Error Propagation Formula

Consider the variance of the energy, $E(p, x)$, which is a function of the momentum, p, and the coordinate, x:

$$\begin{aligned}\left\langle (\Delta E)^2 \right\rangle &= \left\langle \left(\frac{\partial E}{\partial p}\Delta p + \frac{\partial E}{\partial x}\Delta x \right)\left(\frac{\partial E}{\partial p}\Delta p + \frac{\partial E}{\partial x}\Delta x \right) \right\rangle \\ &= \left\langle \left| \frac{\partial E}{\partial p} \right|^2 (\Delta p)^2 + 2\frac{\partial E}{\partial p}\frac{\partial E}{\partial x}\Delta p \Delta x + \left| \frac{\partial E}{\partial x} \right|^2 (\Delta x)^2 \right\rangle \\ &= \left| \frac{\partial E}{\partial p} \right|^2 \left\langle (\Delta p)^2 \right\rangle + 2\frac{\partial E}{\partial p}\frac{\partial E}{\partial x}\left\langle \Delta p \Delta x \right\rangle + \left| \frac{\partial E}{\partial x} \right|^2 \left\langle (\Delta x)^2 \right\rangle \end{aligned} ,$$

where $\Delta E = E - \langle E \rangle$, etc., and the bracket denotes an expectation value.

If the momentum and coordinate are statistically independent,

$$\langle \Delta p \Delta x \rangle = \langle \Delta p \rangle \langle \Delta x \rangle = 0$$

and hence

$$\left\langle (\Delta E)^2 \right\rangle = \left| \frac{\partial E}{\partial p} \right|^2 \left\langle (\Delta p)^2 \right\rangle + \left| \frac{\partial E}{\partial x} \right|^2 \left\langle (\Delta x)^2 \right\rangle .$$