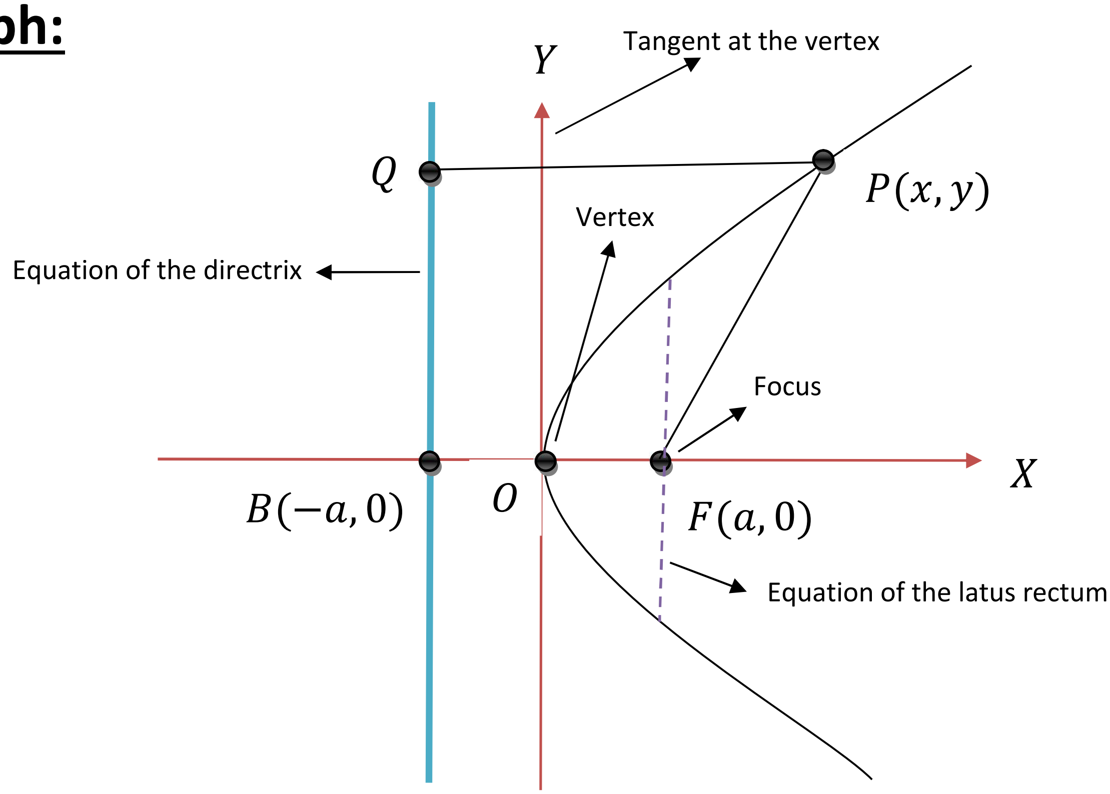
Topic-4: The Parabola

The general equation of a parabola is $Y^2 = 4aX$

Graph:



Note: $FP = PQ$

Properties:

(i) The coordinates of the vertex $(0,0)$.

(ii) The coordinates of the focus $(a, 0)$.

(iii) Equation of the axis $Y = 0$

(iv) Equation of the tangent at the vertex $X = 0$

(v) Equation of the directrix $X = -a$.

(vi) Length of the latus rectum $|4a|$

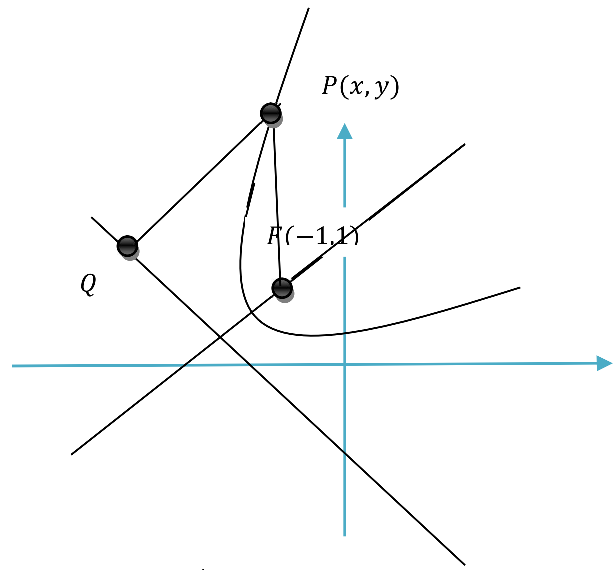
(vii) Equation of the latus rectum $X = a$

Special Information:

The straight line $y = mx + c$ touches the parabola $y^2 = 4ax$ if $c = \frac{a}{m}$.

Solved Problems

1. Obtain The equation of the parabola whose focus is at the point $(-1,1)$ whose directrix is the straight line $x + y + 1 = 0$.



Solution:

We have,

$$FP = PQ$$

$$\Rightarrow \quad \sqrt{(x+1)^2 + (y-1)^2} = \frac{x+y+1}{\sqrt{2}}$$

$$\Rightarrow \quad \sqrt{x^2 + 2x + 1 + y^2 - 2y + 1} = \frac{x+y+1}{\sqrt{2}}$$

$$\Rightarrow \quad x^2 + y^2 + 2x - 2y + 2 = \frac{(x+y+1)^2}{2}$$

$$\Rightarrow \quad 2x^2 + 2y^2 + 4x - 4y + 4 = x^2 + 2x(y+1) + (y+1)^2$$

$\Rightarrow \quad 2x^2 + 2y^2 + 4x - 4y + 4 = x^2 + 2xy + 2x + y^2 + 2y + 1$

$\Rightarrow \quad x^2 + y^2 + 2x - 6y - 2xy + 3 = 0$

$\Rightarrow \quad x^2 - 2xy + y^2 + 2x - 6y + 3 = 0$

2. Find the equation of the tangent to the parabola $\mathrm{y}^2 = 6\mathrm{x}$ which is parallel to the line $4\mathrm{y} - 3\mathrm{x} + 7 = 0.$

<u>Solution</u>:

Consider, the required equation of the tangent is,

$y = mx + c$ ……………………………………(1)

Given,

$4y - 3x + 7 = 0$

$\Rightarrow \quad y = \frac{3}{4}x - \frac{7}{4}$

So, $m = \frac{3}{4}$

From (1),

$$y = \frac{3}{4}x + c \text{ ....(2)}$$

Since equation (2) touches the parabola,

$$y^2 = 6x$$

$$\Rightarrow \quad y^2 = 4.\frac{3}{2}x$$

So we have

$$c = \frac{a}{m}$$

$$\Rightarrow \quad c = \frac{3/2}{3/4}$$

$$\Rightarrow \quad c = 2$$

From (2),

$$y = \frac{3}{4}x + 2$$

3. Show that, the equation $4\mathrm{y}^2 + 12\mathrm{x} - 20\mathrm{y} + 67 = 0$ represents a parabola. Find it's vertex, focus, tangent at the vertex, directrix, latus rectum and the length of the latus rectum.

<u>Solution</u>:

Given,

$$4y^2 + 12x - 20y + 67 = 0 \quad \text{...............(1)}$$

Here,

$$a = 0$$

$$h = 0$$

$$b = 4$$

$g = 6$

$f = -10$

$c = 67$

Now,

$$\Delta = abc + 2fgh - af^2 - bg^2 - ch^2$$

⇨ $\Delta = -144$

And,

$$h^2 - ab$$

$$= 0$$

Since $\Delta \neq 0$ *and* $h^2 - ab = 0$, so the equation (1) represents a parabola.

From (1), we can write

$$4y^2 + 12x - 20y + 67 = 0$$

$$\Rightarrow \quad y^2 + 3x - 5y + \frac{67}{4} = 0$$

$$\Rightarrow \quad y^2 - 2.\frac{5}{2}.y + \left(\frac{5}{2}\right)^2 - \left(\frac{5}{2}\right)^2 + 3x + \frac{67}{4} = 0$$

$$\Rightarrow \quad \left(y - \frac{5}{2}\right)^2 = -3x - \frac{21}{2}$$

$$\Rightarrow \quad \left(y - \frac{5}{2}\right)^2 = -3\left(x + \frac{7}{2}\right)$$

$$\Rightarrow \quad Y^2 = 4aX$$

$$\begin{bmatrix} Y = y - \frac{5}{2} \\ X = x + \frac{7}{2} \\ 4a = -3 \\ \therefore \ a = -\frac{3}{4} \end{bmatrix}$$

Vertex,

$$X = 0$$

$$\Rightarrow \quad x + \frac{7}{2} = 0$$

$$\Rightarrow \quad x = -\frac{7}{2}$$

$$Y = 0$$

$$\Rightarrow \quad y - \frac{5}{2} = 0$$

$$\Rightarrow \quad y = \frac{5}{2}$$

$$\therefore (x, y) = \left(-\frac{7}{2}, \frac{5}{2}\right)$$

Focus,

$$X = a$$

$$\Rightarrow \quad x + \frac{7}{2} = -\frac{3}{4}$$

$$\Rightarrow \quad x = -\frac{17}{4}$$

$$Y = 0$$

$$\Rightarrow \quad y - \frac{5}{2} = 0$$

$$\Rightarrow \quad y = \frac{5}{2}$$

$$\therefore (x, y) = \left(-\frac{17}{4}, \frac{5}{2}\right)$$

Equation of the tangent at the vertex,

$$X = 0$$

$$\Rightarrow \quad x + \frac{7}{2} = 0$$

$$\Rightarrow \quad x = -\frac{7}{2}$$

Equation of the directrix,

$$X = -a$$

$$\Rightarrow \quad x + \frac{7}{2} = -\left(-\frac{3}{4}\right)$$

$$\Rightarrow \qquad x = -\frac{11}{4}$$

Equation of the latus rectum,

$$X = a$$

$$\Rightarrow \qquad x + \frac{7}{2} = -\frac{3}{4}$$

$$\Rightarrow \qquad x = -\frac{17}{4}$$

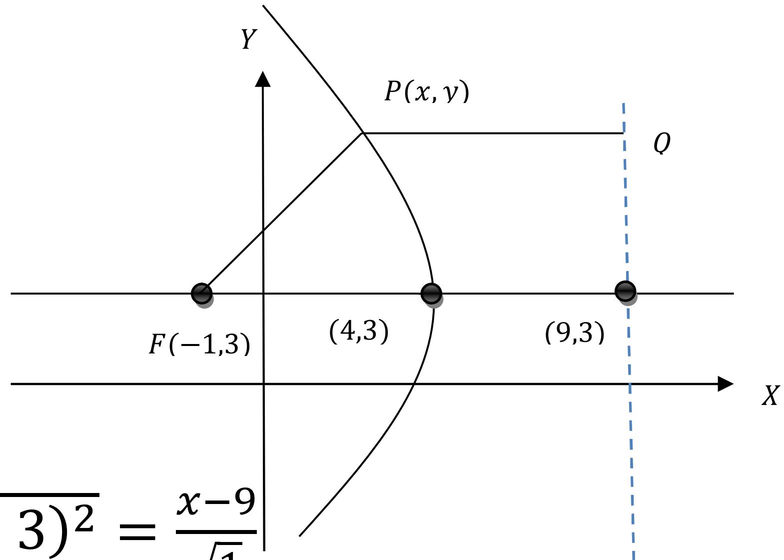
Length of the latus rectum,

$$|4a|$$

$$= \left|4.\left(-\frac{3}{4}\right)\right|$$

$$= 3 \text{ unit}$$

4. Determine the equation of the parabola whose focus is at the point $(-1,3)$ and whose vertex is the at the point $(4,3)$.



Solution:

We have,

$$FP = PQ$$

$$=> \sqrt{(x+1)^2 + (y-3)^2} = \frac{x-9}{\sqrt{1}}$$

$$=> x^2 + 2x + 1 + y^2 - 6y + 9 = x^2 - 18x + 81$$

$$=> y^2 + 20x - 6y - 71 = 0$$

<u>**Homework Problems**</u>

1. Obtain The equation of the parabola whose focus is at the point $(-1,0)$ whose directrix is the straight line $\mathrm{y} = \mathrm{x}$.

2. Find the equation of the tangent to the parabola $\mathrm{y}^2 = 6\mathrm{x}$ which is perpendicular to the line $4\mathrm{y} - 3\mathrm{x} + 7 = 0$.

3. Show that, the equation $2\mathrm{x}^2 + \mathrm{y} - \mathrm{x} + 4 = 0$ represents a parabola. Find it's vertex, focus, tangent at the vertex, directrix, latus rectum, length of the latus rectum. Also sketch the above parabola.

4. Determine the equation of the parabola whose vertex is at the point $(2,1)$ and whose directrix is the line $y = 5$.