

# Circuits and Transforms

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**Abstract**—This manual provides a simple introduction to Transforms

## 1 DEFINITIONS

1. The unit step function is

$$u(t) = \begin{cases} 1 & t > 0 \\ \frac{1}{2} & t = 0 \\ 0 & t < 0 \end{cases} \quad (1.1)$$

2. The Laplace transform of  $g(t)$  is defined as

$$G(s) = \int_{-\infty}^{\infty} g(t)e^{-st} dt \quad (1.2)$$

## 2 LAPLACE TRANSFORM

1. In the circuit, the switch S is connected to position P for a long time so that the charge on the capacitor becomes  $q_1 \mu C$ . Then S is switched to position Q. After a long time, the charge on the capacitor is  $q_2 \mu C$ .

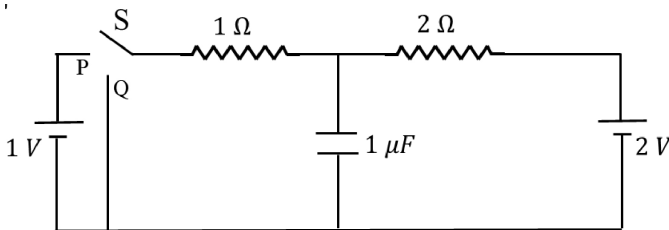
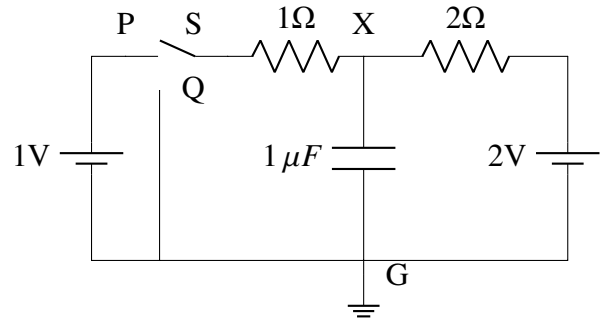


Fig. 2.1

2. Draw the circuit using latex-tikz.

**Solution:**

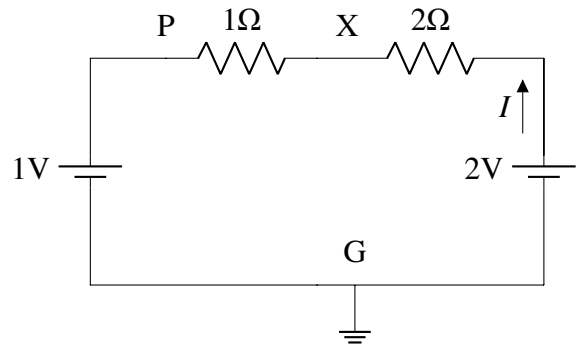


3. Find  $q_1$ .

**Solution:**

After a long time in position P, the charge on the capacitor is  $q_1 \mu C$ .

Current through the capacitor is 0 at steady state. So, current only flows in the outer loop. Let the total current in the circuit be  $I$ .



$$I = \frac{(2 - 1)V}{(1 + 2)\Omega} \quad (2.1)$$

$$I = \frac{1}{3}A \quad (2.2)$$

Potential difference across the capacitor (across the points G and X) is

$$V_{C_0} = 2 - 2 \times \frac{1}{3} = 1.33V \quad (2.3)$$

$$q_1 = CV_{C_0} \quad (2.4)$$

$$q_1 = 1\mu F \times 1.33V \quad (2.5)$$

$$q_1 = 1.33\mu C \quad (2.6)$$

4. Show that the Laplace transform of  $u(t)$  is  $\frac{1}{s}$  and find the ROC.

**Solution:**

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By definition of Laplace transform of  $u(t)$ , we have

$$\mathcal{L}\{u(t)\} = \int_{-\infty}^{\infty} u(t)e^{-st} dt \quad (2.7)$$

$$= \int_0^{\infty} e^{-st} dt \quad (2.8)$$

$$= \left( -\frac{e^{-st}}{s} \right)_0^{+\infty} \quad (2.9)$$

$$\mathcal{L}\{u(t)\} = \frac{1}{s} \quad (2.10)$$

$\lim_{s \rightarrow \infty} e^{-st} = 0$  only when  $\text{Re}(s) > 0$ .

The ROC is  $\text{Re}(s) > 0$ .

5. Show that

$$e^{-at}u(t) \xleftrightarrow{\mathcal{H}} L \frac{1}{s+a}, \quad a > 0 \quad (2.11)$$

and find the ROC.

**Solution:**

$$\mathcal{L}\{e^{-at}u(t)\} = \int_{-\infty}^{\infty} e^{-at}u(t)e^{-st} dt \quad (2.12)$$

$$= \int_0^{\infty} e^{-at}e^{-st} dt \quad (2.13)$$

$$= \left( -\frac{e^{-at}e^{-st}}{s+a} \right)_0^{+\infty} \quad (2.14)$$

$$\mathcal{L}\{e^{-at}u(t)\} = \frac{1}{s+a} \quad (2.15)$$

$\lim_{s \rightarrow \infty} e^{-(s+a)t} = 0$  only when  $\text{Re}(s) > -a$ .

The ROC is  $\text{Re}(s) > -a$ .

6. Now consider the following resistive circuit transformed from Fig. 2.1 where

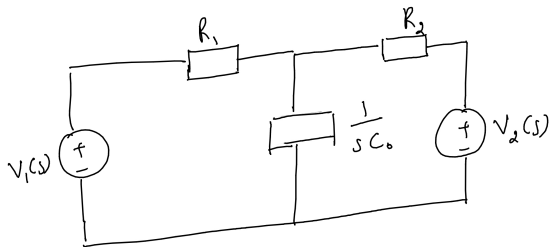


Fig. 2.2

$$u(t) \xleftrightarrow{\mathcal{H}} LV_1(s) \quad (2.16)$$

$$2u(t) \xleftrightarrow{\mathcal{H}} LV_2(s) \quad (2.17)$$

Find the voltage across the capacitor  $V_{C_0}(s)$ .

**Solution:**

$$V_1(s) = \mathcal{L}(u(t)) = \frac{1}{s} \quad (2.18)$$

$$V_2(s) = \mathcal{L}(2u(t)) = \frac{2}{s} \quad (2.19)$$

Potential across the capacitor is  $V_{C_0}(s)$  and assume that the bottom is grounded.

Applying Kirchoff's circuital law at the junction, we have

$$\frac{V_{C_0}(s) - V_1(s)}{R_1} + \frac{V_{C_0}(s) - V_2(s)}{R_2} + \frac{V_{C_0}(s)}{\frac{1}{sC_0}} = 0 \quad (2.20)$$

$$V_{C_0}(s) \left( \frac{1}{R_1} + \frac{1}{R_2} + sC_0 \right) = \frac{V_1(s)}{R_1} + \frac{V_2(s)}{R_2} \quad (2.21)$$

$$V_{C_0}(s) = \frac{V_1(s)R_2 + V_2(s)R_1}{R_1 + R_2 + sC_0R_1R_2} \quad (2.22)$$

$$V_{C_0}(s) = \frac{R_2 + 2R_1}{s(R_1 + R_2 + sC_0R_1R_2)} \quad (2.23)$$

7. Find  $v_{C_0}(t)$ . Plot using python.

**Solution:** Factoring  $V_{C_0}(s)$ , we have

$$V_{C_0}(s) = \frac{\left( \frac{R_2 + 2R_1}{R_1 + R_2} \right)}{s} - \frac{R_1R_2C_0 \left( \frac{R_2 + 2R_1}{R_1 + R_2} \right)}{R_1 + R_2 + sC_0R_1R_2} \quad (2.24)$$

$$= \left( \frac{R_2 + 2R_1}{R_1 + R_2} \right) \left( \frac{1}{s} - \frac{R_1R_2C_0}{R_1 + R_2 + sC_0R_1R_2} \right) \quad (2.25)$$

$$= \left( \frac{R_2 + 2R_1}{R_1 + R_2} \right) \left( \frac{1}{s} - \frac{1}{\frac{1}{R_1C_0} + \frac{1}{R_2C_0} + s} \right) \quad (2.26)$$

Applying inverse Laplace transform on both sides, we obtain

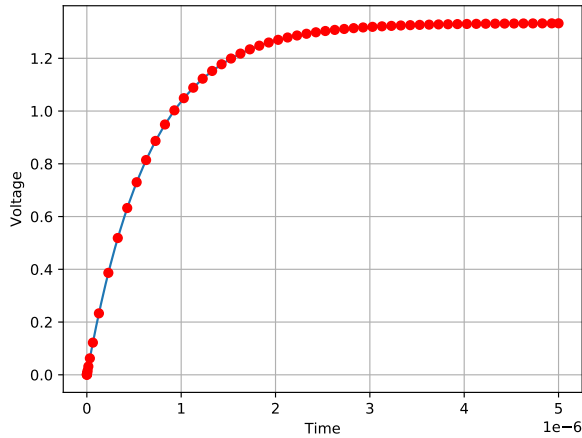
$$v_{C_0}(t) = \frac{R_2 + 2R_1}{R_1 + R_2} \left( u(t) - e^{-\left( \frac{1}{R_1C_0} + \frac{1}{R_2C_0} \right)t} u(t) \right) \quad (2.27)$$

Using the values  $R_1 = 1 \Omega$ ,  $R_2 = 2 \Omega$ ,  $C_0 = 1 \mu F$ , we have

$$v_{C_0}(t) = \frac{4}{3} u(t) \left( 1 - e^{-1.5 \times 10^6 t} \right) \quad (2.28)$$

Download the following python code for plot of  $v_{C_0}(t)$ .

wget 2\_7.py



8. Verify your result using ngspice.

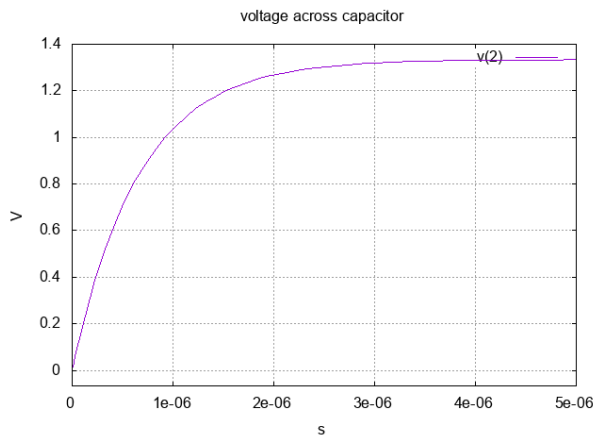
**Solution:** Download the following code for ngspice simulation.

```
wget 2_8.spice
```

Run using the command

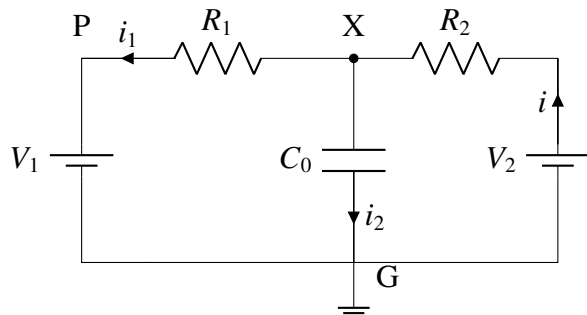
```
ngspice 2_8.spice
```

The plot obtained is show below



9. Obtain Fig. 2.2 using the equivalent differential equation.

**Solution:**



Applying Kirchoff's loop law, we have

$$V_{C_0} = i_1 R_1 + V_1 \quad (2.29)$$

$$V_2 - i R_2 = V_{C_0} \quad (2.30)$$

$$V_2 - (i_1 + i_2) R_2 = V_{C_0} \quad (2.31)$$

$$i_2 = \frac{d(C_0 V_{C_0})}{dt} \quad (2.32)$$

To find resistance of capacitor in  $s$ -domain, we calculate the value of  $\frac{V_{C_0}}{s}$

$$\mathcal{L}\left(\frac{dV_{C_0}}{dt}\right) = \int_{-\infty}^{\infty} \frac{dV_{C_0}}{dt} e^{-st} dt \quad (2.33)$$

$$= e^{-st} V_{C_0}(t) + \int_{-\infty}^{\infty} s e^{-st} V_{C_0}(t) dt \quad (2.34)$$

$$= s V_{C_0}(s) - V_{C_0}(0) \quad (2.35)$$

$$= s V_{C_0}(s) \quad (2.36)$$

To find resistance of capacitor in  $s$ -domain, we calculate the value of

$$\frac{V_{C_0}(s)}{i_2(s)} = \frac{V_{C_0}(s)}{s C_0 V_{C_0}(s)} \quad (2.37)$$

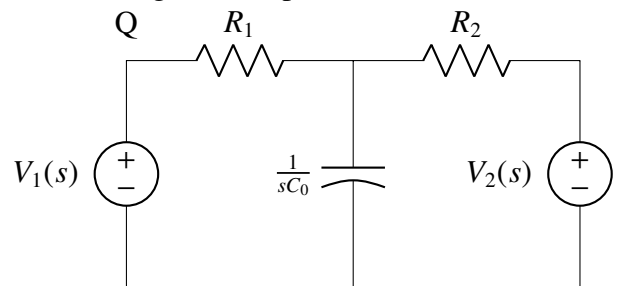
$$= \frac{1}{s C_0} \quad (2.38)$$

The circuit elements will be as follows

$$V_1(t) \xleftrightarrow{\mathcal{L}} \frac{1}{s} \quad V_2(t) \xleftrightarrow{\mathcal{L}} \frac{2}{s} \quad (2.39)$$

$$C_0 \xleftrightarrow{\mathcal{L}} \frac{1}{s C_0} \quad (2.40)$$

Resistors will remain unchanged in  $s$ -domain  
Obtained figure is depicted below

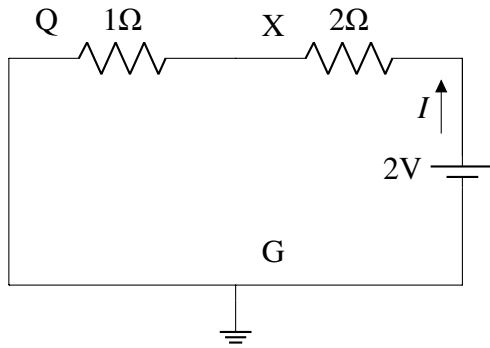


### 3 INITIAL CONDITIONS

1. Find  $q_2$  in Fig. 2.1.

**Solution:** After a long time in position  $Q$ , the charge on the capacitor is  $q_2 \mu C$ .

Current through the capacitor is 0 in steady state. So, current only flows in the outer loop. Let the total current in the circuit be  $I$ .



$$I = \frac{2V}{(1+2)\Omega} = \frac{2}{3}A \quad (3.1)$$

So, the potential difference across the capacitor (across the points G and X) is

$$V_{C_0} = 2 - \left(\frac{2}{3} \times 2\right) = 0.67V \quad (3.2)$$

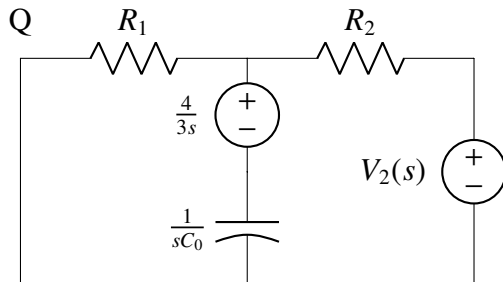
$$q_2 = CV_{C_0} \quad (3.3)$$

$$q_2 = 1\mu C \times 0.67V \quad (3.4)$$

$$q_2 = 0.67\mu C \quad (3.5)$$

2. Draw the equivalent  $s$ -domain resistive circuit when S is switched to position Q. Use variables  $R_1, R_2, C_0$  for the passive elements. Use latex-tikz.

**Solution:**



3.  $V_{C_0}(s) = ?$

**Solution:**

$$V_{C_0}(s) \left( \frac{R_1 + R_2 + sC_0R_1R_2}{R_1R_2} \right) = \frac{6 + 4C_0sR_2}{3sR_2} \quad (3.6)$$

$$V_{C_0}(s) = \frac{\left(\frac{2R_1}{s}\right) + \left(\frac{4C_0}{3}\right)R_1R_2}{(R_1 + R_2 + sC_0R_1R_2)} \quad (3.7)$$

4.  $v_{C_0}(t) = ?$  Plot using python.

**Solution:** Splitting into partial fractions, we

have

$$V_{C_0}(s) = \frac{\left(\frac{2R_1}{s}\right) + \left(\frac{4C_0}{3}\right)R_1R_2}{(R_1 + R_2 + sC_0R_1R_2)} \quad (3.8)$$

$$V_{C_0}(s) = \frac{2R_1}{s(R_1 + R_2 + sC_0R_1R_2)} + \quad (3.9)$$

$$\frac{4C_0R_1R_2}{3(R_1 + R_2 + sC_0R_1R_2)} \quad (3.10)$$

$$= \frac{2R_1}{R_1 + R_2} \left( \frac{1}{s} - \frac{1}{\left(\frac{1}{R_1C_0} + \frac{1}{R_2C_0} + s\right)} \right) + \quad (3.11)$$

$$\frac{4}{3 \left( \frac{1}{R_1C_0} + \frac{1}{R_2C_0} + s \right)} \quad (3.12)$$

Applying inverse Laplace transform,

$$v_{C_0}(t) = \frac{2R_1}{R_1 + R_2} \left( 1 - e^{-\frac{t}{C_0} \left( \frac{1}{R_1} + \frac{1}{R_2} \right)} \right) u(t) + \frac{4}{3} e^{-\frac{t}{C_0} \left( \frac{1}{R_1} + \frac{1}{R_2} \right)} u(t) \quad (3.13)$$

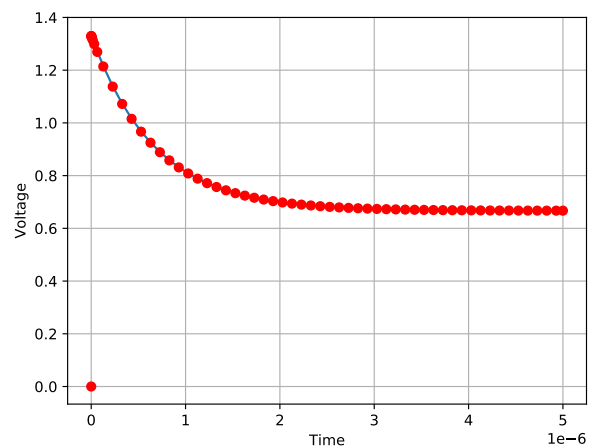
Substituting the values, we have

$$v_{C_0}(t) = \frac{2}{3} \left( 1 - e^{-1.5t \times 10^6} \right) u(t) + \frac{4}{3} e^{-1.5t \times 10^6} u(t) \quad (3.14)$$

$$v_{C_0}(t) = \frac{2}{3} \left( 1 + e^{-1.5t \times 10^6} \right) u(t) \quad (3.15)$$

Download the following python code for the plot

wget 3\_4.py



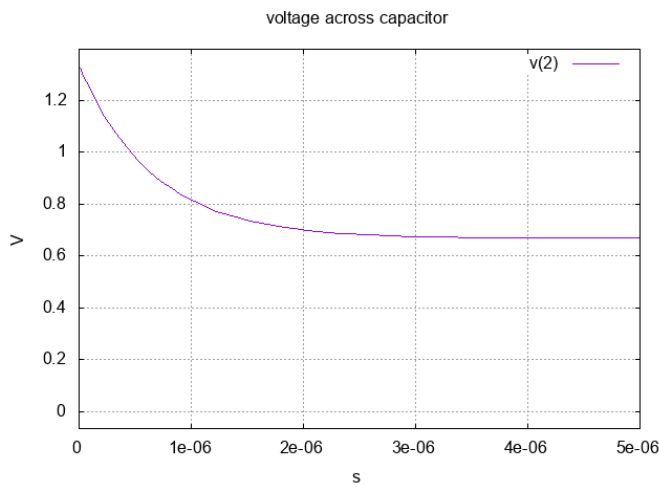
5. Verify your result using ngspice.  
Download the following code for ngspice simulation.

```
wget 3_4.spice
```

Run using the command

```
ngspice 3_4.spice
```

The plot obtained is show below



6. Find  $v_{C_0}(0^-)$ ,  $v_{C_0}(0^+)$  and  $v_{C_0}(\infty)$ .

**Solution:** At  $t = 0^-$ , the switch is in position  $P$  for a long time

$$v_{C_0}(0^-) = \frac{4}{3}V \quad (3.16)$$

At  $t = 0^+$ , the switch has just shifted to position  $Q$

$$\lim_{t \rightarrow 0^+} v_{C_0}(t) = \lim_{t \rightarrow 0^+} \frac{2}{3} (1 + e^{-1.5t \times 10^6}) u(t) \quad (3.17)$$

$$v_{C_0}(0^+) = \frac{4}{3}V \quad (3.18)$$

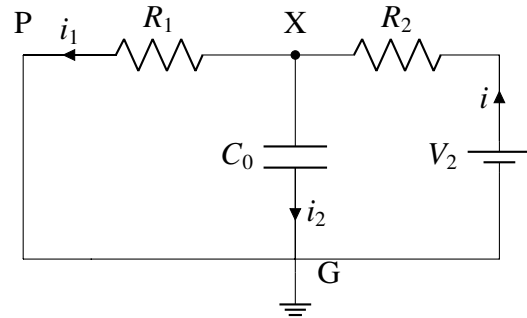
At  $t = \infty$ , the switch is in position  $Q$  for a long time

$$\lim_{t \rightarrow \infty} v_{C_0}(t) = \lim_{t \rightarrow \infty} \frac{2}{3} (1 + e^{-1.5t \times 10^6}) u(t) \quad (3.19)$$

$$v_{C_0}(\infty) = \frac{2}{3}V \quad (3.20)$$

7. Obtain the Fig. in problem 3.2 using the equivalent differential equation.

**Solution:**



Applying Kirchoff's loop law, we have

$$V_{C_0} = i_1 R_1 \quad (3.21)$$

$$V_2 - i R_2 = V_{C_0} \quad (3.22)$$

$$V_2 - (i_1 + i_2) R_2 = V_{C_0} \quad (3.23)$$

$$i_2 = \frac{d(C_0 V_{C_0})}{dt} \quad (3.24)$$

To find resistance of capacitor in  $s$ -domain, we calculate the value of  $\frac{V_{C_0}}{s}$

$$\mathcal{L}\left(\frac{dV_{C_0}}{dt}\right) = \int_{-\infty}^{\infty} \frac{dV_{C_0}}{dt} e^{-st} dt \quad (3.25)$$

$$= e^{-st} V_{C_0}(t) + \int_{-\infty}^{\infty} s e^{-st} V_{C_0}(t) dt \quad (3.26)$$

$$= s V_{C_0}(s) - V_{C_0}(0) \quad (3.27)$$

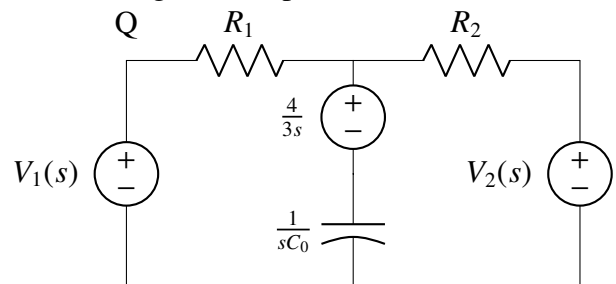
$$= s V_{C_0}(s) - \frac{4}{3} \quad (3.28)$$

The circuit elements will be as follows

$$V_1(t) \xleftrightarrow{\mathcal{L}} \frac{1}{s} \quad V_2(t) \xleftrightarrow{\mathcal{L}} \frac{2}{s} \quad (3.29)$$

$$C_0 \xleftrightarrow{\mathcal{L}} \frac{1}{sC_0} \quad V_{C_0}(0^-) \xleftrightarrow{\mathcal{L}} \frac{4}{3s} \quad (3.30)$$

Resistors will remain unchanged in  $s$ -domain  
Obtained figure is depicted below

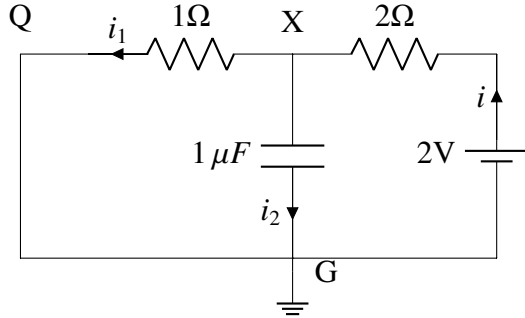


#### 4 BILINEAR TRANSFORM

1. In Fig. 2.1, Consider the case when  $S$  is switched to  $Q$  right in the beginning. Formulate the differential equation.

**Solution:**

The circuit for the case is shown below,



The differential equation can be formulated as follows,

$$i_1(R_1) = \int_0^t \frac{i_2(t)}{C_0} dt \quad (4.1)$$

$$V_2 - iR_2 - \int_0^t \frac{i_2(t)}{C_0} dt = 0 \quad (4.2)$$

$$V_2 - (i_1 + i_2)R_2 = \int_0^t \frac{i_2(t)}{C_0} dt \quad (4.3)$$

$$V_2 - (i_2 R_2) = \left( \frac{R_2}{R_1} + 1 \right) \int_0^t \frac{i_2(t)}{C_0} dt \quad (4.4)$$

Expressing the differential equation in terms of  $V_{C_0}$ , we have

$$V_{C_0} = \int_0^t \frac{i_2(t)}{C_0} dt \quad (4.5)$$

$$\frac{dV_{C_0}}{dt} = \frac{i_2(t)}{C_0} \quad (4.6)$$

$$V_2 - R_2 C_0 \frac{dV_{C_0}}{dt} = \left( \frac{R_1 + R_2}{R_1} \right) V_{C_0} \quad (4.7)$$

$$R_2 C_0 \frac{dV_{C_0}}{dt} + \left( \frac{R_1 + R_2}{R_1} \right) V_{C_0} - V_2 = 0 \quad (4.8)$$

2. Find  $H(s)$  considering the output voltage at the capacitor.

**Solution:**

Applying Laplace transform on both sides of the equation,

$$R_2 C_0 s V_{C_0}(s) + \frac{R_1 + R_2}{R_1} V_{C_0}(s) - V_2(s) = 0 \quad (4.9)$$

$$\frac{V_2(s)}{V_{C_0}(s)} = s R_2 C_0 + 1 + \frac{R_2}{R_1} \quad (4.10)$$

By definition of transfer function, we have

$$H(s) = \frac{V_{C_0}(s)}{V_2(s)} = \frac{1}{s R_2 C_0 + 1 + \frac{R_2}{R_1}} \quad (4.11)$$

On substituting the values, we get

$$H(s) = \frac{5 \times 10^5}{s + 1.5 \times 10^6} \quad (4.12)$$

3. Plot  $H(s)$ . What kind of filter is it?

**Solution:** Download the following Python code that plots Fig. 4.1

```
wget 4_3.py
```

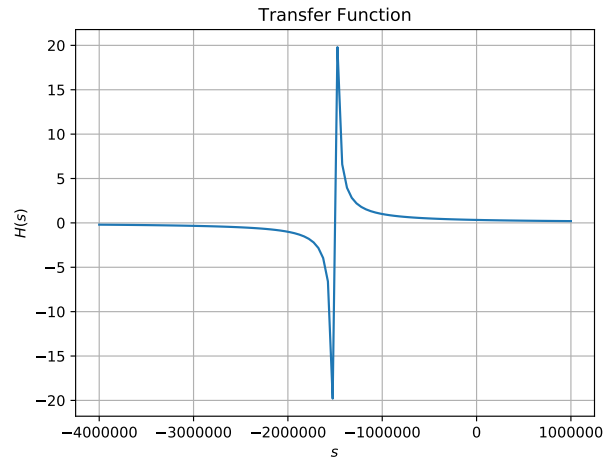


Fig. 4.1: Plot of  $H(s)$

Consider the frequency-domain transfer function by putting  $s = j\omega$

$$H(j\omega) = \frac{5 \times 10^5}{j\omega + 1.5 \times 10^6} \quad (4.13)$$

$$|H(j\omega)| = \frac{5 \times 10^5}{\sqrt{\omega^2 + 2.25 \times 10^{12}}} \quad (4.14)$$

Since maximum value of transfer function occurs when  $\omega = 0$ , we have the following relation,

$H(j\omega)$  decreases as  $\omega$  increases.

So, frequency ranges from 0 to cut-off frequency.

Hence, the given filter is a low-pass filter.

4. Using trapezoidal rule for integration, formulate the difference equation by considering

$$y(n) = y(t)|_{t=n} \quad (4.15)$$

**Solution:**

$$R_2 C_0 \frac{dV_{C_0}}{dt} + \left( \frac{R_1 + R_2}{R_1} \right) V_{C_0} - V_2 = 0 \quad (4.16)$$

$$C_0 \frac{dV_{C_0}}{dt} = \frac{2V_2 - V_{C_0}(t)}{R_2} - \frac{V_{C_0}(t)}{R_1} \quad (4.17)$$

$$V_{C_0}(t)_{n+1} = \int_n^{n+1} \left( \frac{2V_2(t) - V_{C_0}(t)}{R_2 C_0} - \frac{V_{C_0}(t)}{R_1 C_0} \right) dt \quad (4.18)$$

By the trapezoidal rule of integration, we approximate the following integral

$$\int_a^b f(t) dt = \frac{b-a}{2} (f(a) + f(b)) \quad (4.19)$$

Consider  $y(t) = V_{C_0}(t)$ , and simplifying using trapezoid rule on right hand side of the equation, we have

$$\begin{aligned} y(n+1) - y(n) &= \frac{1}{R_2 C_0} (u(n) + u(n+1)) \\ &- \frac{1}{2} (y(n+1) + y(n)) \left( \frac{1}{R_1 C_0} + \frac{1}{R_2 C_0} \right) \end{aligned} \quad (4.20)$$

Thus, the difference equation is

$$\begin{aligned} &y(n+1) \left( 1 + \frac{1}{2R_1 C_0} + \frac{1}{2R_2 C_0} \right) \\ &= y(n) \left( 1 - \frac{1}{2R_1 C_0} - \frac{1}{2R_2 C_0} \right) \\ &+ \frac{1}{R_2 C_0} (u(n) + u(n+1)) \end{aligned} \quad (4.21)$$

5. Find  $H(z)$ .

**Solution:** Applying  $\mathcal{Z}$ -transform to the (4.21) on both sides, we have

$$\begin{aligned} &zY(z) \left( \frac{2R_1 R_2 C_0 + R_1 + R_2}{2R_1 R_2 C_0} \right) = \\ &Y(z) \left( \frac{2R_1 R_2 C_0 - R_1 - R_2}{2R_1 R_2 C_0} \right) + \\ &\frac{1}{R_2 C_0} \left( \frac{1}{1-z^{-1}} + \frac{z}{1-z^{-1}} \right) \end{aligned} \quad (4.22)$$

$$Y(z) = \frac{\left( \frac{1+z}{R_2 C_0 (1-z^{-1})} \right) 2R_1 R_2 C_0}{(2R_1 R_2 C_0 (z-1)) + (z+1)(R_1 + R_2)} \quad (4.23)$$

$$X(z) = \mathcal{Z}[x(n)] = \mathcal{Z}[2u(n)] = \frac{2}{1-z^{-1}} \quad (4.24)$$

Using  $H(z) = \frac{Y(z)}{X(z)}$ , and substituting values

$$H(z) = \frac{(1+z)}{(4C_0(z-1)) + 3(z+1)} \quad (4.25)$$

$$H(z) = \frac{1+z^{-1}}{(3+4C_0) + (3-4C_0)z^{-1}} \quad (4.26)$$

Region of convergence (ROC) for the  $\mathcal{Z}$ -transform is

$$|z| > \max \left( 1, \left| \frac{3+4 \times 10^{-6}}{4 \times 10^{-6} - 3} \right| \right) \quad (4.27)$$

6. How can you obtain  $H(z)$  from  $H(s)$ ?

**Solution:** To convert  $H(z)$  to  $H(s)$ , we use the bilinear transform.

It maps transfer function of  $\mathcal{Z}$ -transform to transfer function of  $s$ -domain.

$$s \longleftrightarrow \frac{2}{T} \frac{1-z^{-1}}{1+z^{-1}} \quad (4.28)$$

$T = 1$  is the step size in trapezoidal rule

From (4.11), we have

$$H(s) = \frac{1}{sR_2 C_0 + 1 + \frac{R_2}{R_1}} \quad (4.29)$$

$$H(z) = \frac{1}{\left( \frac{1-z^{-1}}{1+z^{-1}} \right) 2R_2 C_0 + 1 + \frac{R_2}{R_1}} \quad (4.30)$$

$$H(z) = \frac{1+z^{-1}}{4C_0(1-z^{-1}) + 3(1+z^{-1})} \quad (4.31)$$

$$H(z) = \frac{1+z^{-1}}{(3+4C_0) + (3-4C_0)z^{-1}} \quad (4.32)$$

The equation obtained is the same as the equation (4.26).

7. Plot  $y(n)$ ,  $y(t)$ , and voltage through ngspice. Verify that they are the same.

**Solution:** To obtain  $y(n)$  from  $Y(z)$ , we apply inverse  $\mathcal{Z}$ -transform to (4.23) and substitute values.

$$Y(z) = \frac{\left( \frac{1+z}{(1-z^{-1})} \right) 2R_1}{(2R_1 R_2 C_0 (z-1)) + (z+1)(R_1 + R_2)} \quad (4.33)$$

$$Y(z) = \frac{2(1+z)}{4C_0(1-z^{-1})(z-1) + 3(1+z)(1-z^{-1})} \quad (4.34)$$

$$Y(z) = \frac{2(1+z)}{4C_0(z+z^{-1}-2) + 3(z-z^{-1})} \quad (4.35)$$

$$= \frac{2}{3} \left( \frac{1}{1-z^{-1}} - \frac{4C_0}{(4C_0+3) + (3-4C_0)z^{-1}} \right) \quad (4.36)$$

$$y(n) = \frac{2}{3} \left( 1 - \frac{(1-7.5 \times 10^5)^n}{(1+7.5 \times 10^5)^{n+1}} \right) u(n) \quad (4.37)$$

Using binomial approximation because  $n \approx 10^{-7}$ ,

$$y(n) = \frac{2}{3} \left( 1 - \frac{(1-7.5 \times 10^5)^n}{(1+7.5 \times 10^5)^n} \right) u(n) \quad (4.38)$$

Download the following python code to verify the plots of  $y(t)$ , ngspice plot and  $y(n) = y(t)|_{t=n}$

wget 4\_7.py

