Tatsuki Odake
(Dated: February 2024)

I. BACKGROUND

Suppose that we have a SDP

$$\begin{aligned}\text{maximize } & C \cdot \Omega \\ \text{s.t. } & C \cdot A_j = b_j \ (j = 1, \ldots, m) \\ & C \geq 0.\end{aligned} \tag{1}$$

corresponding to the problem of finding comb C for N-dimensional Hermitian matrices $\Omega, A_1, \ldots, A_m$. Suppose that the task is to confirm whether the answer is exactly 1 or less than 1 (e.g. checking the existence of deterministic exact protocol). Then we only need to check whether a semidefinite $C \neq 0$ satisfying $C \cdot \Omega = 1$ and $C \cdot A_j = b_j$ exists. By defining an orthonormal basis $A'_0, A'_1, \ldots, A'_m, A'_{m+1}, \ldots, A'_{N^2-1}$ of the space of $N \times N$ matrix such that $\mathrm{span}(A'_0, A'_1, \ldots, A'_m) = \mathrm{span}(\Omega, A_1, \ldots, A_m)$, $A'_j \cdot A'_k = \delta_{j,k}$ $(j, k \in \{0, \ldots, N^2 - 1\})$, and all A'_j are Hermitian, this problem is rephrased as the problem of finding a semidefinite matrix $C \neq 0$ expressed as

$$C = \sum_{j=0}^{m} b'_j A'_j + \sum_{k=m+1}^{N^2-1} c_k A'_k \tag{2}$$

where b'_j is the coefficient defined from $C \cdot A'_j = b'_j$ $(j \in \{0, \ldots, m\})$ which is calculated from $C \cdot A_j = b_j$ and $C \cdot \Omega = 1$ and the definition of A'_j, and c_j is an arbitrary real number. This problem is rephrased as checking the following statement:

$$\exists C \geq 0 \text{ s.t. } C \in \mathrm{span}(B, A'_{m+1}, \ldots A'_{N^2-1}) \backslash \{0\} \qquad \left(B := \sum_{j=0}^{m} b'_j A'_j\right). \tag{3}$$

Indeed, when the conditions $C \cdot A_j = b_j$ correspond to the comb condition, then $I \in \mathrm{span}(A'_0, \ldots, A'_m)$ thus $\mathrm{tr}B > 0$ and A'_k $(k \in \{m+1, \ldots, N^2-1\})$ will be all traceless. Thus, the only way in which there exists a semidefinite $C \neq 0$ in Eq. (3) is that the coefficient of B is positive.

Let us denote the projection of matrices M onto the subspace $\mathrm{span}(A_1, \ldots, A_m)$ as $Q'(M)$. Also, let us denote the projection of M onto the subspace $\mathrm{span}(B, A'_{m+1}, \ldots, A'_{N^2-1})$ as $P(M)$ and define $Q(M) := M - P(M)$. Now, suppose that we can calculate the projection $Q'(M)$ for any matrix M in a small runtime by using the symmetry in the subspace $\mathrm{span}(A_1, \ldots, A_m)$. Then, if we know at least one matrix B_{pre} satisfying $B_{\mathrm{pre}} \cdot \Omega = 1$ and $B_{\mathrm{pre}} \cdot A_j = b_j$ $(j = 1, \ldots, m)$, we can calculate $P(M), Q(M)$ of any matrix M as

$$\begin{aligned}\Omega' &= (\Omega - Q'(\Omega))/\|(\Omega - Q'(\Omega))\|_2 \\ B &= Q'(B_{\mathrm{pre}}) + \Omega' \mathrm{tr}(\Omega' \cdot B_{\mathrm{pre}}) \\ B' &= B/\|B\|_2 \\ P(M) &= M - (Q'(M) + \Omega' \mathrm{tr}(\Omega' \cdot M)) + B' \mathrm{tr}(B' M) \\ Q(M) &= M - P(M)\end{aligned} \tag{4}$$

where $\Omega' := \Omega/\|\Omega\|_2$ and $B' := B/\|B\|_2$.

II. METHOD

Lemma 1. *For the function $F(\rho) := \rho \cdot Q(\rho) = \|Q(\rho)\|_2^2$ on the space of the quantum state $\{\rho \geq 0 \mid \mathrm{tr}\rho = 1\}$ (introduced not for the physical reason, but for mathematical convenience), the necessary and sufficient condition that Eq. (3) holds is* $\min_\rho F(\rho) = 0$

Indeed, if $F(\rho_0) = 0$ for a ρ_0, then $Q(\rho_0) = 0$, thus a nonzero semidefinite matrix ρ_0 is in $\mathrm{span}(B, A'_{m+1}, \ldots, A'_{N^2-1})$. Also, if $\min_\rho F(\rho) > 0$, then for ρ_0 giving the minimum, $Q(\rho_0)$ is a nonzero semidefinite matrix in the complementary subspace of $\mathrm{span}(B, A'_{m+1}, \ldots, A'_{N^2-1})$ (since $\rho_0 \in \mathrm{argmin}_\rho(\rho \cdot Q(\rho)) \to \rho_0 \in \mathrm{argmin}_\rho(\rho \cdot Q(\rho_0))$), thus there cannot be a semidefinite matrix in $\mathrm{span}(B, A'_{m+1}, \ldots, A'_{N^2-1})$.

For the function f of a Hermitian matrix H defined as

$$f(H) := \frac{\mathrm{tr}[Q(H^2)^2]}{[\mathrm{tr}(H^2)]^2}, \tag{5}$$

(which corresponds to $\|Q(\rho)\|_2^2$ of $\rho := H^2/\mathrm{tr}(H^2)$), its derivative along axis M (another Hermitian matrix) is calculated as

$$\left.\frac{\mathrm{d}}{\mathrm{d}x}\right|_{x=0} f(H + xM) = 2\left(\frac{\{H, Q(H^2)\}}{[\mathrm{tr}(H^2)]^2} - \frac{2\mathrm{tr}[Q(H^2)^2]H}{[\mathrm{tr}(H^2)]^3}\right) \cdot M := G(H) \cdot M. \tag{6}$$

As an experiment, I try to conduct an SDP to check whether simulating U^T of qutrit unitary U using ordered comb with input $U^* \to U \to U^*$ is possible or not. The condition

$$\forall U,\ \mathrm{tr}\left[\{(J_{U^*})_{12} \otimes (J_U)_{34} \otimes (J_{U^*})_{56}\}^T C_{01234567}\right] = (J_{U^T})_{07} \tag{7}$$

is equivalent to the following simpler condition

$$\begin{aligned} C \cdot \Omega &= 1 \\ \Omega &:= \frac{1}{9}\sum_j J_{U_j} \otimes J_{U_j^*} \otimes J_{U_j} \otimes J_{U_j^T} \end{aligned} \tag{8}$$

for a set of U_j for which $J_{U_j} \otimes J_{U_j^*} \otimes J_{U_j} \otimes J_{U_j^T}$ spans the full space spanned by $J_U \otimes J_{U^*} \otimes J_U \otimes J_{U^T}$ with all unitary U.

Also, the condition for ordered comb is expressed as

$$\begin{aligned} C \cdot (H_{012345} \otimes L_6 \otimes I_7) &= 0 \\ C \cdot (H_{0123} \otimes L_4 \otimes I_{567}) &= 0 \\ C \cdot (H_{01} \otimes L_2 \otimes I_{34567}) &= 0 \\ C \cdot (L_0 \otimes I_{1234567}) &= 0, \end{aligned} \tag{9}$$

where L refers to all matrices that are orthogonal to the identity (namely traceless) and H to all matrices (both with/without trace). together with the normalization condition

$$C \cdot I_{01234567} = 81. \tag{10}$$

Now, the projection Q' of matrix $M_{01234567}$ onto the subspace spanned by $H_{012345} \otimes L_6 \otimes I_7, H_{0123} \otimes L_4 \otimes I_{567}, H_{01} \otimes L_2 \otimes I_{34567}, L_0 \otimes I_{1234567}$ in Eq. (9) and $I_{01234567}$ in Eq. (10) is written as

$$\begin{aligned} Q'(M) &:= Q'_7(M) + Q'_5(M) + Q'_3(M) + Q'_1(M) + T(M) =: Q'_{\mathrm{comb}}(M) + T(M) \\ Q'_7(M) &:= {}_7M - {}_{67}M \\ Q'_5(M) &:= {}_{567}M - {}_{4567}M \\ Q'_3(M) &:= {}_{34567}M - {}_{234567}M \\ Q'_1(M) &:= {}_{1234567}M - {}_{01234567}M \\ T(M) &:= {}_{01234567}M, \end{aligned} \tag{11}$$

which is efficiently computable.

Now, in order to find a matrix B_{pre} such that $B_{\mathrm{pre}} \cdot \Omega = 1$, $Q'_{\mathrm{comb}}(B_{\mathrm{pre}}) = 0$, and $B_{\mathrm{pre}} \cdot I = 81$, the following equations can be used:

$$\begin{aligned} [H_1 - Q'_{\mathrm{comb}}(H_1)] \cdot \Omega &= a \\ [H_1 - Q'_{\mathrm{comb}}(H_1)] \cdot I &= b \\ [H_2 - Q'_{\mathrm{comb}}(H_2)] \cdot \Omega &= c \\ [H_2 - Q'_{\mathrm{comb}}(H_2)] \cdot I &= d \end{aligned} \tag{12}$$

where a, b, c, d are real numbers and H_1, H_2 are randomly chosen Hermitian matrices. Note that both $[H_1 - Q'_{\mathrm{comb}}(H_1)]$ and $[H_2 - Q'_{\mathrm{comb}}(H_2)]$ have zero Q'_7, Q'_5, Q'_3, and Q'_1 values. By adding $[H_1 - Q'_{\mathrm{comb}}(H_1)]$ and $[H_2 - Q'_{\mathrm{comb}}(H_2)]$ with coefficients h_1, h_2 obtained by

$$\begin{pmatrix} h_1 \\ h_2 \end{pmatrix} := \begin{pmatrix} a & c \\ b & d \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 81 \end{pmatrix}, \tag{13}$$

we can obtain a B_{pre}.

Now, we can obtain the projector Q onto the subspace defined as [(subspace on which Q' projects)$\oplus\{\Omega\}]^{\perp} \oplus \{B\}$ as

$$\begin{aligned}
\Omega' &= (\Omega - Q'(\Omega))/\|\Omega - Q'(\Omega)\|_2 \\
B &= Q'(B_{\mathrm{pre}}) + (B_{\mathrm{pre}} \cdot \Omega')\Omega' \\
B' &= B/\|B\|_2 \\
P(M) &= M - (Q'(M) + (M \cdot \Omega')\Omega') + (M \cdot B')B' \\
Q(M) &= M - P(M) = Q'(M) + (M \cdot \Omega')\Omega' - (M \cdot B')B'.
\end{aligned} \tag{14}$$