It suffices to begin with some definitions and notations we see in Quantum Mechanics. First we will denote a quantum state like $|\psi\rangle$. A quantum state is an element of our (possibly infinite) dimensional Hilbert space V. Similarly we have $\langle\psi|$ which is member of the dual space V^*. We can define an inner product on V called $(\cdot,\cdot)$. An important property of this inner product is its anti-symmetry, that is $(\phi,\psi) = (\psi^*,\phi)$. Now for some operator H, we will say it is Hermitian in the case that $(\phi, H\psi) = (H\phi, \psi)$.

Statement. *The eigenvalues of* H *are real*

Let χ be a normalized vector such that $H\chi = \lambda\chi$. Then $(\chi, H\chi) = (\chi, \lambda\chi) = (\lambda\chi, \chi))$ it becomes clear that

$$\lambda^*(\chi,\chi) = \lambda(\chi,\chi)$$

and thus $\lambda^* = \lambda$.

Statement. *The eigenstates corresponding to different eigenvalues of Hermitian operator H are orthogonal.*

Let $|\phi\rangle$ and $|\psi\rangle$ be eigenstates corresponding to different eigenvalues of H. Knowing $\langle\phi| H |\psi\rangle = \langle\phi| H^* |\psi\rangle$ this tells us that $\lambda \langle\phi|\psi\rangle = \chi \langle\phi|\psi\rangle$ since $\lambda \neq \chi$ this tell us that $\langle\phi|\psi\rangle = 0$.

We should note that this Hilbert space is complete, this is given some $|\psi\rangle$ we have $|\psi\rangle = \sum_n c_n |\psi_n\rangle$. It turns out we can explicitly calculate c_n using this condition. $\langle\psi_n|\psi\rangle = \sum_n c_m \langle\psi_n|\psi_m\rangle$ using the orthonormality of the $|\phi_n\rangle$'s we have that $\langle\psi_n|\psi_m\rangle = \delta_{mn}$ so finally $\langle\psi_n|\psi\rangle = c_n$.

Statement. $\sum_n |\psi_n\rangle \langle\psi_n| = 1$

$|\psi\rangle = \sum_n \langle\psi_n|\psi\rangle |\psi_n\rangle = \sum_n |\psi_n\rangle \langle\psi_n|\psi\rangle$ which of course implies that $\sum_n |\psi_n\rangle \langle\psi_n| = 1$.