CLAYTON COLLEGE & STATE UNIVERSITY

PHYS 2211L - Principles of Physics Laboratory I
Propagation of Errors Supplement

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1. **Introduction.** Whenever two or more quantities are measured directly in order to indirectly determine the value of another, the question of propagation of errors must be addressed. This supplement provides a general method for determining how to propagate errors in all cases in which a mathematical expression for the relationship between directly measured and derived quantities can be written.

2. **Development of the general formula for error propagation.** A mathematical technique necessary for the development of error propagation is that of partial differentiation. Consider a general function of two variables:

$$q = q(x, y)$$

An expression for the total differential of this function is:

$$dq = \frac{\partial q}{\partial x} dx + \frac{\partial q}{\partial y} dy \qquad \textbf{(1)}$$

where the partial derivatives are defined as:

$$\frac{\partial q(x,y)}{\partial x} = \frac{dq(x,y)}{dx}\Big|_{y=constant}$$

and

$$\frac{\partial q(x,y)}{\partial y} = \frac{dq(x,y)}{dy}\Big|_{x=constant}$$

If the measurement error in the x variable, δx, and the measurement error in the y variable, δy, are small,

$$\delta x << x$$

and

$$\delta y << y$$

then equation 1 is approximately

$$\delta q = \frac{\partial q(x,y)}{\partial x}\delta x + \frac{\partial q(x,y)}{\partial y}\delta y \quad \textbf{(2)}$$

where δq is the total experimental error. It can be shown (Taylor, p. 121) that if the measurement errors, δx and δy, are independent, random errors with a normal (gaussian) distribution, then the proper method for combining them is not simple summation as is given by equation 2, but by the method of equation 3 below. Since each error term in equation 2 is as likely to be too high or too low (positive or negative), simple summation of the terms would overestimate the actual total error. The proper form is

$$\delta q = \sqrt{\left[\frac{\partial q(x,y)}{\partial x}\delta x\right]^2 + \left[\frac{\partial q(x,y)}{\partial y}\delta y\right]^2} \quad \textbf{(3)}$$

3. **Formulas for addition and subtraction, and multiplication and division.** Frequently, textbooks provide formulae for special cases of equation 3 for error propagation involving the fundamental mathematical operations of addition, subtraction, multiplication and division. This paragraph derives these special formulae from equation 3.

a. **Addition and subtraction.** Consider the determination of the value of a quantity, q, which requires the sum of two direct measurements. An example of such a case is measuring a length which is longer than the measuring instrument so that two separate measurements are required to determine the entire length. For such a case,

$$q = q(x, y) = x + y$$

and

$$\frac{\partial q}{\partial x} = 1 \qquad \frac{\partial q}{\partial y} = 1$$

Substituting into equation 3,

$$\delta q = \sqrt{[(1)\delta x]^2 + [(1)\delta y]^2}$$

or

$$\delta q = \sqrt{(\delta x)^2 + (\delta y)^2}$$

It is easy to show that this formula also applies to subtraction.

b. **Multiplication and division.** Consider the determination of the value of a quantity which requires the multiplication of two directly measured quantities. An example of such a case is the determination of the area of a rectangle from measurements of its length and width. For such a case,

$$q = f(x, y) = xy$$

and

$$\frac{\partial q}{\partial x} = y \qquad \frac{\partial q}{\partial y} = x$$

Substituting into equation 3 after it has been squared,

$$(\delta q)^2 = [y(\delta x)]^2 + [x(\delta y)]^2$$

Dividing this result by q^2, gives

$$\frac{(\delta q)^2}{q^2} = \frac{[y(\delta x)]^2}{(xy)^2} + \frac{[x(\delta y)]^2}{(xy)^2}$$

Simplifying and taking the square root of each side of this equation yields

$$\frac{\delta q}{q} = \sqrt{\left(\frac{\delta x}{x}\right)^2 + \left(\frac{\delta y}{y}\right)^2}$$

This same formula can be shown to hold for division.

c. **General applicability.** Hopefully, you are now convinced of the general applicability of equation 3 for functions involving two variables. The extension to more than two variables is simply a matter of adding additional terms of the same form as those contained within the square root of equation 3.

4. **Example.** Find the volume of a right circular cylinder which has a measured diameter of 10.0 ± 0.2 cm and a measured height of 15.0 ± 0.1 cm. The formula for the volume of a cylinder is

$$V = \pi r^2 h$$

where r is the radius and h is the height of the cylinder. Note that the radius was not measured. In order to propagate errors, the variables involved in the calculations <u>must</u> be those that were actually measured. Therefore,

$$V = \pi \left(\frac{d}{2} \right)^2 h = \frac{\pi}{4} d^2 h$$

Taking the partial derivatives with respect to d and h:

$$\frac{\partial V}{\partial d} = \frac{\pi}{2} d\, h$$

and

$$\frac{\partial V}{\partial h} = \frac{\pi}{4} d^2$$

Then

$$\frac{\partial V}{\partial d}\, \delta d = \left(\frac{\pi}{2} \right) (10.0\, cm)(15.0\, cm)(0.2\, cm) = 47.1238898\, cm^3$$

and

$$\frac{\partial V}{\partial h}\, \delta h = \left(\frac{\pi}{4} \right) (10.0\, cm)^2 (0.1\, cm) = 7.853981634\, cm^3$$

Then

$$\delta V = \sqrt{(47.1238898\, cm^3)^2 + (7.853981634\, cm^3)^2} = 47.77390519\, cm^3$$

$$V = \frac{\pi}{4} d^2 h = \frac{\pi}{4} (10.0\, cm)^2 (15.0\, cm) = 1178.097245\, cm^3$$

5. **<u>Significant figures rules associated with propagation of errors</u>.** Clearly the number of significant figures in the results for V and dV in the example problem are inappropriate. The following rules should be used to determine the reported results:

> **a. Do not apply any rounding or significant figures rules until you have determined the values of the quantity determined from your measurements and its uncertainty.**

b. The number of significant figures in any reported uncertainty value is one, unless the first significant digit is 1. In this case, the number of significant figures in the uncertainty reported is two.

c. Round off the value reported to be the same order of magnitude (same decimal place) as that in the value of the uncertainty's significant figure.

Examples:

before applying significant figures rules	after applying significant figures rules
(23.65789 ± 0.23576) cm	(23.7 ± 0.2) cm
(23.65789 ± 0.00532) cm	(23.658 ± 0.005) cm
(23.65789 ± 0.13579) cm	(23.66 ± 0.14) cm
(23.65789 ± 2.37859) cm	(24 ± 2) cm

d. If calculating a fractional error, use the values of the measured quantity and its uncertainty prior to applying the significant figures rules described above. Report three significant figures for the fractional error.

Applying these rules to the example problem of paragraph 4:

$$V = 1180 \pm 50\, cm^3$$

$$FE = \frac{47.77390519\, cm^3}{1178.097245\, cm^3} = 0.0406$$

6. **Random and systematic error.** Errors in measurement can usually be classified as either random or systematic. All discussion thus far in this supplement has concerned random error and its propagation.

a. **Random error.** A random error will cause repeated measurements of the same quantity to scatter on both sides (high and low) of the best estimate for the quantity. Random error affects the precision of your results. Random error may be reduced, but never eliminated. Examples of its causes include limits on the precision of measuring instruments, fluctuations in experimental conditions and random processes in nature. Random error is quantified by calculation of the percent fractional error in a result:

$$\%\ FE = \frac{\delta x}{\bar{x}}\ x\ 100$$

b. **Systematic error.** A systematic error will affect all measurements of a quantity in the same way; all will be either too high or too low when compared to the true or accepted value of the quantity. Systematic error affects the accuracy of your results. Theoretically, sources of systematic error can be identified and eliminated. Systematic error is quantified by calculation of the percent discrepancy in a result:

$$\%\ Disc = \left| \frac{x_{best} - x_{accepted}}{x_{accepted}} \right| x\ 100\%$$

Clearly, systematic error cannot be quantified without some knowledge of the true or accepted value for the quantity being measured.

c. **Analyzing error.** In cases where both random and systematic error can be quantified, valuable information about improving measurements can be obtained. If the percent fractional error exceeds the percent discrepancy, random error is dominant. If the percent discrepancy exceeds the percent fractional error, systematic error is dominant. Improvements to the experiment should first concentrate on the dominant form of error. If systematic error is dominant, a careful examination of the underlying assumptions or simplifications of the theoretical model for the measurement should be made. If this does not prove to be the source of systematic error, the measuring instruments should be examined for accuracy. If random error dominates, the magnitudes of the individual terms within the square root of equation 3 should be examined to determine which is the largest contributor to the uncertainty in the measured quantity.

Reference:

Taylor, John R. An Introduction to Error Analysis. Mill Valley, CA: University Science Books, 1982.

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