17/10

Recall If G is a finite Abelian group, $|G| = p_1^{r_1} \cdots\cdots p_s^{r_s}$ where $p_1, \ldots, p_s$ are distinct primes, then

$$G = H_1 \oplus \cdots \oplus H_s$$

where H_i is a Sylow-p subgroup for $i = 1, \ldots, s$

$$G = G(p^n) \oplus G(m) \qquad \text{when } |G| = p^n m$$

Induction for $p_1^{r_1} p_2^{r_2} \cdots p_s^{r_s}$

$$G = H_1 \oplus G(p_2^{r_2} \cdots p_s^{r_s})$$

If $|G| = p_1^{r_1}$, then we are done!

Assume that the theorem is true for all G s.t $|G|$ has atmost $s-1$ distinct prime factors.

$$\Rightarrow |G| = H_1 \oplus (H_2 \oplus \cdots \oplus H_s)$$

Statement

Indecomposable p-groups (Abelian) are cyclic

(Proof)

(Trying to prove this!) If G is a finite Abelian group, then G can be written uniquely as a direct sum

$$G = \oplus\, \mathbb{Z}/p^n\mathbb{Z}$$

If we prove statement, we can prove the existence of such a decomposition.

Proposition

Any non-trivial finite abelian p-group having a unique cyclic subgroup of order p is cyclic [Proved previously!]

Proposition

If G is a finite Abelian p-group and $g \in G$ is an element of maximal order, then

$$G = \langle g \rangle \oplus H$$

for some subgroup $H \leq G$

Proof

Let $|G| = p^n$, we prove this by induction

$n = 1 \rightarrow$ (Trivial !)

Let $n \geq 2$ and the assertion be true for all groups G of order $\leq p^{n-1}$

Suppose that G is not cyclic and $o(g) = p^m$, $m < n$

By Proposition - 1, G has atleast two subgroups of order p.

Suppose one of them is contained in $\langle g \rangle$ and the other one be $\langle h \rangle$

Claim: $\langle g \rangle \cap \langle h \rangle = (0)$

Note that

$$|\langle g \rangle \cap \langle h \rangle| = \begin{cases} 1 \\ p \Rightarrow \langle h \rangle \subseteq \langle g \rangle \text{ and} \end{cases}$$

Since there is a unique subgroup of $\langle g \rangle$ of order p, $\langle h \rangle = K_1$

- Consider the projection map

$$\pi : G \rightarrow G/\langle h \rangle$$
$$a \rightarrow a + \langle h \rangle$$

Claim! : $o(g) = o(g + \langle h \rangle)$

$$p^m(g+\langle h\rangle) = p^m g + \langle h\rangle$$
$$= \langle h\rangle$$

$\Rightarrow \quad o(g+\langle h\rangle) \mid p^m$

Suppose $\quad o(g+\langle h\rangle) = p^i, \quad i<m$

$$p^i(g+\langle h\rangle) = p^i g + \langle h\rangle = \langle h\rangle$$

$\Rightarrow p^i g \in \langle h\rangle$

But $\quad \langle g\rangle \cap \langle h\rangle = \emptyset \quad (\cancel{\Rightarrow\Leftarrow})$

$\Rightarrow \quad o(g+\langle h\rangle) = p^m$

$$\left| G/\langle h\rangle \right| = |G|/|\langle h\rangle| = p^n/p = p^{n-1}$$

We know that $\forall\, a \in G$, $o(a) \leq p^m$, then
$o(a+\langle h\rangle) \leq p^m$

$\Rightarrow$ $g+\langle h\rangle$ is an element of maximal order in $G/\langle h\rangle$

By induction hypothesis,

$G/\langle h\rangle = \langle g+\langle h\rangle\rangle \oplus K_1$ where $K_1 \leq G/\langle h\rangle$

$G/\langle h\rangle = \langle g+\langle h\rangle\rangle \oplus H_1/\langle h\rangle$ where $H_1 \leq G$ containing $\langle h\rangle$

[Use correspondence theorem!]

Claim :– $\quad G = \langle g\rangle \oplus H_1$

Take $x \in G$

$x+\langle h\rangle = (tg+\langle h\rangle) + (h_1+\langle h\rangle)$ from $(g+\langle h\rangle)\oplus H_1/\langle h\rangle$

$x - tg - h_1 \in \langle h\rangle$

$x = tg + (h_1 + nh)$

$x \in \langle g\rangle + \cancel{\in} H_1$

We show that $\langle g \rangle \cap H_1 = (0)$

Let $x \in \langle g \rangle \cap H_1$

$x + \langle h \rangle \in \langle g + \langle h \rangle \rangle \cap H_1 / \langle h \rangle$

$\in \{ \langle h \rangle \}$

$x \in \langle g \rangle$

$x = mg \Rightarrow x + \langle h \rangle = mg + \langle h \rangle$

$\Rightarrow x + \langle h \rangle \in \langle g + \langle h \rangle \rangle$

$x \in H_1$

$x + \langle h \rangle \in H_1 / \langle h \rangle$

$\Rightarrow x + \langle h \rangle = \langle h \rangle \Rightarrow x \in \langle h \rangle$

But $\langle g \rangle \cap \langle H \rangle = (g)$

$\Rightarrow x \in \langle g \rangle \cap \langle h \rangle = (0)$

Corollary

Indecomposable finite Abelian p-groups are cyclic

(Proof)

Suppose $|G| = p^n$

If G is not cyclic, then an element $g \in G$ of maximal order satisfies $o(g) = p^m$, $m < n$

By Proposition - 2

$G = \langle g \rangle \oplus H$

Since $H \cong G / \langle g \rangle$, we have $|H| = p^n / p^m > 1$

$\Rightarrow G$ is decomposable $(\Rightarrow\Leftarrow)$

Lemma

If G is a finite Abelian p-group, then

$G = K_1 \oplus \dots \oplus K_m$ where K_i's are uniquely determined cyclic groups of order p^i

Suppose $|G| = p^k$

Essentially

$\mathbb{Z}/4\mathbb{Z} \not\cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$

$\mathbb{Z}/8\mathbb{Z} \not\cong \mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$

$$G = H_1 \oplus \cdots \oplus H_m$$
$$= K_1 \oplus \cdots \oplus K_n$$

with $|H_1| \geq |H_2| \geq \cdots \geq |H_m|$

$|K_1| \geq |K_2| \geq \cdots \geq |K_n|$

Notation

If G is an Abelian group

$$G^p = \{x^p \mid x \in G\} \qquad f : G \to G^p,\ x \mapsto x^p$$

$$f(xy) = (xy)^p = x^p y^p = f(x) f(y)$$

$$\ker(f) = \{x \in G \mid x^p = 1\}$$

Now, let G be an Abelian p-group

$|\ker(f)| \geq p$ [There is a subgroup of order p]

$|\ker(f)| = p$

$|G^p| = [G : \underbrace{\cancel{G^p}}_{\ker(f)}] \leq p^{n-1}$ (abelian)

$|G^p| = [G : \ker(f)] = p^{n-1}$ if G is cyclic

- $G = H_1 \oplus \cdots \oplus H_m \cong K_1 \oplus \cdots \oplus K_n$

$G^p = H_1^p \oplus \cdots \oplus H_m^p \cong K_1^p \oplus \cdots \oplus K_n^p$

$p|H_i^p| = |H_i|$

$p|K_j^p| = |K_j|$

Let $m' = \max\{i \mid |H_i| > p\}$

$n' = \max\{j \mid |K_j| > p\}$

$$G^p = H_1^p \oplus \cdots \oplus H_{m'}^p = K_1^p \oplus \cdots \oplus K_{n'}^p$$

Since $|G^p| \leq p^{k-1}$

$\Rightarrow m' = n'$ and $|H_i| = |K_i|$ for $i \leq m' = n'$

$$G = H_1 \oplus H_2 \oplus \cdots \oplus H_{m'} \oplus \underbrace{(\mathbb{Z}/p\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/p\mathbb{Z})}_{m-m'}$$

$$= K_1 \oplus K_2 \oplus \cdots \oplus K_{n'} \oplus \underbrace{(\mathbb{Z}/p\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/p\mathbb{Z})}_{n-n'}$$

$$|G| = |H_1| \cdots\cdots |H_{m'}| \; p^{m-m'}$$

$$= |K_1| \cdots |K_{n'}| \; p^{n-n'} \Rightarrow \boxed{m=n}$$

Classifying groups of order 8

Abelian $\rightarrow$ (Done!)

Non-abelian $\rightarrow$ There are two!

$m = \max\{ o(g) \mid g \in G \}$

$x \in G$, $o(x) = 4$ $\left[\begin{matrix} 1 & \\ 2 & \times \\ 4 & \times \\ 8 & \times \end{matrix} \right] \rightarrow$ (Prove!)

Let $y \in G \setminus H$

$\Rightarrow y^2 \in H \rightarrow$ $\left[\begin{matrix} H = \langle x \rangle \cong \mathbb{Z}/4\mathbb{Z} \\ [G:H] = 2 \\ \Rightarrow H \trianglelefteq G \end{matrix} \right.$

$o(yxy^{-1}) = o(x) = 4$

$\left[\begin{matrix} G/H \\ (yH)^2 = y^2H = H \\ \Rightarrow y^2 \in H \end{matrix} \right.$