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Field of fractions of integral domains

Definition

Let R be an integral domain. We define an equivalence relation $\sim$ on $R \times (R \setminus \{0\})$ as follows

$$(a, b) \sim (c, d) \iff ad = bc$$

(Exercise!)

- Prove that $\sim$ is an equivalence relation

Let $K = \dfrac{R \times (R \setminus \{0\})}{\sim}$

$$= \left\{ \frac{a}{b} \mid a \in R,\ b \neq 0 \right\}$$

Define

$$\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd}$$

$$\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$$

(Exercise!)

(i) Check that $+, \cdot$ are well-defined

(ii) Check that $(K, +, \cdot)$ is a field

(iii) There is an injective ring homomorphism from $i : R \to K$ given by $a \mapsto \frac{a}{1}$

(iv) For any field L, and an injective map $\varphi : R \to L$, there exists a unique map $\tilde{\varphi} : K \to L$ such that the following diagram commutes

$$\begin{tikzcd} R \arrow[r, "\varphi"] \arrow[d] & L \\ K \arrow[ur, "\tilde{\varphi}"'] & \end{tikzcd}$$

$$\tilde{\varphi}\left(\frac{a}{b}\right) = \frac{\varphi(a)}{\varphi(b)}$$

$\tilde{\varphi} \to$ ring homomorphism ✓

Examples

(i) $\mathbb{Z} \rightarrow \mathbb{Q}$

(ii) $K[x]$

$$K(x) = \left\{ \frac{f(x)}{g(x)} : f(x), g(x) \in K[x], \; g(x) \neq 0 \right\}$$

$\uparrow$

field of rational functions in one variable

(iii) $K[x,y]$

$$K(x,y) = \left\{ \frac{f(x,y)}{g(x,y)} : f, g \in K[x,y] \; , \; g(x,y) \neq 0 \right\}$$

(*) Check whether a homomorphism from a field to a ring is a zero map (or) one-one

UFD

R is a UFD $\Rightarrow$ $R[x]$ is a UFD

(Exercise!)

If R is a UFD, and $a, b \in R$, then $\gcd(a,b)$ exists in R

Definition

Let $f(x) = a_0 + a_1 x + \dots + a_n x^n \in R[x]$

We define the content of f denoted by

$$c(f) = \gcd(a_0, a_1, \dots, a_n)$$

A polynomial $f(x) \in R[x]$ is said to be primitive if $c(f) = 1$

Irreducible $\Rightarrow$ primitive

(Converse is not true!)

Gauss' Lemma

Gauss → Gauß

Let R be a UFD and let $f(x), g(x) \in R[x]$ be primitive. Then $f(x) \cdot g(x)$ is also primitive.

(Proof)

Lemma

If R is a UFD, and $a \in R$ is irreducible then a is prime.

(Proof) $p | ab \Rightarrow p | (\prod p_i)(\prod q_j)$

$p \in \{p_i\}$ or $p \in \{q_j\}$

$\Rightarrow p | a$ or $p | b$

Suppose $c(fg) \neq 1$

Then $\exists\, p \in R$, p is irreducible such that $p | c(fg)$

We consider the ring $R/(p)[x]$.

Then we have a map

$$\varphi_K : R[x] \longrightarrow R/(p)[x]$$

→ Homomorphism

$$a_0 + a_1 x + \dots + a_n x^n \longmapsto a_0 + (p) + (a_1 + (p))x + \dots$$

$$\longmapsto \bar{a}_0 + \bar{a}_1 x + \dots + \bar{a}_n x^n$$

$$\varphi_K(fg) = \varphi_K(f)\,\varphi_K(g)$$

$$\bar{0} = \varphi_K(f)\,\varphi_K(g)$$

$\Rightarrow \varphi_K(f) = \bar{0}$ or $\varphi_K(g) = \bar{0}$

$\Rightarrow p | c(f)$ or $p | c(g)$

Corollary

$$c(fg) = c(f)\, c(g)$$

(Proof)

Suppose $c(f) = a,\ c(g) = b$

$\Rightarrow f = a f_1,\ g = b g_1$

$\Rightarrow fg = ab\, f_1 g_1$

By Gauss lemma, $f_1 g_1$ is primitive

$\Rightarrow c(fg) = ab = c(f)\, c(g)$

Proposition

If R is a UFD and K its quotient field and let $f, g \in R[x]$ be primitive polynomials such that $f(x), g(x)$ are associates in $K[x]$ then $f(x), g(x)$ are associates in $R[x]$

(Proof)

$f(x) = \left(\frac{a}{b}\right) g(x)$

$\Leftrightarrow b f(x) = a\, g(x)$

$\Leftrightarrow c(b f(x)) = c(a\, g(x))$

$\Leftrightarrow b = a \cdot u$ (or) $b u' = a$

$\Rightarrow f(x) = u' g(x)$

$\Rightarrow f(x), g(x)$ are associates in $R[x]$

If R is a UFD, $f(x) \in R[x]$ is irreducible in $R[x]$, then $f(x)$ is irreducible in $K[x]$

(Proof)

suppose $f(x)$ is reducible in $K[x]$

$f(x) = g(x)\, h(x)$ $\quad g(x), h(x) \in K[x]$

we express

$g(x) = \frac{a}{b}\, g_1(x)$ $\quad$ where

$h(x) = \frac{c}{d}\, h_1(x)$ $\quad$ g_1, h_1 are primitive in $R[x]$

$f(x) = \frac{ac}{bd}\, \underbrace{g_1(x)\, h_1(x)}_{\text{primitive}}$

$\Rightarrow f(x) = u\, g_1(x)\, h_1(x)$

$f(x)$ is reducible in $R[x]$

Proof for $R[x]$ is UFD

$f(x) \in R[x]$

Write $f(x) = c\, f_1(x)$ $\quad$ $f_1(x)$ is primitive

$= (\prod p_i)\, f_1(x)$

$f_1(x)$ is primitive in $R[x]$

Write $f_1(x) = \frac{c}{d}\,(g_1(x) \dots g_r(x))$

primitive

$f_1(x) = g_1'(x) \dots g_r'(x)$, $g_i'(x) \in K[x]$