Solutions to the Homework Problems

1. A particle moves so that it's position vector is given by $\vec{r} = \cos\omega t\hat{\imath} + \sin\omega t\hat{\jmath}$, where ω is a constant. Show that

(a) The vector of the particle $\vec{v}$ is perpendicular to $\vec{r}$

(b) $\vec{r} \times \vec{v}$ = a constant vector

Solution:

(a) Given,

$$\vec{r} = \cos\omega t\hat{\imath} + \sin\omega t\hat{\jmath}$$

$$\vec{v} = \frac{d\vec{r}}{dt} = -\omega sin\ \omega t\hat{\imath} + \omega\cos\omega t\hat{\jmath}$$

Now,

$$\vec{r}.\vec{v} = -\ \omega sin\ \omega t\cos\omega t + \omega sin\ \omega t\cos\omega t$$

$= 0$

So, the vector of the particle $\vec{v}$ is perpendicular to $\overrightarrow{r}$

(b) $\vec{r} \times \vec{v} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \cos\omega t & \sin\omega t & 0 \\ -\omega\sin\omega t & \omega\cos\omega t & 0 \end{vmatrix}$

$= (\omega\cos^2\omega t + \omega\sin^2\omega t)\,\hat{k}$
$= \omega(\sin^2\omega t + \cos^2\omega t)\hat{k}$
$= \omega\hat{k}$

$\therefore \vec{r} \times \vec{v} = a$ constant vector.

2. Find the value of a for which the vector

$\vec{A} = (axy - z^3)\hat{\imath} + (a-2)x^2\hat{\jmath} + (1-a)xz^2\hat{k}$
have it's curl identically equal to zero.

Solution:

Given,

$$\vec{A} = (axy - z^3)\hat{\imath} + (a-2)x^2\hat{\jmath} + (1-a)xz^2\hat{k}$$

Now,

$$\vec{\nabla} \times \overrightarrow{A} = \vec{0}$$

$$=> \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ axy - z^3 & (a-2)x^2 & (1-a)xz^2 \end{vmatrix}$$

$$= 0.\hat{\imath} + 0.\hat{\jmath} + 0.\hat{k}$$

$$=> -\left((1-a)z^2 + 3z^2\right)\hat{\jmath} + \left((a-2)2x - ax\right)\hat{k}$$
$$= 0.\hat{\imath} + 0.\hat{\jmath} + 0.\hat{k}$$

$$=> -(z^2 - az^2 + 3z^2)\hat{\jmath} + (2ax - 4x - ax)\hat{k}$$
$$= 0.\hat{\imath} + 0.\hat{\jmath} + 0.\hat{k}$$

$$=> (az^2 - 4z^2)\hat{\jmath} + (ax - 4x)\hat{k} = 0.\hat{\imath} + 0.\hat{\jmath} + 0.\hat{k}$$

$$=> z^2(a-4)\hat{\jmath} + x(a-4)\hat{k} = 0.\hat{\imath} + 0.\hat{\jmath} + 0.\hat{k}$$

Comparing both sides we get,

$$z^2(a-4) = 0$$
$$=> a = 4 \; [z \neq 0]$$
$$x(a-4) = 0$$
$$=> a = 4 \; [x \neq 0]$$

3. Show that $\vec{A} = (6xy + z^3)\hat{\imath} + (3x^2 - z)\hat{\jmath} + (3xz^2 - y)\hat{k}$ is irrotational. Find φ such that $\overrightarrow{A} = \vec{\nabla}\varphi$.

<u>Solution:</u>

curl $\overrightarrow{A} = \vec{\nabla} \times \overrightarrow{A}$

$$= \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 6xy+z^3 & 3x^2 - z & 3xz^2 - y \end{vmatrix}$$

$$= (-1+1)\hat{\imath} - (3z^2 - 3z^2)\hat{\jmath} + (6x - 6x)\hat{k}$$

$$= \vec{0}$$

∴ $\overrightarrow{A}$ is irrotational.

Now,

$$\overrightarrow{A} = \vec{\nabla}\varphi$$

$$=> (6xy + z^3)\hat{\imath} + (3x^2 - z)\hat{\jmath} + (3xz^2 - y)\hat{k} = \frac{\partial\varphi}{\partial x}\hat{\imath} + \frac{\partial\varphi}{\partial y}\hat{\jmath} + \frac{\partial\varphi}{\partial z}\hat{k}$$

Comparing both sides, we get,

$$\frac{\partial\varphi}{\partial x} = (6xy + z^3), \qquad \varphi = 3x^2y + z^3x + f(y,z)$$

$$\frac{\partial\varphi}{\partial y} = (3x^2 - z), \qquad \varphi = 3x^2y - zy + f(x,z)$$

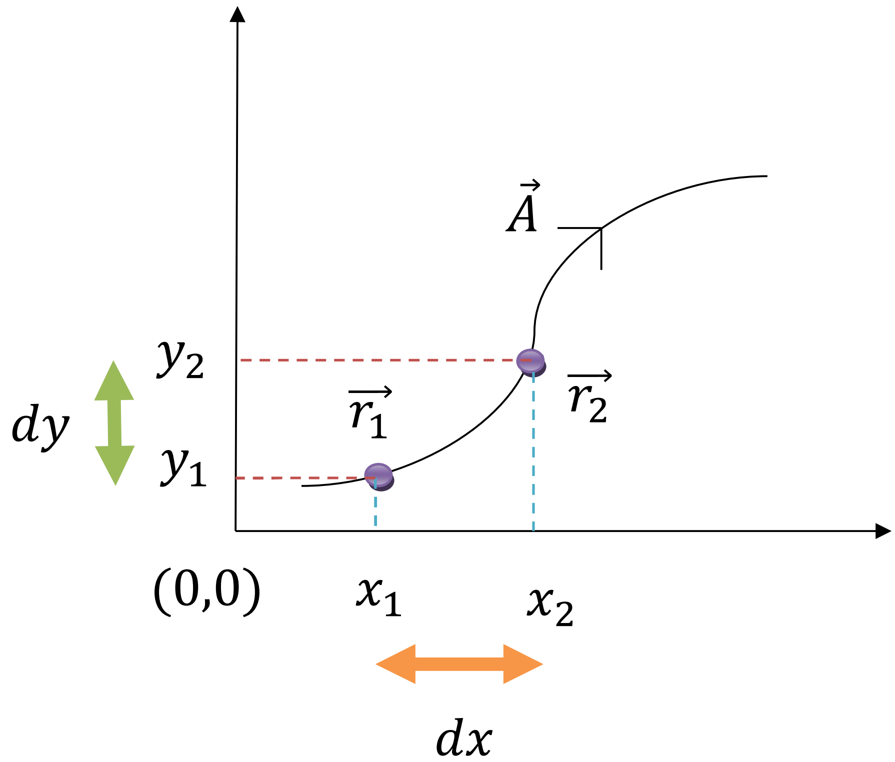
$$\frac{\partial\varphi}{\partial z} = (3xz^2 - y), \qquad \varphi = xz^3 - yz + f(x,y)$$

Here,

$$f(y,z) = -yz$$
$$f(x,z) = xz^3$$
$$f(x,y) = 3x^2y$$

$$\therefore \varphi = 3x^2y - yz + xz^3$$

Topic-10: Vector Integration



Vector field $\vec{A}$

Position of a particle (x, y)

Position vector, $\vec{r} = x\hat{\imath} + y\hat{\jmath}$

Displacement Vector, $d\vec{r} = dx\ \hat{\imath} + dy\ \hat{\jmath}$

$$=> \vec{r_2} - \vec{r_1} = (x_2 - x_1)\hat{\imath} + (y_2 - y_1)\hat{\jmath}$$

Line Integrals

Line integrals deal with a vector along a specified path. As parameterization is used, line integrals can be interpreted as work done by a force along a path.

$$\boxed{\textbf{Line Integral} = \int_C \vec{\boldsymbol{A}} \cdot \boldsymbol{d\vec{r}}}$$

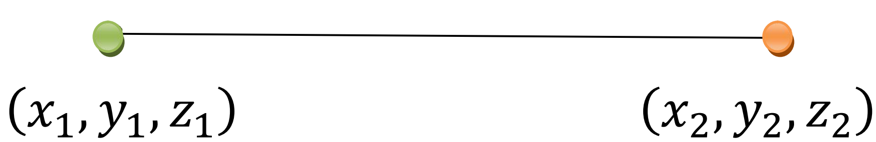
where $\vec{A}$ = vector field

$d\vec{r}$ = dispacement vector

Path Information

(1) $x = f(t), y = f(t), z = f(t)$

(2) straight path from one point to another point



Formula:

$$< (x_1, y_1, z_1) + t[(x_2, y_2, z_2) - (x_1, y_1, z_1)] >$$

where $0 \leq t \leq 1$

(3) path can be defined by an equation

Example: $y = x^2$

Consider

$$x = t$$

So, $y = t^2$

Solved Problems

Problem 1: Find the work done in moving a particle in a force field is given by $\vec{F} = 3xy\hat{\imath} - 5z\hat{\jmath} + 10x\hat{k}$ along the curve $x = t^2 + 1$, $y = 2t^2, z = t^3$ from $t = 1$ to $t = 2$.

Solution: Since $x = t^2 + 1, y = 2t^2, z = t^3$

So $dx = 2t\ dt$, $dy = 4t\ dt$ and $dz = 3t^2 dt$

So we get

$$\vec{F} = 3(t^2 + 1)(2t^2)\hat{\imath} - 5t^3\hat{\jmath} + 10(t^2 + 1)\hat{k}$$

and

$$d\vec{r} = dx\hat{\imath} + dy\hat{\jmath} + dz\hat{k}$$

$$= 2t\ dt\hat{\imath} + 4t\ dt\ \hat{\jmath} + 3t^2 dt\hat{k}$$

Now, $\int_C \vec{F} \cdot d\vec{r} = \int_1^2 (12t^5 + 10t^4 + 12t^3 + 30t^2)\, dt$

$= 303.$

<u>Problem 2:</u> If $\vec{A} = (3x^2 + 6y)\hat{\imath} - 14zy\hat{\jmath} + 20xz^2\hat{k}$, then evaluate $\int_C \vec{A} \cdot d\vec{r}$ from $(0,0,0)$ to $(1,1,1)$ along the following path C:

C: the straight lines from $(0,0,0)$ to $(1,0,0)$, then to $(1,1,0)$ and then to $(1,1,1)$.

<u>Solution</u>:

Along the straight line from $(0,0,0)$ to $(1,0,0)$ we get,

$< (0,0,0) + (1,0,0)t >=< (t, 0,0) >$ where $0 \leq t \leq 1$

So that, $x = t, y = 0$ and $z = 0$

that is $dx = dt, dy = 0$ and $dz = 0$

So we get

$$\vec{A} = 3t^2\,\hat{\imath}$$

and $d\vec{r} = dx\hat{\imath} + dy\hat{\jmath} + dz\hat{k} = dt\hat{\imath}$

Now, $\int_C \vec{A} \cdot d\vec{r} = \int_0^1 3t^2\,dt = 1$

Along the straight line from $(1,0,0)$ to $(1,1,0)$ we get,

$< (1,0,0) + (0,1,0)t >=< (1, t, 0) >$ where $0 \leq t \leq 1$

So that, $x = 1, y = t$ and $z = 0$

that is $dx = 0, dy = dt$ and $dz = 0$

So we get

$$\vec{A} = (3 + 6t)\,\hat{\imath}$$

and $d\vec{r} = dx\hat{\imath} + dy\hat{\jmath} + dz\hat{k} = dt\hat{\jmath}$

Now, $\int_C \vec{A} \cdot d\vec{r} = 0$

Along the straight line from $(1,1,0)$ to $(1,1,1)$ we get,

$< (1,1,0) + (0,0,1)t > = < (1,1,t) >$ where $0 \leq t \leq 1$

So that, $x = 1, y = 1$ and $z = t$

that is $dx = 0$, $dy = 0$ and $dz = dt$

So we get

$$\vec{A} = 9\,\hat{\imath} - 14t\hat{\jmath} + 20t^2\hat{k}$$

and $d\vec{r} = dx\hat{\imath} + dy\hat{\jmath} + dz\hat{k} = dt\hat{k}$

Now, $\int_C \vec{A} \cdot d\vec{r} = \int_0^1 20t^2 dt = \frac{20}{3}$

Adding, $\int_C \vec{A}\cdot d\vec{r} = 1 + 0 + \frac{20}{3} = \frac{23}{3}$

Problem 3: Find the work done in moving a particle once around a circle C in the xy plane, if the circle has center at the origin and radius 3 and if the force field is given by $\vec{F} = (2x - y + z)\hat{\imath} + (x + y - z^2)\hat{\jmath} + (3x - 2y + 4z)\hat{k}$

Solution: Here, $x = 3\ cost, y = 3sint$ and $z = 0$

where $0 \le t \le 2\pi$

that is $dx = -3sint\ dt$, $dy = 3cost\ dt$ and $dz = 0$

So we get

$$\vec{F} = (6cost - 3sint)\hat{\imath} + (3cost + 3sint)\hat{\jmath} + (9cost - 6sint)\hat{k}$$

and $d\vec{r} = dx\hat{\imath} + dy\hat{\jmath} + dz\hat{k} = -3sint\ dt\hat{\imath} + 3cost\ dt\ \hat{\jmath}$

$$\int_C \vec{F} \cdot d\vec{r} = \int_0^{2\pi} (9 - 9sint\ cost)\, dt$$

$$= \int_0^{2\pi} \left(9 - \frac{9}{2} sin2t\right) dt = 18\pi$$

Homework Problems

1. Calculate the work done when a force $\vec{F} = 3xy\hat{\imath} - y^2\hat{\jmath}$ moves a particle in the xy-plane from $(0, 0)$ to $(1, 2)$ along the parabola $y = 2x^2$.

2. Evaluate $\oint_C (y^2dx + x^2dy)$ where C is the triangle with vertices (1, 0), (1, 1), (0, 0).

3. If $\emptyset = 2xyz^2, \vec{F} = xy\hat{\imath} - z\hat{\jmath} + x^2\hat{k}$ and C is the curve $x = t^2, y = 2t, z = t^3$ from $t = 0$ to $t = 1$, evaluate the line integrals (a) $\oint_C \emptyset\, d\vec{r}$ (b) $\oint_C \vec{F} \times d\vec{r}$.