Check PML formulation

Tatsuya Usuki

Scattering MATRix ANalysis

Abstract

This checks Berenger formulation.

I. 3D MAXWELL EQUATIONS

The discrete Maxwell equations are given by

$$
\begin{pmatrix} 0 & -\triangle_2 & \triangle_1 \\ \triangle_2 & 0 & -\triangle_0 \\ -\triangle_1 & \triangle_0 & 0 \end{pmatrix}
\begin{pmatrix} H_0 \\ H_1 \\ H_2 \end{pmatrix}
= \frac{\partial}{\partial t}
\begin{pmatrix} \varepsilon_{00} E_0 \\ \varepsilon_{11} E_1 \\ \varepsilon_{22} E_2 \end{pmatrix} ,
$$

$$
-\begin{pmatrix} 0 & -\triangledown_2 & \triangledown_1 \\ \triangledown_2 & 0 & -\triangledown_0 \\ -\triangledown_1 & \triangledown_0 & 0 \end{pmatrix}
\begin{pmatrix} E_0 \\ E_1 \\ E_2 \end{pmatrix}
= \frac{\partial}{\partial t}
\begin{pmatrix} \mu_{00} H_0 \\ \mu_{11} H_1 \\ \mu_{22} H_2 \end{pmatrix} .
$$

From $\exp(-i\omega t)$, frequency domain equations show that

$$
\boldsymbol{R}\boldsymbol{H} = \frac{\partial}{\partial t}\Re\boldsymbol{\varepsilon}\boldsymbol{E} + \omega\Im\boldsymbol{\varepsilon}\boldsymbol{E} ,
$$

$$
-\boldsymbol{R}^{\dagger}\boldsymbol{E} = \frac{\partial}{\partial t}\Re\boldsymbol{\mu}\boldsymbol{H} + \omega\Im\boldsymbol{\mu}\boldsymbol{H} .
$$

The rotation operators $\boldsymbol{R}$ and $\boldsymbol{R}^{\dagger}$ are defined by

$$
\boldsymbol{R} = \boldsymbol{R}_{+1} + \boldsymbol{R}_{-1}, \quad \boldsymbol{R}^{\dagger} = \boldsymbol{R}^{\dagger}_{+1} + \boldsymbol{R}^{\dagger}_{-1} ,
$$

$$
\begin{aligned}
\boldsymbol{R}_{+1} &= \begin{pmatrix} 0 & 0 & \triangle_1 \\ \triangle_2 & 0 & 0 \\ 0 & \triangle_0 & 0 \end{pmatrix} , \quad
\boldsymbol{R}_{-1} = \begin{pmatrix} 0 & -\triangle_2 & 0 \\ 0 & 0 & -\triangle_0 \\ -\triangle_1 & 0 & 0 \end{pmatrix} , \\
\boldsymbol{R}^{\dagger}_{+1} &= \begin{pmatrix} 0 & 0 & \triangledown_1 \\ \triangledown_2 & 0 & 0 \\ 0 & \triangledown_0 & 0 \end{pmatrix} , \quad
\boldsymbol{R}^{\dagger}_{-1} = \begin{pmatrix} 0 & -\triangledown_2 & 0 \\ 0 & 0 & -\triangledown_0 \\ -\triangledown_1 & 0 & 0 \end{pmatrix} .
\end{aligned}
\tag{1}
$$

The $\boldsymbol{\varepsilon}$ and $\boldsymbol{\mu}$ are diagonal matrices:

$$
\boldsymbol{\varepsilon} = \begin{pmatrix} \varepsilon_{00} & 0 & 0 \\ 0 & \varepsilon_{11} & 0 \\ 0 & 0 & \varepsilon_{22} \end{pmatrix} , \quad
\boldsymbol{\mu} = \begin{pmatrix} \mu_{00} & 0 & 0 \\ 0 & \mu_{11} & 0 \\ 0 & 0 & \mu_{22} \end{pmatrix} .
$$

II. BERENGER'S FORMULATION

For PML, we set

$$
\varepsilon_{jj} = (\Re\varepsilon_{jj})\,\frac{s_{j+1}s_{j+2}}{s_j}, \quad \text{as} \quad s_j\,(l_j) = 1 + i\hat{\sigma}_j\,(l_j) \ . \tag{2}
$$

By using eq. (1), Berenger scheme for FDTD shows that

$$\begin{aligned}\boldsymbol{R}_{\pm 1}\left(\boldsymbol{H}_{+1}+\boldsymbol{H}_{-1}\right) &= \frac{\partial}{\partial t}\Re\boldsymbol{\varepsilon}\boldsymbol{E}_{\pm 1}+\omega\Re\boldsymbol{\varepsilon}\hat{\boldsymbol{\sigma}}_{\pm 1}\boldsymbol{E}_{\pm 1}\,,\\ -\boldsymbol{R}^{\dagger}_{\pm 1}\left(\boldsymbol{E}_{+1}+\boldsymbol{E}_{-1}\right) &= \frac{\partial}{\partial t}\Re\boldsymbol{\mu}\boldsymbol{H}_{\pm 1}+\omega\Re\boldsymbol{\mu}\hat{\boldsymbol{\sigma}}_{\pm 1}\boldsymbol{H}_{\pm 1}\,,\end{aligned} \tag{3}$$

where

$$\boldsymbol{H}_{+1}=\begin{pmatrix}H_{01}\\ H_{12}\\ H_{20}\end{pmatrix},\quad \boldsymbol{H}_{-1}=\begin{pmatrix}H_{02}\\ H_{10}\\ H_{21}\end{pmatrix},\quad \boldsymbol{E}_{+1}=\begin{pmatrix}E_{01}\\ E_{12}\\ E_{20}\end{pmatrix},\quad \boldsymbol{E}_{-1}=\begin{pmatrix}E_{02}\\ E_{10}\\ E_{21}\end{pmatrix},$$

and

$$\hat{\boldsymbol{\sigma}}_{+1}=\begin{pmatrix}\hat{\sigma}_1 & 0 & 0\\ 0 & \hat{\sigma}_2 & 0\\ 0 & 0 & \hat{\sigma}_0\end{pmatrix},\quad \hat{\boldsymbol{\sigma}}_{-1}=\begin{pmatrix}\hat{\sigma}_2 & 0 & 0\\ 0 & \hat{\sigma}_0 & 0\\ 0 & 0 & \hat{\sigma}_1\end{pmatrix}.$$

From eq. (2), frequency domain formulae for (3) are given by

$$\begin{aligned}\boldsymbol{R}_{\pm 1}\left(\boldsymbol{H}_{+1}+\boldsymbol{H}_{-1}\right) &= -i\omega\Re\boldsymbol{\varepsilon}\boldsymbol{s}_{\pm 1}\boldsymbol{E}_{\pm 1}\,,\\ -\boldsymbol{R}^{\dagger}_{\pm 1}\left(\boldsymbol{E}_{+1}+\boldsymbol{E}_{-1}\right) &= -i\omega\Re\boldsymbol{\mu}\boldsymbol{s}_{\pm 1}\boldsymbol{H}_{\pm 1}\,,\end{aligned} \tag{4}$$

where

$$\boldsymbol{s}_{+1}=\begin{pmatrix}s_1 & 0 & 0\\ 0 & s_2 & 0\\ 0 & 0 & s_0\end{pmatrix},\quad \boldsymbol{s}_{0}=\begin{pmatrix}s_0 & 0 & 0\\ 0 & s_1 & 0\\ 0 & 0 & s_2\end{pmatrix},\quad \boldsymbol{s}_{-1}=\begin{pmatrix}s_2 & 0 & 0\\ 0 & s_0 & 0\\ 0 & 0 & s_1\end{pmatrix}.$$

The operators $\boldsymbol{R}_{\pm 1}$ and $\boldsymbol{s}_j\left(l_j\right)$ satisfy that $\boldsymbol{R}_{\pm 1}\boldsymbol{s}_0^{-1} = \boldsymbol{s}_{\mp 1}^{-1}\boldsymbol{R}_{\pm 1}$ and $\boldsymbol{R}^{\dagger}_{\pm 1}\boldsymbol{s}_0^{-1} = \boldsymbol{s}_{\mp 1}^{-1}\boldsymbol{R}^{\dagger}_{\pm 1}$. Therefore, equation (4) can be modified as

$$\begin{aligned}\boldsymbol{R}_{\pm 1}\left(\boldsymbol{s}_0\boldsymbol{H}_{+1}+\boldsymbol{s}_0\boldsymbol{H}_{-1}\right) &= -i\omega\Re\boldsymbol{\varepsilon}\frac{\boldsymbol{s}_{\pm 1}\boldsymbol{s}_{\mp 1}}{\boldsymbol{s}_0}\boldsymbol{s}_0\boldsymbol{E}_{\pm 1}\,,\\ -\boldsymbol{R}^{\dagger}_{\pm 1}\left(\boldsymbol{s}_0\boldsymbol{E}_{+1}+\boldsymbol{s}_0\boldsymbol{E}_{-1}\right) &= -i\omega\Re\boldsymbol{\mu}\frac{\boldsymbol{s}_{\pm 1}\boldsymbol{s}_{\mp 1}}{\boldsymbol{s}_0}\boldsymbol{s}_0\boldsymbol{H}_{\pm 1}\,.\end{aligned}$$

Here we set

$$\begin{aligned}\boldsymbol{H} &= \boldsymbol{s}_0\boldsymbol{H}_{+1}+\boldsymbol{s}_0\boldsymbol{H}_{-1}\,,\\ \boldsymbol{E} &= \boldsymbol{s}_0\boldsymbol{E}_{+1}+\boldsymbol{s}_0\boldsymbol{E}_{-1}\,,\end{aligned}$$

and then

$$\begin{aligned}\boldsymbol{R}\boldsymbol{H} &= -i\omega\Re\boldsymbol{\varepsilon}\frac{\boldsymbol{s}_{\pm 1}\boldsymbol{s}_{\mp 1}}{\boldsymbol{s}_0}\boldsymbol{H}\,,\\ -\boldsymbol{R}^{\dagger}\boldsymbol{E} &= -i\omega\Re\boldsymbol{\mu}\frac{\boldsymbol{s}_{\pm 1}\boldsymbol{s}_{\mp 1}}{\boldsymbol{s}_0}\boldsymbol{E}\,.\end{aligned}$$

The above equations show that the Berenger's scheme of eq. (3) is equivalent to the setting as eq. (2).

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