AI5002 - Assignment 1

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1. Code
2. LaTeX

Problem

6.1.5 Find an expression for $p_A(x)$ using the definition.

Solution

Given $X_1 \sim N(0,1)$, $X_2 \sim N(0,1)$, *and* $V = X_1^2 + X_2^2$.

So we can write A as
$A = \sqrt{V}$
$or,\ A = \sqrt{X_1^2 + X_2^2}$

Since joint distribution is not mentioned so we assume X_1 *and* X_2 to be independent otherwise the distribution of A would be unknown.

By definition, the distribution of A is Chi with two degrees of freedom or Rayleigh.

The CDF given by $F_A(x)$ can be written as

$= P\ (A \le x)$

$= P(\sqrt{X_1^2\ +\ X_2^2} \le x)$

$= \iint_{\sqrt{X_1^2\ +\ X_2^2}\ \le\ x} F(x_1, x_2)\ \ dx_1\, dx_2$
$= \iint_{\sqrt{X_1^2\ +\ X_2^2}\ \le\ x} F(x_1).F(x_2)\ \ dx_1\, dx_2$
$= \iint_{\sqrt{X_1^2\ +\ X_2^2}\ \le\ x} \frac{1}{\sqrt{2\pi\sigma^2}}.\ \frac{1}{\sqrt{2\pi\sigma^2}}.\ e^{\frac{-x_1^2}{2\sigma^2}}.\ e^{\frac{-x_2^2}{2\sigma^2}}\ \ dx_1\, dx_2$
$= \iint_{\sqrt{X_1^2\ +\ X_2^2}\ \le\ x} \frac{1}{2\pi\sigma^2}.\ e^{\frac{-(x_1^2\ +\ x_2^2)}{2\sigma^2}}\ \ dx_1\, dx_2 \qquad (1)$

Now, we perform a transformation of variables to simplify solution.
Let $x_1 = r\cos\ \theta\ and,\ \ x_2 = r\sin\ \theta$.

$\implies \sqrt{x_1^2 + x_2^2} = r$

Also, $\theta = \tan^{-1}\frac{x_2}{x_1}$

Since we're using transformation of two variables, we use Jacobian matrix here

$$J_{x_1,x_2} = \begin{bmatrix} \frac{\partial x_1}{\partial r} & \frac{\partial x_1}{\partial \theta} \\ \frac{\partial x_2}{\partial r} & \frac{\partial x_2}{\partial \theta} \end{bmatrix} = \begin{bmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{bmatrix}$$

$\det(\mathbf{J}_{x_1,x_2}) = r\ \cos^2\theta + r\ \sin^2\theta\ = r$

Thus, $F(r,\ \theta) = r.F(x_1, x_2)$
where (x_1, x_2) *are written in terms of* (r, θ).

Now, we can write (1) as

$$\iint_{r\ \le\ x} \frac{1}{2\pi\sigma^2}.\ e^{\frac{-r^2}{2\sigma^2}}.\ r\, dr\, d\theta$$

or,

$$\int_{\theta=0}^{2\pi}\int_{r=0}^{x} \frac{1}{2\pi\sigma^2}.\ e^{\frac{-r^2}{2\sigma^2}}.\ r\, dr\, d\theta \qquad (2)$$

$$\begin{aligned} Let\ u &= -\frac{r^2}{2\sigma^2} \\ du &= -\frac{2r}{2\sigma^2}\, dr \\ r\, dr &= -\sigma^2\, du \end{aligned}$$

$$where\ 0 \le\ r\ \le\ x\ and\ 0 \le\ u\ \le\ -\frac{x^2}{2\sigma^2}$$

Substituting u and r dr in (2), we get

$$\int_{\theta=0}^{2\pi}\int_{u=0}^{-\frac{x^2}{2\sigma^2}} \frac{1}{2\pi\sigma^2}.\ e^{u}.\ (-\sigma^2)\ \ du\, d\theta$$

$$= \int_{\theta=0}^{2\pi} -\frac{1}{2\pi} \int_{u=0}^{-\frac{x^2}{2\sigma^2}} e^u \; du\, d\theta = -\frac{1}{2\pi} \int_{\theta=0}^{2\pi} (e^{-\frac{x^2}{2\sigma^2}} - 1) \; d\theta$$

$= 1$ - exp $(-x^2/2\sigma^2)$

Thus PDf of the above CDF expression is derived as

$\mathrm{p}_A(x) = \frac{d}{dx} F_A(x)$

$=$ - $\mathrm{e}^{-\frac{x^2}{2\sigma^2}}$. $(-\frac{2x}{2\sigma^2})$

$= \boldsymbol{e^{-\frac{x^2}{2\sigma^2}} \cdot \frac{x}{\sigma^2}}$