Pingala Series

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Contents

***Abstract*—This manual provides a simple introduction to Transforms**

1 JEE 2019

Let

$$a_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \quad n \geq 1 \tag{1.1}$$

$$b_n = a_{n-1} + a_{n+1}, \quad n \geq 2, \quad b_1 = 1 \tag{1.2}$$

Verify the following using a python code.

1.1
$$\sum_{k=1}^{n} a_k = a_{n+2} - 1, \quad n \geq 1 \tag{1.3}$$

1.2
$$\sum_{k=1}^{\infty} \frac{a_k}{10^k} = \frac{10}{89} \tag{1.4}$$

1.3
$$b_n = \alpha^n + \beta^n, \quad n \geq 1 \tag{1.5}$$

1.4
$$\sum_{k=1}^{\infty} \frac{b_k}{10^k} = \frac{8}{89} \tag{1.6}$$

Solution: Download the following python code and run it to verify the above equations

```
wget 1.py
```

2 Pingala Series

2.1 The *one sided* Z-transform of $x(n)$ is defined as

$$X^+(z) = \sum_{n=0}^{\infty} x(n) z^{-n}, \quad z \in \mathbb{C} \tag{2.1}$$

2.2 The *Pingala* series is generated using the difference equation

$$x(n+2) = x(n+1) + x(n), \tag{2.2}$$

$$x(0) = x(1) = 1, n \geq 0 \tag{2.3}$$

Generate a stem plot for $x(n)$.
Solution: Download the following python code for generating the stem plot of $x[n]$

```
wget 2_2.py
```

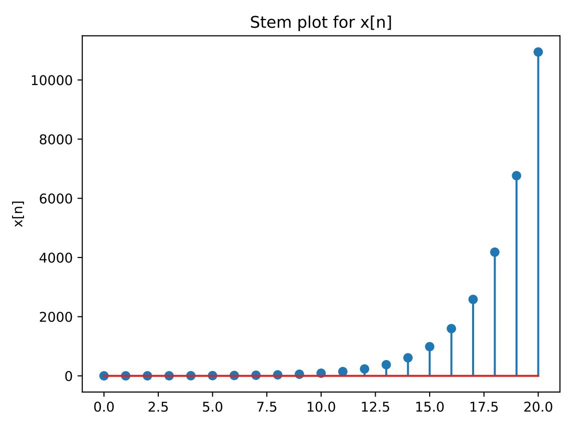

Fig. 2.2: Stem plot of $x[n]$

2.3 Find $X^+(z)$.
Solution:

$$\mathcal{Z}^+[x(n+2)] = \mathcal{Z}^+[x(n+1)] + \mathcal{Z}^+[x(n)] \tag{2.4}$$

$$z^2(X^+(z) - x(0)) - zx(1) = (z+1)(X^+(z)) - zx(0) \tag{2.5}$$

$$\left(z^2 - z - 1\right) X^+(z) = z^2 \tag{2.6}$$

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$$X^{+}(z) = \frac{1}{1 - z^{-1} - z^{-2}} \tag{2.7}$$

2.4 Find $x(n)$.
Solution: Using one-sided Z-transform $X^{+}(z)$, we can find the signal $x(n)$ using inverse Z-transform

$$X^{+}(z) = \frac{1}{1 - z^{-1} - z^{-2}} \tag{2.8}$$

$$X^{+}(z) = \frac{1}{\left(1 - \alpha z^{-1}\right)\left(1 - \beta z^{-1}\right)} \tag{2.9}$$

$$X^{+}(z) = \frac{\left(\frac{\alpha}{\alpha-\beta}\right)}{1 - \alpha z^{-1}} + \frac{\left(\frac{-\beta}{\alpha-\beta}\right)}{1 - \beta z^{-1}} \tag{2.10}$$

Applying inverse Z-transform and using

$$a^{n}u(n) \overset{\mathcal{Z}}{\rightleftharpoons} \frac{1}{1 - \alpha z^{-1}} \tag{2.11}$$

We have

$$x(n) = \left(\frac{\alpha}{\alpha - \beta}\right)\alpha^{n}u[n] - \left(\frac{\beta}{\alpha - \beta}\right)\beta^{n}u[n] \tag{2.12}$$

$$x(n) = \left(\frac{\alpha^{n+1} - \beta^{n+1}}{\alpha - \beta}\right)u[n] \tag{2.13}$$

2.5 Sketch

$$y(n) = x\left(n - 1\right) + x\left(n + 1\right), \quad n \geq 0 \tag{2.14}$$

Solution: Download the following python code for the stem plot of $y[n]$

```
wget 2_5.py
```

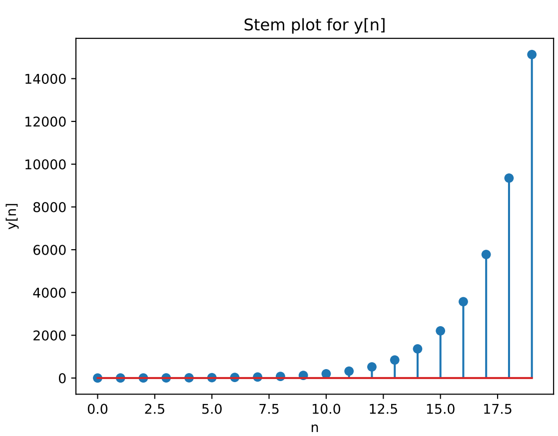

Fig. 2.5: Stem plot of $y[n]$

2.6 Find $Y^{+}(z)$.
Solution: Taking the one-sided Z-transform on both sides of (2.14),

$$\mathcal{Z}^{+}\left[y(n)\right] = \mathcal{Z}^{+}\left[x(n + 1)\right] + \mathcal{Z}^{+}\left[x(n - 1)\right] \tag{2.15}$$

Using $x(n) = 0 \ \forall \ n < 0$ and the one-sided Z-transform,

$$Y^{+}(z) = zX^{+}(z) - zx(0) + z^{-1}X^{+}(z) \tag{2.16}$$

$$= \frac{z + z^{-1}}{1 - z^{-1} - z^{-2}} - z \tag{2.17}$$

$$= \frac{1 + 2z^{-1}}{1 - z^{-1} - z^{-2}} \tag{2.18}$$

2.7 Find $y(n)$
Solution: Factor the one-sided Z-transform as follows,

$$Y^{+}(z) = \frac{1 + 2z^{-1}}{1 - z^{-1} - z^{-2}} \tag{2.19}$$

$$= \frac{1 + 2z^{-1}}{(1 - \alpha z^{-1})(1 - \beta z^{-1})} \tag{2.20}$$

$$= \frac{\left(\frac{\alpha+2}{\alpha-\beta}\right)}{1 - \alpha z^{-1}} - \frac{\left(\frac{\beta+2}{\alpha-\beta}\right)}{1 - \beta z^{-1}} \tag{2.21}$$

$$= \left(\frac{1}{\alpha - \beta}\right)\left(\frac{\alpha + 2}{1 - \alpha z^{-1}} - \frac{\beta + 2}{1 - \beta z^{-1}}\right) \tag{2.22}$$

Applying inverse Z-transform,

$$y(n) = \frac{(\alpha + 2)(\alpha^{n}u[n]) - (\beta + 2)(\beta^{n}u[n])}{\alpha - \beta} \tag{2.23}$$

$$= \frac{\alpha^{n+1} - \beta^{n+1} + 2\left(\alpha^{n} - \beta^{n}\right)}{\alpha - \beta} \cdot u[n] \tag{2.24}$$

$$= \frac{\alpha^{n+1} - \beta^{n+1}}{\alpha - \beta} \cdot u[n] + \frac{2\left(\alpha^{n} - \beta^{n}\right)}{\alpha - \beta} \cdot u[n] \tag{2.25}$$

$$= x(n) + 2x(n - 1) \tag{2.26}$$

$$= (x(n) + x(n - 1)) + x(n - 1) \tag{2.27}$$

The final expression for $y(n)$ is given by

$$y(n) = x(n + 1) + x(n - 1) \quad n > 0 \tag{2.28}$$

3 Power of the Z transform

3.1 Show that

$$\sum_{k=1}^{n} a_k = \sum_{k=0}^{n-1} x(k) = x(n) * u(n-1) \tag{3.1}$$

Solution: We observe the following relation from the expressions for $a[n]$ and $x[n]$

$$a[n]u[n] = \left(\frac{\alpha^n - \beta^n}{\alpha - \beta}\right) u[n] \tag{3.2}$$

$$x(n) = \left(\frac{\alpha^{n+1} - \beta^{n+1}}{\alpha - \beta}\right) u[n] \tag{3.3}$$

So, $a[n+1] \cdot u[n] = x(n)$

$$\sum_{k=1}^{n} a_k = \sum_{k=1}^{n} x(k-1) \tag{3.4}$$

Replacing k with $k+1$ in the above expression

$$\sum_{k=1}^{n} a_k = \sum_{k=0}^{n-1} x(k+1-1) \tag{3.5}$$

$$\sum_{k=1}^{n} a_k = \sum_{k=0}^{n-1} x(k) \tag{3.6}$$

By the definition of convolution,

$$x(n) * u(n-1) = \sum_{k=0}^{n-1} x(k)u(n-1-k) \tag{3.7}$$

$$= \sum_{k=0}^{n-1} x(k)u(n-k) \tag{3.8}$$

$$u(n-k) > 0, \quad k = 0 \dots n-1 \tag{3.9}$$

$$x(n) * u(n-1) = \sum_{k=0}^{n-1} x(k) \tag{3.10}$$

Hence, we have the final result

$$\sum_{k=1}^{n} a_k = \sum_{k=0}^{n-1} x(n) = x(n) * u(n-1) \tag{3.11}$$

3.2 Show that

$$a_{n+2} - 1, \quad n \geq 1 \tag{3.12}$$

can be expressed as

$$[x(n+1) - 1]\, u(n) \tag{3.13}$$

Solution:
Using (2.13), we have

$$x(n) = a(n+1) \quad n \geq 0 \tag{3.14}$$

$$a_{n+2} - 1 = x(n+1) - 1 \tag{3.15}$$

and from the definition of $u[n]$,

$$a_{n+2} - 1 = [x(n+1) - 1] \times u[n] \tag{3.16}$$

3.3 Show that

$$\sum_{k=1}^{\infty} \frac{a_k}{10^k} = \frac{1}{10} \sum_{k=0}^{\infty} \frac{x(k)}{10^k} = \frac{1}{10} X^+(10) \tag{3.17}$$

Solution: Using (2.13), we have

$$x(n) = a(n+1) \quad n \geq 0 \tag{3.18}$$

$$\sum_{k=1}^{\infty} \frac{a_k}{10^k} = \sum_{k=1}^{\infty} \frac{x(k-1)}{10^k} \tag{3.19}$$

$$= \sum_{k=0}^{\infty} \frac{x(k+1-1)}{10^{k+1}} \tag{3.20}$$

$$= \frac{1}{10} \sum_{k=0}^{\infty} \frac{x(k)}{10^k} \tag{3.21}$$

Using the definition of $X^+(z)$, we have

$$\sum_{k=1}^{\infty} \frac{a_k}{10^k} = \frac{1}{10} X^+(10) \tag{3.22}$$

3.4 Show that

$$\alpha^n + \beta^n, \quad n \geq 1 \tag{3.23}$$

can be expressed as

$$w(n) = \left(\alpha^{n+1} + \beta^{n+1}\right) u(n) \tag{3.24}$$

and find $W(z)$.
Solution:

$$\alpha^n + \beta^n \quad n \geq 1 \tag{3.25}$$

Replace n with $n+1$ in the above expression

$$\alpha^{n+1} + \beta^{n+1} \quad n \geq 0 \tag{3.26}$$

$$\text{Also, } u[n] = 1 \quad n \geq 0 \tag{3.27}$$

So, the given expression can be written as

$$\left(\alpha^{n+1}+\beta^{n+1}\right)\cdot u[n] \tag{3.28}$$

$$W(z)=\mathcal{Z}\left(\left(\alpha^{n+1}+\beta^{n+1}\right)\cdot u[n]\right) \tag{3.29}$$

$$=\mathcal{Z}\left(\alpha^{n+1}\cdot u[n]\right)+\mathcal{Z}\left(\beta^{n+1}\cdot u[n]\right) \tag{3.30}$$

$$=\frac{\alpha}{1-\alpha z^{-1}}+\frac{\beta}{1-\beta z^{-1}} \tag{3.31}$$

$$=\frac{\alpha+\beta-2\alpha\beta z^{-1}}{1-\alpha z^{-1}-\beta z^{-1}+\alpha\beta z^{-2}} \tag{3.32}$$

Using $\alpha+\beta=1$ and $\alpha\beta=-1$

$$W(z)=\frac{1+2z^{-1}}{1-z^{-1}-z^{-2}} \tag{3.33}$$

3.5 Show that

$$\sum_{k=1}^{\infty}\frac{b_k}{10^k}=\frac{1}{10}\sum_{k=0}^{\infty}\frac{y(k)}{10^k}=\frac{1}{10}Y^+(10) \tag{3.34}$$

Solution:

Using (2.28) and (3.18), we have

$$y(n)=x(n+1)+x(n-1) \tag{3.35}$$

$$y(n)=a(n+2)+a(n) \tag{3.36}$$

$$y(n)=b(n+1) \tag{3.37}$$

Now, using (3.37), we have

$$\sum_{k=1}^{\infty}\frac{b_k}{10^k}=\sum_{k=1}^{\infty}\frac{y(k-1)}{10^k} \tag{3.38}$$

$$=\sum_{k=0}^{\infty}\frac{y(k+1-1)}{10^{k+1}} \tag{3.39}$$

$$=\frac{1}{10}\sum_{k=0}^{\infty}\frac{y(k)}{10^k} \tag{3.40}$$

From the definition of one-sided Z-transform, we have

$$\frac{1}{10}\sum_{k=0}^{\infty}\frac{y(k)}{10^k}=\frac{1}{10}Y^+(10) \tag{3.41}$$

3.6 Solve the JEE 2019 problem.

Solution:

Option - (A) : True

$$\sum_{k=1}^{n}a_k=a_{n+2}-1,\quad n\geq 1 \tag{3.42}$$

$$\mathcal{Z}(x(n)*u(n-1))=\mathcal{Z}(x(n))\cdot\mathcal{Z}(u(n-1)) \tag{3.43}$$

$$=X^+(z)\cdot\frac{z^{-1}}{1-z^{-1}} \tag{3.44}$$

$$=\frac{z^{-1}}{(1-z^{-1})(1-z^{-1}-z^{-2})} \tag{3.45}$$

$$=z\left(\frac{1}{1-z^{-1}-z^{-2}}-\frac{1}{1-z^{-1}}\right) \tag{3.46}$$

$$=z\sum_{n=0}^{\infty}\left(x(n)-1\right)z^{-n} \tag{3.47}$$

$$=\sum_{n=1}^{\infty}\left(x(n)-1\right)z^{1-n}\quad ,x(0)=1 \tag{3.48}$$

$$=\sum_{n=0}^{\infty}\left(x(n+1)-1\right)z^{-n} \tag{3.49}$$

$$=\mathcal{Z}(x(n+1)-1)u(n) \tag{3.50}$$

$$\text{So, } (x(n)*u(n-1))=(x(n+1)-1)\quad n\geq 0 \tag{3.51}$$

From (3.1), we have

$$\sum_{k=1}^{n}a_k=x(n)*u(n-1) \tag{3.52}$$

$$=x(n+1)-1 \tag{3.53}$$

$$=a_{n+2}-1 \tag{3.54}$$

Option - (B) : True

$$\sum_{n=1}^{\infty}\frac{a_n}{10^n}=\frac{10}{89} \tag{3.55}$$

From (2.7) and (3.22), we have

$$\sum_{k=1}^{\infty}\frac{a_k}{10^k}=\frac{1}{10}X^+(10) \tag{3.56}$$

$$X^+(10)=\frac{1}{1-\frac{1}{10}-\frac{1}{100}}=\frac{100}{89} \tag{3.57}$$

$$\sum_{k=1}^{\infty}\frac{a_k}{10^k}=\frac{1}{10}\cdot\frac{100}{89}=\frac{10}{89} \tag{3.58}$$

Option - (C) : False

$$\sum_{n=1}^{\infty}\frac{b_n}{10^n}=\frac{10}{89} \tag{3.59}$$

From (2.18) and (3.41), we have

$$\sum_{k=1}^{\infty} \frac{b_k}{10^k} = \frac{1}{10} Y^+(10) \tag{3.60}$$

$$Y^+(10) = \frac{1 + \frac{2}{10}}{1 - \frac{1}{10} - \frac{1}{100}} = \frac{120}{89} \tag{3.61}$$

$$\sum_{k=1}^{\infty} \frac{b_k}{10^k} = \frac{1}{10} \cdot \frac{120}{89} = \frac{12}{89} \tag{3.62}$$

Option - (D) : True

$$b_n = \alpha^n + \beta^n \quad n \geq 1 \tag{3.63}$$

From definitions of a_n and b_n, we have

$$b_n = a_{n-1} + a_{n+1} \quad n \geq 2 \tag{3.64}$$

$$= \frac{\alpha^{n-1} + \alpha^{n+1}}{\alpha - \beta} - \frac{\beta^{n-1} + \beta^{n+1}}{\alpha - \beta} \tag{3.65}$$

$$= \frac{(\alpha^n + \beta^n)(\alpha - \beta)}{\alpha - \beta} = \alpha^n + \beta^n \tag{3.66}$$

Using $b_1 = 1$ and $\alpha + \beta = 1$
So,

$$b_n = \alpha^n + \beta^n \quad n \geq 1 \tag{3.67}$$