

AI5002 - Assignment 14

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Problem NET_JUNE_2012_Q104

Which of the following conditions imply independence of the random variables X and Y ?

- 1) $\Pr(X > a | Y > a) = \Pr(X > a) \forall a \in \mathbb{R}$
- 2) $\Pr(X > a | Y < b) = \Pr(X > a) \forall a, b \in \mathbb{R}$
- 3) X and Y are uncorrelated.
- 4) $E[(X-a)(Y-b)] = E(X-a)E(Y-b) \forall a, b \in \mathbb{R}$

Solution

We analyze the options one by one and see which option best implies that random variables X and Y are independent.

- 1) Let's assume continuous r.v.s X and Y are not independent and,

$$\begin{aligned} X &\in \{0, 1\} \\ Y &= X + 2 \\ \Pr(X = 0) &= \Pr(X = 1) = \frac{1}{2} \end{aligned} \quad (1)$$

Now since Y is always greater than X therefore $\Pr(X > a | Y > a)$ equals $\Pr(X > a) \forall a \in \mathbb{R}$ and thus in spite of the option (1) being true, it fails to imply that X and Y are independent random variables and hence option (1) is false.

- 2) Let us denote the cumulative distribution functions of X , Y and (X, Y) , as below,

$$\begin{aligned} F_X(a) &= \Pr(X \leq a), \\ F_Y(b) &= \Pr(Y \leq b) \text{ and} \end{aligned} \quad (2)$$

$$F_{X,Y}(a, b) = \Pr(X \leq a \text{ and } Y \leq b)$$

To show independence, we want to prove that,

$$F_X(a)F_Y(b) = F_{X,Y}(a, b) \forall a, b \in \mathbb{R} \quad (3)$$

Conditional probability tells us that:

$$\Pr(X > a | Y < b) = \frac{\Pr(X > a \text{ and } Y < b)}{\Pr(Y < b)} \quad (4)$$

and so by the assumptions of the option (1),

$$\Pr(X > a) = \frac{\Pr(X > a \text{ and } Y < b)}{\Pr(Y < b)}$$

$$\Pr(X > a) \Pr(Y < b) = \Pr(X > a \text{ and } Y < b) \quad (5)$$

Now since

$$\begin{aligned} \Pr(X > a) &= 1 - F_X(a) \text{ and,} \\ \Pr(Y < b) &= F_Y(b) \end{aligned} \quad (6)$$

we may rewrite the above equation as:

$$\begin{aligned} F_Y(b) - F_X(a)F_Y(b) &= \Pr(X > a \text{ and } Y < b) \\ F_X(a)F_Y(b) &= F_Y(b) - \Pr(X > a \text{ and } Y < b) \end{aligned} \quad (7)$$

Also, note that

$$F_Y(b) = \Pr(X > a \text{ and } Y < b) + \Pr(X < a \text{ and } Y < b) \quad (8)$$

Thus putting value of $F_Y(b)$ from (8) into (7) proves (2) ,

$$F_X(a)F_Y(b) = \Pr(X < a \text{ and } Y < b) \quad (9)$$

Thus option (2) seems to be always true.

- 3) Given random variables X and Y are

uncorrelated which means that their correlation is 0, or, equivalently, $\text{Cov}(X, Y) = 0$.

$$\begin{aligned} \text{Cov}(X, Y) &= E[XY] - E[X]E[Y] \\ [\because \text{Cov}(X, Y) &= 0] \\ E[XY] &= E[X]E[Y] \end{aligned} \quad (10)$$

We have to prove that uncorrelated implies independence.

Let's take X and Y to exist as an ordered pair at the points $(-1,1)$, $(0,0)$, and $(1,1)$ with probabilities $\frac{1}{4}, \frac{1}{2}$, and $\frac{1}{4}$. The expected values of X and Y is

$$\begin{aligned} E[X] &= -1 \cdot \frac{1}{4} + 0 \cdot \frac{1}{2} + 1 \cdot \frac{1}{4} = 0 = E[Y] \\ E[XY] &= -1 \cdot \frac{1}{4} + 0 \cdot \frac{1}{2} + 1 \cdot \frac{1}{4} = 0 = E[X]E[Y] \end{aligned} \quad (11)$$

Now let's look at the marginal distributions of X and Y . X and Y both take on the values $-1, 0, 1$ and the probability it takes for each of those are given by $\frac{1}{4}, \frac{1}{2}, \frac{1}{4}$. Then looping through the possibilities, we have to check if

$$\Pr(X = x, Y = y) = \Pr(X = x) \Pr(Y = y)$$

Let's take the first point $(-1, 1)$ and examine,

$$\begin{aligned} \Pr(X = -1, Y = 1) &= \frac{1}{4} \neq \\ \frac{1}{16} &= \Pr(X = -1) \Pr(Y = 1) \end{aligned} \quad (12)$$

We loop through the other two points, and see that X and Y do not meet the definition of independent.

Thus it proves that uncorrelated random variables are not always independent. Hence option (3) is false.

4) We extend L.H.S and R.H.S to compare,

$$\begin{aligned} E[(X - a)(Y - b)] &= E[XY] \\ &\quad - aE[Y] - bE[X] + ab \end{aligned} \quad (13)$$

Also,

$$\begin{aligned} E(X - a)E(Y - b) &= E[X]E[Y] \\ &\quad - aE[Y] - bE[X] + ab \end{aligned} \quad (14)$$

We see (13) = (14), i.e. independent iff $E[XY] = E[X]E[Y]$ but in this case independence of X and Y cannot be inferred. Hence option (4) is false.