Project 1 – Visualize ODE With SciPy

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CST-305: Project 2 – Runge-Kutta-Fehlberg (RKF) for ODE

Professor Citro

September 29, 2024

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CST-305: Project 2 – Runge-Kutta-Fehlberg (RKF) for ODE

Responsibilities and completed tasks by each team member
Grant: Code
Ben: Documentation

Specific problem solved
The problem was to solve an ODE using the Runge Kutta Fehlberg.

The mathematical approach for solving the problem
To solve the problem, we used the formula for Runge Kutta Fehlberg.

$$y_{n+1} = y_n + (h/6)(k_1 + 2k_2 + 3k_3 + k_4)$$

$$x_{n+1} = x_n + h$$

$$k_1 = f(x_n, y_n)$$

$$k_2 = f(x_n + (h/2), y_n + ((hk_1)/2)$$

$$k_3 = f(x_n + (h/2), y_n + ((hk_2)/2)$$

$$k_4 = f(x_n + h, y_n + (hk_3))$$

which is equivalent to:

$$y_{n+1} = y_n + (1/6)(k_1 + 2k_2 + 3k_3 + k_4)$$

$$x_{n+1} = x_n + h$$

$$k_1 = h * f(x_n, y_n)$$

$$k_2 = h * f(x_n + (h/2), y_n + ((hk_1)/2)$$

$$k_3 = h * f(x_n + (h/2), y_n + ((hk_2)/2)$$

$$k_4 = h * f(x_n + h, y_n + (hk_3))$$

which we used in the code.

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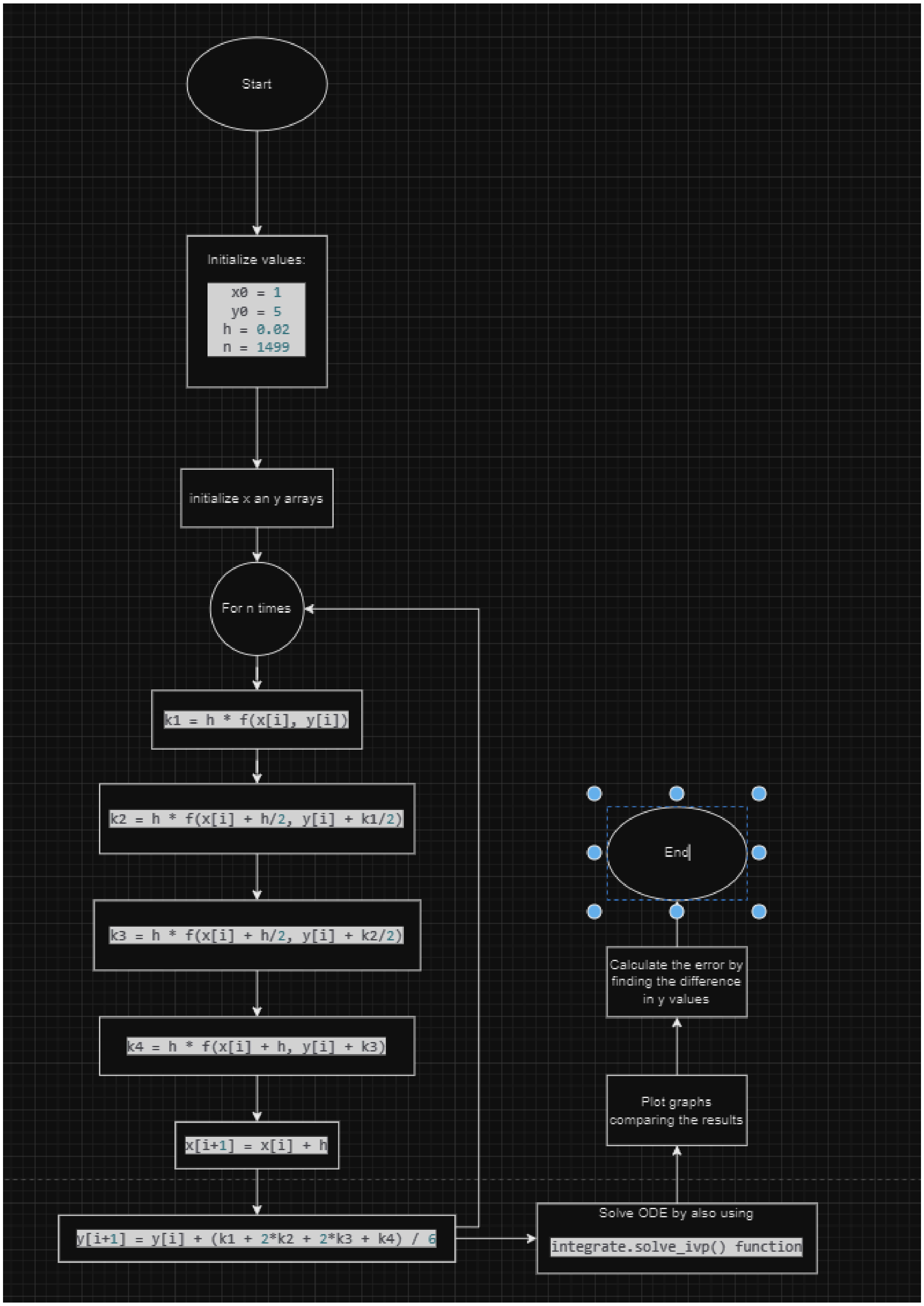
Flowchart:



Screenshots depicting key phases in the program execution

Comparing results:

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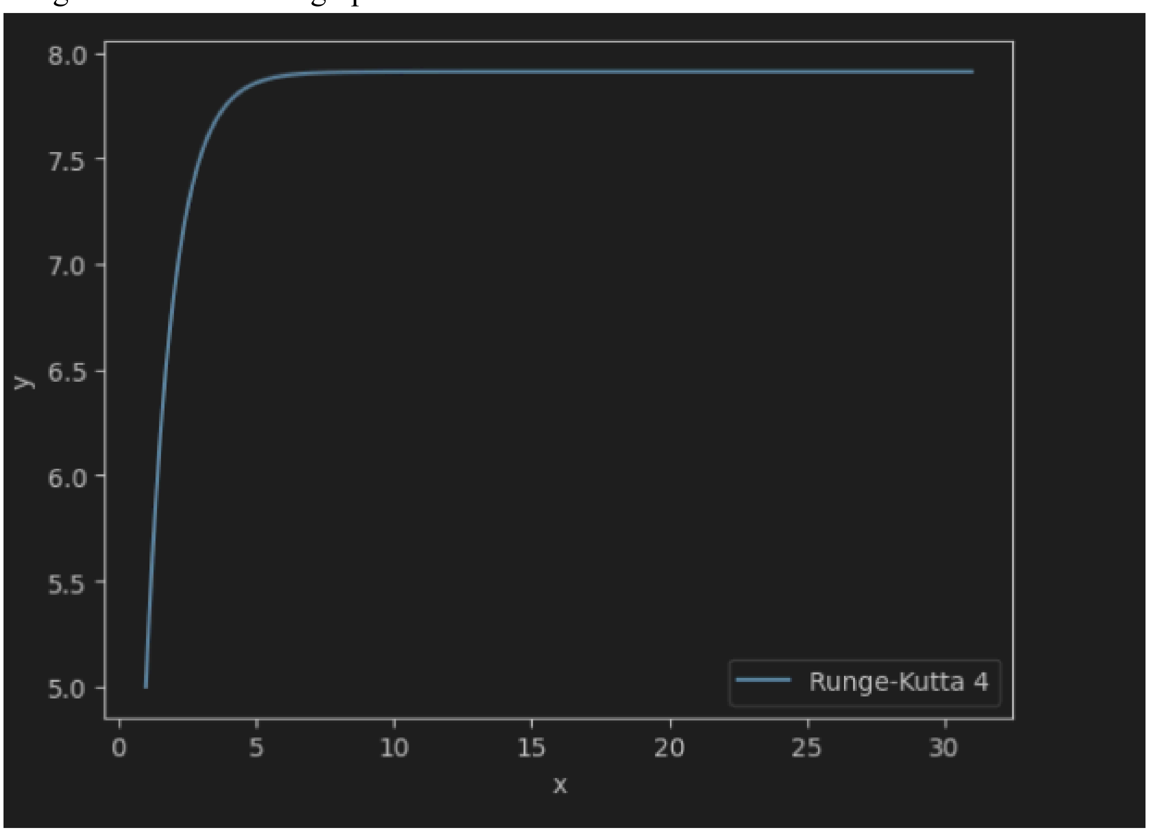
```
31.00    7.9109
Compare:
x        x_actual     y        y_actual
1.00     1.00         5.00     5.0000
1.02     1.02         5.06     5.0577
1.04     1.04         5.11     5.1142
1.06     1.06         5.17     5.1696
1.08     1.08         5.22     5.2239
1.10     1.10         5.28     5.2771
1.12     1.12         5.33     5.3293
1.14     1.14         5.38     5.3804
1.16     1.16         5.43     5.4305
1.18     1.18         5.48     5.4797
1.20     1.20         5.53     5.5279
1.22     1.22         5.57     5.5751
1.24     1.24         5.62     5.6214
```

Runge-Kutta calculated graph:



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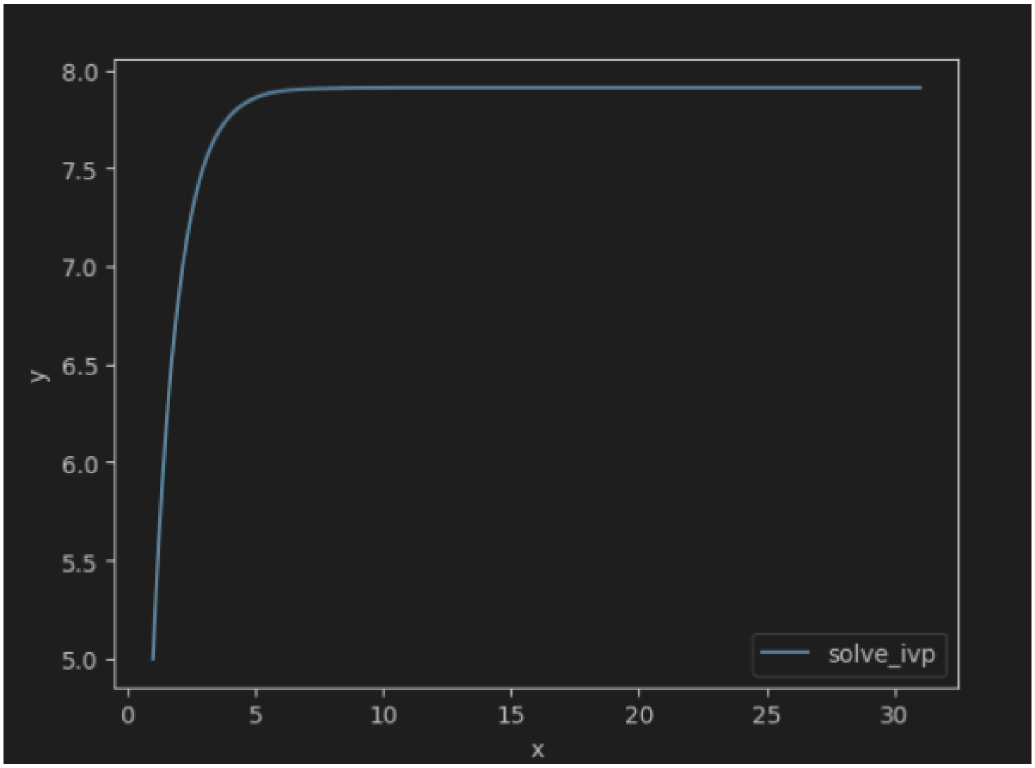
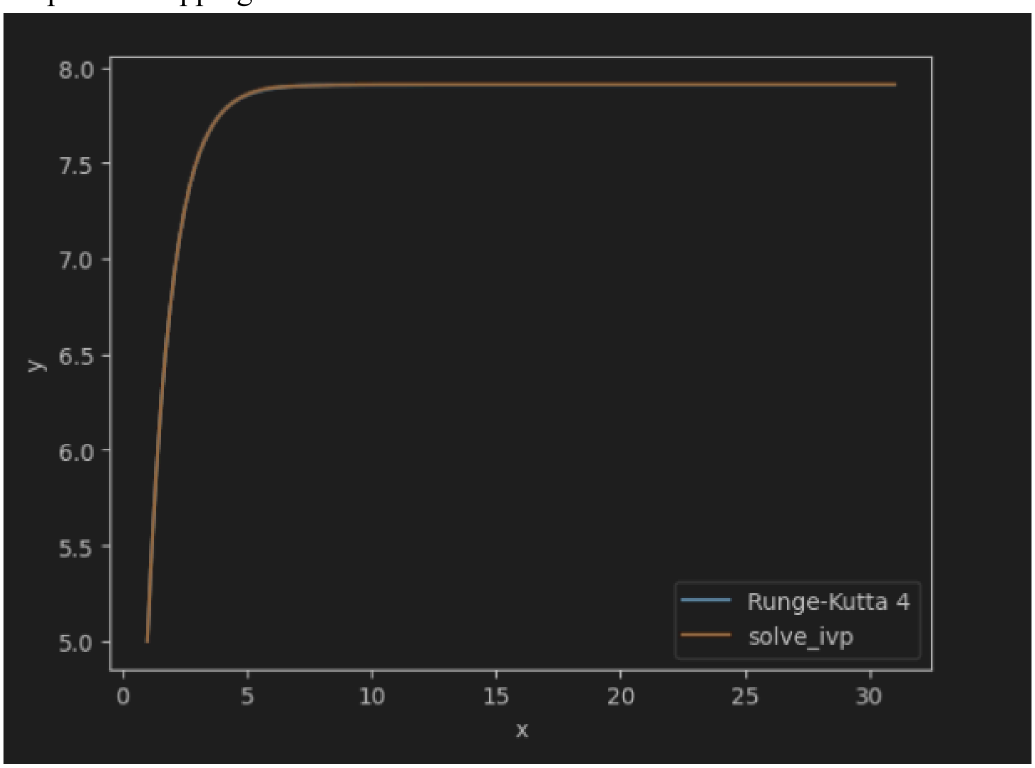
Library calculated graph:



Graphs overlapping each other:



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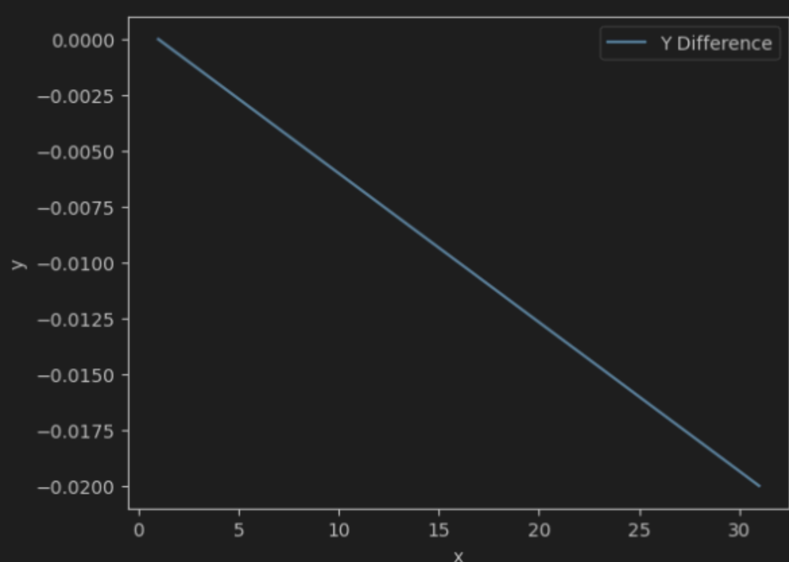
Printing difference in y values and graphing the difference:

```
Difference between y values:  -0.019759839893826125
Difference between y values:  -0.019773182121976873
Difference between y values:  -0.019786524350131174
Difference between y values:  -0.01979986657828192
Difference between y values:  -0.019813208806436222
Difference between y values:  -0.01982655103458697
Difference between y values:  -0.01983989326274127
Difference between y values:  -0.01985323549089202
Difference between y values:  -0.01986657771904632
Difference between y values:  -0.019879919947197067
Difference between y values:  -0.019893262175351367
Difference between y values:  -0.019906604403502115
Difference between y values:  -0.019919946631656416
Difference between y values:  -0.019933288859807163
Difference between y values:  -0.019946631087961464
Difference between y values:  -0.01995997331611221
Difference between y values:  -0.019973315544266512
Difference between y values:  -0.01998665777241726
Difference between y values:  -0.02000000000057156
```



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References for theory and code sources

The reference was the textbook for the class as it contained the RKF formula we used to build the program.

Objective: Use RKF to assess the power of a computing system

Description: This assignment has two parts: theoretical and practical. You will first solve a mathematical problem using the RKF method. Then, you will write a computer program that solves the same computational problem using RKF and measure the performance of your own computer for this task.

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Part 1: Your instructor will assign an ODE to solve, for example *y′ = 1 + y²*, using RKF. The instructor may choose to assign the same problem to the entire class or different problems to each student or team. Solve the ODE manually using the RKF method (e.g., RKF45), and show all the steps leading up to the solution. Estimate and explain the precision of the calculations using this method. Use formal mathematical rigor when writing and discussing your solution and intermediary steps. Then write a Python program that will implement the algorithm. Check the answers obtained manually with the ones obtained from the program.

The ODE is $f(x, y) = \frac{y}{e^x - 1}$, $x_0 = 1,\ y_0 = 5,\ h = 0.02$. Calculate manually for $\left(x_0, y_0\right)$, $\left(x_1,\ y_1\right)$, $\left(x_2,\ y_2\right)$, $\left(x_3,\ y_3\right)$, $\left(x_4, y_4\right) and\ (x_5, y_5)$. Populate the table below accordingly.

$$y_{n+1} = y_n + (h/6)(k_1 + 2k_2 + 3k_3 + k_4)$$

$$x_{n+1} = x_n + h$$

$$k_1 = f(x_n, y_n)$$

$$k_2 = f(x_n + (h/2), y_n + ((hk_1)/2)$$

$$k_3 = f(x_n + (h/2), y_n + ((hk_2)/2)$$

$$k_4 = f(x_n + h, y_n + (hk_3))$$

For x1 and y1:

$$k_1 = f(1, 5) = \frac{5}{e^1 - 1} = 2.909883534$$

$$k_2 = f(1 + 0.01, 5 + 0.01 * 2.909883534) = f(1.01, 5.029098835) = \frac{5.029098835}{e^{1.01} - 1}$$

$= 2.881012781$

$$k_3 = f(1 + 0.01, 5 + 0.01 * 2.881012781) = f(1.01, 5.028810128) = \frac{5.028810128}{e^{1.01} - 1}$$

= 2.88084739

$$k_4 = f(1 + 0.02, 5 + 0.02 * 2.88084739) = f(1.02, 5.057616948) = \frac{5.057616948}{e^{1.02} - 1}$$

= 2.85226251

$$x_1 = 1 + 0.02 = 1.02$$

$$y_1 = 5 + \frac{0.02}{6}(2.909883534 + 2(2.881012781) + 2(2.88084739) + 2.85226251)$$

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= 5.057619555

For x2 and y2:

$k_1 = f(1.02, 5.057619555) = \frac{5.057619555}{e^{1.02}-1} = 2.85226398$

$k_2 = f(1.02 + 0.01, 5.057619555 + 0.01 * 2.85226398) = f(1.03, 5.086142195) = \frac{5.086142195}{e^{1.03}-1}$
= 2.823962399

$k_3 = f(1.03, 5.057619555 + 0.01 * 2.823962399) = f(1.03, 5.085859179) = \frac{5.085859179}{e^{1.03}-1}$
= 2.823805261

$k_4 = f(1.02 + 0.02, 5.057619555 + 0.02 * 2.823805261) = f(1.04, 5.11409566) = \frac{5.11409566}{e^{1.04}-1}$
= 2.795783999

$x_2 = 1.02 + 0.02 = 1.04$

$y_2 = 5.057619555 + \frac{0.02}{6}(2.85226398 + 2(2.823962399) + 2(2.823805261) + 2.795783999)$

= 5.114098166

For x3 and y3:

$k_1 = f(1.04, 5.114098166) = \frac{5.114098166}{e^{1.04}-1} = 2.795785369$

$k_2 = f(1.04 + 0.01, 5.114098166 + 0.01 * 2.795785369) = f(1.05, 5.14205602) = \frac{5.14205602}{e^{1.05}-1}$
= 2.76804184

$k_3 = f(1.05, 5.114098166 + 0.01 * 2.76804184) = f(1.05, 5.141778584) = \frac{5.141778584}{e^{1.05}-1}$

= 2.767892493

$k_4 = f(1.04 + 0.02, 5.114098166 + 0.02 * 2.767892493) = f(1.06, 5.169456016) = \frac{5.169456016}{e^{1.06}-1}$

= 2.740423832

$x_3 = 1.04 + 0.02 = 1.06$

$y_3 = 5.114098166 + \frac{0.02}{6}(2.795785369 + 2(2.76804184) + 2(2.767892493) + 2.740423832)$

= 5.169458426

For x4 and y4:

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$k_1 = f(1.06, 5.169458426) = \frac{5.169458426}{e^{1.06}-1} = 2.74042511$

$k_2 = f(1.06 + 0.01, 5.169458426 + 0.01 * 2.74042511) = f(1.07, 5.196862677) = \frac{5.196862677}{e^{1.07}-1}$
$= 2.713228724$

$k_3 = f(1.07, 5.169458426 + 0.01 * 2.713228724) = f(1.07, 5.196590713) = \frac{5.196590713}{e^{1.07}-1}$

$= 2.713086735$

$k_4 = f(1.06 + 0.02, 5.169458426 + 0.02 * 2.713086735) = f(1.08, 5.223720161) = \frac{5.223720161}{e^{1.08}-1}$

$= 2.686159865$

$x_4 = 1.06 + 0.02 = 1.08$

$y_4 = 5.169458426 + \frac{0.02}{6}(2.74042511 + 2(2.713228724) + 2(2.713086735) + 2.686159865)$
$= 5.223722479$

For x5 and y5:

$k_1 = f(1.08, 5.223722479) = \frac{5.223722479}{e^{1.08}-1} = 2.686161057$

$k_2 = f(1.08 + 0.01, 5.223722479 + 0.01 * 2.686161057) = f(1.09, 5.25058409) = \frac{5.25058409}{e^{1.09}-1}$

$= 2.659501111$

$k_3 = f(1.09, 5.223722479 + 0.01 * 2.659501111) = f(1.09, 5.25031749) = \frac{5.25031749}{e^{1.09}-1}$
$= 2.659366074$

$k_4 = f(1.08 + 0.02, 5.223722479 + 0.02 * 2.659366074) = f(1.1, 5.2769098) = \frac{5.2769098}{e^{1.1}-1}$
$= 2.632970391$

$x_5 = 1.08 + 0.02 = 1.1$

$y_5 = 5.223722479 + \frac{0.02}{6}(2.686161057 + 2(2.659501111) + 2(2.659366074) + 2.632970391)$
$= 5.276912032$

Method: RUNGE-KUTTA METHOD			
Problem: $y' = \frac{y}{e^x-1}$; $x_0 = 1,\ y_0 = 5$			
n	$h = 0.02$		True Solution
	x_n	y_n	

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0	1	5	
1	1.02	5.057619555	
2	1.04	5.114098166	
3	1.06	5.169458426	
4	1.08	5.223722479	
5	1.1	5.276912032	

Part 2: Write a computer program, using appropriate mathematical software packages discussed in this class, which performs the calculations necessary to solve the ODE above. Estimate and explain the precision of complex computer calculations when using this method. Your program should calculate and display the number of computational steps performed, as well as the actual computing time. Test the capabilities of your own computer on two additional variations of the initial problem you were tasked to solve. (In class, you will compare the performance of your computer with those of your classmates). Explain how accuracy of results can be improved and the computation time tradeoff.

Your program should provide 1,000 to 2,000 solutions, that is, the program should stop when it reaches $\left(x_{1000}, y_{1000}\right) or \ (x_{2000}, y_{2000})$. Solve the original equation by using the traditional numerical programming and compare with your Runge-Kutta solution. Is there any error(s)?

Note: *The Lab Questions in this topic provide the opportunity to reflect and experiment with mathematical and programming implementation of concepts referred to in this assignment.*

Deliverables:

1. Documentation
    a.
    b. System performance context description
    c. Specific problem solved
    d. The mathematical approach for solving it
    e. The approach for implementation in code (e.g., algorithm, flowchart)
    f. Screenshots depicting key phases in the program execution
    g. References for theory and code sources
    h. README document written in Markdown detailing how to install and run the program
2. Code
    a. Full code submitted to GitHub
    b. Code executes correctly:
        i. Reads ODE as input
        ii. Solves the ODE

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  iii. Displays the solution
  iv. Solution is presented using terminology and units in the context of the problem

 c. Include a header comment, including name of programmers, code packages used, and approach to implementation
 d. Include comments in key areas of the code

3. Note: Refer to the instructor for additional clarifications for the requirements of this assignment.

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