# Fermi-Operator Expansions for Linear Scaling Electronic Structure Calculations

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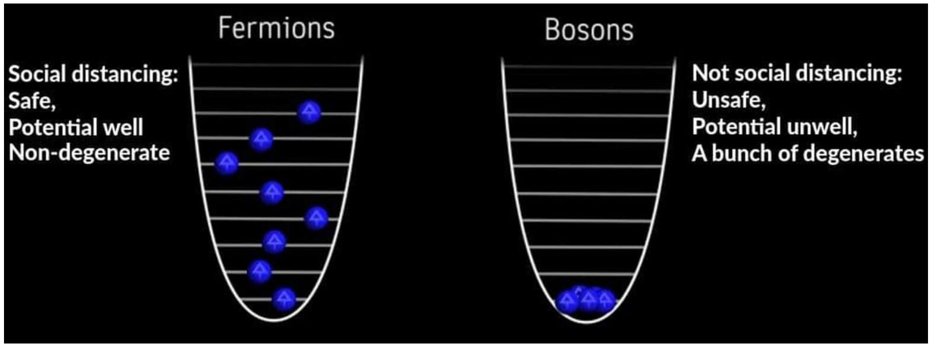
***O*(*N*) sparse matrix representation**

# Fermi Operator




- **Fermi operator**

$$F(\hat{H}) = \frac{2}{\exp\left(\frac{\hat{H}-\mu}{k_B T}\right)+1}$$

- **Projection to the occupied subspace**

$$|\psi_{\text{proj}}\rangle = F(\hat{H})|\psi\rangle$$

- **The expectation value of any operator *A* is obtained by**

$$\langle \hat{A} \rangle = \text{tr}\left[\hat{A}\hat{F}\right]$$

- **Widely used in *O*(*N*) electronic structure calculations (*N* = number of electrons) through its sparse representation**

*cf.* $O(N^3)$ way

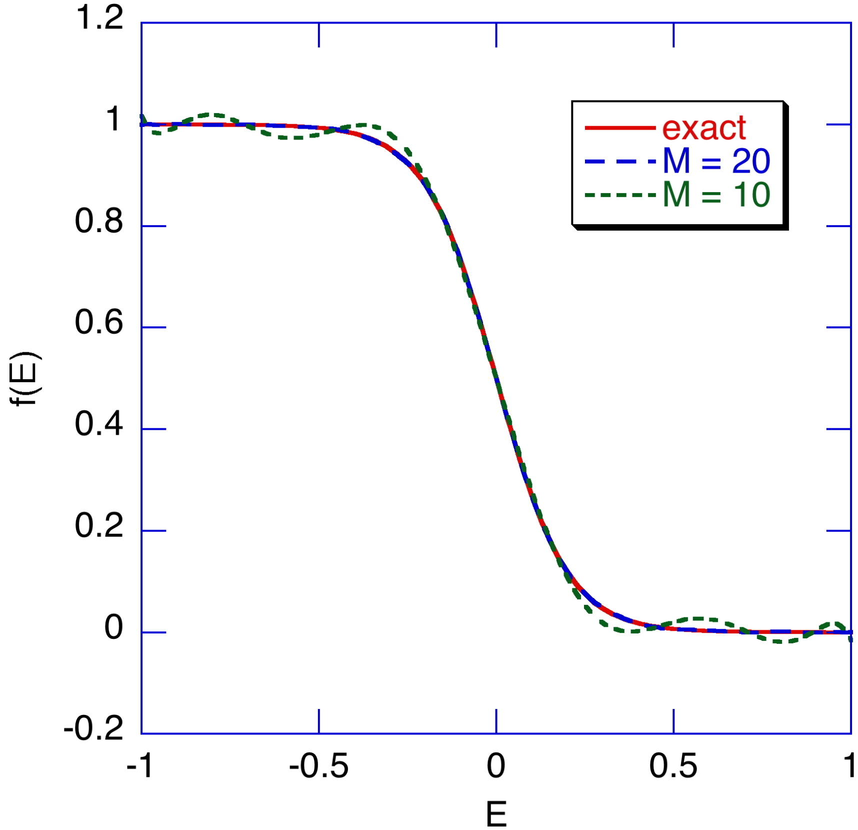
$$\hat{H}|n\rangle = \varepsilon_n |n\rangle$$

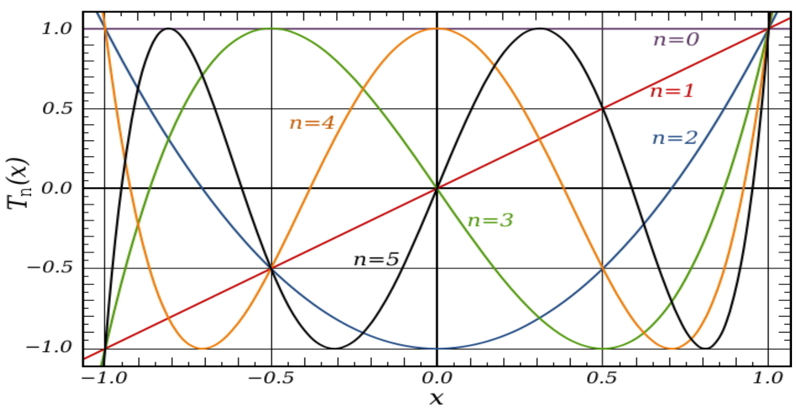
$$\langle \hat{A} \rangle = \sum_n \frac{2}{\exp\left(\frac{\varepsilon_n - \mu}{k_B T}\right)+1} \langle n|\hat{A}|n\rangle$$

S. Goedecker, *Rev. Mod. Phys*. **71**, 1085 ('99)

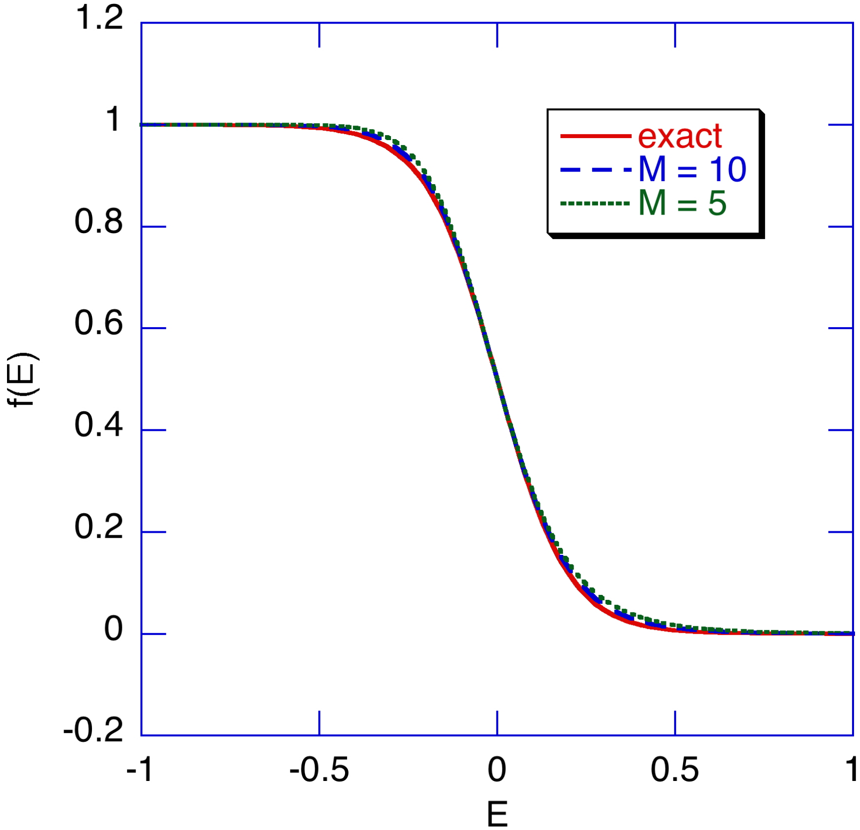
# Fermi-Operator Approximations

## Chebyshev polynomial




## Rational




$$F(\hat{H}) \cong \sum_{\nu=1}^{M} \frac{R_\nu}{\hat{H} - z_\nu}$$

$$\left(\hat{H} - z_\nu\right)\left|\psi_{\text{out}}^{\nu}\right\rangle = R_\nu \left|\psi_{\text{in}}\right\rangle$$

See note on [Fermi-operator expansion]

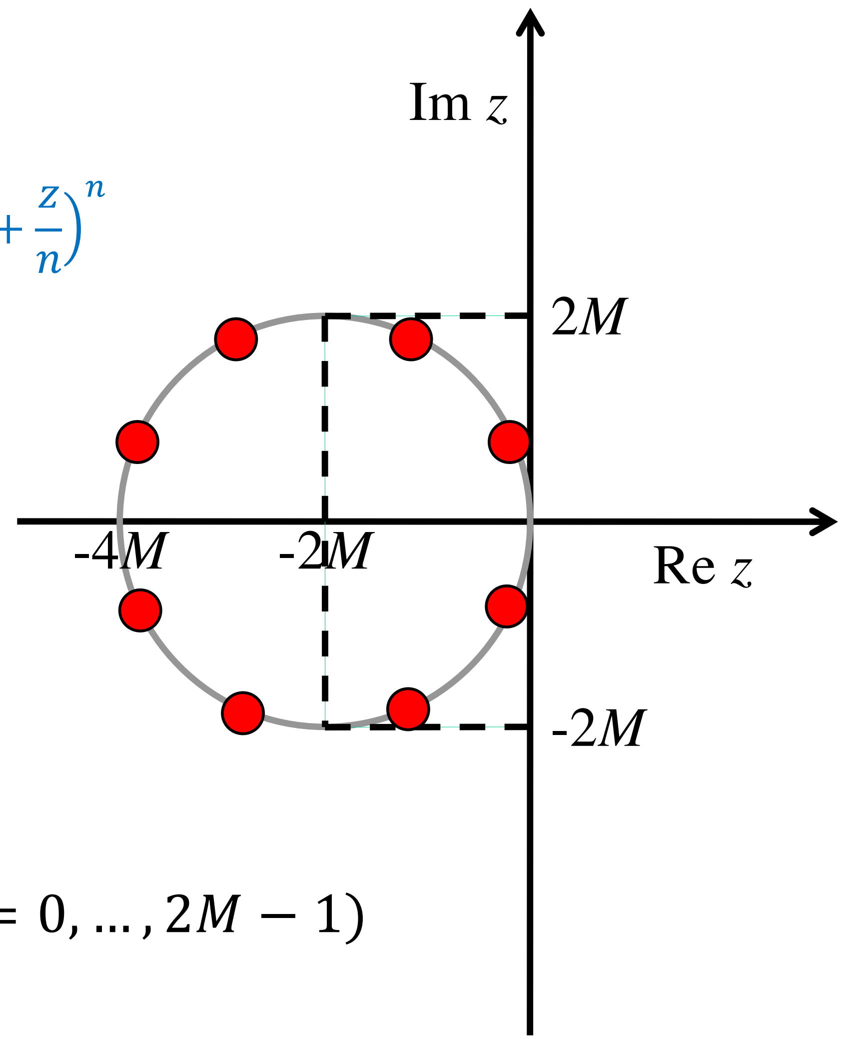
# Rational Fermi-Operator Expansion

$$f(z) = \frac{1}{\exp(z)+1}$$

$$\cong \frac{1}{\left(1+\frac{z}{2M}\right)^{2M}+1}$$

$$\cong \sum_{\nu=0}^{2M-1} \frac{R_\nu}{z-z_\nu}$$

$$e^z = \lim_{n\to\infty}\left(1+\frac{z}{n}\right)^n$$




**Poles**

$$\begin{cases} z_\nu = 2M\left(\exp\left(i\frac{(2\nu+1)\pi}{2M}\right)-1\right) \\ R_\nu = -\exp\left(i\frac{(2\nu+1)\pi}{2M}\right) \end{cases} \quad (\nu = 0, \ldots, 2M-1)$$

**Residues**

D. M. C. Nicholson *et al.*, *Phys. Rev. B* **50**, 14686 ('94); A. P. Horsfield *et al.*, *Phys. Rev. B* **53**, 12694 ('96); L. Lin *et al.*, *J. Phys. Condes. Matter* **25**, 1295501 ('13)

See note on Fermi-operator expansion

# $O(N)$ Fermi Operator Expansion

- **Truncated expansion of Fermi-operator by Chebyshev polynomial $\{T_p\}$**

$$F(\hat{H}) \cong \sum_{p=0}^{P} c_p T_p(\hat{H})$$

- ***O*(*N*) algorithm**

prepare a basis set of size $O(N)$
(let the size be $N$ for simplicity)

for $l = 1, N$

let an $N$-dimensional unit vector be $|e_l\rangle = \begin{array}{cc} \begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix} & \begin{array}{c} \\ \\ l \\ \\ \\ \end{array} \end{array}$

recursively construct the $l^{\text{th}}$ column of matrix $T_p$, $\left|t_l^p\right\rangle$, keeping only $O(1)$ off-diagonal elements (*cf*. quantum nearsightedness)

$$\begin{cases} \left|t_l^0\right\rangle = \left|e_l\right\rangle \\ \left|t_l^1\right\rangle = \hat{H}\left|e_l\right\rangle \\ \left|t_l^{p+1}\right\rangle = 2\hat{H}\left|t_l^p\right\rangle - \left|t_l^{p-1}\right\rangle \end{cases}$$

*cf*. Legendre polynomial by recursion

build a sparse representation of the $l^{\text{th}}$ column of $F$ as

$$\left|f_l\right\rangle \cong \sum_{p=0}^{P} c_p \left|t_l^p\right\rangle$$