Solutions to the Homework Problems

1. Find the value(s) of k so that the equation

$$3x^2 + 10xy + 8y^2 - kx - 26y + 21 = 0$$

represents a pair of straight lines and then find the angle between the lines.

Solution:

Given,

$3x^2 + 10xy + 8y^2 - kx - 26y + 21 = 0$ ....(1)

Here,

$a = 3$

$h = 5$

$b = 8$

$g = -\frac{k}{2}$

$f = -13$

$c = 21$

Now,

$$\Delta = 0$$

$\Rightarrow \quad abc + 2fgh - af^2 - bg^2 - ch^2 = 0$

$\Rightarrow \quad 3.8.21 + 2.(-13)\left(-\frac{k}{2}\right).5 - 3.169 - 8.\frac{k^2}{4} - 21.25 = 0$

$\Rightarrow \quad 504 + 65k - 507 - 2k^2 - 525 = 0$

$\Rightarrow \quad -2k^2 + 65k - 528 = 0$

$\Rightarrow \quad 2k^2 - 65k + 528 = 0$

$\therefore k = \frac{33}{2}, 16$

Angle between the lines,

$$tan\theta = \frac{2\sqrt{h^2 - ab}}{a+b}$$

$\Rightarrow \quad tan\theta = \frac{2\sqrt{25-24}}{3+8}$

$\Rightarrow \quad \theta = \tan^{-1}\left(\frac{2}{11}\right)$

2. Find the values of p and q if the equation

$$\mathrm{px^2 - 8xy + 3y^2 + 14x + 2y + q = 0}$$

represents a pair of perpendicular lines.

Solution:

Given,

$$px^2 - 8xy + 3y^2 + 14x + 2y + q = 0$$

Here,

$$a = p$$

$$h = -4$$

$$b = 3$$

$$g = 7$$

$$f = 1$$

$$c = q$$

Now, $\Delta = 0$

$\Rightarrow abc + 2fgh - af^2 - bg^2 - ch^2 = 0$

$\Rightarrow p.3.q + 2.1.7.(-4) - p - 3.49 - q.16 = 0$

$\Rightarrow 3pq - 56 - p - 147 - 16q = 0$

$\Rightarrow 3pq - p - 16q - 203 = 0$ ....(1)

Also,

$a + b = 0$

$\Rightarrow \quad p + 3 = 0$

$\Rightarrow \quad p = -3$

Putting $p = -3$ in equation (1),

$3.(-3)q - (-3) - 16q - 203 = 0$

$\Rightarrow \quad -9q - 16q - 200 = 0$

$\Rightarrow \quad -25q = 200$

$\Rightarrow \quad q = -8$

$\therefore$ The values of $p = -3$ and $q = -8$

3. Find the equation to the pair of straight lines through the origin perpendicular to the pair given by

$$2x^2 + 5xy + 2y^2 + 10x + 5y = 0$$

Solution:

Given,

$$2x^2 + 5xy + 2y^2 + 10x + 5y = 0$$

$$\Rightarrow \quad 2x^2 + (5y + 10)x + (2y^2 + 5y) = 0$$

$$\Rightarrow \quad x = \frac{-(5y+10) \pm \sqrt{(5y+10)^2 - 4.2(2y^2+5y)}}{2.2}$$

$$\Rightarrow \quad x = \frac{-(5y+10) \pm \sqrt{25y^2+100y+100-16y^2-40y}}{4}$$

$$\Rightarrow \quad x = \frac{-(5y+10) \pm \sqrt{9y^2+60y+100}}{4}$$

$$\Rightarrow \quad x = \frac{-(5y+10) \pm \sqrt{(3y+10)^2}}{4}$$

$$\Rightarrow \quad x = \frac{-(5y+10) \pm (3y+10)}{4}$$

$$\therefore \quad x = \frac{-(5y+10) + (3y+10)}{4}$$

$$\therefore 4x = -2y$$

$$\Rightarrow \quad 4x + 2y = 0$$

$$\Rightarrow \quad 2x + y = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots(1)$$

Or,

$$x = \frac{-(5y+10) - (3y+10)}{4}$$

$$4x = -8y - 20$$

$$\Rightarrow \quad 4x + 8y + 20 = 0$$

$$\Rightarrow \quad x + 2y + 5 = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots(2)$$

Since our required pair of straight lines are perpendicular to the lines (1) and (2), so we get

$$x - 2y + a = 0 \quad \ldots\ldots\ldots\ldots(3)$$

$$2x - y + b = 0 \quad \ldots\ldots\ldots\ldots(4)$$

Again, (3) and (4) passes through the origin, so we get,

$$a = 0 \text{ and } b = 0$$

From (3) and (4), we finally get our pair of straight lines

$$x - 2y = 0$$

$$2x - y = 0$$

So, our required equation is

$$(x - 2y)(2x - y) = 0$$

$$\Rightarrow \ 2x^2 - 5xy + 2y^2 = 0$$

4. Show that the equation

$$x^2 + 4xy - 2y^2 + 6x - 12y - 15 = 0$$

represents a pair of straight lines which together with $x^2 + 4xy - 2y^2 = 0$ form a rhombus.

Solution:

Given,

$$x^2 + 4xy - 2y^2 + 6x - 12y - 15 = 0 \quad ....(1)$$

Here,

$$a = 1$$

$$h = 2$$

$$b = -2$$

$$g = 3$$

$f = -6$

$c = -15$

Now,

$$\Delta = abc + 2fgh - af^2 - bg^2 - ch^2$$

$$\Rightarrow \quad \Delta = 0$$

∴ The equation (1) represents a pair of straight lines.

Now,

$$x^2 + 4xy - 2y^2 + 6x - 12y - 15 = 0$$

$$\Rightarrow \quad x^2 + (4y + 6)x + (-2y^2 - 12y - 15) = 0$$

$$\Rightarrow \quad x = \frac{-(4y+6) \pm \sqrt{(4y+6)^2 - 4\cdot(-2y^2 - 12y - 15)}}{2}$$

$$\Rightarrow \quad x = \frac{-(4y+6) \pm \sqrt{24y^2 + 96y + 96}}{2}$$

$\Rightarrow \quad x = \frac{-(4y+6) \pm 2\sqrt{6}\sqrt{y^2+4y+4}}{2}$

$\Rightarrow \quad x = \frac{-(4y+6) \pm 2\sqrt{6}\sqrt{(y+2)^2}}{2}$

$\Rightarrow \quad x = \frac{-(4y+6) \pm 2\sqrt{6}(y+2)}{2}$

$\Rightarrow \quad x = -(2y+3) \pm \sqrt{6}(y+2)$

$\therefore \quad x = -(2y+3) + \sqrt{6}(y+2)$

$\Rightarrow \quad x + \left(2-\sqrt{6}\right)y - (2\sqrt{6}-3) = 0$………….(2)

Or,

$x = -(2y+3) - \sqrt{6}(y+2)$

$\Rightarrow \quad x + \left(2+\sqrt{6}\right)y + (2\sqrt{6}+3) = 0$ ………..(3)

Again, we have

$$x^2 + 4xy - 2y^2 = 0$$

$\Rightarrow \quad x = \frac{-4y \pm \sqrt{16y^2 + 8y^2}}{2}$

$\Rightarrow \quad x = \frac{-4y \pm \sqrt{24y^2}}{2}$

$\Rightarrow \quad x = -2y \pm \sqrt{6}y$

$\therefore \quad x = -2y + \sqrt{6}y$

$\Rightarrow \quad x + (2 - \sqrt{6})y = 0$…………(4)

Or, $\quad x = -2y - \sqrt{6}y$

$\Rightarrow \quad x + (2 + \sqrt{6})y = 0$ ……….(5)

Solving (2) and (3) we get, $x = 1$ and $y = -2$

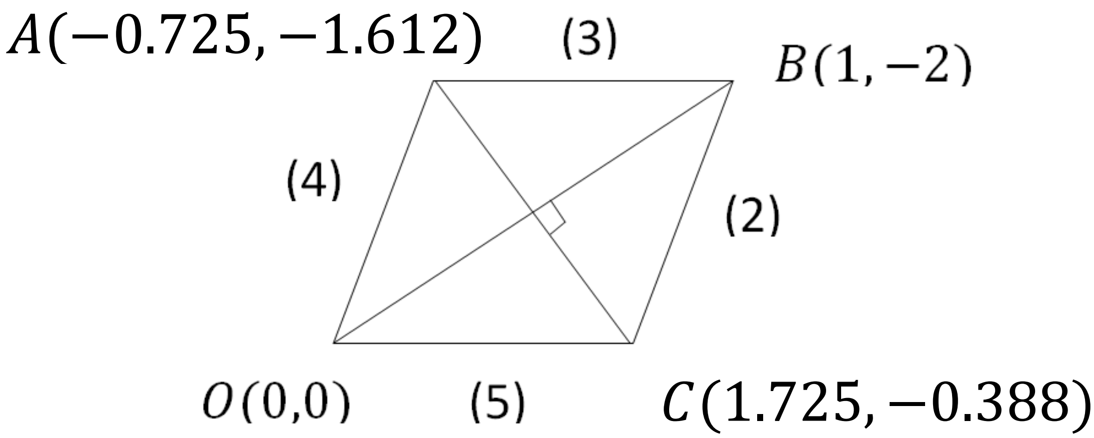
Solving (4) and (5) we get, $x = 0$ and $y = 0$

Now solving (3) and (4) we get,

$$x = -0.725 \text{ and } y = -1.612$$

Now solving (2) and (5) we get

$$x = 1.725 \text{ and } y = -0.388$$



Now the parallelogram $OABC$ will be a rhombus if all sides are equal.

$$OA = 1.77$$

$$AB = 1.77$$

$$BC = 1.77$$

$$OC = 1.77$$

Hence our given equation represents a rhombus.

Topic 3: System of Circles

Important Formulas:

Consider, the general equations of two circles are

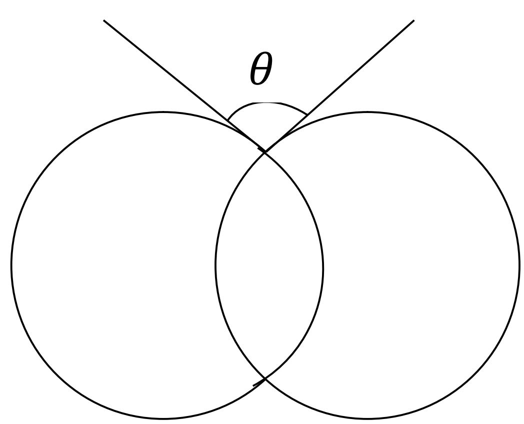
$$x^2 + y^2 + 2g_1x + 2f_1y + c_1 = 0 \ldots(1) \text{ and}$$

$$x^2 + y^2 + 2g_2x + 2f_2y + c_2 = 0 \ldots.(2)$$

1. Angle of Intersection

The angle of intersection between two circles (1) and (2) is,

$$\theta = \cos^{-1}\left\{\frac{2g_1g_2+2f_1f_2-c_1-c_2}{2\sqrt{g_1^2+f_1^2-c_1}\cdot\sqrt{g_2^2+f_2^2-c_2}}\right\}$$



Note:

(i) The two circles touch each other internally if $\theta = 0°$, that means

$$2g_1g_2 + 2f_1f_2 - c_1 - c_2 = 2\sqrt{g_1^2 + f_1^2 - c_1}\,.\sqrt{g_2^2 + f_2^2 - c_2}$$

(ii) The two circles touch each other externally if $\theta = 180°$, that means

$$2g_1g_2 + 2f_1f_2 - c_1 - c_2 = -2\sqrt{g_1^2 + f_1^2 - c_1}\,.\sqrt{g_2^2 + f_2^2 - c_2}$$

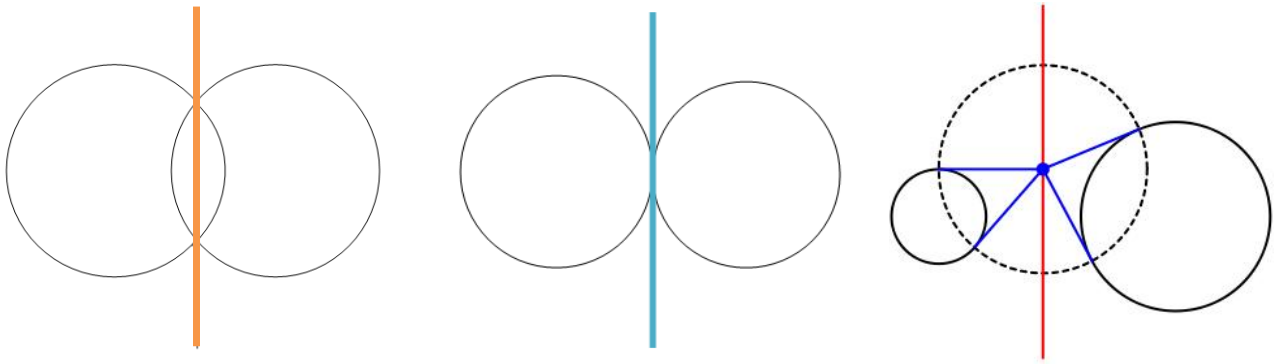
(iii) The two circles cut orthogonally when the tangents at their points of intersection are at right angles if $\theta = 90°$, that means

$$2g_1g_2 + 2f_1f_2 - c_1 - c_2 = 0$$

2. Radical Axis

If the circles cross, their radical axis is the line through their two crossing points, and if they are tangent, it is their line of tangency. For two disjoint circles, the radical axis is the locus of points at which tangents drawn to both circles have equal lengths.

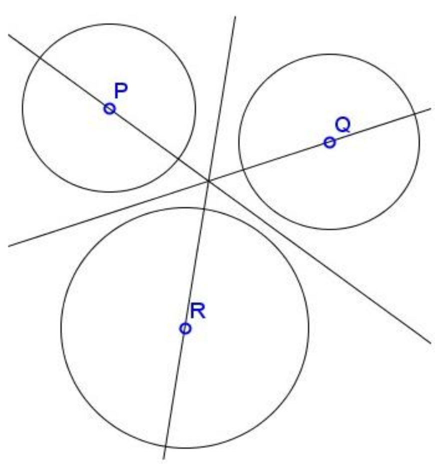
Note: The radical axis is always a straight line and always perpendicular to the line connecting the centers of the circles. We will get a radical axis by subtracting equations (1) and (2).



3. Radical Center

The point in which radical axes of the three circles meet is called the radical center of the three circles.

<u>Note:</u> The lengths of the tangents from the radical center to the three circles are equal.



Solved Problems

1. Find the radical axis of the following circles

$$x^2 + y^2 - 3x - 4y + 5 = 0$$

$$3x^2 + 3y^2 - 7x + 8y + 11 = 0$$

Solution:

Given,

$$x^2 + y^2 - 3x - 4y + 5 = 0 \text{ ………(1)}$$

And, $3x^2 + 3y^2 - 7x + 8y + 11 = 0$

$\Rightarrow \quad x^2 + y^2 - \frac{7}{3}x + \frac{8}{3}y + \frac{11}{3} = 0$………….(2)

The radical axis of the circles (1) and (2) is,

$$-\frac{2}{3}x - \frac{20}{3}y + \frac{4}{3} = 0$$

$\Rightarrow \quad -\frac{2}{3}(x + 10y - 2) = 0$

⇨ $x + 10y - 2 = 0$

2. Find the radical center of the following three circles

$$x^2 + y^2 + 3x - 2y - 4 = 0$$

$$x^2 + y^2 - 2x - y - 6 = 0$$

$$x^2 + y^2 - 1 = 0$$

Solution:

Given,

$x^2 + y^2 + 3x - 2y - 4 = 0$...............(1)

$x^2 + y^2 - 2x - y - 6 = 0$.................(2)

$x^2 + y^2 - 1 = 0$ ....(3)

The radical axis of the circles (1) and (2) is,

$5x - y + 2 = 0$.........................(4)

The radical axis of the circles (1) and (3) is,

$$3x - 2y - 3 = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(5)$$

The radical axis of the circles (2) and (3) is,

$$-2x - y - 5 = 0$$

$$\Rightarrow \quad 2x + y + 5 = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(6)$$

Solving (4) and (5), we get,

$$(x, y) = (-1, -3)$$

Now putting $x = -1$ and $y = -3$ in equation (6), we get

$$0 = 0.$$

∴ The radical center of the circles (1), (2) and (3) is,

$$(x, y) = (-1, -3)$$

3. Find the equation to the circle which cuts orthogonally each of the circles $x^2 + y^2 - 4x + 6y + 10 = 0$, $x^2 + y^2 + 12y + 6 = 0$ and passes through the origin.

Solution:

Given,

$$x^2 + y^2 - 4x + 6y + 10 = 0 \quad \text{..........(1)}$$

And, $x^2 + y^2 + 12y + 6 = 0$ ....(2)

From (1) and (2),

$g_1 = -2$ and, $g_2 = 0$

$f_1 = 3$ $f_2 = 6$

$c_1 = 10$ $c_2 = 6$

Consider, the general equation of a circle is,

$$x^2 + y^2 + 2gx + 2fy + c = 0 \quad \text{..........(3)}$$

Since (1) and (3) cut orthogonally, so we get,

$$2.(-2).g + 2.3.f - 10 - c = 0$$

⇨ $$-4g + 6f - 10 - c = 0 \quad \text{................(4)}$$

Since (2) and (3) cut orthogonally, so we get,

$$2.0.g + 2.6.f - 6 - c = 0$$

⇨ $$12f - 6 - c = 0 \quad \text{.........................(5)}$$

Since (3) passes through the origin, so we get,

$$(x, y) = (0,0)$$

From (3),

$$0^2 + 0^2 + 2.g.0 + 2.f.0 + c = 0$$

⇨ $$c = 0$$

Putting the value of c in (5)

$$12f - 6 - 0 = 0$$

⇨ $$f = \frac{6}{12}$$

⇨ $$f = \frac{1}{2}$$

Putting the value of f and c in (4),

$$-4g + 6.\frac{1}{2} - 10 - 0 = 0$$

⇨ $$g = -\frac{7}{4}$$

Putting the value of g , f and c in (3),

$$x^2 + y^2 + 2\left(-\frac{7}{4}\right)x = 2.\frac{1}{2}.y + 0 = 0$$

⇨ $$x^2 + y^2 - \frac{7}{2}x + y = 0$$

⇨ $$2x^2 + 2y^2 - 7x + 2y = 0$$

4. Prove that the circles $x^2 + y^2 - 3x + 8y - 2 = 0$ and $x^2 + y^2 + 4x - 5y - 24 = 0$ cut orthogonally.

Solution:

Given,

$$x^2 + y^2 - 3x + 8y - 2 = 0 \quad \text{………..(1)}$$

And, $x^2 + y^2 + 4x - 5y - 24 = 0$……..(2)

From (1) and (2),

$g_1 = -\frac{3}{2}$ and, $g_2 = 2$

$f_1 = 4$ $f_2 = -\frac{5}{2}$

$c_1 = -2$ $c_2 = -24$

We know, the two circles cut orthogonally if,

$$2g_1g_2 + 2f_1f_2 - c_1 - c_2$$

$$= 2\left(-\frac{3}{2}\right).2 + 2.4.\left(-\frac{5}{2}\right) - (-2) - (-24)$$

$$= 0$$

5. Find the angle of intersection of the circles $x^2 + y^2 + 4x - 5 = 0$ and $x^2 + y^2 + 4x - 14y - 28 = 0$.

Solution:

Given,

$$x^2 + y^2 + 4x - 5 = 0 \quad \text{................(1)}$$

And, $x^2 + y^2 + 4x - 14y - 28 = 0$........(2)

From (1) and (2),

$g_1 = 2$ and, $g_2 = 2$

$f_1 = 0$ $f_2 = -7$

$c_1 = -5$ $c_2 = -28$

The angle of intersection between two circles is,

$$\theta = \cos^{-1}\left\{\frac{2g_1g_2+2f_1f_2-c_1-c_2}{2\sqrt{g_1^2+f_1^2-c_1}\,.\sqrt{g_2^2+f_2^2-c_2}}\right\}$$

⇨ $$\theta = \cos^{-1}\left\{\frac{2.2.2+2.0.(-7)-(-5)-(-28)}{2\sqrt{2^2+0^2-(-5)}.\sqrt{2^2+(-7)^2-(-28)}}\right\}$$

⇨ $$\theta = \cos^{-1}\left(\frac{41}{54}\right)$$

6. Find the equation to the circle which intersect the circles $x^2+y^2-6y+1=0$ and $x^2+y^2-4y+1=0$ orthogonally and touch the line $3x+4y+5=0$

<u>Solution:</u>

Given,

$$x^2+y^2-6y+1=0 \quad \text{.....................(1)}$$

And, $x^2 + y^2 - 4y + 1 = 0$ ....(2)

From (1) and (2),

$g_1 = 0$ and, $g_2 = 0$

$f_1 = -3$ $f_2 = -2$

$c_1 = 1$ $c_2 = 1$

Consider, the general equation of a circle is,

$x^2 + y^2 + 2gx + 2fy + c = 0$ ....(3)

Since (1) and (3) cut orthogonally, so we get,

$$2.0.g + 2.(-3).f - 1 - c = 0$$

⇨ $-6f - c - 1 = 0$

⇨ $6f + c + 1 = 0$ ....(4)

Since (2) and (3) cut orthogonally, so we get,

$$2.0.g + 2.(-2).f - 1 - c = 0$$

⇨ $-4f - c - 1 = 0$

⇨ $4f + c + 1 = 0$ ……………………(5)

Solving (4) and (5), we get,

$f = 0$

$c = -1$

Putting the value of f and c in (3),

$x^2 + y^2 + 2gx + 2.0. y + (-1) = 0$

⇨ $x^2 + y^2 + 2gx - 1 = 0$ ……………(6)

Whose, center $= (-g, 0)$

And, radius $= \sqrt{g^2 + 1}$

If the circle (6) touches the line $3x + 4y + 5 = 0$, so we get,

$\sqrt{g^2 + 1} = \frac{3.(-g)+4.0+5}{\sqrt{3^2+4^2}}$

⇨ $\sqrt{g^2+1} = \frac{-3g+5}{\sqrt{25}}$

⇨ $g^2 + 1 = \frac{(-3g+5)^2}{25}$

⇨ $25g^2 + 25 = 9g^2 - 30g + 25$

⇨ $16g^2 + 30g = 0$

⇨ $2g(8g + 15) = 0$

$\therefore \; g = 0, -\frac{15}{8}$

Putting the values of g in (6), we get our required circles.

$$x^2 + y^2 - 1 = 0 \qquad [g = 0]$$

$$x^2 + y^2 - \frac{15}{4}x - 1 = 0 \qquad \left[g = -\frac{15}{8}\right]$$

Homework Problems

1. Prove that the two circles $x^2 + y^2 + 2ax + c^2 = 0$ and $x^2 + y^2 + 2by + c^2 = 0$ touch if $\frac{1}{a^2} + \frac{1}{b^2} = \frac{1}{c^2}$

2. Find the equation of the circle which cuts orthogonally the three circles,

$$x^2 + y^2 + 4x + 7 = 0$$

$$2x^2 + 2y^2 + 3x + 5y + 9 = 0$$

$$x^2 + y^2 + y = 0$$

3. Find the radical center of the three circles,

$$x^2 + y^2 + x + 2y + 3 = 0$$

$$x^2 + y^2 + 2x + 4y + 5 = 0$$

$$x^2 + y^2 - 7x - 8y - 9 = 0$$