29/08

Permutation $\longrightarrow$ Product of disjoint cycles

Try general theorem for conjugacy (Q.16)

Number of k-cycles : $\frac{{}^nP_k}{k}$

To every permutation σ, we assigned an "action" of σ on a polynomial

$$(\sigma f)(x_1, \dots, x_n) = f(x_{\sigma(1)}, \dots, x_{\sigma(n)})$$

$$\Delta(x_1, \dots, x_n) = \prod_{1 \le i \le j \le n} (x_i - x_j)$$

$$= \det \begin{pmatrix} 1 & \cdots & 1 \\ x_1 & & x_n \\ \vdots & & \vdots \\ x_1^{n-1} & & x_n^{n-1} \end{pmatrix}$$

Let $\sigma_1 \in S_n$ be a transposition

Then $(\sigma_1 \Delta)(x_1, \dots, x_n) = (-1)\, \Delta(x_1, \dots, x_n)$

If $\sigma = \sigma_1 (\sigma_2 \dots, \sigma_m)$ where $\sigma_1, \sigma_2, \dots, \sigma_m$ are transpositions then

$$\sigma \Delta = (-1)^m \Delta(x_1, \dots, x_n)$$

If $\sigma = \tau_1 \tau_2 \cdots \tau_r$

$$(\sigma \Delta)(x_1, \dots, x_n) = (-1)^r \Delta(x_1, \dots, x_n)$$

$\Rightarrow (-1)^m = (-1)^r$

$\Rightarrow m \equiv r \pmod 2$

It thus makes sense to define even permutations and odd permutations.

$$A_n = \{\sigma \in S_n \mid \sigma \text{ is an even permutation}\}$$

- Non-empty :- Id
- Closed :- $\sigma\tau = \sigma_1 \cdots \sigma_n \tau_1 \cdots \tau_k$

  $n+k$ is even because n, k are even

- If $\sigma \in A_n$, then $\sigma = \sigma_1 \sigma_2 \dots \sigma_m$ where σ_i are 2-cycles.

$$\sigma^{-1} = \sigma_m^{-1} \dots \sigma_1^{-1}$$
$$= \sigma_m \dots \sigma_1 \quad (\text{they are 2-cycles})$$

Property: $|A_n| = \frac{n!}{2}$

Proof: Let $X = \{\sigma \in S_n \mid \sigma \text{ is odd permutation}\}$

For a transposition $(1\ 2) \in S_n$

Define $\phi : A_n \longrightarrow X$
$$\sigma \longmapsto (1\ 2)\sigma$$

Injective: $(1\ 2)\sigma = (1\ 2)\tau \Rightarrow \sigma = \tau$

Surjective: Let $\sigma \in X$
Then $(1\ 2)\sigma \in A_n$

$(1\ 2)(\sigma_1) = \sigma$

Thus ϕ is a bijection