$g \in \ker \phi$

$\Rightarrow \phi(g) = Id$

$\Rightarrow \phi(g)(h) = h \quad \forall h \in G$

$\Rightarrow \tau_g(h) = h \quad \forall h \in G$

$\Rightarrow ghg^{-1} = h \quad \forall h \in G$

$\Rightarrow gh = hg \quad \forall h \in G$

$\Rightarrow g \in Z(G)$

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Second isomorphism theorem

If $H, N \leq G$ with $N \trianglelefteq G$, then

$H \cap N \trianglelefteq H$ and

$$\frac{H}{H \cap N} \cong \frac{HN}{N}$$

Proof

$G/N = \{gN \mid g \in G\}$

$HN \leq G$

$HN/N = \{hnN \mid h \in H, n \in N\}$
$= \{hN \mid h \in H\}$

Define a group homomorphism (map)

$$\phi : H \longrightarrow \frac{HN}{N}$$
$$h \longmapsto hN$$

Homomorphism

$\phi(h_1 h_2) = h_1 h_2 N$
$= (h_1 N)(h_2 N)$
$= \phi(h_1)\phi(h_2)$

ϕ is also surjective

Any element of HN/N is of the form $hN = \phi(h)$

$\ker(\phi) = \{h \in H \mid hN = N\}$
$= \{h \in H \mid h \in N\}$
$= H \cap N$

By First Isomorphism Theorem,

$$H/_{H\cap N} \cong HN/N$$

Third Isomorphism Theorem

If H, N are subgroups of a group G with $N \trianglelefteq H \trianglelefteq G$, then

$$\frac{G/N}{H/N} \cong G/H$$

(Proof)

Define $\phi : G/N \to G/H$

$gN \mapsto gH$

well-defined

Let $gN = g'N$

$\Rightarrow g^{-1}g' \in N \subseteq H$

$\Rightarrow gH = g'H$

The map is clearly surjective

$\ker\phi = \{gN \mid gH = H\}$

$= \{gN \mid g \in H\}$

$= H/N$

$$\Rightarrow \frac{G/N}{H/N} \cong G/H$$

(Exercise!)

(i) If G is a group and $G/Z(G)$ is a cyclic group, then G is Abelian.

(ii) If G is a finite Abelian group and $d \mid |G|$ then G has a subgroup of order d.

Group Actions

Let G be a group and S be a set. An action of G on S is a map

$G \times S \longrightarrow S$ satisfying the following

$(g, s) \longmapsto gs$

$gs \to g(s)$?

(i) $1s = s \quad \forall s \in S$

(ii) $g_1(g_2 s) = (g_1 g_2)s \quad \forall g_1, g_2 \in G, s \in S$

Examples

(i) $S_n \curvearrowright [n] = \{1, 2, \dots, n\}$

$S_n \times [n] \to [n]$

$(\sigma, s) \mapsto \sigma s$

$\sigma_1(\sigma_2 x) = (\sigma_1 \circ \sigma_2) x$

(ii) Left translation

$H \leq G$

$H \curvearrowright G$

$H \times G \to G$

$(h, g) \mapsto hg$

(iii) $K \leq G \quad G/K = \{gK \mid g \in G\}$

If $H \leq G$

$H \curvearrowright G/K$

$H \times G/K \to G/K$

$(h, gK) \mapsto hgK$

* (iv) (conjugation)

$G \curvearrowright G$ by conjugation

$G \times G \longrightarrow G$

$(g, h) \longmapsto ghg^{-1}$

$g_1(g_2 h) = g_1(g_2 h g_2^{-1})$

$= g_1(g_2 h g_2^{-1}) g_1^{-1}$

$= (g_1 g_2) h (g_1 g_2)^{-1}$

(v) Let S be the set of all subgroup of G

$G \curvearrowright S$ by "conjugation"

$$G \times S \longrightarrow S$$
$$(g, H) \mapsto gHg^{-1}$$

Let G be a group acting on a set S

We define a relation $\sim$ on S as

$$X \sim X' \iff \exists\, g \in G \text{ such that } X' = gX$$

Reflexive : (Trivial !)

Symmetric : $X' = gX$

$$\Rightarrow g^{-1}X' = g^{-1}(gX) = (g^{-1}g)X$$
$$= 1X$$
$$= X$$

Transitive : $X \sim X'$ $\quad X' \sim X''$

$X' = gX$ $\quad X'' = g'X'$

$$X'' = g'(gX) = (g'g)X$$

Let G be a group acting on a set S and $x \in S$. The equivalence class of x, given by

$$O(x) = \{x' \in S \mid x \sim x'\}$$
$$= \{gx \mid g \in G\}$$

Let $G_x = \{g \in G \mid gx = x\} \leq G$

stabilizer of x

$$g^{-1}(gX) = g^{-1}(X)$$
$$X = g^{-1}X$$

- $G_x \neq \emptyset$
- $g_1(g_2 x) \to (g_1 g_2)(x)$

$$= g_1(x)$$
$$= x$$

Conjugacy

$G \curvearrowright G$ by conjugation

$o(x) = \{gxg^{-1} \mid g \in G\} = C_G(x)$ (Not this notation!)

conjugacy class of x

$G_x = \{g \in G \mid gxg^{-1} = x\} = C_G(x)$

$= \{g \in G \mid gx = xg\}$ = Centraliser of x

$G \curvearrowright S :=$ (subgroups of G)

$o(H) = \{gHg^{-1} \mid g \in G\}$

$N_G(H) = G_H = \{g \in G \mid gHg^{-1} = H\}$

$= \{g \in G \mid gH = Hg\}$ normalizer of H in G

$H \trianglelefteq G \iff N_G(H) = G$

Orbit Stabilizer Theorem

Let G act on a set S and $x \in S$.

Let G/G_x denote set of all left cosets of G_x in G. Then there is a bijection,

$$G/G_x \longrightarrow o(x)$$

In particular, if G is a finite group then $|o(x)| = \dfrac{|G|}{|G_x|}$

(Proof)

$gG_x \longmapsto gx$

Let $gG_x = g'G_x$

$\iff g^{-1}g' \in G_x$

$\iff g^{-1}g'x = x$

$\iff gx = g'x$

one-one

well defined

Trivially surjective

$G \curvearrowright S$

$S = \bigsqcup O(X)$

$$|S| = \sum |O(X)| \overset{OST}{\Longleftrightarrow} \sum [G : G_x]$$

If $g \in G$ and $x \in S$, then $g G_x g^{-1} = G_{gx}$

• $G_{gx} = \{ g'X \in G \mid g g X = g' X \}$ (??)

If $h \in g G_x g^{-1}$

$hgx = gx \Rightarrow gG_xg^{-1} = G_{gx}$

$\Leftrightarrow g^{-1}hgx = x$

$\Leftrightarrow g^{-1}hg \in G_x$

$\Leftrightarrow h \in gG_xg^{-1}$

• $G \curvearrowright G$ by conjugation

$|G| < \infty$

$$|G| = \sum_{i=1}^{n} |O(x_i)|$$ where $O(x_1), \dots, O(x_n)$ are distinct orbits

$$= \sum_{O(x_i)=1} |O(x_i)| + \sum_{O(x_i)>1} |O(x_i)|$$

$|O(x_i)| = 1 \Rightarrow \{ g x_i g^{-1} \mid g \in G \} = \{x_i\}$

$\downarrow$

$\{x_i\}$

$\Rightarrow \{ g x_i = x_i g \} \quad \forall g \in G$

$\Rightarrow x_i \in Z(G)$

$$|G| = |Z(G)| + \sum_{O(x_i)>1} |O(x_i)|$$

$$|G| = |Z(G)| + \sum_{|O(x_i)|>1} [G : C_G(x_i)]$$

class equation for finite groups

If G is a group, $|G| = p^n$, p is a prime

$$|o(x_i)| = [G : C_G(x_i)] = \frac{|G|}{|C_G(x_i)|}$$

$$\Rightarrow |C_G(x_i)| = p^r$$

$r = 0, 1, \ldots, n-1$

$$p \mid |G| = p^n$$

$$p \mid \sum_{|o(x_i)| > 1} [G : C_G(x_i)]$$

$$\Rightarrow p \mid Z(G)$$

Any group of order p^2 is Abelian

$|Z(G)| \to 1$ ⊗ (atleast >1 element)

$\to p$

$\to p^2 \to Z(G) = G$

$$|G/Z(G)| = p \Rightarrow \text{cyclic}$$

- $\sigma \in S_n$

  $\sigma = \sigma_1 \cdots \sigma_r$

  $\sigma_i \to$ disjoint cycles

  $o(\sigma_i) = k_i$

  $1 \leq k_1 \leq \cdots \leq k_r$

  $k_1 + \cdots + k_r = n$

$$\{ c(\sigma) \mid \sigma \in S_n \} \longleftrightarrow \{ (k_1, \ldots, k_r) \mid 1 \leq k_1 \leq \cdots \leq k_r, \ \textstyle\sum k_i = n \}$$