CHAPTER

RECURRENCE RELATION

DEFINITION OF RECURRENCE RELATIONS & SOLUTION

A **recurrence relation** for the sequence $\{a_n\}$ is an equation that expresses a_n is terms of one or more of the previous terms of the sequence, namely, a_0, a_1, …, a_{n-1}, a_n for all integers n with $n \geq n_0$, where n_0 is a nonnegative integer. **For example**

The ***Fibonacci sequence***, is defined by the initial conditions $f_0 = 0$, $f_1 = 1$,and the recurrence relation

$$\boldsymbol{f_n = f_{n-1} + f_{n-2} \text{ for } n = 2, 3, 4, \ldots.}$$

A sequence is called a **solution** of a recurrence relation if it terms satisfy the recurrence relation.

Example:

Consider the recurrence relation **$a_n = 2a_{n-1} – a_{n-2}$ for n = 2, 3, 4, …**

Is the sequence $\{a_n\}$ with $a_n=3n$ a solution of this recurrence relation?

For $n \geq 2$ we see that

$2a_{n-1} – a_{n-2} = 2(3(n – 1)) – 3(n – 2) = 3n = a_n$.

Therefore, $\{a_n\}$ with $a_n=3n$ is a solution of the recurrence relation.

DEFINITION OF RECURRENCE RELATIONS & SOLUTION

Example: Consider the recurrence relation $\mathbf{a_n = 2a_{n-1} - a_{n-2}}$ **for n = 2, 3, 4, …**

Is the sequence $\{a_n\}$ with $a_n=5$ a solution of the same recurrence relation?

For $n \geq 2$ we see that

$$2a_{n-1} - a_{n-2} = 2 \cdot 5 - 5 = 5 = a_n.$$

Therefore, $\{a_n\}$ with $a_n=5$ is also a solution of the recurrence relation.

Practice Example :

Determine whether the sequence $\{an\}$, where $\boldsymbol{a_n = 3n}$ for every non negative integer n, is a solution of the recurrence relation $\boldsymbol{a_n = 2a_{n-1} - a_{n-2}}$ for $n = 2, 3, 4, \ldots$. Answer the same question where $\boldsymbol{a_n = 2n}$ and where $\boldsymbol{a_n = 5}$.

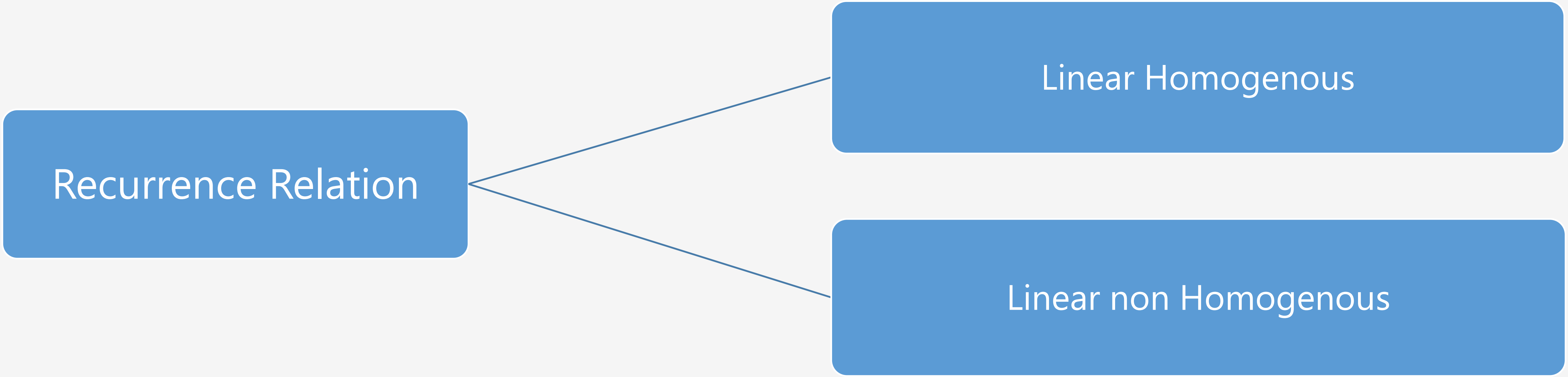
RECURRENCE RELATIONS & SOLUTION

In other words, a recurrence relation is like a recursively defined sequence, but **without specifying any initial values (initial conditions)**.

Therefore, the same recurrence relation can have (and usually has) **multiple solutions**.

If **both** the initial conditions and the recurrence relation are specified, then the sequence is **uniquely** determined.

TYPE OF RECURRENCE RELATION



TYPES OF RECURRENCE RELATIONS

1.Linear Homogenous Recurrence Relation

Definition A linear homogeneous recurrence relation of degree k with constant coefficients is a recurrence relation of the form:

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + c_3 a_{n-3} + \dots\dots\dots + c_k a_{n-k}$$

where c_1, c_2, …, c_k are real numbers, and $c_k \neq 0$.

- it is ***linear*** because the right-hand side is a sum of the previous terms of the sequence each multiplied by **constant**.
- it is ***homogeneous*** because no terms occur that are not multiples of the a_js. Each coefficient is a constant i.e. we can say that each term must at contain one term of sequence
- the ***degree*** is k because a_n is expressed in terms of the previous k terms of the sequence.

EXAMPLE OF LINEAR HOMOGENOUS RECURRENCE REALATION

$P_n = (1.11)P_{n-1}$ linear homogeneous recurrence relation of degree one

$f_n = f_{n-1} + f_{n-2}$ linear homogeneous recurrence relation of degree two

$H_n = 2H_{n-1} + 1$ not homogeneous

$B_n = nB_{n-1}$ coefficients are not constants

Example: **Which of the following are linear homogenous recurrence relation ?**

1.Which relations are of order 1 and 2 ?

2.Which relation are linear ?

3.Which are Homogenous ?

4.Which are Linear Homogenous Equation of Order 2 ?

$$a_n = \sqrt{a_{n-2}}$$

$$a_n = na_{n-1} + 5a_{n-2}$$

$$a_n = 3a_{n-1}$$

$$a_n = a_{n-1} + 2 - 3a_{n-2}$$

$$a_n = 4n + a_{n-1}$$

$$a_n = 5a_{n-1} - 6a_{n-2}$$

TYPE OF LINEAR HOMOGENOUS RECURRENCE RELATION (LHRR)

1. Linear Homogenous 1^{st} order Recurrence Relation
2. Linear Homogenous 2nd order Recurrence Relation

Further can be divided on basis of solution

a. Having real and distinct root.
b. Having real and repeated root
c. Complex root

3. Multi Order Recurrence Relation

LINEAR NON HOMOGENOUS RECURRENCE REALATIONS WITH CONSTANT COEFFICIENTS

The General form of **Linear nonhomogeneous recurrence relation with constant coefficients is**

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \dots\dots\dots + c_k a_{n-k} + f(n)$$

f(n) is independent of a_i .

Remarks : Each Non Homogenous Recurrence Relation has Associated Homogenous Recurrence Relation.

Example of Linear nonhomogeneous recurrence relation with constant coefficients

$a_n = 3a_{n-1} + 2n$

$a_n = 3a_{n-1} + 3^n$

SOLUTION METHODS OF RECURRENCE RELATION

1. Characteristic Equation and Root Method

2. Generating Functions

3. Iteration

METHOD 1: CHARCTERSTIC EQUATION AND ROOTS

Basically, when solving recurrence relations, we try to find solutions of the form $\mathbf{a_n = r^n}$, where r is a constant.

$a_n = r^n$ is a solution of the recurrence relation $a_n = c_1a_{n-1} + c_2a_{n-2} + \ldots + c_k a_{n-k}$ if and only if $r^n = c_1r^{n-1} + c_2r^{n-2} + \ldots + c_k r^{n-k}$.

Divide this equation by r^{n-k} and subtract the right-hand side from the left:

$$r^k - c_1r^{k-1} - c_2r^{k-2} - \ldots - c_{k-1}r - c_k = 0$$

This is called the **characteristic equation** of the recurrence relation.

The solution of **characteristic equation** of the recurrence relation are called **characteristic root** of the recurrence relation.

Theorem: Let c_1 and c_2 be real numbers. Suppose that $r^2 - c_1r - c_2 = 0$ has two distinct roots r_1 and r_2.

Then the sequence $\{a_n\}$ is a solution of the recurrence relation $a_n = c_1a_{n-1} + c_2a_{n-2}$ if and only if $a_n = \alpha_1r_1^n + \alpha_2r_2^n$ for $n = 0, 1, 2, \ldots$, where α_1 and α_2 are constants.

SOLUTION TECHNIQUE :SOLVING 1ST AND 2ND ORDER LINEAR HOMOGENOUS RECURRENCE REALTION

REQUIRED INPUT: A linear 1st and 2nd order linear Homogenous recurrence relation with **initial condition**

Step1 : Find the characteristic equation for the recurrence relation.

Step2 : Use algebra to find the characteristic roots **say r_1 and r_2** .

Step 3: Set up the following frame work **(use the relation only when characteristic equation has distinct root)**

$$a_n = c_1 r_1^{\ n} + c_2 r_2^{\ n}$$

Use the initial condition to find the specific value of C_1 and C_2.

Example :Find the solution of the recurrence relation

1 $a_n = 5a_{n-1} - 6a_{n-2}$ with initial condition $a_0=1$, $a_1=0$.

2 $a_n = 3a_{n-1} - 2a_{n-2}$ with initial condition $a_0=1$, $a_1=0$.

$b_n = 10b_{n-1} - 25b_{n-2}$ with initial condition $b_0=3$, $b_1=5$.

But what happens if the characteristic equation has only one root?

How can we then match our equation with the initial conditions a_0 and a_1 ?

Theorem: Let c_1 and c_2 be real numbers with $c_2 \neq 0$. Suppose that $r^2 - c_1 r - c_2 = 0$ has only one root r_0 which is repeated two times.

A sequence $\{a_n\}$ is a solution of the recurrence relation

$$a_n = c_1 a_{n-1} + c_2 a_{n-2}$$

if and only if

,

$$a_n = c_1 r_0^{\ n} + c_2 n r_0^{\ n}$$

for n = 0, 1, 2, …, where α_1 and α_2 are constants.

Example:

Suppose that the roots of the characteristic equation of a linear homogeneous recurrence relation are 2, 2, 2, 5, 5, and 9 . What is the form of the general solution?

$$a_n = \left(\alpha_1 (2)^n + \alpha_2 n (2)^n + \alpha_3 n^2 (2)^n\right) + \left(\alpha_3 (5)^n + \alpha_4 n (5)^n\right) + \alpha_5 (9)^n$$

Example: What is the solution of the recurrence relation $a_n = 6a_{n-1} - 9a_{n-2}$ with $a_0 = 1$ and $a_1 = 6$?

The only root of $r^2 - 6r + 9 = 0$ is $r_0 = 3$.Hence, the solution to the recurrence relation is

$$\mathbf{a_n = \alpha_1 3^n + \alpha_2 n 3^n}$$

for some constants α_1 and α_2.

To match the initial condition, we need

$a_0 = 1 = \alpha_1$
$a_1 = 6 = \alpha_1 \cdot 3 + \alpha_2 \cdot 3$

Solving these equations yields $\alpha_1 = 1$ and $\alpha_2 = 1$.

Consequently, the overall solution is given by

$$a_n = 3^n + n3^n.$$

Example : Find the solution to the recurrence relation $a_n = -3a_{n-1} - 3a_{n-2} - a_{n-3}$ with initial conditions $a_0 = 1$, $a_1 = -2$ and $a_2 = -1$.

Solution : $r^3 + 3r^2 + 3r + 1 = 0$ has a single root $r_0 = -1$ of multiplicity three.

$$\therefore a_n = (\alpha_1+\alpha_2 n+\alpha_3 n^2)\, r_0^{\,n} = (\alpha_1+\alpha_2 n+\alpha_3 n^2)(-1)^n$$

initial conditions are given $a_0 = 1$, $a_1 = -2$ and $a_2 = -1$.

$$a_0 = \alpha_1 = 1$$

$$a_1 = (\alpha_1+\alpha_2+\alpha_3)\cdot(-1) = -2$$

$$a_2 = \alpha_1+2\alpha_2+4\alpha_3 = -1$$

$$\therefore \alpha_1 = 1,\ \alpha_2 = 3,\ \alpha_3 = -2 \Rightarrow a_n = (1+3n-2n^2)\cdot(-1)^n$$

Linear Nonhomogeneous Recurrence Relations with Constant Coefficients

The General form of **Linear nonhomogeneous recurrence relation with constant coefficients is**

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \dots\dots\dots + c_k a_{n-k} + f(n)$$

Every solution of a linear nonhomogeneous recurrence relation is the sum of

- a particular relation and
- a solution to the associated linear homogeneous recurrence relation

Solving Linear Non Homogeneous Recurrences

Theorem : If $\{a_n^{(p)}\}$ is a particular solution of the nonhomogeneous linear recurrence relation with constant coefficients

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \dots\dots\dots + c_k a_{n-k} + f(n)$$

then every solution is of the form

$$a_n^{(p)} + a_n^{(h)}$$

where $\{a_n^{(h)}\}$ is a solution of the associated homogeneous recurrence relation

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + c_3 a_{n-3} + \dots\dots\dots + c_k a_{n-k}$$

Solution Technique For Linear Non Homogenous Equation

Step 1: Solve the Associated Homogenous Recurrence Relation .

Step 2: Find the Particular Solution of the recurrence relation .

If f(n)= polynomial **For example** f(n)=n ,then $a_n^{(p)} = c\,n + d$ *(find value of c and d)*

f(n)=(constant)^n then $a_n^{(p)} = c$. (constant)^n *(find value of c)*

Step 3: Find the solution of the recurrence relation a_n .Using the general solution

$$a_n^{(p)} + a_n^{(h)}$$

Step 4: Find Explicit solution using initial condition.

Some standard form for particular solution

Some substitution

$f(n)$	$a_n^{(p)}$
a constant and characteristic root is not 1	B
a constant and characteristic root is 1 with multiplicity m	Bn^m
An+b	B_1n+B_0
an^2	$B_2{}^{n2}+B_1n+B_0$
a^n	$B.\ a^n$

Example. Find all solutions of the recurrence relation $a_n = 3a_{n-1} + 2n$. What is the solution with $a_1=3$?

Solution : Associated homogeneous recurrence relation is $\boldsymbol{a_n = 3a_{n-1}}$

Characteristic equation: $\boldsymbol{r - 3 = 0 \Rightarrow r = 3 \Rightarrow a_n^{(h)} = \alpha \times 3^n}$.

Particular Solution $\boldsymbol{\because F(n) = 2n}$

$\therefore$ Let $a_n^{(p)} = c\,n + d$, where $c, d \in \mathbf{R}$.

If $a_n^{(p)} = c\,n + d$ is a solution to $a_n = 3a_{n-1} + 2n$, then $c\,n + d = 3(c(n-1)+d)+2n$

$cn + d = 3cn - 3c + 3d + 2n \Rightarrow 2cn - 3c + 2d + 2n = 0 \Rightarrow (2c+2)n + (2d-3c) = 0n+0$

$\therefore$ By comparing coefficients of n and constant We get $2c+2 = 0$, and $2d-3c = 0$

$\Rightarrow c = -1, d = -3/2$ $\Rightarrow a_n^{(p)} = -n - 3/2$

$a_n = a_n^{(h)} + a_n^{(p)} = \alpha \times 3^n - n - 3/2$ Given that $a_1=3$ If $a_1 = \alpha \times 3 - 1 - 3/2 = 3$ $\Rightarrow \alpha = 11/6$

$\Rightarrow a_n = (11/6) \times 3^n - n - 3/2$

Example . Find all solutions of the recurrence relation $a_n = 5a_{n-1} - 6a_{n-2} + 7^n$.

Solution: **Associated homogeneous recurrence relation** is $a_n = 5a_{n-1} - 6a_{n-2}$

Characteristic equation: $r^2 - 5r + 6 = 0 \quad \Rightarrow r_1 = 3, r_2 = 2 \quad \Rightarrow a_n^{(h)} = \alpha_1 \times 3^n + \alpha_2 \times 2^n$.

Particular solution $\because F(n) = 7^n \quad \therefore$ Let $a_n^{(p)} = c \cdot 7^n$, where $c \in \mathbf{R}$.

If $a_n^{(p)} = c \cdot 7^n$ is a solution to $a_n = 5a_{n-1} - 6a_{n-2} + 7^n$, then $c \cdot 7^n = 5c \cdot 7^{n-1} - 6c \cdot 7^{n-2} + 7^n$

$c \cdot 7^n = 5c \cdot 7^{n-1} - 6c \cdot 7^{n-2} + 7^n \quad \Rightarrow c \cdot 7^2 = 5c \cdot 7^1 - 6c + 7^2$

$\Rightarrow 49c = 35c - 6c + 49$

$\Rightarrow c = 49/20$

$\Rightarrow a_n^{(p)} = (49/20) \cdot 7^n \quad \Rightarrow a_n = a_n^{(h)} + a_n^{(p)}$

$a_n = \alpha_1 \times 3^n + \alpha_2 \times 2^n + (49/20) \cdot 7^n$

Example : What form does a particular solution of the linear nonhomogeneous recurrence relation $a_n = 6a_{n-1} - 9a_{n-2} + F(n)$ have when $F(n) = 3^n$, $F(n) = n3^n$, $F(n) = n^2 2^n$, and $F(n) = (n^2+1)3^n$?

Solution :

The associated linear homogeneous recurrence relation is $a_n = 6a_{n-1} - 9a_{n-2}$.

characteristic equation: $r^2 - 6r + 9 = 0 \Rightarrow r = 3$ (Multiple root)

$F(n) = 3^n$, and 3 is a root $\Rightarrow a_n^{(p)} = p_0 n^2 3^n$

$F(n) = n3^n$, and 3 is a root $\Rightarrow a_n^{(p)} = n^2(p_1 n + p_0)\, 3^n$

$F(n) = n^2 2^n$, and 2 is not a root $\Rightarrow a_n^{(p)} = (p_2 n^2 + p_1 n + p_0) 2^n$

$F(n) = (n^2+1)3^n$,

and 3 is a root $\Rightarrow a_n^{(p)} = n^2\, (p_2 n^2 + p_1 n + p_0)\, 3^n$

Generating Functions

Definition 1. The generating function for the sequence $a_0, a_1, a_2, \ldots$ of real numbers is the infinite series

$$G(x) = a_0 + a_1x + a_2x^2 + \ldots\ldots\ldots\ldots\ldots\ldots + a_nx^n + \ldots \qquad = \sum_{k=0}^{\infty} a_k x^k$$

Example Find the generating functions for the sequences $\{a_k\}$ with

(1) $a_k = 3$ (2) $a_k = k+1$ (3) $a_k = 2^k$

Solution :

$$G(x) = \sum_{k=0}^{\infty} a_k x^k = \sum_{k=0}^{\infty} 3x^k$$

$$G(x) = \sum_{k=0}^{\infty} a_k x^k = \sum_{k=0}^{\infty} (k+1)\, x^k$$

$$G(x) = \sum_{k=0}^{\infty} a_k x^k = \sum_{k=0}^{\infty} 2^k x^k$$

Example What is the generating function for the sequence 1,1,1,1,1,1 ?

Solution : $G(x) = \sum_{k=0}^{\infty} a_k x^k = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots$

$$= 1 + x + x^2 + x^3 + x^4 + x^5 \quad (Expansion)$$

$$= \frac{x^6 - 1}{x - 1} \qquad (Closed\ \ Form)$$

Using Characteristic Equation & Root Method

Example Solving the recurrence relation $a_k = 3a_{k-1}$ for k=1,2,3,… and initial condition $a_0 = 2$.

Solution : Characteristic Equation & Root Method

$$r - 3 = 0 \Rightarrow r = 3 \Rightarrow a_n = \alpha \cdot 3^n$$

$$\because a_0 = 2 = \alpha$$

$$\therefore a_n = 2 \cdot 3^n$$

Another Method to solve recurrence relation is using

Generating Function

Solution Using Generating Function

Example Solving the recurrence relation $a_k = 3a_{k-1}$ for $k=1,2,3,\ldots$ and initial condition $a_0 = 2$.

Let $G(x) = a_0 + a_1x + a_2x^2 + \ldots\ldots\ldots\ldots = \sum_{k=0}^{\infty} a_k x^k$ be the generating function for $\{a_k\}$.

First note that $a_k = 3a_{k-1}$

$$\sum_{k=1}^{\infty} a_k x^k = 3\sum_{k=1}^{\infty} a_{k-1} x^k = 3x\sum_{k=1}^{\infty} a_{k-1} x^{k-1} = 3x\sum_{k=0}^{\infty} a_k x^k$$

$$G(x) - a_0 = 3x.G(x)$$

$$G(x) - 2 = 3x.G(x) \quad (\because a_0 = 2)$$

$$G(x) - 3x\,G(x) = 2$$

$$G(x) = \frac{2}{1-3x} = 2\sum_{k=0}^{\infty} (3x)^k = \sum_{k=0}^{\infty} 2.3^k$$

$$a_k = 2.3^k$$

Example : Solving $a_k = 8a_{k-1} + 10^{k-1}$ for $k = 1,2,3,\ldots$ and initial condition $a_0 = 1$ and $a_1 = 9$?

Let $G(x) = a_0 + a_1 x + a_2 x^2 + \ldots\ldots\ldots\ldots = \sum_{k=0}^{\infty} a_k x^k$ be the generating function for $\{a_k\}$.

$$G(x) - 1 = \sum_{k=1}^{\infty} a_k x^k = \sum_{k=1}^{\infty} \left(8a_{k-1} + 10^{k-1}\right) x^k = \sum_{k=1}^{\infty} 8a_{k-1} x^k + \sum_{k=1}^{\infty} 10^{k-1} x^k$$

$$= 8x \sum_{k=0}^{\infty} a_k x^k + \sum_{k=0}^{\infty} 10^k x^k$$

$$= 8x\ G(x) + x \sum_{k=0}^{\infty} 10^k x^k \quad = 8x\ G(x) + \frac{x}{1-10x}$$

$$(1-8x) G(x) = \frac{1-9x}{(1-10x)}$$

$$G(x) = \frac{1-9x}{(1-10x)(1-8x)} = \frac{1}{2}\left[\frac{1}{1-10x} + \frac{1}{1-8x}\right] = \frac{1}{2}\left[\sum_{k=0}^{\infty} 10^k x^k + \sum_{k=0}^{\infty} 8^k x^k\right]$$

$$= \sum_{k=0}^{\infty} \frac{\left(10^k + 8^k\right)}{2} x^k$$

Practice Example

Example: Solve the following recurrence relation using Generating Function

1. $a_n=3a_{n-1}-2$; $a_0=0$

2. $u_n=2u_{n-1}+n$; $u_0=1$

Example: Use generating functions to find an explicit formula for the Fibonacci numbers.