Stable formulation

Tatsuya Usuki

Scattering MATRix ANalysis

Abstract

This checks Berenger formulation.

I. 3D MAXWELL EQUATIONS

The discrete Maxwell equations are given by

$$\begin{pmatrix} 0 & -\triangle_2 & \triangle_1 \\ \triangle_2 & 0 & -\triangle_0 \\ -\triangle_1 & \triangle_0 & 0 \end{pmatrix} \begin{pmatrix} H_0 \\ H_1 \\ H_2 \end{pmatrix} = \frac{\partial}{\partial t} \begin{pmatrix} \varepsilon_{00} E_0 \\ \varepsilon_{11} E_1 \\ \varepsilon_{22} E_2 \end{pmatrix},$$

$$-\begin{pmatrix} 0 & -\bigtriangledown_2 & \bigtriangledown_1 \\ \bigtriangledown_2 & 0 & -\bigtriangledown_0 \\ -\bigtriangledown_1 & \bigtriangledown_0 & 0 \end{pmatrix} \begin{pmatrix} E_0 \\ E_1 \\ E_2 \end{pmatrix} = \frac{\partial}{\partial t} \begin{pmatrix} \mu_{00} H_0 \\ \mu_{11} H_1 \\ \mu_{22} H_2 \end{pmatrix}.$$

From $\exp(-i\omega t)$, frequency domain equations show that

$$\begin{aligned} \boldsymbol{R}\boldsymbol{H} &= \frac{\partial}{\partial t} \Re\boldsymbol{\varepsilon}\boldsymbol{E} + \omega \Im\boldsymbol{\varepsilon}\boldsymbol{E}, \\ -\boldsymbol{R}^{\dagger}\boldsymbol{E} &= \frac{\partial}{\partial t} \Re\boldsymbol{\mu}\boldsymbol{H} + \omega \Im\boldsymbol{\mu}\boldsymbol{H}. \end{aligned} \tag{1}$$

The $\boldsymbol{\varepsilon}$ and $\boldsymbol{\mu}$ are diagonal matrices:

$$\boldsymbol{\varepsilon} = \begin{pmatrix} \varepsilon_{00} & 0 & 0 \\ 0 & \varepsilon_{11} & 0 \\ 0 & 0 & \varepsilon_{22} \end{pmatrix}, \quad \boldsymbol{\mu} = \begin{pmatrix} \mu_{00} & 0 & 0 \\ 0 & \mu_{11} & 0 \\ 0 & 0 & \mu_{22} \end{pmatrix}.$$

II. CASE THAT $\Im\varepsilon/\Re\varepsilon > 0$

For the case, we set

$$\boldsymbol{R}\boldsymbol{H}(l_t) = \frac{\Re\boldsymbol{\varepsilon}}{\Delta t}\left(\boldsymbol{E}(l_t) - \boldsymbol{E}(l_t - 1)\right) + \frac{\omega\Im\boldsymbol{\varepsilon}}{2}\boldsymbol{E}\left(\boldsymbol{E}(l_t) + \boldsymbol{E}(l_t - 1)\right).$$

Then

$$\left[\frac{\Re\boldsymbol{\varepsilon}}{\Delta t} - \frac{\omega\Im\boldsymbol{\varepsilon}}{2}\right]\boldsymbol{E}(l_t - 1) + \boldsymbol{R}\boldsymbol{H}(l_t) = \left[\frac{\Re\boldsymbol{\varepsilon}}{\Delta t} + \frac{\omega\Im\boldsymbol{\varepsilon}}{2}\right]\boldsymbol{E}(l_t).$$

$$\boldsymbol{E}(l_t) = \frac{\frac{\Re\boldsymbol{\varepsilon}}{\Delta t} - \frac{\omega\Im\boldsymbol{\varepsilon}}{2}}{\frac{\Re\boldsymbol{\varepsilon}}{\Delta t} + \frac{\omega\Im\boldsymbol{\varepsilon}}{2}}\boldsymbol{E}(l_t - 1) + \frac{1}{\frac{\Re\boldsymbol{\varepsilon}}{\Delta t} + \frac{\omega\Im\boldsymbol{\varepsilon}}{2}}\boldsymbol{R}\boldsymbol{H}(l_t).$$

III. CASE THAT $\Im\varepsilon/\Re\varepsilon < 0$

We set that

$$\boldsymbol{E}' \triangleq \frac{\partial \boldsymbol{E}}{\partial t} = -i\omega\boldsymbol{E}.$$

$$\frac{\partial \boldsymbol{E}^{'}}{\partial t} = -i\omega \boldsymbol{E}^{'} = -\omega^2 \boldsymbol{E}\,.$$

$$\boldsymbol{RH} = \Re\varepsilon \boldsymbol{E}^{'} + \omega \Im\varepsilon \boldsymbol{E}\,,$$

From eq. (1),

$$\boldsymbol{RH} = \Re\varepsilon \boldsymbol{E}^{'} - \frac{\Im\varepsilon}{\omega}\frac{\partial \boldsymbol{E}^{'}}{\partial t}\,,$$

$$\boldsymbol{RH}\left(l_t\right) = \frac{\Re\varepsilon}{2}\left(\varDelta\boldsymbol{E}\left(l_t\right) + \varDelta\boldsymbol{E}\left(l_t - 1\right)\right) - \frac{\Im\varepsilon}{\omega\varDelta t}\left(\varDelta\boldsymbol{E}\left(l_t\right) - \varDelta\boldsymbol{E}\left(l_t - 1\right)\right)\,.$$

$$\varDelta\boldsymbol{E}\left(l_t\right) = -\frac{\frac{\Re\varepsilon}{2} + \frac{\Im\varepsilon}{\omega\varDelta t}}{\frac{\Re\varepsilon}{2} - \frac{\Im\varepsilon}{\omega\varDelta t}}\varDelta\boldsymbol{E}\left(l_t - 1\right) + \frac{1}{\frac{\Re\varepsilon}{2} - \frac{\Im\varepsilon}{\omega\varDelta t}}\boldsymbol{RH}\left(l_t\right)\,.$$

$$\boldsymbol{E}\left(l_t\right) \triangleq \boldsymbol{E}\left(l_t - 1\right) + \frac{\varDelta\boldsymbol{E}\left(l_t\right) + \varDelta\boldsymbol{E}\left(l_t - 1\right)}{2}\,.$$

Acknowledgments

The author would like to thank my family.