Quantitative Finance

Case Study

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Black-Scholes SDE: Stock Price Valuation

Part 1

Let us denote the total portfolio value at time V_t. Likewise, it is given that at time 0 the investor has initial wealth V_0 and he invest a fixed fraction m a stock S and (1-m) in money market account denoted by B. This strategy is self-financing, meaning that no extra cash-inflows or cash-outflows take place. The investor does not remove any cash for consumption and he does not inject any cash for added investment. At time t+h we have:

$$V_{t+h} = \phi_{t+h} S_{t+h} + \psi_{t+h} B_{t+h} = \phi_t S_{t+h} + \psi_t B_{t+h} \tag{0.1}$$

At time t we have:

$$V_t = \phi_t S_t + \psi_t B_t \tag{0.2}$$

By subtracting (2) from (1) we get:

$$\begin{aligned} V_{t+h} - V_t &= \phi_t S_{t+h} + \psi_t B_{t+h} - \phi_t S_t - \psi_t B_t \\ &= \phi_t (S_{t+h} - S_t) + \psi_t (B_{t+h} - B_t) \end{aligned} \tag{0.3}$$

Therefore we observe that:

$$dV_t = \phi_t dS_t + \psi_t dB_t \tag{0.4}$$

Where we ϕ and ψ are the amount of portfolio holdings of stock and bond respectively. Using (4) and the fact that we invest a fraction m in stock S and (1-m) in B we observe:

$$
\begin{aligned}
&dV_t = \phi_t dS_t + \psi_t dB_t \\
&\phi_t S_t = mV_t \\
&\psi_t S_t = (1-m)V_t
\end{aligned}
\tag{0.5}
$$

We divide all parts of (4) to V_t and we use (5) to obtain:

$$
\begin{aligned}
\frac{dV_t}{V_t} &= \frac{\phi_t dS_t}{V_t} + \frac{\psi_t dB_t}{V_t} \\
V_t &= \frac{\phi_t S_t}{m} \\
V_t &= \frac{\psi_t S_t}{1-m} \\
\frac{dV_t}{V_t} &= \frac{\phi_t dS_t}{\frac{\phi_t S_t}{m}} + \frac{\psi_t dB_t}{\frac{\psi_t B_t}{1-m}} \\
\frac{dV_t}{V_t} &= \frac{m}{S_t} dS_t + \frac{1-m}{B_t} dB_t
\end{aligned}
\tag{0.6}
$$

Therefore we can say that the wealth of our investor follows following SDE:

$$
dV_t = \frac{mV_t}{S_t} dS_t + \frac{(1-m)V_t}{B_t} dB_t
$$

Which means that the relative returns of the portfolio (dV_t/V_t) is the weighted average of the relative returns of the stock(dS_t/S_t) and bond(dB_t/B_t). The relative change of V_t is the weighted average of the relative changes of S_t and the B_t. So,these strategy are constant proportional investment strategy and the portfolio is continuously re-balanced such that the investment proportions are kept constant. We assume that we are in Black-Scholes world with following setting:

$$
\begin{aligned}
dB_t &= rB_t dt \\
dS_t &= \mu S_t dt + \sigma S_t dW_t
\end{aligned}
\tag{0.7}
$$

Filling in the SDE's from (7) in the last equation of (6) we obtain:

$$
\begin{aligned}
&dV_t = (m\mu + (1-m)r)V_t dt + m\sigma V_t dW_t \\
&V_T = V_0 * exp((m\mu + (1-m)r - \frac{1}{2}m^2\sigma^2))T + m\sigma W_T)
\end{aligned}
\tag{0.8}
$$

Where W_T is Brownian Motion. From (8) we can conclude that V_T is lognormally distributed. For the mean and variance of V_T we get:

$$
\begin{aligned}
&E[V_T] = V_0 * e^{(m\mu+(1-m)r)*T} \\
&Var[V_T] = V_0^2 e^{2(m\mu+(1-m)r)T}\left(e^{m^2\sigma^2 T} - 1\right)
\end{aligned}
\tag{0.9}
$$

For the mean and variance of log(V_T) we first compute dlog(V_T) with Itô's lemma.

$$
\begin{aligned}
dlog(V_t) &= \frac{1}{V_t}dV_t - \frac{1}{2V_t^2}d[V,V]_t \\
&= \frac{1}{V_t} * \big((m\mu + (1-m)r)V_t dt + m\sigma V_t dW_t\big) - \frac{1}{2V_t^2} * m^2\sigma^2 V_t^2 dt \\
&= (m\mu + (1-m)r - \frac{1}{2}m^2\sigma^2\big)dt + m\sigma dW_t \\
log(V_T) &= log(V_0) + (m\mu + (1-m)r - \frac{1}{2}m^2\sigma^2)T + m\sigma W_T
\end{aligned}
\tag{0.10}
$$

Where we use the fact that d[V,V]$_t$= $m^2\sigma^2 V_t{}^2$dt.We can conclude that log(V_T) follows log-normal distribution:

$$
\begin{aligned}
E[log(V_T)] &= log(V_0) + \big(m\mu + (1-m)r - \frac{1}{2}m^2\sigma^2\big) * T \\
Var[log(V_T)] &= m^2\sigma^2 T
\end{aligned}
\tag{0.11}
$$

In order to find closed-forms for the derivatives of E[V_t] we use the previous part:

$$
\begin{aligned}
E[V_T] &= V_0 * e^{(m\mu + r - mr)*T} \\
\frac{dE(V_t)}{m} &= V_0 * e^{(m\mu + r - mr)*T} * (\mu - r)T \\
\frac{dE(V_t)}{r} &= V_0 * e^{(m\mu + r - mr)*T} * (1 - m)T \\
\frac{dE(V_t)}{\mu} &= V_0 * e^{(m\mu + r - mr)*T} * mT
\end{aligned}
\tag{0.12}
$$

Part 2

We assume that the stock and bond follows following SDE's:

$$
\begin{aligned}
dS_t &= \mu S_t dt + \sigma S_t^\beta dW_t^S \\
dB_t &= r_t B_t dt \\
dr_t &= a(b - r_t)dt + \sigma_r dW_t^r \\
dV_t &= (m\mu + (1-m)r_t)V_t dt + m\sigma V_t S^{\beta-1} dW_t^S
\end{aligned}
\tag{0.13}
$$

Where the instantaneous correlation between W^r and W^S is denoted by ρ.

This is the CEV process which is a particular version of the geometric Brownian motion. We have assumed that β is 0.9. If this would be 1, we would have the classical Black-Scholes SDE. The β is the elasticity parameter of the volatility σ.

Even if this process is assumed to be positive, the discretized version of it can reach negative values. There is a positive probability for the SDE of S to be absorbed in zero. This is a problem in the process of simulation, since we assume that the investor stops investing any fraction in stocks once it's value becomes negative and invests solely in money-market account, but in case we don't check and set the negative values to be equal to zero we will get wrong simulation for both stocks and bonds. Namely, we will get in the end wrong simulation results (not precisely right answer), because if one of the S's is negative in any of N time steps and we add this value to our next realization of S, this will be less value of S than it should actually be if we would set it equal to 0 which is pretty obvious since we know that the investor doesn't invest in negative return stocks.

Therefore, if in one of the possible scenarios crosses the line of 0, we have to stop the simulation and consider this path as being equal to 0.

We will do this by checking for each scenario whether S is positive or negative by simply setting it equal to (S + |S|)/2. Which means that we get S when it is positive and 0 when it is negative.

We observe that, the the stock price volatility increases as the stock price declines. Consequently the stock price volatility is an increasing function of the stock price.

Monte Carlo Simulation

For the Euler Scheme simulations of V we will use the SDE's from (17) with given parameters. Note that this is the Vasicek model under probability measure P.

$$\begin{aligned} S_{t+\Delta t} &= S_t + \mu S_t \Delta t - \sigma S_t^{\beta} \Delta W_t^S \\ r_{t+\Delta t} &= r_t + a(b - r_t)\Delta t - \sigma_r \Delta W_t^r \end{aligned} \tag{0.14}$$

Where W_t^S and W_t^r are random draws from bivariate normal distribution with correlation coefficient ρ. We set $\sigma_r = 0$ and $\beta = 1$ in parameter values we observe that:

$$\begin{aligned} dS_t &= \mu S_t dt + \sigma S_t dW_t^S \\ dr_t &= a(b - r_t)dt \\ dB_t &= r_t B_t dt \\ dV_t &= (m\mu + (1 - m)r_t)V_t dt + m\sigma V_t dW_t^S \end{aligned} \tag{0.15}$$

We see that this is the same as in part (b), the only difference is r and rt. We then first perform simulation with Euler scheme by setting beta to be equal to 1 and σ_r to 0. Then, using the closed form of V_t from expression (8).

We should get two comparable results. We get:

We will first simulate V_t with Euler scheme based on (13) ,then we will use as exact value for $E[V_t]$

```
>> Assignment2
Closed form V(b):1252.5289 +/- 0.82596
MC with adjusted parameters:1252.5485 +/- 8.499
>>
```

the closed form expression derived in (b):(9). For the exact value of $E[V_t]$ we will use constant r equal to $r_0 = 0.02$.

Exact model:

$$E[V_T] = V_0 * e^{(m\mu + (1-m)r)*T}$$

Exact simulation model:

$$V_T = V_0 * exp((m\mu + (1 - m)r - \frac{1}{2}m^2\sigma^2))T + m\sigma W_T)$$

Direct MC simulation:

$$dV_t = \frac{mV_t}{S_t}dS_t + \frac{(1-m)V_t}{B_t}dB_t$$

We call the MC estimate V, the estimate from the exact simulation we call Vex and the exact value from (b). We then using the control variate estimate with following way:

$$V^{cv} = V - \alpha(V_{ex} - V_{exact})$$

Where we firstly calculate the Control Variate estimate with alpha equals to 1.

$$\frac{std(V)}{std(V^{cv})} = 10.3099 \quad \frac{var(V)}{var(V^{cv})} = 106.2946$$

The length of the confidence interval is approximately reduced by a factor 10.31. This corresponds to approximately 106.3 times more scenarios in the plain MC model.
Notice that above calculations have been done using $\alpha = 1$. In order to be sure that this is not a wrong assumption we compute the optimal value of α: α^*.

$$\alpha^* = Corr(V, V_{ex})\frac{std(V)}{std(V_{ex})} = 1.0473$$

So we could say that α^* is not significantly different from 1, so that the assumption that we have used for our calculations is just fine. But in order to see the difference between two results with α and α^* we calculate the estimate of V^{cv*} and corresponding ratio of standard deviations of V and V^{cv*} and we observe that in this case the length of the confidence is approximately reduced by a factor 11.6406 which is more effective than in earlier case.

We obtained following results in Matlab:

```
Part (f):
Plain MC estimation:1255.1652 +/- 5.5289
MC estimate with CV:1254.9535 +/- 0.53627
Ratio of standard deviations MC and CV:10.3099
Ratio of variances MC and CV:106.2946
Alpha_star:1.0473
MC estimate with CV_star:1254.9434 +/- 0.47496
Ratio of standard deviations MC and CV_star:11.6406
fx >>
```

To approximate sensitivities of $E[V_t]$ with respect to b, μ and m we will apply one-sided Bump-and-Reprice method.

Sensitivity with respect to b

We need to find:

$$\frac{dE[V_t]}{b}$$

As we did it in part (f), we denote the MC estimate with V. Then the one-sided finite-difference estimate of the derivative is:

$$\frac{V(b+h)-V(b)}{h} \tag{0.16}$$

Where the h is small number,so we assume it is equal to 0.01.

Notice that for simulations we will use common random numbers like we did it earlier.

So, for the calculation for b-sensitivity we create extra SDE's which contain b or respectively r_t and we use there as parameter b+h.We simulate the model and compute in the end the mean of the vector (V-Vh)/h.

Sensitivity with respect to μ

We need to find:

$$\frac{dE[V_t]}{\mu}$$

$$\frac{V(\mu+h)-V(\mu)}{h} \tag{0.17}$$

Sensitivity with respect to m

We need to find:

$$\frac{dE[V_t]}{m}$$

$$\frac{V(m+h)-V(m)}{h} \tag{0.18}$$

We obtained following results in Matlab: In order to analyse the probability that V_t lies below V_0

```
>> ass
Part (g):
bump&repr b: 1348.5996 +/- 5.7303
>> sensmu
bump&repr mu: 3165.8354 +/- 12.2974
>> msens
bump&repr m: 322.9592 +/- 11.6726
>>
```

we first calculate dlog(V_t) , by applying Itô's lemma.

$$dV_t = (m\mu + (1-m)r_t)V_t dt + m\sigma V_t S^{\beta-1} dW_t^S$$

$$dlog(V_t) = \frac{1}{V_t}dV_t - \frac{1}{V_t^2}d[V,V]_t$$

$$= \frac{1}{V_t} * \left((m\mu + (1-m)r_t)V_t dt + m\sigma V_t S^{\beta-1} dW_t^S\right) - \frac{1}{V_t^2} * \left(m^2\sigma^2 V_t^2 S_t^{2(\beta-1)} dt\right)$$

$$= \left((m\mu + (1-m)r_t - \frac{1}{2}m^2\sigma^2 S_t^{2(\beta-1)}\right)dt + m\sigma S_t^{\beta-1} dW_t^S$$

Therefore, we observe that:

$$\begin{aligned} &log(V_T) = log(V_0) + \big((m\mu + (1-m)r_T - \frac{1}{2}m^2\sigma^2 S_t^{2(\beta-1)}\big)T + m\sigma S_T^{\beta-1}W_T^S \\ &V_T = V_0 * exp\big[\big((m\mu + (1-m)r_T - \frac{1}{2}m^2\sigma^2 S_t^{2(\beta-1)}\big)T + m\sigma S_T^{\beta-1}W_T^S\big] \end{aligned} \tag{0.19}$$

V$_T$ is V$_0$ is the same as stating that exponent part of e-power becomes smaller than 1.

$$\begin{aligned} &V_T < V_0 \\ &exp\big[\big((m\mu + (1-m)r_T - \frac{1}{2}m^2\sigma^2 S_t^{2(\beta-1)}\big)T + m\sigma S_T^{\beta-1}W_T^S\big] < 1 \\ &\big((m\mu + (1-m)r_T - \frac{1}{2}m^2\sigma^2 S_t^{2(\beta-1)}\big)T + m\sigma S_T^{\beta-1}W_T^S < 0 \\ &\frac{-W_T^S}{\sqrt{T}} < \frac{((m\mu + (1-m)r_T - \frac{1}{2}m^2\sigma^2 S_t^{2(\beta-1)})T}{m\sigma S_T^{\beta-1}\sqrt{T}} \end{aligned} \tag{0.20}$$

The left hand side is standard normally distributed,therefore we get:

$$Pr[V_T < V_0] = \Phi\Big(\frac{((m\mu + (1-m)r_T - \frac{1}{2}m^2\sigma^2 S_t^{2(\beta-1)})T}{m\sigma S_T^{\beta-1}\sqrt{T}}\Big) \tag{0.21}$$

Where Φ is the cdf of standard normal distribution.

(i) Certainty Equivalent analysis

The certainty equivalent of an investor with utility function u is given by:

$$CE = u^{-1}(E[u(V_T)]) \tag{0.22}$$

We will consider CRRA (Constant Relative Risk Aversion) utility function:

$$\begin{aligned} u(x) &= \frac{x^{1-\gamma}}{1-\gamma} \qquad \gamma > 1 \\ u^{-1}(y) &= [y(1-\gamma)]^{1/(1-\gamma)} \end{aligned} \tag{0.23}$$

The *certainty equivalent* is a guaranteed return that the investor would accept rather than taking a chance on a higher, but uncertain, return. So,it is the guaranteed amount of money that would yield the same exact expected utility as a given risky asset with absolute certainty. Put it differently, CE is the maximal certain amount of money an investor is willing to pay for a uncertain stock.

$$\begin{aligned} CE &= \Big(E\big[\frac{V_T^{1-\gamma}}{1-\gamma}\big] * (1-\gamma)\Big)^{1/(1-\gamma)} = \Big(E\big[V_T^{1-\gamma}\big]\Big)^{1/(1-\gamma)} \\ &= \Big(E\Big[V_0^{(1-\gamma)} * exp\big[(1-\gamma) * \Big(\big((m\mu + (1-m)r_T - \frac{1}{2}m^2\sigma^2 S_t^{2(\beta-1)}\big)T + m\sigma S_T^{\beta-1} W_T^S\Big)\big]\Big]\Big)^{1/(1-\gamma)} \end{aligned} \tag{0.24}$$

According to (24) higher γ lowers the CE and large m increases the CE.
So,for large values for gamma's certainty equivalent decreases,keeping everything else constant. The opposite relation holds between m and CE. When m decreases CE decreases as well, keeping everything constant. Notice, that this relation excludes the $\gamma = 1$ case.

$$\begin{aligned} u(V_T) &= \frac{V_0^{(1-\gamma)} * exp\big[(1-\gamma)\big(\big((m\mu + (1-m)r_T - \frac{1}{2}m^2\sigma^2 S_t^{2(\beta-1)}\big)T + m\sigma S_T^{\beta-1} W_T^S\big)\big]}{1-\gamma} \\ u(V_T(m=0)) &= \frac{V_0^{(1-\gamma)} * exp\big[(1-\gamma)r_T * T\big]}{1-\gamma} \\ CE &= \Big(E\Big[V_0^{(1-\gamma)} * exp\big[(1-\gamma)r_T * T\big]\Big)^{1/(1-\gamma)} \end{aligned} \tag{0.25}$$

We see that for constant and large value of γ,which obviously indicates that the investor is risk averse the utility is not maximized when m is zero.
When m is larger than zero, the utility can take larger values, which is obvious from the comparison of first and second equations of (25): the second component of exponential power might be larger than 0 for corresponding positive stock returns, leading to higher exponent value(higher utility value).
From the intuitive point of view, m being equal to zero indicates that the investor invests solely in the money-market account which doesn't always provide the maximized utility for the risk-averse investor, since it doesn't minimize the risk but the opposite.
Higher fraction of investment in the risk-free bond lead to higher sensitiveness for the investor to the selection of re-balancing interval.

So, when the fraction of the investment in the risky asset decreases, the volatility of the implied risk aversion coefficient increases ,leading to more risk for the investor.
Therefore claiming that "for a risk-averse agent the utility - maximizing strategy is m = 0 because this strategy minimizes risk" is not correct.

Expectations

We would expect that high risk aversion coefficient, which indicates that the investor is more risk-averse, will lower the certainty equivalent, since the maximal certain amount of money that this investor is willing to pay for an uncertain stock is lower than the one of less risk averse person.
The opposite relation we would expect for m.Large m indicates that the investor invests large fraction in the risky stock which means that the maximum amount that he is willing to pay for an uncertain stock is high.
Following graph describes the relation between CE and gamma.

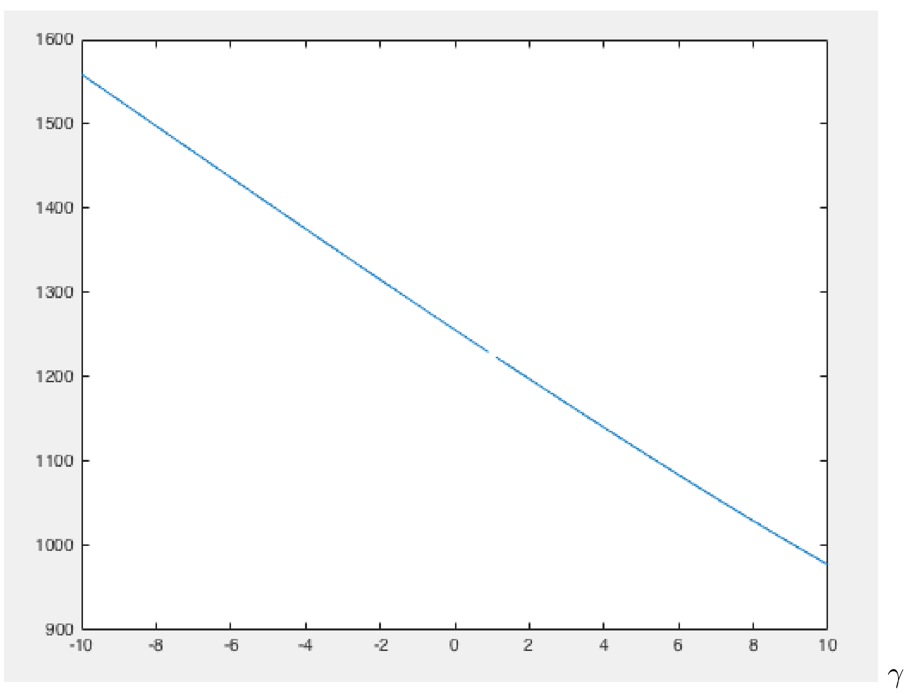

Figure 1: CE as a function of γ

From the figure 1 we see that as expected, the CE is decreasing as the risk-aversion coefficient increases.