AI5002 - Assignment 14

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1. LaTeX

Problem NET_JUNE_2012_Q104

Which of the following conditions imply independence of the random variables X and Y ?

1) $\Pr(X > a|Y > a) = \Pr(X > a) \; \forall \; a \in \mathbb{R}$
2) $\Pr(X > a|Y < b) = \Pr(X > a) \; \forall \; a, \; b \in \mathbb{R}$
3) X and Y are uncorrelated.
4) $E[(X-a)(Y-b)] = E(X-a)\,E(Y-b) \; \forall \; a, \; b \in \mathbb{R}$

Solution

We analyze the options one by one and see which option best implies that random variables X and Y are independent.

1) Let's assume continuous r.v.s X and Y are not independent and,

$$\begin{aligned} X &\in \{0, 1\} \\ Y &= X + 2 \\ \Pr(X = 0) = \Pr(X = 1) &= \frac{1}{2} \end{aligned} \tag{1}$$

Now since Y is always greater than X therefore $\Pr(X > a\,|Y > a)$ equals $\Pr(X > a) \;\; \forall \; a \;\; \in \mathbb{R}$ and thus in spite of the option (1) being true, it fails to imply that X and Y are independent random variables and hence option (1) is false.

2) Let us denote the individual C.D.F.s of the continuous random variables X, Y and the joint C.D.F (X, Y), as below,

$$\begin{aligned} F_X(a) &= \Pr(X \le a) . = \Pr(X < a), \\ F_Y(b) &= \Pr(Y \le b) = \Pr(Y < b) \;\; and \\ F_{X,Y}(a, b) &= \Pr(X \le a, Y \le b) \\ &= \Pr(X < a, Y < b) \end{aligned} \tag{2}$$

To show independence, we want to prove that,

$$F_X(a)F_Y(b) = F_{X,Y}(a, b) \; \forall \; a, \;\; b \;\; \in \mathbb{R} \tag{3}$$

From conditional probability we know:

$$\Pr(X > a|Y < b) = \frac{\Pr(X > a, Y < b)}{\Pr(Y < b)} \tag{4}$$

and so using the given condition in option 2), we can write (4) as

$$\begin{aligned} \Pr(X > a) &= \frac{\Pr(X > a, Y < b)}{\Pr(Y < b)} \\ \implies \Pr(X > a)\Pr(Y < b) &= \Pr(X > a, Y < b) \end{aligned} \tag{5}$$

We can write the C.D.F as

$$\begin{aligned} \Pr(X > a) &= 1 - F_X(a) \;\; and, \\ \Pr(Y < b) &= F_Y(b) \end{aligned} \tag{6}$$

We may rewrite (5) using (6) as:

$$\begin{aligned} (1 - F_X(a))(F_Y(b)) &= \Pr(X > a, Y < b) \\ F_Y(b) - F_X(a)F_Y(b) &= \Pr(X > a, Y < b) \\ F_X(a)F_Y(b) &= F_Y(b) - \Pr(X > a, Y < b) \end{aligned} \tag{7}$$

Note that since X is continuous so we can write,

$$P(X \le a) = \Pr(X < a) \tag{8}$$

Regardless of the value of X the marginal C.D.F F_Y(b) is given by

$$F_Y(b) = \Pr(Y < b) \tag{9}$$

Now let us define two events

$$Event\ A : (Y < b \cap X < a)$$

$$Event\ B : (Y < b \cap X > a)$$

We can also think of the event (Y < b) as

$$(Y < b) = (\text{Event A}) \cup (\text{Event B}) \tag{10}$$

So it implies

$$\Pr(A, B) = \Pr(Y < b) \tag{11}$$

Since X cannot both be less than a and greater than a, we have

$$\begin{aligned}\Pr(A, B) &= \Pr(A) + \Pr(B) \\ &= \Pr(Y < b, X < a) + \\ &\quad \Pr(Y < b, X > a)\end{aligned} \tag{12}$$

∴ We can write

$$\begin{aligned}F_Y(b) = \Pr(X > a, Y < b) + \\ \Pr(X < a, Y < b)\end{aligned} \tag{13}$$

Now putting value of $F_Y(b)$ from (13) into (7) proves (3) ,

$$\begin{aligned}F_X(a)F_Y(b) &= \Pr(X < a, Y < b) \\ &= F_{X,Y}(a, b)\end{aligned} \tag{14}$$

Thus (14) imply X and Y to be independent as the joint p.d.f is the product of the two individual p.d.f. given the condition in option 2). So option 2) seems to be always true.

3) Given random variables X and Y are uncorrelated which means that their correlation is 0, or, equivalently, Cov(X, Y) = 0.

$$\begin{aligned}Cov(X, Y) = E[XY] - E[X]E[Y] \\ [\because Cov(X, Y) = 0] \\ E[XY] = E[X]E[Y]\end{aligned} \tag{15}$$

We have to prove that uncorrelated implies independence.
Let's take X and Y to exist as an ordered pair at the points (-1,1), (0,0), and (1,1) with probabilities $\frac{1}{4}$,$\frac{1}{2}$, and $\frac{1}{4}$. The expected values of X and Y is

$$\begin{aligned}E[X] = -1 \cdot \frac{1}{4} + 0 \cdot \frac{1}{2} + 1 \cdot \frac{1}{4} = 0 = E[Y] \\ E[XY] = -1 \cdot \frac{1}{4} + 0 \cdot \frac{1}{2} + 1 \cdot \frac{1}{4} = 0 = E[X]E[Y]\end{aligned} \tag{16}$$

Now let's look at the marginal distributions of X and Y. X and Y both take on the values −1, 0, 1 and the probability it takes for each of those are given by $\frac{1}{4}$, $\frac{1}{2}$, $\frac{1}{4}$. Then looping through the possibilities, we have to check if

$$\Pr(X = x, Y = y) = \Pr(X = x)\Pr(Y = y) \tag{17}$$

Let's take the first point (-1, 1) and examine,

$$\begin{aligned}\Pr(X = -1, Y = 1) = \frac{1}{4} \\ \neq \frac{1}{16} = \Pr(X = -1)\Pr(Y = 1)\end{aligned} \tag{18}$$

We loop through the other two points, and see that X and Y do not meet the definition of independent.
Thus it proves that uncorrelated random variables are not always independent. Hence option (3) is false.

4) We extend L.H.S

$$\begin{aligned}E[(X - a)(Y - b)] = E[XY] \\ -aE[Y] - bE[X] + ab\end{aligned} \tag{19}$$

and R.H.S to compare,

$$\begin{aligned}E(X - a)E(Y - b) = E[X]E[Y] \\ -aE[Y] - bE[X] + ab\end{aligned} \tag{20}$$

We see (13) = (14), i.e. independent iff $E[XY] = E[X]E[Y]$ but in this case independence of X and Y cannot be inferred.

Let us take a counter example to understand further as to why this condition not always true.
Let X be a standard normal random variable and $Y = X^2$.
Then, since $E(X) = E(X^3) = 0$, we have

$$E(XY) = E(X^3) = 0 = E(X)E(Y) \tag{21}$$

However, they are not independent:

$$\begin{aligned}\Pr(0 < X < 1, Y > 1) = 0 \neq \\ \Pr(0 < X < 1)\Pr(Y > 1)\end{aligned} \tag{22}$$

Hence option (4) is false.