Topic 1: Transformation of Coordinates

The process of changing the coordinates of a point or the equation of the curve is called "Transformation of Coordinates."

(1) **Translation:** (By shifting the origin without rotating of coordinate axes)

$$x = x + \alpha \quad \text{and}$$

$$y = y + \beta$$

where $(\alpha, \beta) \rightarrow$ Translated Point

(2) **Rotation**: (By rotating of coordinate axes without shifting the origin)

$$x = xcos\theta - ysin\theta \quad \text{and}$$

$$y = xsin\theta + ycos\theta$$

where $\theta \rightarrow$ angle of rotation

Special Information:

The general equation of second degree can be written as-

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0 \quad \text{----(1)}$$

where a, b, c, f, g, h are constants.

(1) By shifting the origin to the particular point, equation (1) transforms to one in which the first degree terms are removed.

(2) By rotating the co-ordinate axes to the particular angle, equation (1) transforms to one in which the xy term is removed.

The idea of the problems:

Translation $\rightarrow$ given (α, β)

Translation $\rightarrow$ find (α, β)

Rotation $\rightarrow$ given θ

Rotation $\rightarrow$ find θ

Solved Problems:

1. Determine the equation of the curve $9x^2 + 24xy + 2y^2 - 6x + 20y + 41 = 0$, when the origin is transferred to the point $(-1,1)$, where the direction of axes remains unaltered.

Solution:

Given,

$9x^2 + 24xy + 2y^2 - 6x + 20y + 41 = 0$ …………(1)

Putting $x = x - 1$ and $y = y + 1$ in equation (1),

$$9(x-1)^2 + 24(x-1)(y+1) + 2(y+1)^2 - 6(x-1) + 20(y+1) + 41 = 0$$

$$\Rightarrow \quad 9(x^2 - 2x + 1) + 24(xy + x - y - 1) + 2(y^2 + 2y + 1) - 6(x-1) + 20(y+1) + 41 = 0$$

$$\Rightarrow \quad 9x^2 + 24xy + 2y^2 + x(-18 + 24 - 6) + y(-24 + 4 + 20) + (9 - 24 + 2 + 6 + 20 + 41) = 0$$

$$\Rightarrow \quad 9x^2 + 24xy + 2y^2 + 54 = 0$$

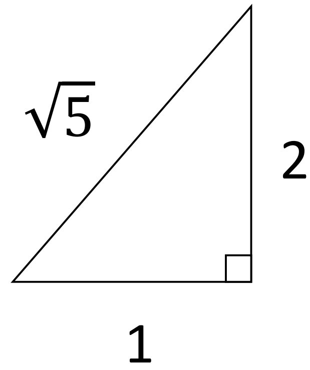
This is our required transformed equation.

2. If the axes be turned through an angle $\tan^{-1} 2$, then what does the equation $4xy - 3x^2 = 4$ become?

Solution:

Given, $\qquad 4xy - 3x^2 = 4 \qquad$ ....(1)

and $\quad \theta = \tan^{-1} 2$



$$\Rightarrow \quad tan\theta = 2$$

$$\therefore sin\theta = \frac{2}{\sqrt{5}} \text{ and } cos\theta = \frac{1}{\sqrt{5}}$$

We know,

$$x = xcos\theta - ysin\theta$$

$$\therefore x = \frac{x}{\sqrt{5}} - \frac{2y}{\sqrt{5}}$$

And $y = xsin\theta + ycos\theta$

$$\therefore y = \frac{2x}{\sqrt{5}} + \frac{y}{\sqrt{5}}$$

Putting $x = \frac{x}{\sqrt{5}} - \frac{2y}{\sqrt{5}}$ and $y = \frac{2x}{\sqrt{5}} + \frac{y}{\sqrt{5}}$ in equation (1),

$$4\left(\frac{x}{\sqrt{5}} - \frac{2y}{\sqrt{5}}\right)\left(\frac{2x}{\sqrt{5}} + \frac{y}{\sqrt{5}}\right) - 3\left(\frac{x}{\sqrt{5}} - \frac{2y}{\sqrt{5}}\right)^2 - 4 = 0$$

$$\Rightarrow \quad 4\left(\frac{2x^2}{5} + \frac{xy}{5} - \frac{4xy}{5} - \frac{2y^2}{5}\right) - 3\left(\frac{x^2}{5} - 2.\frac{x}{\sqrt{5}}.\frac{2y}{\sqrt{5}} + \frac{4y^2}{5}\right) - 4 = 0$$

$$\Rightarrow \quad 4\left(\frac{2x^2}{5}+\frac{xy}{5}-\frac{4xy}{5}-\frac{2y^2}{5}\right)-3\left(\frac{x^2}{5}-\frac{4xy}{5}+\frac{4y^2}{5}\right)-4=0$$

$$\Rightarrow \quad x^2\left(\frac{8}{5}-\frac{3}{5}\right)+xy\left(\frac{4}{5}-\frac{16}{5}+\frac{12}{5}\right)-y^2\left(\frac{8}{5}+\frac{12}{5}\right)-4=0$$

$$\Rightarrow \quad x^2-4y^2-4=0$$

This is our required transformed equation.

3. Transform the equation $9x^2+24xy+2y^2-6x+20y+41=0$ in rectangular coordinates so as to remove the terms in x, y and xy.

<u>Solution</u>:

Given,

$$9x^2+24xy+2y^2-6x+20y+41=0 \text{ ....(1)}$$

Putting $x=x+\alpha$ and $y=y+\beta$ in equation (1),

$$9(x+\alpha)^2 + 24(x+\alpha)(y+\beta) + 2(y+\beta)^2 - 6(x+\alpha) + 20(y+\beta\ ) + 41 = 0$$

⇨ $9(x^2 + 2x\alpha + \alpha^2) + 24(xy + x\beta + y\alpha + \alpha\beta) + 2(y^2 + 2y\beta + \beta^2) - 6(x+\alpha) + 20$-$(y+\beta\ ) + 41 = 0$

⇨ $9x^2 + 24xy + 2y^2 + x(18\alpha + 24\beta - 6) + y(24\alpha + 4\beta + 20) + 9\alpha^2 + 24\alpha\beta + 2\beta^2 - 6\alpha + 20\beta + 41 = 0$ ……………………………(2)

The terms of x and y in equation (2) will be absent if

$$18\alpha + 24\beta - 6 = 0$$

And $\quad 24\alpha + 4\beta + 20 = 0$

Solving these two equations we get,

$\therefore \alpha = -1$ and $\beta = 1$

Putting $\alpha = -1$ and $\beta = 1$ in equation (2),

$$9x^2 + 24xy + 2y^2 + 9 - 24 + 2 + 6 + 20 + 41 = 0$$

⇨ $9x^2 + 24xy + 2y^2 + 54 = 0$…………(3)

Now Putting $x = xcos\theta - ysin\theta$ and $y = xsin\theta + ycos\theta$ in equation (3),

$$9(xcos\theta - ysin\theta)^2 + 24(xcos\theta - ysin\theta)$$

$$(xsin\theta + ycos\theta) + 2(xsin\theta + ycos\theta)^2 + 54 = 0$$

$\Rightarrow$ $9(x^2\cos^2\theta - 2xysin\theta cos\theta + y^2\sin^2\theta) + 24(x^2sin\theta cos\theta + xycos^2\theta - xysin^2\theta - y^2sin\theta cos\theta) + 2(x^2\sin^2\theta + 2xysin\theta cos\theta + y^2\cos^2\theta) + 54 = 0$

$\Rightarrow$ $x^2(9\cos^2\theta + 24sin\theta cos\theta + 2\sin^2\theta) + xy(-18sin\theta cos\theta + 24\cos^2\theta - 24\sin^2\theta + 4sin\theta cos\theta) + y^2(9\sin^2\theta - 24sin\theta cos\theta + 2\cos^2\theta) + 54 = 0$ ………………………….(4)

To remove the xy term in equation (4), we can write,

$$-18sin\theta cos\theta + 24\cos^2\theta - 24\sin^2\theta + 4sin\theta cos\theta = 0$$

⇨ $24\cos^2\theta - 14sin\theta cos\theta - 24\sin^2\theta = 0$

⇨ $12\cos^2\theta - 7sin\theta cos\theta - 12\sin^2\theta = 0$

⇨ $12\cos^2\theta - 16sin\theta cos\theta + 9sin\theta cos\theta - 12\sin^2\theta = 0$

⇨ $4cos\theta(3cos\theta - 4sin\theta) + 3sin\theta(3cos\theta - 4sin\theta) = 0$

⇨ $(3cos\theta - 4sin\theta)(4cos\theta + 3sin\theta) = 0$

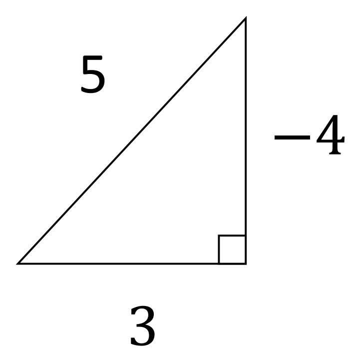
$\therefore (4cos\theta + 3sin\theta) = 0$

⇨ $tan\theta = -\frac{4}{3}$

Or, $(3cos\theta - 4sin\theta) = 0$

⇨ $tan\theta = \frac{3}{4}$

when $tan\theta = -\frac{4}{3}$



$\therefore sin\theta = -\frac{4}{5}$ and $cos\theta = \frac{3}{5}$

Putting $sin\theta = -\frac{4}{5}$ and $cos\theta = \frac{3}{5}$ in equation (4),

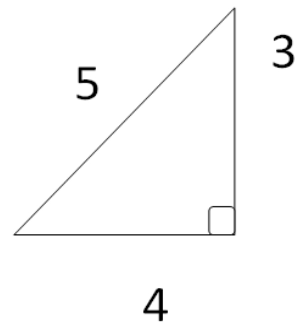
$$x^2\left(9.\frac{9}{25} + 24\left(-\frac{4}{5}\right)\frac{3}{5} + 2\frac{16}{25}\right) + xy\left(-18\left(-\frac{4}{5}\right)\frac{3}{5} + 24.\frac{9}{25} - 24\left(-\frac{4}{5}\right)^2 + 4\left(-\frac{4}{5}\right).\left(\frac{3}{5}\right)\right) + y^2\left(9.\frac{16}{25} - 24\left(-\frac{4}{5}\right).\frac{3}{5} + 2\left(\frac{3}{5}\right)^2\right) + 54 = 0$$

⇨ $-7x^2 + 18y^2 + 54 = 0$

⇨ $18y^2 - 7x^2 + 54 = 0$

This is our required transformed equation.

Or, when $tan\theta = \frac{3}{4}$



$\therefore sin\theta = \frac{3}{5}$ and $cos\theta = \frac{4}{5}$

Putting $sin\theta = \frac{3}{5}$ and $cos\theta = \frac{4}{5}$ in equation (4),

$x^2\left(9.\frac{16}{25} + 24\ \frac{3}{5}.\frac{4}{5} + 2\frac{9}{25}\right) + xy\left(-18.\frac{3}{5}.\frac{4}{5} + 24.\frac{16}{25} - 24\ \frac{9}{25} + 4.\frac{3}{5}.\frac{4}{5}\right) + y^2\left(9.\frac{9}{25} - 24.\frac{3}{5}.\frac{4}{5} + 2.\frac{16}{25}\right) + 54 = 0$

$\Rightarrow \quad 18x^2 - 7y^2 + 54 = 0$

This is our required transformed equation.

Homework Problems

1. Transform the equation $11x^2 + 24xy + 4y^2 - 20x - 40y - 5 = 0$ to rectangular axes through the point $(2, -1)$ and inclined at an angle $\tan^{-1}\left(-\frac{4}{3}\right)$.

2. Through what angle must the axes be rotated to remove the term containing xy in $11x^2 + 4xy + 14y^2 = 5$? Find the transformed equation.

3. Determine the equation of the parabola $x^2 - 2xy + y^2 + 2x - 4y + 3 = 0$ after rotating of axes through $45°$.

4. Verify that when the axes are turned through an angle $\frac{\pi}{4}$, the equation $5x^2 + 4xy + 5y^2 - 10 = 0$ transforms to one in which the term xy is absent.

5. Transform the equation $17x^2 + 18xy - 7y^2 - 16x - 32y - 18 = 0$ to one in which there is no term involving x, y and xy.