Check discrete Maxwell equations

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Scattering MATRix ANalysis

Abstract

This checks discretized Maxwell equations.

I. 3D MAXWELL EQUATIONS

The discrete Maxwell equations are given by

$$
\begin{pmatrix} 0 & -\triangle_2 & \triangle_1 \\ \triangle_2 & 0 & -\triangle_0 \\ -\triangle_1 & \triangle_0 & 0 \end{pmatrix} \begin{pmatrix} H_0 \\ H_1 \\ H_2 \end{pmatrix} = \frac{\partial}{\partial t} \begin{pmatrix} \varepsilon_{00} E_0 \\ \varepsilon_{11} E_1 \\ \varepsilon_{22} E_2 \end{pmatrix} ,
$$

$$
- \begin{pmatrix} 0 & -\triangledown_2 & \triangledown_1 \\ \triangledown_2 & 0 & -\triangledown_0 \\ -\triangledown_1 & \triangledown_0 & 0 \end{pmatrix} \begin{pmatrix} E_0 \\ E_1 \\ E_2 \end{pmatrix} = \frac{\partial}{\partial t} \begin{pmatrix} \mu_{00} H_0 \\ \mu_{11} H_1 \\ \mu_{22} H_2 \end{pmatrix} .
$$

From $\exp(-i\omega t)$, frequency domain equations show that

$$
\begin{pmatrix} 0 & -\triangle_2 & \triangle_1 \\ \triangle_2 & 0 & -\triangle_0 \\ -\triangle_1 & \triangle_0 & 0 \end{pmatrix} \begin{pmatrix} H_0 \\ H_1 \\ H_2 \end{pmatrix} = -i\omega \begin{pmatrix} \varepsilon_{00} E_0 \\ \varepsilon_{11} E_1 \\ \varepsilon_{22} E_2 \end{pmatrix} , \tag{1}
$$

$$
\begin{pmatrix} 0 & -\triangledown_2 & \triangledown_1 \\ \triangledown_2 & 0 & -\triangledown_0 \\ -\triangledown_1 & \triangledown_0 & 0 \end{pmatrix} \begin{pmatrix} E_0 \\ E_1 \\ E_2 \end{pmatrix} = i\omega \begin{pmatrix} \mu_{00} H_0 \\ \mu_{11} H_1 \\ \mu_{22} H_2 \end{pmatrix} . \tag{2}
$$

The E_2 and H_2 are given by

$$
\begin{aligned}
-i\omega\varepsilon_{22} E_2 &= -\triangle_1 H_0 + \triangle_0 H_1 , \\
i\omega\mu_{22} H_2 &= -\triangledown_1 E_0 + \triangledown_0 E_1 .
\end{aligned} \tag{3}
$$

From eqs. (1) and (3),

$$
\begin{aligned}
-\triangle_2 H_1 + \triangle_1 H_2 &= -i\omega\varepsilon_{00} E_0 \\
&= -\triangle_2 H_1 + \triangle_1 \frac{1}{i\omega\mu_{22}} \left(-\triangledown_1 E_0 + \triangledown_0 E_1\right)
\end{aligned}
$$

and

$$
\begin{aligned}
\triangle_2 H_0 - \triangle_0 H_2 &= -i\omega\varepsilon_{11} E_1 \\
&= \triangle_2 H_0 - \triangle_0 \frac{1}{i\omega\mu_{22}} \left(-\triangledown_1 E_0 + \triangledown_0 E_1\right) .
\end{aligned}
$$

Then,

$$
\begin{aligned}
\triangle_2 H_0 &= -i\omega\varepsilon_{11} E_1 + \triangle_0 \frac{i}{\omega\mu_{22}} \left(\triangledown_1 E_0 - \triangledown_0 E_1\right) , \\
\triangle_2 H_1 &= i\omega\varepsilon_{00} E_0 + \triangle_1 \frac{i}{\omega\mu_{22}} \left(\triangledown_1 E_0 - \triangledown_0 E_1\right) .
\end{aligned} \tag{4}
$$

From eqs. (2) and (3),

$$
\begin{aligned}
- \bigtriangledown_2 E_1 + \bigtriangledown_1 E_2 &= i\omega\mu_{00} H_0 \\
&= - \bigtriangledown_2 E_1 + \bigtriangledown_1 \frac{1}{-i\omega\varepsilon_{22}} \left(- \bigtriangleup_1 H_0 + \bigtriangleup_0 H_1\right)
\end{aligned}
$$

and

$$
\begin{aligned}
\bigtriangledown_2 E_0 - \bigtriangledown_0 E_2 &= i\omega\mu_{11} H_1 \\
&= \bigtriangledown_2 E_0 - \bigtriangledown_0 \frac{1}{-i\omega\varepsilon_{22}} \left(- \bigtriangleup_1 H_0 + \bigtriangleup_0 H_1\right) .
\end{aligned}
$$

Then

$$
\begin{aligned}
- \bigtriangledown_2 E_1 &= i\omega\mu_{00} H_0 - \bigtriangledown_1 \frac{i}{\omega\varepsilon_{22}} \left(- \bigtriangleup_1 H_0 + \bigtriangleup_0 H_1\right) \\
\bigtriangledown_2 E_0 &= i\omega\mu_{11} H_1 + \bigtriangledown_0 \frac{i}{\omega\varepsilon_{22}} \left(- \bigtriangleup_1 H_0 + \bigtriangleup_0 H_1\right)
\end{aligned}
\tag{5}
$$

From eqs. (4) and (5),

$$
\begin{aligned}
-i \bigtriangleup_2 \begin{pmatrix} H_0 \\ H_1 \end{pmatrix} &= \begin{pmatrix} \omega\varepsilon_{11} + \bigtriangleup_0 \frac{1}{\omega\mu_{22}} \bigtriangledown_0 & \bigtriangleup_0 \frac{1}{\omega\mu_{22}} \bigtriangledown_1 \\ \bigtriangleup_1 \frac{1}{\omega\mu_{22}} \bigtriangledown_0 & \omega\varepsilon_{00} + \bigtriangleup_1 \frac{1}{\omega\mu_{22}} \bigtriangledown_1 \end{pmatrix} \begin{pmatrix} -E_1 \\ E_0 \end{pmatrix} , \\
-i \bigtriangledown_2 \begin{pmatrix} -E_1 \\ E_0 \end{pmatrix} &= \begin{pmatrix} \omega\mu_{00} + \bigtriangledown_1 \frac{1}{\omega\varepsilon_{22}} \bigtriangleup_1 & - \bigtriangledown_1 \frac{1}{\omega\varepsilon_{22}} \bigtriangleup_0 \\ - \bigtriangledown_0 \frac{1}{\omega\varepsilon_{22}} \bigtriangleup_1 & \omega\mu_{11} + \bigtriangledown_0 \frac{1}{\omega\varepsilon_{22}} \bigtriangleup_0 \end{pmatrix} \begin{pmatrix} H_0 \\ H_1 \end{pmatrix} .
\end{aligned}
$$

$$
\begin{aligned}
-i \bigtriangleup_2 \boldsymbol{H}_{2D} &= \boldsymbol{m}_{bb} \boldsymbol{E}_{2D} , \\
-i \bigtriangledown_2 \boldsymbol{E}_{2D} &= \boldsymbol{m}_{aa} \boldsymbol{H}_{2D} .
\end{aligned}
\tag{6}
$$

where

$$
\begin{aligned}
\boldsymbol{H}_{2D} &\triangleq \begin{pmatrix} H_0 \\ H_1 \end{pmatrix}, \quad \boldsymbol{m}_{aa} \triangleq \begin{pmatrix} \omega\mu_{00} + \bigtriangledown_1 \frac{1}{\omega\varepsilon_{22}} \bigtriangleup_1 & - \bigtriangledown_1 \frac{1}{\omega\varepsilon_{22}} \bigtriangleup_0 \\ - \bigtriangledown_0 \frac{1}{\omega\varepsilon_{22}} \bigtriangleup_1 & \omega\mu_{11} + \bigtriangledown_0 \frac{1}{\omega\varepsilon_{22}} \bigtriangleup_0 \end{pmatrix} , \\
\boldsymbol{E}_{2D} &\triangleq \begin{pmatrix} -E_1 \\ E_0 \end{pmatrix}, \quad \boldsymbol{m}_{bb} \triangleq \begin{pmatrix} \omega\varepsilon_{11} + \bigtriangleup_0 \frac{1}{\omega\mu_{22}} \bigtriangledown_0 & \bigtriangleup_0 \frac{1}{\omega\mu_{22}} \bigtriangledown_1 \\ \bigtriangleup_1 \frac{1}{\omega\mu_{22}} \bigtriangledown_0 & \omega\varepsilon_{00} + \bigtriangleup_1 \frac{1}{\omega\mu_{22}} \bigtriangledown_1 \end{pmatrix} .
\end{aligned}
$$

II. EIGEN-VALUE EQUATION

We set $\boldsymbol{H}_{2D}(l_2+1) = e^{i\theta} \boldsymbol{H}_{2D}(l_2)$ and $\boldsymbol{E}_{2D}(l_2-1) = e^{-i\theta} \boldsymbol{E}_{2D}(l_2)$. Equation (6) is that

$$
\begin{aligned}
-i \left(e^{i\theta} - 1\right) \boldsymbol{H}_{2D} &= \boldsymbol{m}_{bb} \boldsymbol{E}_{2D} , \\
-i \left(1 - e^{-i\theta}\right) \boldsymbol{E}_{2D} &= \boldsymbol{m}_{aa} \boldsymbol{H}_{2D} .
\end{aligned}
$$

Then

$$2\sin\left(\frac{\theta}{2}\right)e^{i\theta/2}\boldsymbol{H}_{2D} = \boldsymbol{m}_{bb}\boldsymbol{E}_{2D}\,,$$
$$2\sin\left(\frac{\theta}{2}\right)\boldsymbol{E}_{2D} = \boldsymbol{m}_{aa}e^{i\theta/2}\boldsymbol{H}_{2D}\,.$$

We can set an eigenvalue equation:

$$\begin{pmatrix} \boldsymbol{m}_{aa} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{m}_{bb} \end{pmatrix}\begin{pmatrix} \boldsymbol{h}_m \\ \boldsymbol{e}_m \end{pmatrix} = 2\sin\left(\frac{\theta_m}{2}\right)\begin{pmatrix} \boldsymbol{0} & \boldsymbol{1} \\ \boldsymbol{1} & \boldsymbol{0} \end{pmatrix}\begin{pmatrix} \boldsymbol{h}_m \\ \boldsymbol{e}_m \end{pmatrix}\,,$$

where

$$\begin{pmatrix} \boldsymbol{H}_{2D} \\ \boldsymbol{E}_{2D} \end{pmatrix} = \begin{pmatrix} e^{-i\theta_m/2} & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \boldsymbol{h}_m \\ \boldsymbol{e}_m \end{pmatrix}.$$

III. 3D MAXWELL EQUATIONS FOR FDTD

We represent equations as

$$\boldsymbol{R}\boldsymbol{H}_{3D} = \frac{1}{\tau_R}\boldsymbol{\varepsilon}\,\bigtriangledown_t\,\boldsymbol{E}_{3D}\,,$$
$$-\boldsymbol{R}^\dagger\boldsymbol{E}_{3D} = \frac{1}{\tau_R}\boldsymbol{\mu}\,\triangle_t\,\boldsymbol{H}_{3D}\,,$$

where

$$\boldsymbol{R} = \begin{pmatrix} 0 & -\triangle_2 & \triangle_1 \\ \triangle_2 & 0 & -\triangle_0 \\ -\triangle_1 & \triangle_0 & 0 \end{pmatrix}, \quad \boldsymbol{H}_{3D} = \begin{pmatrix} H_0 \\ H_1 \\ H_2 \end{pmatrix}, \quad \boldsymbol{\varepsilon} = \begin{pmatrix} \varepsilon_{00} & 0 & 0 \\ 0 & \varepsilon_{11} & 0 \\ 0 & 0 & \varepsilon_{22} \end{pmatrix},$$
$$\boldsymbol{R}^\dagger = \begin{pmatrix} 0 & -\bigtriangledown_2 & \bigtriangledown_1 \\ \bigtriangledown_2 & 0 & -\bigtriangledown_0 \\ -\bigtriangledown_1 & \bigtriangledown_0 & 0 \end{pmatrix}, \quad \boldsymbol{E}_{3D} = \begin{pmatrix} E_0 \\ E_1 \\ E_2 \end{pmatrix}, \quad \boldsymbol{\mu} = \begin{pmatrix} \mu_{00} & 0 & 0 \\ 0 & \mu_{11} & 0 \\ 0 & 0 & \mu_{22} \end{pmatrix}.$$

$$\tau_R\boldsymbol{R}\boldsymbol{H}_{3D}(l_t+1) = \boldsymbol{\varepsilon}\left(\boldsymbol{E}_{3D}(l_t+1) - \boldsymbol{E}_{3D}(l_t)\right)\,,$$
$$-\tau_R\boldsymbol{R}^\dagger\boldsymbol{E}_{3D}(l_t) = \boldsymbol{\mu}\left(\boldsymbol{H}_{3D}(l_t+1) - \boldsymbol{H}_{3D}(l_t)\right)\,.$$

Acknowledgments

The author would like to thank my family.