15/09

Example

$G = S_3$

- $H = \langle (1\ 2\ 3) \rangle$

  $= \langle e, (1\ 2\ 3), (1\ 3\ 2) \rangle$

Left coset $(1\ 2)H = \{(1\ 2), (2\ 3), (1\ 3)\}$
Right coset $H(1\ 2) = \{(1\ 2), (1\ 3), (2\ 3)\}$

They are equal

- $H = \langle (1\ 2) \rangle$

  $= \{e, (1\ 2)\}$

$H(1\ 2\ 3) = \{(1\ 2\ 3), (2\ 3)\}$

$(1\ 2\ 3)H = \{(1\ 2\ 3), (1\ 3)\}$

They are not equal !

In general, left cosets need not be equal to right cosets.

Definition

A subgroup H of G is said to be a normal subgroup if $gH = Hg \quad \forall g \in G$

Example

(i) If G is abelian, then any subgroup $H \leq G$ is a normal subgroup

$gH = \{gh \mid h \in H\}$. $Hg = \{hg \mid h \in H\}$

$gh = hg \quad \forall g, h \in G$

(ii) Let G, G' be groups and $\phi : G \to G'$ be a group homomorphism

Then $\ker \phi$ is a normal subgroup of G

Notation

$H \trianglelefteq G \rightarrow H$ is a normal subgroup of G

$\{ghg^{-1} \mid h \in H\}$

Lemma

$H \trianglelefteq G \iff gHg^{-1} = H \quad \forall g \in G$

(Proof)

($\Rightarrow$) Take $g \in G$ and $h \in H$

$ghg^{-1} = (h_1 g) g^{-1} = h_1 \in H$

$\downarrow$

$gH = Hg$

Hence, $gHg^{-1} \subseteq H \quad \forall g \in G$

Fix $g \in G$. Applying ① to the element g^{-1}, we see that

$g^{-1}Hg \subseteq H$

$\Rightarrow H \subseteq gHg^{-1}$

Thus, $gHg^{-1} = H \quad \forall g \in G$

($\Leftarrow$)

Suppose that $gHg^{-1} = H$ for all $g \in G$

Take $x \in gH$

$\Rightarrow x = gh$ for some $h \in H$

$x = x g^{-1} g$

$= (ghg^{-1}) g$

$\in Hg$ as $ghg^{-1} \in gHg^{-1} = H$

$\Rightarrow gH \subseteq Hg$

(Similarly prove $Hg \subseteq gH$)

$\Rightarrow y = hg$ for some $h \in H$

$y =$

Remark

From the proof of the Lemma, it is clear that in order to check if $H \trianglelefteq G$, it is enough to check whether $gHg^{-1} \subseteq H \quad \forall g \in G$

- Ker ϕ is a normal subgroup of G

In view of the remark, it is enough to show that

$$g(\ker\phi)g^{-1} \subseteq \ker\phi$$

Let $k \in \ker\phi$

$$\phi(gkg^{-1}) = \phi(g)\,\phi(k)\,\phi(g^{-1})$$
$$= \phi(g)\,\phi(k)(\phi(g))^{-1}$$
$$= 1_{G'}$$

$\Rightarrow \quad gkg^{-1} \in \ker\phi$

Proposition

Let G be a finite group and $H \leq G$.
Then $gHg^{-1} \leq G$

(Proof)

Since $H \leq G$, $1 \in H$
Thus $1 = g1g^{-1} \in gHg^{-1}$

Let $x, y \in ghg^{-1}$
Then $x = gh_1g^{-1}$, $y = gh_2g^{-1}$ for some $h_1, h_2 \in H$

Then $xy = g(h_1h_2)g^{-1} \in gHg^{-1}$
because $h_1h_2 \in H \quad (H \leq G)$

For $x = ghg^{-1}$, $x^{-1} = gh^{-1}g^{-1} \in gHg^{-1} \quad (H \leq G)$

Thus $gHg^{-1} \leq G$

(ii) If there exists only one subgroup H of G of a given order, then $H \trianglelefteq G$.

It is enough to show that $|H| = |gHg^{-1}|$ for all $g \in G$

Define $\phi : H \longrightarrow gHg^{-1}$

$h \longmapsto ghg^{-1}$

(one - one and onto)

$\Rightarrow H = gHg^{-1} \quad \forall g \in G$

Proposition

If $H \leq G$ s.t $[G \mid H] = 2$, then $H \triangle G$

Proof Fix $g \in G$

If $g \in H$, then $ghg^{-1} \in H$ $\left[\begin{array}{l} gh \in H \\ (gh)g^{-1} \in H \end{array}\right]$

If $g \notin H$, then

$G = H \sqcup gH$

But $G = H \sqcup Hg$ (because $[G \mid H] = 2$)

Then $gH = G \setminus H = Hg$

Correspondence Theorem

Let $\phi : G \rightarrow G'$ be a group homomorphism

Let $H' \leq G'$

We define $f^{-1}(H') = \{g \in G \mid f(g) \in H'\}$

Let $K = \ker \phi$

ⓐ $K \leq \phi^{-1}(H') \leq G$

ⓑ $H \leq G \Rightarrow \phi(H) \leq G'$

ⓒ $H' \trianglelefteq G' \Rightarrow \phi^{-1}(H') \trianglelefteq G$

Proofs

(a) $K \subseteq \phi^{-1}(H')$

Take $k \in \ker \phi$

Then $\phi(k) = 1_{G'} \in H'$ as $H' \leq G'$

$\Rightarrow$ ~~$k = \phi^{-1}(1_{G'})$~~

$\Rightarrow k \in H'$

$\Rightarrow K \subseteq \phi^{-1}(H')$

$\phi^{-1}(H') \leq G$

since $K \subseteq \phi^{-1}(H')$ and $K \leq G$, $id \in K$

$\Rightarrow \phi^{-1}(H') \neq \emptyset$

Take $x, y \subseteq \phi^{-1}(H')$

Then $\phi(x), \phi(y) \in H'$

$\Rightarrow \phi(xy) = \phi(x)\phi(y) \in H'$

$\Rightarrow xy \in \phi^{-1}(H')$

If $x \in \phi^{-1}(H')$

$\phi(x) \in H' \Rightarrow (\phi(x))^{-1} \in H'$

$\Rightarrow \phi(x)(\phi(x))^{-1} \in H'$

$\Rightarrow \phi(x x^{-1}) \in H'$

$\Rightarrow x^{-1} \in \phi^{-1}(H')$

(b) $H \neq \emptyset$ $(H \leq G)$

Suppose $h \in H \Rightarrow \phi(h) \in \phi(H) \neq \emptyset$

Take $x, y \in \phi(H)$

$\exists\, h_1, h_2 \in H$ such that

$x = \phi(h_1)$

$y = \phi(h_2)$

$xy = \phi(h_1 h_2)$ $[h_1 h_2 \in H]$

$xy \in \phi(H)$

If $x \in \phi(H)$ and $x = \phi(h)$

$$x^{-1} = (\phi(h))^{-1} = \phi(h^{-1}) \in \phi(H)$$

(c) $\phi^{-1}(H') \trianglelefteq G$

Take $g \in G$ and $h \in \phi^{-1}(H')$

To show that $ghg^{-1} \in \phi^{-1}(H')$

Equivalently, $\phi(ghg^{-1}) \in H'$

$$\phi(ghg^{-1}) = \phi(g)\ \underset{H'}{\underset{\downarrow}{\phi(h)}}\ \phi(g^{-1}) \in H'$$

(Continued theorem ...)

If ϕ is surjective

(i) $\phi^{-1}(H') \trianglelefteq G \Rightarrow H' \trianglelefteq G'$

(ii) there is a bijection

$$\{H \leq G \mid K \leq H\} \longrightarrow \{H' \leq G'\}$$

Proof – (i)

$\phi^{-1}(H') \trianglelefteq G$

$\phi(\phi^{-1}(H')) = H'$ [claim]

Let $x \in \phi(\phi^{-1}(H'))$

$\Rightarrow \exists\, y \in \phi^{-1}(H')$ s.t $\phi(y) = x$ [from definition]

$\Rightarrow x = \phi(y) \in H'$

$\Rightarrow \phi(\phi^{-1}(H')) \subseteq H'$

Take $x \in H'$

Since ϕ is surjective

$\exists\, y \in \phi^{-1}(H')$ s.t $\phi(y) = x$

$\Rightarrow x \in \phi(\phi^{-1}(H'))$

$\Rightarrow H' \subseteq \phi(\phi^{-1}(H'))$

$\Rightarrow \phi(\phi^{-1}(H')) = H'$