Topic-5: The Ellipse

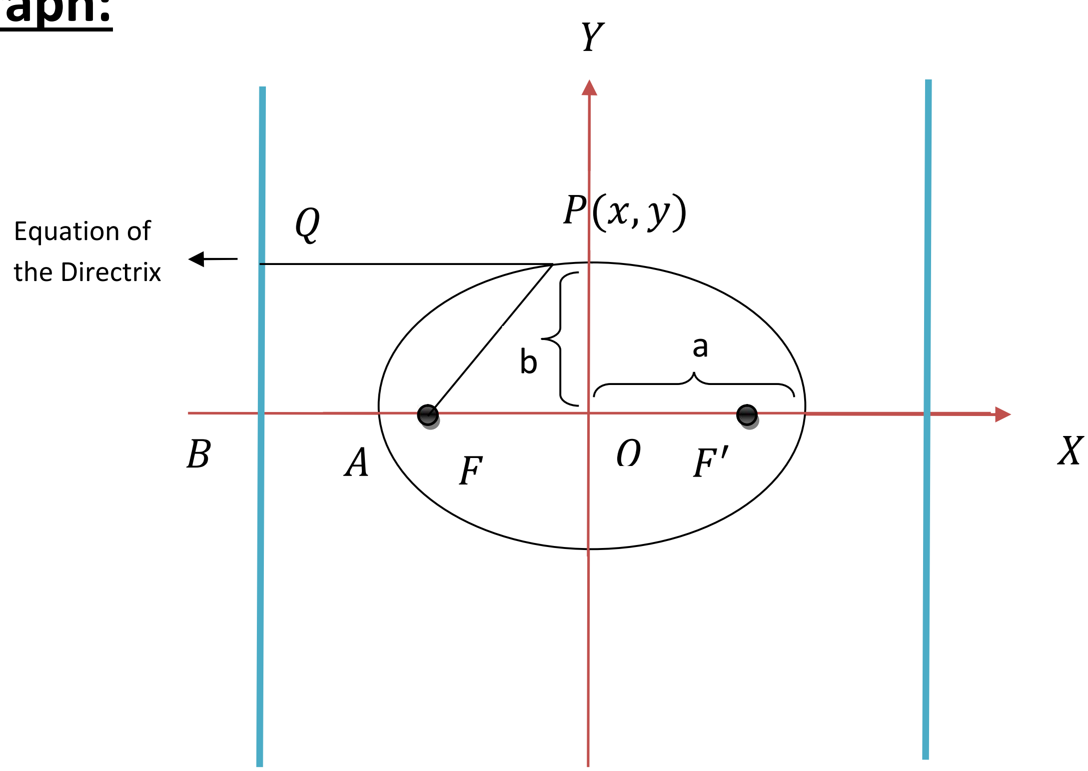
The general equation of an ellipse with center (α, β) is

$$\frac{(x-\alpha)^2}{a^2} + \frac{(y-\beta)^2}{b^2} = 1, a > b$$

In Particular, the general equation of an ellipse is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, a > b$$

Graph:



Note:

1. $FA = e \cdot AB$

2. $FP = e \cdot PQ$

3. $FF' = 2ae$

4. $FP + F'P = 2a$

where eccentricity, $e^2 = 1 - \frac{b^2}{a^2}$, $a > b$ [e<1]

Properties:

(i) The coordinates of the center $(0,0)$.

(ii) The coordinates of the vertices $(\pm a, 0)$.

(iii) The coordinates of the focus $(\pm ae, 0)$.

(iv) Equation of the major axis $Y = 0$

(v) Equation of the minor axis $X = 0$

(vi) Equation of the directrices $x = \pm \frac{a}{e}$.

(vii) Length of the major axis $2a$

(viii) Length of the minor axis $2b$

(ix) Length of the latus rectum $2\frac{b^2}{a}$

(x) Equation of the latus rectum $x = \pm ae$

Special Information:

The straight line $y = mx + c$ touches the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, a > b$ if

$$c = \pm\sqrt{a^2m^2 + b^2}.$$

Solved Problems

1. Identify the conic given by the following equation

$$11x^2 + 4xy + 14y^2 - 5 = 0$$

Then reduce this conic to its standard form.

Solution:

Given,

$$11x^2 + 4xy + 14y^2 - 5 = 0$$

Here,

$\mathrm{a} = 11$

$\mathrm{h} = 2$

$\mathrm{b} = 14$

$\mathrm{g} = 0$

$\mathrm{f} = 0$

$$c = -5$$

Now,

$$\Delta = abc + 2fgh - af^2 - bg^2 - ch^2$$

⇨ $\Delta = -750$

and, $h^2 - ab$

$$= (2)^2 - 11 \cdot 14$$

$$= -150 < 0$$

Since, $\Delta \neq 0$ and $h^2 - ab < 0$, so the given equation represents an ellipse.

Given,

$$11x^2 + 4xy + 14y^2 = 5 \text{ ....(1)}$$

Putting $x = xcos\theta - ysin\theta$ and $y = xsin\theta + ycos\theta$ in equation (1),

$11(xcos\theta - ysin\theta)^2 + 4(xcos\theta - ysin\theta)(xsin\theta + ycos\theta) + 14(xsin\theta + ycos\theta)^2 - 5 = 0$

⇨ $11(x^2\cos^2\theta - 2xysin\theta cos\theta + y^2\sin^2\theta) + 4(x^2sin\theta cos\theta + xycos^2\theta - xysin^2\theta - y^2sin\theta cos\theta) + 14(x^2\sin^2\theta + 2xysin\theta cos\theta + y^2\cos^2\theta) - 5 = 0$

⇨ $x^2(11\cos^2\theta + 4sin\theta cos\theta + 14\sin^2\theta) + xy(-22sin\theta cos\theta + 4\cos^2\theta - 4\sin^2\theta + 28sin\theta cos\theta) + y^2(11\sin^2\theta - 4sin\theta cos\theta + 14\cos^2\theta) - 5 = 0$ ....(2)

To remove the xy term in equation (2), we can write,

$$-22sin\theta cos\theta + 4\cos^2\theta - 4\sin^2\theta + 28sin\theta cos\theta = 0$$

⇨ $4\cos^2\theta + 6sin\theta cos\theta - 4\sin^2\theta = 0$

⇨ $4\cos^2\theta + 8sin\theta cos\theta - 2sin\theta cos\theta - 4\sin^2\theta = 0$

⇨ $4cos\theta(cos\theta + 2sin\theta) - 2sin\theta(cos\theta + 2sin\theta) = 0$

⇨ $(cos\theta + 2sin\theta)(4cos\theta - 2sin\theta) = 0$

$\therefore (4cos\theta - 2sin\theta) = 0$

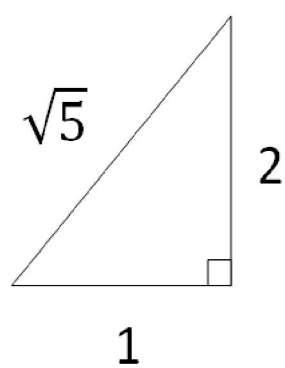
⇨ $tan\theta = 2$

Or, $(cos\theta + 2sin\theta) = 0$

⇨ $tan\theta = -\frac{1}{2}$

when $tan\theta = 2$

$\therefore sin\theta = \frac{2}{\sqrt{5}}$ and $cos\theta = \frac{1}{\sqrt{5}}$



Putting $sin\theta = \frac{2}{\sqrt{5}}$ and $cos\theta = \frac{1}{\sqrt{5}}$ in equation (2),

$$x^2\left(11.\frac{1}{5} + 4\,.\frac{2}{\sqrt{5}}\,.\frac{1}{\sqrt{5}} \;+ 14\left(\frac{2}{\sqrt{5}}\right)^2\right) +$$
$$xy\left(-22.\frac{2}{\sqrt{5}}\,.\frac{1}{\sqrt{5}} \;+ 4.\frac{1}{5} - 4.\frac{4}{5} + 28\,\frac{2}{\sqrt{5}}\;.\frac{1}{\sqrt{5}}\right) +$$
$$y^2\left(11.\frac{4}{5} - 4\frac{2}{\sqrt{5}}\,.\frac{1}{\sqrt{5}} \;+ 14.\frac{1}{5}\right) - 5 = 0$$

⇨ $15x^2 + 10y^2 - 5 = 0$

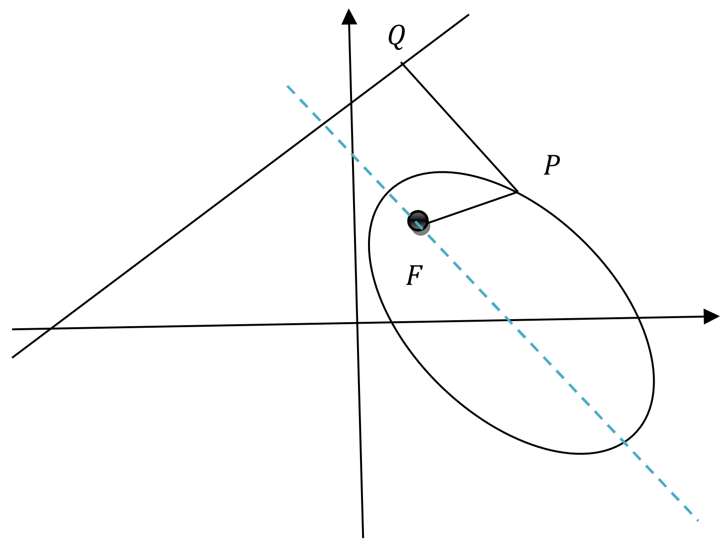
⇨ $3x^2 + 2y^2 - 1 = 0$

⇨ $3x^2 + 2y^2 = 1$

$$\therefore \frac{x^2}{1/3} + \frac{y^2}{1/2} = 1$$

This is the standard equation of an ellipse.

2. Find the equation of the ellipse whose focus is $(2,3)$, eccentricity $\frac{1}{\sqrt{3}}$ and directrix $\mathrm{x}-\mathrm{y}+13=0$.



Solution:

We have,

$$FP = e.PQ$$

$$\Rightarrow \quad \sqrt{(x-2)^2+(y-3)^2} = \frac{1}{\sqrt{3}} \cdot \frac{x-y+13}{\sqrt{2}}$$

$$\Rightarrow \quad x^2-4x+4+y^2-6y+9 = \frac{(x-y+13)^2}{6}$$

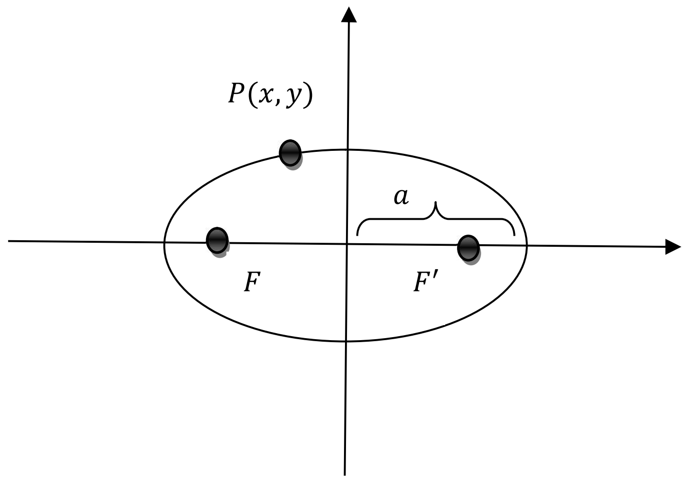
$$\Rightarrow \quad 6x^2+6y^2-24x-36y+24+54 = x^2-2x(y-13)(y-13)^2$$

$$\Rightarrow \quad 6x^2+6y^2-24x-36y+78 = x^2-2xy+26x+y^2-26y+169$$

$$\Rightarrow \quad 5x^2-2xy+5y^2-50x-10y-91=0$$

3. Find the equation of the ellipse whose center is at the origin and whose foci are $(1,0), (-1,0)$ and eccentricity $\frac{1}{2}$.

Solution:



We have,

$$FF' = 2ae$$

⇨ $$\sqrt{(-1-1)^2} = 2a.\frac{1}{2}$$

⇨ $$a = \sqrt{4}$$

⇨ $$a = 2$$

Then,

$$FP + F'P = 2a$$

⇨ $$\sqrt{(x+1)^2 + y^2} + \sqrt{(x-1)^2 + y^2} = 4$$

⇨ $\sqrt{x^2 + 2x + 1 + y^2} = 4 - \sqrt{x^2 - 2x + 1 + y^2}$

⇨ $x^2 + 2x + 1 + y^2 = 16 - 8\sqrt{x^2 - 2x + 1 + y^2} + x^2 - 2x + 1 + y^2$

⇨ $4x - 16 = -8\sqrt{x^2 - 2x + 1 + y^2}$

⇨ $16x^2 - 128x + 256 = 64x^2 - 128x + 64 + 64y^2$

⇨ $48x^2 + 64y^2 - 192 = 0$

⇨ $3x^2 + 4y^2 - 12 = 0$

Or,

$e^2 = 1 - \frac{b^2}{a^2}$

⇨ $\frac{1}{4} = 1 - \frac{b^2}{4}$

⇨ $b^2 = 3$

Then,

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

⇨ $$\frac{x^2}{4} + \frac{y^2}{3} = 1$$

⇨ $$\frac{3x^2+4y^2}{12} = 1$$

⇨ $$3x^2 + 4y^2 - 12 = 0$$

4. Find the equation of the tangent to the ellipse $\frac{x^2}{16} + \frac{y^2}{9} = 1$ which is perpendicular to the line $x + y + 2 = 0$.

Solution:

Consider, the required equation of the tangent is

$$y = mx + c \text{ ....(1)}$$

Given,

$$x + y + 2 = 0$$

$\Rightarrow \quad y = -x - 2$

So, $m = 1$

From (1),

$y = x + c$ ....(2)

Since equation (2) touches the ellipse,

$$\frac{x^2}{16} + \frac{y^2}{9} = 1$$

$$\therefore c = \pm\sqrt{a^2m^2 + b^2}$$

$\Rightarrow \quad c = \pm\sqrt{16 + 9}$

$\Rightarrow \quad c = \pm 5$

From (2) ,

$$y = x + 5$$

and $\quad y = x - 5$

Homework Problems

1. Find the eccentricity and latus rectum of the ellipse $3x^2 + 4y^2 + 6x - 8y - 5 = 0$.

2. Find the equation of the ellipse whose latus rectum is 5 and eccentricity is $\frac{2}{3}$.

3. Find the equations of the tangent to the ellipse $\frac{x^2}{8} + \frac{y^2}{4} = 1$ which is parallel to the line $y = 2x + 1$.

4. Find the equation of the ellipse whose foci are (1, 0), (0, 0) and length of the major axis is 2.