Solutions to the Homework Problems

1. Prove that the two circles $x^2 + y^2 + 2ax + c^2 = 0$ and $x^2 + y^2 + 2by + c^2 = 0$ touch if $\frac{1}{a^2} + \frac{1}{b^2} = \frac{1}{c^2}$.

Solution:

Given,

$$x^2 + y^2 + 2ax + c^2 = 0 \quad \text{................(1)}$$

And, $$x^2 + y^2 + 2by + c^2 = 0 \quad \text{................(2)}$$

The radical axis of the two circles is

$$2ax + c^2 - 2by - c^2 = 0$$

⇨

The circles will touch each other if the radical axis touches them. That is, the distance from the center of the circles is equal to the radius of that circle.

The center of (1) is $(-a, 0)$ and radius is $\sqrt{a^2 - c^2}$

So we can write,

$$\frac{a(-a) + b(0)}{\sqrt{a^2 + b^2}} = \sqrt{a^2 - c^2}$$

$$\Rightarrow \frac{a^4}{a^2+b^2} = a^2 - c^2$$

$$\Rightarrow a^4 = a^4 + a^2b^2 - a^2c^2 - b^2c^2$$

$$\Rightarrow a^2b^2 - a^2c^2 - b^2c^2 = 0$$

$$\Rightarrow \frac{1}{c^2} - \frac{1}{b^2} - \frac{1}{a^2} = 0 \quad [\text{dividing by } a^2b^2c^2]$$

$$\Rightarrow \frac{1}{a^2} + \frac{1}{b^2} = \frac{1}{c^2}$$

2. Find the equation of the circle which cuts orthogonally the three circles,

$$\mathrm{x}^2 + \mathrm{y}^2 + 4\mathrm{x} + 7 = 0$$

$$2\mathrm{x}^2 + 2\mathrm{y}^2 + 3\mathrm{x} + 5\mathrm{y} + 9 = 0$$

$$\mathrm{x}^2 + \mathrm{y}^2 + \mathrm{y} = 0$$

Solution:

Given

$$x^2 + y^2 + 4x + 7 = 0 \text{ ....(1)}$$

$$2x^2 + 2y^2 + 3x + 5y + 9 = 0$$

$$\Rightarrow \quad x^2 + y^2 + \frac{3}{2}x + \frac{5}{2}y + \frac{9}{2} = 0 \text{ ....(2)}$$

$$x^2 + y^2 + y = 0 \text{ ....(3)}$$

Consider, our required equation of a circle is,

$$x^2 + y^2 + 2gx + 2fy + c = 0 \text{ ....(4)}$$

Since (1) and (4) cut orthogonally, so we get,

$$2.2.g + 2.0.f - 7 - c = 0$$

$\Rightarrow \quad 4g - c - 7 = 0$ ....(5)

Since (2) and (4) cut orthogonally, so we get,

$$2.\frac{3}{4}.g + 2.\frac{5}{4}.f - \frac{9}{2} - c = 0$$

$\Rightarrow \quad \frac{3}{2}g + \frac{5}{2}f - c - \frac{9}{2} = 0$ ....(6)

Since (3) and (4) cut orthogonally, so we get,

$$2.0.g + 2.\frac{1}{2}.f - 0 - c = 0$$

$\Rightarrow \quad f - c = 0$ ....(7)

Solving (5), (6) and (7) ,we get,

$$g = 2$$

$$f = 1$$

$$c = 1$$

Putting the value of g, f and c in (4),

$$x^2 + y^2 + 2.2.x + 2.1.y + 1 = 0$$

$$\Rightarrow \quad x^2 + y^2 + 4x + 2y + 1 = 0$$

This is our required equation of the circle.

3. Find the radical center of the three circles,

$$\mathrm{x}^2 + \mathrm{y}^2 + \mathrm{x} + 2\mathrm{y} + 3 = 0$$

$$\mathrm{x}^2 + \mathrm{y}^2 + 2\mathrm{x} + 4\mathrm{y} + 5 = 0$$

$$\mathrm{x}^2 + \mathrm{y}^2 - 7\mathrm{x} - 8\mathrm{y} - 9 = 0$$

<u>Solution:</u>

Given,

$$x^2 + y^2 + x + 2y + 3 = 0 \text{.....................(1)}$$

$$x^2 + y^2 + 2x + 4y + 5 = 0 \text{....................(2)}$$

$$x^2 + y^2 - 7x - 8y - 9 = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots(3)$$

The radical axis of the circles (1) and (2) is,

$$-x - 2y - 2 = 0$$

$$\Rightarrow \quad x + 2y + 2 = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4)$$

The radical axis of the circles (1) and (3) is,

$$8x + 10y + 12 = 0$$

$$4x + 5y + 6 = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(5)$$

The radical axis of the circles (2) and (3) is,

$$9x + 12y + 14 = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(6)$$

Solving (4) and (5), we get,

$$(x, y) = \left(-\frac{2}{3}, -\frac{2}{3}\right)$$

Now putting $x = -\frac{2}{3}$ and $y = -\frac{2}{3}$ in equation (6), we get

$$0 = 0$$

$\therefore$ The radical center of the circles (1), (2) and (3) is,

$$(x, y) = \left(-\frac{2}{3}, -\frac{2}{3}\right)$$