B. Tech Project Report

Rendezvous of two Dubins cars

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1 Abstract

2 Introduction

Motion planning is one of the fundamental problem in robotics. Broadly it is a problem of selecting a set of actions which, on execution will take a bot from an initial state to a given final state. Rendezvous search problem was first introduced by Alpern in [**?**]. Here two identical cars attempt to meet each other by moving with speed bounded by a given maximum until the first meeting time T. The paths are chosen such that T is minimized. We study the problem for cars with bounded curvature which can move with positive velocity only. Car with these constraints was first studied by L. E. Dubins [1]. Reachability sets have been used to ... Focus is on situations when the vehicles are located at a distance in the range of minimum radius of turning.

3 Dubins Vehicle

Dubins car can move only in forward direction with a maximum speed up to v_s. The curvature of the path at any point, $u(t)$, is not greater than $u_{max} > 0$.For motion in a plane, $1/curvature$ is the turning radius. So, in the Dubins problem, the turning radius is constrained to be at least $1/u_{max}$. Let the position of vehicle in plane at time t be given by $z(t) = (x(t), y(t)) \in \mathbb{R}^2$ and the orientation with respect to the x-axis be $\theta(t) \in [-\pi, \pi]$. Then motion for such a vehicle is governed by following dynamics:

$$\begin{aligned} \dot{x}(t) &= v\sin(\theta(t)), \\ \dot{y}(t) &= v\cos(\theta(t)), \\ \dot{\theta}(t) &= u(t)v(t) \end{aligned} \tag{1}$$

Control variable, $u(t)$ is chosen from the interval $[-u_{max}, u_{max}]$. Speed of the vehicle $v(t)$ is chosen from the interval $[0, v_s]$. Initial coordinates for both vehicles be given as p_1^i and p_2^i and orientation as θ_1^i and θ_2^i. Let the first meeting time for both vehicles be T. Then rendezvous problem can be given by

$$\begin{aligned} &min\, T \\ &s.t.\; z_1(0) = p_1^i, z_2(0) = p_2^i, \theta 1(0) = \theta_1^i, \theta_2^i(0) = \theta_2^i, \\ &\quad\;\; z_1(T) = z_2(T) \end{aligned} \tag{2}$$

When curvature, $u(t)$ is zero, vehicle moves in straight line, denoted by S. When curvature is maximum, i.e. $|u(t)| = u_{max}$, vehicle moves along a circle of minimum radius, R. This circular path, denoted by C, can be of two types depending on sign of $u(t)$, L and R. Let C_L denote the circle of minimum radius on left side of initial orientation and C_R denote the circle on right.
It is proved in [1] that shortest path from an initial position and orientation to a final position and orientation for such a vehicle can be expressed as a combination of at most

three parts, which can be arcs of C_L or C_R or straight line (S). This leads to six possible cases: $\{RSR, LSL, RSL, LSR, RLR, LRL\}$. Further, if the constraint on final orientation is relaxed, the shortest paths are changed. [2] proved that in such a case they are reduced to type CS or CC. This generates four possible trajectories: $\{RS, LS, RL, LR\}$. For the rendezvous problem the final orientation constraint is relaxed.

4 Reachability Sets

Reachability set, R_T, is defined as the set of points a vehicle can reach in time less than or equal to T. Boundary of reachability set can be used to identify the time optimal trajectories. For time much greater than r_0/v_s, the reachability set can be approximated as a circle for Dubins vehicle. But for smaller time intervals it's shape is dependent on the orientation of the vehicle and amount of time. The sets of CS and CC are plotted separately in figure 1 and figure 2.

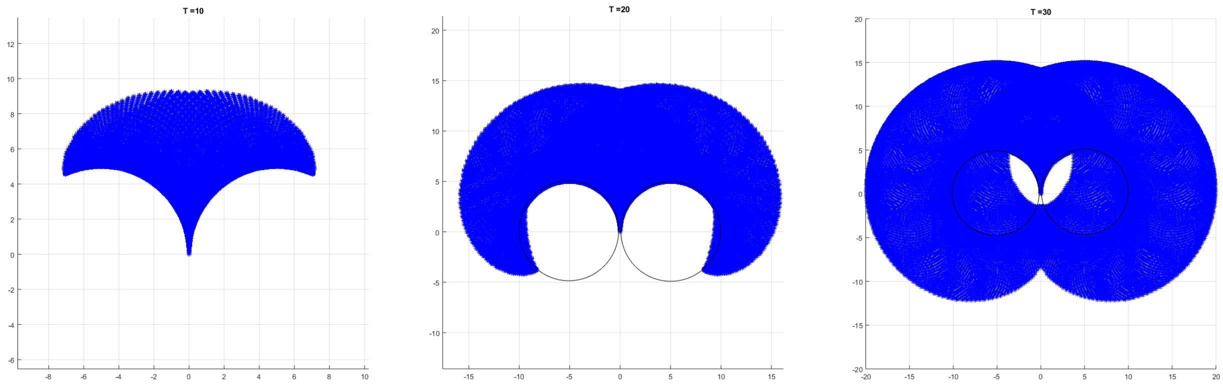

Figure 1: Reachability set for CC trajectories

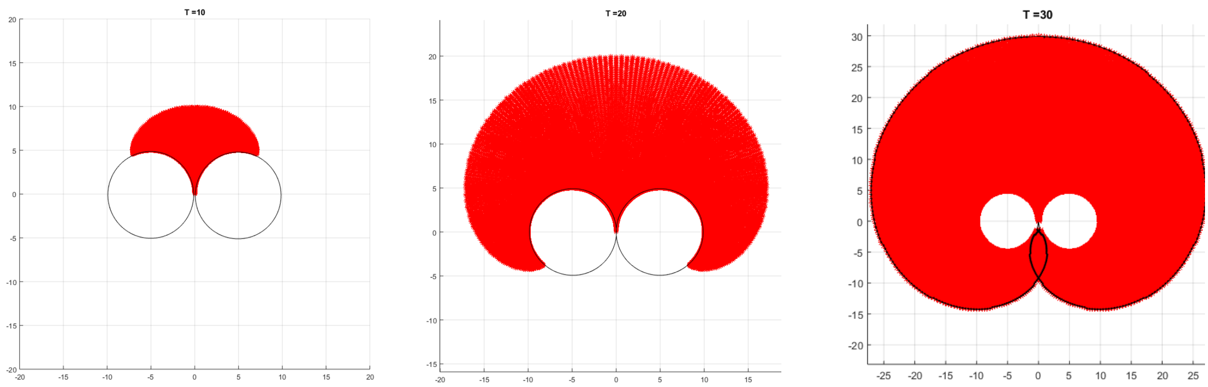

Figure 2: Reachability set for CS trajectories

4.1 Single agent case

The car was initially programmed to reach the origin, starting from any initial position (x_0, y_0, θ_0), without constraints on angle of arrival. The initial position was taken such that its distance from origin is sufficiently larger than minimum radius of turning. In this case, according to [2], the shortest path will be a CS trajectory. This can be described as a combination of an arc and a tangent to one of the circles of minimum radius. This is was implemented using the feedback strategy:

Algorithm 1 Reach (0,0)

```
while √(x² + y²) ≥ ϵ do
  if θ − tan⁻¹(y/x) ≤ ρ then
    u = 0
  else if θ − tan⁻¹(y/x) ≤ 0 then
    u = −u_max
  else
    u = u_max
  end if
  update θ, x and y
end while
```

But this feedback law fails in the situation when final position lie within C_R or C_L. Thus to look at the nearby points we need to modify our solution.

4.2 Characterization of 2-D plane

The strategy for reaching any point in the 2-D plane can be described as follows. Any point outsideC_R and C_L can be reached time optimally by a CS trajectory. Whereas, any point within them can be reached optimally by a CC trajectory. This can be proven as follows:

I Take a point outside the circles. CS and CC trajectory are common till point A. Beyond that the shortest path from A to P is a straight line.

II Take a point inside C_R. Draw a circle of minimum radius passing through B and tangent to C_L. This intersects C_L at P. Thus, CS and CC trajectory are common till point P (Figure 3). To prove that CC is optimal we need to show

$$\begin{aligned} R(2\pi - \phi) + l &\geq r(2\pi - \theta)) \\ \text{or } l &\geq R(\phi - \theta) = R\alpha \\ \text{Since } \phi \geq \theta, \text{we have } l &\geq R\,tan\alpha \\ \text{As } \alpha \in [0, \pi/2], R\,tan\alpha &\geq R\alpha \end{aligned} \tag{3}$$

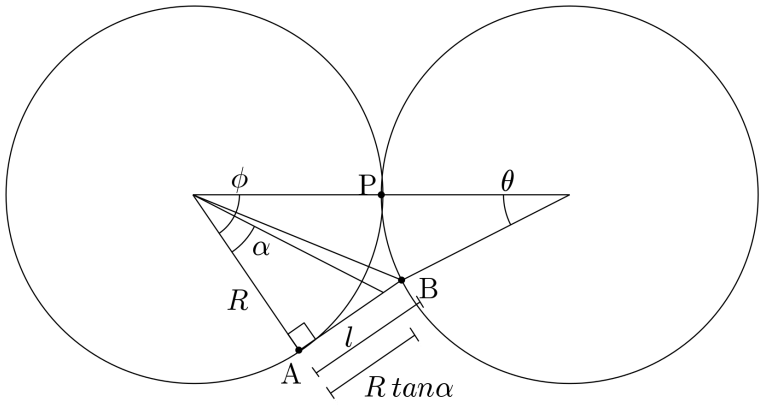

Figure 3: Proof of CC case

5 Two vehicle case

Following a similar approach as for one vehicle case, the rendezvous problem of two vehicles can be looked at by dividing the plane in different sections based on their initial positions and searching for time optimal trajectories of both the vehicles. Let d be the distance between the two vehicles. A rectangular coordinate system is chosen such that its origin is at position of vehicle A, $P_A = (0, 0)$ and the positive direction of x-axis is towards vehicle B, $P_B = (d, 0)$. Let α and β be the angles their initial orientations make with the positive x-axis when measured counter-clockwise.

5.1 Equivalency Groups

The classification of plane can be proceeded by first dividing the range of possible orientations α and β in four quadrants. Quadrant 1 corresponds to the range $[0, \pi/2]$, quadrant 2 to $[\pi/2, \pi]$, quadrant 3 to $[\pi/3, \pi/2]$, and quadrant 4 to the range $[3\pi/2,\, 2\pi]$. Since α and β

can be in any of the four quadrants, there are 16 different possible combinations of quadrants. We represent those 16 by a 4×4 matrix, $\{a_{ij}\}$, where index i corresponds to the quadrant number of vehicle A, and index j of vehicle B.

Element a_{ij} therefore describes the class of all paths whose initial and final orientation angles (α, β) belong to the quadrants i and j , respectively. For example, the case $\alpha \in [0, \pi/2]$, $\beta \in [\pi/2, \pi]$ corresponds to the element a_{12} and covers all those paths whose orientation angles belong to the first and second quadrant, respectively.

It will be shown below that these 16 classes can be reduced to six independent clusters, called equivalency groups. We consider two trajectories to be equivalent if one can be converted into another by taking a reflection along x or y axis.

Proposition: Let $T_{\alpha,\beta}$ be the optimum trajectories for vehicle A and B together, then following trajectories are equivalent:

$T_{\alpha,\beta} \simeq T_{\pi-\beta,\pi-\alpha}$

$T_{\alpha,\beta} \simeq T_{-\alpha,-\beta}$.
This leads to following equivalency groups:

$$\begin{array}{ll} a_{11} \simeq a_{22} \simeq a_{33} \simeq a_{44} & a_{14} \simeq a_{23} \simeq a_{32} \simeq a_{41} \\ a_{12} \simeq a_{43} & a_{21} \simeq a_{34} \\ a_{13} \simeq a_{42} & a_{24} \simeq a_{31} \end{array}$$

Each of these six equivalent groups are discussed by varying the distance between the vehicles.
There are three possible scenarios for rendezvous. One is both vehicles move for the same duration till they meet at a point. Another is when one vehicle moves for a longer duration than another vehicle. We call this trajectory to be degenerate in second vehicle. Last possibility is for one vehicle to reach the initial position of the other directly. The first two cases are denoted by T[CS CS] and third one is denoted by T[CS 0] or T[0 CS].

5.2 a_{11}

Let d_1 be the minimum distance at which direct path to initial position of B by vehicle A is optimal. Similarly let d_2 be the minimum distance at which T[LS LS] is the optimal trajectory and d_3 be the minimum distance at which T[RS LS] is the optimal trajectory. If the distance between the two vehicles, d, is less than d_1 then the path followed by A and B is RS and LS respectively.
Now if d_2 is less than d_3, following sequence of optimal trajectories is followed:

- T[RS 0] for $d_1 \leq d \leq d_2$
- T[LS LS] for $d_2 \leq d \leq d_3$
- T[RS LS] for $d_3 \leq d \leq d_4$

Else if $d_3 \geq d_4$, then sequence is:

- T[RS 0] for $d_1 \leq d \leq d_3$
- T[RS LS] for $d_3 \leq d \leq d_4$

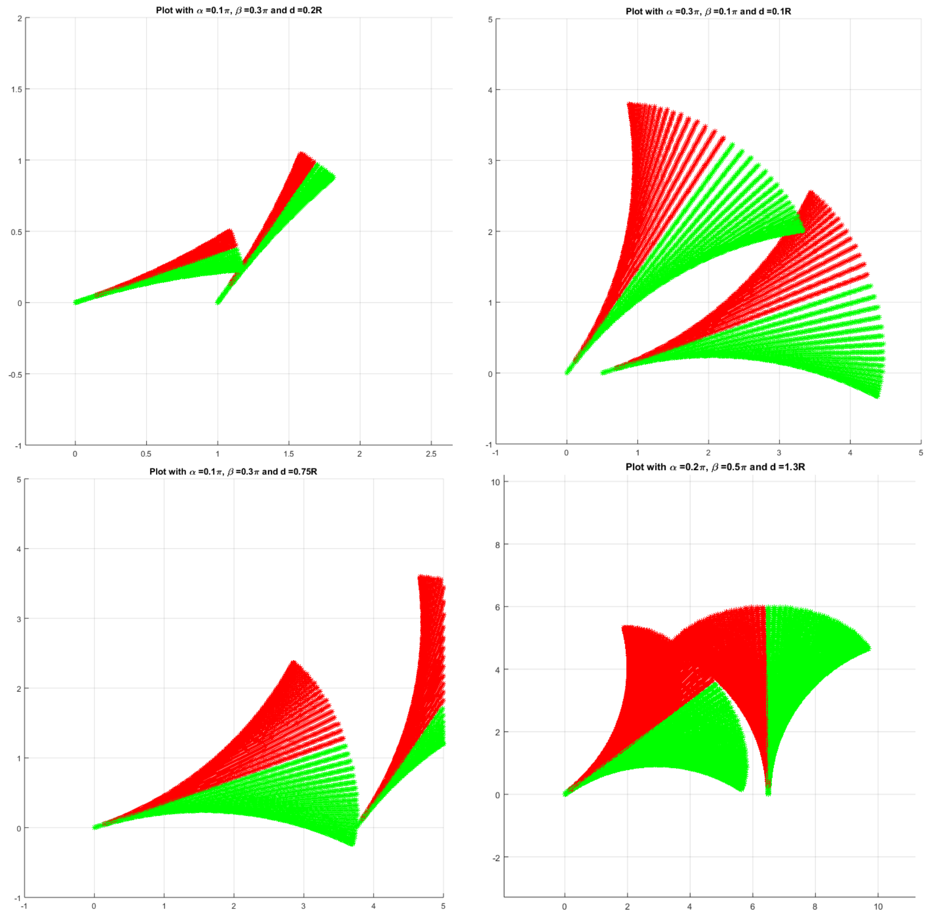

Figure 4: Trajectories for a_{11}

For all values of d, T[RS,LS] $\leq$ T[RS,RS] as any point on the right of line passing through B and at angle β is reached by A later than the corresponding point on the line. Similarly we can prove T[RS,LS] $\leq$ T[LS,RS].

5.3 a_{12}

Let d_1 be the distance at which straight line path by one vehicle and CS by other is optimal. Then we can write the sequence of optimal trajectories as:

- For $0 < d \leq d_1$
  - For $\alpha \leq \pi - \beta$ T[LS LS]
  - For $\alpha \geq \pi - \beta$ T[RS RS]
- For $d_1 \leq d$ T[RS LS]

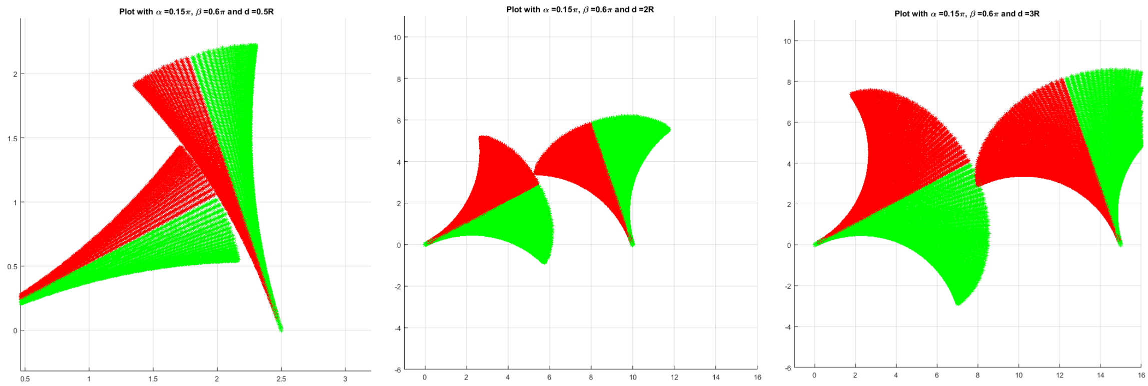

Figure 5: A12

5.4 a_{13}

- If both are inside each other's circles of minimum radius:
  - For $\alpha \leq -\pi + \beta$ T[LS RS]
  - For $\alpha \geq -\pi + \beta$ T[RS LS]
- If both are outside each other's circles:
  - T[RS RS]
- If one is inside and other is outside:
  - For B inside A's circle: T[RS RS] or T[0 RS]
  - For A inside B's circle: T[LS L] or T[LS 0]

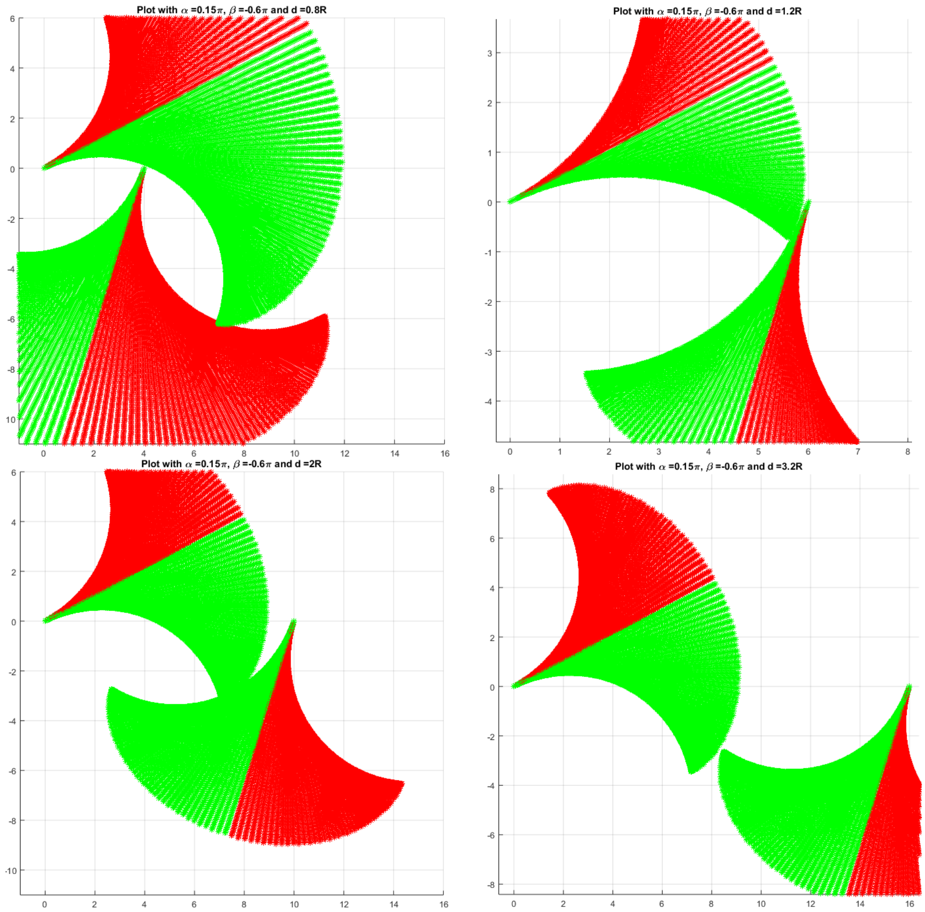

Figure 6: A13

5.5 a_{14}

- If B lies in the circle of A:
  - T[RS L]
- If B lies outside the circle of A, in increasing order of distance:
  - T[RS 0]
  - T[RS R]
  - T[RS RS]

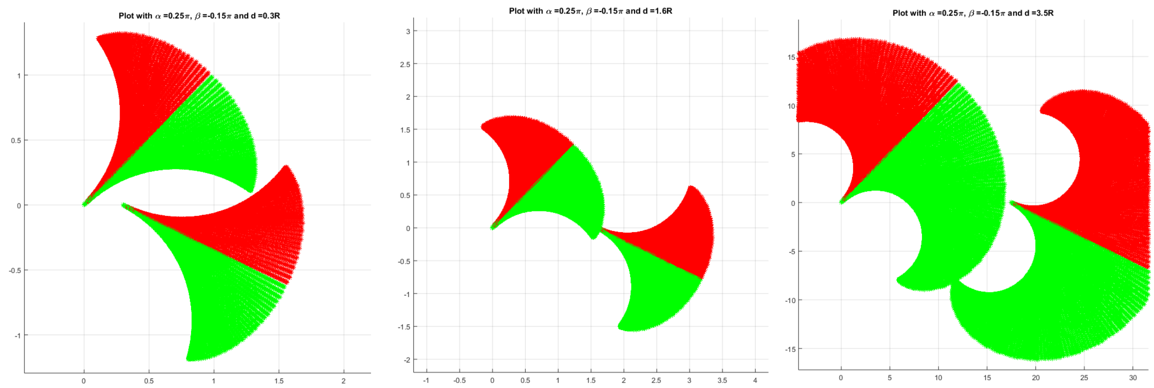

Figure 7: A14

5.6 a_{21}

The optimal trajectory is T[RS LS] for all distances.
This can be proved as any CS curve starting from A will intersect the straight line passing through B (S_B) at angle β before any point on the right of S_B. Thus T[RS LS] $\leq$ T[RS RS]. Similarly T[RS LS] $\leq$ T[LS RS] and T[RS LS] $\leq$ T[LS RS].

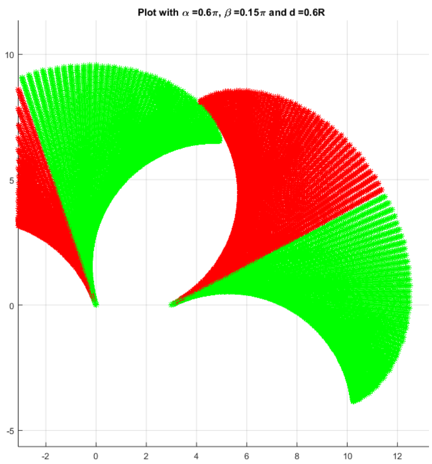

Figure 8: A21

5.7 a_{24}

For $\pi - \alpha \leq 2\pi - \beta$ T[LS RS]
For $\pi - \alpha \geq 2\pi - \beta$ T[RS LS]

6 Curvature

Meeting of two vehicles can be seen as the intersection of their rechability sets. Radius of curvature of the reachability set can be used to find intersection them.
Radius of curvature decreases uniformly as turning time increases from 0 to T. It reduces to 0 at t = T, implying that is a corner point.

7 Types of paths

The meeting of two vehicles can occur by one of the four possible paths:

- Along the common tangent
- At the intersection of two circles of minimum radius
- At the initial position of one of the vehicles : direct
- At a point on minimum radius circle of one, reached by a CS path by another, with nonzero S: CSC

Proposition: Time optimal point of intersection for two vehicles cannot lie within any of the four circles of minimum radius. In other words trajectotry for anyone of them cannot be CC.

Proposition: Intersection of CS paths by both vehicles is along common tangent.

Proof: For a time optimal meeting, intersection of paths occurs as the the boundaries of reachability set of both vehicles. Let the boundary made by CS paths be called outer boundary and that made by C alone be called inner boundary. Thus intersection of CS paths by both vehicles happens on intersection of outer boundaries of both at one point.

The above four paths can be seen from these two propositions. Since CC is ruled out, we have four possible trajectories for each vehicle: CS (with both C and L of non-zero length), C, S and zero (both C and S are zero). This gives following cases:

	CS	**C**	**S**	**zero**
CS	common tangent	CSC	common tangent	direct
C	CSC	circle intersection	CSC	direct
S	common tangent	CSC	common tangent	direct
zero	direct	direct	direct	-

- If both the vehicles are in each other's circles of minimum radius, only intersection of circles and CSC are possible
- If the distance between vehicles is greater that 4R the intersection is always along common tangent
- a_{12} and a_{21} do not have a direct path for any value of d

8 Rendezvous point calculation

Optimal path can be determined by calculating the path length for the four possible cases given in above section and choosing the minimum time taking one. For the case of common tangent, total path length along circles and tangent need to be calculated and divided in two to give time of one vehicle. For the case of intersection of circles is intersection point is computed and length along the arc for both is calculated separately. Maximum of the two is used for calculating final time. Length along direct path can be calculated by treating as a single vehicle programmed to reach a predetermined point.

CSC requires a minimization routine. Here the outer boundary intersects with a point on inner boundary or at corner point. Since the outer boundary is always the time limiting path, we focus on the CS path only. Thus we need to minimize the time taken by CS path to intersect a point on the arc of the circle reached by other vehicle in lesser time.

9 Future Work

The observed sequences for each of the six cases remains to be proved. Next step would be to calculate the exact rendezvous point from knowledge of the initial state alone. Building on these, then further feedback based laws for the two vehicle case can be stated. We would further investigate the ways to generalize these results and methods for more than two vehicles.

References

[1] L. Dubins, "On curves of minimal length with a constraint on average curvature, and with prescribed initial and terminal positions and tangents," *American Journal of Mathematics*, vol. 79, pp. 497 – 516, 1957.

[2] G. J. Cockayne, E & W. C. Hall, "Plane motion of a particle subject to curvature constraints," *Siam Journal on Control*, vol. 13, p. 13. 10.1137/0313012, 1975.