AI5002 - Assignment 15

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1. LaTeX

Problem NET_JUNE_2012_Q107

Let X_1, X_2, ... be independent random variables with
X_n ~ U(-n, 3n) where n = 1, 2, ...

Let $S_N = \frac{1}{\sqrt{N}} \sum_{n=1}^{N} \frac{X_n}{n} \; for \; N = 1, 2, .. \infty$

Let F_N be the distribution function of S_N
Also let ϕ denote the distribution function of a standard normal random variable. Which of the following is/are true?

1) $\lim_{N\to\infty} F_N(0) \leq \phi(0)$
2) $\lim_{N\to\infty} F_N(0) \geq \phi(0)$
3) $\lim_{N\to\infty} F_N(1) \leq \phi(1)$
4) $\lim_{N\to\infty} F_N(1) \geq \phi(1)$

Solution

Let

$$Y_n = \frac{X_n - n}{3n} \tag{1}$$

And by a simple computation we can see,

$$Y_n \; is \; i.i.d. \; \sim U(-\frac{2}{3}, \frac{2}{3}) \tag{2}$$

Also from (1), we can write

$$X_n = 3nY_n + n \tag{3}$$

Now,

$$S_N = \frac{1}{\sqrt{N}} \sum_{n=1}^{N} \frac{X_n}{n}$$

$$From \; (3) \; we \; can \; write,$$

$$\begin{aligned} &= \frac{1}{\sqrt{N}} \sum_{n=1}^{N} \frac{n + 3nY_n}{n} \\ &= \frac{1}{\sqrt{N}} \sum_{n=1}^{N} 1 + 3Y_n \\ &= 3\frac{1}{\sqrt{N}} \sum_{n=1}^{N} Y_n + \sqrt{N} \end{aligned} \tag{4}$$

By Central Limit Theorem, the first term tends to a normal distribution and the whole R.H.S. tends to,

$$\begin{aligned} &As \; (N \to \infty) \\ &S_N = 3\frac{1}{\sqrt{N}} \sum_{n=1}^{N} Y_n + \sqrt{N} \longrightarrow \infty \end{aligned} \tag{5}$$

$\because S_N \to \infty$, it implies

$$\begin{aligned} \lim_{N\to\infty} F_N(0) = 0 \\ \lim_{N\to\infty} F_N(1) = 0 \end{aligned} \tag{6}$$

By definition we know, a standard normal random variable is a normally distributed random variable with mean $\mu = 0$ and standard deviation $\sigma = 1$. It is denoted by the letter Z. From Z-distribution table we know,

$$\begin{aligned} P(Z \leqslant 0) = \phi(0) = 0.5 \\ P(Z \leqslant 1) = \phi(1) = 0.84134 \end{aligned} \tag{7}$$

Hence by comparing (6) and (7) we see only options 1) and 3) are true.