# *Solutions to the Homework Problems*

**1.** Find the equation of the tangent to the hyperbola $\frac{x^2}{25} - \frac{y^2}{1} = 1$ which is perpendicular to the $-2y = x + 1$.

**Solution:**

Consider the required equation of the tangent is,

$$y = mx + c \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

Given,

$$-2y = x + 1$$

$$\Rightarrow \quad y = -\frac{1}{2}x - \frac{1}{2}$$

So, $m = 2$

From (1),

$$y = 2x + c \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

Since the equation (2) touches the hyperbola $\frac{x^2}{25} - \frac{y^2}{1} = 1$,

$$\therefore c = \pm\sqrt{a^2m^2 - b^2}$$

$$= \pm\sqrt{25 \cdot 2^2 - 1}$$

$$= \pm 3\sqrt{11}$$

From (2),

$$y = 2x + 3\sqrt{11}$$

And $y = 2x - 3\sqrt{11}$

**2.** Find the center of the following hyperbolas:

a. $2x^2 - 3xy + y^2 - 5x + 4y + 6 = 0$

b. $x^2 - 6xy + y^2 - 10x - 10y - 19 = 0$

c. $xy + 3ax - 3ay = 0$

**Solution:**

**(a)**

Given,

$$2x^2 - 3xy + y^2 - 5x + 4y + 6 = 0$$

Where $\Delta = -\frac{3}{4}$ and $h^2 - ab = \frac{1}{4}$

Now the equation of the asymptotes can be written as

$$2x^2 - 3xy + y^2 - 5x + 4y + 6 + \frac{\Delta}{h^2 - ab} = 0$$

$$=> 2x^2 - 3xy + y^2 - 5x + 4y + 6 - 3 = 0$$

$$=> 2x^2 - 3xy + y^2 - 5x + 4y + 3 = 0$$

$$=> 2x^2 + (-3y - 5)x + (y^2 + 4y + 3) = 0$$

$$=> x = \frac{(3y+5) \pm \sqrt{(3y+5)^2 - 4.2(y^2+4y+3)}}{2.2}$$

$$=> x = \frac{(3y+5) \pm \sqrt{y^2-2y+1}}{2.2}$$

$$=> x = \frac{(3y+5) \pm \sqrt{(y-1)^2}}{4}$$

$$=> x = \frac{(3y+5) \pm (y-1)}{4}$$

So, the asymptotes are

$$\therefore 4x = 4y + 4$$

$$=> x - y - 1 = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots(1)$$

and

$$4x = 2y + 6$$

$$=> 2x - y - 3 = 0 \ldots\ldots\ldots\ldots\ldots\ldots(2)$$

Solving (1) and (2), we get

$$\therefore x = 2 \text{ and } y = 1$$

Hence, the centre of the given hyperbola is (2,1).

**(b)**

Given,

$$x^2 - 6xy + y^2 - 10x - 10y - 19 = 0$$

Where $\Delta = -48$ and $h^2 - ab = 8$

Now the equation of the asymptotes can be written as

$$x^2 - 6xy + y^2 - 10x - 10y - 19 + \frac{\Delta}{h^2 - ab} = 0$$

$$=> x^2 - 6xy + y^2 - 10x - 10y - 19 - 6 = 0$$

$$=> x^2 - 6xy + y^2 - 10x - 10y - 25 = 0$$

$$=> x^2 + (-6y - 10)x + (y^2 - 10y - 25) = 0$$

$$=> x = \frac{(6y+10) \pm \sqrt{(6y+10)^2 - 4 \cdot (y^2 - 10y - 25)}}{2}$$

$$=> x = \frac{(6y+10) \pm \sqrt{32y^2 + 160y + 200}}{2}$$

$$=> x = \frac{(6y+10) \pm \sqrt{2}\sqrt{16y^2+80y+100}}{2}$$

$$=> x = \frac{(6y+10) \pm \sqrt{2}\,\sqrt{(4y+10)^2}}{2}$$

$$=> x = \frac{(6y+10) \pm \sqrt{2}\,(4y+10)}{2}$$

So, the asymptotes are

$$\therefore 2x = \left(6 + 4\sqrt{2}\right)y + \left(10 + 10\sqrt{2}\right)$$

$$=> x - \left(3 + 2\sqrt{2}\right)y - \left(5 + 5\sqrt{2}\right) = 0 \quad .......(1)$$

and

$$2x = \left(6 - 4\sqrt{2}\right)y + \left(10 - 10\sqrt{2}\right)$$

$$=> x - \left(3 - 2\sqrt{2}\right)y - \left(5 - 5\sqrt{2}\right) = 0 \quad .......(1)$$

Solving (1) and (2), we get

$$\therefore x = -\frac{5}{2} \text{ and } y = -\frac{5}{2}$$

Hence, the centre of the given hyperbola is $(-\frac{5}{2}, -\frac{5}{2})$.

**(c)**

Given,

$$xy + 3ax - 3ay = 0$$

Where $\Delta = -\frac{9a^2}{4}$ and $h^2 - ab = \frac{1}{4}$

Now the equation of the asymptotes can be written as

$$xy + 3ax - 3ay + \frac{\Delta}{h^2 - ab} = 0$$

$$=> xy + 3ax - 3ay - 9a^2 = 0$$

$$=> x(y + 3a) - 3a(y + 3a) = 0$$

$$=> (x - 3a)(y + 3a) = 0$$

So, the asymptotes are

$$x - 3a = 0 \text{ and}$$

$$y + 3a = 0$$

Hence, the center of the given hyperbola is (3a,-3a).

**3.** Find the asymptotes of the following hyperbolas:

a. $x^2 - y^2 + 3x - 7y - 3 = 0$

b. $3x^2 + 8xy - 3y^2 + 6x + 8y + 4 = 0$

c. $2x^2 + 9xy - 5y^2 + 2y - 7 = 0$

**Solution:**

**(a)**

Given,

$$x^2 - y^2 + 3x - 7y - 3 = 0$$

Where $\Delta = -7$ and $h^2 - ab = 1$

Now the equation of the asymptotes can be written as

$$x^2 - y^2 + 3x - 7y - 3 + \frac{\Delta}{h^2 - ab} = 0$$

$$=> x^2 - y^2 + 3x - 7y - 3 - 7 = 0$$

$$=> x^2 - y^2 + 3x - 7y - 10 = 0$$

$$=> x^2 + 3x - (y^2 + 7y + 10) = 0$$

$$=> x = \frac{-3 \pm \sqrt{9 + 4 \cdot (y^2 + 7y + 10)}}{2}$$

$$=> x = \frac{-3 \pm \sqrt{4y^2 + 28y + 49}}{2}$$

$$=> x = \frac{-3 \pm \sqrt{(2y+7)^2}}{2}$$

$$=> x = \frac{-3 \pm (2y+7)}{2}$$

So, the asymptotes are

$$\therefore 2x = 2y + 4$$

$$=> x - y - 2 = 0$$

and

$$2x = -2y - 10$$

$$=> x + y + 5 = 0$$

**(b)**

Given,

$$3x^2 + 8xy - 3y^2 + 6x + 8y + 4 = 0$$

Where $\Delta = -25$ and $h^2 - ab = 25$

Now the equation of the asymptotes can be written as

$$3x^2 + 8xy - 3y^2 + 6x + 8y + 4 + \frac{\Delta}{h^2 - ab} = 0$$

$$=> \ 3x^2 + 8xy - 3y^2 + 6x + 8y + 4 - 1 = 0$$

$$=> \ 3x^2 + 8xy - 3y^2 + 6x + 8y + 3 = 0$$

$$=> 3x^2 + \ (8y + 6)x + (-3y^2 + 8y + 3) = 0$$

$$=> x = \frac{-(8y+6) \pm \sqrt{(8y+6)^2 - 4.3.(-3y^2+8y+3)}}{2.3}$$

$$=> x = \frac{-(8y+6) \pm \sqrt{100y^2}}{6}$$

$$=> x = \frac{-(8y+6) \pm 10y}{6}$$

So, the asymptotes are

$$\therefore 6x = \ 2y - 6$$

$$=> 3x - y + 3 = 0$$

and

$$6x = \ -18y - 6$$

$$=> x + 3y + 1 = 0$$

**(c)**

Given,

$$2x^2 + 9xy - 5y^2 + 2y - 7 = 0$$

Where $\Delta = \frac{839}{4}$ and $h^2 - ab = \frac{121}{4}$

Now the equation of the asymptotes can be written as

$$2x^2 + 9xy - 5y^2 + 2y - 7 + \frac{\Delta}{h^2 - ab} = 0$$

$$=> 2x^2 + 9xy - 5y^2 + 2y - 7 + \frac{839}{121} = 0$$

$$=> 2x^2 + 9xy - 5y^2 + 2y - \frac{8}{121} = 0$$

$$=> 2x^2 + (9y)x + \left(-5y^2 + 2y - \frac{8}{121}\right) = 0$$

$$=> x = \frac{-9y \pm \sqrt{81y^2 - 4.2.(-5y^2 + 2y - \frac{8}{121})}}{2.2}$$

$$=> x = \frac{-9y \pm \frac{1}{11}\sqrt{14641y^2 - 1936y + 64}}{4}$$

$$=> x = \frac{-9y \pm \frac{1}{11}\sqrt{(121y-8)^2}}{4}$$

$$=> x = \frac{-9y \pm \frac{1}{11}(121y-8)}{4}$$

So, the asymptotes are

$$4x = -9y + \frac{1}{11}(121y - 8)$$

$$=> 44x = -99y + 121y - 8$$

$$=> 22x - 11y + 4 = 0$$

and

$$4x = -9y - \frac{1}{11}(121y - 8)$$

$$=> 44x = -99y - 121y + 8$$

$$=> 11x + 55y - 2 = 0$$

**4.** Find the equation of the hyperbola whose asymptotes are the straight lines $2x + 3y - 5 = 0$ and $5x + 3y - 8 = 0$ and which passes through the point $(1, -1)$.

**Solution:**

We can write the equation of the hyperbola,

$$(2x + 3y - 5)(5x + 3y - 8) + k = 0 \text{............(1)}$$

Since equation (1) passes through the point $(1, -1)$, so we get

⇨ $(2 - 3 - 5)(5 - 3 - 8) + k = 0$

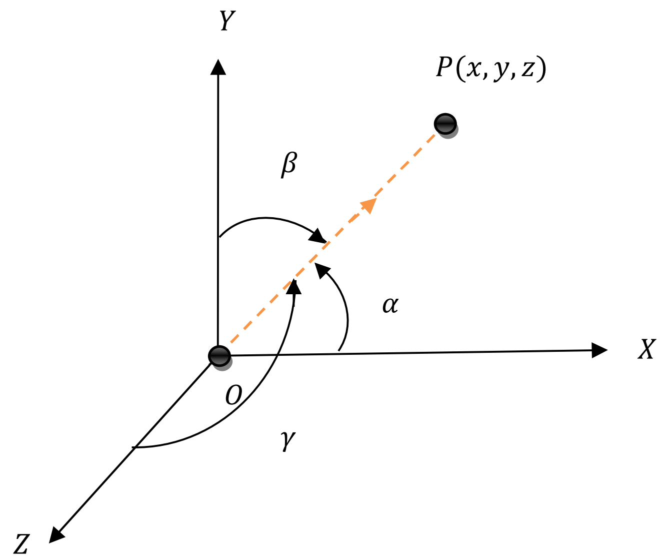
⇨ $k = -36$

Putting the value of $k$ in equation (1),

$$(2x + 3y - 5)(5x + 3y - 8) = 36$$

# Topic-7: Three Dimensional Straight Line

## Direction Cosines of a Line:




Axes make a positive angle with the line $\overrightarrow{OP}$.

$$l = cos\alpha$$

$$m = cos\beta$$

$$n = cos\gamma$$

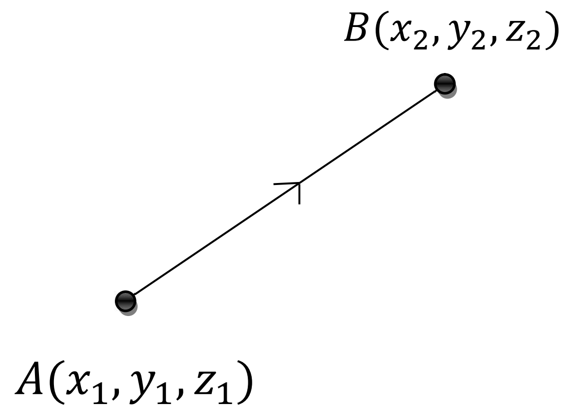
**Note**: $l^2 + m^2 + n^2 = 1$

## Direction Ratios of a Line:

$$a = x_2 - x_1$$

$$b = y_2 - y_1$$

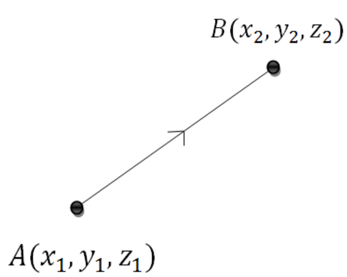
$$c = z_2 - z_1$$




$$\overrightarrow{AB} = a\hat{\imath} + b\hat{\jmath} + c\hat{k}$$

**Note:** $l \propto a,\ m \propto b$ and $n \propto c$

## Relation between Direction Cosines and Direction Ratios of a line:



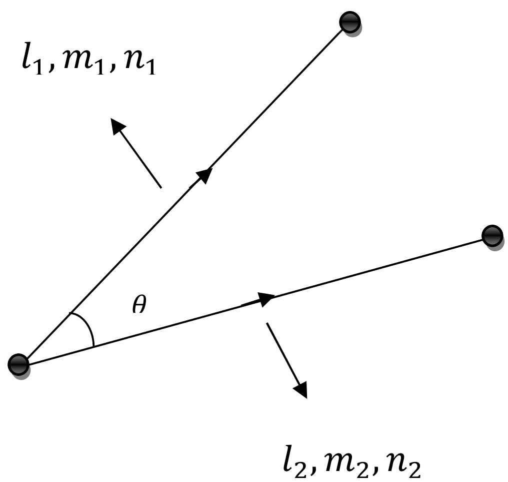

For the line $\overrightarrow{AB}$,

$$l = \frac{a}{\sqrt{a^2+b^2+c^2}}, \quad m = \frac{b}{\sqrt{a^2+b^2+c^2}}, n \quad = \frac{c}{\sqrt{a^2+b^2+c^2}}$$

For the line $\overrightarrow{BA}$,

$$l = \frac{-a}{\sqrt{a^2+b^2+c^2}}, \quad m = \frac{-b}{\sqrt{a^2+b^2+c^2}}, \quad n = \frac{-c}{\sqrt{a^2+b^2+c^2}}$$

## Angle between Two Lines:




The angle $\theta$ between two lines is

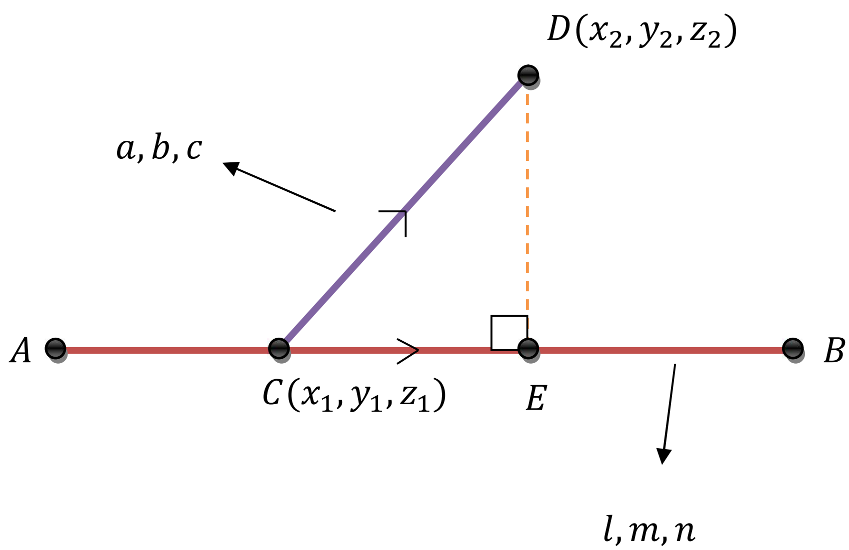
$$\cos\theta = l_1 l_2 + m_1 m_2 + n_1 n_2$$

**Note:**

$$l_1 l_2 + m_1 m_2 + n_1 n_2 = 0 \text{ (Perpendicularity)}$$

$$l_1 l_2 + m_1 m_2 + n_1 n_2 = 1 \text{ (Parallelism)}$$

## Projection of a Line on a Line:




Direction cosines $l, m, n$ for the the line $\overrightarrow{AB}$

Direction ratios $a, b, c$ for the the line $\overrightarrow{CD}$

Projection of $\overrightarrow{CD} = \left|\overrightarrow{CE}\right| = CE = |al + bm + cn|$

## *Solved Problems*

**1.** Find the angle between the two lines with direction cosines $\frac{4}{9}, -\frac{8}{9}, -\frac{1}{9}$ and $-\frac{1}{3}, -\frac{2}{3}, -\frac{2}{3}$.

**Solution:**

For the first line,

$$l_1 = \frac{4}{9}$$

$$m_1 = -\frac{8}{9}$$

$$n_1 = -\frac{1}{9}$$

For the second line,

$$l_2 = -\frac{1}{3}$$

$$m_2 = -\frac{2}{3}$$

$$n_2 = -\frac{2}{3}$$

We know that the angle between two lines is

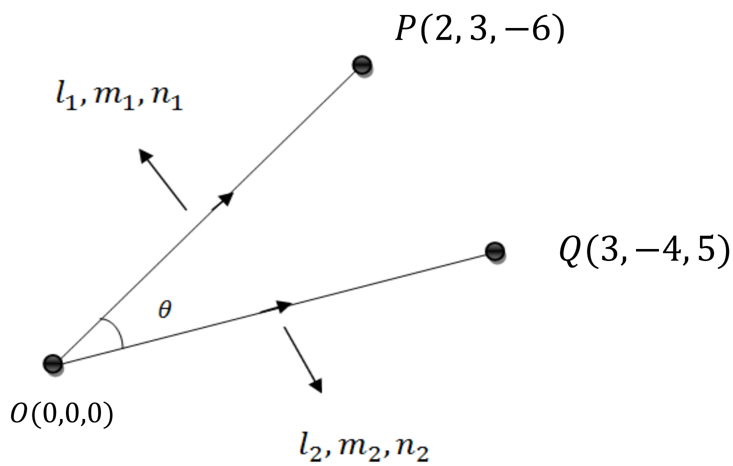
$$cos\theta = l_1 l_2 + m_1 m_2 + n_1 n_2$$

⇨ $$cos\theta = \frac{4}{9} \times \left(-\frac{1}{3}\right) + \left(-\frac{8}{9}\right) \times \left(-\frac{2}{3}\right) + \left(-\frac{1}{9}\right) \times \left(-\frac{2}{3}\right)$$

⇨ $$\theta = \cos^{-1}\left(\frac{14}{27}\right)$$

**2.** The coordinates of the two points $P$ and $Q$ are $(2, 3, -6)$ and $(3, -4, 5)$ respectively. Find the angle between the two lines $\overrightarrow{OP}$ and $\overrightarrow{OQ}$, where $O$ is the origin.

**Solution:**




We have,

$$a_1 = x_2 - x_1 = 2 - 0 = 2$$

$$b_1 = y_2 - y_1 = 3 - 0 = 3$$

$$c_1 = z_2 - z_1 = -6 - 0 = -6$$

$$a_2 = x_2 - x_1 = 3 - 0 = 3$$

$$b_2 = y_2 - y_1 = -4 - 0 = -4$$

$$c_2 = z_2 - z_1 = 5 - 0 = 5$$

Now,

For the first line,

$$l_1 = \frac{a_1}{\sqrt{a_1^2+b_1^2+c_1^2}} = \frac{2}{7}$$

$$m_1 = \frac{b_1}{\sqrt{a_1^2+b_1^2+c_1^2}} = \frac{3}{7}$$

$$n_1 = \frac{c_1}{\sqrt{a_1^2+b_1^2+c_1^2}} = -\frac{6}{7}$$

For the second line,

$$l_2 = \frac{a_2}{\sqrt{a_2^2+b_2^2+c_2^2}} = \frac{3\sqrt{2}}{10}$$

$$m_2 = \frac{b_2}{\sqrt{a_2^2+b_2^2+c_2^2}} = -\frac{2\sqrt{2}}{5}$$

$$n_2 = \frac{c_2}{\sqrt{a_2^2+b_2^2+c_2^2}} = \frac{\sqrt{2}}{2}$$

We know that the angle between two lines is

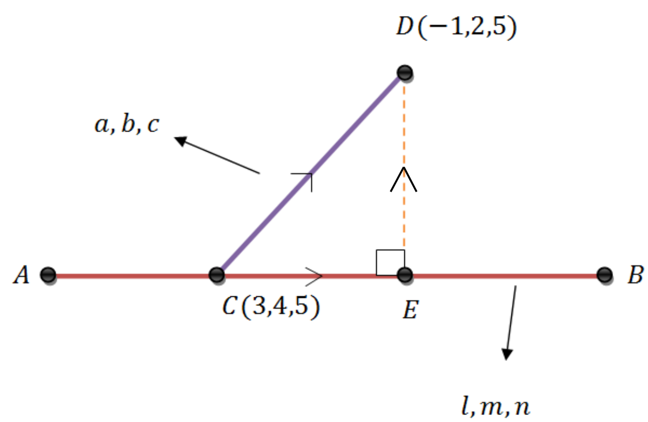
$$cos\theta = l_1 l_2 + m_1 m_2 + n_1 n_2$$

⇨ $$cos\theta = \frac{2}{7} \times \frac{3\sqrt{2}}{10} + \frac{3}{7} \times \left(-\frac{2\sqrt{2}}{5}\right) + \left(-\frac{6}{7}\right) \times \frac{\sqrt{2}}{2}$$

⇨ $$\theta = \cos^{-1}\left(-\frac{18\sqrt{2}}{35}\right)$$

**3.** Find the distance of the point $(-1, 2, 5)$ from the line through the point $(3, 4, 5)$ whose direction cosines are proportional to $2, -3, 6$.

**Solution:**




Now,

$$CD = \sqrt{(-1-3)^2 + (2-4)^2 + (5-5)^2}$$

$$\Rightarrow \quad CD = 2\sqrt{5}$$

Since the direction cosines are proportional to $2, -3, 6$ of the line $\overrightarrow{AB}$.

That is, the direction ratios of the line $\overrightarrow{AB}$ are

$$a = 2$$

$$b = -3$$

$$c = 6$$

Now the direction cosines of the line $\overrightarrow{AB}$ are

$$l = \frac{2}{7}$$

$$m = -\frac{3}{7}$$

$$n = \frac{6}{7}$$

Direction ratios of the line $\overrightarrow{CD}$ are

$$a = -4$$

$$b = -2$$

$$c = 0$$

Now,

The Projection of $\overrightarrow{CD} = CE = |al + bm + cn|$

$$= \left|(-4) \times \frac{2}{7} + \ (-2) \times \left(-\frac{3}{7}\right)\right|$$

$$= \left|-\frac{2}{7}\right|$$

$$= \frac{2}{7}$$

From the right-angled triangle CED we can write

$$CD^2 = ED^2 + CE^2$$

$$\Rightarrow \quad ED^2 = CD^2 - CE^2$$

⇨ $$ED = \sqrt{\left(2\sqrt{5}\right)^2 - \left(\frac{2}{7}\right)^2}$$

⇨ $$ED = \frac{4\sqrt{61}}{7}$$ unit.

## Homework Problems

**1.** Find the direction cosines of the line which is equally inclined to the axes.

**2.** Show that the line joining the two points (0, 1, 2) and (3, 4, 6) is parallel to the line joining the two points $(-4, 3, -6)$ and (5, 12, 6).

**3.** Show that the three lines whose direction cosines are proportional to 2, 1, 1 ; 4, $\sqrt{3}-1, -\sqrt{3}-1$ and 4, $-\sqrt{3}-1, \sqrt{3}-1$ respectively are inclined to each other at an angle $\frac{\pi}{3}$.

**4.** Find the distance of the point $(-2, 3, 4)$ from the line through the point $(-1, 3, 2)$ whose direction cosines are proportional to $12, 3, -4$.