29/09

Group Actions

Let $G \curvearrowright X$. This action induces a group homomorphism $\phi : G \to S_X$

$$\phi : G \longrightarrow S_X \qquad m_g : X \longrightarrow X$$
$$g \longmapsto m_g \qquad x \longmapsto gx$$

$$\phi(gh) = \phi(g) \circ \phi(h)$$
$$\downarrow \qquad\qquad \downarrow$$
$$m_{gh} \qquad\qquad m_g \circ m_h$$

$$m_{gh}(x) = (gh)x = g(hx) \quad \text{[By group action]}$$
$$= m_g \circ m_h$$

- $\mathbb{F} \to$ field

$$GL_n(\mathbb{F}) \curvearrowright \mathbb{F}^n$$

$$GL_n(\mathbb{F}) \times \mathbb{F}^n \longmapsto \mathbb{F}^n$$
$$(A, u) \longmapsto Au$$

$\mathbb{F} \to \mathbb{F}_2$ = field with 2 elements
$= \mathbb{Z}/2\mathbb{Z}$

$$|GL_2(\mathbb{F}_2)| = 6 \leftarrow (2^2-1)(2^2-2)$$

$$GL_2(\mathbb{F}_2) \curvearrowright \{e_1, e_2, e_1+e_2\}$$
non-zero elements in $\mathbb{F}_2^2$

$$\phi : GL_2(\mathbb{F}_2) \longrightarrow S_X \cong S_3$$
$$A \longmapsto \sigma_A$$

$$\sigma_A : X \longrightarrow X$$
$$u \longmapsto Au$$

$$\ker \phi = \{A \mid \sigma_A = id\}$$
$$= \{A \mid \sigma_A e_1 = e_1, \sigma_A e_2 = e_2\}$$
$$= \{Id\}$$

$\Rightarrow \phi$ is injective

since $|GL_2(\mathbb{F}_2)| = |S_3|$, ϕ is a bijection

$\Rightarrow GL_2(\mathbb{F}_2) \cong S_3$

- $H \leq G$, $H \curvearrowright G/H$

$$H \times G/H \longrightarrow G/H$$
$$(h, xH) \longrightarrow hxH$$

$\Rightarrow$ The kernel of the induced homomorphism is contained in H.

$\phi : H \to S_X$ $\qquad X = G/H$

$$\phi(g) : X \longrightarrow X$$
$$xH \longmapsto gxH$$

If $g \in \ker \phi$

$\Rightarrow \phi(g) = Id$ on X

$\Rightarrow \phi(g)(xH) = xH \quad \forall x \in G$

$\Rightarrow gxH = xH \quad \forall x \in G$

$\Rightarrow x^{-1}gx \in H \quad \forall x \in G$

$\Rightarrow g \in H$ when $x = 1_G$

- $H \leq G$, $[G:H] = n$ and H does not contain any non-identity ~~element~~ normal subgroup. Then H is isomorphic to a subgroup of S_n

- If $H \leq G$ and $[G:H] = p$, where p is the smallest prime that divides $|G|$, then $H \trianglelefteq G$

$G \curvearrowright G/H \quad X$

$\phi : G \longrightarrow S_X \cong S_p \qquad |S_p| = p!$

$K = \ker\phi \subseteq H$

$|G/K|$

$= [G:K]$

$= [G:H][H:K] \qquad ([G:H] = p)$

$\geq p$

$|G/K| \mid p! = 1.2, 3. \ldots. (p-1).p$

If $|G/K| > p$, it should still divide $p!$,

so any multiple would contain a factor

$1 \leq x \leq p-1$

That will have a unique factorization

with smallest prime less than p

$\Rightarrow |G/K| = p$

$\Rightarrow [H:K] = 1$

$\Rightarrow H = K \Rightarrow H \trianglelefteq G$

** $G \curvearrowright S$. If $|G| = p^n$, then $|S| \equiv |S_0| \pmod p$

$S_0 = \{x \in G \mid |O(x)| = 1\}$

$|S| = |S_0| + \sum_{|O(x)|>1} |O(x)|$

$S_0' = \{x \in S \mid gx = x \ \forall g \in G\}$

$p \mid |O(x)| = [G:G_x]$ (orbit stabilizer theorem)

$= \dfrac{|G|}{|G_x|}$

$|G_x| \neq p^n$ [otherwise $x \in S_0$]

$\Rightarrow p \mid \sum_{|O(x)|>1} |O(x)| \Rightarrow p \mid (|S| - |S_0|)$

Cauchy's Theorem

If G is a finite group and $p \mid |G|$ then G has an element of order p.

$$S = \{ (a_1, a_2, \dots, a_p) \in G' \mid a_1 \dots a_p = 1 \}$$

If $|G| = n$, then $|S| = n^{p-1}$

$\mathbb{Z}/p\mathbb{Z} \curvearrowright S$

$$0(a_1, \dots, a_p) = (a_1, \dots, a_p)$$

$$1(a_1, \dots, a_p) = (a_2, \dots, a_p, a_1)$$

$$k(a_1, \dots, a_p) = (a_{k+1}, \dots, a_p, a_1, a_2, \dots, a_k)$$

$(a_1, \dots, a_p) \in S$

$(a_2, \dots, a_p, a_1) \in S$

$x \in S_0$

$\Leftrightarrow \quad x = (a, \dots, a) \in G \times \dots \times G$

$|S| = |S_0| \pmod p$

$\Rightarrow p \mid |S_0|$

But $(1, \dots, 1) \in S_0$

$\Rightarrow \exists\, a \in G \setminus \{1\}$ s.t $(a, \dots\dots a) \in S_0 \subseteq S$

$\Rightarrow a = 1$