Check discrete Maxwell equations

 ${\it Tatsuya~Usuki}$ ${\it Scattering~MATRix~ANalysis}$

Abstract

This checks discretized Maxwell equations.

3D MAXWELL EQUATIONS

The discrete Maxwell equations are given by

$$\begin{pmatrix} 0 & -\triangle_2 & \triangle_1 \\ \triangle_2 & 0 & -\triangle_0 \\ -\triangle_1 & \triangle_0 & 0 \end{pmatrix} \begin{pmatrix} H_0 \\ H_1 \\ H_2 \end{pmatrix} = \frac{\partial}{\partial t} \begin{pmatrix} \varepsilon_{00} E_0 \\ \varepsilon_{11} E_1 \\ \varepsilon_{22} E_2 \end{pmatrix},$$

$$-\begin{pmatrix} 0 & -\nabla_2 & \nabla_1 \\ \nabla_2 & 0 & -\nabla_0 \\ -\nabla_1 & \nabla_0 & 0 \end{pmatrix} \begin{pmatrix} E_0 \\ E_1 \\ E_2 \end{pmatrix} = \frac{\partial}{\partial t} \begin{pmatrix} \mu_{00} H_0 \\ \mu_{11} H_1 \\ \mu_{22} H_2 \end{pmatrix}.$$

From $\exp(-i\omega t)$, frequency domain equations show that

$$\begin{pmatrix} 0 & -\triangle_2 & \triangle_1 \\ \triangle_2 & 0 & -\triangle_0 \\ -\triangle_1 & \triangle_0 & 0 \end{pmatrix} \begin{pmatrix} H_0 \\ H_1 \\ H_2 \end{pmatrix} = -i\omega \begin{pmatrix} \varepsilon_{00}E_0 \\ \varepsilon_{11}E_1 \\ \varepsilon_{22}E_2 \end{pmatrix}, \tag{1}$$

$$\begin{pmatrix} 0 & -\nabla_2 & \nabla_1 \\ \nabla_2 & 0 & -\nabla_0 \\ -\nabla_1 & \nabla_2 & 0 \end{pmatrix} \begin{pmatrix} E_0 \\ E_1 \\ E_2 \end{pmatrix} = i\omega \begin{pmatrix} \mu_{00}H_0 \\ \mu_{11}H_1 \\ \mu_{22}H_2 \end{pmatrix}. \tag{2}$$

$$\begin{pmatrix}
0 & -\nabla_2 & \nabla_1 \\
\nabla_2 & 0 & -\nabla_0 \\
-\nabla_1 & \nabla_0 & 0
\end{pmatrix}
\begin{pmatrix}
E_0 \\
E_1 \\
E_2
\end{pmatrix} = i\omega \begin{pmatrix}
\mu_{00}H_0 \\
\mu_{11}H_1 \\
\mu_{22}H_2
\end{pmatrix}.$$
(2)

The E_2 and H_2 are given by

$$-i\omega\varepsilon_{22}E_2 = -\Delta_1 H_0 + \Delta_0 H_1,$$

$$i\omega\mu_{22}H_2 = -\nabla_1 E_0 + \nabla_0 E_1.$$
(3)

From eqs. (1) and (3),

$$- \triangle_2 H_1 + \triangle_1 H_2 = -i\omega \varepsilon_{00} E_0$$
$$= - \triangle_2 H_1 + \triangle_1 \frac{1}{i\omega \mu_{22}} \left(- \nabla_1 E_0 + \nabla_0 E_1 \right)$$

and

$$\Delta_2 H_0 - \Delta_0 H_2 = -i\omega \varepsilon_{11} E_1$$

$$= \Delta_2 H_0 - \Delta_0 \frac{1}{i\omega \mu_{22}} \left(-\nabla_1 E_0 + \nabla_0 E_1 \right).$$

Then,

$$\triangle_2 H_0 = -i\omega \varepsilon_{11} E_1 + \triangle_0 \frac{i}{\omega \mu_{22}} \left(\nabla_1 E_0 - \nabla_0 E_1 \right) ,$$

$$\triangle_2 H_1 = i\omega \varepsilon_{00} E_0 + \triangle_1 \frac{i}{\omega \mu_{22}} \left(\nabla_1 E_0 - \nabla_0 E_1 \right) .$$
(4)

From eqs. (2) and (3),

$$- \nabla_2 E_1 + \nabla_1 E_2 = i\omega \mu_{00} H_0$$
$$= - \nabla_2 E_1 + \nabla_1 \frac{1}{-i\omega \varepsilon_{22}} \left(- \Delta_1 H_0 + \Delta_0 H_1 \right)$$

and

$$\nabla_2 E_0 - \nabla_0 E_2 = i\omega \mu_{11} H_1$$

$$= \nabla_2 E_0 - \nabla_0 \frac{1}{-i\omega \varepsilon_{22}} \left(-\Delta_1 H_0 + \Delta_0 H_1 \right) .$$

Then

$$- \nabla_2 E_1 = i\omega \mu_{00} H_0 - \nabla_1 \frac{i}{\omega \varepsilon_{22}} \left(- \Delta_1 H_0 + \Delta_0 H_1 \right)$$

$$\nabla_2 E_0 = i\omega \mu_{11} H_1 + \nabla_0 \frac{i}{\omega \varepsilon_{22}} \left(- \Delta_1 H_0 + \Delta_0 H_1 \right)$$
(5)

From eqs. (4) and (5),

$$-i \triangle_{2} \begin{pmatrix} H_{0} \\ H_{1} \end{pmatrix} = \begin{pmatrix} \omega \varepsilon_{11} + \triangle_{0} \frac{1}{\omega \mu_{22}} \nabla_{0} & \triangle_{0} \frac{1}{\omega \mu_{22}} \nabla_{1} \\ \triangle_{1} \frac{1}{\omega \mu_{22}} \nabla_{0} & \omega \varepsilon_{00} + \triangle_{1} \frac{1}{\omega \mu_{22}} \nabla_{1} \end{pmatrix} \begin{pmatrix} -E_{1} \\ E_{0} \end{pmatrix},$$

$$-i \nabla_{2} \begin{pmatrix} -E_{1} \\ E_{0} \end{pmatrix} = \begin{pmatrix} \omega \mu_{00} + \nabla_{1} \frac{1}{\omega \varepsilon_{22}} \triangle_{1} & -\nabla_{1} \frac{1}{\omega \varepsilon_{22}} \triangle_{0} \\ -\nabla_{0} \frac{1}{\omega \varepsilon_{22}} \triangle_{1} & \omega \mu_{11} + \nabla_{0} \frac{1}{\omega \varepsilon_{22}} \triangle_{0} \end{pmatrix} \begin{pmatrix} H_{0} \\ H_{1} \end{pmatrix}.$$

$$-i \triangle_{2} \mathbf{H}_{2D} = \mathbf{m}_{bb} \mathbf{E}_{2D},$$

$$-i \nabla_{2} \mathbf{E}_{2D} = \mathbf{m}_{aa} \mathbf{H}_{2D}.$$

$$(6)$$

where

$$egin{aligned} oldsymbol{H}_{2D} & riangleq egin{aligned} oldsymbol{H}_{2D} & riangleq egin{aligned} oldsymbol{H}_{1} \end{pmatrix}, & oldsymbol{m}_{aa} & riangleq egin{aligned} \omega \mu_{00} + igtriangledown_{1} rac{1}{\omega arepsilon_{22}} igtriangledown_{1} & - igtriangledown_{1} rac{1}{\omega arepsilon_{22}} igtriangledown_{0} \ - igtriangledown_{0} rac{1}{\omega arepsilon_{22}} igtriangledown_{1} & \omega \mu_{11} + igtriangledown_{0} rac{1}{\omega arepsilon_{22}} igtriangledown_{0} \ oldsymbol{E}_{2D} & egin{aligned} \Delta_{1} rac{1}{\omega \mu_{22}} igtriangledown_{0} & igtriangledown_{0} rac{1}{\omega \mu_{22}} igtriangledown_{1} \ egin{aligned} \Delta_{1} rac{1}{\omega \mu_{22}} igtriangledown_{0} & \omega arepsilon_{00} + igtriangledown_{1} rac{1}{\omega \mu_{22}} igtriangledown_{1} \ egin{aligned} \Delta_{1} rac{1}{\omega \mu_{22}} igtriangledown_{0} & \omega arepsilon_{00} + igtriangledown_{1} rac{1}{\omega \mu_{22}} igtriangledown_{1} \ egin{aligned} \Delta_{1} rac{1}{\omega \mu_{22}} igtriangledown_{1} & \omega arepsilon_{0} + igtriangledown_{1} rac{1}{\omega \mu_{22}} igtriangledown_{1} \ egin{aligned} \Delta_{1} rac{1}{\omega \mu_{22}} igtriangledown_{1} & \omega arepsilon_{0} + igtriangledown_{1} rac{1}{\omega \mu_{22}} igtriangledown_{1} \ egin{aligned} \Delta_{1} rac{1}{\omega \mu_{22}} igtriangledown_{1} & \omega arepsilon_{0} + igtriangledown_{1} rac{1}{\omega \mu_{22}} igtriangledown_{1} \ egin{aligned} \Delta_{1} rac{1}{\omega \mu_{22}} igtriangledown_{2} & \omega arepsilon_{0} + oldsymbol{\Delta_{1}} rac{1}{\omega \mu_{22}} igtriangledown_{2} \ egin{aligned} \Delta_{1} & \omega \mu_{1} & \omega \mu_{2} & \omega \mu_{2} \ egin{aligned} \Delta_{1} & \omega \mu_{2} & \omega \mu_{2} & \omega \mu_{2} \ egin{aligned} \Delta_{1} & \omega \mu_{2} & \omega \mu_{2} & \omega \mu_{2} \ egin{aligned} \Delta_{2} & \omega \mu_{2} & \omega \mu_{2} \ egin{aligned} \Delta_{1} & \omega \mu_{2} & \omega \mu_{2} \ egin{aligned} \Delta_{2} & \omega \mu_{2} & \omega \mu_{2} \ egin{aligned} \Delta_{2} & \omega \mu_{2} & \omega \mu_{2} \ egin{aligned} \Delta_{2} & \omega \mu_{2} & \omega \mu_{2} \ egin{aligned} \Delta_{2} & \omega \mu_{2} & \omega \mu_{2} \ egin{aligned} \Delta_{2} & \omega \mu_{2} & \omega \mu_{2} \ egin{aligned} \Delta_{2} & \omega \mu_{2} & \omega \mu_{2} \ egin{aligned} \Delta_{2} & \omega \mu_{2} & \omega \mu_{2} \ egin{aligned} \Delta_{2} & \omega \mu_{2} & \omega \mu_{$$

II. EIGEN-VALUE EQUATION

We set
$$\boldsymbol{H}_{2D}\left(l_{2}+1\right)=e^{i\theta}\boldsymbol{H}_{2D}\left(l_{2}\right)$$
 and $\boldsymbol{E}_{2D}\left(l_{2}-1\right)=e^{-i\theta}\boldsymbol{E}_{2D}\left(l_{2}\right)$. Equation (6) is that
$$-i\left(e^{i\theta}-1\right)\boldsymbol{H}_{2D}=\boldsymbol{m}_{bb}\boldsymbol{E}_{2D}\,,$$

$$-i\left(1-e^{-i\theta}\right)\boldsymbol{E}_{2D}=\boldsymbol{m}_{aa}\boldsymbol{H}_{2D}\,.$$

Then

$$2\sin\left(rac{ heta}{2}
ight)e^{i heta/2}oldsymbol{H}_{2D} = oldsymbol{m}_{bb}oldsymbol{E}_{2D}\,, \ 2\sin\left(rac{ heta}{2}
ight)oldsymbol{E}_{2D} = oldsymbol{m}_{aa}e^{i heta/2}oldsymbol{H}_{2D}\,.$$

We can set an eigenvalue equation:

$$egin{pmatrix} egin{pmatrix} m{m}_{aa} & \mathbf{0} \ \mathbf{0} & m{m}_{bb} \end{pmatrix} egin{pmatrix} m{h}_m \ m{e}_m \end{pmatrix} = 2\sin\left(rac{ heta_m}{2}
ight) egin{pmatrix} \mathbf{0} & \mathbf{1} \ \mathbf{1} & \mathbf{0} \end{pmatrix} egin{pmatrix} m{h}_m \ m{e}_m \end{pmatrix} \, ,$$

where

$$\left(egin{array}{c} oldsymbol{H}_{2D} \ oldsymbol{E}_{2D} \end{array}
ight) = \left(egin{array}{c} e^{-i heta_m/2} & 0 \ 0 & 1 \end{array}
ight) \left(egin{array}{c} oldsymbol{h}_m \ oldsymbol{e}_m \end{array}
ight).$$

III. 3D MAXWELL EQUATIONS FOR FDTD

We represent equations as

$$egin{aligned} m{R}m{H}_{3D} &= rac{1}{ au_R}m{arepsilon} m{arepsilon}_t m{E}_{3D} \,, \ -m{R}^\daggerm{E}_{3D} &= rac{1}{ au_R}m{\mu} igtriangle_t m{H}_{3D} \,, \end{aligned}$$

where

$$\mathbf{R} = \begin{pmatrix}
0 & -\triangle_2 & \triangle_1 \\
\triangle_2 & 0 & -\triangle_0 \\
-\triangle_1 & \triangle_0 & 0
\end{pmatrix}, \quad \mathbf{H}_{3D} = \begin{pmatrix}
H_0 \\
H_1 \\
H_2
\end{pmatrix}, \quad \boldsymbol{\varepsilon} = \begin{pmatrix}
\varepsilon_{00} & 0 & 0 \\
0 & \varepsilon_{11} & 0 \\
0 & 0 & \varepsilon_{22}
\end{pmatrix},$$

$$\mathbf{R}^{\dagger} = \begin{pmatrix}
0 & -\nabla_2 & \nabla_1 \\
\nabla_2 & 0 & -\nabla_0 \\
-\nabla_1 & \nabla_0 & 0
\end{pmatrix}, \quad \mathbf{E}_{3D} = \begin{pmatrix}
E_0 \\
E_1 \\
E_2
\end{pmatrix}, \quad \boldsymbol{\mu} = \begin{pmatrix}
\mu_{00} & 0 & 0 \\
0 & \mu_{11} & 0 \\
0 & 0 & \mu_{22}
\end{pmatrix}.$$

$$au_{R}\mathbf{R}\mathbf{H}_{3D}\left(l_{t}+1\right)=\boldsymbol{\varepsilon}\left(\mathbf{E}_{3D}\left(l_{t}+1\right)-\mathbf{E}_{3D}\left(l_{t}\right)\right)\,,$$

$$- au_{R}\mathbf{R}^{\dagger}\mathbf{E}_{3D}\left(l_{t}\right)=\boldsymbol{\mu}\left(\mathbf{H}_{3D}\left(l_{t}+1\right)-\mathbf{H}_{3D}\left(l_{t}\right)\right)\,.$$

Acknowledgments

The author would like to thank my family.