AI5002 - Assignment 14

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1. LaTeX

Problem NET_JUNE_2012_Q104

Which of the following conditions imply independence of the random variables X and Y ?

1) p(X > a | Y > a) = P(X > a) for all a ∈ $\mathbb{R}$
2) p(X > a | Y < b) = P(X > a) for all a, b ∈ $\mathbb{R}$
3) X and Y are uncorrelated.
4) $E[(X-a)(Y-b)] = E(X-a)\,E(Y-b)\ \forall\ a,\ b \in\ \mathbb{R}$

Solution

We analyze the options one by one and see which option best implies that random variables X and Y are independent.

1) In case of option (1)
Let's assume continuous r.v.s X and Y are not independent and,

$$\begin{aligned} X &\in \{0,1\} \\ Y &= X+2 \\ \Pr(X=0) = \Pr(X=1) &= \frac{1}{2} \end{aligned} \tag{1}$$

Now since Y is always greater than X therefore $\Pr(X > a\,|Y > a)$ equals $\Pr(X > a)\ \ \forall\ a\ \ \in \mathbb{R}$ and thus in spite of the option (1) being true, it fails to imply that X and Y are independent random variables and hence option (1) is false.

2) In case of option (2),
Let us denote the cumulative distribution functions of X, Y and (X, Y), as below,

$$\begin{aligned} F_X(a) &= P(X \le a), \\ F_Y(b) &= P(Y \le b)\ and \\ F_{X,Y}(a,b) &= P(X \le a \text{ and } Y \le b) \end{aligned} \tag{2}$$

To show independence, we want to prove that,

$$F_X(a)F_Y(b) = F_{X,Y}(a,b)\ \forall\ a,\ \ b\ \ \in \mathbb{R} \tag{3}$$

Conditional probability tells us that:

$$P(X > a|Y < b) = \frac{P(X > a \text{ and } Y < b)}{P(Y < b)} \tag{4}$$

and so by the assumptions of the option (1),

$$\begin{aligned} P(X > a) &= \frac{P(X > a \text{ and } Y < b)}{P(Y < b)} \\ P(X > a)P(Y < b) &= P(X > a \text{ and } Y < b) \end{aligned} \tag{5}$$

Now since

$$\begin{aligned} &P(X > a) = 1 - F_X(a)\ and, \\ &P(Y < b) = F_Y(b) \end{aligned} \tag{6}$$

we may rewrite the above equation as:

$$\begin{aligned} F_Y(b) - F_X(a)F_Y(b) &= P(X > a \text{ and } Y < b) \\ F_X(a)F_Y(b) &= F_Y(b) - P(X > a \text{ and } Y < b) \end{aligned} \tag{7}$$

Also, note that

$$F_Y(b) = P(X > a \text{ and } Y < b) + P(X < a \text{ and } Y < b) \tag{8}$$

Thus putting value of $F_Y(b)$ from (8) into (7) proves (2) ,

$$F_X(a)F_Y(b) = P(X < a \text{ and } Y < b) \tag{9}$$

Thus option (2) seems to be always true.

3) In case of option (3),
Given random variables X and Y are uncorrelated which means that their correlation

is 0, or, equivalently, Cov(X, Y) = 0.

$$\begin{aligned} Cov(X, Y) &= E[XY] - E[X]E[Y] \\ &\qquad [\because Cov(X, Y) = 0] \\ E[XY] &= E[X]E[Y] \end{aligned} \tag{10}$$

We have to prove that uncorrelated implies independence.
Let's take X and Y to exist as an ordered pair at the points (-1,1), (0,0), and (1,1) with probabilities $\frac{1}{4}$,$\frac{1}{2}$, and $\frac{1}{4}$. The expected values of X and Y is

$$\begin{aligned} E[X] = -1 \cdot \frac{1}{4} + 0 \cdot \frac{1}{2} + 1 \cdot \frac{1}{4} = 0 = E[Y] \\ E[XY] = -1 \cdot \frac{1}{4} + 0 \cdot \frac{1}{2} + 1 \cdot \frac{1}{4} = 0 = E[X]E[Y] \end{aligned} \tag{11}$$

Now let's look at the marginal distributions of X and Y. X and Y both take on the values $-1,\ 0,\ 1$ and the probability it takes for each of those are given by $\frac{1}{4}$, $\frac{1}{2}$, $\frac{1}{4}$. Then looping through the possibilities, we have to check if

$$P(X = x, Y = y) = P(X = x)P(Y = y)$$

Let's take the first point (-1, 1) and examine,

$$P(X = -1, Y = 1) = \frac{1}{4} \neq \frac{1}{16} = P(X = -1)\ P(Y = 1) \tag{12}$$

We loop through the other two points, and see that X and Y do not meet the definition of independent.
Thus it proves that uncorrelated random variables are not always independent. Hence option (3) is false.

4) In case of option (4),we have

$$E[(X - a)(Y - b)] = E[XY] - aE[Y] - bE[X] + ab \tag{13}$$

Also,

$$E(X - a)E(Y - b) = E[X]E[Y] - aE[Y] - bE[X] + ab \tag{14}$$

We see (13) = (14), i.e. independent iff $E[XY] = E[X]E[Y]$ but in this case independence of X and Y cannot be inferred. Hence option (4) is false.