Solutions to the Homework Problems

1. Find the volume of the parallelepiped whose edges are represented by $\vec{a} = 2\hat{\imath} - 3\hat{\jmath} + 4\hat{k}$, $\vec{b} = \hat{\imath} + 2\hat{\jmath} - \hat{k}$, $\vec{c} = 3\hat{\imath} - \hat{\jmath} + 2\hat{k}$.

Solution:

Given,

$$\vec{a} = 2\hat{\imath} - 3\hat{\jmath} + 4\hat{k}$$

$$\vec{b} = \hat{\imath} + 2\hat{\jmath} - \hat{k},$$

$$\vec{c} = 3\hat{\imath} - \hat{\jmath} + 2\hat{k}$$

We know,

$$\text{Volume} = |\vec{a} \cdot (\vec{b} \times \vec{c})|$$

$$= 7\ unit^3$$

2. The position vectors of A, B, C and D are $2\hat{\imath} + 4\hat{k}$, $5\hat{\imath} + 3\sqrt{3}\hat{\jmath} + 4\hat{k}$, $-2\sqrt{3}\hat{\jmath} + \hat{k}$ and $2\hat{\imath} + \hat{k}$ respectively. Show that $\overrightarrow{AB}$ and $\overrightarrow{CD}$ are parallel and $CD = \frac{2}{3}AB$.

Solution:

Given,

$$\overrightarrow{OA} = 2\hat{\imath} + 4\hat{k}$$

$$\overrightarrow{OB} = 5\hat{\imath} + 3\sqrt{3}\hat{\jmath} + 4\hat{k}$$

$$\overrightarrow{OC} = -2\sqrt{3}\hat{\jmath} + \hat{k}$$

$$\overrightarrow{OD} = 2\hat{\imath} + \hat{k}.$$

Now,

$$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$$

$$= 3\hat{\imath} + 3\sqrt{3}\,\hat{\jmath}$$

$$\overrightarrow{CD} = \overrightarrow{OD} - \overrightarrow{OC}$$

$$= 2\hat{\imath} + 2\sqrt{3}\hat{\jmath}$$

$$\therefore\ \overrightarrow{AB} \times \overrightarrow{CD} = \vec{0}$$

$\therefore \overrightarrow{AB}$ and $\overrightarrow{CD}$ are parallel.

Then,

$$\left|\overrightarrow{AB}\right| = AB = 6$$

$$\left|\overrightarrow{CD}\right| = CD = 4$$

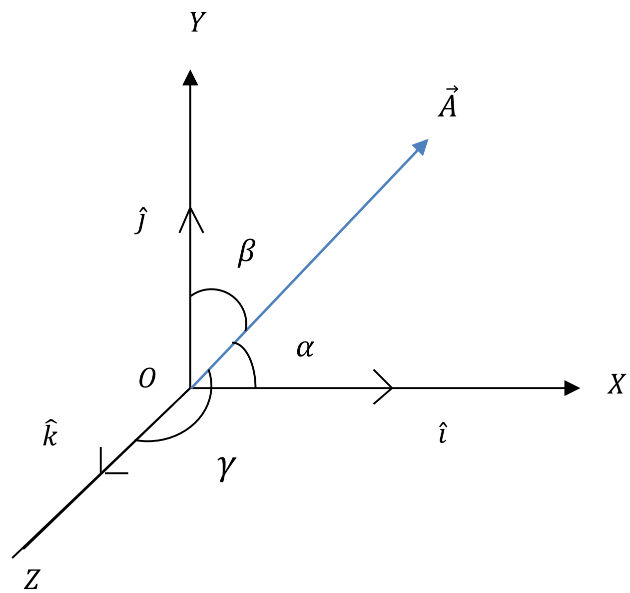
Now,

$$CD = {}^{2}/_{3} \times A\mathrm{B}$$

$$\Rightarrow \quad 4 = {}^{2}/_{3} \times 6$$

$$\Rightarrow \quad 4 = 4$$

3. Find the angles α, β, γ, which the vector $\vec{A} = 3\hat{\imath} - 6\hat{\jmath} + 2\hat{k}$ makes with the coordinates axes and also show that $cos^2\alpha + cos^2\beta + cos^2\gamma = 1.$



<u>Solution:</u>

Now,

$$\cos\alpha = \frac{\vec{A}.\hat{\imath}}{|\vec{A}||\hat{\imath}|}$$

$$= {}^{3}/_{7}$$

$$\therefore \alpha = \cos^{-1} {}^{3}/_{7}$$

$$\cos\beta = \frac{\vec{A}.\hat{\jmath}}{|\vec{A}||\hat{\jmath}|}$$

$$= -{}^{6}/_{7}$$

$$\therefore \beta = \cos^{-1}(-{}^{6}/_{7})$$

$$\cos\gamma = \frac{\vec{A}.\hat{k}}{|\vec{A}||\hat{k}|}$$

$= {}^{2}/_{7}$

$\therefore \gamma = \cos^{-1} {}^{2}/_{7}$

Now,

$cos^2\alpha + cos^2\beta + cos^2\gamma$

$= ({}^{3}/_{7})^2 + ({}^{-6}/_{7})^2 + ({}^{2}/_{7})^2$

= 1

4. If the position vectors of the three points A, B and C are (2, 4, -1), (1, 2, -3) and (3, 1, 2) respectively. Find a vector perpendicular to the plane ABC.

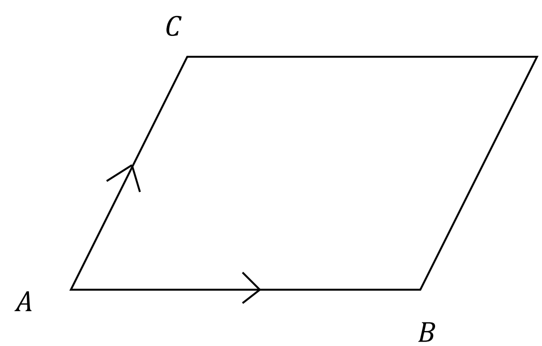
<u>Solution:</u>

Given,

$$\overrightarrow{OA} = 2\hat{\imath} + 4\hat{\jmath} - \hat{k}$$

$$\overrightarrow{OB} = \hat{\imath} + 2\hat{\jmath} - 3\hat{k}$$

$$\overrightarrow{OC} = 3\hat{\imath} + \hat{\jmath} + 2\hat{k}$$



Now,

$$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$$

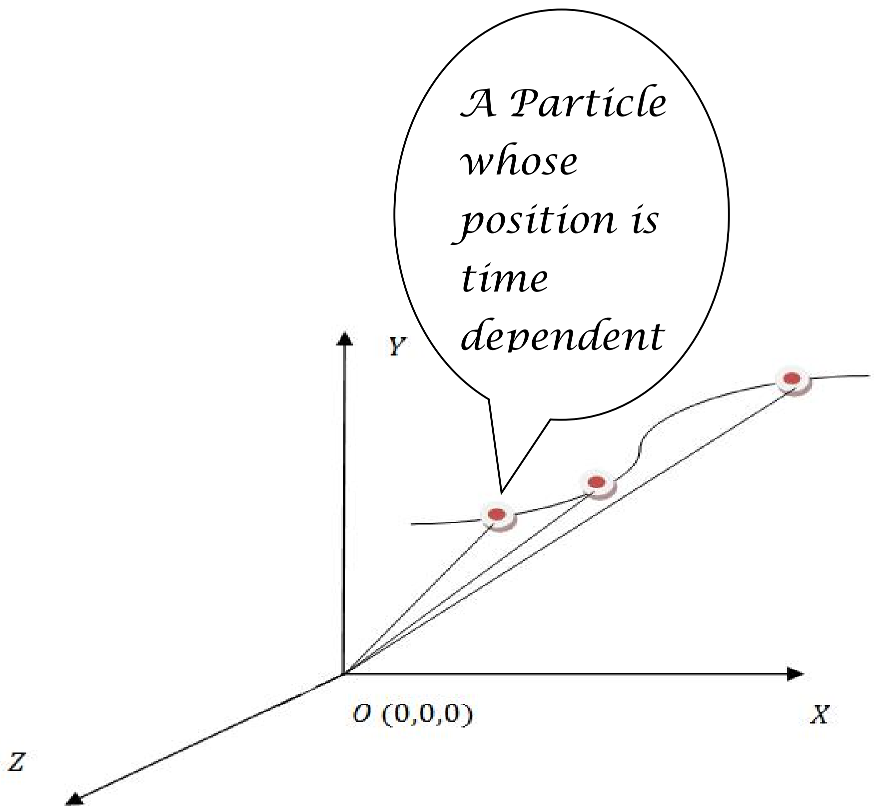
$$= -\hat{\imath} - 2\hat{\jmath} - 2\hat{k}$$

$$\overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA}$$

$$= \hat{\imath} - 3\hat{\jmath} + 3\hat{k}$$

$$\therefore \overrightarrow{AB} \times \overrightarrow{AC} = -12\hat{\imath} + \hat{\jmath} + 5\hat{k}$$

Topic-9: Vector Differentiation



Position of a particle (x, y, z)

Position vector, $\vec{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$

Where,

$$x = f(t)$$
$$y = g(t)$$
$$z = h(t)$$

Velocity vector/Tangent vector,

$$\vec{v} = \frac{d\vec{r}}{dt} = dx\,\hat{\imath} + dy\,\hat{\jmath} + dz\,\hat{k}$$

Acceleration vector, $\vec{a} = \dfrac{d\vec{v}}{dt}$

Solved Problems

1. A particle moves along a curve whose parametric equations are $x = e^{-t},\ y = 2cos3t,\ z = 2sin3t$, where t is the time.

(a) Determine its velocity and acceleration at any time.

(b) Find the magnitude of the velocity and acceleration at $t = 0$.

Solution:

(a) The position vector of a particle is,

$$\vec{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$$

$$=> \vec{r} = e^{-t}\hat{\imath} + 2\cos 3t\hat{\jmath} + 2\sin 3t\hat{k}$$

Now,

Velocity, $\vec{v} = \frac{d\vec{r}}{dt}$

$$= -e^{-t}\hat{\imath} - 6\sin 3t\hat{\jmath} + 6\cos 3t\hat{k}$$

Acceleration, $\vec{a} = \frac{d\vec{v}}{d\vec{t}}$

$$= e^{-t}\hat{\imath} - 18cos3t\hat{\jmath} - 18sin3t\hat{k}$$

(b) At $t = 0$,

Velocity, $\vec{v} = -\hat{\imath} + 6\hat{k}$

Acceleration, $\vec{a} = \hat{\imath} - 18\hat{\jmath}$

$$|\vec{v}| = \sqrt{37}$$

$$|\vec{a}| = 5\sqrt{13}$$

2. Find the unit tangent vector to any point on the curve $x = t^2 + 1,\ y = 4t - 3,\ z = 2t^2 - 6t$. Determine the unit tangent vector at the point where $t = 2$.

Solution:

The position vector of the particle is ,

$$\vec{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$$

⇨ $\vec{r} = (t^2 + 1)\hat{\imath} + (4t - 3)\hat{\jmath} + (2t^2 - 6t)\hat{k}$

Tangent vector,

$$\vec{v} = \frac{d\vec{r}}{d\vec{t}}$$

$$= 2t\hat{\imath} + 4\hat{\jmath} + (4t - 6)\hat{k}$$

∴ Unit tangent vector = $\frac{\vec{v}}{|\vec{v}|}$

$$= \frac{2t\hat{\imath}+4\hat{\jmath}+(4t-6)\hat{k}}{\sqrt{4t^2+16+(4t-6)^2}}$$

At t=2,

Unit tangent vector $= \frac{4\hat{\imath}+4\hat{\jmath}+2\hat{k}}{6} = \frac{2}{3}\hat{\imath} + \frac{2}{3}\hat{\jmath} + \frac{1}{3}\hat{k}$

3. If $\emptyset(x, y, z) = xy^2z$ and $\vec{A} = xz\hat{\imath} - xy^2\hat{\jmath} + yz^2\hat{k}$, find $\frac{\partial^3}{\partial x^2 \partial z}\left(\emptyset\vec{A}\right)$ at the point (2, -1, 1).

<u>Solution:</u>

Given, $\varphi\,(x, y, z) = \ xy^2z$ and $\vec{A} = xz\hat{\imath} - xy^2\hat{\jmath} + yz^2\hat{k}$

Now,

$$\varphi\vec{A} \ = (xy^2z)(xz\hat{\imath} - xy^2\hat{\jmath} + yz^2\hat{k})$$

$$= x^2y^2z^2\hat{\imath} - x^2y^4z\hat{\jmath} + xy^3z^3\hat{k}$$

Then, $\frac{\partial}{\partial z}\left(\varphi\vec{A}\right) = \ \frac{\partial}{\partial z}(x^2y^2z^2\hat{\imath} - x^2y^4z\hat{\jmath} + xy^3z^3\hat{k})$

$$= 2x^2y^2z\hat{\imath} - x^2y^4\hat{\jmath} + 3xy^3z^2\hat{k}$$

$$\frac{\partial}{\partial x}\left(\frac{\partial}{\partial z}\left(\varphi\vec{A}\right)\right) = \ \frac{\partial}{\partial x}(2x^2y^2z\hat{\imath} - x^2y^4\hat{\jmath} + 3xy^3z^2\hat{k})$$

$$=> \ \frac{\partial^2}{\partial x \partial z}\left(\emptyset\vec{A}\right) = \ 4xy^2z\hat{\imath} - 2xy^4\hat{\jmath} + 3y^3z^2\hat{k}$$

$$\frac{\partial}{\partial x}\left(\frac{\partial^2}{\partial x \partial z}(\varphi \vec{A})\right) = \frac{\partial^3}{\partial x^2 \partial z}(\emptyset \vec{A}) =$$

$$\frac{\partial}{\partial x}\ (4xy^2z\hat{\imath} - 2xy^4\hat{\jmath} + 3y^3z^2\hat{k}) = 4y^2z\hat{\imath} - 2y^4\hat{\jmath}$$

Now, $\frac{\partial^3}{\partial x^2 \partial z}\ (\varphi \vec{A}\,)$ at the point (2,-1,1) is,

$$4y^2z\hat{\imath} - 2y^4\hat{\jmath} = 4(-1)^2.\,1\ \hat{\imath} - \ 2(-1)^4\hat{\jmath} = 4\hat{\imath} - 2\hat{\jmath}$$

Gradient, Divergence and Curl

Vector Differential Operator/Del → $\vec{\nabla}$

$$\vec{\nabla} \equiv \frac{\partial}{\partial x}\hat{\imath} + \frac{\partial}{\partial y}\hat{\jmath} + \frac{\partial}{\partial z}\hat{k} \quad \text{[3D]}$$

Gradient

Gradient of φ/grad$\varphi = \vec{\nabla}\varphi$, where φ →scalar quantity

grad φ represents a vector quantity.

Divergence

Divergence of $\vec{A}$ / div$\vec{A} = \vec{\nabla} \cdot \vec{A}$, where $\vec{A}$ →vector quantity

div$\vec{A}$ represents a scalar quantity.

Note: div$\vec{A} > 0$ → divergent (source),

div$\vec{A} < 0$ → convergent (sink)

div$\vec{A} = 0$ → neither source nor sink

Curl

Curl of $\vec{A}$ / curl$\vec{A} = \vec{\nabla} \times \vec{A}$, where $\vec{A}$ →vector quantity

curl$\vec{A}$ represents a vector quantity.

Note: curl$\vec{A} = \vec{0}$ → irrotational

Solved Problems

1. Find the constants a, b, c so that, $\vec{V} = (x + 2y + az)\hat{\imath} + (bx - 3y - z)\hat{\jmath} + (4x + cy + 2z)\hat{k}$ is irrotational.

Solution:

According to the question,

Curl $\vec{V} = \vec{0}$ .... (1)

Curl $\vec{V} = \vec{\nabla} \times \vec{V}$

$$= \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x + 2y + az & bx - 3y - z & 4x + cy + 2z \end{vmatrix}$$

$$= (c + 1)\hat{\imath} - (4 - a)\hat{\jmath} + (b - 2)\hat{k}$$

$$= (c + 1)\hat{\imath} + (a - 4)\hat{\jmath} + (b - 2)\hat{k}$$

From (1),

$$(c+1)\hat{\imath} + (a-4)\hat{\jmath} + (b-2)\hat{k}$$
$$= 0.\hat{\imath} + 0.\hat{\jmath} + 0.\hat{k}$$

So,

$$\mathrm{c} + 1 = 0$$

$$=> c = -1$$

$$a - 4 = 0$$

$$=> a = 4$$

and $b - 2 = 0$

$$=> b = 2$$

2. If $\vec{\nabla}\varphi = 2xyz^3\hat{\imath} + x^2z^3\hat{\jmath} + 3x^2yz^2\hat{k}$. Find $\varphi\,(x, y, z)$ when $\varphi\,(1, -2{,}2) = 4.$

Solution:

Given,

$$\vec{\nabla}\varphi = 2xyz^3\hat{\imath} + x^2z^3\hat{\jmath} + 3x^2yz^2\hat{k}$$

$$=> \left(\frac{\partial\varphi}{\partial x}\hat{\imath} + \frac{\partial\varphi}{\partial y}\hat{\jmath} + \frac{\partial\varphi}{\partial z}\hat{k}\right) = 2xyz^3\hat{\imath} + x^2z^3\hat{\jmath} + 3x^2yz^2\hat{k}$$

Comparing both sides, we get,

$$\frac{\partial\varphi}{\partial x} = 2xyz^3\,, \qquad \varphi = x^2yz^3 + f(y,z)$$

$$\frac{\partial\varphi}{\partial y} = x^2z^3\,, \qquad \varphi = x^2yz^3 + f(x,z)$$

$$\frac{\partial\varphi}{\partial z} = 3x^2yz^2\,, \qquad \varphi = x^2yz^3 + f(x,y)$$

Applying $\varphi\,(1, -2{,}2) = 4$, we get,

$$4 = -16 + f(y,z)$$

$$=> f(y,z) = 20$$

$$4 = -16 + f(x,z)$$

$$=> f(x,z) = 20$$

and

$$4 = -16 + f(x,y)$$

$$=> f(x,y) = 20$$

$$\therefore \varphi = x^2yz^3 + 20$$

3. If $\overrightarrow{A} = 3xyz^2\hat{\imath} + 2xy^3\hat{\jmath} - x^2yz\hat{k}$ and $\varphi = 2x^3y^2z^4$. Find div curl $\vec{A}$ and curl gradφ.

Solution:

Given,

$$\overrightarrow{A} = 3xyz^2\hat{\imath} + 2xy^3\hat{\jmath} - x^2yz\hat{k}$$

$$\varphi = 2x^3y^2z^4$$

Now,

$$\text{curl } \overrightarrow{A} = \vec{\nabla} \times \overrightarrow{A}$$

$$= \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 3xyz^2 & 2xy^3 & -x^2yz \end{vmatrix}$$

$$= -x^2z\hat{\imath} - (-2xyz - 6xyz)\hat{\jmath} + (2y^3 - 3xz^2)\hat{k}$$

$$= -x^2z\hat{\imath} + 8xyz\hat{\jmath} + (2y^3 - 3xz^2)\hat{k}$$

$$\text{div curl } \overrightarrow{A} = \vec{\nabla}.\left(\vec{\nabla} \times \overrightarrow{A}\right)$$

$$= \frac{\partial}{\partial x}(-x^2z) + \frac{\partial}{\partial y}(8xyz) + \frac{\partial}{\partial z}(2y^3 - 3xz^2)$$

$$= -2xz + 8xz - 6xz$$

$$= 0$$

$$\text{div curl } \overrightarrow{A} = 0$$

Now,

grad $\varphi = \vec{\nabla}\varphi$

$$= \frac{\partial}{\partial x}(2x^3y^2z^4)\hat{\imath} + \frac{\partial}{\partial y}(2x^3y^2z^4)\hat{\jmath} + \frac{\partial}{\partial z}(2x^3y^2z^4)\hat{k}$$

$$= 6x^2y^2z^4\hat{\imath} + 4x^3yz^4\hat{\jmath} + 8x^3y^2z^3\hat{k}$$

curl grad $\varphi = \vec{\nabla} \times \left(\vec{\nabla}\varphi\right)$

$$= \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 6x^2y^2z^4 & 4x^3yz^4 & 8x^3y^2z^3 \end{vmatrix}$$

$$= (16x^3yz^3 - 16x^3yz^3)\hat{\imath} - (24x^2y^2z^3 - 24x^2y^2z^3)\hat{\jmath} + (12x^2yz^4 - 12x^2yz^4)\hat{k}$$

$$= \vec{0}$$

$\therefore$ curl grad $\varphi = \vec{0}$

Homework Problems

1. A particle moves so that it's position vector is given by $\vec{r} = \cos\omega t\hat{\imath} + \sin\omega t\hat{\jmath}$, where ω is a constant. Show that

(a) The vector of the particle $\vec{v}$ is perpendicular to $\overrightarrow{r}$

(b) $\overrightarrow{r} \times \vec{v}$ = a constant vector

2. Find the value of a for which the vector

$\vec{A} = (axy - z^3)\hat{\imath} + (a-2)x^2\hat{\jmath} + (1-a)xz^2\hat{k}$

have it's curl identically equal to zero.

3. Show that $\vec{A} = (6xy + z^3)\hat{\imath} + (3x^2 - z)\hat{\jmath} + (3xz^2 - y)\hat{k}$ is irrotational. Find φ such that $\overrightarrow{A} = \vec{\nabla}\varphi$.