Inverse

$(hk)^{-1} = k^{-1}h^{-1} \in k^{-1}H$

$k^{-1}h^{-1} \in Hk^{-1}$ for some $h_1 \in H$

$k^{-1}h^{-1} = h_1 k^{-1}$

$\Rightarrow h_1 k^{-1} \in HK$

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(d) Suppose ϕ is an isomorphism

($\Rightarrow$) $\Rightarrow H \cap K = \{1\}$ and $HK = G$

First we show that $H \trianglelefteq G$

Take $g \in G$ and $h \in H$

Since $G = HK$, $\exists\, h_1 \in H$ and $k_1 \in K$ such that $g = h_1 k_1$

Now $ghg^{-1} = h_1 k_1 h k_1^{-1} h_1^{-1}$

$= h_1 h k_1 k_1^{-1} h_1^{-1} = h_1 h h_1^{-1} \in H$

Isomorphism

$\downarrow$

Homomorphism

$hk = kh$

$\forall\, h, k \in H \times K$

Injective

$\{1\} = H \cap K$

onto

$HK = G$

($\Leftarrow$) Let $h \in H$ and $k \in K$

consider $hkh^{-1}k^{-1} \in K \cap H$

$\Rightarrow hkh^{-1}k^{-1} = 1$

$\Rightarrow hk = kh$

Proposition

There are exactly two isomorphism classes of groups of order 4.

(Proof)

Let G be a group of order 4

If G has an element of order 4, then

$G = \langle x \rangle \cong \mathbb{Z}/4\mathbb{Z}$

otherwise, every non-identity element of G has order 2.

Let $x, y \in G \setminus \{1\}$

and $H = \langle x \rangle$, $K = \langle y \rangle$

Then $|H| = |K| = 2$

If $[G : H] = 2 \Rightarrow H \trianglelefteq G \Rightarrow \left[\begin{array}{l} G = H \amalg Ha \\ \quad = H \amalg aH \end{array} \right]$

If $|H \cap K| = 2 \Rightarrow |H| = |K| = 2$
$$\Downarrow$$
$$H = K$$

Thus, $H \cap K = \{1\}$

$\varphi : H \times K \longrightarrow G$

$H, K \trianglelefteq G$

$\Rightarrow \varphi$ is an homomorphism

$H \cap K = \{1\}$

$\Rightarrow \varphi$ is injective

Since $|H \times K| = |G| = 4$, the map φ is a bijection

Thus $G \cong H \times K$
$$\cong (\mathbb{Z}/2\mathbb{Z}) \times (\mathbb{Z}/2\mathbb{Z})$$

Quotient Groups

Let G be a group, and $N \trianglelefteq G$

Define $G/N := \{gN \mid g \in G\}$

Define a binary operation$^{(?)}$ on G/N as follows

$$(g_1 N)(g_2 N) = g_1 g_2 N$$

If $g_1 N = g_1' N$ and $g_2 N = g_2' N$, is $g_1 g_2 N = g_1' g_2' N$?

- We check if the operation is well-defined,

Let $g_1 N = g_1' N \Rightarrow g_1'^{-1} g_1 \in N$

$g_2 N = g_2' N \Rightarrow g_2'^{-1} g_2 \in N$

We need to prove that $(g_1' g_2')^{-1} g_1 g_2 \in N$

$\downarrow$

$(N g_1 = g_2 N)$

But $(g_1' g_2')^{-1} g_1 g_2 \in N$

because $g_2'^{-1} (g_1'^{-1} g_1) g_2 = g_2'^{-1} n g_2$

$= g_2'^{-1} g_2 n' \in N$

Thus $g_1 g_2 N = g_1' g_2' N$

- We check if the operation is associative

$((g_1 N)(g_2 N))(g_3 N)$

$= (g_1 g_2 N)(g_3 N)$

$= (g_1 g_2) g_3 N$

- Identity

  N is the identity element of G/N

- Inverse

  For $gN \in G/N$, the inverse is $g^{-1}N$

  $(gN)(g^{-1}N) = (gg^{-1})N = N$

Thus, G/N is a group and it is called the quotient group of G by N.

Proposition

If G is a group and $N \trianglelefteq G$, then there is a natural projection $\boxed{\text{map}}$ $\pi : G \to G/N$

$g \mapsto gN$

group homomorphism

Moreover π is surjective and $\ker(\pi) = N$

Homomorphism

$$\pi(g_1 g_2) = g_1 g_2 N = (g_1 N)(g_2 N) = \pi(g_1)\pi(g_2)$$

Surjective

Trivial !

Kernel

$$\begin{aligned} Ker(\pi) &= \{g \in G \mid \pi(g) = N\} \\ &= \{g \in G \mid gN = N\} \\ &= \{g \in G \mid g \in N\} \\ &= N \qquad \text{(Because } N \trianglelefteq G) \end{aligned}$$

First Isomorphism Theorem

Let G, G' be groups and $\phi : G \to G'$ be a group homomorphism. If $N = ker(\phi)$, then ϕ induces an isomorphism

$$\tilde{\phi} : G/N \to Im(\phi)$$

such that $\tilde{\phi} \circ \pi = \phi$

$$\begin{tikzcd} G \arrow[r, "\phi"] \arrow[d, "\pi"'] & G' \\ G/N \arrow[ur, "\tilde{\phi}"'] & \end{tikzcd}$$

Equivalent to saying, this diagram commutes

Proof

Define $\tilde{\phi} : G/N \to G'$

$$\tilde{\phi}(gN) \mapsto \phi(g)$$

Well-defined

Suppose $gN = g'N$

$\Rightarrow g^{-1}g' N \in N$ and $ker(\phi) = N$

$\Rightarrow \phi(g^{-1}g') = 1_{G'}$

$\Rightarrow \phi(g') = \phi(g)$

Homomorphism

$$\begin{aligned} \tilde{\phi}((g_1 N)(g_2 N)) &= \tilde{\phi}(g_1 g_2 N) = \phi(g_1 g_2) \\ &\qquad\qquad = \phi(g_1)\phi(g_2) \\ &= \tilde{\phi}(g_1 N)\tilde{\phi}(g_2 N) \end{aligned}$$

- Injective

$$\tilde{\phi}(g_1 N) = \tilde{\phi}(g_2 N)$$

$$\phi(g_1) = \phi(g_2)$$

$$\phi(g_1^{-1} g_2) = 1 \quad \Rightarrow \quad g_1^{-1} g_2 \in N$$

$$\Rightarrow (g_1 N = g_2 N)$$

- onto

The map $\tilde{\phi} : G/N \rightarrow \text{Im}\,\phi$ is surjective as every element $\phi(g)$ has a preimage $\tilde{\phi}(gN)$ $[\phi(g) \in \text{Im}\,\phi]$

$$\Rightarrow \boxed{G/N \cong \text{Im}\,\phi}$$

Examples

(i) $\mathbb{Z}/n\mathbb{Z}$

(ii) $GL_n(\mathbb{F}) \longmapsto \mathbb{F}^*$

$M \longmapsto \det M$ ← onto $\begin{pmatrix} c & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix}$

$c \in \det M$

$SL_n(\mathbb{F}) \longmapsto$ defined as $\{M \in GL_n(\mathbb{F}) \mid \det M = 1\}$

$$GL_n(\mathbb{F}) / SL_n(\mathbb{F}) \cong \mathbb{F}^*$$

(iii) $S_n \rightarrow \mathbb{Z}/2\mathbb{Z}$

$$\sigma \longmapsto \text{sgn}\,\sigma = \begin{cases} 1 & \text{if } \sigma \text{ is even} \\ -1 & \text{if } \sigma \text{ is odd} \end{cases}$$

$$S_n/A_n \cong \mathbb{Z}/2\mathbb{Z} \rightarrow \{-1, 1\}$$

(iv) $\mathbb{R} \rightarrow S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$

$x \mapsto e^{2\pi i x}$

$$\mathbb{R}/\mathbb{Z} \cong S^1$$

(v) $\mathbb{C}^{\times} \longrightarrow \mathbb{R}_{>0}$

$z \longmapsto |z|$

homomorphism $\rightarrow$ $|z_1 z_2| = |z_1||z_2|$

onto $\rightarrow$ ✓

$\Rightarrow \mathbb{C}^{\times}/S^1 \cong \mathbb{R}_{>0}$

Application

$\{ H \leq G \mid \ker\phi \leq H \} \leftrightarrow \{ H' \leq G' \}$

$\pi : G \longrightarrow G/N$

$\{ H \leq G \mid N \leq H \} \leftrightarrow \{ H' \leq G/N \}$

Proposition

If $K \trianglelefteq G$ and $K' \trianglelefteq G'$, then $K \times K' \trianglelefteq G \times G'$

and $\dfrac{G \times G'}{K \times K'} \cong G/K \times G'/K'$

Proof

$\phi : G \times G' \longrightarrow G/K \times G'/K'$

$(g, g') \longmapsto (gK, g'K')$

Homomorphism ✓ Surjective ✓

$\ker\phi = \{(g, g') \mid (gK, g'K') = (K, K')\}$

$= \{(g, g') \mid g \in K, g' \in K'\}$

$= K \times K'$

Proposition

$G/Z(G) \cong \text{Inn } G = \{ \tau_g : G \rightarrow G \mid \tau_g(h) = g h g^{-1} \}$

$\phi : G \longrightarrow \text{Inn}(G)$

$g \longmapsto \tau(g)$

Homomorphism : $\phi(g_1 g_2) = \tau_{g_1 g_2}(h) = g_1 (g_2) h g_2^{-1} g_1^{-1}$

$= \tau_{g_1}(\tau_{g_2}(h))$