Solutions to the Homework Problems

1. Obtain the equation of the parabola whose focus is at the point $(-1,0)$ whose directrix is the straight line $y = x$.
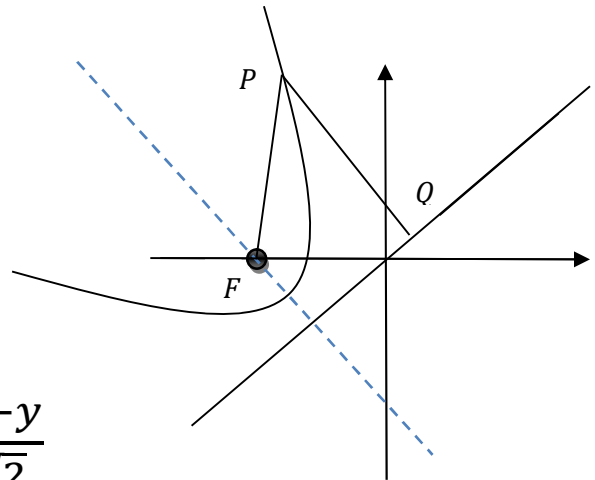
Solution:

We have,

$$FP = PQ$$

$\Rightarrow \quad \sqrt{(x+1)^2 + y^2} = \frac{x-y}{\sqrt{2}}$

$\Rightarrow \quad \sqrt{x^2 + 2x + 1 + y^2} = \frac{x-y}{\sqrt{2}}$

$\Rightarrow \quad x^2 + y^2 + 2x + 1 = \frac{x^2 - 2xy + y^2}{2}$

$\Rightarrow \quad 2x^2 + 2y^2 + 4x + 2 = x^2 - 2xy + y^2$

$\Rightarrow \quad x^2 - 2xy + y^2 + 4x + 2 = 0$

2. Find the equation of the tangent to the parabola $y^2 = 6x$ which is perpendicular to the line $4y - 3x + 7 = 0$.

Solution:

Consider, the required equation of the tangent is,

$$y = mx + c \quad \text{..(1)}$$

Given,

$$4y - 3x + 7 = 0$$

$$\Rightarrow \quad y = \frac{3}{4}x - \frac{7}{4}$$

So, $\quad m = -\frac{4}{3}$

From (1),

$$y = -\frac{4}{3}x + c \quad \text{..(2)}$$

Since equation (2) touches the parabola,

$$y^2 = 6x$$

$$\Rightarrow \quad y^2 = 4.\frac{3}{2}x$$

$$\therefore c = \frac{a}{m}$$

$$\Rightarrow \quad c = \frac{3/2}{-4/3}$$

$$\Rightarrow \quad c = -\frac{9}{8}$$

From (2),

$$y = -\frac{4}{3}x - \frac{9}{8}$$

3. Show that, the equation $2x^2 + y - x + 4 = 0$ represents a parabola. Find it's vertex, focus, tangent at the vertex, directrix, latus rectum, length of the latus rectum. Also sketch the above parabola.

Solution:

Given,

$$2x^2 + y - x + 4 = 0 \quad \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots(1)$$

Here,

$$a = 2$$

$$h = 0$$

$$b = 0$$

$$g = -\frac{1}{2}$$

$$f = \frac{1}{2}$$

$$c = 4$$

Now,

$$\Delta = abc + 2fgh - af^2 - bg^2 - ch^2$$

$$\Rightarrow \quad \Delta = -\frac{1}{2}$$

And,

$$h^2 - ab$$

$$= 0$$

Hence, the equation (1) represents a parabola.

From (1),

$$2x^2 + y - x + 4 = 0$$

$$\Rightarrow \quad x^2 + \frac{1}{2}y - \frac{1}{2}x + 2 = 0$$

$$\Rightarrow \quad x^2 - 2 \cdot \frac{1}{4} \cdot x + \left(\frac{1}{4}\right)^2 - \left(\frac{1}{4}\right)^2 + \frac{1}{2}y + 2 = 0$$

$\Rightarrow \quad \left(x - \frac{1}{4}\right)^2 = -\frac{1}{2}y - \frac{31}{16}$

$\Rightarrow \quad \left(x - \frac{1}{4}\right)^2 = -\frac{1}{2}\left(y + \frac{31}{8}\right)$

$\Rightarrow \quad X^2 = 4aY$

where

$$X = x - \frac{1}{4}$$
$$Y = y + \frac{31}{8}$$
$$4a = -\frac{1}{2}$$
$$\therefore \ a = -\frac{1}{8}$$

Vertex,

$Y = 0$

$\Rightarrow \quad y + \frac{31}{8} = 0$

$\Rightarrow \quad y = -\dfrac{31}{8}$

$X = 0$

$\Rightarrow \quad x - \dfrac{1}{4} = 0$

$\Rightarrow \quad x = \dfrac{1}{4}$

The Coordinates of the vertex are $\left(\dfrac{1}{4}, -\dfrac{31}{8}\right)$

Focus,

$\quad X = 0$

$\Rightarrow \quad x - \dfrac{1}{4} = 0$

$\Rightarrow \quad x = \dfrac{1}{4}$

$$Y = a$$

$$\Rightarrow \quad y + \frac{31}{8} = -\frac{1}{8}$$

$$\Rightarrow \quad y = -4$$

$$\therefore (x, y) = \left(\frac{1}{4}, -4\right)$$

Equation of the tangent at the vertex,

$$Y = 0$$

$$\Rightarrow \quad y + \frac{31}{8} = 0$$

$$\Rightarrow \quad y = -\frac{31}{8}$$

Equation of the directrix,

$$Y = -a$$

$$\Rightarrow \quad y + \frac{31}{8} = -\left(-\frac{1}{8}\right)$$

$$\Rightarrow \quad y = -\frac{15}{4}$$

Equation of the latus rectum,

$$Y = a$$

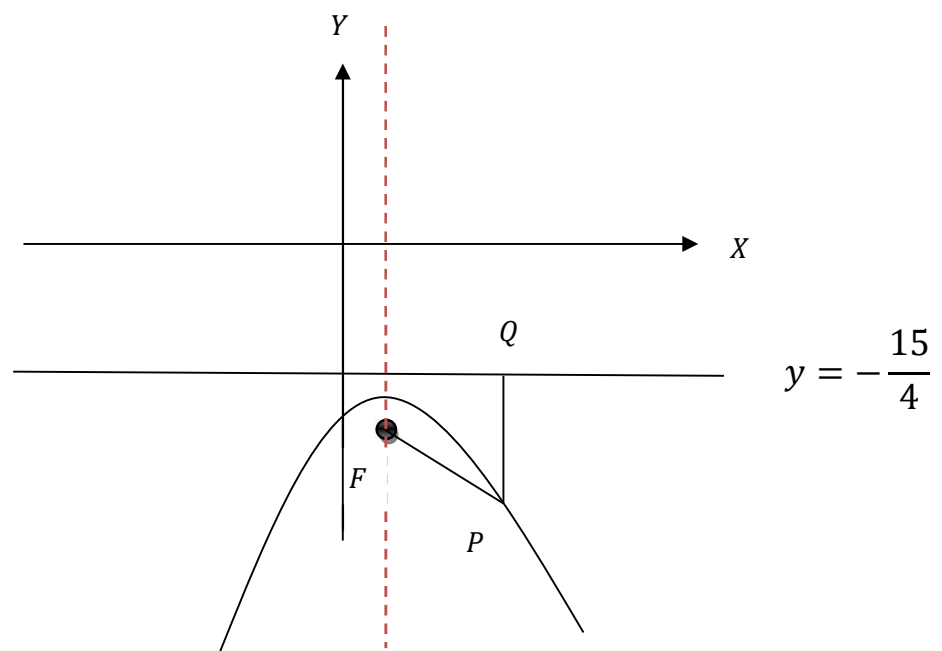$$\Rightarrow \quad y + \frac{31}{8} = -\frac{1}{8}$$

$$\Rightarrow \quad y = -4$$

Length of the latus rectum,

$$|4a|$$

$$= \left| 4 \cdot \left(-\frac{1}{8} \right) \right|$$

$$= \frac{1}{2} \ unit$$

4. Determine the equation of the parabola whose vertex is at the point $(2,1)$ and whose directrix is the line $y = 5$.

<u>Solution:</u>

We have,

$$FP = PQ$$

$$\Rightarrow \sqrt{(x-2)^2 + (y+3)^2} = \frac{y-5}{\sqrt{1}}$$

$$\Rightarrow \sqrt{x^2 - 4x + 4 + y^2 + 6y + 9} = y - 5$$

$$\Rightarrow x^2 - 4x + 4 + y^2 + 6y + 9 = y^2 - 10y + 25$$

$$\Rightarrow x^2 - 4x + 16y - 12 = 0$$