

# Numerical Integration

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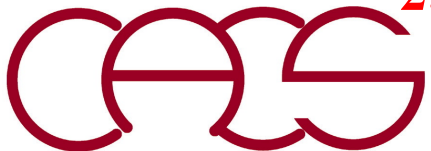
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**New toolbox (use it! it's user friendly):**

- 1. Gaussian quadratures (orthogonal functions)**
- 2. Newton's method**



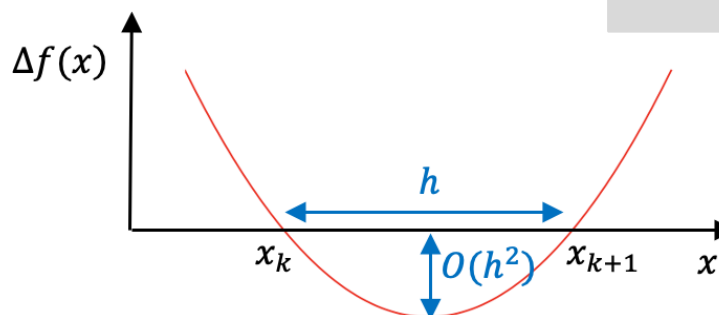
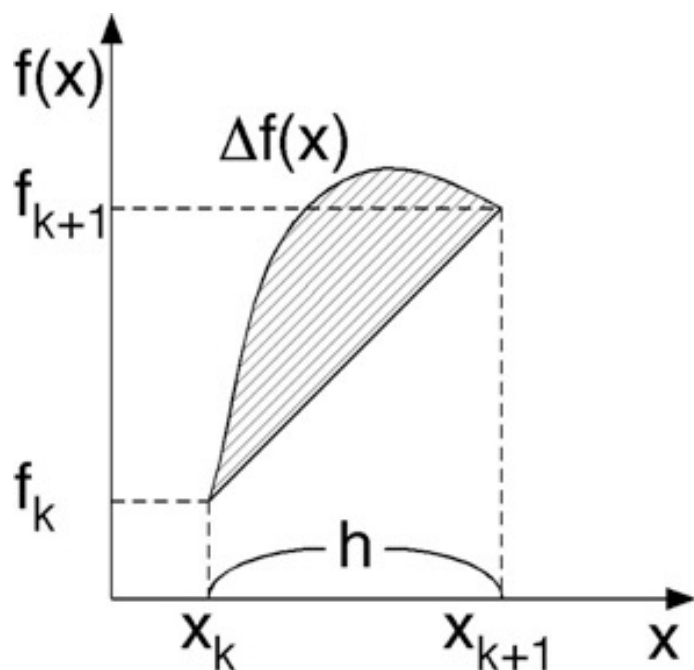
# Numerical Integration

- Numerical integration = weighted sum of function values**

$$S = \int_a^b f(x)dx \cong \sum_{k=0}^{n-1} w_k f(x_k)$$

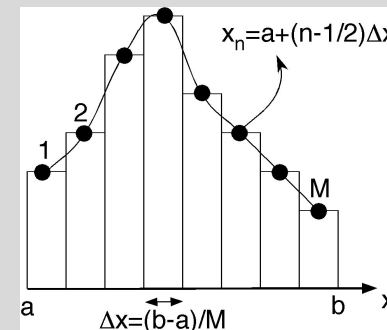
- Trapezoid quadrature: Piecewise linear approximation**

$$f(x) \cong f_k + (x - x_k)(f_{k+1} - f_k)/h \quad x \in [x_k, x_{k+1}]$$



$$\text{error} = \text{height } (O(h^2)) \times \text{base } (h) \times \# \text{ of bins } \left( \frac{b-a}{h} \right) = O(h^2)$$

**Resulting area:**  $\int_a^b f(x)dx \cong \frac{h}{2} \sum_{k=0}^{n-1} (f_k + f_{k+1}) + O(h^2)$



Piecewise constant  
 $O(h)$  approximation

$$\begin{cases} x_k = kh = (b-a)k/n \\ w_k = h \end{cases}$$

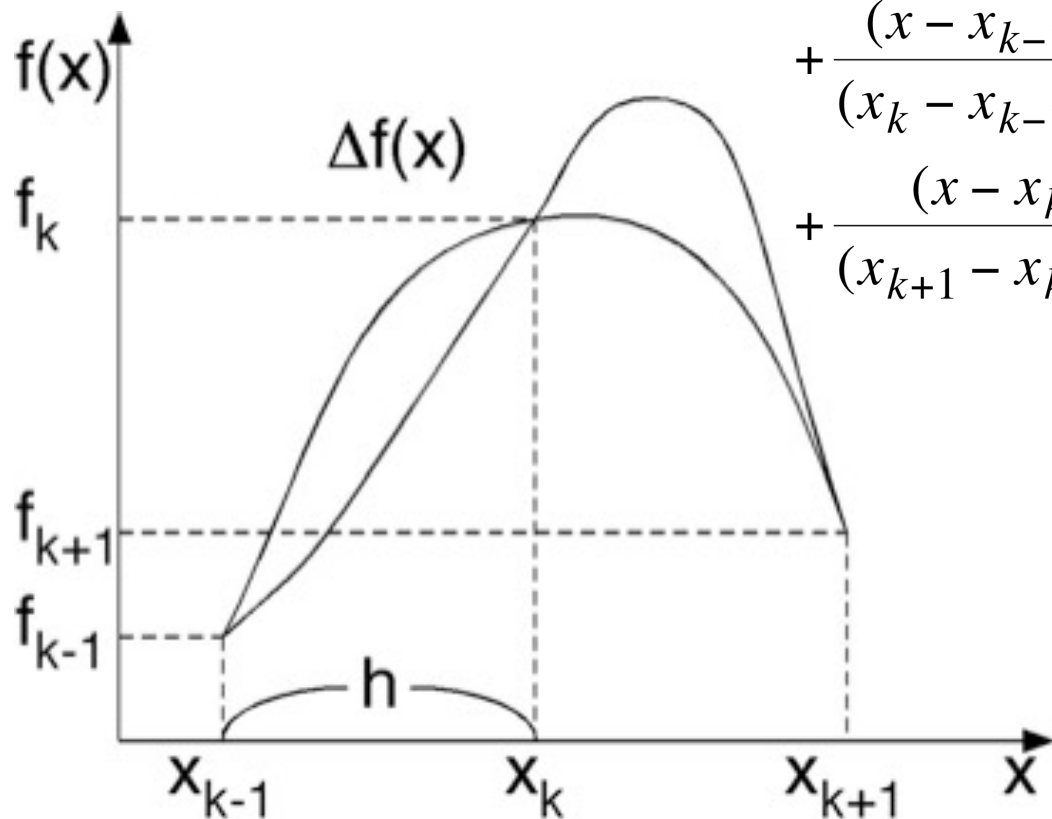
# Simpson Rule

- **Simpson quadrature: Piecewise quadratic approximation**

- **Lagrange interpolation:**  $f(x) \cong \frac{(x - x_k)(x - x_{k+1})}{(x_{k-1} - x_k)(x_{k-1} - x_{k+1})} f_{k-1}$

$$+ \frac{(x - x_{k-1})(x - x_{k+1})}{(x_k - x_{k-1})(x_k - x_{k+1})} f_k$$

$$+ \frac{(x - x_{k-1})(x - x_k)}{(x_{k+1} - x_{k-1})(x_{k+1} - x_k)} f_{k+1}$$



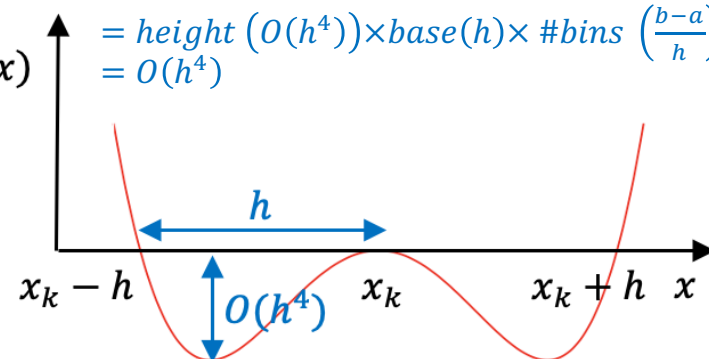
$$\int_{x_{k-1}}^{x_{k+1}} \Delta f(x) dx = 0$$

$\Delta f(x) = c(x-h)x(x+h)$

error  
 $= \text{height } (O(h^4)) \times \text{base } (h) \times \text{\#bins } \left(\frac{b-a}{h}\right)$   
 $= O(h^4)$

$$\int_a^b f(x) dx \cong \frac{h}{3} \sum_{l=0}^{n/2-1} (f_{2l} + 4f_{2l+1} + f_{2l+2}) + O(h^4)$$

**Nonuniform weight!**



# Gaussian Quadratures

- **Idea of Gaussian quadrature:** Freedom to choose both weighting coefficients & the location of abscissas to evaluate the function
- **Gaussian quadrature:** Chooses the weight & abscissas to make the integral exact for a class of integrands “polynomials times some known function  $W(x)$ ”.

> **Gauss-Legendre:**  $W(x) = 1; -1 < x < 1$

> **Gauss-Chebyshev:**  $W(x) = (1 - x^2)^{-1/2}; -1 < x < 1$

...

$$\int_a^b W(x) f(x) dx = \sum_{k=1}^n w_k f(x_k)$$

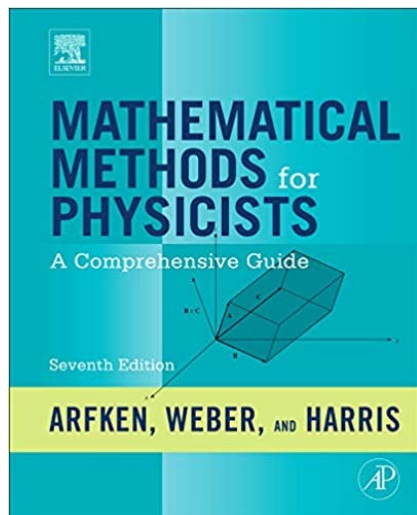
- **New toolbox:** (1) **orthogonal functions** (recursive generation via a generating function); (2) **Newton method** for root finding

See [gauleg-driver.c](#) & [gauleg.c](#)

W.H. Press, B.P. Flannery, S.A. Teukolsky, & W.T. Vetterling,  
*Numerical Recipes, 2nd Ed.* (Cambridge U Press, '93), Sec. 4.5

# Orthogonal Functions

- Gaussian quadratures are defined through orthogonal functions
- Orthogonal functions are often introduced as solutions to differential equations
- **Examples:** Legendre, Bessel, Laguerre, Hermite, Chebyshev, ...
- Operationally well-defined to compute the function values & derivatives
- Efficiently computable through **recursive relations** (more than elementary functions like  $\sin(x)$ ,  $\exp(x)$ , ...)



13	Gamma Function	599
14	Bessel Functions	643
15	Legendre Functions	715

# Orthogonal Functions

- **Scalar product (vector space):**

$$\langle f | g \rangle \equiv \int_a^b W(x) f(x) g(x) dx$$

- **Orthonormal set of functions: Mutually orthogonal & normalized**

$$\langle p_m | p_n \rangle = \delta_{m,n} = \begin{cases} 1, & m = n \\ 0, & m \neq n \end{cases}$$

- **Recurrence relation to construct an orthonormal set:**

$$p_{-1}(x) \equiv 0$$

$$p_0(x) \equiv 1$$

$$p_{j+1}(x) = (x - a_j)p_j(x) - b_j p_{j-1}(x) \quad j = 0, 1, 2, \dots$$

$$a_j = \frac{\langle x p_j | p_j \rangle}{\langle p_j | p_j \rangle} \quad j = 0, 1, \dots$$

$$b_j = \frac{\langle p_j | p_j \rangle}{\langle p_{j-1} | p_{j-1} \rangle} \quad j = 1, 2, \dots$$

**(Theorem)**  $p_j(x)$  has exactly  $j$  distinct roots in  $(a,b)$ , & the roots interleave the  $j-1$  roots of  $p_{j-1}(x)$

# Legendre Polynomial

$$W(x) = 1 \quad -1 < x < 1$$

- Recursive function evaluation**

$$(j+1)P_{j+1} = (2j+1)xP_j - jP_{j-1} \quad P_0 = 1 \quad P_1 = x$$

- Generating function:** The recurrence may be obtained through the Taylor expansion of the following function with respect to  $t$

$$g(t, x) = \frac{1}{\sqrt{1 - 2xt + t^2}} \equiv \sum_{j=0}^{\infty} P_j(x) t^j$$

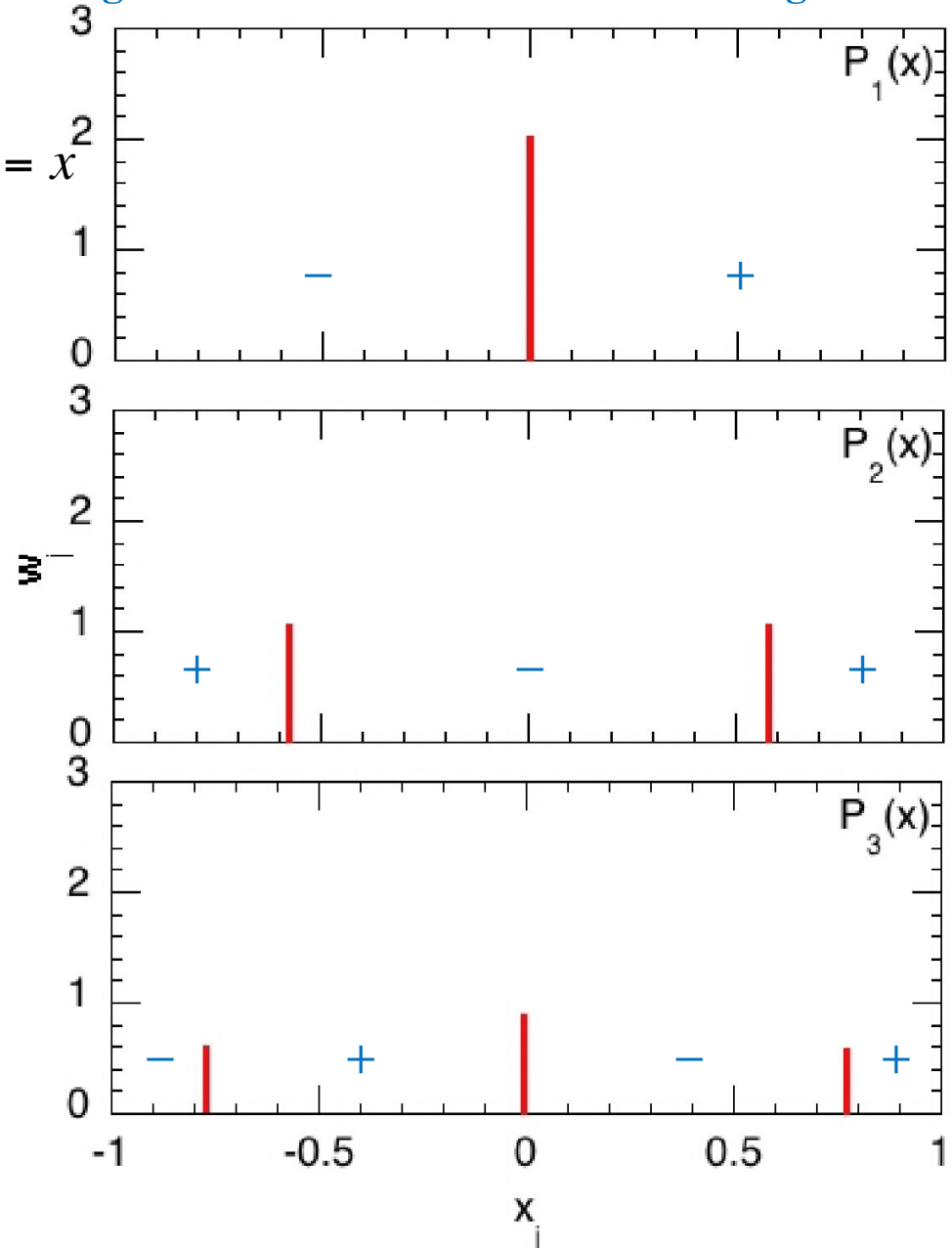
**(Hint)** Differentiate both sides by  $t$  & compare the coefficients of  $t^j$

- Function derivative:** A recurrence derived by differentiating  $g$  by  $x$

$$(x^2 - 1)P'_j = jxP_j - jP_{j-1}$$

See lecture on [recursive formula for Legendre polynomials](#)

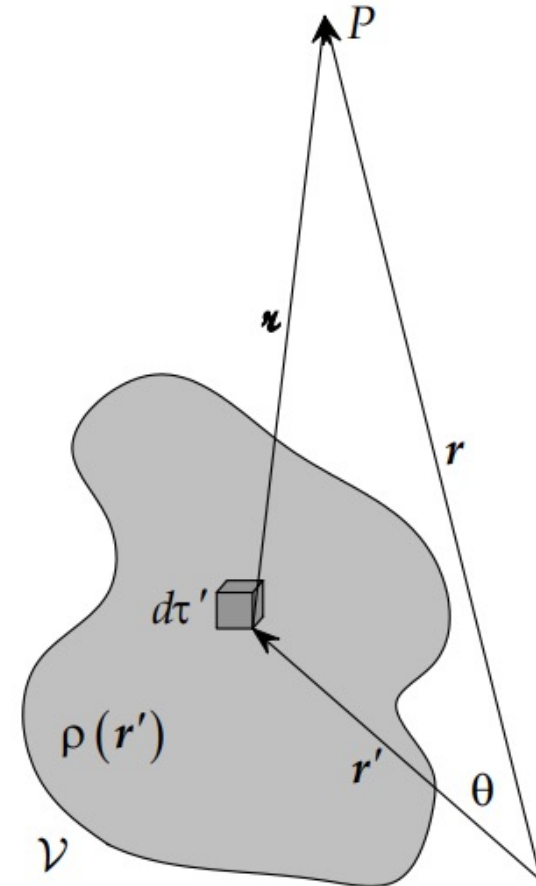
Orthogonalization necessitates interleaving nodes



# Origin of Legendre Polynomial

- Generating function of the Legendre polynomial is used for multipole expansion in electrostatics

$$\begin{aligned}\frac{1}{|\vec{r} - \vec{r}'|} &= \frac{1}{\sqrt{r^2 - 2rr'\cos\theta + r'^2}} \\ &= \frac{1}{r \sqrt{1 - 2\frac{r'}{r}\cos\theta + \left(\frac{r'}{r}\right)^2}} \\ &= \frac{1}{r} \sum_{j=0}^{\infty} P_j(\cos\theta) \left(\frac{r'}{r}\right)^j\end{aligned}$$



See lecture note on [O\(N\) fast multipole method](#)

Open-source code: S. Ogata *et al.*, [Comput. Phys. Commun](#) **153**, 445 ('03)



# Gauss-Legendre Quadrature

$$\int_{-1}^1 W(x) f(x) dx = \sum_{k=1}^n w_k f(x_k)$$

- **Abscissae from roots,  $x_k$**

$$P_n(x_k) = 0 \quad k = 1, \dots, n$$

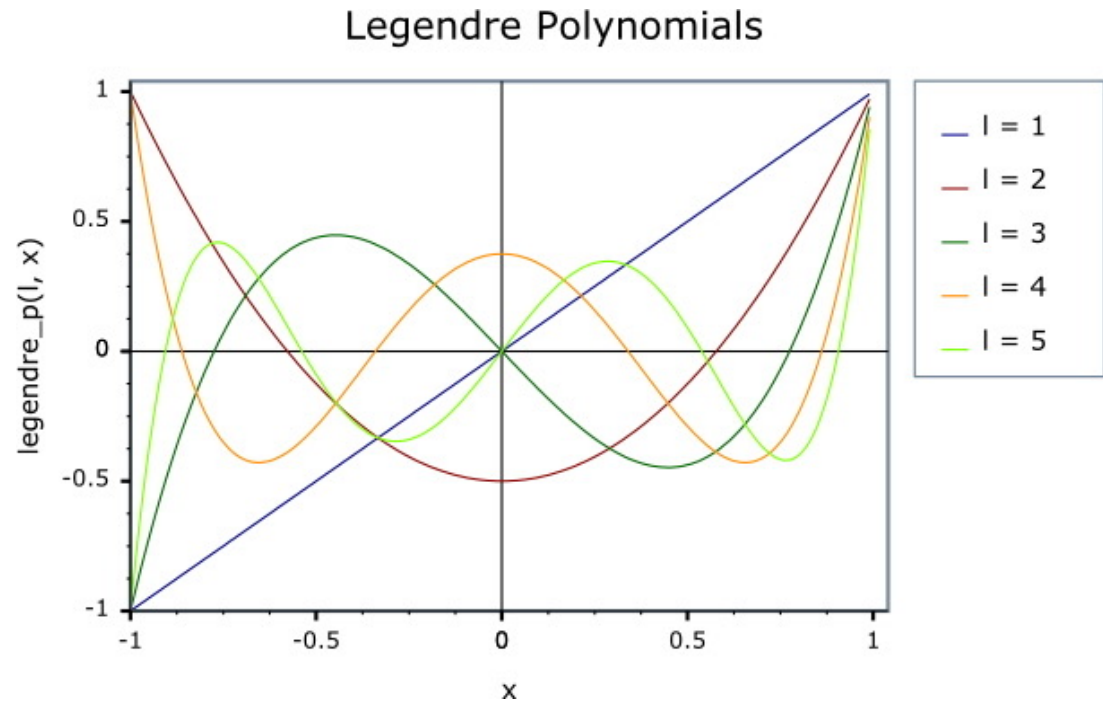
- **Weights,  $w_k$ : To reproduce some integrals exactly (linear equation)**

$$\int_{-1}^1 P_0(x) P_n(x) dx = \frac{2}{2n+1} \delta_{0,n} = \sum_{k=1}^n w_k P_n(x_k)$$

or

$$w_k = \frac{2}{n P_{n-1}(x_k) P'_n(x_k)} = \frac{2}{(1-x_k^2) [P'_n(x_k)]^2}$$

Note  $(x^2 - 1)P'_n = nxP_n - nP_{n-1}$  0 at root



# Newton's Method for Root Finding

- **Problem:** Find a root of a function

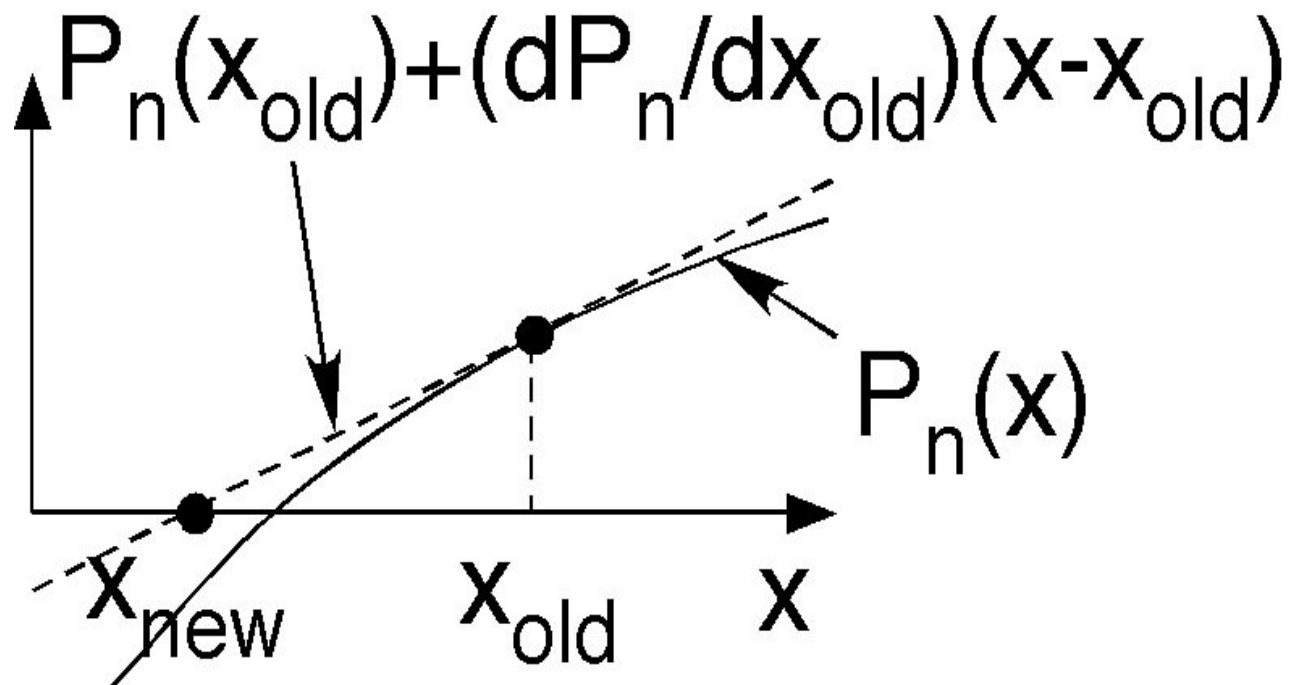
$$P_n(x) = 0$$

- **Newton iteration:** Successive linear approximation of the function

1. Start from an initial guess,  $x_0$ , of the root
2. Given the  $k$ -th guess,  $x_k$ , obtain a refined guess,  $x_{k+1}$ , from the linear fit:

$$P_n(x) \cong P'_n(x_k)(x - x_k) + P_n(x_k) = 0$$

$$\rightarrow x_{k+1} = x_k - \frac{P_n(x_k)}{P'_n(x_k)}$$



# Gauss-Legendre Program

- Given the lower & upper limits ( $x_1$  &  $x_2$ ) of integration &  $n$ , returns the abscissas & weights of the Gauss-Legendre  $n$ -point quadrature in  $x[1:n]$  &  $w[1:n]$ .

```
void gauleg(float x1,float x2,float x[],float w[],int n) {
    int m,j,i;
    double z1,z,xm,xl,pp,p3,p2,p1;
    m=(n+1)/2; // Find only half the roots because of symmetry
    xm=0.5*(x2+x1);
    xl=0.5*(x2-x1);
    for (i=1;i<=m;i++) {
        z=cos(3.141592654*(i-0.25)/(n+0.5));
        do {
            p1=1.0; p2=0.0;
            for (j=1;j<=n;j++) { // Recurrence relation
                p3=p2; p2=p1;
                p1=((2.0*j-1.0)*z*p2-(j-1.0)*p3)/j;
            }
            pp=n*(z*p1-p2)/(z*z-1.0); // Derivative
            z1=z;
            z=z1-p1/pp; // Newton's method
        } while (fabs(z-z1) > EPS); // EPS=3.0e-11
        x[i]=xm-xl*z;
        x[n+1-i]=xm+xl*z;
        w[i]=2.0*xl/((1.0-z*z)*pp*pp);
        w[n+1-i]=w[i]; // Weights
    }
}
```

$$\begin{cases} P_{-1} = 0 \\ P_0 = 1 \\ jP_j = (2j-1)zP_{j-1} - (j-1)P_{j-2} \end{cases}$$

$$(z^2 - 1)P'_j = jzP_j - jP_{j-1}$$

$$z \leftarrow z - \frac{P_n(z)}{P'_n(z)}$$

$$w_i = \frac{2}{(1 - x_i^2)[P'_n(x_i)]^2}$$

# How to Use the Gauss-Legendre Program

\$ cc -o gauleg-driver gauleg-driver.c gauleg.c -lm    See [gauleg-driver.c](#) & [gauleg.c](#)

```
//gauleg-driver.c
#include <stdio.h>
#include <math.h>

double *dvector(int, int);
void gauleg(double, double, double *, double *, int);

int main() {
    double *x,*w;
    double x1= -1.0,x2=1.0,sum;
    int N,i;
    printf("Input the number of quadrature points\n");
    scanf("%d",&N);
    x = dvector(1,N); // Allocate & use array elements x[1], ..., x[N]
    w = dvector(1,N); // It's Numerical Recipe's utility function (in gauleg.c)
    gauleg(x1,x2,x,w,N);
    sum=0.0;
    for (i=1; i<=N; i++)
        sum += w[i]*2.0/(1.0 + x[i]*x[i]);
    printf("Integration = %f\n", sum);
}
```

$$\pi = \int_{-1}^1 dx \frac{2}{x^2 + 1} \cong \sum_{k=1}^N w_k \frac{2}{x_k^2 + 1}$$

cf. [Gauss-Laguerre](#):  $x \in [0, \infty]$   
[Gauss-Hermite](#):  $x \in [-\infty, \infty]$

# Recursive Function Evaluation

- Legendre function

```
p1=1.0; p2=0.0;
for (j=1;j<=n;j++) { // Recurrence relation
    p3=p2;
    p2=p1;
    p1=((2.0*j-1.0)*z*p2-(j-1.0)*p3)/j;
}
pp=n*(z*p1-p2)/(z*z-1.0); // Derivative
```

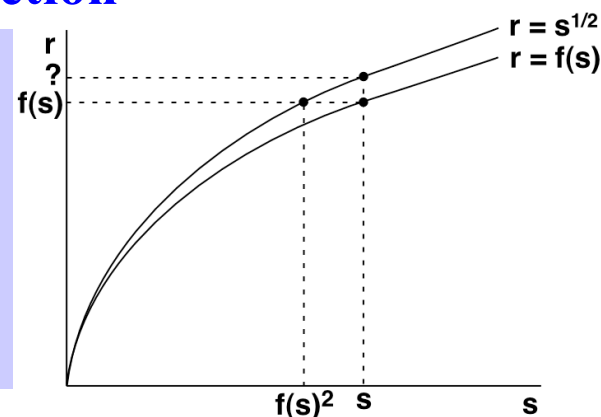
$$\begin{cases} P_{-1} = 0 \\ P_0 = 1 \\ jP_j = (2j-1)zP_{j-1} - (j-1)P_{j-2} \end{cases}$$

$$(z^2 - 1)P'_j = jzP_j - jP_{j-1}$$

- Compare it with a (low-accuracy) square-root function

```
#define C0 0.188030699
#define C1 1.48359853
#define C2 (-1.0979059)
#define C3 0.430357353

fs = C0+x*(C1+x*(C2+x*C3)); // Polynomial approximation
sr = fs+0.5*(x/fs-fs); // Newton correction
```



$$r - f(s) \approx \frac{dr}{ds} (s - f(s)^2)$$

$$\frac{dr}{ds} = \frac{d}{ds} s^{1/2} = \frac{1}{2} s^{-1/2} \approx \frac{1}{2f(s)}$$

$$\therefore r - f(s) = \frac{1}{2f(s)} (s - f(s)^2) = \frac{1}{2} \left( \frac{s}{f(s)} - f(s) \right)$$

# Where to Go from Here?

- **Gaussian quadrature for multiscale simulations?** *cf.* quasicontinuum method, where each function evaluation is an expensive quantum-mechanical calculation  
*cf.* Knap & Ortiz, *J. Mech. Phys. Solids* **49**, 1899 (2001)
- **Adaptive Gaussian quadrature?** *cf.* power of Metropolis importance sampling:  $2 \times 10^6 \ll 2^{400} \sim 10^{120}$  configurations

Lepage, *J. Comput. Phys.* **27**, 192 (1978)

Evila *et al.*, *IEEE T. Signal Process.* **69**, 474 (2021)

- **Related technique: Bayesian optimization** (active learning, kriging), using Gaussian process regression with minimal number of function evaluations (trade-off between *exploration & exploitation*)

Bassman *et al.*, *npj Comput. Mater.* **4**, 74 (2018)

Shields *et al.*, *Nature* **590**, 89 (2021)

