}}{dt}\right) = \int_{-\infty}^{\infty} \frac{dV_{C_0}}{dt} e^{-st}dt \tag{3.25}$$

$$= e^{-st}V_{C_0}(t) + \int_{-\infty}^{\infty} se^{-st}V_{C_0}(t)dt \tag{3.26}$$

$$= sV_{C_0}(s) - V_{C_0}(0) \tag{3.27}$$

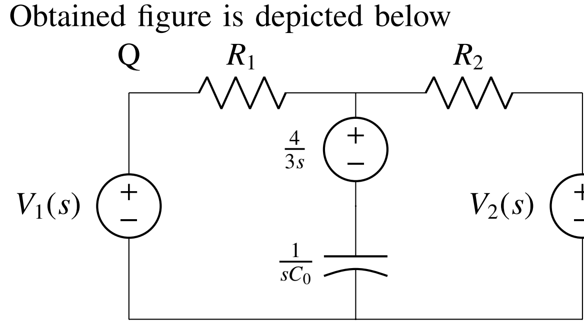
$$= sV_{C_0}(s) - \frac{4}{3} \tag{3.28}$$

The circuit elements will be as follows

$$V_1(t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{1}{s} \quad V_2(t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{2}{s} \tag{3.29}$$

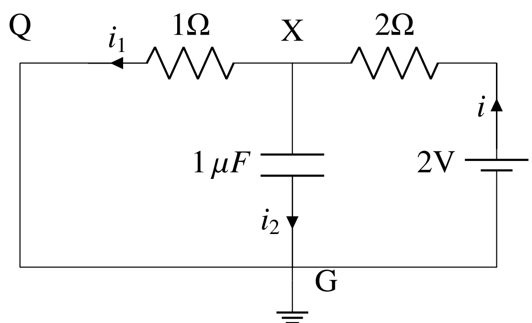
$$C_0 \overset{\mathcal{L}}{\longleftrightarrow} \frac{1}{sC_0} \quad V_{C_0}(0^-) \overset{\mathcal{L}}{\longleftrightarrow} \frac{4}{3s} \tag{3.30}$$

Resistors will remain unchanged in $s-$domain
Obtained figure is depicted below



4 Bilinear Transform

1. In Fig. 2.1, Consider the case when S is switched to Q right in the beginning. Formulate the differential equation.
Solution:
The circuit for the case is shown below,



The differential equation can be formulated as follows,

$$i_1(R_1) = \int_0^t \frac{i_2(t)}{C_0} dt \tag{4.1}$$

$$V_2 - iR_2 - \int_0^t \frac{i_2(t)}{C_0} dt = 0 \tag{4.2}$$

$$V_2 - (i_1 + i_2)R_2 = \int_0^t \frac{i_2(t)}{C_0} dt \tag{4.3}$$

$$V_2 - (i_2 R_2) = \left(\frac{R_2}{R_1} + 1\right) \int_0^t \frac{i_2(t)}{C_0} dt \tag{4.4}$$

Expressing the differential equation in terms of V_{C_0}, we have

$$V_{C_0} = \int_0^t \frac{i_2(t)}{C_0} dt \tag{4.5}$$

$$\frac{dV_{C_0}}{dt} = \frac{i_2(t)}{C_0} \tag{4.6}$$

$$V_2 - R_2\, C_0 \frac{dV_{C_0}}{dt} = \left(\frac{R_1 + R_2}{R_1}\right) V_{C_0} \tag{4.7}$$

$$R_2 C_0 \frac{dV_{C_0}}{dt} + \left(\frac{R_1 + R_2}{R_1}\right) V_{C_0} - V_2 = 0 \tag{4.8}$$

2. Find $H(s)$ considering the output voltage at the capacitor.
Solution:
Applying Laplace transform on both sides of the equation,

$$R_2 C_0 s V_{C_0}(s) + \frac{R_1 + R_2}{R_1} V_{C_0}(s) - V_2(s) = 0 \tag{4.9}$$

$$\frac{V_2(s)}{V_{C_0}(s)} = sR_2 C_0 + 1 + \frac{R_2}{R_1} \tag{4.10}$$

By definition of transfer function, we have

$$H(s) = \frac{V_{C_0}(s)}{V_2(s)} = \frac{1}{sR_2 C_0 + 1 + \frac{R_2}{R_1}} \tag{4.11}$$

On substituting the values, we get

$$H(s) = \frac{5 \times 10^5}{s + 1.5 \times 10^6} \tag{4.12}$$

3. Plot $H(s)$. What kind of filter is it?

Solution: Download the following Python code that plots Fig. 4.1

```
wget 4_3.py
```

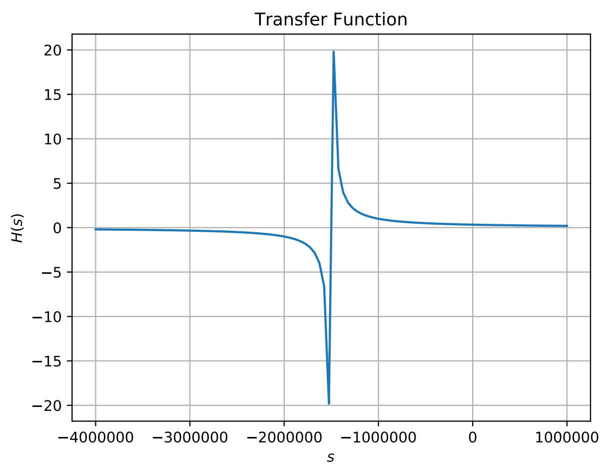


Fig. 4.1: Plot of $H(s)$

Consider the frequency-domain transfer function by putting $s = j\omega$

$$H(j\omega) = \frac{5 \times 10^5}{j\omega + 1.5 \times 10^6} \tag{4.13}$$

$$|H(j\omega)| = \frac{5 \times 10^5}{\sqrt{\omega^2 + 2.25 \times 10^{12}}} \tag{4.14}$$

Since maximum value of transfer function occurs when $\omega = 0$, we have the following relation,
$H(j\omega)$ decreases as ω increases.
So, frequency ranges from 0 to cut-off frequency.
Hence, the given filter is a low-pass filter.

4. Using trapezoidal rule for integration, formulate the difference equation by considering

$$y(n) = y(t)|_{t=n} \tag{4.15}$$

Solution:

$$R_2C_0\frac{dV_{C_0}}{dt} + \left(\frac{R_1 + R_2}{R_1}\right)V_{C_0} - V_2 = 0 \tag{4.16}$$

$$C_0\frac{dV_{C_0}}{dt} = \frac{2V_2 - V_{C_0}(t)}{R_2} - \frac{V_{C_0}(t)}{R_1} \tag{4.17}$$

$$V_{C_0}(t)_n^{n+1} = \int_n^{n+1}\left(\frac{2V_2(t) - V_{C_0}(t)}{R_2C_0} - \frac{V_{C_0}(t)}{R_1C_0}\right)dt \tag{4.18}$$

By the trapezoidal rule of integration, we approximate the following integral

$$\int_a^b f(t)dt = \frac{b-a}{2}\Big(f(a) + f(b)\Big) \tag{4.19}$$

Consider $y(t) = V_{C_0}(t)$, and simplifying using trapezoid rule on right hand side of the equation, we have

$$\begin{aligned} y(n+1) - y(n) &= \frac{1}{R_2C_0}\left(u(n) + u(n+1)\right) \\ &- \frac{1}{2}(y(n+1) + y(n))\left(\frac{1}{R_1C_0} + \frac{1}{R_2C_0}\right) \end{aligned} \tag{4.20}$$

Thus, the difference equation is

$$\begin{aligned} & y(n+1)\left(1 + \frac{1}{2R_1C_0} + \frac{1}{2R_2C_0}\right) \\ =& y(n)\left(1 - \frac{1}{2R_1C_0} - \frac{1}{2R_2C_0}\right) \\ +& \frac{1}{R_2C_0}\left(u(n) + u(n+1)\right) \end{aligned} \tag{4.21}$$

5. Find $H(z)$.
Solution: Applying $\mathcal{Z}$-transform to the (4.21) on both sides, we have

$$\begin{aligned} & zY(z)\left(\frac{2R_1R_2C_0 + R_1 + R_2}{2R_1R_2C_0}\right) = \\ & Y(z)\left(\frac{2R_1R_2C_0 - R_1 - R_2}{2R_1R_2C_0}\right) + \\ & \frac{1}{R_2C_0}\left(\frac{1}{1 - z^{-1}} + \frac{z}{1 - z^{-1}}\right) \end{aligned} \tag{4.22}$$

$$Y(z) = \frac{\left(\frac{1+z}{R_2C_0(1 - z^{-1})}\right)2R_1R_2C_0}{(2R_1R_2C_0(z-1)) + (z+1)(R_1 + R_2)} \tag{4.23}$$

$$X(z) = \mathcal{Z}[x(n)] = \mathcal{Z}[2u(n)] = \frac{2}{1 - z^{-1}} \tag{4.24}$$

Using $H(z) = \frac{Y(z)}{X(z)}$, and substituting values

$$H(z) = \frac{(1+z)}{(4C_0(z-1)) + 3(z+1)} \tag{4.25}$$

$$H(z) = \frac{1 + z^{-1}}{(3 + 4C_0) + (3 - 4C_0)\,z^{-1}} \tag{4.26}$$

Region of convergence (ROC) for the $\mathcal{Z}$-transform is

$$|z| > \max\left(1, \left|\frac{3 + 4\times 10^{-6}}{4\times 10^{-6} - 3}\right|\right) \tag{4.27}$$

6. How can you obtain $H(z)$ from $H(s)$?
Solution: To convert $H(z)$ to $H(s)$, we use the bilinear transform.
It maps transfer function of $\mathcal{Z}$-transform to transfer function of s-domain.

$$s \longleftrightarrow \frac{2}{T}\frac{1 - z^{-1}}{1 + z^{-1}} \tag{4.28}$$

$T = 1$ is the step size in trapezoidal rule
From (4.11), we have

$$H(s) = \frac{1}{sR_2C_0 + 1 + \frac{R_2}{R_1}} \tag{4.29}$$

$$H(z) = \frac{1}{\left(\frac{1 - z^{-1}}{1 + z^{-1}}\right)2R_2C_0 + 1 + \frac{R_2}{R_1}} \tag{4.30}$$

$$H(z) = \frac{1 + z^{-1}}{4C_0\left(1 - z^{-1}\right) + 3\left(1 + z^{-1}\right)} \tag{4.31}$$

$$H(z) = \frac{1 + z^{-1}}{(3 + 4C_0) + (3 - 4C_0)\,z^{-1}} \tag{4.32}$$

The equation obtained is the same as the equation (4.26).

7. Plot y(n), y(t), and voltage through ngspice. Verify that they are the same.
Solution: