Check discrete Maxwell equations

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Scattering MATRix ANalysis

Abstract

This checks discretized Maxwell equations.

I.   3D MAXWELL EQUATIONS

The discrete Maxwell equations are given by

$$
\begin{pmatrix} 0 & -\triangle_2 & \triangle_1 \\ \triangle_2 & 0 & -\triangle_0 \\ -\triangle_1 & \triangle_0 & 0 \end{pmatrix} \begin{pmatrix} H_0 \\ H_1 \\ H_2 \end{pmatrix} = \frac{\partial}{\partial t} \begin{pmatrix} \varepsilon_{00} E_0 \\ \varepsilon_{11} E_1 \\ \varepsilon_{22} E_2 \end{pmatrix} ,
$$

$$
- \begin{pmatrix} 0 & -\triangledown_2 & \triangledown_1 \\ \triangledown_2 & 0 & -\triangledown_0 \\ -\triangledown_1 & \triangledown_0 & 0 \end{pmatrix} \begin{pmatrix} E_0 \\ E_1 \\ E_2 \end{pmatrix} = \frac{\partial}{\partial t} \begin{pmatrix} \mu_{00} H_0 \\ \mu_{11} H_1 \\ \mu_{22} H_2 \end{pmatrix} .
$$

From $\exp\left(-i\omega t\right)$, frequency domain equations show that

$$
\begin{pmatrix} 0 & -\triangle_2 & \triangle_1 \\ \triangle_2 & 0 & -\triangle_0 \\ -\triangle_1 & \triangle_0 & 0 \end{pmatrix} \begin{pmatrix} H_0 \\ H_1 \\ H_2 \end{pmatrix} = -i\omega \begin{pmatrix} \varepsilon_{00} E_0 \\ \varepsilon_{11} E_1 \\ \varepsilon_{22} E_2 \end{pmatrix} , \tag{1}
$$

$$
\begin{pmatrix} 0 & -\triangledown_2 & \triangledown_1 \\ \triangledown_2 & 0 & -\triangledown_0 \\ -\triangledown_1 & \triangledown_0 & 0 \end{pmatrix} \begin{pmatrix} E_0 \\ E_1 \\ E_2 \end{pmatrix} = i\omega \begin{pmatrix} \mu_{00} H_0 \\ \mu_{11} H_1 \\ \mu_{22} H_2 \end{pmatrix} . \tag{2}
$$

The E_2 and H_2 are given by

$$
\begin{aligned}
-i\omega\varepsilon_{22} E_2 &= -\triangle_1 H_0 + \triangle_0 H_1 , \\
i\omega\mu_{22} H_2 &= -\triangledown_1 E_0 + \triangledown_0 E_1 .
\end{aligned} \tag{3}
$$

From eqs. (1) and (3),

$$
\begin{aligned}
-\triangle_2 H_1 + \triangle_1 H_2 &= -i\omega\varepsilon_{00} E_0 \\
&= -\triangle_2 H_1 + \triangle_1 \frac{1}{i\omega\mu_{22}} \left(-\triangledown_1 E_0 + \triangledown_0 E_1 \right)
\end{aligned}
$$

and

$$
\begin{aligned}
\triangle_2 H_0 - \triangle_0 H_2 &= -i\omega\varepsilon_{11} E_1 \\
&= \triangle_2 H_0 - \triangle_0 \frac{1}{i\omega\mu_{22}} \left(-\triangledown_1 E_0 + \triangledown_0 E_1 \right) .
\end{aligned}
$$

Then,

$$
\begin{aligned}
\triangle_2 H_0 &= -i\omega\varepsilon_{11} E_1 + \triangle_0 \frac{i}{\omega\mu_{22}} \left(\triangledown_1 E_0 - \triangledown_0 E_1 \right) , \\
\triangle_2 H_1 &= i\omega\varepsilon_{00} E_0 + \triangle_1 \frac{i}{\omega\mu_{22}} \left(\triangledown_1 E_0 - \triangledown_0 E_1 \right) .
\end{aligned} \tag{4}
$$

From eqs. (2) and (3),

$$-\bigtriangledown_2 E_1 + \bigtriangledown_1 E_2 = i\omega\mu_{00}H_0$$

$$= -\bigtriangledown_2 E_1 + \bigtriangledown_1 \frac{1}{-i\omega\varepsilon_{22}}\left(-\bigtriangleup_1 H_0 + \bigtriangleup_0 H_1\right)$$

and

$$\bigtriangledown_2 E_0 - \bigtriangledown_0 E_2 = i\omega\mu_{11}H_1$$

$$= \bigtriangledown_2 E_0 - \bigtriangledown_0 \frac{1}{-i\omega\varepsilon_{22}}\left(-\bigtriangleup_1 H_0 + \bigtriangleup_0 H_1\right).$$

Then

$$-\bigtriangledown_2 E_1 = i\omega\mu_{00}H_0 - \bigtriangledown_1 \frac{i}{\omega\varepsilon_{22}}\left(-\bigtriangleup_1 H_0 + \bigtriangleup_0 H_1\right)$$

$$\bigtriangledown_2 E_0 = i\omega\mu_{11}H_1 + \bigtriangledown_0 \frac{i}{\omega\varepsilon_{22}}\left(-\bigtriangleup_1 H_0 + \bigtriangleup_0 H_1\right) \tag{5}$$

From eqs. (4) and (5),

$$-i\bigtriangleup_2 \begin{pmatrix} H_0 \\ H_1 \end{pmatrix} = \begin{pmatrix} \omega\varepsilon_{11} + \bigtriangleup_0 \frac{1}{\omega\mu_{22}}\bigtriangledown_0 & \bigtriangleup_0 \frac{1}{\omega\mu_{22}}\bigtriangledown_1 \\ \bigtriangleup_1 \frac{1}{\omega\mu_{22}}\bigtriangledown_0 & \omega\varepsilon_{00} + \bigtriangleup_1 \frac{1}{\omega\mu_{22}}\bigtriangledown_1 \end{pmatrix} \begin{pmatrix} -E_1 \\ E_0 \end{pmatrix},$$

$$-i\bigtriangledown_2 \begin{pmatrix} -E_1 \\ E_0 \end{pmatrix} = \begin{pmatrix} \omega\mu_{00} + \bigtriangledown_1 \frac{1}{\omega\varepsilon_{22}}\bigtriangleup_1 & -\bigtriangledown_1 \frac{1}{\omega\varepsilon_{22}}\bigtriangleup_0 \\ -\bigtriangledown_0 \frac{1}{\omega\varepsilon_{22}}\bigtriangleup_1 & \omega\mu_{11} + \bigtriangledown_0 \frac{1}{\omega\varepsilon_{22}}\bigtriangleup_0 \end{pmatrix} \begin{pmatrix} H_0 \\ H_1 \end{pmatrix}.$$

$$-i\bigtriangleup_2 \boldsymbol{H}_{2D} = \boldsymbol{m}_{bb}\boldsymbol{E}_{2D},$$

$$-i\bigtriangledown_2 \boldsymbol{E}_{2D} = \boldsymbol{m}_{aa}\boldsymbol{H}_{2D}. \tag{6}$$

where

$$\boldsymbol{H}_{2D} \triangleq \begin{pmatrix} H_0 \\ H_1 \end{pmatrix}, \quad \boldsymbol{m}_{aa} \triangleq \begin{pmatrix} \omega\mu_{00} + \bigtriangledown_1 \frac{1}{\omega\varepsilon_{22}}\bigtriangleup_1 & -\bigtriangledown_1 \frac{1}{\omega\varepsilon_{22}}\bigtriangleup_0 \\ -\bigtriangledown_0 \frac{1}{\omega\varepsilon_{22}}\bigtriangleup_1 & \omega\mu_{11} + \bigtriangledown_0 \frac{1}{\omega\varepsilon_{22}}\bigtriangleup_0 \end{pmatrix},$$

$$\boldsymbol{E}_{2D} \triangleq \begin{pmatrix} -E_1 \\ E_0 \end{pmatrix}, \quad \boldsymbol{m}_{bb} \triangleq \begin{pmatrix} \omega\mu_{00} + \bigtriangledown_1 \frac{1}{\omega\varepsilon_{22}}\bigtriangleup_1 & -\bigtriangledown_1 \frac{1}{\omega\varepsilon_{22}}\bigtriangleup_0 \\ -\bigtriangledown_0 \frac{1}{\omega\varepsilon_{22}}\bigtriangleup_1 & \omega\mu_{11} + \bigtriangledown_0 \frac{1}{\omega\varepsilon_{22}}\bigtriangleup_0 \end{pmatrix}.$$

II.   EIGEN-VALUE EQUATION

We set $\boldsymbol{H}_{2D}\left(l_2 + 1\right) = e^{i\theta}\boldsymbol{H}_{2D}\left(l_2\right)$ and $\boldsymbol{E}_{2D}\left(l_2 - 1\right) = e^{-i\theta}\boldsymbol{E}_{2D}\left(l_2\right)$. Equation (6) is that

$$-i\left(e^{i\theta} - 1\right)\boldsymbol{H}_{2D} = \boldsymbol{m}_{bb}\boldsymbol{E}_{2D},$$

$$-i\left(1 - e^{-i\theta}\right)\boldsymbol{E}_{2D} = \boldsymbol{m}_{aa}\boldsymbol{H}_{2D}.$$

Then

$$2\sin\left(\frac{\theta}{2}\right)e^{i\theta/2}\boldsymbol{H}_{2D} = \boldsymbol{m}_{bb}\boldsymbol{E}_{2D}\,,$$

$$2\sin\left(\frac{\theta}{2}\right)\boldsymbol{E}_{2D} = \boldsymbol{m}_{aa}e^{i\theta/2}\boldsymbol{H}_{2D}\,.$$

We can set an eigenvalue equation:

$$\begin{pmatrix} \boldsymbol{m}_{aa} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{m}_{bb} \end{pmatrix}\begin{pmatrix} \boldsymbol{h}_m \\ \boldsymbol{e}_m \end{pmatrix} = 2\sin\left(\frac{\theta_m}{2}\right)\begin{pmatrix} \boldsymbol{0} & \boldsymbol{1} \\ \boldsymbol{1} & \boldsymbol{0} \end{pmatrix}\begin{pmatrix} \boldsymbol{h}_m \\ \boldsymbol{e}_m \end{pmatrix}\,,$$

where

$$\begin{pmatrix} \boldsymbol{H}_{2D} \\ \boldsymbol{E}_{2D} \end{pmatrix} = \begin{pmatrix} e^{-i\theta_m/2} & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \boldsymbol{h}_m \\ \boldsymbol{e}_m \end{pmatrix}\,.$$