Shlonotes

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Contents

I	Quantum Groups											3									
1	Pre	liminaries																			4
	1	The Affine Line and Plane																			4

Part I Quantum Groups

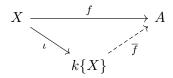
Chapter 1

Preliminaries

1 Co-Algebra

[1.1] (Free Algebras). Fix a field k and a set X. The k-vector space with basis the set of all words (called monomials) in X including the empty word is denoted $k\{X\}$. An associative multiplication can be defined on $k\{X\}$ by concatenation of words. The empty word plays the role of 1. Hence $k\{X\}$ is a k-algebra called the free algebra on X.

We have the following UMP: given a k-algebra A and a set map $f: X \to A$, there is an unique algebra map $\overline{f}: k\{X\} \to A$ such that $f = \overline{f}\iota$, where $\iota: X \to k\{X\}$ is the natural inclusion.



In other words, we have a natural bijection $\operatorname{Hom}(k\{X\},A) \cong A^X$. Combined with the UMP of quotient rings, any two-sided ideal I induces a (useful) bijection

$$\operatorname{Hom}(k\{X\}/I, A) \cong \left\{ f \in A^X \mid \overline{f}(I) = 0 \right\}.$$

As an application, the polynomial algebra k[X] can be realized as the quotient $k\{X\}/I$, where I is the two-sided ideal generated by terms xx' - x'x with $x, x' \in X$. We have a natural bijection

$$\operatorname{Hom}(k[X], A) \cong \{(a_i) \in A^X \mid a_i a_j = a_j a_i \text{ for all } i, j \in X\}.$$

In particular, if A is a commutative k-algebra, then $\operatorname{Hom}(k[X],A)\cong A^X$. So k[X] is the free commutative k-algebra.

[1.2] (The Affine Line and Plane).

[1.3]. Yoneda lemma says that $Nat(Hom(C, -), F) \cong F(C)$ via $\alpha \mapsto \alpha_C(id_C)$.

Let A be a commutative k-algebra. The addition operation on A then induces a natural transformation $\eta: \operatorname{Hom}(k[x] \otimes k[x], -) \to \operatorname{Hom}(k[x], -)$ given by

$$\eta_A: (x\otimes 1\mapsto a_1, 1\otimes x\mapsto a_2)\mapsto (x\mapsto a_1+a_2)$$

for all $a_1, a_2 \in A$. Under the identification $\operatorname{Hom}(k[x]^{\otimes n}) = A^n$, η_A is just the addition map.

Yoneda lemma tells us that this natural transformation corresponds (naturally) to some $\Delta \in \text{Hom}(k[x], k[x] \otimes k[x])$. In fact, $\Delta = \eta_{k[x] \otimes k[x]}(\text{id}_{k[x] \otimes k[x]})$. The map $\text{id}_{k[x] \otimes k[x]}$ sends $x \otimes 1 \mapsto x \otimes 1$ and $1 \otimes x \mapsto 1 \otimes x$, so Δ is the map which sends $x \mapsto x \otimes 1 + 1 \otimes x$.