RELIABLE QUASI-MONTE CARLO WITH CONTROL VARIATES

BY

DA LI

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ABSTRACT

Recently Quasi Monte Carlo (QMC) methods have been implemented in a guaranteed adaptive algorithm. This raises the possibility of combining adaptive QMC with efficiency improvement techniques for IID Monte Carlo (MC) such as control variates.

The challenge for adding control variates to QMC is that optimal control variate coefficient for QMC is generally not the same as that for MC. Here we propose a method for computing the optimal control variate coefficients with a guaranteed adaptive QMC algorithm. One merit of control variates is that it is theoretically no worse than using no control variates. Our method is implemented in an efficient way so that the extra cost for control variates is not significant.

Our new adaptive QMC algorithm with control variates is illustrated by two financial problems. One is pricing an arithmetic mean Asian option with geometric mean Asian option as control variates and the other is barrier option with European option as control variates. Our results show that with good control variates, the cost of adaptive QMC is greatly reduced compared to vanilla QMC.

CHAPTER 1
INTRODUCTION

1.1 The idea

Recently there are some great results from construction of Quasi Monte Carlo (QMC) methods that can adaptively choose a sample size for given error tolerances [1]. Our work is trying to combine reliable QMC methods with control variates. We will justify the theory behind it, construct a practical algorithm which can be implemented and tested through high dimensional integration examples. Control Variates (CV) is a variance reduction technique for IID MC methods. QMC can be viewed as a deterministic version of IID MC, which outperforms MC for many integrals [2]. Naturally we wonder if QMC can also benefit from the CV technique. If that is possible, it can be especially useful for problems where we can easily find good control variates.

1.2 The challenge

The challenge is that for QMC the quadrature points are deterministic instead of random, so the variance minization can not be used. Even if one try to use random QMC, the optimal control variate coefficient for QMC is generally not the same as for simple Monte Carlo as explained by Hickernell, Lemieux, and Owen [3]. This requires us to figure out a new way to get the optimal coefficients for control variates with Quasi-Monte Carlo.

1.3 Outline

In chapter 2 we first briefly talk about QMC rule and it's difference between Monte-Carlo. Then we introduce control variates and the reliable QMC algorithm that our work is based on. In Chapter 3 we show the derivations and theories of our methods along with the corresponding algorithm. In chapter 4 we demonstrate results from several numerical experiments on option pricing problems. For the final chapter we will discuss the results and possible extensions or improvemnts for the method.

CHAPTER 2
BACKGROUND

2.1 Problem setup

Numerical integration problems are involved in fields such as physics, mathematical finance, biology, computer graphics, and many others fields. It usually happens when it is hard to solve some integral analytically. Therefore, one has to use numerical methods for such problems. MC method is a general way to solve problems in such case [4]. The method can be simply explained in the following way.

Suppose we have the following standard integration approximation problem whose format is:

$$I = \int_{[0,1)^d} f(x)dx. \tag{2.1}$$

Then we take sample of n points $\{\mathbf{x_0}, \ldots, \mathbf{x_n}\} \in [0,1)^d$ follow the uniform distribuation randomly, and construct the following MC estimator:

$$\hat{I}(f) = \frac{1}{n}\sum_{i=1}^{n} f(X_i).$$

However, there are several problems with IID MC method [5]. First, it is difficult to generate truly random samples. Second, error bound for IID MC works only in probabilistic sense. Last, in many applications the convergence rate of MC is considered not fast enough.

Hence, QMC method were introduced to address these problems. For QMC method the estimator is almost the same with MC. The difference is that the sample points are taken from low discrepency sequence, which is determinsticly chosen instead of random. We will briefly rewiew one type of such sequence that we used for implementation of our method.

2.2 Sobol sequence

Like we mentioned earlier, for QMC we do not use random points, so the question become how to choose such points? Naturally, in order to make good approximation we want the error to go to 0 as sample size N increases, which is:

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} f(x_n) = \int_{[0,1]^s} f(x)dx.$$

A sequence $\mathcal{S} = (x_n)_{n\geq 0}$ satisfies the above for all Riemann intregrable function f is called uniformly distributed modulo one. That is the kind of sequence from which we should draw our sample points to approximate the integral. However, in practice we can use only finite set of points and a finite sequence can never be uniformly distributed modulo one. Thus, we introduce some quantitative measure for the deviation of a finite point set from uniform distribution. Such measures are called discrepancies. [6].

Definition 2.2.1. Let $\mathscr{P} = \{x_0, \ldots, x_{N1}\}$ be a finite point set in $[0,1)^s$. The extreme discrepancy D_N of this point set is defined as:

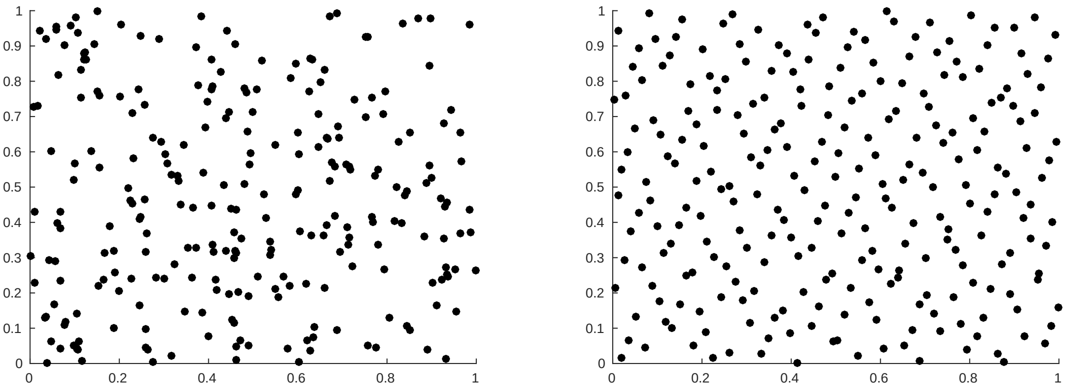
$$D_N(\mathscr{P}) := \sup_J \left|\frac{A(J,N)}{N} - \lambda_s(J)\right|.$$

where the supremum is extended over all sub-intervals $J \subseteq [0,1)^s$ of the form $J = [\mathbf{a}, \mathbf{b})$. It is a quantitative measure for the deviation of a finite point set from uniform distribution. It is proven that a sequence $\mathcal{S}$ is uniformly distributed modulo one, if and only if $\lim_{N\to\infty} D_N(S) = 0$ But ofen we use a weaker version of extreme discrepancy called the star-discrepancy and is defined as:

$$D_N^*(\mathscr{P}) := \sup_{x\in[0,1)^d} |\Delta_{\mathscr{P}}(x)|$$

$$\Delta_{\mathscr{P}}(x) := A([0,x), N, P)/N - \lambda_s([0,x)).$$

Hence our goal is to find sequences of which this star discrepanc are minimized. Initially low-discrepancy sequences were not designed with digital arithmetic in mind.



(a) 256 random generated points (b) 256 sobol points

Figure 2.1: Comparision between MC and QMC sample points

The Van der Corput sequence [7] is an example of such sequences. Modern low-discrepancy sequences are called digital sequences [8]. This is essentially because they are constructed using binary operations and are therefore well-suited to efficient implementations on computers.

Sobol sequence was the first constructed digital sequences in base 2 in 1967 [6]. Figure 2.1 shows a comparision between random and sobol points. The latter looks more evenly scattered. Infact if we draw a $16x16$ grid in the plot, for each grid there is exactly only 1 points in it.

2.3 Control variates

CV is a well known variance reduction technique used in MC simulation. It is ofen used when a 'simpler' version of the origin problem can be solved explicitly. In this section we briefly review the ideas and main results of the method.

Suppose we want to solve the integration problem (2.1) showed earlier, now we have a known function h and its value on the interval $\int_{[0,1)^d} h(x)dx = \theta$. We then construct a new estimator as the following:

$$\hat{I}_{\text{CV}}(f) = \frac{1}{n}\sum_{i=1}^{n}\Big[f(X_i) - \beta_{\text{MC}}[h(X_i) - \theta]\Big] \quad s.t.\ X_i \sim \mathcal{U}[0,1),\ i.i.d.$$

We can easily see it's an unbiased estimator, i.e. $\mathbb{E}(\hat{I}_{\text{CV}}) = I$. Now the question is how should we give β_{MC} and why is that. The idea is rather straitforward. We know the mean square error of MC estimator is $\text{Var}(\hat{I}) + \text{Bias}(\hat{I}^2$ []. CV method aims at efficiency improvment, so we need to reduce mean square error. Since the estimator is unbiased, we only need to minimize its variance. Hence, the optimal β_{MC} should be the one that minmize the variance of esimator. Here we give a simple derivation of optimal β_{MC} for single CV. First, the variance of $\hat{I}_{\text{CV}}$ is:

$$\begin{aligned}
\text{Var}(\hat{I}_{\text{CV}}) =& \text{Var}\Big(\frac{1}{n}\sum_{i=1}^{n}\big[f(X_i) - \beta_{\text{MC}}[h(X_i) - \theta]\big]\Big) \\
=& \frac{1}{n}\text{Var}\Big(f(X_i) - \beta_{\text{MC}}[h(X_i) - \theta]\Big) \quad \text{by } X_i \text{ i.i.d} \\
=& \frac{1}{n}\mathbb{E}\Big(\big[f(X_i) - \beta_{\text{MC}}[h(X_i) - \theta] - I\big]^2\Big) \\
=& \frac{1}{n}\mathbb{E}\Big(\big[[f(X_i) - I] - \beta_{\text{MC}}[h(X_i) - \theta]\big]^2\Big) \\
=& \frac{1}{n}\mathbb{E}\big([f(X_i) - I]^2 - 2\beta_{\text{MC}}[f(X_i) - I][h(X_i) - \theta] + \beta_{\text{MC}}^2[h(X_i) - \theta]^2\Big) \\
=& \frac{1}{n}\Big(\text{Var}[f(X_i)] - 2\beta_{\text{MC}}\text{Cov}[f(X_i), h(X_i)] + \beta_{\text{MC}}^2\text{Var}[h(X_i)]\Big) \\
=& \frac{1}{n}\Big(\text{Var}[h(X_i)](\beta_{\text{MC}} - \frac{\text{Cov}[f(X_i), h(X_i)]}{\text{Var}[h(X_i)]})^2 + \\
& \quad \text{Var}[f(X_i] - \frac{\text{Cov}^2[f(X_i), h(X_i)]}{\text{Var}[h(X_i)]}\Big),
\end{aligned}$$

then the optimal β_{MC} is given by:

$$\beta_{\text{MC}}^* = \frac{\text{Cov}[f(X_i), h(X_i)]}{\text{Var}[h(X_i)]}. \tag{2.2}$$

In this case the variance become:

$$\text{Var}(\hat{I}_{\text{CV}}) = \frac{\text{Var}[f(X_i)]}{n}\big(1 - \text{corr}^2[f(X_i), h(X_i)]\big),$$

and note we always have:

$$\mathrm{Var}(\hat{I}_{\mathrm{CV}}) \leq \frac{\mathrm{Var}[f(X_i)]}{n} = \mathrm{Var}(\hat{I}).$$

Now we can see the merit of control variates as a variance reduction method. In the worst case, we get a completely uncorrelated g that leads correlation to zero, and we have variance exactly the same as not using control variates. On the other hand, the more correlated our control variates is to the target function, the more variance we can get rid of by using the method.

2.4 Reliable adaptive QMC with digital sequence

2.4.1 Idea of adaptive cubature algorithm.

One practical problem for QMC method is that how to get the sample size big enough for a required error tolerance. The idea in work of Hickernell and Lluís (2014) [1] is to construct a QMC algorithm with reliable error estimation on digital sequence. Here we briefly summarize their results.

The error of QMC method on digital sequence can be expressed in terms of Walsh coefficients of the integrand on certain cone conditions.

$$\text{if } f \in \mathscr{C} \text{ then } \left| \int_{[0,1)^d} f(x)dx - \hat{I}_m(f) \right| \leq a(r,m) \sum_{\lfloor 2^{m-r-1} \rfloor}^{2^{m-r}-1} |\tilde{f}_{m,k}| \tag{2.3}$$

$$\hat{I}_m(f) := \frac{1}{b^m} \sum_{i=0}^{b^m-1} f(z_i \oplus \Delta)$$

$$\tilde{f}_{m,k} = \text{ discrete Walsh coefficients of } f$$

$$a(r,m) = \text{ inflation factor that depends on } \mathscr{C}.$$

Here is the defination of the cone condition:

$$\mathscr{C} := \left\{ f \in L^2[0,1)^d : \bigcirc \leq \hat{\omega}(m-l)\Diamond,\ l \leq m; \quad \Diamond \leq \mathring{\omega}(m-l)\Box,\ l^* \leq l \leq m \right\}$$

$$\bigcirc := \sum_{\kappa=\lfloor b^{l-1} \rfloor}^{b^l-1} \sum_{\lambda=1}^{\infty} |\hat{f}_{\kappa+\lambda b^m}|, \quad \Box := \sum_{\kappa=b^{l-1}}^{b^l-1} |\hat{f}_\kappa|, \quad \Diamond := \sum_{\kappa=b^m}^{\infty} |\hat{f}_\kappa| \tag{2.4}$$

$$l^* \in \mathbb{N} \text{ be fixed };\forall m \in \mathbb{N}, \hat{\omega}(m), \mathring{\omega}(m) \geq 0, \text{ and } \lim_{m\to\infty} \hat{\omega}(m) = 0,\ \lim_{m\to\infty} \mathring{\omega}(m) = 0.$$

The first inequality($\bigcirc \leq \Diamond$) means the sum of the larger indexed Walsh coefficients bounds a partial sum of the same coeffcients. Take $l = 0, m = 12$ for example, in Figure 2.2 the sum of circles should be bounded by some factor times the sum of diamonds. The second inequality($\Diamond \leq \Box$) requires the sum of the larger Walsh coefficients be bounded by the sum of smaller indexed Walsh coefficients. Take $l = 8$ at this time, which means in Figure 2.2 the sum of diamonds should be bounded by some relax factor times the squares.

The cone gives some meanings for the functions about how they should behave to get the error bound formula (2.3). This means that $|\hat{f}_\kappa|$ does not dramatically bounce back as κ goes to infinity. Note that in Figure 2.2 we call circles the err bound, this is proven to be true and under the cone conditions we can estimate it using discrete Walsh coefficients instead of true Walsh coefficients.

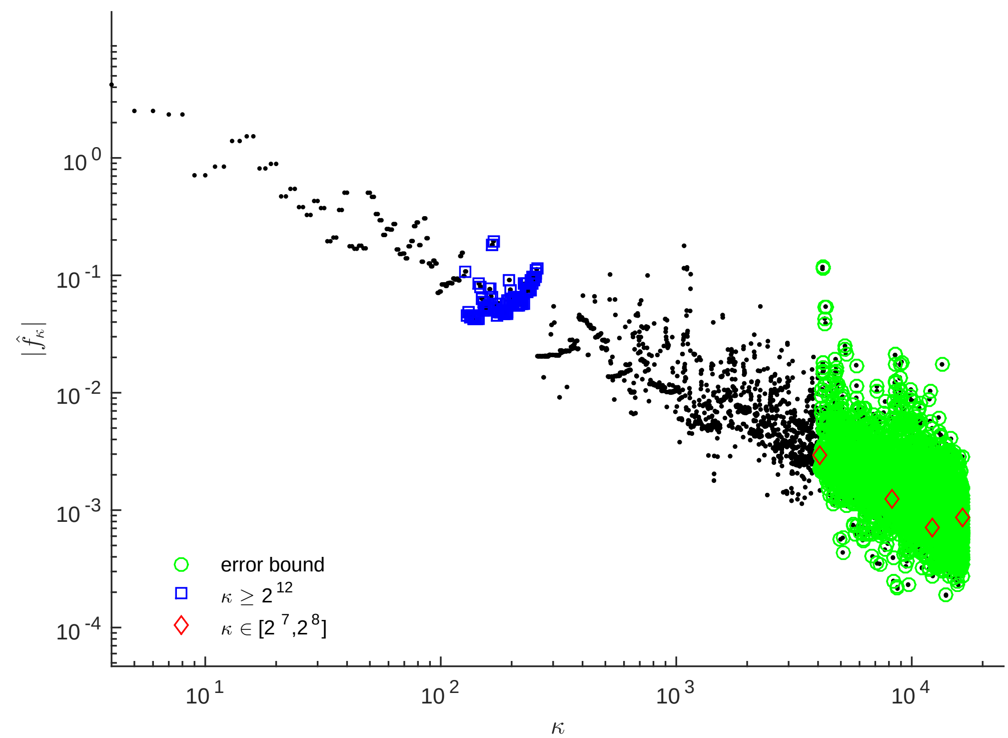


Figure 2.2: Cone condition for reliable adpative QMC algorithm

CHAPTER 3

RELIABLE ADAPTIVE QMC WITH CV

3.1 Idea to add control variates to reliable adaptive QMC

The whole method starts with an idea similar to traditional control variates technique for MC. If we know the integration of a function $\boldsymbol{h} = (h_1, \dots, h_J)$ on the interval same as our f, say $\int_{[0,1)^d} h_j dx = \theta_j$, then we can define a new function g:

$$g := f - (\boldsymbol{h} - \boldsymbol{\theta})\boldsymbol{\beta} \tag{3.1}$$
$$\text{s.t. } \boldsymbol{\theta} = (\theta_1, \dots, \theta_J), \boldsymbol{\beta} = (\beta_1, \dots, \beta_J)^T.$$

Then easily we can find that if we replace f with g, the integration stays the same:

$$\int_{[0,1)^d} g dx = \int_{[0,1)^d} f - (\boldsymbol{h} - \boldsymbol{\theta})\boldsymbol{\beta} dx = \int_{[0,1)^d} f dx.$$

Now we wonder if we can still use the same method as MC, the answer is no and the reason is in the next section.

3.2 The problem of CV with random QMC

The problem is that QMC is not a random process and we simply can't use the minimizing mean squre error trick as shown earlier anymore. However, one can use random QMC instead to 'restore' the randomness to QMC [9]. Random QMC use a different way for generating X_i, they are still identical(i.e. from same distribution) but not independent, which will make it different from CV with MC.

Suppose $X_1, \dots, X_n$ are generated by QMC rule, the estimator stays the same

$$\hat{I}_{cv}(f) = \frac{1}{n} \sum_{i=1}^{n} \Big[f(X_i) - \beta_{qmc}[h(X_i) - \theta] \Big] \quad X_i \in \mathcal{U}(0, 1)$$

We can easily prove it is still unbiased

$$\mathbb{E}(\hat{I}_{cv}) = \mathbb{E}\Big(\frac{1}{n} \sum_{i=1}^{n} \Big[f(X_i) - \beta_{mc}[h(X_i) - \theta] \Big] \Big) = I$$

However, it's not the same case as MC like we presented before, because we do not have i.i.d for X_i this time

$$\mathrm{Var}_{qmc}(\hat{I}_{cv}) \neq \frac{1}{n}\mathrm{Var}\Big(f(X_i) - \beta_{mc}[h(X_i) - \theta]\Big)$$

Instead the variance become

$$\begin{aligned}
\mathrm{Var}\hat{I}_{cv}) &= \mathrm{Var}\Big(\hat{I} - \beta_{qmc}\hat{H}\Big) \quad s.t.\ \hat{I} = \sum_{i=1}^{n} f(X_i),\ \hat{H} = \sum_{i=1}^{n}[h(X_i) - \theta] \\
&= \mathrm{Var}(\hat{I}) - 2\beta_{qmc}\mathrm{Cov}(\hat{I}, \hat{H}) + \beta_{qmc}^2\mathrm{Var}(\hat{H}) \\
&= \mathrm{Var}(\hat{H})\Big(\beta_{qmc} - \frac{\mathrm{Cov}(\hat{I}, \hat{H})}{\mathrm{Var}(\hat{H})}\Big)^2 + \mathrm{Var}(\hat{I}) - \frac{\mathrm{Cov}(\hat{I}, \hat{H})}{\mathrm{Var}(\hat{H})}
\end{aligned}$$

The optimal β_{qmc} is

$$\beta_{qmc}^* = \mathrm{Var}(\hat{H})^{-1}\mathrm{Cov}(\hat{I}, \hat{H}) \tag{3.2}$$

which leave the variance to be

$$\mathrm{Var}_{qmc}(\hat{I}_{cv}) = \mathrm{Var}(\hat{I})\big(1 - \mathrm{corr}^2[\hat{I}, \hat{H}]\big)$$

Now we are interested that if our previous formula for $\hat{\beta}_{mc}$ could be an estimation for $\hat{\beta}_{qmc}$. The fact is that they are generally not the same. Let's take the covariance part of formula (3.2) and (2.2) to see the difference.

$$\begin{aligned}
\mathrm{Cov}(\hat{I}, \hat{H}) &= \int [f(X_1) + \cdots + f(X_n)][h(X_1) + \cdots + h(X_n)]\ d\mathbf{X} \\
&= \int [\sum_{i=1}^{n} f(X_i)h(X_i) + \sum_{i,j=1}^{i\neq j} f(X_i)h(X_j)]\ d\mathbf{X} \\
&\neq \int f(X_i)h(X_i)dX_i \\
&= \mathrm{Cov}[(f(X_i), h(X_i)].
\end{aligned}$$

There is also a very good example from Hickernell and Lemieux (2005) [3]'s paper, showing that β_{mc} and β_{qmc} can make a quite different results in some cases.

3.3 A new way to find β

As we stated in previous section, we can not find optimal β by minimizing variance of estimator like MC. However, if using the reliable adaptive QMC method introduced in chapter 2, we may have another way to find β.

Recall equation (2.3) the error bound for new estimator of g still holds

$$\Big|\int_{[0,1)^d} gdx - \hat{I}_m(g)\Big| \leq a(r,m) \sum_{\lfloor 2^{m-r-1} \rfloor}^{2^{m-r}-1} |\tilde{g}_{m,k}| \tag{3.3}$$

Naturally, the new estimator become

$$\hat{I}_m(g) := \frac{1}{b^m} \sum_{i=0}^{b^m-1} g(z_i + \Delta) \tag{3.4}$$

From (3.3) it is clear that the optimal β is the one that minimize the error term.

$$\begin{aligned}
\boldsymbol{\beta}^* &= \min_{\boldsymbol{\beta}} \sum_{\kappa=b^{m-r-1}}^{b^{m-r}-1} |\hat{g}_\kappa| && && (3.5)\\
&= \min_{\boldsymbol{\beta}} \sum_{\kappa=b^{m-r-1}}^{b^{m-r}-1} |\hat{f}_\kappa - (\hat{\boldsymbol{h}}_\kappa - \hat{\boldsymbol{\theta}})\boldsymbol{\beta}| && \hat{\boldsymbol{h}}_\kappa = (\hat{h}_{\kappa,1}, \dots, \hat{h}_{\kappa,J}), \hat{\boldsymbol{\theta}} = (\hat{\theta}_{\kappa,1}, \dots, \hat{\theta}_{\kappa,J}) && (3.6)\\
&= \min_{\boldsymbol{\beta}} \|\hat{\boldsymbol{f}} - \hat{\boldsymbol{H}}\boldsymbol{\beta}\|_1 && \hat{\boldsymbol{f}} = (\hat{f}_{b^{m-r-1}}, \dots, \hat{f}_{b^{m-r}-1)})^T && (3.7)\\
&\approx \min_{\boldsymbol{\beta}} \|\hat{\boldsymbol{f}} - \hat{\boldsymbol{H}}\boldsymbol{\beta}\|_2 && \hat{\boldsymbol{H}} = (\hat{\boldsymbol{H}}_1, \dots, \hat{\boldsymbol{H}}_J) && (3.8)\\
& && \hat{\boldsymbol{H}}_j = (\hat{h}_{b^{m-r-1},j} - \hat{\theta}_j, \dots, \hat{h}_{b^{m-r}-1,j} - \hat{\theta}_j)^T.
\end{aligned}$$

The second equivalence is not hard to get, but the third one may not be so obvious. Let's consider it backwards. Suppose we have a vector A and it's $\mathcal{L}_1$-norm.

$$A = \begin{pmatrix} f_1 - z_1 \\ f_2 - z_2 \\ \dots \\ f_n - z_n \end{pmatrix}, \quad \|A\|_1 = \sum_{i=1}^n |f_i - z_i|, \quad z_i := (\boldsymbol{h}_i - \boldsymbol{\theta})$$

If we replace the index,A is exactly what's inside the $\mathcal{L}_1$-norm in (3.7). Hence we justified the third equivalence. The reason we use an approximation instead, i.e. the $\mathcal{L}_1$-norm, is because there is no efficient way to solve it compared to existing least square methods.

3.4 The problem with θ

We noticed a problem in solution for optimal β, which is we do a lot of subtractions with θ. This could be a large cost when we have difficult functions which means b^{m-r} could be very large number. Therefore we present a way to avoid that part.

The idea is form a observation that Walsh transform of θ in (3.6) is actually zero, since $\hat{h}_\theta = \theta\delta_{\kappa,0}$ and the summation is not start from $\kappa = 0$.

This simplifies (3.6) to the following. Note that we only need the information of function f and h to calculate β^*, θ has been get rid of the optimization process.

$$\begin{aligned}
\boldsymbol{\beta}^* &= \min_{\boldsymbol{\beta}} \sum_{\kappa=b^{m-r-1}}^{b^{m-r}-1} |\hat{f}_\kappa - (\hat{\boldsymbol{h}}_\kappa - \hat{\boldsymbol{\theta}})\boldsymbol{\beta}| && (3.9)\\
&= \min_{\boldsymbol{\beta}} \sum_{\kappa=b^{m-r-1}}^{b^{m-r}-1} |\hat{f}_\kappa - (\hat{\boldsymbol{h}}_\kappa - \boldsymbol{\theta}\delta_{\kappa,0})\boldsymbol{\beta}| \\
&= \min_{\boldsymbol{\beta}} \sum_{\kappa=b^{m-r-1}}^{b^{m-r}-1} |\hat{f}_\kappa - \hat{\boldsymbol{h}}_\kappa\boldsymbol{\beta}| && \hat{\boldsymbol{f}} = (\hat{f}_{b^{m-r-1}}, \dots, \hat{f}_{b^{m-r}-1)})^T \\
&= \min_{\boldsymbol{\beta}} \|\hat{\boldsymbol{f}} - \hat{\boldsymbol{H}}\boldsymbol{\beta}\|_1 && \hat{\boldsymbol{H}} = (\hat{\boldsymbol{H}}_1, \dots, \hat{\boldsymbol{H}}_J) \\
&\approx \min_{\boldsymbol{\beta}} \|\hat{\boldsymbol{f}} - \hat{\boldsymbol{H}}\boldsymbol{\beta}\|_2 && \hat{\boldsymbol{H}}_j = (\hat{h}_{b^{m-r-1},j}, \dots, \hat{h}_{b^{m-r}-1,j})^T
\end{aligned}$$

The same problem happened with the estimator (3.4). We have the similar solution for that.

$$\begin{aligned}
\hat{I}_m(g) &= \frac{1}{b^m}\sum_{i=0}^{b^m-1} g(z_i+\Delta) \\
&= \frac{1}{b^m}\sum_{i=0}^{b^m-1} f(z_i+\Delta) - (\boldsymbol{h}(z_i+\Delta)-\boldsymbol{\theta})\boldsymbol{\beta} \\
&= \frac{1}{b^m}\sum_{i=0}^{b^m-1} [f(z_i+\Delta) - \boldsymbol{h}(z_i+\Delta)\boldsymbol{\beta}] + \boldsymbol{\theta}\boldsymbol{\beta} \qquad (3.10)
\end{aligned}$$

After organize it the in format of (3.10), θ is eliminated from the summation part. From these two parts of work on θ we managed to save $\frac{b-1}{b}b^{m-r}+b^m$ operations of subtraction.

3.5 The modified method

Now we make the following changes:

$$\begin{aligned}
g &= f - \beta h \\
\hat{I}_m(g) &= \frac{1}{b^m}\sum_{i=0}^{b^m-1} g(z_i+\Delta).
\end{aligned}$$

And we have the following equivalence:

$$\begin{aligned}
\int_{[0,1)^d} f dx &= \int_{[0,1)^d} g dx + \theta\beta \\
\hat{I}_m(f) &= \hat{I}_m(g) + \theta\beta.
\end{aligned}$$

So the estimation error becomes:

$$\Big|\int_{[0,1)^d} f dx - \hat{I}_m(f)\Big| = \Big|\int_{[0,1)^d} g dx - \hat{I}_m(g)\Big|.$$

Here if our g is in the cone we introduced earlier (2.4), then we can use the results from Hickernell and Jimnez Rugama(2014) [1], the error is bounded by:

$$\Big|\int_{[0,1)^d} g dx - \hat{I}_m(g)\Big| \le a(r,m)\sum_{\lfloor 2^{m-r-1}\rfloor}^{2^{m-r}-1} |\tilde{g}_{m,k}|.$$

This leads to the same algorithm suggested by Hickernell and Jimnez Rugama(2014) [1], but since we are using control variates, several modifications have to be made.

3.6 The Algorithm

We now give the algorithm for reliable adaptive QMC with control variates using digital sequence.

Algorithm 1: Reliable Adaptive QMC with control variates

Data: function f and $\boldsymbol{H}$; value of $\int_{[0,1)^d} h_j dx = \theta_j$; tolerance ε

Result: $\hat{I}(f)$; samples size; optimal β

begin

1 $m, r =$ start numbers, $x = 2^m$ sobolset points

2 get kappa map($\tilde{\kappa}$) and Walsh coefficients($\tilde{f}, \tilde{\boldsymbol{H}}$) using algorithm 2

3 $\boldsymbol{\beta} = \tilde{H}\{\tilde{\kappa}[x(a:b)]\} \backslash \tilde{f}\{\tilde{\kappa}[x(a:b)]\}, (a:b) = (2^{m-r-1} : 2^{m-r} - 1)$

4 $g = f - \boldsymbol{H}\boldsymbol{\beta}$, repeat step 2 on g

5 $\tilde{S}_{m-r,m}(g) = \sum_a^b \left|\tilde{g}\{\tilde{\kappa}[x(a:b)]\}\right|$

6 check whether g is in the cone

7 **if** $a(m,r)\tilde{S}_{m-r,m}(g) \le \varepsilon$ **then**

  return $\hat{I}_m(g) = \sum_{i=0}^{2^m-1} f[x(i)] + \boldsymbol{\theta}\boldsymbol{\beta}$

  return $\boldsymbol{\beta}, n = 2^m$

8 **for** $m = m+1 : mmax$ **do**

  $xnext =$ next 2^{m-1} sobolset points

  repeat step 2 on $[x, xnext]$

  repeat step 5, 6, 7

Note that for generating kappa map, i.e. step 2 in Algorithm 1, we used an explicit way to generate it. One can find the details for that in appendix from Hickernell and Lluís(2014) [1]. Here we reorganize it and show it in algorithm 2.

Another important point need to be mentioned is that in our algorithm, we used an iterative way, which may require recalculating *beta* for each iteration.

Algorithm 2: kappa map and discrete Walsh coefficients

Data: function f; $Y_v^{(m)}$; $m \in \mathbb{N}_0$

Result: $\tilde{\kappa}$; $\tilde{S}_{m-r,m}(f)$

begin

- **if** $m = 0$ **then**
  - $\mathring{\mathbf{v}}(0) = 0$
- **if** $m \geq 1$ **then**
  - **for** $m : 1 : -1$ **do**
    - $\mathring{\mathbf{v}}_m(\mathbf{k}) = \mathring{\mathbf{v}}_m - 1(\mathbf{k})$
    - $\mathring{\mathbf{v}}_m(\mathbf{k}) = \mathbf{k},\ \mathbf{k} = b^{m-1}, \dots, b^m - 1$
- **for** $l = m - 1 : max(1, m - r) : -1$ **do**
  - **for** $k = 1 : b^l - 1$ **do**
    - $\forall a \in \mathbb{F}_b$, find a s.t. $|\,Y^{(m)}_{\mathring{\mathbf{v}}(k+a^*b^l)}| \geq |\,Y^{(m)}_{\mathring{\mathbf{v}}(k+ab^l)}|$

CHAPTER 4

NUMERICAL EXPERIMENT

Option Pricing has always been a challenging topic in financial mathematics. Although there are several other methods for pricing options, Monte Carlo performs better when solving high dimension problems. In this chapter we make several tests our reliable QMC with CV algorithm with option pricing problems. Note all our experiments are implemented under brownian motion (GBM) pricing model on non dividend paying stock. Since the GBM model is a well known model for option pricing, we only lay out the parameters for option formulas we use later and not dig into the model itself.

$$\begin{aligned}
S(jT/d) &= \text{current asset price at time } jT/d,\ j = 1, \dots, d \\
K &= \text{strike price} \\
T &= \text{expiration time} \\
\sigma &= \text{volatility} \\
r &= \text{interest rate} \\
d &= \text{number of time steps.}
\end{aligned}$$

4.1 Accuracy

The first thing we want to test is whether our algorithm provides the 'accurate' solution. This means if the function satisfies the cone condition, the difference between our estimation and true value should be bounded by the pre-defined error tolerance. We will test our algorithm for accuracy in three cases: original reliable adaptive QMC (RAQMC), reliable adaptive QMC with CV (RAQMC_CV) for single CV and RAQMC_CV for double CV.

To do this we have to know the exact value of our integral to calculate the exact error of our results. Therefore, we choose geomatric mean Asian option as our target function because they have exact solution for it under GBM model [10]. We choose to test call options, so for geomatric mean Asian call option the payoff function is:

$$C_T^{\text{gmean}} = \max\Big(\Big[\prod_{j=1}^{d} S(jT/d)\Big]^{\frac{1}{d}} - K, 0\Big)e^{-rT}.$$

The closed formula for exact price of it under GBM model is:

$$\begin{aligned} C_T^{\text{gmeanExact}} &= S(0)e^{-(r+\sigma^2/2)T/2}\Phi(\tilde{d}_1) + Ke^{-rT}\Phi(\tilde{d}_2) \\ \tilde{d}_1 &= \frac{\ln(S(0)/K) + (r + \frac{n-1}{6(n+1)}\sigma^2/2)T/2}{\sqrt{\frac{2n+1}{6(n+1)}}\sigma\sqrt{T}} \\ \tilde{d}_2 &= \frac{\ln(S(0)/K) + (r - \sigma^2/2)T/2}{\sqrt{\frac{2n+1}{6(n+1)}}\sigma\sqrt{T}}. \end{aligned} \tag{4.1}$$

Then we need a couple control variates of which the exact solution must have clear formulas. Here we pick European call option and stock price at expiration as control variates. For European call option, the payoff function is:

$$C_T = \max(S(T) - K, 0)e^{-rT}.$$

The exact price of it under GBM model is given by [11]:

$$\begin{aligned} C_T^{\text{euroExact}} &= S(0)\Phi(d_1) - Ke^{-rT}\Phi(d_2) \\ d_1 &= \frac{\ln(S_0/K) + (r + \sigma^2/2)T}{\sigma\sqrt{T}} \\ d_2 &= \frac{\ln(S_0/K) + (r - \sigma^2/2)T}{\sigma\sqrt{T}}. \end{aligned}$$

For stock price, the payoff function is:

$$C_T = S(T)e^{-rT},$$

while exact price of it under GBM model is just $S(0)$.

Table 4.1 shows the set up for all options in the accuracy test. We use in the money call option for both Asian and European options. They are monitored weekly

Table 4.1: Parameter Setup for accuracy test

S0	K	TimeVector	r	volatility
120	100	1/52:1/52:4/52	0.01	0.5

for one month period with 1% interest and 50% volatility anunally. The test is then perform with three different absolute error tolerances as shown in table 4.2. First we caculated the exact price for the geomatric mean Asian call option using formula (4.1). Then we run the same parameters through our reliable QMC algorithm to get the estimates. Finally, we caluate the absolute value of difference between estimates and true prices as errors. We used three different absolute error tolerance set ups for this: $10^{-2}, 10^{-3}, 10^{-6}$. From the results we can see all errors lied within the preset tolerance. This give us confidence for our algorithm as accurate as guranteed. Note for second test, which the tolerance is 10^{-3}, while the error seems all within a lesser bound 10^{-4}. We believe this is not an error for the algorithm or the code. It happened since each iteration our sample size is doubled and this could be more than satisfied for the predefined tolerance level.

4.2 Efficiency

Now that we know our algorithm provides the desired results, we will go on test the time efficiency of our algorithm. Here we can not talk about time complexity using big O notation for the results depend not only on dimention but also on target function itself. Instead, we do experiments to test the 'efficiency' of our algorithm by compare the sample size used for caculation and the corresponding time cost. Note the results do not stay the same for each time, to compensate that we take an average of 10 consecutive runs with the same set-up. This test breaks into two parts. The first one is that we want to know how our algorithm performs without using control

Table 4.2: Accuracy Test of RAQMC_CV algorithm

abstol = 10^{-2}	RAQMC	RAQMC_CV$_1$	RAQMC_CV$_2$
exact price	1.2926641930	1.2926641930	1.2926641930
estimate price	1.2938177355	1.2937890386	1.2930809161
err=|exact-estimate|	1.1535e-3	1.124e-3	4.167e-4
abstol = 10^{-3}	RAQMC	RAQMC_CV$_1$	RAQMC_CV$_2$
exact price	1.2926641930	1.2926641930	1.2926641930
estimate price	1.2926529108	1.2925687148	1.2925793049
err=|exact-estimate|	1.12821e-5	9.54782e-5	8.48881e-5
abstol = 10^{-6}	RAQMC	RAQMC_CV$_1$	RAQMC_CV$_2$
exact price	1.2926641930	1.2926641930	1.2926641930
estimate price	1.2926643000	1.2926639635	1.296646297
err=|exact-estimate|	1.07e-7	2.295e-7	4.367e-7

variates. Naturally the RAQMC algorithm we introduced in chapter 3 become a good reference. We know our algorithm is a bit slower compared to it when not using control variates. The question is how small the gap is. If there is no significant difference, then it means we will not waste too much time on cases without control variates or with poor control variates. The second one, which is more important, is that we want to see how much time it can save for using good control variates. Of course this depends on quality of the control variates. Fortunately, several good CV is known to be use for option pricing under GBM model [12].

4.2.1 RAQMC_CV without CV. Parameter set up for efficiency test is shown in table 4.3. This time we try to price a daily monitored four months period option, which increases the dimension to 112. Note we will keep using this set up for all the folloing efficienty tests. We test our algorithm through four scenarios: RAQMC,

Table 4.3: Parameter Setup for efficienty test

S0	K	TimeVector	r	volatility	abstol	reltol
120	130	1/354:1/354:112/354	0.01	0.5	1e-3	0

RAQMC_CV without CV, RAQMC_CV with one poor CV and RAQMC_CV with two poor CV. For target option we still use geomtric mean Asian option European option. For single CV we use European option and add stock price for double CV. As

Table 4.4: Efficiency test I of RAQMC_CV

abstol= 10^{-3}	RAQMC	RAQMC_CV$_0$	RAQMC_CV$_1$	RAQMC_CV$_2$
Sample Size	65535	65535	65535	65535
Time Cost	1.8365	1.8379	1.8808	1.8703

shown in table 4.4, all test end up with same sample size, which means that choice of CV didn't help at all. But what we concern is that how much extra cost can it bring if our choice of CV is bad? We can see the time cost just increased around 2% for QMC with bad CV. Now we can say our method did make the cost of CV to a minimal.

4.2.2 RAQMC_CV with CV.

There are two types of asian options, depends on which types of mean one

Table 4.5: Efficiency test II of RAQMC_CV algorithm with Asian Option

	Estimate Price	Sample Size	Time Cost	β^*
RAQMC	4.2011	65535	1.9054	null
RAQMC_CV$_1$	4.2008	15564.8	0.49966	0.995
RAQMC_CV$_2$	4.2018	13926.4	0.43793	[0.958;0.005]

use. In last section we introduced geomatric mean Asian option, this time we choose arithematic mean asian call option as our target function, whose payoff function is:

$$C_T^{Amean} = \max\left(\frac{1}{d}\sum_{j=1}^{d} S(jT/d) - K, 0\right)e^{-rT}.$$

For this option there is no close formula for exact price under GBM model. However, recall we introduced another Asian option earlier of which the price has a clear formula. It is known that geomatric mean Asian option was first used as a CV for pricing arithmatic mean Asian option [10]. Hence, for single CV we choose geomatric mean Asian call option as CV. We also want to test it for double CV so we add European call option as a second CV. Results are shown in table **??**. We can see this choice of CV is really good. The sample size is approximately 1/4 of original size and same with time cost. The second CV didn't make an obvious improvement compared to single CV but seems helped a bit.

Figure 4.1 is a plot of the walsh coefficients for the target function and control variates in this Asian options experiments. Recall our error of approximation is bound on this exact same coeffcients. What we'd expect is that on both method it can decrease fast and for CV method we expect it to decrease faster. The horizontal axis in the figure can be interpreted as sample size, the vertical axis can be view as the error bound. What our algorithm does is that for a give error bound, we monitor

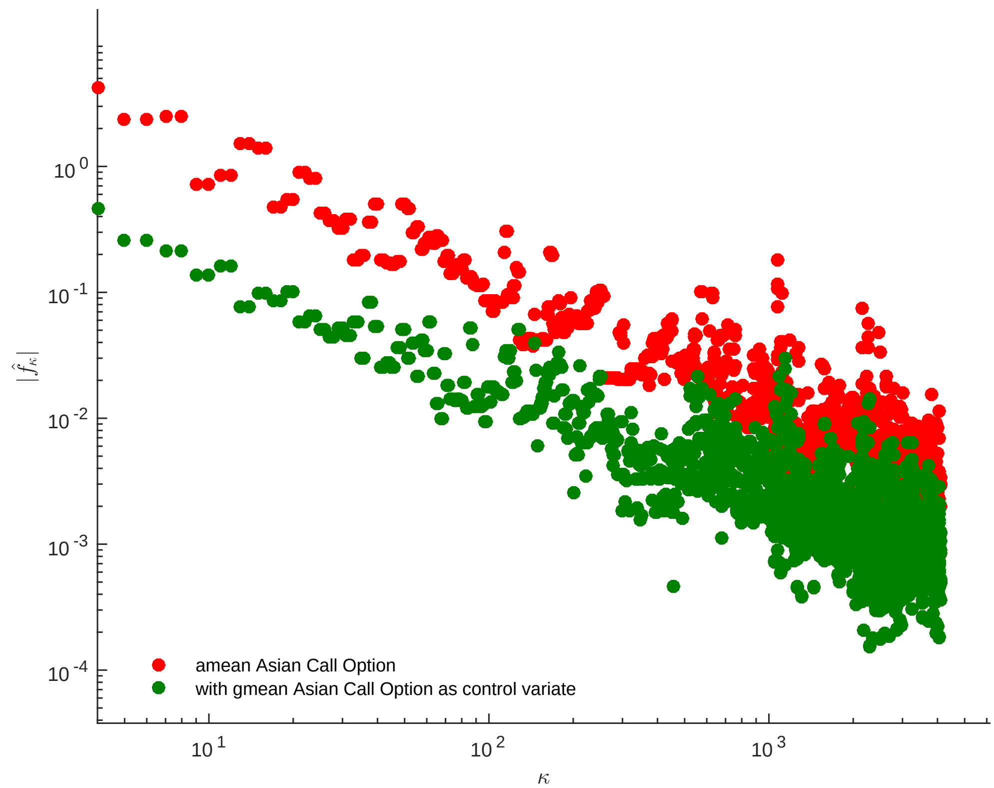


Figure 4.1: Walsh coefficients of f

these Walsh coefficients untill it reaches the bound. Obviously, the CV method will stop early in this case which means less compuational cost for it.

4.2.3 Barrier Option. Another good example is barrier option. This is the payoff function for up and in barrier call option:

$$C_T^{\mathrm{U\&I}} = [S(T) - K]^+ 1_{\{\max S(jT/d) \geq \mathrm{Barrier}\}}.$$

The reason we choose it is for its similaritis to the European option. Note if the barrier equals strike price (130), then it is just a European call option, which makes european option a good control variates. For this test we take four different barriers that gradually decrease to the strick price. As same from last test, we compared oringinal reliable QMC algorithm and our CV version on sample size and time cost.

One merit for this example is that we can see how the cost and value of *beta* change

Table 4.6: Efficiency test II of RAQMC_CV algorithm with Barrier Option

Barrier	Sample Size		Time Cost		β^*
	RAQMC	RAQMC_CV	RAQMC	RAQMC_CV	
160	1048576	1048576	49.9861	50.8245	0.6613
150	1048576	524288	47.8742	23.2743	0.8999
140	524288	32768	22.7936	1.0146	0.9935
130	524288	1024	22.3675	0.0826	1.0000

as we change the barrier. In this example we know the optimal value should be close to 1 as shown in table 4.6 when barrier=130. Also, note we set the start sample size for iteration to be 2^{10}, which is exactly the sample size used in this case. This is consistant with our statement earlier, it is just a European option. When we move barrier further from 130, we assume the European option will not be as good as the pervious one as a CV. Our results confirmed this assumption, we can see in table 4.6 both the time cost and sample size rise as we increase the barrier. At last when barrier=160, European option completely lost its function as a CV and cost of two methods are almost the same.

CHAPTER 5

CONCLUSION

5.1 Discussion

So far there are only few QMC algorithms that can adaptively determine the sample size needed based on integrand values. This is because the estimation of error for QMC is hard. Several studies show that if using quasi standard error there will be some serious drawbacks [9]. There is also a way using internal replications of i.i.d. Randomized QMC rules, but the number of replications are not known [3].

For control variates with QMC the research progress is also limited since its hard to estimate the value of β as we stated in chapter 3. Hickernell and Lluís (2014) [1]'s work on building a QMC method provided a reliable and adaptive way to use QMC, as well as gave us insights into combining reliable QMC with control variates. The main idea of this reliable QMC is to bound the error of estimation using summation of part Walsh coefficients. We utilized the same idea for calculating the optimal coefficient for control variates. In order to compensate the extra computation cost for control variates, we used several technique in our design to keep those cost minimal.

We tested our algorithm on several option pricing problem under Black-Scholes scheme. We found the accuracy of algorithm is consistent and reliable. Comparison of QMC with CV and normal QMC is performed on different pairs of options, the results show that using the proper control variates can have great improvement over normal QMC methods.

5.2 Future work

Currently we only have the method implemented in sobol sequences. One further possible work is that we can extend this method on rank-1 lattices [13]. The idea for getting CV coefficients is the same, but due to the different structure for digital sequences, some effort has to be done for adaption of the method.

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