

## Taylor's Expansion

We use *recursive telescoping* to derive Taylor's expansion. Assume that function  $f(x)$  is continuous up to the  $n$ -th derivative  $f^{(n)}(x)$  in the range  $[a, x]$ . Then

$$\int_a^x f^{(n)}(x) dx = f^{(n-1)}(x) \Big|_a^x = f^{(n-1)}(x) - f^{(n-1)}(a). \quad (1)$$

Note this is telescoping: Let  $\Delta = (x-a)/N$  for a large integer  $N$ , then

$$\begin{aligned} \int_a^x f^{(n)}(x) dx &\approx \sum_{i=0}^{N-1} \frac{f^{(n-1)}(a + (i+1)\Delta) - f^{(n-1)}(a + i\Delta)}{\Delta} \Delta \\ &= f^{(n-1)}(a + \Delta) - f^{(n-1)}(a) + f^{(n-1)}(a + 2\Delta) - f^{(n-1)}(a + \Delta) + \dots + f^{(n-1)}(x) - f^{(n-1)}(x - \Delta). \\ &= f^{(n-1)}(x) - f^{(n-1)}(a) \end{aligned}$$

With the use of telescoping again,  $\int_a^x dx \times$  Eq. (1) yields

$$\begin{aligned} \int_a^x dx \int_a^x dx f^{(n)}(x) &= \int_a^x f^{(n-1)}(x) dx - (x-a)f^{(n-1)}(a) \\ &= f^{(n-2)}(x) - f^{(n-2)}(a) - (x-a)f^{(n-1)}(a). \end{aligned} \quad (2)$$

$\int_a^x dx \times$  Eq. (2) then yields

$$\int_a^x dx \int_a^x dx \int_a^x dx f^{(n)}(x) = f^{(n-3)}(x) - f^{(n-3)}(a) - (x-a)f^{(n-2)}(a) - \frac{(x-a)^2}{2} f^{(n-1)}(a). \quad (3)$$

Here, we have used

$$\int_a^x \frac{(x-a)^{n-1}}{(n-1)!} dx = \frac{(x-a)^n}{n!} \Big|_a^x = \frac{(x-a)^n}{n!}. \quad (4)$$

By repeating the same procedure, we eventually reach

$$\begin{aligned} \underbrace{\int_a^x dx \dots \int_a^x dx}_n f^{(n)}(x) &= f(x) - f(a) - (x-a)f^{(1)}(a) - \frac{(x-a)^2}{2} f^{(2)}(a) \\ &\quad - \dots - \frac{(x-a)^{n-1}}{(n-1)!} f^{(n-1)}(a). \end{aligned}$$

By rearranging the terms, we finally obtain Taylor's expansion:

$$f(x) = \sum_{k=0}^{n-1} \frac{(x-a)^k}{k!} f^{(k)}(a) + \underbrace{\int_a^x dx \dots \int_a^x dx}_n f^{(n)}(x). \quad (5)$$

If we neglect the last integration in the right-hand side of Eq. (5), the error bound is give by

$$\left| \underbrace{\int_a^x dx \dots \int_a^x dx}_n f^{(n)}(x) \right| \leq \max_{[a,x]} |f^{(n)}(x)| \underbrace{\int_a^x dx \dots \int_a^x dx}_n 1 = \max_{[a,x]} |f^{(n)}(x)| \frac{(x-a)^n}{n!}. \quad (6)$$