25/08

Exercise !!

If X, Y are two sets of same cardinalities then $S_X \leftrightarrow S_Y$

Lemma

Every permutation is a product of 2-cycles.

Proof

It is enough to show that every cycle can be written as a product of transpositions.

- Transposition is a 2-cycle.

Let $(a_1 \dots a_k)$ be a k-cycle.

claim: $(a_1 \dots a_k) = (a_1 \; a_k) \dots (a_1 \; a_2)$

We prove the claim by induction.

$k = 2 \rightarrow$ Trivial

Enough to prove that

$$(a_1 \dots a_k) = (a_1 \; a_k)(a_1 \dots a_{k-1})$$

$$= \begin{pmatrix} a_1 & a_2 & \cdots & a_{k-1} & a_k & \cdots & a_n \\ a_k & \downarrow & & & a_1 & & a_n \end{pmatrix} \begin{pmatrix} a_1 & \cdots & a_{k-2} & a_{k-1} & a_k & \cdots & a_n \\ a_2 & & a_{k-1} & a_1 & \downarrow & & \downarrow \end{pmatrix}$$

$$= \begin{pmatrix} a_1 & \cdots & a_{k-2} & a_{k-1} & a_k & \cdots & a_n \\ \downarrow & & \downarrow & & & & \\ a_2 & & a_{k-1} & a_k & a_1 & & a_n \end{pmatrix}$$

$$= (a_1 \dots a_k)$$

Definition

Let G be a group and $a_1, \dots, a_n \in G$.
We say that a subgroup H is generated by $a_1, \dots, a_n$ if any element of H could be written as finite products of elements in $\{a_1, \dots, a_n\}$

Notation

$$H = \langle a_1, \dots, a_n \rangle$$

Proposition

(i) $S_n = \langle \text{transpositions} \rangle$

(ii) $S_n = \langle (1,2), (1,3), (1,4), \dots (1,n) \rangle$

(iii) $S_n = \langle (1,2), (2,3), \dots, (n-1, n) \rangle$

(iv) $S_n = \langle (1,2), (1 \dots n-1) \rangle$

Lemma

For $\sigma \in S_n$, we have

$$\sigma (a_1 \dots a_k) \sigma^{-1} = (\sigma(a_1) \dots \sigma(a_k))$$

(Formal proof exercise!)

Proof of proposition

(ii) It is enough to show that every transposition can be written as a product of elements in $\{(1,2), \dots, (1,n)\}$

Let $(a\ b) \in S_n$

If $1 \in a, b$, we are done

Else $(a, b) = (1\ a)(1\ b)(1\ a)$

(iii)

It is enough to show that

$(1\ k) \in \langle (1\ 2), \ldots, (n-1, n) \rangle$

For $k = 2$, we are done.

$(1\ k) = (k-1\ \ k)(1\ \ k-1)(k-1\ \ k)$

(iv) It is enough to show that

$(k-1, k) \in \langle (1\ 2), (1, 2, \ldots n) \rangle$

$(k-1\ \ k) = \sigma (k-2\ \ k-1) \sigma^{-1}$ [Trivial]

Even and odd permutations

A permutation $\sigma \in S_n$ is said to be an even permutation if σ can be written as product of even number of transpositions.

(Proof!)

- Let $\sigma \in S_n$ and f be any polynomial in n variables $x_1, \ldots, x_n$

We define

$$(\sigma f)(x_1, \ldots, x_n) = f(x_{\sigma(1)}, \ldots, x_{\sigma(n)})$$

Note:

If $\sigma = (1\ 3\ 2)$

$f = x_1^3 + x_2^4 + x_3^5$

$\sigma f = x_3^3 + x_1^4 + x_2^5$

(i) If $\sigma = (1)$, then $\sigma f = f$

(ii) If $\sigma, \tau \in S_n$, then

$(\sigma\tau) f = \sigma(\tau f)$

$$\begin{aligned}(\sigma\tau) f(x_1, \ldots, x_n) &= f(x_{\sigma\tau(1)}, \ldots, x_{\sigma\tau(n)}) \\ &= \sigma f(x_{\tau(1)}, \ldots, x_{\tau(n)}) \\ &= \sigma(\tau f)\end{aligned}$$

- Let $\Delta(x_1, \ldots, x_n) = \prod_{1 \le i < j \le n} (X_i - X_j)$

$$= \det \begin{pmatrix} 1 & \cdots & 1 \\ x_1 & & x_n \\ \vdots & & \vdots \\ x_1^{n-1} & & x_n^{n-1} \end{pmatrix} \quad V_n \rightarrow \text{Vandermonde matrix}$$

Proof

$n=2 \rightarrow$ trivial $\begin{vmatrix} 1 & 1 \\ x_1 & x_2 \end{vmatrix} = x_2 - x_1$

Assume result is true for V_k $(k \le n-1)$

$$\Delta(x, x_2, \ldots, x_n) = \begin{vmatrix} 1 & 1 & & 1 \\ x & x_2 & \cdots & x_n \\ \vdots & \vdots & & \vdots \\ x^{n-1} & x_2^{n-1} & & x_n^{n-1} \end{vmatrix}$$

$$= (x - x_2) \ldots (x - x_n) \, f(x_2, \ldots, x_n)$$

x^{n-1} expanded inside det will have the coefficient $\Delta(x_2, \ldots, x_n)$

$$\Rightarrow f(x_2 \ldots x_n) = \Delta(x_2, \ldots, x_n) (-1)^{n-1}$$