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Part I

Quantum Groups

Chapter 1

Preliminaries

1 Co-Algebra

[1.1] (Free Algebras)**.** Fix a field k and a set X. The k-vector space with basis the set of all words (called monomials) in X including the empty word is denoted $k\{X\}$. An associative multiplication can be defined on $k\{X\}$ by concatenation of words. The empty word plays the role of 1. Hence $k\{X\}$ is a k-algebra called the free algebra on X.

We have the following UMP: given a k-algebra A and a set map $f : X \to A$, there is an unique algebra map $\overline{f} : k\{X\} \to A$ such that $f = \overline{f}\iota$, where $\iota : X \to k\{X\}$ is the natural inclusion.

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "\iota"'] & & A \\ & k\{X\} \arrow[ur, dashed, "\overline{f}"'] & \end{tikzcd}$$

In other words, we have a natural bijection $\mathrm{Hom}(k\{X\}, A) \cong A^X$. Combined with the UMP of quotient rings, any two-sided ideal I induces a (useful) bijection

$$\mathrm{Hom}(k\{X\}/I, A) \cong \left\{ f \in A^X \;\middle|\; \overline{f}(I) = 0 \right\}.$$

As an application, the polynomial algebra $k[X]$ can be realized as the quotient $k\{X\}/I$, where I is the two-sided ideal generated by terms $xx' - x'x$ with $x, x' \in X$. We have a natural bijection

$$\mathrm{Hom}(k[X], A) \cong \left\{ (a_i) \in A^X \;\middle|\; a_i a_j = a_j a_i \text{ for all } i, j \in X \right\}.$$

In particular, if A is a *commutative* k-algebra, then $\mathrm{Hom}(k[X], A) \cong A^X$. So $k[X]$ is the free commutative k-algebra.

[1.2] (The Affine Line and Plane)**.**

[1.3]. Yoneda lemma says that $\mathrm{Nat}(\mathrm{Hom}(C, -), F) \cong F(C)$ via $\alpha \mapsto \alpha_C(\mathrm{id}_C)$.

Let A be a commutative k-algebra. The addition operation on A then induces a natural transformation $\eta : \mathrm{Hom}(k[x] \otimes k[x], -) \to \mathrm{Hom}(k[x], -)$ given by

$$\eta_A : (x \otimes 1 \mapsto a_1, 1 \otimes x \mapsto a_2) \mapsto (x \mapsto a_1 + a_2)$$

for all $a_1, a_2 \in A$. Under the identification $\mathrm{Hom}(k[x]^{\otimes n}) = A^n$, η_A is just the addition map.

Yoneda lemma tells us that this natural transformation corresponds (naturally) to some $\Delta \in \mathrm{Hom}(k[x], k[x] \otimes k[x])$. In fact, $\Delta = \eta_{k[x]\otimes k[x]}(\mathrm{id}_{k[x]\otimes k[x]})$. The map $\mathrm{id}_{k[x]\otimes k[x]}$ sends $x \otimes 1 \mapsto x \otimes 1$ and $1 \otimes x \mapsto 1 \otimes x$, so Δ is the map which sends $x \mapsto x \otimes 1 + 1 \otimes x$.