

AI5002 - Assignment 14

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Problem NET_JUNE_2012_Q104

Which of the following conditions imply independence of the random variables X and Y ?

- 1) $\Pr(X > a | Y > a) = \Pr(X > a) \quad \forall a \in \mathbb{R}$
- 2) $\Pr(X > a | Y < b) = \Pr(X > a) \quad \forall a, b \in \mathbb{R}$
- 3) X and Y are uncorrelated.
- 4) $E[(X-a)(Y-b)] = E(X-a) E(Y-b) \quad \forall a, b \in \mathbb{R}$

Solution

We analyze the options one by one and see which option best implies that random variables X and Y are independent.

- 1) Let us take a counter example to understand if option 1) being true implies X and Y to be independent random variables,

$$\begin{aligned} \text{Let } X &\in \{0, 1\} \\ \text{Let } Y &= X + 2 \end{aligned} \quad (1)$$

$$\text{Let } \Pr(X = 0) = \Pr(X = 1) = \frac{1}{2}$$

Now since Y is always greater than X , so for any given $a \in \mathbb{R}$, $X > a$ does not depend on $Y > a$, so we can write

$$\Pr(X > a | Y > a) = \Pr(X > a) \quad \forall a \in \mathbb{R} \quad (2)$$

Although the condition in option 1) is true in our example but we can see X and Y are dependent random variables. Hence option 1)

does not imply X and Y to be independent.

- 2) Let us denote the individual C.D.F.s of the continuous random variables X , Y and the joint C.D.F (X, Y) , as below,

$$\begin{aligned} F_X(a) &= \Pr(X \leq a) = \Pr(X < a), \\ F_Y(b) &= \Pr(Y \leq b) = \Pr(Y < b) \text{ and} \\ F_{X,Y}(a, b) &= \Pr(X \leq a, Y \leq b) \\ &= \Pr(X < a, Y < b) \end{aligned} \quad (3)$$

To show independence, we want to prove that,

$$F_X(a)F_Y(b) = F_{X,Y}(a, b) \quad \forall a, b \in \mathbb{R} \quad (4)$$

From conditional probability we know:

$$\Pr(X > a | Y < b) = \frac{\Pr(X > a, Y < b)}{\Pr(Y < b)} \quad (5)$$

and so using the given condition in option 2), we can write (4) as

$$\begin{aligned} \Pr(X > a) &= \frac{\Pr(X > a, Y < b)}{\Pr(Y < b)} \\ \implies \Pr(X > a) \Pr(Y < b) &= \Pr(X > a, Y < b) \end{aligned} \quad (6)$$

We can write the C.D.F as

$$\begin{aligned} \Pr(X > a) &= 1 - F_X(a) \text{ and,} \\ \Pr(Y < b) &= F_Y(b) \end{aligned} \quad (7)$$

We may rewrite (5) using (6) as:

$$\begin{aligned} (1 - F_X(a))(F_Y(b)) &= \Pr(X > a, Y < b) \\ F_Y(b) - F_X(a)F_Y(b) &= \Pr(X > a, Y < b) \\ F_X(a)F_Y(b) &= F_Y(b) - \Pr(X > a, Y < b) \end{aligned} \quad (8)$$

Note that since X is continuous so we can write,

$$\Pr(X \leq a) = \Pr(X < a) \quad (9)$$

Regardless of the value of X the marginal

C.D.F $F_Y(b)$ is given by

$$F_Y(b) = \Pr(Y < b) \quad (10)$$

Now let us define two events

$$\text{Event } A : (Y < b \cap X < a)$$

$$\text{Event } B : (Y < b \cap X > a)$$

We can also think of the event $(Y < b)$ as

$$(Y < b) = (\text{Event } A) \cup (\text{Event } B) \quad (11)$$

So it implies

$$\Pr(A, B) = \Pr(Y < b) \quad (12)$$

Since X cannot both be less than a and greater than a , we have

$$\begin{aligned} \Pr(A, B) &= \Pr(A) + \Pr(B) \\ &= \Pr(Y < b, X < a) + \Pr(Y < b, X > a) \end{aligned} \quad (13)$$

\therefore We can write

$$\begin{aligned} F_Y(b) &= \Pr(X > a, Y < b) + \\ &\Pr(X < a, Y < b) \end{aligned} \quad (14)$$

Now putting value of $F_Y(b)$ from (14) into (8) proves (4) ,

$$\begin{aligned} F_X(a)F_Y(b) &= \Pr(X < a, Y < b) \\ &= F_{X,Y}(a, b) \end{aligned} \quad (15)$$

Thus (15) shows that the joint C.D.F. is the product of the two individual C.D.F. Hence using the given the condition in option 2) we have proved that X and Y to be independent random variables.

- 3) Given random variables X and Y are uncorrelated which means that their correlation is 0, or, equivalently, $\text{Cov}(X, Y) = 0$.

$$\begin{aligned} \text{Cov}(X, Y) &= E(XY) - E(X)E(Y) \\ [\because \text{Cov}(X, Y) &= 0] \\ E(XY) &= E(X)E(Y) \end{aligned} \quad (16)$$

We have to prove that uncorrelated implies independence.

Let's take X and Y to exist as an ordered pair at the points $(-1, 1)$, $(0, 0)$, and $(1, 1)$ with

probabilities $\frac{1}{4}, \frac{1}{2}$, and $\frac{1}{4}$. The expected values of X and Y is

$$\begin{aligned} E[X] &= -1\left(\frac{1}{4}\right) + 0\left(\frac{1}{2}\right) + 1\left(\frac{1}{4}\right) = 0 = E[Y] \\ E[XY] &= -1\left(\frac{1}{4}\right) + 0\left(\frac{1}{2}\right) + 1\left(\frac{1}{4}\right) = 0 = E[X]E[Y] \end{aligned} \quad (17)$$

Now let's look at the marginal distributions of X and Y . X and Y both take on the values $-1, 0, 1$ and the probability it takes for each of those are given by $\frac{1}{4}, \frac{1}{2}, \frac{1}{4}$. Then looping through the possibilities, we have to check if

$$\Pr(X = x, Y = y) = \Pr(X = x)\Pr(Y = y) \quad (18)$$

Let's take the first point $(-1, 1)$ and examine,

$$\begin{aligned} \Pr(X = -1, Y = 1) &= \frac{1}{4} \\ &\neq \frac{1}{16} = \Pr(X = -1)\Pr(Y = 1) \end{aligned} \quad (19)$$

We loop through the other two points, and see that X and Y do not meet the definition of independent.

Hence it proves that uncorrelated random variables are not always independent. Thus condition in option 3) fail to imply X and Y are independent random variables.

- 4) We extend L.H.S

$$\begin{aligned} E[(X - a)(Y - b)] &= E[XY] \\ &\quad - aE[Y] - bE[X] + ab \end{aligned} \quad (20)$$

and R.H.S to compare,

$$\begin{aligned} E(X - a)E(Y - b) &= E[X]E[Y] \\ &\quad - aE[Y] - bE[X] + ab \end{aligned} \quad (21)$$

We see (20) = (21), i.e. independent iff $E[XY] = E[X]E[Y]$ but in this case independence of X and Y cannot be inferred.

Let us take a counter example to understand further as to why this condition is not always true.

Let $X \sim \mathcal{N}(0, 1)$

$$Y = X^2$$

Multiply X on both sides,

$$XY = X^3$$

Let us calculate the expected values,

$$E(XY) = E(X^3) = E(X)E(Y)$$

[By property of expectations]

$$E(XY) = E(X^3) = 0.E(Y) = 0$$

$$[\because E(X) = 0]$$

$$\therefore E(XY) = E(X)E(Y) \quad (22)$$

From (22) we see they are uncorrelated but not independent because

a) We have a case when $X \in (0, 1)$ and at the same time $Y = X^2 > 1$. This will never happen, because if $X^2 > 1$ then $X > 1$. So the probability is 0.

$$\begin{aligned} \text{b) } \Pr(0 < X < 1) &= F_X(1) - F_X(0) \\ &= 0.84134475 - 0.5 \\ &= 0.34134475 \end{aligned}$$

$$\begin{aligned} \text{c) } P(Y > 1) &= P(X^2 > 1) \\ &= P(X > 1) + P(X < -1) \\ &= 2 * (1 - P(X \leq 1)) \\ &= 2 * (1 - F_X(1)) \\ &= 2 * (1 - 0.84134475) = 0.31731 \end{aligned}$$

Hence we can write,

$$\begin{aligned} \Pr(0 < X < 1, Y > 1) &= 0 \neq \\ \Pr(0 < X < 1) \Pr(Y > 1) \end{aligned} \quad (23)$$

Thus condition in option 4) fail to imply X and Y are independent random variables.