

RELIABLE QUASI-MONTE CARLO
WITH CONTROL VARIATES

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ABSTRACT

Recently, Quasi-Monte Carlo (QMC) methods have been implemented in a guaranteed adaptive algorithm. This raises the possibility of combining adaptive QMC with efficiency improvement techniques for independent and identically distributed (i.i.d.) Monte Carlo (MC) such as control variates (CV).

The challenge for adding control variates to QMC is that optimal control variate coefficient for QMC is generally not the same as that for MC. Here we propose a method for implementing control variates in a guaranteed adaptive QMC algorithm. One merit of using CV with MC is that theoretically the efficiency is always no worse than vanilla MC. Our method is implemented in an efficient way so that the extra cost for CV is tolerable.

We test our algorithm on various problems including option pricing and multivariate normal probability for dimensions from 4 to 64. Same tests are performed on adaptive QMC algorithm without CV as a comparison. Our results show that with good CV, the cost of adaptive QMC is greatly reduced compared to vanilla QMC.

CHAPTER 1

INTRODUCTION

1.1 The idea

Recently there are some great results from construction of Quasi-Monte Carlo (QMC) methods that can adaptively choose a sample size for given error tolerances [1]. Our work try to combine the reliable adaptive QMC (RAQMC) methods with control variates (CV). We will justify the theory behind it, construct a practical algorithm which can be implemented and tested through high dimensional integration examples. CV is a variance reduction technique for independent and identically distributed (i.i.d.) Monte Carlo (MC) methods. QMC can be viewed as a deterministic version of IID MC, which outperforms MC for many integrals [2]. Naturally we wonder if QMC can also benefit from the CV technique. If that is possible, it can be especially useful for problems where we can easily find good CV.

1.2 The challenge

The challenge is that for QMC the quadrature points are deterministic instead of random, so the variance minization can not be used. Even if one tries to use the randomized QMC, the optimal control variate coefficient for QMC is generally not the same as for IID MC as explained by Hickernell, Lemieux, and Owen [3]. This requires us to figure out a new way to get the optimal coefficients for CV with QMC.

1.3 Outline

In chapter 2 we first briefly talk about QMC rules and their difference from IID MC. Then we introduce CV and the RAQMC algorithm. In Chapter 3 we derive our method and provide its justification. In chapter 4 we demonstrate results from several numerical experiments. For the final chapter we conclude the results and discuss unsolved problems as well as possible improvements to our method.

CHAPTER 2

BACKGROUND

2.1 Problem setup

Numerical integration problems are involved in fields such as physics, mathematical finance, biology, computer graphics, and many others fields. It usually happens when it is hard to solve some integral analytically. Therefore, one has to use numerical methods for such problems. MC method is a general way to solve problems in such case [4]. The method can be simply explained in the following way.

Suppose we have the following standard integration approximation problem whose format is:

$$I = \int_{[0,1]^d} f(\mathbf{x}) \, d\mathbf{x}. \quad (2.1)$$

Then we take a sample of N IID points $\{\mathbf{X}_1, \dots, \mathbf{X}_N\} \in [0, 1]^d$ following a uniform distribution randomly, and construct the following MC estimator:

$$\hat{I}(f) = \frac{1}{N} \sum_{i=1}^N f(\mathbf{X}_i).$$

However, there are several problems with the IID MC method [5]. First, it is difficult to generate truly random samples. Second, an error bound for IID MC works only in probabilistic sense. Last, in many applications the convergence rate of MC method is considered not fast enough.

Hence, the QMC method were introduced to address these problems. The QMC estimator is almost the same as MC estimator. The difference is that the sample points are taken from low discrepancy sequence, which is deterministically chosen instead of random. We will briefly review one type of such sequence that we used for implementation of our method.

2.2 Sobol sequence

Like we mentioned earlier, for QMC we do not use random points, so the question become how to choose such points? Naturally, in order to make good approximation we want the error to go to 0 as sample size N increases, which is:

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{i=1}^N f(\mathbf{x}_i) = \int_{[0,1]^d} f(\mathbf{x}) \, d\mathbf{x}.$$

The answer for how to choose such sample points leads to the theory of uniformly distributed modulo one [6]. One important property of sequences satisfy such theory is called ‘fair’ interval.

Definition 2.2.1. Let $\mathcal{P} = \{\mathbf{X}_1, \dots, \mathbf{X}_N\}$ be a finite point set in $[0, 1]^d$. For any given subset $J = [\mathbf{a}, \mathbf{b}]$ of $[0, 1]^d$, we say J is *fair* if:

$$\frac{\sum_{i=1}^N 1_{[\mathbf{a}, \mathbf{b}]}(\mathbf{X}_i)}{N} = \prod_{j=1}^d (b_j - a_j)$$

This simply means that a measurable subset J of a unit cube is fair if the number of sample points in it (left side of equation) is equal to its volume (right side of equation). Similarly, we say a sequence \mathcal{P} is fair if any given subset J is fair. Such fair sequences are ideal for our purpose. Unfortunately, it is not attainable unless the demand for all intervals to be fair is weakened. This motivated the definition of (t, d) -sequence.

Definition 2.2.2. For a given dimension $d \geq 1$, an integer base $b \geq 2$, a positive integer m , and an integer t with $0 \leq t < m$, a point set $\mathcal{P} = \{\mathbf{X}_1, \mathbf{X}_2, \dots\}$ in $[0, 1]^d$ is called a (t, d) -sequence in base b if the point set $\{\mathbf{X}_{kb^m+1}, \dots, \mathbf{X}_{kb^m+b^m}\}$ with all $m > t$, $k \geq 0$ is fair with respect to all intervals of the following form:

$$J = \prod_{i=1}^d \left[\frac{a_i}{b^{s_i}}, \frac{a_i + 1}{b^{s_i}} \right),$$

where $s_1, \dots, s_d \in \mathbb{N}_0$ with $s_1 + \dots + s_d = k$ and $0 \leq a_i < b^{s_i}$

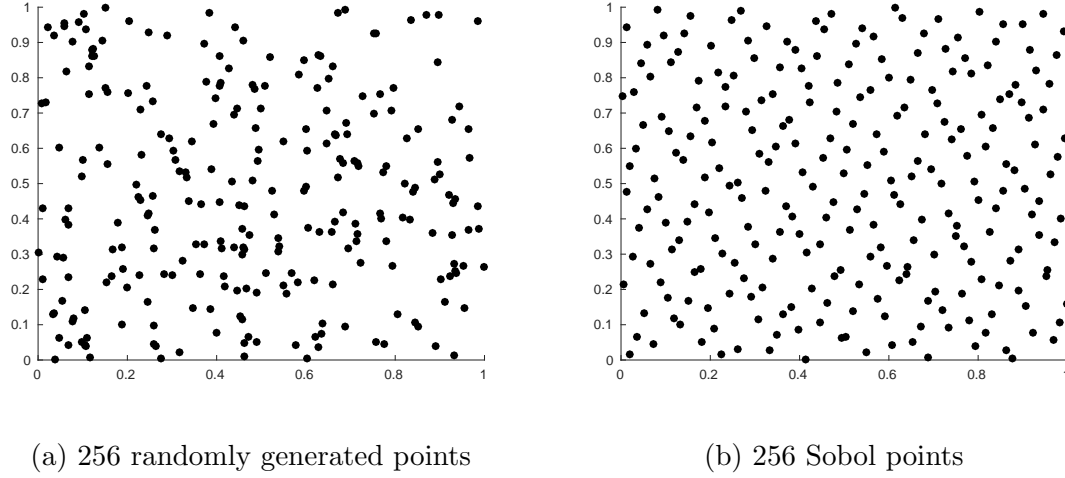


Figure 2.1: Comparison between MC and QMC sample points

Sobol sequence is the first constructed (t, d) -sequence in base 2 [6]. Another well known (t, d) -sequence is Faure sequence, which is a $(0, d)$ -sequences in a base $\geq d$. The advantage of Sobol sequence is that it has a small base. Later in next chapter we will see this is a great help since the sample size is in form of b^m . Also since we use computer for simulations, we gain some advantages on bit-level operations in base 2. Figure 2.1 shows a comparison between randomly generated sample points and Sobol sequence points. The latter looks more evenly scattered. In fact by the definition of (t, d) -sequence if we divide the unit square into 256 grids evenly ($1 \times 256, 2 \times 128, \dots$), there will be exactly one point in each grid.

2.3 Control variates

CV is a well known variance reduction technique used in MC simulation. It is often used when a 'simpler' version of the origin problem can be solved explicitly. In this section we briefly review the ideas and main results of the method.

Suppose we want to solve the integration problem (2.1) showed earlier, now we have a known function h and its value on the interval $\int_{[0,1]^d} h(\mathbf{x}) \, d\mathbf{x} = \theta$. We then

construct a new estimator as the following:

$$\hat{I}_{CV}(f) = \frac{1}{N} \sum_{i=1}^N [f(\mathbf{X}_i) - \beta_{MC}[h(\mathbf{X}_i) - \theta]] \quad s.t. \mathbf{X}_i \sim \mathcal{U}[0, 1]^d, IID.$$

We can easily see it's an unbiased estimator, i.e. $\mathbb{E}(\hat{I}_{CV}) = I$. Now the question is how should we choose β_{MC} and why. The idea is rather straightforward. We know that the mean square error of the MC estimator is $\text{Var}(\hat{I}) + \text{Bias}(\hat{I}^2)$. CV method aims at efficiency improvement, so we need to reduce the mean square error. Since the estimator is unbiased, we only need to minimize its variance. Hence, the optimal β_{MC} should be the one that minimize the variance of estimator. Here we give a simple derivation of optimal β_{MC} for single CV. First, the variance of \hat{I}_{CV} is:

$$\begin{aligned} \text{Var}(\hat{I}_{CV}) &= \text{Var}\left(\frac{1}{N} \sum_{i=1}^N [f(\mathbf{X}_i) - \beta_{MC}[h(\mathbf{X}_i) - \theta]]\right) \\ &= \frac{1}{N} \text{Var}\left(f(\mathbf{X}_i) - \beta_{MC}[h(\mathbf{X}_i) - \theta]\right) \quad \text{by } X_i \text{ i.i.d} \\ &= \frac{1}{N} \mathbb{E}\left([f(\mathbf{X}_i) - \beta_{MC}[h(\mathbf{X}_i) - \theta] - I]^2\right) \\ &= \frac{1}{N} \mathbb{E}\left([f(\mathbf{X}_i) - I] - \beta_{MC}[h(\mathbf{X}_i) - \theta]\right)^2 \\ &= \frac{1}{N} \mathbb{E}\left([f(\mathbf{X}_i) - I]^2 - 2\beta_{MC}[f(\mathbf{X}_i) - I][h(\mathbf{X}_i) - \theta] + \beta_{MC}^2[h(\mathbf{X}_i) - \theta]^2\right) \\ &= \frac{1}{N} \left(\text{Var}[f(\mathbf{X}_i)] - 2\beta_{MC}\text{Cov}[f(\mathbf{X}_i), h(\mathbf{X}_i)] + \beta_{MC}^2\text{Var}[h(\mathbf{X}_i)]\right) \\ &= \frac{1}{N} \left(\text{Var}[h(\mathbf{X}_i)]\left(\beta_{MC} - \frac{\text{Cov}[f(\mathbf{X}_i), h(\mathbf{X}_i)]}{\text{Var}[h(\mathbf{X}_i)]}\right)^2 + \right. \\ &\quad \left. \text{Var}[f(\mathbf{X}_i)] - \frac{\text{Cov}^2[f(\mathbf{X}_i), h(\mathbf{X}_i)]}{\text{Var}[h(\mathbf{X}_i)]}\right), \end{aligned}$$

then the optimal β_{MC} is given by:

$$\beta_{MC}^* = \frac{\text{Cov}[f(\mathbf{X}_i), h(\mathbf{X}_i)]}{\text{Var}[h(\mathbf{X}_i)]}. \quad (2.2)$$

In this case the variance become:

$$\text{Var}(\hat{I}_{CV}) = \frac{\text{Var}[f(\mathbf{X}_i)]}{N} (1 - \text{corr}^2[f(\mathbf{X}_i), h(\mathbf{X}_i)]),$$

and note we always have:

$$\text{Var}(\hat{I}_{\text{CV}}) \leq \frac{\text{Var}[f(\mathbf{X}_i)]}{N} = \text{Var}(\hat{I}).$$

Now we can see the merit of control variates as a variance reduction method. In the worst case, we get a completely uncorrelated g that leads correlation to zero, and we have variance exactly the same as not using control variates. On the other hand, the more correlated our control variates is to the target function, the more variance we can get rid of by using the method.

2.4 Reliable adaptive QMC with digital sequence

One practical problem for QMC method is that how to get the sample size big enough for a required error tolerance. The idea in work of Hickernell and Jiménez Rugama (2014) [1] is to construct a QMC algorithm with reliable error estimation on digital sequence. Here we briefly summarize their results.

The error of QMC method on digital sequence can be expressed in terms of Walsh coefficients of the integrand on certain cone conditions.

$$\text{if } f \in \mathcal{C} \text{ then } \left| \int_{[0,1]^d} f(\mathbf{x}) \, d\mathbf{x} - \hat{I}_m(f) \right| \leq a(r, m) \sum_{[2^{m-r-1}]}^{2^{m-r}-1} |\tilde{f}_{m,k}| \quad (2.3)$$

$$\hat{I}_m(f) := \frac{1}{b^m} \sum_{i=0}^{b^m-1} f(\mathbf{X}_i) \quad \mathbf{X}_i \text{ are Sobol points}$$

$$\tilde{f}_{m,k} = \text{discrete Walsh coefficients of } f$$

$$a(r, m) = \text{inflation factor that depends on } \mathcal{C}.$$

Here is the defination of the cone condition:

$$\begin{aligned} \mathcal{C} &:= \left\{ f \in L^2[0, 1]^d : \bigcirc \leq \hat{\omega}(m-l)\diamond, l \leq m; \quad \diamond \leq \hat{\omega}(m-l)\square, l^* \leq l \leq m \right\} \\ \bigcirc &:= \sum_{\kappa=\lfloor b^{l-1} \rfloor}^{b^l-1} \sum_{\lambda=1}^{\infty} |\hat{f}_{\kappa+\lambda b^m}|, \quad \square := \sum_{\kappa=b^{l-1}}^{b^l-1} |\hat{f}_{\kappa}|, \quad \diamond := \sum_{\kappa=b^m}^{\infty} |\hat{f}_{\kappa}| \end{aligned} \quad (2.4)$$

$l^* \in \mathbb{N}$ be fixed ; $\forall m \in \mathbb{N}, \hat{\omega}(m), \hat{\omega}(m) \geq 0$, and $\lim_{m \rightarrow \infty} \hat{\omega}(m) = 0$, $\lim_{m \rightarrow \infty} \hat{\omega}(m) = 0$.

The first inequality($\bigcirc \leq \diamond$) means the sum of the larger indexed Walsh coefficients bounds a partial sum of the same coefficients. Take $l = 0, m = 12$ for example, in Figure 2.2 the sum of circles should be bounded by some factor times the sum of diamonds. The second inequality($\diamond \leq \square$) requires the sum of the larger Walsh coefficients be bounded by the sum of smaller indexed Walsh coefficients. Take $l = 8$ at this time, which means in Figure 2.2 the sum of diamonds should be bounded by some relax factor times the squares.

The cone gives some meanings for the functions about how they should behave to get the error bound formula (2.3). This means that $|\hat{f}_{\kappa}|$ does not dramatically bounce back as κ goes to infinity. Note that in Figure 2.2 we call circles the error bound, this is proven to be true and under the cone conditions we can estimate it using discrete Walsh coefficients instead of true Walsh coefficients.

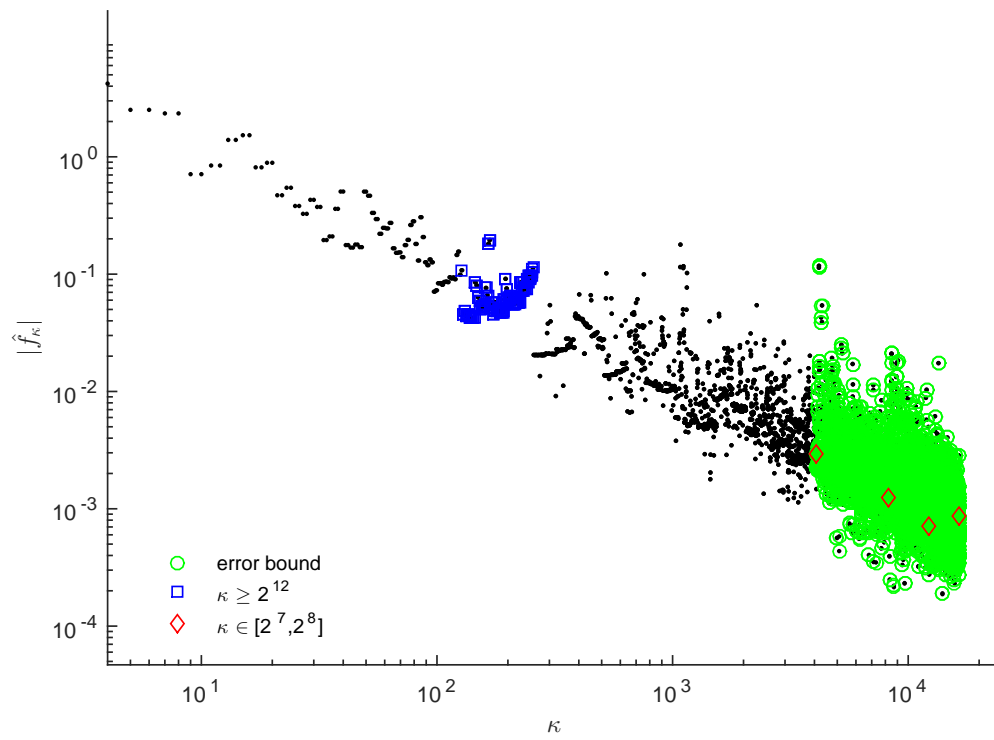


Figure 2.2: Cone condition for reliable adaptive QMC algorithm

CHAPTER 3

RELIABLE ADAPTIVE QMC WITH CV

3.1 Idea to add control variates to reliable adaptive QMC

The whole method starts with an idea similar to traditional control variates technique for MC. If we know the integration of a function $\mathbf{h} = (h_1, \dots, h_J)$ on the interval same as our f , say $\int_{[0,1]^d} h_j d\mathbf{x} = \theta_j$, then we can define a new function g :

$$\begin{aligned} g &:= f - (\mathbf{h} - \boldsymbol{\theta})\boldsymbol{\beta} \\ \text{s.t. } \boldsymbol{\theta} &= (\theta_1, \dots, \theta_J), \boldsymbol{\beta} = (\beta_1, \dots, \beta_J)^T. \end{aligned} \tag{3.1}$$

Then easily we can find that if we replace f with g , the integration stays the same

$$\int_{[0,1]^d} g d\mathbf{x} = \int_{[0,1]^d} f - (\mathbf{h} - \boldsymbol{\theta})\boldsymbol{\beta} d\mathbf{x} = \int_{[0,1]^d} f d\mathbf{x}.$$

Now we wonder if we can still use the same method as MC, the answer is no and the reason is in the next section.

3.2 The problem of CV with random QMC

The problem is that QMC is not a random process and we simply can't use the minimizing mean square error trick as shown earlier anymore. However, one can use random QMC instead to 'restore' the randomness to QMC [7]. Randomized QMC use a different way for generating \mathbf{X}_i , they are still identical(i.e. from same distribution) but not independent, which will make it different from CV with MC.

Suppose $\{\mathbf{X}_1, \dots, \mathbf{X}_N\}$ are generated by QMC rule, the estimator stays the same

$$\hat{I}_{cv}(f) = \frac{1}{N} \sum_{i=1}^N \left[f(\mathbf{X}_i) - \beta_{qmc}[h(\mathbf{X}_i) - \theta] \right] \quad \mathbf{X}_i \in \mathcal{U}(0, 1).$$

We can easily prove it is still unbiased

$$\mathbb{E}(\hat{I}_{cv}) = \mathbb{E}\left(\frac{1}{n} \sum_{i=1}^N \left[f(\mathbf{X}_i) - \beta_{mc}[h(\mathbf{X}_i) - \theta] \right]\right) = I.$$

However, it's not the same case as MC like we presented before, because we do not have IID sample points this time, i.e.

$$\text{Var}_{qmc}(\hat{I}_{cv}) \neq \frac{1}{N} \text{Var}\left(f(\mathbf{X}_i) - \beta_{mc}[h(\mathbf{X}_i) - \theta]\right).$$

Instead the variance becomes

$$\begin{aligned} \text{Var}(\hat{I}_{cv}) &= \text{Var}\left(\hat{I} - \beta_{qmc}\hat{H}\right) \quad s.t. \quad \hat{I} = \sum_{i=1}^N f(\mathbf{X}_i), \quad \hat{H} = \sum_{i=1}^N [h(\mathbf{X}_i) - \theta] \\ &= \text{Var}(\hat{I}) - 2\beta_{qmc}\text{Cov}(\hat{I}, \hat{H}) + \beta_{qmc}^2 \text{Var}(\hat{H}) \\ &= \text{Var}(\hat{H}) \left(\beta_{qmc} - \frac{\text{Cov}(\hat{I}, \hat{H})}{\text{Var}(\hat{H})} \right)^2 + \text{Var}(\hat{I}) - \frac{\text{Cov}(\hat{I}, \hat{H})}{\text{Var}(\hat{H})}. \end{aligned}$$

The optimal β_{qmc} is

$$\beta_{qmc}^* = \text{Var}(\hat{H})^{-1} \text{Cov}(\hat{I}, \hat{H}), \quad (3.2)$$

which leave the variance to be

$$\text{Var}_{qmc}(\hat{I}_{cv}) = \text{Var}(\hat{I})(1 - \text{corr}^2[\hat{I}, \hat{H}]).$$

Now we are interested that if our previous formula for $\hat{\beta}_{mc}$ could still be an estimation for $\hat{\beta}_{qmc}$. The fact is that they are generally not the same. Let's take the covariance part of formula (3.2) and (2.2) to see the difference.

$$\begin{aligned} \text{Cov}(\hat{I}, \hat{H}) &= \int [f(\mathbf{X}_1) + \dots + f(\mathbf{X}_n)][h(\mathbf{X}_1) + \dots + h(\mathbf{X}_n)] d\mathbf{X} \\ &= \int \left[\sum_{i=1}^N f(\mathbf{X}_i)h(\mathbf{X}_i) + \sum_{i,j=1}^{i \neq j} f(\mathbf{X}_i)h(\mathbf{X}_j) \right] d\mathbf{X} \\ &\neq \int f(\mathbf{X}_i)h(\mathbf{X}_i) d\mathbf{X}_i \\ &= \text{Cov}[f(\mathbf{X}_i), h(\mathbf{X}_i)]. \end{aligned}$$

There is also a very good example from Hickernell and Lemieux (2005) [3]'s paper, showing that β_{mc} and β_{qmc} can give a quite different results in some cases.

3.3 A new way to find β

As we stated in previous section, we can not find optimal β by minimizing variance of estimator like MC. However, if using the reliable adaptive QMC method introduced in chapter 2, we may have another way to find β .

Recall equation (2.3) the error bound for new estimator of g still holds

$$\left| \int_{[0,1]^d} g dx - \hat{I}_m(g) \right| \leq a(r, m) \sum_{[2^{m-r-1}]^{2^{m-r-1}}} |\tilde{g}_{m,k}|. \quad (3.3)$$

Naturally, the new estimator becomes

$$\hat{I}_m(g) := \frac{1}{b^m} \sum_{i=0}^{b^m-1} g(z_i + \Delta). \quad (3.4)$$

From (3.3) it is clear that the optimal β is the one that minimize the error term.

$$\beta^* = \min_{\beta} \sum_{\kappa=b^{m-r-1}}^{b^{m-r}-1} |\hat{g}_{\kappa}| \quad (3.5)$$

$$= \min_{\beta} \sum_{\kappa=b^{m-r-1}}^{b^{m-r}-1} |\hat{f}_{\kappa} - (\hat{\mathbf{h}}_{\kappa} - \hat{\boldsymbol{\theta}})\beta| \quad \hat{\mathbf{h}}_{\kappa} = (\hat{h}_{\kappa,1}, \dots, \hat{h}_{\kappa,J}), \hat{\boldsymbol{\theta}} = (\hat{\theta}_{\kappa,1}, \dots, \hat{\theta}_{\kappa,J}) \quad (3.6)$$

$$= \min_{\beta} \|\hat{\mathbf{f}} - \widehat{\mathbf{H}}\beta\|_1 \quad \hat{\mathbf{f}} = (\hat{f}_{b^{m-r-1}}, \dots, \hat{f}_{b^m-1})^T \quad (3.7)$$

$$\approx \min_{\beta} \|\hat{\mathbf{f}} - \widehat{\mathbf{H}}\beta\|_2 \quad \widehat{\mathbf{H}} = (\widehat{\mathbf{H}}_1, \dots, \widehat{\mathbf{H}}_J) \quad (3.8)$$

$$\widehat{\mathbf{H}}_j = (\hat{h}_{b^{m-r-1},j} - \hat{\theta}_j, \dots, \hat{h}_{b^m-1,j} - \hat{\theta}_j)^T.$$

The second equivalence is not hard to get, but the third one may not be so obvious. Let's consider it backwards. Suppose we have a vector A and it's \mathcal{L}_1 -norm.

$$A = \begin{pmatrix} f_1 - z_1 \\ f_2 - z_2 \\ \dots \\ f_n - z_n \end{pmatrix}, \quad \|A\|_1 = \sum_{i=1}^n |f_i - z_i|, \quad z_i := (\mathbf{h}_i - \boldsymbol{\theta})$$

If we replace the index, A is exactly what's inside the \mathcal{L}_1 -norm in (3.7). Hence we justified the third equivalence. The reason we use the least square approximate the \mathcal{L}_1 regression, is because there is no efficient way to solve it compared to existing least square methods.

3.4 The problem with θ

We noticed a problem in solution for optimal β , which is we do a lot of subtractions with θ . This could be a large cost when we have difficult functions which means b^{m-r} could be very large number. Therefore we present a way to avoid that part.

The idea is form a observation that Walsh transform of θ in (3.6) is actually zero, since $\hat{h}_\theta = \theta\delta_{\kappa,0}$ and the summation is not start from $\kappa = 0$.

This simplifies (3.6) to the following. Note that we only need the information of function f and h to calculate β^* , θ has been get rid of the optimization process.

$$\begin{aligned}
\beta^* &= \min_{\beta} \sum_{\kappa=b^{m-r-1}}^{b^{m-r}-1} |\hat{f}_\kappa - (\hat{h}_\kappa - \hat{\theta})\beta| & (3.9) \\
&= \min_{\beta} \sum_{\kappa=b^{m-r-1}}^{b^{m-r}-1} |\hat{f}_\kappa - (\hat{h}_\kappa - \theta\delta_{\kappa,0})\beta| \\
&= \min_{\beta} \sum_{\kappa=b^{m-r-1}}^{b^{m-r}-1} |\hat{f}_\kappa - \hat{h}_\kappa\beta| & \hat{\mathbf{f}} = (\hat{f}_{b^{m-r-1}}, \dots, \hat{f}_{b^{m-r}-1})^T \\
&= \min_{\beta} \|\hat{\mathbf{f}} - \hat{\mathbf{H}}\beta\|_1 & \hat{\mathbf{H}} = (\hat{\mathbf{H}}_1, \dots, \hat{\mathbf{H}}_J) \\
&\approx \min_{\beta} \|\hat{\mathbf{f}} - \hat{\mathbf{H}}\beta\|_2 & \hat{\mathbf{H}}_j = (\hat{h}_{b^{m-r-1},j}, \dots, \hat{h}_{b^{m-r}-1,j})^T
\end{aligned}$$

The same problem happened with the estimator (3.4). We have the similar

solution for that.

$$\begin{aligned}
\hat{I}_m(g) &= \frac{1}{b^m} \sum_{i=0}^{b^m-1} g(z_i + \Delta) \\
&= \frac{1}{b^m} \sum_{i=0}^{b^m-1} f(z_i + \Delta) - (\mathbf{h}(z_i + \Delta) - \boldsymbol{\theta})\boldsymbol{\beta} \\
&= \frac{1}{b^m} \sum_{i=0}^{b^m-1} [f(z_i + \Delta) - \mathbf{h}(z_i + \Delta)\boldsymbol{\beta}] + \boldsymbol{\theta}\boldsymbol{\beta}.
\end{aligned} \tag{3.10}$$

After organize it the in format of (3.10), θ is eliminated from the summation part. From these two parts of work on θ we managed to save $\frac{b-1}{b}b^{m-r} + b^m$ operations of subtraction.

3.5 The modified method

Now we make the following changes:

$$\begin{aligned}
g &= f - \beta h \\
\hat{I}_m(g) &= \frac{1}{b^m} \sum_{i=0}^{b^m-1} g(z_i + \Delta).
\end{aligned}$$

And we have the following equivalence:

$$\begin{aligned}
\int_{[0,1]^d} f dx &= \int_{[0,1]^d} g dx + \theta \beta \\
\hat{I}_m(f) &= \hat{I}_m(g) + \theta \beta.
\end{aligned}$$

So the estimation error becomes:

$$\left| \int_{[0,1]^d} f dx - \hat{I}_m(f) \right| = \left| \int_{[0,1]^d} g dx - \hat{I}_m(g) \right|.$$

Here if our g is in the cone we introduced earlier (2.4), then we can use the results from Hickernell and Jimnez Rugama(2014) [1], the error is bounded by:

$$\left| \int_{[0,1]^d} g dx - \hat{I}_m(g) \right| \leq a(r, m) \sum_{[2^{m-r-1}]}^{2^{m-r}-1} |\tilde{g}_{m,k}|.$$

This leads to the same algorithm suggested by Hickernell and Jimnez Rugama(2014) [1], but since we are using control variates, several modifications have to be made.

3.6 The Algorithm

We now give the algorithm for reliable adaptive QMC with control variates using digital sequence.

Algorithm 1: Reliable Adaptive QMC with control variates

Data: function f and \mathbf{H} ; value of $\int_{[0,1]^d} h_j dx = \theta_j$; tolerance ε

Result: $\hat{I}(f)$; samples size; optimal β

begin

```

1    $m, r =$  start numbers,  $x = 2^m$  sobolset points
2   get kappa map( $\tilde{\kappa}$ ) and Walsh coefficients( $\tilde{f}, \tilde{\mathbf{H}}$ ) using algorithm 2
3    $\beta = \tilde{H}\{\tilde{\kappa}[x(a : b)]\} \setminus \tilde{f}\{\tilde{\kappa}[x(a : b)]\}, (a : b) = (2^{m-r-1} : 2^{m-r} - 1)$ 
4    $g = f - \mathbf{H}\beta$ , repeat step 2 on  $g$ 
5    $\tilde{S}_{m-r,m}(g) = \sum_a^b \left| \tilde{g}\{\tilde{\kappa}[x(a : b)]\} \right|$ 
6   check whether  $g$  is in the cone
7   if  $a(m, r)\tilde{S}_{m-r,m}(g) \leq \varepsilon$  then
    |   return  $\hat{I}_m(g) = \sum_{i=0}^{2^m-1} f[x(i)] + \theta\beta$ 
    |   return  $\beta, n = 2^m$ 
8   for  $m = m + 1 : mmax$  do
    |    $xnext =$  next  $2^{m-1}$  sobolset points
    |   repeat step 2 on  $[x, xnext]$ 
    |   repeat step 5, 6, 7

```

Note that for generating kappa map, i.e. step 2 in Algorithm 1, we used an explicit way to generate it. One can find the details for that in appendix from Hickernell and Jiménez Rugama (2014) [1]. Here we reorganize it and show it in

algorithm 2. Another important point need to be mentioned is that in our algorithm, we used an iterative way, which may require recalculating β for each iteration.

Algorithm 2: kappa map and discrete Walsh coefficients

Data: function f ; $Y_v^{(m)}$; $m \in \mathbb{N}_0$

Result: $\tilde{\kappa}$; $\tilde{S}_{m-r,m}(f)$

begin

if $m = 0$ **then**

$\mathring{\mathbf{v}}(0) = 0$

if $m \geq 1$ **then**

for $m : 1 : -1$ **do**

$\mathring{\mathbf{v}}_m(\mathbf{k}) = \mathring{\mathbf{v}}_m - 1(\mathbf{k})$

$\mathring{\mathbf{v}}_m(\mathbf{k}) = \mathbf{k}, \mathbf{k} = b^{m-1}, \dots, b^m - 1$

for $l = m - 1 : \max(1, m - r) : -1$ **do**

for $k = 1 : b^l - 1$ **do**

$\forall a \in \mathbb{F}_b$, find a s.t. $|Y_{\mathring{\mathbf{v}}(k+a \cdot b^l)}^{(m)}| \geq |Y_{\mathring{\mathbf{v}}(k+ab^l)}^{(m)}|$

CHAPTER 4

NUMERICAL EXPERIMENT

4.1 Option Pricing

Option Pricing has always been a challenging topic in financial mathematics. Although there are several other methods for pricing options, Monte Carlo performs better when solving high dimension problems. In this chapter we make several tests our reliable QMC with CV algorithm with option pricing problems. Note all our experiments are implemented under brownian motion (GBM) pricing model on non dividend paying stock. Since the GBM model is a well known model for option pricing, we only lay out the parameters for option formulas we use later and not dig into the model itself.

$S(jT/d)$ = current asset price at time jT/d , $j = 1, \dots, d$

K = strike price

T = expiration time

σ = volatility

r = interest rate

d = number of time steps.

4.1.1 Accuracy. The first thing we want to test is whether our algorithm provides the ‘accurate’ solution. This means if the function satisfies the cone condition, the difference between our estimation and true value should be bounded by the pre-defined error tolerance. We will test our algorithm for accuracy in three cases: original reliable adaptive QMC (RAQMC), reliable adaptive QMC with CV (RAQMC_CV) for single CV and RAQMC_CV for double CV.

To do this we have to know the exact value of our integral to calculate the

exact error of our results. Therefore, we choose the geometric mean Asian option as our target function because we have the exact pricing formula under GBM model [8]. We choose to test the call options, for the geometric mean Asian call option the payoff function is:

$$C_T^{\text{gmean}} = \max \left(\left[\prod_{j=1}^d S(jT/d) \right]^{\frac{1}{d}} - K, 0 \right) e^{-rT}.$$

The closed formula for the exact price under the GBM model is:

$$\begin{aligned} C_T^{\text{gmeanExact}} &= S(0)e^{-(r+\sigma^2/2)T/2}\Phi(\tilde{d}_1) + Ke^{-rT}\Phi(\tilde{d}_2) \\ \tilde{d}_1 &= \frac{\ln(S(0)/K) + (r + \frac{n-1}{6(n+1)}\sigma^2/2)T/2}{\sqrt{\frac{2n+1}{6(n+1)}}\sigma\sqrt{T}} \\ \tilde{d}_2 &= \frac{\ln(S(0)/K) + (r - \sigma^2/2)T/2}{\sqrt{\frac{2n+1}{6(n+1)}}\sigma\sqrt{T}}. \end{aligned} \quad (4.1)$$

Then we need several CV of which the exact solution must have clear formulas. Here we pick the European call option and the stock price at expiration as control variates. For the European call option, the payoff function is:

$$C_T = \max(S(T) - K, 0)e^{-rT}.$$

The exact price under the GBM model is given by [9]:

$$\begin{aligned} C_T^{\text{euroExact}} &= S(0)\Phi(d_1) - Ke^{-rT}\Phi(d_2) \\ d_1 &= \frac{\ln(S_0/K) + (r + \sigma^2/2)T}{\sigma\sqrt{T}} \\ d_2 &= \frac{\ln(S_0/K) + (r - \sigma^2/2)T}{\sigma\sqrt{T}}. \end{aligned}$$

For the stock price, the expected discounted price is $S(0)e^{-rT}$, while the exact price under the GBM model is just $S(0)$.

Table 4.1 shows the set up for all options in the accuracy test. We use the in the money call option for both Asian and European options. They are monitored weekly for a one month period with 1% interest and 50% volatility annually. The

Table 4.1: Parameter Setup for accuracy test

S0	K	TimeVector	r	volatility
120	100	1/52:1/52:4/52	0.01	0.5

test is on three scenarios: RAQMC, RAQMC_CV with one CV (RAQMC_CV₁) and RAQMC_CV with two CV (RAQMC_CV₂). The test is then performed with three different absolute error tolerances as shown in table 4.2. First we calculated the exact price for the geometric mean Asian call option using formula (4.1). Then we run the same parameters through RAQMC_CV to get the estimates. Finally, we calculate the absolute value of difference between estimates and true prices as errors. We used three different absolute error tolerance set ups for this: 10^{-2} , 10^{-3} , 10^{-6} . From the results we can see all errors lied within the preset tolerance. This give us confidence for our algorithm as accurate as guranteed. Note for the second test, which the tolerance is 10^{-3} , while the error seems all within a lesser bound 10^{-4} . We believe this is not an error for the algorithm or the code. It happened since each iteration our sample size is doubled and this could be more than satisfied for the predefined tolerance level.

4.1.2 Efficiency. Now that we know our algorithm provides the desired results, we continue to test the time efficiency of the algorithm. Here we do not use time complexity like big O notation. This is because the results of our algorithm depend not only on dimation but also on target function itself. Since we use an iterative way to stop we can not know the anwer apriori. Instead, we do experiments to test the ‘efficiency’ of our algorithm by compare the sample size used for caculation and the corresponding time cost. Note the results do not stay the same for each time, to compensate that we take an average of 10 consecutive runs with the same set-up. This test breaks into two parts. The first one is that we want to know

Table 4.2: Accuracy Test of RAQMC_CV algorithm

abstol = 10^{-2}	RAQMC	RAQMC_CV ₁	RAQMC_CV ₂
exact price	1.2926641930	1.2926641930	1.2926641930
estimate price	1.2938177355	1.2937890386	1.2930809161
err= exact-estimate	1.1535e-3	1.124e-3	4.167e-4
abstol = 10^{-3}	RAQMC	RAQMC_CV ₁	RAQMC_CV ₂
exact price	1.2926641930	1.2926641930	1.2926641930
estimate price	1.2926529108	1.2925687148	1.2925793049
err= exact-estimate	1.12821e-5	9.54782e-5	8.48881e-5
abstol = 10^{-6}	RAQMC	RAQMC_CV ₁	RAQMC_CV ₂
exact price	1.2926641930	1.2926641930	1.2926641930
estimate price	1.2926643000	1.2926639635	1.296646297
err= exact-estimate	1.07e-7	2.295e-7	4.367e-7

how our algorithm performs without using CV. Naturally the RAQMC algorithm we introduced in chapter 3 become a good reference. We know our algorithm is a bit slower compared to it when not using CV. The question is how small the gap is. If there is no significant difference, then it means we will not waste too much time on cases without CV or with poor CV. The second one, which is more important, is that we want to see how much time it can save by using good CV. Of course this depends on quality of the CV. Fortunately, several good CV is known to be use for option pricing under GBM model [10].

4.1.2.1 RAQMC_CV without CV. Parameter set up for efficiency test is shown in table 4.3. This time we try to price a daily monitored four months period option, which increases the dimension to 64. Note we will keep using this set up for all the following efficiency tests. We test our algorithm through four scenarios:

Table 4.3: Parameter Setup for all efficiency tests

S0	K	TimeVector	r	volatility	abstol	reitol
120	130	1/250:1/250:64/250	0.01	0.5	1e-3	0

RAQMC, RAQMC_CV without CV (RAQMC_CV₀), RAQMC_CV with one poor CV (RAQMC_CV₁) and RAQMC_CV with two poor CV (RAQMC_CV₂). For target option we still use geometric mean Asian option European option. For single CV we use European option and add stock price for double CV. As shown in table 4.4, all test

Table 4.4: Efficiency test I of RAQMC_CV

abstol= 10^{-3}	RAQMC	RAQMC_CV ₀	RAQMC_CV ₁	RAQMC_CV ₂
Sample Size	65535	65535	65535	65535
Time Cost	1.8365	1.8379	1.8808	1.8703

end up with same sample size, which means that choice of CV didn't help at all. But what we concern is that how much extra cost can it bring if our choice of CV is bad? We can see the time cost just increased around 2% for QMC with bad CV. Now we can say our method did make the cost of CV to a minimal.

4.1.2.2 RAQMC_CV with CV. There are two types of asian options, depends on which types of mean one use. In last section we introduced geometric mean Asian

Table 4.5: Efficiency test II of RAQMC_CV algorithm with Asian Option

	Estimate Price	Sample Size	Time Cost	β^*
RAQMC	4.2011	65535	1.9054	null
RAQMC_CV ₁	4.2008	15564.8	0.49966	0.995
RAQMC_CV ₂	4.2018	13926.4	0.43793	$(0.958, 0.005)^T$

option, this time we choose arithmetic mean asian call option as our target function, whose payoff function is:

$$C_T^{Amean} = \max\left(\frac{1}{d} \sum_{j=1}^d S(jT/d) - K, 0\right) e^{-rT}.$$

For this option there is no close formula for exact price under GBM model. However, recall we introduced another Asian option earlier of which the price has a clear formula. It is known that geometric mean Asian option was first used as a CV for pricing arithmetic mean Asian option [8]. Hence, for single CV we choose geometric mean Asian call option as CV. We also want to test it for double CV so we add European call option as a second CV. Results are shown in table 4.5. We can see this choice of CV is really good. The sample size is approximately 1/4 of original size and same with time cost. The second CV didn't make an obvious improvement compared to single CV but seems helped a bit.

Figure 4.1 is a plot of the walsh coefficients for the target function and control variates in this Asian options experiments. Recall our error of approximation is bound on this exact same coefficients. What we'd expect is that on both method it can decrease fast and for CV method we expect it to decrease faster. The horizontal axis in the figure can be interpreted as sample size, the vertical axis can be view as the error bound. What our algorithm does is that for a give error bound, we monitor

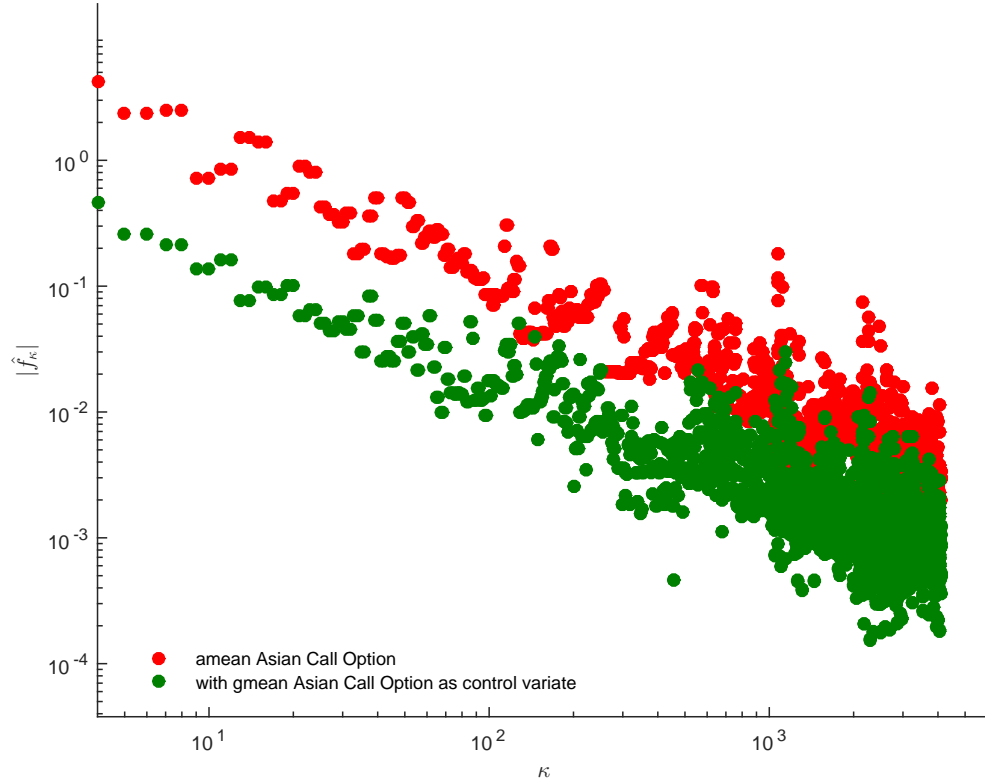


Figure 4.1: Walsh coefficients of f

these Walsh coefficients until it reaches the bound. Obviously, the CV method will stop early in this case which means less computational cost for it.

Another good example is barrier option. This is the payoff function for up and in barrier call option:

$$C_T^{U\&I} = [S(T) - K]^+ 1_{\{\max_{j \leq T/d} S(jT/d) \geq \text{Barrier}\}}.$$

The reason we choose it is for its similarity to the European option. Note if the barrier equals strike price (130), then it is just a European call option, which makes European option a good control variate. For this test we take four different barriers that gradually decrease to the strike price. We use the same parameter set-up as the previous one which is shown in table 4.3. As same as the last test, we compared

original RAQMC algorithm and RAQMC_CV on sample size and time cost. One

Table 4.6: Efficiency test II of RAQMC_CV algorithm with Barrier Option

Barrier	Sample Size		Time Cost		β^*
	RAQMC	RAQMC_CV	RAQMC	RAQMC_CV	
160	1048576	1048576	49.9861	50.8245	0.6613
150	1048576	524288	47.8742	23.2743	0.8999
140	524288	32768	22.7936	1.0146	0.9935
130	524288	1024	22.3675	0.0826	1.0000

merit for this example is that we can see how the cost and value of β^* change as we change the barrier. In this example we know the optimal value should be close to 1 as shown in table 4.6 when barrier=130. Also, note we set the start sample size for iteration to be 2^{10} , which is exactly the sample size used in this case. This is consistent with our statement earlier, it is just a European option. When we move barrier further from 130, we assume the European option will not be as good as the previous one as a CV. Our results confirmed this assumption, we can see in table 4.6 both the time cost and sample size rise as we increase the barrier. At last when barrier=160, European option completely lost its value as CV and cost of two methods are almost the same.

4.2 Multivariate normal probabilities

For this part tests, we are interested in computing the probability of a multivariate normal distribution in dimension 6

$$F([0, 1]^6) = \int_{[0,1]^6} \frac{1}{\sqrt{|\Sigma|(2\pi)^6}} e^{\frac{1}{2}\mathbf{x}^T \Sigma^{-1} \mathbf{x}} d\mathbf{x},$$

where Σ is a symmetric positive definite covariance matrix. So this time the target function is multivariate normal density function and to make it simple we set the covariance matrix in the following form

$$f(\mathbf{x}) = \frac{1}{\sqrt{|\Sigma|(2\pi)^6}} e^{\frac{1}{2}\mathbf{x}^T \Sigma^{-1} \mathbf{x}}, \quad \Sigma = \begin{pmatrix} 1 & \sigma & \sigma & \sigma & \sigma & \sigma \\ \sigma & 1 & \sigma & \sigma & \sigma & \sigma \\ \sigma & \sigma & 1 & \sigma & \sigma & \sigma \\ \sigma & \sigma & \sigma & 1 & \sigma & \sigma \\ \sigma & \sigma & \sigma & \sigma & 1 & \sigma \\ \sigma & \sigma & \sigma & \sigma & \sigma & 1 \end{pmatrix}.$$

Then we choose the two multivariate normal distribution with diagonal covariance matrices as CV

$$h_1(\mathbf{x}) = \frac{1}{\sqrt{|\Sigma_1|(2\pi)^6}} e^{\frac{1}{2}\mathbf{x}^T \Sigma_1^{-1} \mathbf{x}}, \quad \Sigma_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$h_2(\mathbf{x}) = \frac{1}{\sqrt{|\Sigma_2|(2\pi)^6}} e^{\frac{1}{2}\mathbf{x}^T \Sigma_2^{-1} \mathbf{x}}, \quad \Sigma_2 = \begin{pmatrix} \sigma & 0 & 0 & 0 & 0 & 0 \\ 0 & \sigma & 0 & 0 & 0 & 0 \\ 0 & 0 & \sigma & 0 & 0 & 0 \\ 0 & 0 & 0 & \sigma & 0 & 0 \\ 0 & 0 & 0 & 0 & \sigma & 0 \\ 0 & 0 & 0 & 0 & 0 & \sigma \end{pmatrix}.$$

The reason we pick these as CV is that they are from the same multivariate gaussian family and correlate to the target function. Computation of the integrals on these CV are rather simple since each dimension is independent in this case. But still the θ we computed is an approximation and this introduce a bias into our estimator. We

think the results can still be useful since the error for θ is rather small and we can do it fast.

We choose three different value for σ (0.1, 0.4 and 0.7) and run the test for three senarios: RAQMC, RAQMC-CV with h1 as CV (RAQMC-CV₁), RAQMC-CV with h1 and h2 as CV (RAQMC-CV₂). The sample size and time cost in table 4.7 are average value from 20 identical runs. Because the exact value of the target integral is unknown, we use an approximation for ‘exact’ value to compute the error. These approximations are computed using a method developed by Genz and Bretzwith an error of 10^{-10} [11]. Similarly, we computed the mean of CV with the method developed by Drezner and Wesolowsky and by Genz [12] [13]. The time cost for computation of means of CV is included in the results.

We start the test with $\sigma = 0.1$. With this setting the targe function should be close to the first CV. As expected, both single and double CV perform great. The sample size is reduced to 4096 from 27852 and time cost is about half down compared to RAQMC. As we increase the value of σ , the first CV start to lose its value. When $\sigma = 0.7$ the first CV become not helpful and cost more time than RAQMC. In the mean time, the double CV method show some good effect, at $\sigma = 0.7$, the sample size is half of the other 2 and time cost is about 30% lesser.

Table 4.7: Test of RAQMC_CV with multivariate normal probability

$\sigma = 0.1, \text{ abstol}=10^{-7}$	RAQMC	RAQMC_CV ₁	RAQMC_CV ₂
estimate value	0.0020706728	0.0020706626	0.0020706649
err= ‘exact’-estimate	1.8e-9	4.3e-9	7.0e-9
sample size	27852	4096	4096
time cost	0.0414	0.0206	0.0242
β^*	null	1.1093	$(0.9649, 4.47\text{e-}5)^T$
$\sigma = 0.4, \text{ abstol}=10^{-7}$	RAQMC	RAQMC_CV ₁	RAQMC_CV ₂
estimate value	0.0046138363	0.0046138346	0.0046138341
err= ‘exact’-estimate	1.9e-9	1.9e-9	2.6e-9
sample size	65536	32768	32768
time cost	0.0805	0.0595	0.0628
β^*	null	2.2582	$(0.9673, 0.0408)^T$
$\sigma = 0.7, \text{ abstol}=10^{-7}$	RAQMC	RAQMC_CV ₁	RAQMC_CV ₂
estimate value	0.0169509335	0.0169509327	0.0169509342
err= ‘exact’-estimate	1.8e-9	6.1e-9	3.6e-9
sample size	524288	511180	262144
time cost	0.5235	0.6187	0.3626
β^*	null	8.6074	$(-18.95, 7.78)^T$

CHAPTER 5

CONCLUSION

5.1 Discussion

So far there are only few QMC algorithms that can adaptively determine the sample size needed basing on integrand values. This is because the estimation of error for QMC is hard. Several studies show that if using quasi-standard error there will be some serious drawbacks [7]. There is also a way using internal replications of IID randomized QMC rules, but the number of replications are not known [3].

For CV with QMC the research progress is also limited since its hard to estimate the value of CV coefficients as we stated in chapter 3. Hickernell and Jiménez Rugama (2014) [1]’s work on building a QMC method provide a reliable and adaptive way to use QMC, as well as give us insights into combining reliable adaptive QMC with CV. The main idea of the reliable adaptive QMC is to bound the error of estimation using summation of part Walsh coefficients. We utilize the same idea for calculating the optimal coefficient for CV. In order to compensate the extra computation cost for CV, we used several techniques in our design to keep those cost minimal.

We test our algorithm on several option pricing problems under Black-Scholes scheme. We found the accuracy of algorithm is consistent and reliable. Comparison of QMC with CV and normal QMC is performed on different pairs of options, the results show that using the proper CV can make improvement over vanilla QMC methods. We also provide an example of computing multivariate normal probability using CV with estimated mean. It shows in certain case multiple CV outperforms single CV.

5.2 Future work

There are several unsolved problems in our work, of which we haven't found the answers yet. To start with, we use the least square regression solution to approximate the least absolute error regression for CV coefficients. Is there a way to do \mathcal{L}_1 regression more efficiently? Can it offer enough improvement on computational cost to compensate its own cost? This leads to the second problem. In our numerical examples we do not update CV coefficients for each iteration. The reason is that we find doing updating requires a lot of recalculations of previous terms, which could reduce the advantage brought by CV. For further research we hope to find a way to update β more efficiently. Another possible work for future research is that we can extend this method on rank-1 lattices [14]. The idea for getting CV coefficients is the same, but due to the different structure for digital sequences, some effort has to be done for adaption of the method.

BIBLIOGRAPHY

- [1] F. J. Hickernell and L. A. J. Rugama, “Reliable adaptive cubature using digital sequences,” *arXiv preprint arXiv:1410.8615*, 2014.
- [2] A. N. Avramidis and J. R. Wilson, “Integrated variance reduction strategies for simulation,” *Operations Research*, vol. 44, no. 2, pp. 327–346, 1996.
- [3] F. J. Hickernell, C. Lemieux, A. B. Owen, *et al.*, “Control variates for quasi-monte carlo,” *Statistical Science*, vol. 20, no. 1, pp. 1–31, 2005.
- [4] G. Fishman, *Monte Carlo: concepts, algorithms, and applications*. Springer Science & Business Media, 2013.
- [5] H. Niederreiter, *Quasi-Monte Carlo Methods*. Wiley Online Library, 2010.
- [6] J. Dick and F. Pillichshammer, *Digital nets and sequences: Discrepancy Theory and Quasi-Monte Carlo Integration*. Cambridge University Press, 2010.
- [7] A. B. Owen, “On the warnock-halton quasi-standard error,” *Monte Carlo Methods and Applications mcma*, vol. 12, no. 1, pp. 47–54, 2006.
- [8] A. G. Kemna and A. Vorst, “A pricing method for options based on average asset values,” *Journal of Banking & Finance*, vol. 14, no. 1, pp. 113–129, 1990.
- [9] P. P. Boyle, “Options: A monte carlo approach,” *Journal of financial economics*, vol. 4, no. 3, pp. 323–338, 1977.
- [10] T. Lidebrandt, *Variance reduction: Three approaches to control variates*. PhD thesis, Citeseer, 2007.
- [11] A. Genz, “Numerical computation of multivariate normal probabilities,” *Journal of computational and graphical statistics*, vol. 1, no. 2, pp. 141–149, 1992.
- [12] Z. Drezner, “Computation of the bivariate normal integral,” *Mathematics of Computation*, pp. 277–279, 1978.
- [13] A. Genz, “Numerical computation of rectangular bivariate and trivariate normal and t probabilities,” *Statistics and Computing*, vol. 14, no. 3, pp. 251–260, 2004.
- [14] L. A. J. Rugama and F. J. Hickernell, “Adaptive multidimensional integration based on rank-1 lattices,” *arXiv preprint arXiv:1411.1966*, 2014.