Topic-6: The Hyperbola

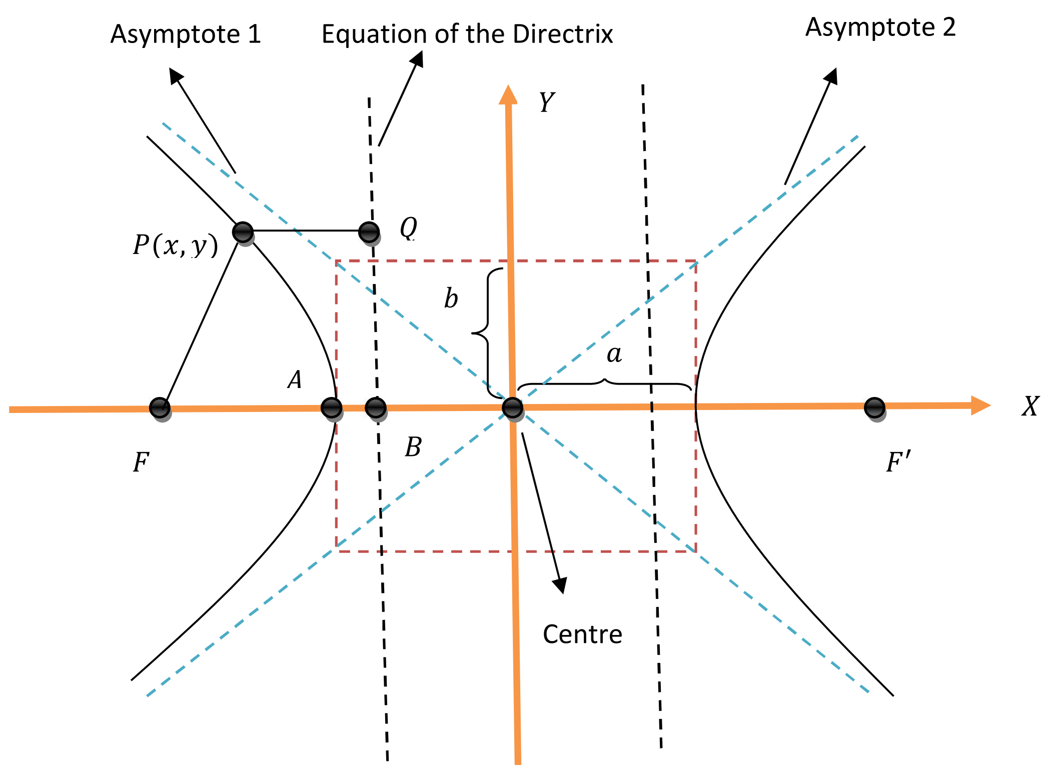
The general equation of a hyperbola with centre (α, β) is

$$\frac{(x-\alpha)^2}{a^2} - \frac{(y-\beta)^2}{b^2} = 1, \qquad a > b$$

In Particular, the general equation of a hyperbola is

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1, \qquad a > b$$

Graph:



Note:

1. $FA = e \cdot AB$

2. $FP = e \cdot PQ$

3. $FP - F'P = 2a$

where eccentricity, $e^2 = 1 + \frac{b^2}{a^2}$ $\quad [e > 1]$

Properties:

(i) The coordinates of the centre $(0,0)$.

(ii) Coordinates of the vertices $(\pm a, 0)$.

(iii) Coordinates of the focus $(\pm ae, 0)$.

(iv) Equation of the transverse axis $Y = 0$

(v) Equation of the conjugate axis $X = 0$

(vi) Equations of the directrices $x = \pm \frac{a}{e}$.

(vii) Length of the transverse axis $2a$

(viii) Length of the conjugate axis $2b$

(ix) Length of the latus rectum $2\frac{b^2}{a}$

(x) Equations of the latus rectum $x = \pm ae$

Special Information:

(1) The straight line $y = mx + c$ touches the ellipse $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ if

$$c = \pm\sqrt{a^2m^2 - b^2}.$$

(2)

(i) Requirement: Hyperbola

Given: Asymptotes of a hyperbola

Formula: (Asymptote 1) $\times$ (Asymptote 2) $+ k = 0$

(ii) Requirement: Asymptotes of a hyperbola

Given: Equation of a hyperbola

Formula: (hyperbola) + $\frac{\Delta}{h^2-ab} = 0$

Solved Problems

1. Identify the conic given by the following equation

$$9x^2 + 24xy + 2y^2 - 6x + 20y + 41 = 0$$

Then reduce this conic to its standard form.

Solution:

Given,

$$9x^2 + 24xy + 2y^2 - 6x + 20y + 41 = 0$$

Here,

$$\mathrm{a} = 9$$

$$h = 12$$

$$b = 2$$

$$g = -3$$

$$f = 10$$

$$c = 41$$

Now,

$$\Delta = abc + 2fgh - af^2 - bg^2 - ch^2$$

$\Rightarrow \quad \Delta = -6804$

and, $\quad h^2 - ab$

$$= 126 > 0$$

Since, $\Delta \neq 0$ and $h^2 - ab > 0$, so the given equation represents a hyperbola.

Given,

$$9x^2 + 24xy + 2y^2 - 6x + 20y + 41 = 0 \quad \ldots\ldots\ldots(1)$$

Putting $x = x + \alpha$ and $y = y + \beta$ in equation (1),

$$9(x+\alpha)^2 + 24(x+\alpha)(y+\beta) + 2(y+\beta)^2 - 6(x+\alpha) + 20(y+\beta) + 41 = 0$$

$$\Rightarrow \quad 9(x^2 + 2x\alpha + \alpha^2) + 24(xy + x\beta + y\alpha + \alpha\beta) + 2(y^2 + 2y\beta + \beta^2) - 6(x+\alpha) + 20(y+\beta) + 41 = 0$$

$$\Rightarrow \quad 9x^2 + 24xy + 2y^2 + x(18\alpha + 24\beta - 6) + y(24\alpha + 4\beta + 20) + 9\alpha^2 + 24\alpha\beta + 2\beta^2 - 6\alpha + 20\beta + 41 = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

The terms of x and y in equation (2) will be absent if

$$18\alpha + 24\beta - 6 = 0$$

And $\quad 24\alpha + 4\beta + 20 = 0$

Solving these two equations we get,

$\therefore \alpha = -1$ and $\beta = 1$

Putting $\alpha = -1$ and $\beta = 1$ in equation (2),

$$9x^2+24xy+2y^2+9-24+2+6+20+41=0$$

⇨ $9x^2+24xy+2y^2+54=0$…………..(3)

Now Putting $x = xcos\theta - ysin\theta$ and $y = xsin\theta + ycos\theta$ in equation (3),

$$9(xcos\theta - ysin\theta)^2 + 24(xcos\theta - ysin\theta)$$

$$(xsin\theta + ycos\theta) + 2(xsin\theta + ycos\theta)^2 + 54 = 0$$

⇨ $9(x^2\cos^2\theta - 2xysin\theta cos\theta + y^2\sin^2\theta) + 24(x^2sin\theta cos\theta + xycos^2\theta - xysin^2\theta - y^2sin\theta cos\theta) + 2(x^2\sin^2\theta + 2xysin\theta cos\theta + y^2\cos^2\theta) + 54 = 0$

⇨ $x^2(9\cos^2\theta + 24sin\theta cos\theta + 2\sin^2\theta) + xy(-18sin\theta cos\theta + 24\cos^2\theta - 24\sin^2\theta + 4sin\theta cos\theta) + y^2(9\sin^2\theta - 24sin\theta cos\theta + 2\cos^2\theta) + 54 = 0$ ……………………….(4)

To remove the xy term in equation (4), we can write,

$$-18sin\theta cos\theta + 24\cos^2\theta - 24\sin^2\theta + 4sin\theta cos\theta = 0$$

⇨ $$24\cos^2\theta - 14sin\theta cos\theta - 24\sin^2\theta = 0$$

⇨ $$12\cos^2\theta - 7sin\theta cos\theta - 12\sin^2\theta = 0$$

⇨ $$12\cos^2\theta - 16sin\theta cos\theta + 9sin\theta cos\theta - 12\sin^2\theta = 0$$

⇨ $$4cos\theta(3cos\theta - 4sin\theta) + 3sin\theta(3cos\theta - 4sin\theta) = 0$$

⇨ $$(3cos\theta - 4sin\theta)(4cos\theta + 3sin\theta) = 0$$

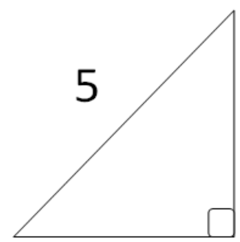
$\therefore$ $(4cos\theta + 3sin\theta) = 0$

⇨ $$tan\theta = -\frac{4}{3}$$

Or, $(3cos\theta - 4sin\theta) = 0$

⇨ $$tan\theta = \frac{3}{4}$$

when $tan\theta = \frac{3}{4}$



$\therefore\ sin\theta = \frac{3}{5}$ and $cos\theta = \frac{4}{5}$

Putting $sin\theta = \frac{3}{5}$ and $cos\theta = \frac{4}{5}$ in equation (4),

$$x^2\left(9.\frac{16}{25} + 24\ \frac{3}{5}.\frac{4}{5}\ + 2\frac{9}{25}\right) + xy\left(-18.\frac{3}{5}.\frac{4}{5}\ + 24.\frac{16}{25} - 24\ \frac{9}{25} + 4.\frac{3}{5}.\frac{4}{5}\right)\ + y^2\left(9.\frac{9}{25} - 24.\frac{3}{5}.\frac{4}{5} + 2.\frac{16}{25}\right) + 54 = 0$$

⇨ $\quad 18x^2 - 7y^2 + 54 = 0$

⇨ $\quad 18x^2 - 7y^2 = -54$

⇨ $\quad \frac{x^2}{-54/18} - \frac{y^2}{-54/7} = 1$

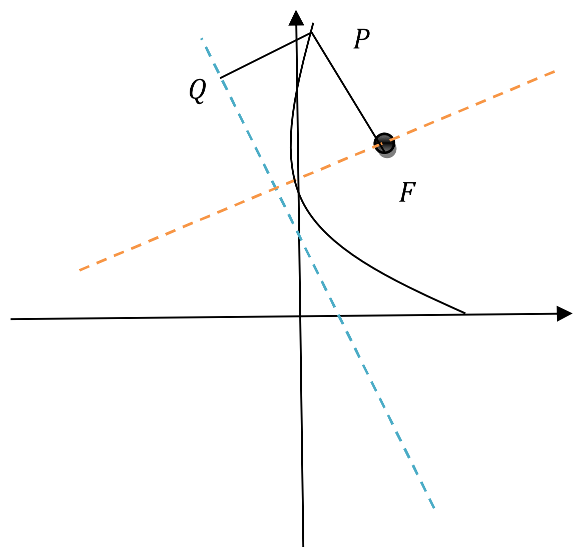
⇨ $\quad -\frac{x^2}{3} + \frac{y^2}{54/7} = 1$

$$\therefore \frac{y^2}{54/7} - \frac{x^2}{3} = 1$$

This is the standard equation of the hyperbola.

2. Find the equation of the hyperbola whose directrix is $2x + y = 1$, focus is $(1,2)$ eccentricity is $\sqrt{3}$.

Solution:



We have

$$FP = e.PQ$$

$$\Rightarrow \quad \sqrt{(x-1)^2 + (y-2)^2} = \sqrt{3}.\frac{2x+y-1}{\sqrt{5}}$$

$$\Rightarrow \quad \sqrt{x^2 - 2x + 1 + y^2 - 4y + 4} = \sqrt{3}.\frac{2x+y-1}{\sqrt{5}}$$

$\Rightarrow \quad x^2 + y^2 - 2x - 4y + 5 = \frac{(2x+y-1)^2}{5} \times 3$

$\Rightarrow \quad 5x^2 + 5y^2 - 10x - 20y + 25 =$
$3\{4x^2 + 4x(y-1) + (y-1)^2\}$

$\Rightarrow \quad 5x^2 + 5y^2 - 10x - 20y + 25 =$
$3(4x^2 + 4xy - 4x + y^2 - 2y + 1)$

$\Rightarrow \quad 5x^2 + 5y^2 - 10x - 20y + 25 = 12x^2 +$
$12xy - 12x + 3y^2 - 6y + 3$

$\Rightarrow \quad 7x^2 + 12xy - 2y^2 - 2x + 14y - 22 = 0$

3. Find the equation of the hyperbola which passes through the point $(5,1)$ and has asymptotes $2\mathrm{x}+\mathrm{y}-1=0$ and $2\mathrm{x}-\mathrm{y}+1=0$.

Solution:

We can write the equation of the hyperbola,

$$(2x+y-1)(2x-y+1)+k=0\text{............(1)}$$

Since Equation (1) passes through the point $(5,1)$, So we can write

$$(2.5+1-1)(2.5-1+1)+k=0$$

$$\Rightarrow \quad k=-100$$

Putting the value of k in equation (1),

$$(2x+y-1)(2x-y+1)=100$$

4. Find the equation of the hyperbola whose asymptotes are parallel to the lines $2\mathrm{x} + 3\mathrm{y} = 0$ and $3\mathrm{x} - 2\mathrm{y} = 0$, whose center is at $(1,2)$ and which passes through the point $(5,3)$.

Solution:

Asymptotes are parallel to the lines $2x + 3y = 0$ and $3x - 2y = 0$

So, the asymptotes are $2x + 3y + a = 0$ and $3x - 2y + b = 0$.

Since the center of the hyperbola is at $(1,2)$

So,

$$2x + 3y + a = 0$$

$$\Rightarrow \quad 2.1 + 3.2 + a = 0$$

$$\Rightarrow \quad a = -8$$

and,

$$3x - 2y + b = 0$$

⇨ $3.1 - 2.2 + b = 0$

⇨ $b = 1$

Now we can write the equation of the hyperbola,

$$(2x + 3y - 8)(3x - 2y + 1) + k = 0 \ldots\ldots\ldots\ldots(1)$$

Since Equation (1) passes through the point $(5,3)$, So we can write

$$(10 + 9 - 8)(15 - 6 + 1) + k = 0$$

⇨ $k = -110$

Putting the value of k in equation (1),

$$(2x + 3y - 8)(3x - 2y + 1) - 110 = 0$$

⇨ $(2x + 3y - 8)(3x - 2y + 1) = 110$

5. Find the centre of the following hyperbola

$$xy - 3x - 2y = 0$$

Solution:

Given,

$$xy - 3x - 2y = 0$$

Where $\Delta = \frac{3}{2}$ and $h^2 - ab = \frac{1}{4}$

Now the equation of the asymptotes can be written as

$$xy - 3x - 2y + \frac{\Delta}{h^2 - ab} = 0$$

$$=> xy - 3x - 2y + 6 = 0$$

$$=> x(y - 3) - 2(y - 3) = 0$$

$$=> (x - 2)(y - 3) = 0$$

So, the asymptotes are

$$x - 2 = 0 \text{ and}$$

$$y - 3 = 0$$

Hence, the centre of the given hyperbola is (2,3).

Homework Problems

1. Find the equation of the tangent to the hyperbola $\frac{x^2}{25} - \frac{y^2}{1} = 1$ which is perpendicular to the $-2y = x + 1$.

2. Find the center of the following hyperbolas:

a. $2x^2 - 3xy + y^2 - 5x + 4y + 6 = 0$

b. $x^2 - 6xy + y^2 - 10x - 10y - 19 = 0$

c. $xy + 3ax - 3ay = 0$

3. Find the asymptotes of the following hyperbolas:

a. $x^2 - y^2 + 3x - 7y - 3 = 0$

b. $3x^2 + 8xy - 3y^2 + 6x + 8y + 4 = 0$

c. $2x^2 + 9xy - 5y^2 + 2y - 7 = 0$

4. Find the equation of the hyperbola whose asymptotes are the straight lines $2x + 3y - 5 = 0$ and $5x + 3y - 8 = 0$ and which passes through the point $(1, -1)$.