Solutions to the Homework Problems

1. Calculate the work done when a force $\vec{F} = 3xy\hat{\imath} - y^2\hat{\jmath}$ moves a particle in the xy-plane from $(0, 0)$ to $(1, 2)$ along the parabola $y = 2x^2$.

Solution: Let, $x = t, y = 2t^2$

that is $dx = dt$, $dy = 4t\ dt$

For the point $(0, 0)$,

$$0 = t, 0 = 2t^2$$

That means we get $t = 0$

Again for the point $(1, 2)$,

$$1 = t, 2 = 2t^2$$

That means we get $t = 1$

Now we get

$$\vec{F} = 6t^3\hat{\imath} - 4t^4\hat{\jmath}$$

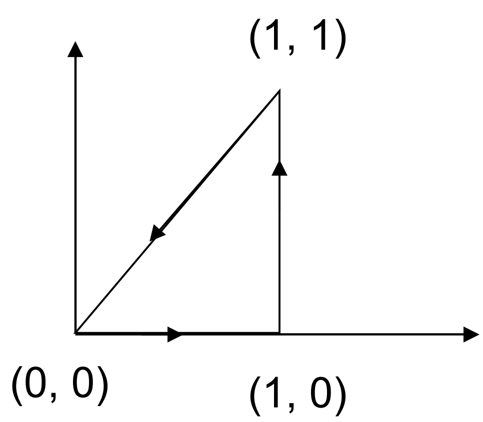
and $d\vec{r} = dx\hat{\imath} + dy\hat{\jmath} + dz\hat{k} = \; dt\hat{\imath} + 4t\; dt\; \hat{\jmath}$

Now, $\int_C \; \vec{F} \cdot d\vec{r} = \; \int_0^1 (6t^3 - 16t^5)dt = -\frac{7}{6}$

So, the work done is $\frac{7}{6}$.

2. Evaluate $\oint_C (y^2 dx + x^2 dy)$ where C is the triangle with vertices (1, 0), (1, 1), (0, 0).

<u>Solution:</u>



Along the straight line from $(0,0)$ to $(1,0)$ we get,

$< (0,0) + (1,0)t >= < (t, 0) >$ where $0 \leq t \leq 1$

So that, $x = t, y = 0$ that is $dx = dt$, $dy = 0$

So we get

$$\oint_C (y^2 dx + x^2 dy) = 0$$

Along the straight line from $(1,0)$ to $(1,1)$ we get,

$< (1,0) + (0,1)t >= < (1, t) >$ where $0 \leq t \leq 1$

So that, $x = 1, y = t$ that is $dx = 0$, $dy = dt$

So we get

$$\oint_C (y^2 dx + x^2 dy) = \oint_0^1 dt = 1$$

Along the straight line from $(1,1)$ to $(0,0)$ we get,

$< (1,1) + (-1,-1)t >= < (1 - t, 1 - t) >$

where $0 \leq t \leq 1$

So that, $x = 1 - t, y = 1 - t$ that is $dx = -dt$, $dy = -dt$

So we get

$$\oint_C (y^2 dx + x^2 dy) = \oint_0^1 \{-(1-t)^2 - (1-t)^2\}\, dt$$

$$= -\frac{2}{3}$$

Adding,

$$\oint_C (y^2 dx + x^2 dy) = 0 + 1 - \frac{2}{3} = \frac{1}{3}$$

3. If $\emptyset = 2xyz^2, \vec{F} = xy\hat{\imath} - z\hat{\jmath} + x^2\hat{k}$ and C is the curve $x = t^2, y = 2t, z = t^3$ from $t = 0$ to $t = 1$, evaluate the line integrals (a) $\oint_C \emptyset \, d\vec{r}$ (b) $\oint_C \vec{F} \times d\vec{r}$.

<u>Solution:</u> Since $x = t^2, y = 2t, z = t^3$

so that $dx = 2tdt,\ y = 2dt,\ dz = 3t^2dt$

$d\vec{r} = dx\hat{\imath} + dy\hat{\jmath} + dz\hat{k} = 2tdt\hat{\imath} + 2dt\,\hat{\jmath} + 3t^2dt\hat{k}$

$$\vec{F} = 2t^3\hat{\imath} - t^3\hat{\jmath} + t^4\hat{k}$$

$$\emptyset = 4t^9$$

(a) $\oint_C \emptyset \, d\vec{r} = \oint_0^1 (4t^9)(2t\hat{\imath} + 2\,\hat{\jmath} + 3t^2\hat{k}\,)\,dt = \frac{8}{11}\hat{\imath} + \frac{4}{5}\hat{\jmath} + \hat{k}$

(b) $\oint_C \vec{F} \times d\vec{r} = \oint_0^1 \Big[(-3t^5 - 2t^4)\hat{\imath} +$

$$(2t^5 - 6t^5)\hat{\jmath} + (4t^3 + 2t^4)\widehat{k}\,\Big]dt$$

$$= -\frac{9}{10}\hat{\imath} - \frac{2}{3}\hat{\jmath} + \frac{7}{5}\widehat{k}$$