BTP Report: Rendezvous of dubins cars in short range

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1 Introduction

Motion planning is one of the fundamental problem in robotics. Broadly it is a problem of selecting a set of actions which, on execution will take a robot from an initial state to a given final state. The aim of project is to provide feedback based solutions for Dubins vehicle's strategies. Focus is on situations when the vehicles are located at a distance in the range of minimum radius of turning. Feedback solutions are aimed to be closed form expressions which can be evaluated in minimal time by the vehicle on-board.

2 Dubins Vehicle

The dubins car can move only in forward direction with a maximum speed of v_s. The minimum turning radius can be $r\,(= 1/u_{max})$.The motion for such a vehicle is governed by following dynamics:

$$\begin{aligned} \dot{x(t)} &= v_s sin(\theta(t)), \\ \dot{y(t)} &= v_s cos(\theta(t)), \\ \dot{\theta(t)} &= u \end{aligned} \tag{1}$$

where $(x(t), y(t)) \in \mathbb{R}^2$ is the position of vehicle at time t and $\theta(t)$ is the orientation. Control variable, u is chosen from the interval $[-u_{max}, u_{max}]$. Any position of the car in a 2-D plane can be described by a 3-D vector $(x(t), y(t), \theta(t))$.

3 Reachability Sets

Let the starting configuration be defined as (x_0, y_0, θ_0) and final configuration as (x_f, y_f, θ_f). Let C denote an arc of circle of minimum radius, r_0, and S denote a straight line path. The circular path C can be of types, L and R, corresponding to vehicle turning as sharply as possible to left and right, respectively. Let C_L denote the circle of minimum radius on left side of initial orientation and C_R denote the circle on right.
Dubins proved in [1] that shortest path between any two configurations for such a vehicle can be expressed as a combination of at most three parts, which can be arcs of C_L or C_R or straight line (S). This leads to six possible cases: $\{RSR, LSL, RSL, LSR, RLR, LRL\}$.

Further, if the constraint on angle of arrival at the final destination is relaxed, the shortest paths are changed. Cockayne [2] proved that in such a case they are reduced to type CS or CC. This generates four possible trajectories: $\{RS, LS, RL, LR\}$. We focus here on the latter case only.

Reachability set, R_T, is defined as the set points a vehicle can reach in time less than or equal to T. For time much greater than r_0/v_s, the reachability set can be approximated as a circle for Dubins's vehicle. But for smaller time intervals its shape is dependent on the orientation of the vehicle and amount of time. The sets of CL and CC are plotted separately in figure 1 and figure 2.

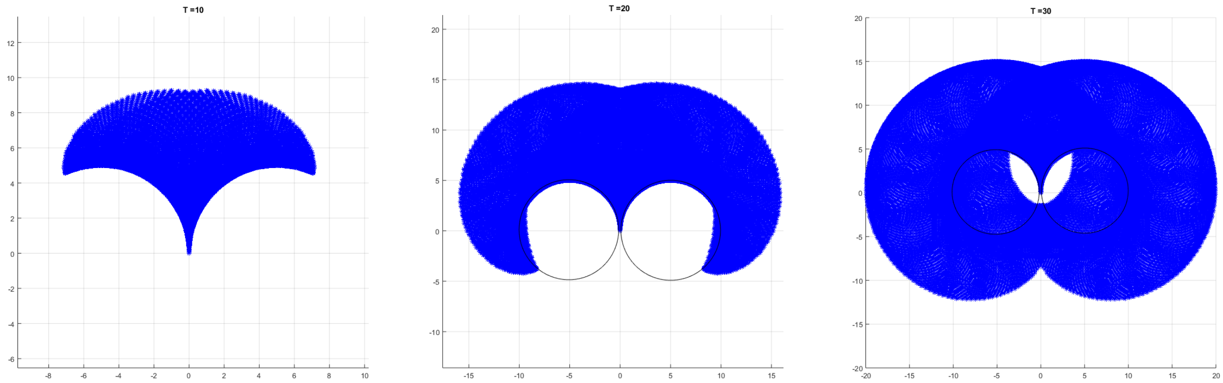

Figure 1: Reachability set for CC trajectories

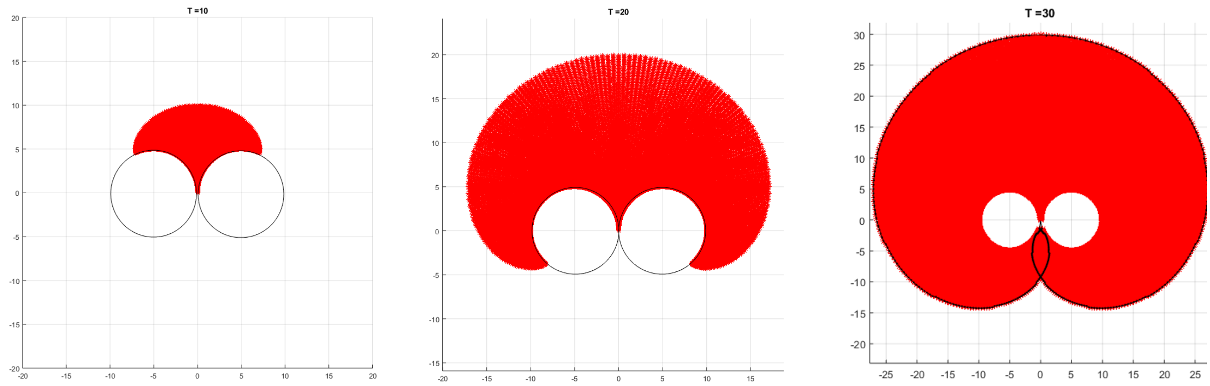

Figure 2: Reachability set for CL trajectories

3.1 Single agent case

The car was initially programmed to reach the origin, starting from any initial position (x_0, y_0, θ_0), without constraints on angle of arrival. The initial position was taken such that its distance from origin is sufficiently larger than minimum radius of turning. In this case, according to Cockayne, the shortest path will be a CS trajectory. This can be described as a combination of an arc and a tangent to one of the circles of minimum radius. This is was implemented using the feedback strategy:

Algorithm 1 Reach (0,0)

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while $\sqrt{x^2 + y^2} \geq \epsilon$ **do**
  if $\theta - \tan^{-1}(y/x) \leq \rho$ **then**
    $u = 0$
  else if $\theta - \tan^{-1}(y/x) \leq 0$ **then**
    $u = -u_{max}$
  else
    $u = u_{max}$
  end if
  update θ, x and y
end while

But this feedback law fails in the situation when final position lie within C_R or C_L. Thus to look at the nearby points we need to modify our solution.

3.2 Characterization of 2-D plane

The strategy for reaching any point in the 2-D plane can be described as follows. Any point outsideC_R and C_L can be reached time optimally by a CS trajectory. Whereas, any point within them can be reached optimally by a CC trajectory. This can be proven as follows:

I Take a point outside the circles. CS and CC trajectory are common till point A. Beyond that the shortest path from A to P is a straight line.

II Take a point inside C_R. Draw a circle of minimum radius passing through B and tangent to C_L. This intersects C_L at P. Thus, CS and CC trajectory are common till point P (Figure 3). To prove that CC is optimal we need to show

$$\begin{aligned} R(2\pi - \phi) + l &\geq r(2\pi - \theta)) \\ \text{or } l &\geq R(\phi - \theta) = R\alpha \\ \text{Since } \phi \geq \theta, \text{we have } l &\geq R\,tan\alpha \\ \text{As } \alpha \in [0, \pi/2], R\,tan\alpha &\geq R\alpha \end{aligned} \tag{2}$$

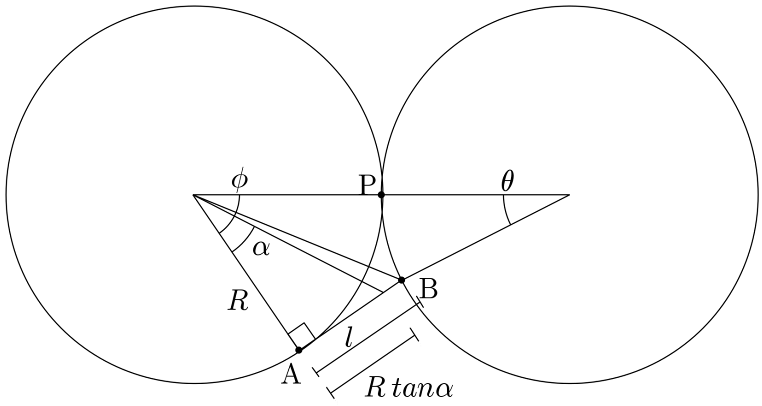

Figure 3: Proof of CC case

4 Two vehicle case

Following a similar approach as for one vehicle case, the rendezvous problem of two vehicles can be looked at by dividing the plane in different sections based on their initial positions and searching for time optimal trajectories of both the vehicles. Let d be the distance between the two vehicles. A rectangular coordinate system is chosen such that its origin is at position of vehicle A, $P_A = (0, 0)$ and the positive direction of x-axis is towards vehicle B, $P_B = (d, 0)$. Let α and β be the angles their initial orientations make with the positive x-axis when measured counter-clockwise.

4.1 Equivalency Groups

The classification of plane can be proceeded by first dividing the range of possible orientations α and β in four quadrants. Quadrant 1 corresponds to the range $[0, \pi/2]$, quadrant 2 to $[\pi/2, \pi]$, quadrant 3 to $[\pi/3, \pi/2]$, and quadrant 4 to the range $[3\pi/2,\ 2\pi]$. Since α and β

can be in any of the four quadrants, there are 16 different possible combinations of quadrants. We represent those 16 by a 4×4 matrix, $\{a_{ij}\}$, where index i corresponds to the quadrant number of vehicle A, and index j of vehicle B.

Element a_{ij} therefore describes the class of all paths whose initial and final orientation angles (α, β) belong to the quadrants i and j , respectively. For example, the case $\alpha \in [0, \pi/2]$, $\beta \in [\pi/2, \pi]$ corresponds to the element a_{12} and covers all those paths whose orientation angles belong to the first and second quadrant, respectively.

It will be shown below that these 16 classes can be reduced to six independent clusters, called equivalency groups. We consider two trajectories to be equivalent if one can be converted into another by taking a reflection along x or y axis.

Proposition: Let $T_{\alpha,\beta}$ be the optimum trajectories for vehicle A and B together, then following trajectories are equivalent:

$T_{\alpha,\beta} \simeq T_{\pi-\beta,\pi-\alpha}$

$T_{\alpha,\beta} \simeq T_{-\alpha,-\beta}$.
This leads to following equivalency groups:

$$\begin{array}{ll} a_{11} \simeq a_{22} \simeq a_{33} \simeq a_{44} & a_{14} \simeq a_{23} \simeq a_{32} \simeq a_{41} \\ a_{12} \simeq a_{43} & a_{21} \simeq a_{34} \\ a_{13} \simeq a_{42} & a_{24} \simeq a_{31} \end{array}$$

Each of these six equivalent groups are discussed by varying the distance between the vehicles.
The possible scenarios for rendezvous can be: Both vehicles move for the same duration till they meet at a point. One vehicle moves for a longer duration than another vehicle. We call this trajectory to be degenerate in second vehicle. Last possibility is for one vehicle to reach the initial position of the other directly. The first two cases are denoted by T[CS CS] and third one is denoted by T[CS 0] or T[0 CS].

4.2 a_{11}

Let d_1 be the minimum distance at which direct path to initial position of B by vehicle A is optimal. Similarly let d_2 be the minimum distance at which T[LS LS] is the optimal trajectory and d_3 be the minimum distance at which T[RS LS] is the optimal trajectory. If the distance between the two vehicles, d, is less than d_1 then the path followed by A and B is RS and LS respectively.
Now if d_2 is less than d_3, following sequence of optimal trajectories is followed:

- T[RS 0] for $d_1 \leq d \leq d_2$
- T[LS LS] for $d_2 \leq d \leq d_3$
- T[RS LS] for $d_3 \leq d \leq d_4$

Else if $d_3 \geq d_4$, then sequence is:

- T[RS 0] for $d_1 \leq d \leq d_3$
- T[RS LS] for $d_3 \leq d \leq d_4$

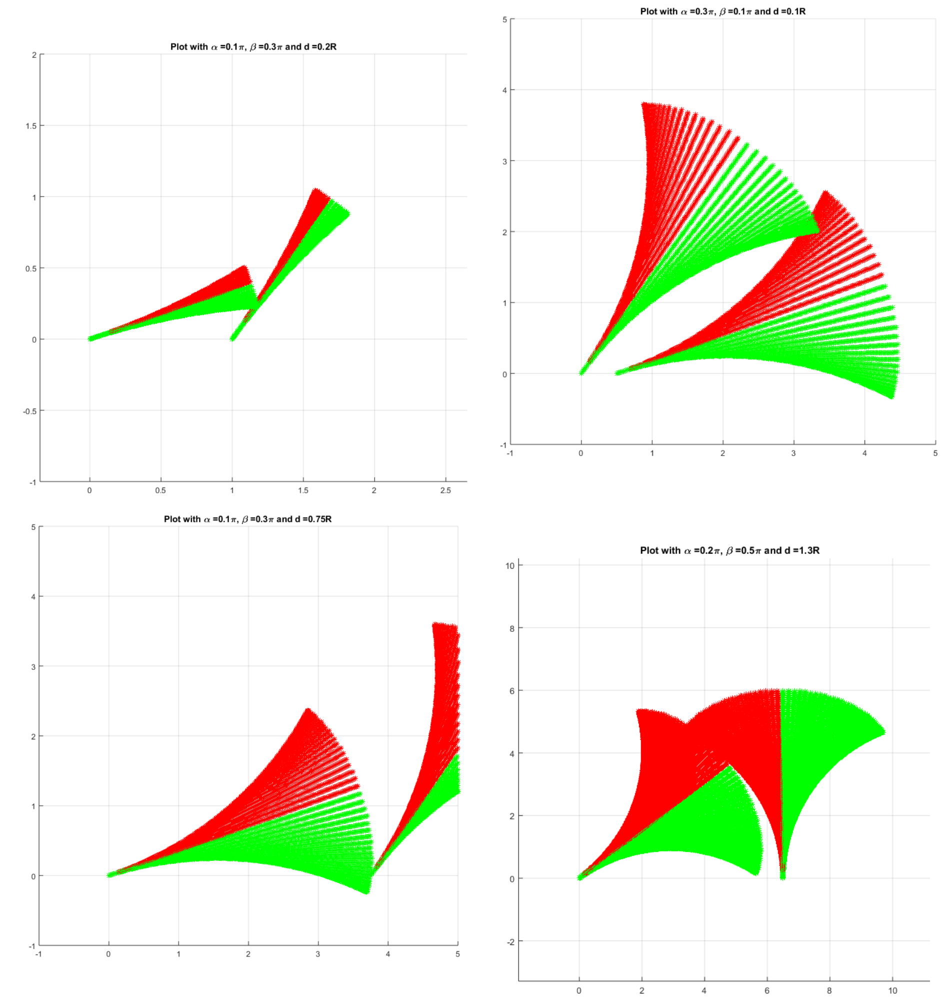

Figure 4: Trajectories for a_{11}

For all values of d, T[RS,LS] $\leq$ T[RS,RS] as any point on the right of line passing through B and at angle β is reached by A later than the corresponding point on the line. Similarly we can prove T[RS,LS] $\leq$ T[LS,RS].

4.3 a_{12}

Let d_1 be the distance at which straight line path by one vehicle and CS by other is optimal. Then we can write the sequence of optimal trajectories as:

- For $0 < d \leq d_1$
  - For $\alpha \leq \pi - \beta$ T[LS LS]
  - For $\alpha \geq \pi - \beta$ T[RS RS]

- For $d_1 \leq d$ T[RS LS]

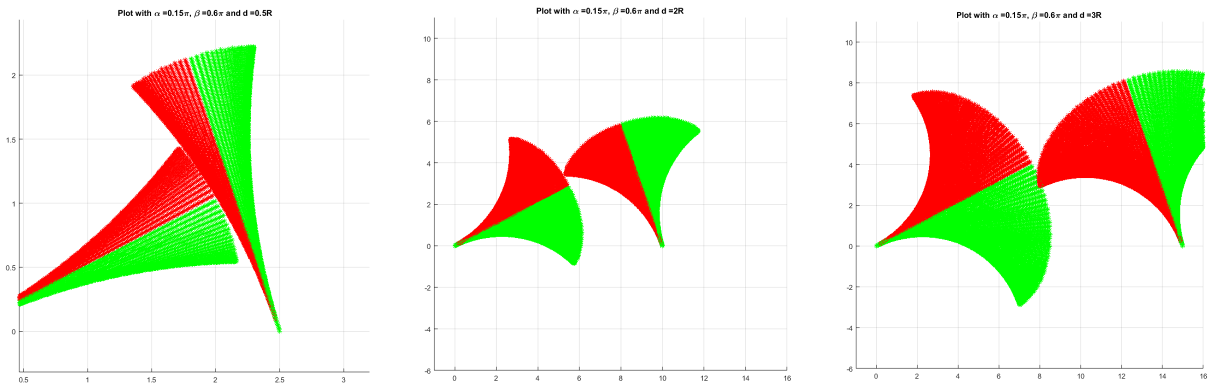

Figure 5: A12

4.4 a_{13}

- If both are inside each other's circles of minimum radius:
  - For $\alpha \leq -\pi + \beta$ T[LS RS]
  - For $\alpha \geq -\pi + \beta$ T[RS LS]
- If both are outside each other's circles:
  - T[RS RS]
- If one is inside and other is outside:
  - For B inside A's circle: T[RS RS] or T[0 RS]
  - For A inside B's circle: T[LS L] or T[LS 0]

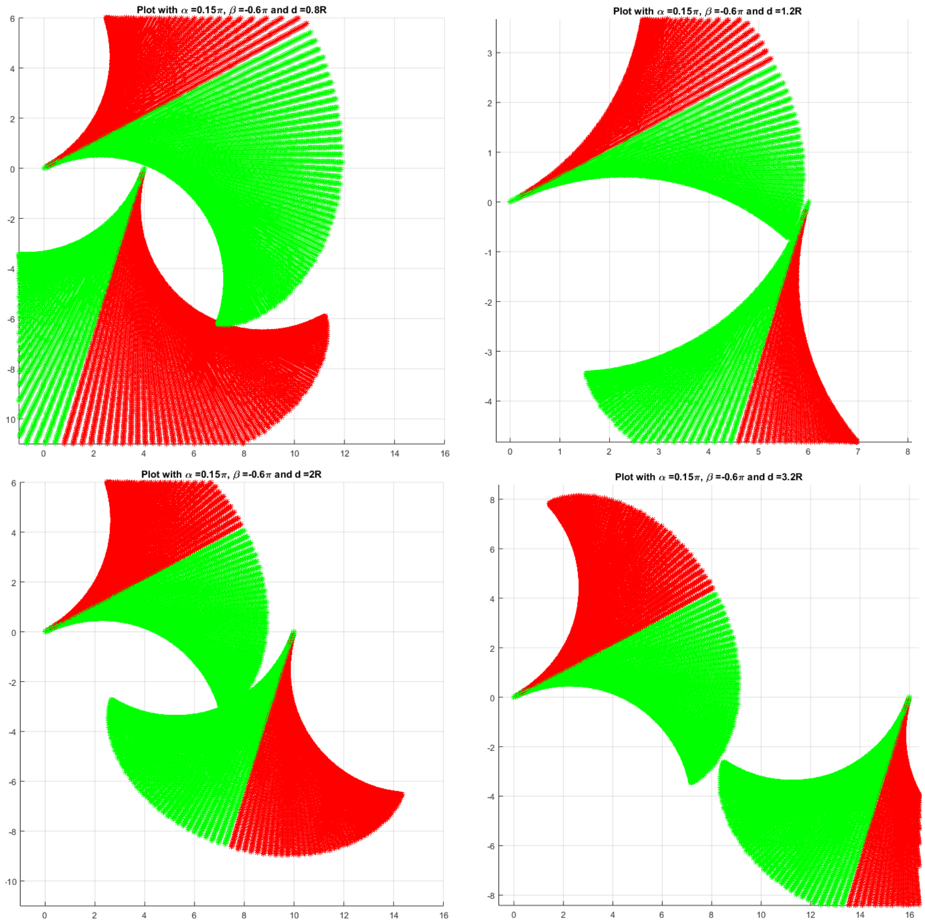

Figure 6: A13

4.5 a_{14}

- If B lies in the circle of A:
  - T[RS L]
- If B lies outside the circle of A, in increasing order of distance:
  - T[RS 0]
  - T[RS R]
  - T[RS RS]

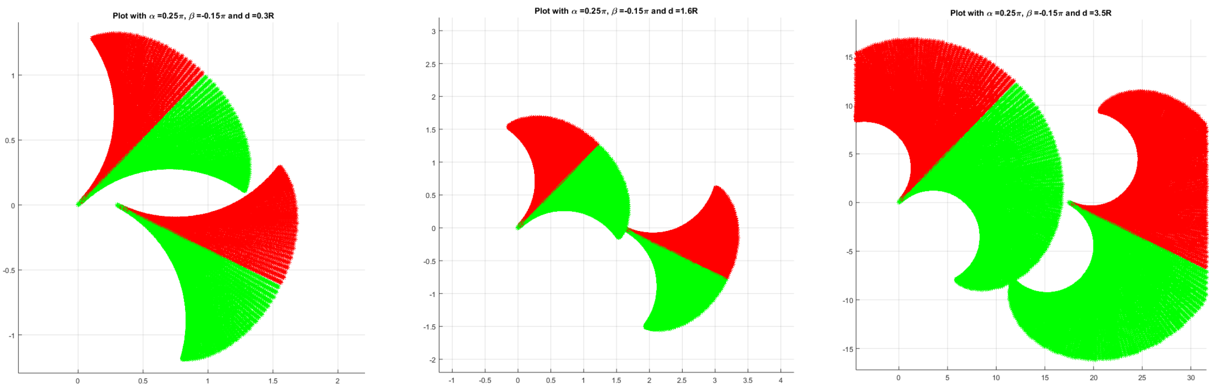

Figure 7: A14

4.6 a_{21}

The optimal trajectory is T[RS LS] for all distances.
This can be proved as any CS curve starting from A will intersect the straight line passing through B (S_B) at angle β before any point on the right of S_B. Thus T[RS LS] $\leq$ T[RS RS]. Similarly T[RS LS] $\leq$ T[LS RS] and T[RS LS] $\leq$ T[LS RS].

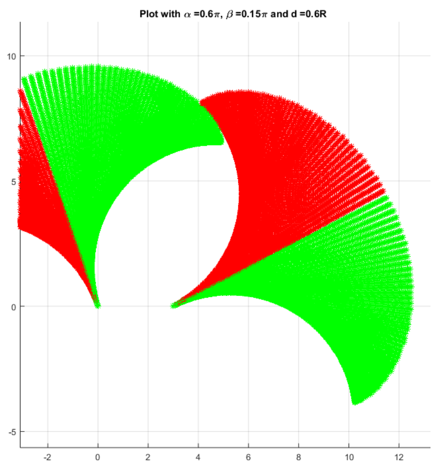

Figure 8: A21

4.7 a_{24}

For $\pi - \alpha \leq 2\pi - \beta$ T[LS RS]
For $\pi - \alpha \geq 2\pi - \beta$ T[RS LS]

5 Future Work

The observed sequences for each of the six cases remains to be proved. Next step would be to calculate the exact rendezvous point from knowledge of the initial state alone. Building on these, then further feedback based laws for the two vehicle case can be stated. We would further investigate the ways to generalize these results and methods for more than two vehicles.

References

[1] L. Dubins, "On curves of minimal length with a constraint on average curvature, and with prescribed initial and terminal positions and tangents," *American Journal of Mathematics*, vol. 79, pp. 497 – 516, 1957.

[2] G. J. Cockayne, E & W. C. Hall, "Plane motion of a particle subject to curvature constraints," *Siam Journal on Control*, vol. 13, p. 13. 10.1137/0313012, 1975.