07/11

$$S = \left\{ \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \mid a, b \in \mathbb{R} \right\}$$

It is closed with respect to addition

Additive inverse $\rightarrow \begin{pmatrix} -a & -b \\ b & -a \end{pmatrix}$ exists

Product :- is closed $\begin{pmatrix} a & b \\ -b & a \end{pmatrix} \begin{pmatrix} c & d \\ -d & c \end{pmatrix} = \begin{pmatrix} ac-bd & ad+bc \\ -(ad+bc) & ac-bd \end{pmatrix}$

$\Rightarrow$ S is a subring with identity $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$S \longrightarrow \mathbb{C}$

$\begin{pmatrix} a & b \\ -b & a \end{pmatrix} \rightarrow a+ib$

ring homomorphism ✓
surjective ✓
one-one $(\ker \phi = \{(0,0)\})$

$\Rightarrow$ S is isomorphic to $\mathbb{C}$

Example

① $C([0,1]) = \{ f : [0,1] \rightarrow \mathbb{R} \mid f \text{ is continuous} \}$

Let $b \in [0,1]$ and $ev_b : C[0,1] \longrightarrow \mathbb{R}$

$f \longmapsto f(b)$

$\ker(ev_b) = \{ f : [0,1] \rightarrow \mathbb{R} \mid f \text{ is continuous and } f(b) = 0 \}$

From first isomorphism theorem,

$$C([0,1]) / \ker(ev_b) \cong \mathbb{R}$$

②

$\mathbb{R}[x] \longrightarrow \mathbb{C}$

$f \longrightarrow f(i)$

It is a surjective ring homomorphism

$\ker(\varphi) = I = \langle x^2+1 \rangle$

Because

$f(x) = g(x)(x^2+1) + ax + b$

$\Rightarrow a = b = 0$

$\Rightarrow \mathbb{R}[x] / \langle x^2+1 \rangle \cong \mathbb{C}$

Prime Ideals and Integral Domains

Definition

A proper ideal I of a ring is said to be a prime ideal if

$ab \in I \Rightarrow a \in I$ (or) $b \in I$

$a \notin I$ and $b \notin I \Rightarrow ab \notin I$

Proposition

Let R be a commutative ring with identity

Then I is a prime ideal $\Leftrightarrow$ R/I is an integral domain

Proof

($\Rightarrow$) $(a+I)(b+I) = I$

$(ab + I) = I$

$\Rightarrow ab \in I$

$\Rightarrow a \in I$ or $b \in I$ [I is prime ideal]

$\Rightarrow R/I$ is an integral domain

($\Leftarrow$) Let $ab \in I$ $(a, b \in R)$

$\Rightarrow ab + I = I$

$\Rightarrow (a+I)(b+I) = I$

$\Rightarrow a + I = I$ (or) $b + I = I$

$\Rightarrow a \in I$ (or) $b \in I$

$\Rightarrow I$ is prime ideal

Example

1. $\mathbb{Z}/n\mathbb{Z}$ is integral domain $\Leftrightarrow$ n is prime

$n\mathbb{Z}$ is prime ideal $\Leftrightarrow$ $\mathbb{Z}/n\mathbb{Z}$ is integral domain

$\Rightarrow p\mathbb{Z}$ are prime ideals in $\mathbb{Z}$

(2) $\mathbb{Z}[i] = \{a+ib \mid a,b \in \mathbb{Z}\}$

$$\phi : \mathbb{Z}[x] \to \mathbb{Z}[i]$$
$$f(x) \mapsto f(i)$$

$$\ker(\phi) = x^2+1$$

$$\mathbb{Z}[x]/\langle x^2+1\rangle \cong \mathbb{Z}[i]$$

$\langle x^2+1\rangle$ is prime

Is $2\mathbb{Z}[i]$ a prime ideal?

$$2 \in 2\mathbb{Z}[i]$$
$$(1+i) \notin 2\mathbb{Z}[i]$$
$$(1-i) \notin 2\mathbb{Z}[i]$$
But $(1+i)(1-i) = 2$

$\Rightarrow 2\mathbb{Z}[i]$ is not a prime ideal of $\mathbb{Z}[i]$

Maximal Ideal and Fields

A proper ideal I of R is said to be a maximal ideal if for any ideal J satisfying $I \subseteq J \subseteq R$ is given by $I = J$ or $J = R$

Proposition

Let R be a commutative ring with identity and $I \trianglelefteq R$

I is maximal ideal $\Leftrightarrow$ R/I is a field

Proof

$(\Rightarrow)$

Let $a + I \neq I$

$\Rightarrow I \subsetneq (a) + I \subseteq R$

$\Rightarrow (a) + I = R$

$\Rightarrow ab + x = 1 \quad [\exists\, b \in R \text{ and } x \in I]$

$\Rightarrow ab + x + I = 1 + I$

$\Rightarrow ab + I = 1 + I \Rightarrow (a+I)(b+I) = (1+I)$

$\Rightarrow R/I$ is a field

(*) In a field, there are only two ideals!

$I \subseteq J \subseteq R$

$\Rightarrow 0 \subseteq J/I \subseteq R/I$

$\Rightarrow I$ is maximal

* Let K be a field and $K[x]$ be the polynomial ring

Theorem

An ideal $I = (f(x))$ is maximal ideal

$\Leftrightarrow f(x)$ is irreducible

Proof

($\Rightarrow$) suppose $f(x) = g(x)\,h(x)$ $\quad 1 \leq \deg(g(x)), \deg(h(x)) < \deg(f(x))$

$(p(x)) \supsetneq (g(x)) \supsetneq (f(x))$

$\Rightarrow f(x)$ is not maximal ($\Rightarrow\Leftarrow$)

($\Leftarrow$) $f(x)$ is irreducible

$I \subseteq J \subseteq K[x]$

$(f(x)) \subseteq (g(x)) \subseteq K[x]$

$\Rightarrow f(x) \in (g(x))$

$\Rightarrow f(x) = g(x)\,h(x)$

But $f(x)$ is irreducible

$\therefore$ $g(x)$ is constant (or) $h(x)$ is constant

$\Rightarrow (g(x)) = K[x]$ $\qquad (g(x)) = (f(x))$

$\Rightarrow (f(x))$ is maximal