Polar and Singular-Value Decompositions

POLAR DECOMPOSITION

Let **A** be a real $N \times M$ matrix, where $N \geq M$ (*i.e.*, mapping from an *M*-dimensional source vector space to a larger *N*-dimensional target vector space). Then, there exists a column-wise orthogonal matrix **S** ($\in \Re^{N\times M}$ and) such that

$$\mathbf{A} = \mathbf{SJ}, \tag{1}$$

$$\mathbf{S}^{\mathrm{T}}\mathbf{S} = \mathbf{I}^{M\times M}, \tag{2}$$

where $\mathbf{I}^{M\times M}$ is the identity matrix and the unique nonnegative matrix **J** is

$$\mathbf{J} = \sqrt{\mathbf{A}^{\mathrm{T}}\mathbf{A}} \in \Re^{M\times M}. \tag{3}$$

(Proof)

Consider a spectral (or eigen) decomposition of **J**:

$$\mathbf{J} = \sum_{i=1}^{M} \lambda_i |i\rangle\langle i|, \tag{4}$$

where $\lambda_i (\geq 0)$ is the *i*-th eigenvalue and $\{|i\rangle \mid i = 1, \ldots, M\}$ is an orthonormal set of eigenvectors. Define

$$|\psi_i\rangle = \mathbf{A}|i\rangle \; (\in \Re^N), \tag{5}$$

then

$$\langle\psi_i|\psi_i\rangle = \langle i|\mathbf{A}^{\mathrm{T}}\mathbf{A}|i\rangle = \langle i|\mathbf{J}^2|i\rangle = \lambda_i^2\;. \tag{6}$$

For those eigenvectors with $\lambda_i \neq 0$, define

$$|e_i\rangle = |\psi_i\rangle/\lambda_i \; (\in \Re^N), \tag{7}$$

so that these vectors are orthonormal. For those eigenvectors with $\lambda_i = 0$, we use the Gram-Schmidt procedure to construct an orthonormal basis set and append it to the above basis set. Define a column-wise orthogonal matrix,

$$\mathbf{U} = \sum_{i=1}^{M} |e_i\rangle\langle i| \in \Re^{N\times M}. \tag{8}$$

When $\lambda_i \neq 0$, we have

$$\mathbf{UJ}|i\rangle = \sum_{j=1}^{M} |e_j\rangle \lambda_i \underbrace{\langle j|i\rangle}_{\delta_{ji}} = \lambda_i|e_i\rangle = |\psi_i\rangle = \mathbf{A}|i\rangle. \tag{9}$$

When $\lambda_i = 0$,

$$\mathbf{UJ}|i\rangle = \sum_{j=1}^{M} |e_j\rangle \underbrace{\lambda_i}_{0} \underbrace{\langle j|i\rangle}_{\delta_{ji}} = 0|e_i\rangle = 0 = |\psi_i\rangle = \mathbf{A}|i\rangle. \tag{10}$$

Namely, **UJ** is identical to **A** as a mapping for the entire *M*-dimensional source vector space. //

SINGULAR VALUE DECOMPOSITION (SVD)

Let **A** be a real $N \times M$ matrix, where $N \geq M$ as above. Then, there exists column-wise orthogonal matrices **U** ($\in \Re^{N\times M}$) and **V** ($\in \Re^{M\times M}$), such that

$$\mathbf{A} = \mathbf{UDV}^{\mathrm{T}}, \tag{11}$$

$$\mathbf{U}^{\mathrm{T}}\mathbf{U} = \mathbf{V}^{\mathrm{T}}\mathbf{V} = \mathbf{I}^{M\times M}, \tag{12}$$

where **D** ($\in \Re^{M\times M}$) is a nonnegative diagonal matrix.

(Proof)

Consider the polar decomposition, **A** = **SJ**, in Eq. (1). We perform the eigen-decompostion of **J** as

$$\mathbf{J} = \mathbf{V}\mathbf{D}\mathbf{V}^{\mathrm{T}}, \tag{13}$$

where **D** is the diagonal matrix such that its matrix elements are

$$D_{ij} = \lambda_i \delta_{ij}, \tag{14}$$

and **V** ($\in \Re^{M\times M}$) is an orthogonal matrix, *i.e.*, $\mathbf{V}^{\mathrm{T}}\mathbf{V} = \mathbf{I}^{M\times M}$. Substituting Eq. (13) in Eq. (1), we have

$$\mathbf{A} = \mathbf{S}\mathbf{V}\mathbf{D}\mathbf{V}^{\mathrm{T}} \equiv \mathbf{U}\mathbf{D}\mathbf{V}^{\mathrm{T}}, \tag{13}$$

Note that **U** = **SV** ($\in \Re^{N\times M}$) is a column-wise orthogonal, since

$$\mathbf{U}^{\mathrm{T}}\mathbf{U} = \mathbf{V}^{\mathrm{T}}\mathbf{S}^{\mathrm{T}}\mathbf{S}\mathbf{V} = \mathbf{V}^{\mathrm{T}} \underbrace{\mathbf{S}^{\mathrm{T}}\mathbf{S}}_{\mathbf{I}\in\mathbf{I}^{M\times M}} \mathbf{V} = \mathbf{V}^{\mathrm{T}}\mathbf{V} = \mathbf{I}^{M\times M}. //$$