GROUP–A

1. $\int_0^{2\pi} \frac{d\theta}{2+\cos\theta}$ **[NUH–93, 2003 (Phy), DUH–93, RUH–200**

2. $\int_0^{2\pi} \frac{d\theta}{5+4\cos\theta}$ **[NUH–02, 04, 06 (Old), 07, 12(Old), 1**

3. $\int_0^{2\pi} \frac{d\theta}{3+2\sin\theta}$ **[NUH–199**

3(a). $\int_0^{2\pi} \frac{d\theta}{3+2\cos\theta}$ **[NUH–201**

4. $\int_0^{2\pi} \frac{d\theta}{5+3\sin\theta}$ **[NUH–2000 2005 (Old), NU(Pre)–200**

5. $\int_0^{2\pi} \frac{d\theta}{2+\sin\theta}$ **[RUH–200**

6. $\int_0^{2\pi} \frac{1}{1+a\sin x}\,dx, 0<a<1$ **[NUH–190**

7. $\int_0^{2\pi} \frac{d\theta}{(5-3\cos\theta)^2}$ **[RUH–199**

8. $\int_0^{2\pi} \frac{d\theta}{(5-3\sin\theta)^2}$ **[RUH–1998, 200**

9. $\int_0^{2\pi} \frac{\cos 2\theta}{5+4\cos\theta}\,d\theta$ **[NUH–1997, DUH–200**

10. $\int_0^{2\pi} \frac{\cos 3\theta}{5-4\cos\theta}\,d\theta$

[NUH–99, 03, 06, 08, NU(Pre)–08, DUH–90, 9

11. $\int_0^{2\pi} \frac{\sin 2\theta}{5-3\cos\theta}\,d\theta$ **[NUH–200**

12. $\int_0^{2\pi} \frac{d\theta}{a+b\cos\theta}, a>|b|$ **[DUH–1975, 1977, 19**

13. $\int_0^{2\pi} \frac{d\theta}{a+b\sin\theta}, a>|b|$ **[DUH–2005, RUH–1995, 19**

14. (i) $\int_0^{2\pi} \frac{d\theta}{(a+b\cos\theta)^2}$ **[DUH–1984, CUH–2000, 20**

(ii) $\int_0^{\pi} \frac{d\theta}{(a+b\cos\theta)^2}$

15. $\int_0^{2\pi} \frac{d\theta}{1+a^2-2a\cos\theta}, 0<a<1$

16. (i) $\int_0^{2\pi} \frac{\cos 2\theta}{1-2a\cos\theta+a^2}\,d\theta, a^2<1$

[DUH–1984, RUH–1973, 1975, 1981]

(ii) $\int_0^{\pi} \frac{\cos 2\theta}{1-2a\cos\theta+a^2}\,d\theta, a^2<1$

[DUH–1976, 1986, CUH–1990]

17. $\int_0^{\pi} \frac{a\,d\theta}{a^2+\cos^2\theta}, a>0$ **[NUH–1998]**

18. $\int_0^{\pi} \frac{a\,d\theta}{a^2+\sin^2\theta}, a>0$ **[DUH–86, 03, 04, CUH–01, 04]**

19. (i) $\int_0^{2\pi} \frac{\sin^2\theta\,d\theta}{a+b\cos\theta}, a>b>0$ **[NU(Pre)–2011]**

(ii) $\int_0^{\pi} \frac{\sin^2\theta\,d\theta}{a+b\cos\theta}, a>b>0$

[DUH–1977, 1983, 1987, 1990, CUH–1982]

20. $\int_0^{2\pi} \frac{\cos^2 3\theta}{5-4\cos 2\theta}\,d\theta$ **[NUH–2004(Old), 2008, 2011, CUH–2003]**

21. By calculas of residue prove that

(i) $\int_0^{2\pi} e^{\cos\theta}\cos(\sin\theta - n\theta)\,d\theta = \frac{2\pi}{n!}$

[CUH–2001, 2003, 2004]

(ii) $\int_0^{2\pi} e^{-\cos\theta}\cos(n\theta+\sin\theta)\,d\theta = \frac{2\pi}{n!}(-1)^n$

GROUP–B

22. $\int_0^{\infty} \frac{1}{(x^2+1)(x^2+4)}\,dx$

23. $\int_{-\infty}^{\infty} \frac{x^2}{(x^2+4)(x^2+9)}\,dx$ **[NUH–1995]**

24. $\int_0^{\infty} \frac{1}{(x^2+1)(x^2+4)^2}\,dx$

[NUH–2000, RUH–1996, NU(Pre)–2006, CUH–2001]

25. $\int_0^\infty \frac{2x^2}{(x^2+9)(x^2+4)^2}\,dx$ [NUH–199

26. $\int_{-\infty}^\infty \frac{1}{(x^2+b^2)(x^2+c^2)^2}\,dx$

[DUH–1980, 1982, 1988, DUH–1984, RUH–197

27. $\int_{-\infty}^\infty \frac{x^2}{(x^2+1)^2(x^2+2x+2)}\,dx$ [RUH–198

28. $\int_0^\infty \frac{dx}{x^4+1}$

[NUH–2002, 2005, 2012, DUH–1986, 88, RUH–98, 200

29. $\int_0^\infty \frac{dx}{x^4+a^4}, a>0$ [NUH–01, 04, 06, 08, 12(Old), DUH–8

30. $\int_0^\infty \frac{1}{(x^2+a^2)^2}\,dx, a>0$ [NUH–199

31. $\int_0^\infty \frac{1}{x^6+1}\,dx$ [NU(Pre)–2011, CUH–200

32. $\int_0^\infty \frac{x^6}{(a^4+x^4)^2}\,dx$ [DUH–1977, 1985, DUH–198

33. $\int_0^\infty \frac{x^6}{(x^4+1)^2}\,dx$ [NUH–199

34. $\int_0^\infty \frac{\log(x^2+1)}{x^2+1}\,dx$ [NUH–97, 03, 06 (Old), NU(Pre)–08, 1

DUH–05, RUH–82, 97, 99, CUH–81, 89, 0

34. (a) $\int_0^\infty \frac{z^a}{(1+z^2)^2}\,dz, 0<a<1$ [NUH–199

(b) $\int_0^\infty \frac{dz}{1+z^2}$ [NU(Phy)–200

GROUP–C

35. $\int_{-\infty}^\infty \frac{\cos x}{(x^2+1)(x^2+9)}\,dx$ [RUH–197

36. $\int_{-\infty}^\infty \frac{\cos x}{(x^2+8)(x^2+12)}\,dx$ [DUH–197

37. $\int_{-\infty}^\infty \frac{\cos x}{(x^2+a^2)(x^2+b^2)}\,dx$ [DUH–1982, 1987, 19

38. $\int_0^\infty \frac{\cos 2\pi x}{x^4+x^2+1}\,dx$ [NUH–1998]

39. (i) $\int_0^\infty \frac{\cos mx}{x^2+a^2}\,dx$ [DUH–1978, 1988, 2006, RUH–1976]

(ii) $\int_{-\infty}^\infty \frac{\sin mx}{x^2+a^2}\,dx$

40. (i) $\int_0^\infty \frac{\cos ax}{x^2+1}\,dx, a>0$

[NUH–96, 05 (Old), 06(Old), 11, NU(Pre)–05]

(ii) $\int_{-\infty}^\infty \frac{\sin ax}{x^2+1}\,dx$

41. $\int_{-\infty}^\infty \frac{\cos 2x}{x^2+1}\,dx$ [NUH–1995, 2004(Old), 2012]

42. $\int_0^\infty \frac{\cos 3x}{x^2+4}\,dx$ [RUH–1999]

43. (i) $\int_0^\infty \frac{\cos mx}{(x^2+a^2)^2}\,dx$ [DUH–1986]

(ii) $\int_{-\infty}^\infty \frac{\sin mx}{(x^2+a^2)^2}\,dx$

44. $\int_0^\infty \frac{\cos mx}{(x^2+1)^2}\,dx, m>0$ [RUH–1995, 1997, CUH–2002]

45. $\int_0^\infty \frac{\cos mx}{x^4+a^4}\,dx$ [DUH–1975, 1986]

46. $\int_0^\infty \frac{x \sin mx}{x^4+a^4}\,dx$ [DUH–1976, 1978, 1986]

47. (i) $\int_0^\infty \frac{x\cos mx}{x^2+a^2}\,dx$

(ii) $\int_0^\infty \frac{x\sin mx}{x^2+a^2}\,dx, a>0, m>0$

[NUH–97, 05 (Old), NU(Phy)–05, DUH–76, 78, 03]

48. $\int_0^\infty \frac{x\sin x}{a^2+x^2}\,dx$ [NUH–2003, CUH–1989]

49. $\int_0^\infty \frac{x\sin x}{x^2+4}\,dx$

[NUH–94, 05,(Old) NU(Pre)–05, DUH–94, RUH–75, CUH–01]

GROUP–D

50. $\int_0^\infty \frac{\sin x}{x}\,dx$ **[NUH-94, 12(Old), DUH-85, 88, 94, 04, 05, RUH-1995, 2002, 2003]**

51. $\int_0^\infty \frac{\sin mx}{x}\,dx,\ m > 0$ **[NUH-93, 03, 04, 06, 07, DUH-93, 02, 04, 06, 07]**

52. $\int_0^\infty \frac{\cos x}{a^2 - x^2}\,dx$ **[NUH-2000, NU(Pre)-2006]**

53. $\int_0^\infty \frac{\sin \pi x}{x(1 - x^2)}\,dx$ **[NUH-2001, 2005 (Old), 2006 (Old), NUP-2005, DUH-2003, RUH-1973, 1974, 1981, CUH-2004]**

54. (i) $\int_0^\infty \frac{(\log x)^2}{1 + x^2}\,dx$ **[RUH-2001, CUH-2004]**

(ii) $\int_0^\infty \frac{\log x}{1 + x^2}\,dx$

55. $\int_0^\infty \frac{\log x}{(1 + x^2)^2}\,dx$ **[RUH-2002]**

GROUP–E

56. If $0 < p < 1$, then show that

(i) $\int_0^\infty \frac{x^{p-1}}{1 + x}\,dx = \frac{\pi}{\sin p\pi}$

[NUH-98, 05, 08, 11, NU(Phy)-06, DUH-86, RUH-96, 01]

(ii) $\Gamma(p)\,\Gamma(1 - p) = \frac{\pi}{\sin p\pi}$

57. Show that $\int_{-\infty}^\infty \frac{e^{px}}{1 + e^x}\,dx = \frac{\pi}{\sin p\pi}$, where $0 < p < 1$ **[NUH-97, NU(Pre)-08, 11, DUH-76, 78, 87, RUH-98, 01, 02, CUH-01]**

58. Show that $\int_0^\infty \sin x^2\,dx = \int_0^\infty \cos x^2\,dx = \frac{1}{2}\sqrt{\frac{\pi}{2}}$.

[NUH-2005, 2007, RUH-1981, 1999]

59. Show that $\int_0^\infty \frac{\cosh ax}{\cosh x}\,dx = \frac{\pi}{2\cos\left(\frac{\pi a}{2}\right)}$ **[DUH-1975, 1989]**

60. By contour integration prove that

$$\int_{-\infty}^\infty \frac{\cos x^2 + \sin x^2 - 1}{x^2}\,dx = 0$$

SOLUTIONS
GROUP–A

Solution–1. Let us consider the unit circle $|z| = 1$ as the contour C. [কন্টুর C কে আমরা একক বৃত্ত $|z| = 1$ বিবেচনা করি]

Then [তখন] $z = e^{i\theta} \Rightarrow dz = ie^{i\theta}\,d\theta = iz\,d\theta \Rightarrow d\theta = \frac{dz}{iz}$ where [যেখানে] $0 \le \theta \le 2\pi$.

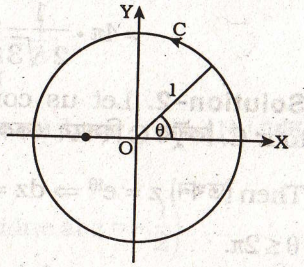
$$\cos\theta = \frac{1}{2}(e^{i\theta} + e^{-i\theta}) = \frac{1}{2}\left(z + \frac{1}{z}\right) = \frac{z^2 + 1}{2z}$$

$$\therefore \int_0^{2\pi} \frac{d\theta}{2 + \cos\theta} = \oint_C \frac{\frac{1}{iz}\,dz}{2 + \frac{z^2 + 1}{2z}}$$

$$= \frac{1}{i}\oint_C \frac{2z/z}{4z + z^2 + 1}\,dz$$

$$= \frac{2}{i}\oint_C \frac{dz}{z^2 + 4z + 1}$$

$$= \frac{2}{i}\oint_C f(z)\,dz, \text{ say } \ldots\ldots (1)$$



where [যেখানে] $f(z) = \frac{1}{z^2 + 4z + 1}$

The poles of $f(z)$ are obtained by solving the equation

$z^2 + 4z + 1 = 0$

[$f(z)$ এর পোল $z^2 + 4z + 1 = 0$ সমীকরণ সমাধান করে পাওয়া যাবে]

$$\Rightarrow z = \frac{-4 \pm \sqrt{16 - 4}}{2} = \frac{-4 \pm 2\sqrt{3}}{2} = -2 \pm \sqrt{3}$$

$|z| = |-2 + \sqrt{3}| = |-2 + 1.73| = |-0.27| = 0.27 < 1$

$|z| = |-2 - \sqrt{3}| = |-2 - 1.73| = |-3.73| = 3.73 > 1$

$\therefore$ The pole $z = -2 + \sqrt{3}$ lies inside the contour C which is a simple pole. [$z = -2 + \sqrt{3}$ পোলটি C কন্টুরের ভিতরে অবস্থিত যাহা একটি সরল পোল।]

Residue at $z = -2 + \sqrt{3}$ is [$z = -2 + \sqrt{3}$ এ অবশেষ]

$$\lim_{z \to -2+\sqrt{3}} (z + 2 - \sqrt{3})\,f(z)$$

$$= \lim_{z \to -2+\sqrt{3}} (z + 2 - \sqrt{3}) \cdot \frac{1}{z^2 + 4z + 1}$$

$$= \lim_{z\to -2+\sqrt{3}} (z+2-\sqrt{3})\cdot \frac{1}{(z+2-\sqrt{3})\,(z+2+\sqrt{3})}$$

$$= \lim_{z\to -2+\sqrt{3}} \frac{1}{z+2+\sqrt{3}} = \frac{1}{-2+\sqrt{3}+2+\sqrt{3}} = \frac{1}{2\sqrt{3}}$$

By Cauchy's residue theorem we have from (1) [কচির অবশেষ উপপাদ্য দ্বারা (1) হতে পাই]

$$\int_0^{2\pi} \frac{d\theta}{2+\cos\theta} = \frac{2}{i}\cdot 2\pi i \cdot (\text{Residue at } z = -2+\sqrt{3})$$

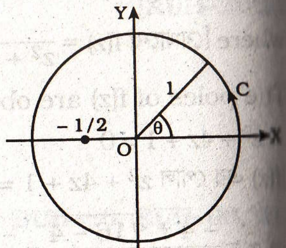
$$= 4\pi \cdot \frac{1}{2\sqrt{3}} = \frac{2\pi}{\sqrt{3}}.\ \text{(Ans)}$$

Solution–2. Let us consider the unit circle $|z| = 1$ as the contour C. [কন্টুর C হিসাবে একক বৃত্ত $|z| = 1$ কে বিবেচনা করি]

Then [তখন] $z = e^{i\theta} \Rightarrow dz = ie^{i\theta}\,d\theta = iz\,d\theta \Rightarrow d\theta = \frac{dz}{iz}$, where [যেখানে] $0 \le \theta \le 2\pi$.

$$\cos\theta = \frac{1}{2}(e^{i\theta}+e^{-i\theta}) = \frac{1}{2}\left(z+\frac{1}{z}\right) = \frac{z^2+1}{2z}$$

$$\therefore \int_0^{2\pi} \frac{d\theta}{5+4\cos\theta} = \oint_C \frac{\frac{1}{iz}\,dz}{5+4\cdot\frac{z^2+1}{2z}}$$



$$= \frac{1}{i}\oint_C \frac{dz}{5z+2z^2+2}$$

$$= \frac{1}{i}\oint_C f(z)\,dz, \text{ say } \cdots\cdots (1)$$

where [যেখানে] $f(z) = \dfrac{1}{2z^2+5z+2}$

The poles of f(z) are obtained by solving the equation

$2z^2+5z+2=0$

[f(z) এর পোল $2z^2 + 5z + 2 = 0$ সমীকরণ সমাধান করে পাওয়া যাবে]

$$\Rightarrow z = \frac{-5\pm\sqrt{25-16}}{4} = \frac{-5\pm 3}{4} = -\frac{1}{2}, -2$$

$$|z| = \left|-\frac{1}{2}\right| = \frac{1}{2} < 1 \text{ and } |z| = |-2| = 2 > 1.$$

Only the pole $z = -\frac{1}{2}$ lies inside the contour C which is a simple pole. [একমাত্র পোল $z = -\frac{1}{2}$ কন্টুর C এর ভিতর অবস্থিত যাহা একটি সরলপোল]

Residue at $z = -\frac{1}{2}$ is [$z = -\frac{1}{2}$ এ অবশেষ]

$$\lim_{z\to -\frac{1}{2}} \left(z+\frac{1}{2}\right)\cdot f(z) = \lim_{z\to -\frac{1}{2}} \left(z+\frac{1}{2}\right)\cdot \frac{1}{2z^2+5z+2}$$

$$= \lim_{z\to -\frac{1}{2}} \left(z+\frac{1}{2}\right)\cdot \frac{1}{2\left(z+\frac{1}{2}\right)(z+2)}$$

$$= \frac{1}{2}\cdot \lim_{z\to -\frac{1}{2}} \frac{1}{z+2}$$

$$= \frac{1}{2}\cdot \frac{1}{-\frac{1}{2}+2} = \frac{1}{2\cdot\frac{3}{2}} = \frac{1}{3}$$

Now by Cauchy's residue theorem we have from (1) [এখন কচির অবশেষ উপপাদ্য দ্বারা পাই]

$$\int_0^{2\pi} \frac{d\theta}{5+4\cos\theta} = \frac{1}{i}\cdot 2\pi i\left(\text{Residue at } z = -\frac{1}{2}\right)$$

$$= 2\pi\cdot\frac{1}{3} = \frac{2\pi}{3} \quad \text{(Ans)}$$

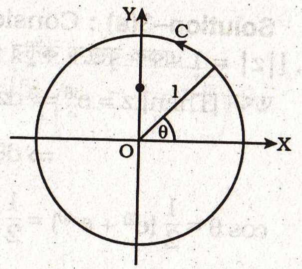
Solution–3. Let us consider the unit circle $|z| = 1$ as the contour C.

Then $z = e^{i\theta} \Rightarrow dz = ie^{i\theta}\,d\theta = iz\,d\theta \Rightarrow d\theta = \frac{1}{iz}\,dz$

where $0 \le \theta \le 2\pi$.

$$\sin\theta = \frac{1}{2i}(e^{i\theta}-e^{-i\theta}) = \frac{1}{2i}\left(z-\frac{1}{z}\right)$$

$$= \frac{z^2-1}{2iz}$$



$$\therefore \int_0^{2\pi} \frac{d\theta}{3+2\sin\theta} = \oint_C \frac{\frac{1}{iz}\,dz}{3+2\cdot\frac{z^2-1}{2iz}}$$

$$= \oint_C \frac{\frac{1}{iz}\,dz}{\frac{3iz+z^2-1}{2iz}} = 2\oint_C \frac{dz}{z^2+3iz-1}$$

$$= 2\oint_C f(z)\,dz, \text{ say } \cdots\cdots (1)$$

where $f(z) = \dfrac{1}{z^2+3iz-1}$

The poles of f(z) are obtained from the equation

$$z^2 + 3iz - 1 = 0$$

$$\Rightarrow z = \frac{-3i \pm \sqrt{9i^2 + 4}}{2} = \frac{-3i \pm \sqrt{-5}}{2} = \frac{(-3 \pm \sqrt{5})i}{2}$$

$$|z| = |(-3 + \sqrt{5})i| = |-3 + \sqrt{5}| < 1$$

and $|z| = |(-3 - \sqrt{5})i| = |-3 - \sqrt{5}| > 1$

$\therefore$ The only pole $z = (-3 + \sqrt{5})i$ lies inside the contour C which is a simple pole.

Residue at $z = (-3 + \sqrt{5})i$ is

$$\lim_{z \to (-3+\sqrt{5})i} \{z - (-3 + \sqrt{5})i\} \cdot \frac{1}{z^2 + 3iz - 1}$$

$$= \lim_{z \to (-3+\sqrt{5})i} \{z - (-3 + \sqrt{5})i\} \cdot \frac{1}{\{z - (-3 + \sqrt{5})i\}\{z - (-3 - \sqrt{5})i\}}$$

$$= \frac{1}{(-3 + \sqrt{5})i - (-3 - \sqrt{5})i} = \frac{1}{(-3 + \sqrt{5} + 3 + \sqrt{5})i} = \frac{1}{2i\sqrt{5}}$$

Therefore, by Cauchy's residue theorem we have from (1)

$$\int_0^{2\pi} \frac{d\theta}{3 + 2\sin\theta} = 2 \cdot 2\pi i(\text{Residue at the pole})$$

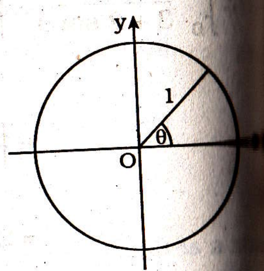
$$= 4\pi i \cdot \frac{1}{2i\sqrt{5}} = \frac{2\pi}{\sqrt{5}} \quad \text{(Ans)}$$

Solution–3(a) : Consider the unit circle $|z| = 1$ as the contour C. [$|z| = 1$ একক বৃত্তকে কন্টুর C হিসাবে বিবেচনা করি]

তখন [Then] $z = e^{i\theta} \Rightarrow dz = ie^{i\theta}\, d\theta = iz\, d\theta$

$$\Rightarrow d\theta = \frac{dz}{iz}, \text{ where [যেখানে] } 0 \le \theta \le 2\pi.$$

$$\cos\theta = \frac{1}{2}(e^{i\theta} + e^{-i\theta}) = \frac{1}{2}\left(z + \frac{1}{z}\right) = \frac{z^2 + 1}{2z}$$



$$\therefore \int_0^{2\pi} \frac{d\theta}{3 + 2\cos\theta} = \oint_C \frac{\frac{dz}{iz}}{3 + 2 \cdot \frac{z^2 + 1}{2z}}$$

$$= \frac{1}{i} \oint_C \frac{dz}{3z + z^2 + 1}$$

$$= \frac{1}{i} \oint_C f(z)\, dz, \text{ say (ধরি) ...... (1)}$$

where [যেখানে] $f(z) = \frac{1}{z^2 + 3z + 1}$

The poles of f(z) can be obtained by solving the equation $z^2 + 3z + 1 = 0$ [f(z) এর পোল $z^2 + 3z + 1 = 0$ সমীকরণ সমাধান করে পাওয়া যাবে]

$$\Rightarrow z = \frac{-3 \pm \sqrt{9 - 4}}{2} = \frac{-3 \pm \sqrt{5}}{2} = \frac{-3 + \sqrt{5}}{2}, \frac{-3 - \sqrt{5}}{2}$$

$$|z| = \left|\frac{-3 + \sqrt{5}}{2}\right| < 1 \text{ and [এবং] } |z| = \left|\frac{-3 - \sqrt{5}}{2}\right| > 1$$

The pole $z = \frac{-3 + \sqrt{5}}{2}$ lies inside the contour which is a simple pole. [$z = \frac{-3 + \sqrt{5}}{2}$ পোলটি কন্টুরের ভিতরে অবস্থিত যাহা একটি সরল পোল]

$z = \frac{-3 + \sqrt{5}}{2}$ [Residue at [এ অবশেষ] $z = \frac{-3 + \sqrt{5}}{2}$]

$$= \lim_{z \to \frac{-3+\sqrt{5}}{2}} \left(z - \frac{-3 + \sqrt{5}}{2}\right) f(z)$$

$$= \lim_{z \to \frac{-3+\sqrt{5}}{2}} \left(z - \frac{-3 + \sqrt{5}}{2}\right) \cdot \frac{1}{z^2 + 3z + 1}$$

$$= \lim_{z \to \frac{-3+\sqrt{5}}{2}} \left(z - \frac{-3 + \sqrt{5}}{2}\right) \cdot \frac{1}{\left(z - \frac{-3 + \sqrt{5}}{2}\right)\left(z - \frac{-3 - \sqrt{5}}{2}\right)}$$

$$= \lim_{z \to \frac{-3+\sqrt{5}}{2}} \frac{1}{\left(z + \frac{3 + \sqrt{5}}{2}\right)}$$

$$= \frac{1}{\frac{-3 + \sqrt{5}}{2} + \frac{3 + \sqrt{5}}{2}} = \frac{1}{\sqrt{5}}$$

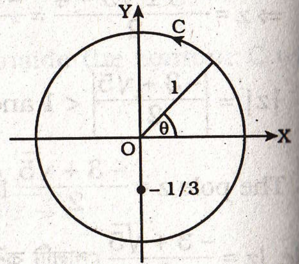
By Cauchy's residue theorem we have form (1) [কচির অবশেষ উপপাদ্য দ্বারা (1) হতে পাই]

$$\int_0^{2\pi} \frac{d\theta}{3 + 2\cos\theta} = \frac{1}{i} \cdot 2\pi i \cdot \left[\frac{1}{\sqrt{5}}\right] = \frac{2\pi}{\sqrt{5}}. \quad \text{(Ans)}$$

Solution–4. Let us consider the unit circle $|z| = 1$ as the contour C. [একক বৃত্ত $|z| = 1$ কে কন্টুর C হিসাবে বিবেচনা করি]

Then [তখন] $z = e^{i\theta} \Rightarrow dz = ie^{i\theta}\, d\theta = iz\, d\theta \Rightarrow d\theta = \dfrac{1}{iz} dz$

where [যেখানে] $0 \le \theta \le 2\pi$.



$$\sin\theta = \frac{1}{2i}(e^{i\theta} - e^{-i\theta}) = \frac{1}{2i}\left(z - \frac{1}{z}\right)$$

$$= \frac{z^2 - 1}{2iz}$$

$$\therefore \int_0^{2\pi} \frac{d\theta}{5 + 3\sin\theta} = \oint_C \frac{\frac{1}{iz} dz}{5 + 3 \cdot \frac{z^2 - 1}{2iz}}$$

$$= \oint_C \frac{2}{10iz + 3z^2 - 3} dz$$

$$= 2\oint_C \frac{1}{3z^2 + 10iz - 3} dz$$

$$= 2\oint_C f(z)\, dz, \text{ say } \cdots\cdots (1)$$

where [যেখানে] $f(z) = \dfrac{1}{3z^2 + 10iz - 3}$.

The poles of f(z) are obtained by solving the equation

$3z^2 + 10iz - 3 = 0$

[$3z^2 + 10iz - 3 = 0$ সমীকরণ সমাধান করে f(z) এর পোল পাওয়া যাবে]

$$\Rightarrow z = \frac{-10i \pm \sqrt{100i^2 + 36}}{6}$$

$$= \frac{-10i \pm 8i}{6} = \frac{-i}{3}, -3i$$

$\left|\dfrac{-i}{3}\right| = \dfrac{1}{3} < 1$ and [এবং] $|-3i| = 3 > 1$

Thus the only pole $z = -\dfrac{i}{3}$ lies inside the contour C. [অতএব কন্টুর C এর মধ্যে একমাত্র পোল $z = -\dfrac{i}{3}$.]

Residue at $z = \dfrac{-i}{3}$ is [$z = \dfrac{-i}{3}$ এ অবশেষ] $\lim\limits_{z \to \frac{-i}{3}} \left(z + \dfrac{i}{3}\right) f(z)$

$$= \lim_{z \to \frac{-i}{3}} \left(z + \frac{i}{3}\right) \cdot \frac{1}{3z^2 + 10iz - 3}$$

$$= \lim_{z \to \frac{-i}{3}} \left(z + \frac{i}{3}\right) \cdot \frac{1}{3\left(z + \frac{i}{3}\right)(z + 3i)}$$

$$= \lim_{z \to \frac{-i}{3}} \frac{1}{3(z + 3i)} = \frac{1}{3\left(\frac{-i}{3} + 3i\right)}$$

$$= \frac{1}{-i + 9i} = \frac{1}{8i}$$

By Cauchy's residue theorem we have from (1) [কচির অবশেষ উপপাদ্য দ্বারা (1) হতে পাই]

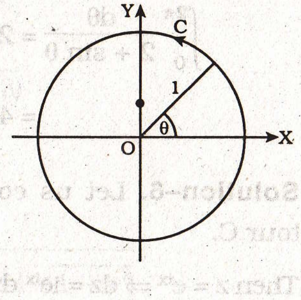
$$\int_0^{2\pi} \frac{d\theta}{5 + 3\sin\theta} = 2 \cdot 2\pi i \left(\text{Residue at } z = \frac{-i}{3}\right)$$

$$= 4\pi i \cdot \frac{1}{8i} = \frac{\pi}{2} \text{ (Ans)}$$

Solution–5. Let us consider the unit circle $|z| = 1$ as the contour C.

Then $z = e^{i\theta} \Rightarrow dz = ie^{i\theta}\, d\theta = iz\, d\theta \Rightarrow d\theta = \dfrac{1}{iz} dz$, where $0 \le \theta \le 2\pi$

$$\sin\theta = \frac{1}{2i}(e^{i\theta} - e^{-i\theta}) = \frac{1}{2i}\left(z - \frac{1}{z}\right) = \frac{z^2 - 1}{2iz}$$



$$\therefore \int_0^{2\pi} \frac{d\theta}{2 + \sin\theta} = \oint_C \frac{\frac{1}{iz} dz}{2 + \frac{z^2 - 1}{2iz}}$$

$$= 2\oint_C \frac{dz}{4iz + z^2 - 1}$$

$$= 2\oint_C f(z)\, dz, \text{ say } \cdots\cdots (1)$$

where $f(z) = \dfrac{1}{z^2 + 4iz - 1}$

The poles of f(z) are obtained by solving the equation

$$z^2 + 4iz - 1 = 0$$

$$\Rightarrow z = \frac{-4i \pm \sqrt{16i^2 + 4}}{2} = \frac{-4i \pm 2\sqrt{3}i}{2}$$

$$\Rightarrow z = -2i \pm \sqrt{3}i = (-2 + \sqrt{3})i,\ (-2 - \sqrt{3})i$$

$$|(-2 + \sqrt{3})i| = |-2 + \sqrt{3}| < 1$$

and $|(-2 - \sqrt{3})i| = |2 + \sqrt{3}| > 1$

$\therefore$ The only pole $z = (-2 + \sqrt{3})i$ lies inside the contour C which is a simple pole.

Residue at $z = (-2 + \sqrt{3})i$ is

$$\lim_{z \to (-2+\sqrt{3})i} \{z - (-2 + \sqrt{3})i\} \cdot f(z)$$

$$= \lim_{z \to (-2+\sqrt{3})i} \{z - (-2 + \sqrt{3})i\} \cdot \frac{1}{z^2 + 4iz - 1}$$

$$= \lim_{z \to (-2+\sqrt{3})i} \{z - (-2 + \sqrt{3})i\} \cdot \frac{1}{\{z - (-2 + \sqrt{3})i\}\{z - (-2 - \sqrt{3})i\}}$$

$$= \lim_{z \to (-2+\sqrt{3})i} \frac{1}{z - (-2 - \sqrt{3})i}$$

$$= \frac{1}{(-2 + \sqrt{3})i - (-2 - \sqrt{3})i} = \frac{1}{(-2 + \sqrt{3} + 2 + \sqrt{3})i} = \frac{1}{2\sqrt{3}i}$$

By Cauchy's residue theorem we have from (1)

$$\int_0^{2\pi} \frac{d\theta}{2 + \sin\theta} = 2 \cdot 2\pi i \{\text{Residue at } z = (-2 + \sqrt{3})i\}$$

$$= 4\pi i \cdot \frac{1}{2\sqrt{3}i} = \frac{2\pi}{\sqrt{3}} \quad \text{(Ans)}$$

Solution–6. Let us consider the unit circle $|z| = 1$ as the contour C.

Then $z = e^{ix} \Rightarrow dz = ie^{ix}\,dx = iz\,dx \Rightarrow dx = \frac{1}{iz}dz$, where $0 \le x \le 2\pi$

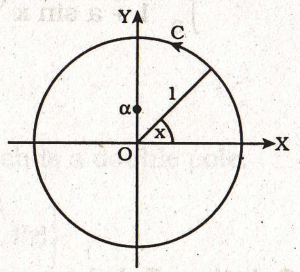
$$\sin x = \frac{1}{2i}(e^{ix} + e^{-ix}) = \frac{1}{2i}\left(z - \frac{1}{z}\right) = \frac{z^2 - 1}{2iz}$$

$$\therefore \int_0^{2\pi} \frac{1}{1 + a\sin x}dx$$

$$= \oint_C \frac{\frac{1}{iz}dz}{1 + a \cdot \frac{z^2 - 1}{2iz}}$$

$$= 2\oint_C \frac{dz}{2iz + az^2 - a}$$

$$= 2\oint_C f(z)\,dz, \text{ say} \cdots\cdots (1)$$



where $f(z) = \frac{1}{az^2 + 2iz - a}$

The poles of f(z) are obtained from the equation

$$az^2 + 2iz - a = 0$$

$$\Rightarrow z = \frac{-2i \pm \sqrt{4i^2 + 4a^2}}{2a} = \frac{-2i \pm \sqrt{4a^2 - 4}}{2a}$$

$$= \frac{-2i \pm 2\sqrt{a^2 - 1}}{2a} = \frac{-i \pm \sqrt{a^2 - 1}}{a} = \frac{-i \pm i\sqrt{1 - a^2}}{a}, \quad \because a < 1$$

Let $\alpha = \frac{-i + i\sqrt{1 - a^2}}{a}$ and $\beta = \frac{-i - i\sqrt{1 - a^2}}{a}$

Since $0 < a < 1$, so only the simple pole $z = \alpha$ lies inside the contour.

Residue at $z = \alpha$ is $\lim_{z \to \alpha} (z - \alpha) \cdot f(z)$

$$= \lim_{z \to \alpha} (z - \alpha) \cdot \frac{1}{az^2 + 2iz - a}$$

$$= \lim_{z \to \alpha} (z - \alpha) \cdot \frac{1}{a(z - \alpha)(z - \beta)}$$

$$= \lim_{z \to \alpha} \frac{1}{a(z - \beta)} = \frac{1}{a(\alpha - \beta)}$$

$$= \frac{1}{a \cdot \frac{1}{a}(-i + i\sqrt{1 - a^2} + i + i\sqrt{1 - a^2})}$$

$$= \frac{1}{2i\sqrt{1 - a^2}}$$

∴ By Cauchy's residue theorem we have from (1)

$$\int_0^{2\pi}\frac{1}{1+a\sin x}\,dx = 2\oint_C f(z)\,dz$$

$$= 2\cdot 2\pi i.\ (\text{Residue at } z=\alpha)$$

$$= 4\pi i\cdot\frac{1}{2i\sqrt{1-a^2}}$$

$$= \frac{2\pi}{\sqrt{1-a^2}}\quad (\text{Ans})$$

Solution–7. Let us consider the unit circle $|z| = 1$ as the contour C.

Then $z = e^{i\theta} \Rightarrow dz = ie^{i\theta}\,d\theta = iz\,d\theta \Rightarrow d\theta = \frac{1}{iz}\,dz$, where $0 \le \theta \le 2\pi$

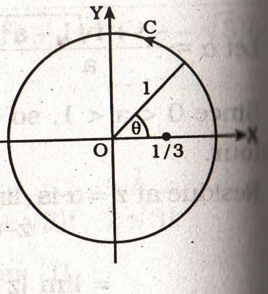
$$\cos\theta = \frac{1}{2}(e^{i\theta}+e^{-i\theta}) = \frac{1}{2}\left(z+\frac{1}{z}\right) = \frac{z^2+1}{2z}$$

$$\therefore \int_0^{2\pi}\frac{d\theta}{(5-3\cos\theta)^2} = \oint_C \frac{\frac{1}{iz}\,dz}{\left(5-3\cdot\frac{z^2+1}{2z}\right)^2}$$

$$= \frac{1}{i}\oint_C \frac{\frac{1}{z}\,dz}{\left(\frac{10z-3z^2-3}{2z}\right)^2}$$

$$= \frac{4}{i}\oint_C \frac{z\,dz}{(-3+10z-3z^2)^2}$$

$$= \frac{4}{i}\oint_C f(z)\,dz, \text{ say } \cdots\cdots (1)$$



where $f(z) = \dfrac{z}{(-3+10z-3z^2)^2} = \dfrac{z}{(3z^2-10z+3)^2}$

The poles of f(z) are obtained by solving the equation

$$(3z^2-10z+3)^2 = 0$$

$$\Rightarrow 3z^2-10z+3=0$$

$$\Rightarrow 3z^2-z-9z+3=0$$

$$\Rightarrow 3z\left(z-\frac{1}{3}\right)-9\left(z-\frac{1}{3}\right)=0$$

$$\Rightarrow \left(z-\frac{1}{3}\right)(3z-9)=0$$

$$\Rightarrow 3\left(z-\frac{1}{3}\right)(z-3)=0$$

$$\Rightarrow z=\frac{1}{3},3$$

$$\left|\frac{1}{3}\right| = \frac{1}{3}<1 \text{ and } |3| = 3>1.$$

$\therefore z=\frac{1}{3}$ lies inside the contour C which is a double pole.

Residue at $z=\frac{1}{3}$ is $\displaystyle\lim_{z\to 1/3}\frac{d}{dz}\left\{\left(z-\frac{1}{3}\right)^2 f(z)\right\}$

$$= \lim_{z\to 1/3}\frac{d}{dz}\left[\left(z-\frac{1}{3}\right)^2\cdot\frac{z}{\left\{3\left(z-\frac{1}{3}\right)(z-3)\right\}^2}\right]$$

$$= \lim_{z\to 1/3}\frac{d}{dz}\left\{\frac{z}{9(z-3)^2}\right\}$$

$$= \frac{1}{9}\cdot\lim_{z\to 1/3}\frac{(z-3)^2\cdot 1 - z\cdot 2(z-3)}{(z-3)^4}$$

$$= \frac{1}{9}\lim_{z\to 1/3}\frac{z-3-2z}{(z-3)^3} = \frac{1}{9}\cdot\frac{\frac{1}{3}-3-\frac{2}{3}}{\left(\frac{1}{3}-3\right)^3}$$

$$= \frac{1}{9}\cdot\frac{-10/3}{-512/27} = \frac{10}{512} = \frac{5}{256}$$

Now by Cauchy's residue theorem we have from (1)

$$\int_0^{2\pi}\frac{d\theta}{(5-3\cos\theta)^2} = \frac{4}{i}\cdot 2\pi i\cdot\left(\text{Residue at } z=\frac{1}{3}\right)$$

$$= 8\pi\cdot\frac{5}{256} = \frac{5\pi}{32}\quad (\text{Ans})$$

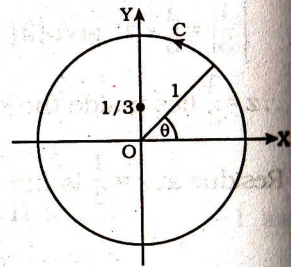
Solution–8. Let us consider the unit circle $|z| = 1$ as the contour C. [একক বৃত্ত $|z| = 1$ কে কন্টুর C হিসাবে বিবেচনা করি ।]

Then [তখন] $z = e^{i\theta} \Rightarrow dz = ie^{i\theta}\,d\theta = iz\,d\theta \Rightarrow d\theta = \frac{1}{iz}\,dz$, where [যেখানে] $0 \le \theta \le 2\pi$

$$\sin\theta = \frac{1}{2i}(e^{i\theta}-e^{-i\theta}) = \frac{1}{2i}\left(z-\frac{1}{z}\right) = \frac{z^2-1}{2iz}$$

$$\therefore \int_0^{2\pi}\frac{d\theta}{(5-3\sin\theta)^2}=\oint_C \frac{\frac{1}{iz}\,dz}{\left(5-3\cdot\frac{z^2-1}{2iz}\right)^2}$$

$$=\frac{1}{i}\oint_C \frac{\frac{1}{z}\,dz}{\left(\frac{10iz-3z^2+3}{i2z}\right)^2}$$



$$=\frac{-4}{i}\oint_C\frac{z\,dz}{(3z^2-10iz-3)^2}$$

$$=\frac{-4}{i}\oint_C f(z)\,dz,\ \text{say} \cdots\cdots (1)$$

where [যেখানে] $f(z)=\dfrac{z}{(3z^2-10iz-3)^2}$

The poles of f(z) are obtained from the equation

$(3z^2-10iz-3)^2=0$

[$(3z^2-10iz-3)^2=0$ সমীকরণ সমাধান করে f(z) এর পোল পাব]

$\Rightarrow 3z^2-10iz-3=0$

$$\Rightarrow z=\frac{10i\pm\sqrt{100i^2+36}}{6}=\frac{10i\pm 8i}{6}=3i,\ \frac{i}{3}$$

$|3i|=3>1$ and $\left|\frac{i}{3}\right|=\frac{1}{3}<1$

Only the pole $z=\frac{i}{3}$ lies inside the contour C which is a double pole (pole of order 2). [শুধু $z=\frac{i}{3}$ পোলটি কন্টুর C এর ভিতর অবস্থিত যাহা একটি দ্বিপোল]

$\therefore$ Residue at $z=\frac{i}{3}$ is [$z=\frac{i}{3}$ অবশেষ] $\lim\limits_{z\to i/3}\cdot\dfrac{1}{1!}\dfrac{d}{dz}\left\{\left(z-\frac{i}{3}\right)^2 f(z)\right\}$

$$=\lim_{z\to i/3}\frac{d}{dz}\left\{\left(z-\frac{i}{3}\right)^2\cdot\frac{z}{(3z^2-10iz-3)^2}\right\}$$

$$=\lim_{z\to i/3}\frac{d}{dz}\left\{\left(z-\frac{i}{3}\right)^2\cdot\frac{z}{9\left(z-\frac{i}{3}\right)^2(z-3i)^2}\right\}$$

$$=\lim_{z\to i/3}\frac{d}{dz}\left\{\frac{z}{9(z-3i)^2}\right\}$$

$$=\frac{1}{9}\lim_{z\to i/3}\frac{(z-3i)^2\cdot 1-z\cdot 2(z-3i)}{(z-3i)^4}$$

$$=\frac{1}{9}\lim_{z\to i/3}\frac{z-3i-2z}{(z-3i)^3}$$

$$=\frac{1}{9}\lim_{z\to i/3}\frac{-z-3i}{(z-3i)^3}$$

$$=\frac{1}{9}\frac{\frac{i}{3}-3i-\frac{2i}{3}}{\left(\frac{i}{3}-3i\right)^3}=\frac{-10i/3}{9\cdot\left(\frac{-8i}{3}\right)^3}=\frac{-10i}{-512i^3}=\frac{-5}{256}$$

By Cauchy's residue theorem we have from (1) [কচির অবশেষ উপপাদ্যে দ্বারা (1) হতে পাই]

$$\int_0^{2\pi}\frac{d\theta}{(5-3\sin\theta)^2}=\frac{-4}{i}\cdot 2\pi i.\left(\text{Residue at } z=\frac{i}{3}\right)$$

$$=-8\pi\left(\frac{-5}{256}\right)=\frac{5\pi}{32}\quad\text{(Ans)}$$

Solution–9. Let us consider the unit circle $|z|=1$ as the contour C.

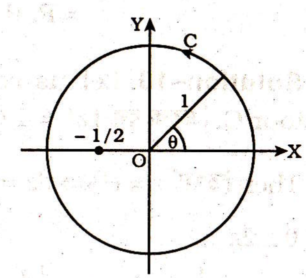
Then $z=e^{i\theta}\Rightarrow dz=ie^{i\theta}\,d\theta=iz\,d\theta\Rightarrow d\theta=\frac{1}{iz}dz$, where $0\le\theta\le 2\pi$

$$\cos\theta=\frac{1}{2}(e^{i\theta}+e^{-i\theta})=\frac{1}{2}\left(z+\frac{1}{z}\right)=\frac{z^2+1}{2z}$$

$$\therefore\int_0^{2\pi}\frac{\cos 2\theta\,d\theta}{5+4\cos\theta}=\text{Real part of}\int_0^{2\pi}\frac{\cos 2\theta+i\sin 2\theta}{5+4\cos\theta}d\theta$$

$=\text{R. P.}\displaystyle\int_0^{2\pi}\frac{e^{i2\theta}}{5+4\cos\theta}d\theta$, where R. P. means real part of

$$=\text{R. P.}\oint_C\frac{z^2}{5+4\frac{z^2+1}{2z}}\cdot\frac{1}{iz}dz$$



$$=\text{R. P.}\frac{1}{i}\oint_C\frac{z^2}{5z+2z^2+2}dz$$

$$=\text{R. P.}\frac{1}{i}\oint_C f(z)\,dz,\ \text{say}\cdots\cdots(1)$$

where $f(z)=\dfrac{z^2}{2z^2+5z+2}$

The poles of f(z) are obtained by solving the equation

$$2z^2+5z+2=0$$

$$\Rightarrow z=\frac{-5\pm\sqrt{25-16}}{4}=\frac{-5\pm3}{4}=\frac{-1}{2},-2$$

$$|-2|=2>1 \text{ and } \left|-\frac{1}{2}\right|=\frac{1}{2}<1$$

The only pole $z=-\frac{1}{2}$ lies inside the contour C which is a simple pole.

Residue at $z=-\frac{1}{2}$ is $\lim\limits_{z\to-1/2}\left(z+\frac{1}{2}\right)\cdot f(z)$

$$=\lim_{z\to-1/2}\left(z+\frac{1}{2}\right)\cdot\frac{z^2}{2z^2+5z+2}$$

$$=\lim_{z\to-1/2}\left(z+\frac{1}{2}\right)\cdot\frac{z^2}{2\left(z+\frac{1}{2}\right)(z+2)}$$

$$=\lim_{z\to-1/2}\frac{z^2}{2(z+2)}=\frac{\left(-\frac{1}{2}\right)^2}{2\left(-\frac{1}{2}+2\right)}$$

$$=\frac{1}{8\times\frac{3}{2}}=\frac{1}{12}$$

Therefore, by Cauchy's residue theorem we have from (1)

$$\int_0^{2\pi}\frac{\cos 2\theta\, d\theta}{5+4\cos\theta}=\text{R. P. }\frac{1}{i}\cdot 2\pi i\left[\text{Residue at } z=-\frac{1}{2}\right]$$

$$=\text{R. P.}\left(2\pi\times\frac{1}{12}\right)=\frac{\pi}{6} \text{ (Ans)}$$

Solution–10. Let us consider the unit circle $|z|=1$ as the contour C. [একক বৃত্ত $|z|=1$ কে কন্টুর C হিসাবে বিবেচনা করি ।]

Then [তখন] $z=e^{i\theta}\Rightarrow dz=ie^{i\theta}\,d\theta=iz\,d\theta\Rightarrow d\theta=\frac{1}{iz}dz$, where [যেখানে]

$0\le\theta\le2\pi$

$$\cos\theta=\frac{1}{2}(e^{i\theta}+e^{-i\theta})=\frac{1}{2}\left(z+\frac{1}{z}\right)=\frac{z^2+1}{2z}$$

$$\therefore\int_0^{2\pi}\frac{\cos 3\theta}{5-4\cos\theta}d\theta=\text{Real part of}\int_0^{2\pi}\frac{\cos3\theta+i\sin3\theta}{5-4\cos\theta}d\theta$$

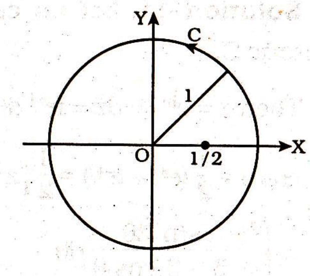
$=\text{R. P.}\int_0^{2\pi}\frac{e^{i3\theta}\,d\theta}{5-4\cos\theta}$, where R. P. means real part of [যেখানে R. P. অর্থ বাস্তব অংশ]

$$=\text{R. P.}\oint_C\frac{z^3}{5-4\cdot\frac{z^2+1}{2z}}\cdot\frac{1}{iz}dz$$

$$=\text{R. P.}\frac{1}{i}\oint_C\frac{z^3\,dz}{5z-2z^2-2}$$

$$=\text{R. P.}\frac{-1}{i}\oint_C\frac{z^3}{2z^2-5z+2}dz$$

$$=\text{R. P.}\frac{-1}{i}\oint_C f(z)\,dz,\ \text{say} \cdots\cdots (1)$$



where [যেখানে] $f(z)=\frac{z^3}{2z^2-5z+2}$

The poles of f(z) are obtained by solving the equation

$$2z^2-5z+2=0$$

[$2z^2-5z+2=0$ সমীকরণ সমাধান করে f(z) এর পোল পাওয়া যাবে]

$$\Rightarrow z=\frac{5\pm\sqrt{25-16}}{4}=\frac{5\pm3}{4}=2,\frac{1}{2}$$

Only the pole $z=\frac{1}{2}$ lie inside the contour C which is a simple pole. [শুধুমাত্র $z=\frac{1}{2}$ পোলটি কন্টুর C এর ভিতর অবস্থিত যাহা একটি সরল পোল]

Residue at $z=\frac{1}{2}$ is [$z=\frac{1}{2}$ এ অবশেষ] $\lim\limits_{z\to1/2}\left(z-\frac{1}{2}\right)\cdot f(z)$

$$=\lim_{z\to1/2}\left(z-\frac{1}{2}\right)\cdot\frac{z^3}{2z^2-5z+2}$$

$$=\lim_{z\to1/2}\left(z-\frac{1}{2}\right)\cdot\frac{z^3}{2\left(z-\frac{1}{2}\right)(z-2)}$$

$$=\lim_{z\to1/2}\frac{z^3}{2(z-2)}=\frac{(1/2)^3}{2\left(\frac{1}{2}-2\right)}$$

$$=\frac{1}{8\times2\times\frac{-3}{2}}=\frac{-1}{24}$$

By Cauchy's residue theorem we have from (1) [কচির অবশে
উপপাদ্য দ্বারা (1) হতে পাই]

$$\int_0^{2\pi} \frac{\cos 3\theta}{5 - 4\cos\theta}\, d\theta = \text{R. P.} \frac{-1}{i} \cdot 2\pi i \left[\text{Residue at } z = \frac{1}{2}\right]$$

$$= \text{R. P.} (-2\pi)\left(\frac{-1}{24}\right) = \frac{\pi}{12} \quad \text{(Ans)}$$

Solution–11. Let us consider the unit circle $|z| = 1$ as the contour C.

Then $z = e^{i\theta} \Rightarrow dz = ie^{i\theta}\, d\theta = iz\, d\theta \Rightarrow d\theta = \frac{1}{iz} dz$, where $0 \le \theta \le 2\pi$.

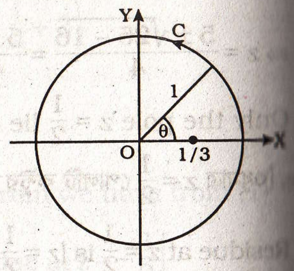
$$\cos\theta = \frac{1}{2}(e^{i\theta} + e^{-i\theta}) = \frac{1}{2}\left(z + \frac{1}{z}\right) = \frac{z^2+1}{2z}$$

$$\therefore \int_0^{2\pi} \frac{\sin 2\theta}{5 - 3\cos\theta}\, d\theta$$

$$= \text{Imaginary part of } \int_0^{2\pi} \frac{\cos 2\theta + i \sin 2\theta}{5 - 3\cos\theta}\, d\theta$$

$$= \text{I. P.} \int_0^{2\pi} \frac{e^{i2\theta}}{5 - 3\cos\theta}\, d\theta, \text{ where I.P. stands for imaginary part}$$

$$= \text{I. P.} \oint_C \frac{z^2}{5 - 3 \cdot \frac{z^2+1}{2z}} \cdot \frac{1}{iz}\, dz$$



$$= \text{I. P.} \frac{2}{i} \oint_C \frac{z^2}{10z - 3z^2 - 3}\, dz$$

$$= \text{I. P.} \frac{-2}{i} \oint_C \frac{z^2}{3z^2 - 10z + 3}\, dz$$

$$= \text{I. P.} \frac{-2}{i} \oint_C f(z)\, dz \cdots\cdots (1)$$

where $f(z) = \frac{z^2}{3z^2 - 10z + 3}$

The poles of f(z) are obtained by solving the equation

$3z^2 - 10z + 3 = 0$

$\Rightarrow 3z^2 - 9z - z + 3 = 0$

$\Rightarrow 3z(z-3) - 1(z-3) = 0$

$\Rightarrow (3z-1)(z-3) = 0$

$\Rightarrow z = \frac{1}{3}, 3$

The only pole $z = \frac{1}{3}$ lies inside the contour C which is a simple pole.

Residue at $z = \frac{1}{3}$ is $\lim\limits_{z \to 1/3} \left(z - \frac{1}{3}\right) \cdot f(z)$

$$= \lim_{z \to 1/3} \left(z - \frac{1}{3}\right) \cdot \frac{z^2}{3z^2 - 10z + 3}$$

$$= \lim_{z \to 1/3} \left(z - \frac{1}{3}\right) \cdot \frac{z^2}{3\left(z - \frac{1}{3}\right)(z-3)}$$

$$= \lim_{z \to 1/3} \frac{z^2}{3(z-3)} = \frac{(1/3)^2}{3\left(\frac{1}{3} - 3\right)} = \frac{-1}{72}$$

By Cauchy's residue theorem we have from (1)

$$\int_0^{2\pi} \frac{\sin 2\theta}{5 - 3\cos\theta}\, d\theta = \text{I. P.} \frac{-2}{i} \cdot 2\pi i \left[\text{Residue at } z = \frac{1}{3}\right]$$

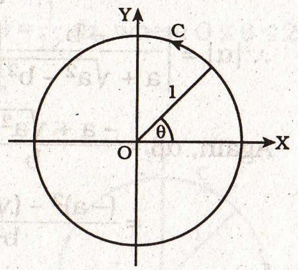
$$= \text{I. P.} \left(-4\pi \times \frac{-1}{72}\right) = 0 \quad \text{(Ans)}$$

Solution–12. Let us consider the unit circle $|z| = 1$ as the contour C.

Then $z = e^{i\theta} \Rightarrow dz = ie^{i\theta}\, d\theta = iz\, d\theta \Rightarrow d\theta = \frac{dz}{iz}$, where $0 \le \theta \le 2\pi$.

$$\cos\theta = \frac{1}{2}(e^{i\theta} + e^{-i\theta}) = \frac{1}{2}\left(z + \frac{1}{z}\right) = \frac{z^2+1}{2z}$$

$$\therefore \int_0^{2\pi} \frac{d\theta}{a + b\cos\theta} = \oint_C \frac{\frac{dz}{iz}}{a + b \cdot \frac{z^2+1}{2z}}$$



$$= \frac{2}{i} \oint_C \frac{1}{2az + bz^2 + b}\, dz$$

$$= \frac{2}{i} \oint_C f(z)\, dz, \text{ say} \cdots\cdots (1)$$

where $f(z) = \frac{1}{bz^2 + 2az + b}$

The poles of f(z) are obtained by solving the equation

$bz^2 + 2az + b = 0 \cdots\cdots (2)$

$$\Rightarrow z = \frac{-2a \pm \sqrt{4a^2 - 4b^2}}{2b} = \frac{-2a \pm 2\sqrt{a^2 - b^2}}{2b}$$

$$\Rightarrow z = \frac{-a \pm \sqrt{a^2 - b^2}}{b}$$

Let $z = \alpha = \dfrac{-a + \sqrt{a^2 - b^2}}{b}$ and $z = \beta = \dfrac{-a - \sqrt{a^2 - b^2}}{b}$

Here $\alpha = \dfrac{-a + \sqrt{a^2 - b^2}}{b} = \dfrac{-a + \sqrt{a^2 - b^2}}{b} \times \dfrac{-a - \sqrt{a^2 - b^2}}{-a - \sqrt{a^2 - b^2}}$

$$= \frac{(-a)^2 - (\sqrt{a^2 - b^2})^2}{-b(a + \sqrt{a^2 - b^2})} = \frac{a^2 - a^2 + b^2}{-b(a + \sqrt{a^2 - b^2})}$$

$$= \frac{b^2}{-b(a + \sqrt{a^2 - b^2})}$$

$$= \frac{-b}{a + \sqrt{a^2 - b^2}}$$

Given that $a > |b| \Rightarrow a^2 > b^2$

$\Rightarrow (a^2 - b^2) > 0$

$\Rightarrow \sqrt{a^2 - b^2} > 0$

$\Rightarrow a + \sqrt{a^2 - b^2} > a > |b|$; adding a both sides

$\Rightarrow \dfrac{1}{a + \sqrt{a^2 - b^2}} < \dfrac{1}{|b|}$

$\Rightarrow \dfrac{|b|}{a + \sqrt{a^2 - b^2}} < 1$

$$\therefore |\alpha| = \left|\frac{-b}{a + \sqrt{a^2 - b^2}}\right| = \frac{|b|}{a + \sqrt{a^2 - b^2}} < 1$$

Again, $\alpha\beta = \dfrac{-a + \sqrt{a^2 - b^2}}{b} \cdot \dfrac{-a - \sqrt{a^2 - b^2}}{b}$

$$= \frac{(-a)^2 - (\sqrt{a^2 - b^2})^2}{b^2} = \frac{a^2 - a^2 + b^2}{b^2} = \frac{b^2}{b^2} = 1$$

[Or, from (2) product of the roots $\alpha\beta = \dfrac{b}{b} = 1$]

$$\Rightarrow \beta = \frac{1}{\alpha} \Rightarrow |\beta| = \frac{1}{|\alpha|} > 1, \text{ since } |\alpha| < 1 \Rightarrow \frac{1}{|\alpha|} > 1$$

Thus only the pole $z = \alpha$ lies inside the contour C which is a simple pole.

Residue at $z = \alpha$ is $\lim\limits_{z \to \alpha} (z - \alpha) \cdot f(z)$

$$= \lim_{z \to \alpha} (z - \alpha) \cdot \frac{1}{bz^2 + 2az + b}$$

$$= \lim_{z \to \alpha} (z - \alpha) \cdot \frac{1}{b(z - \alpha)(z - \beta)}$$

$$= \lim_{z \to \alpha} \frac{1}{b(z - \beta)} = \frac{1}{b(\alpha - \beta)}$$

$$= \frac{1}{b} \cdot \frac{1}{\dfrac{-a + \sqrt{a^2 - b^2}}{b} - \dfrac{-a - \sqrt{a^2 - b^2}}{b}}$$

$$= \frac{1}{-a + \sqrt{a^2 - b^2} + a + \sqrt{a^2 - b^2}} = \frac{1}{2\sqrt{a^2 - b^2}}$$

By Cauchy's residue theorem we have from (1)

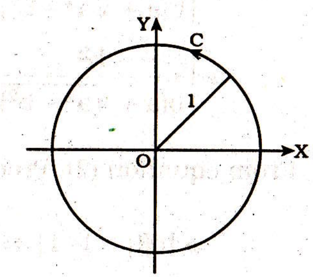
$$\int_0^{2\pi} \frac{d\theta}{a + b\cos\theta} = \frac{2}{i} \cdot 2\pi i \text{ [Residue at } z = \alpha]$$

$$= 4\pi \cdot \frac{1}{2\sqrt{a^2 - b^2}} = \frac{2\pi}{\sqrt{a^2 - b^2}} \quad \text{(Ans)}$$

Solution–13. Let us consider the unit circle $|z| = 1$ as the contour C.

Then $z = e^{i\theta} \Rightarrow dz = ie^{i\theta}\, d\theta = iz\, d\theta \Rightarrow d\theta = \dfrac{1}{iz} dz$, where $0 \le \theta \le 2\pi$

$$\sin\theta = \frac{1}{2i}(e^{i\theta} - e^{-i\theta}) = \frac{1}{2i}\left(z - \frac{1}{z}\right) = \frac{z^2 - 1}{2iz}$$



$$\therefore \int_0^{2\pi} \frac{d\theta}{a + b\sin\theta} = \oint_C \frac{\frac{1}{iz}dz}{a + b \cdot \frac{z^2 - 1}{2iz}}$$

$$= 2\oint_C \frac{dz}{2iaz + bz^2 - b}$$

$$= 2\oint_C f(z)\, dz, \text{ say} \;\cdots\cdots (1)$$

where $f(z) = \dfrac{1}{bz^2 + 2iaz - b}$

The poles of f(z) are obtained by solving the equation

$bz^2 + 2iaz - b = 0$ ...... (2)

$$\Rightarrow z = \frac{-2ia \pm \sqrt{4a^2 i^2 + 4b^2}}{2b} = \frac{-2ia \pm 2\sqrt{b^2 - a^2}}{2b}$$

$$= \frac{-ia \pm i\sqrt{a^2 - b^2}}{b}$$

Let $z = \alpha = \dfrac{i(-a + \sqrt{a^2 - b^2})}{b}$ and $z = \beta = \dfrac{i(-a - \sqrt{a^2 - b^2})}{b}$

Given that $a > |b| \Rightarrow a^2 > b^2$

$$\Rightarrow a^2 - b^2 > 0$$

$$\Rightarrow \sqrt{a^2 - b^2} > 0$$

$$\Rightarrow a + \sqrt{a^2 - b^2} > a > |b|$$

$$\Rightarrow \frac{1}{a + \sqrt{a^2 - b^2}} < \frac{1}{|b|}$$

$$\Rightarrow \frac{|b|}{a + \sqrt{a^2 - b^2}} < 1$$

$$\therefore |\alpha| = \left|\frac{i(-a + \sqrt{a^2 - b^2})}{b}\right| = \left|\frac{-a + \sqrt{a^2 - b^2}}{b}\right| \quad \because |i| = 1$$

$$= \left|\frac{-a + \sqrt{a^2 - b^2}}{b} \times \frac{-a - \sqrt{a^2 - b^2}}{-(a + \sqrt{a^2 - b^2})}\right|$$

$$= \left|\frac{a^2 - (a^2 - b^2)}{b(a + \sqrt{a^2 - b^2})}\right| = \left|\frac{a^2 - a^2 + b^2}{b(a + \sqrt{a^2 - b^2})}\right|$$

$$= \left|\frac{b^2}{b(a + \sqrt{a^2 - b^2})}\right| = \frac{|b|}{a + \sqrt{a^2 - b^2}} < 1$$

From equation (2), Product of the roots $\alpha\beta = \dfrac{-b}{b} = -1$

$$\Rightarrow |\alpha\beta| = |-1| \Rightarrow |\beta| = \frac{1}{|\alpha|} > 1 \quad \because |\alpha| < 1 \Rightarrow \frac{1}{|\alpha|} > 1$$

Thus the only pole $z = \alpha$ lies inside the contour C which is a simple pole.

Residue at $z = \alpha$ is $\lim\limits_{z \to \alpha} (z - \alpha) \cdot f(z)$

$$= \lim_{z \to \alpha} (z - \alpha) \cdot \frac{1}{bz^2 + 2iaz - b}$$

$$= \lim_{z \to \alpha} (z - \alpha) \cdot \frac{1}{b(z - \alpha)(z - \beta)}$$

$$= \frac{1}{b} \cdot \lim_{z \to \alpha} \frac{1}{z - \beta} = \frac{1}{b} \cdot \frac{1}{\alpha - \beta}$$

$$= \frac{1}{b} \cdot \frac{1}{\frac{i}{b}(-a + \sqrt{a^2 - b^2} + a + \sqrt{a^2 - b^2})}$$

$$= \frac{1}{2i\sqrt{a^2 - b^2}}$$

Therefore, by Cauchy's residue theorem we have from (1)

$$\int_0^{2\pi} \frac{d\theta}{a + b\sin\theta} = 2 \cdot 2\pi i \cdot (\text{Residue at } z = \alpha)$$

$$= 4\pi i \cdot \frac{1}{2i\sqrt{a^2 - b^2}}$$

$$= \frac{2\pi}{\sqrt{a^2 - b^2}} \quad \text{(Ans)}$$

Solution–14. (i) & (ii) : Let us consider the unit circle $|z| = 1$ as the contour C.

Then $z = e^{i\theta} \Rightarrow dz = ie^{i\theta}\, d\theta = iz\, d\theta \Rightarrow d\theta = \dfrac{1}{iz}\, dz$, where $0 \le \theta \le 2\pi$

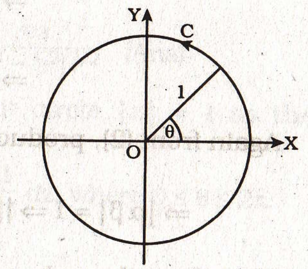
$$\cos\theta = \frac{1}{2}(e^{i\theta} + e^{-i\theta}) = \frac{1}{2}\left(z + \frac{1}{z}\right) = \frac{z^2 + 1}{2z}$$

$$\therefore \int_0^{2\pi} \frac{d\theta}{(a + b\cos\theta)^2}\, d\theta$$

$$= \oint_C \frac{\frac{1}{iz}\, dz}{\left(a + b \cdot \frac{z^2 + 1}{2z}\right)^2}$$

$$= \frac{4}{i} \oint_C \frac{z\, dz}{(2az + bz^2 + b)^2}$$

$$= \frac{4}{i} \oint_C f(z)\, dz, \text{ say ...... (1)}$$



where $f(z) = \dfrac{z}{(bz^2 + 2az + b)^2}$

The poles of f(z) are obtained by solving the equation

$(bz^2 + 2az + b)^2 = 0$

$\Rightarrow bz^2 + 2az + b = 0$ ...... (2)

$$\Rightarrow z = \frac{-2a \pm \sqrt{4a^2 - 4b^2}}{2b} = \frac{-a \pm \sqrt{a^2 - b^2}}{b}$$

$$z = \alpha = \frac{-a + \sqrt{a^2 - b^2}}{b} \text{ and } z = \beta = \frac{-a - \sqrt{a^2 - b^2}}{b}$$

$$|z| = |\alpha| = \left|\frac{-a + \sqrt{a^2 - b^2}}{b}\right|$$

$$= \left|\frac{-a + \sqrt{a^2 - b^2}}{b} \times \frac{-a - \sqrt{a^2 - b^2}}{-a - \sqrt{a^2 - b^2}}\right|$$

$$= \left|\frac{a^2 - (a^2 - b^2)}{-b(a + \sqrt{a^2 - b^2})}\right| = \left|\frac{b}{a + \sqrt{a^2 - b^2}}\right| \quad \text{...... (3)}$$

Given that $a > b > 0 \Rightarrow a^2 > b^2$

$$\Rightarrow a^2 - b^2 > 0$$

$$\Rightarrow \sqrt{a^2 - b^2} > 0$$

$$\Rightarrow a + \sqrt{a^2 - b^2} > a > b$$

$$\Rightarrow \frac{1}{a + \sqrt{a^2 - b^2}} < \frac{1}{b}$$

$$\Rightarrow \frac{b}{a + \sqrt{a^2 - b^2}} < 1$$

$$\Rightarrow \left|\frac{b}{a + \sqrt{a^2 - b^2}}\right| < 1, \text{ since } a > b > 0$$

$$\Rightarrow |\alpha| < 1, \text{ by (3)}$$

Again from (2), product of the roots $\alpha\beta = \frac{b}{b} = 1$

$$\Rightarrow |\alpha\beta| = 1 \Rightarrow |\beta| = \frac{1}{|\alpha|} > 1, \text{ since } |\alpha| < 1 \Rightarrow \frac{1}{|\alpha|} > 1$$

Therefore, the only pole $z = \alpha$ lies inside the contour C which is a double pole (pole of order 2).

Residue at $z = \alpha$ is $\lim\limits_{z \to \alpha} \frac{1}{1!} \frac{d}{dz} \{(z - \alpha)^2 \cdot f(z)\}$

$$= \lim_{z \to \alpha} \frac{d}{dz} \left\{(z - \alpha)^2 \cdot \frac{z}{(bz^2 + 2az + b)^2}\right\}$$

$$= \lim_{z \to \alpha} \frac{d}{dz} \left\{(z - \alpha)^2 \cdot \frac{z}{b^2(z - \alpha)^2 (z - \beta)^2}\right\}$$

$$= \lim_{z \to \alpha} \frac{d}{dz} \left\{\frac{z}{b^2(z - \beta)^2}\right\}$$

$$= \frac{1}{b^2} \lim_{z \to \alpha} \frac{(z - \beta)^2 \cdot 1 - z \cdot 2(z - \beta)}{(z - \beta)^4}$$

$$= \frac{1}{b^2} \lim_{z \to \alpha} \frac{z - \beta - 2z}{(z - \beta)^3}$$

$$= \frac{1}{b^2} \lim_{z \to \alpha} \frac{-z - \beta}{(z - \beta)^3} = \frac{1}{b^2} \cdot \frac{-\alpha - \beta}{(\alpha - \beta)^3}$$

$$= \frac{-1}{b^2} \cdot \frac{\frac{1}{b}\left(-a + \sqrt{a^2 - b^2} - a - \sqrt{a^2 - b^2}\right)}{\left\{\frac{1}{b}\left(-a + \sqrt{a^2 - b^2} + a + \sqrt{a^2 - b^2}\right)\right\}^3}$$

$$= -\frac{-2a}{(2\sqrt{a^2 - b^2})^3} = \frac{9}{4(\sqrt{a^2 - b^2})^3} = \frac{a}{4(a^2 - b^2)^{3/2}}$$

By Cauchy's residue theorem we have from (1)

$$\int_0^{2\pi} \frac{d\theta}{(a + b\cos\theta)^2} = \frac{4}{i} \cdot 2\pi i \cdot (\text{Residue at } z = \alpha)$$

$$= 8\pi \cdot \frac{a}{4(a^2 - b^2)^{3/2}} = \frac{2\pi a}{(a^2 - b^2)^{3/2}} \quad \text{(Ans)}$$

(ii) $$\int_0^{\pi} \frac{d\theta}{(a + b\cos\theta)^2} = \frac{1}{2} \int_0^{2\pi} \frac{d\theta}{(a + b\cos\theta)^2}$$

$$= \frac{1}{2} \cdot \frac{2\pi a}{(a^2 - b^2)^{3/2}} = \frac{\pi a}{(a^2 - b^2)^{3/2}} \quad \text{(Ans)}$$

Solution–15. Let us consider the unit circle $|z| = 1$ as the contour C.

Then $z = e^{i\theta} \Rightarrow dz = ie^{i\theta}\, d\theta = iz\, d\theta \Rightarrow d\theta = \frac{1}{iz} dz$, where $0 \le \theta \le 2\pi$.

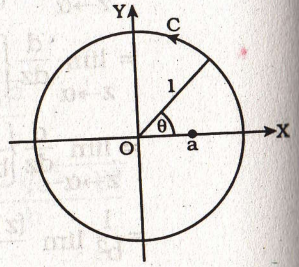
$$\cos\theta = \frac{1}{2}(e^{i\theta} + e^{-i\theta}) = \frac{1}{2}\left(z + \frac{1}{z}\right) = \frac{z^2 + 1}{2z}$$

$$\therefore \int_0^{2\pi} \frac{d\theta}{1+a^2-2a\cos\theta} = \oint_C \frac{\frac{1}{iz}dz}{1+a^2-2a\cdot\frac{z^2+1}{2z}}$$

$$= \frac{1}{i}\oint_C \frac{dz}{(1+a^2)z - az^2 - a}$$

$$= \frac{-1}{i}\oint_C \frac{dz}{az^2-(1+a^2)z+a}$$

$$= \frac{1}{i}\oint_C f(z)\,dz, \text{ say } \cdots\cdots (1)$$



where $f(z) = \dfrac{1}{az^2-(1+a^2)z+a}$

The poles of f(z) will be obtained from the equation

$$az^2-(1+a^2)z+a=0$$

$$\Rightarrow az^2 - z - a^2z + a = 0$$

$$\Rightarrow az\left(z-\frac{1}{a}\right) - a^2\left(z-\frac{1}{a}\right) = 0$$

$$\Rightarrow a\left(z-\frac{1}{a}\right)(z-a) = 0$$

$$\Rightarrow z = \frac{1}{a}, a$$

Since $0 < a < 1$. so $\left|\frac{1}{a}\right| > 1$ and $|a| < 1$.

$\therefore$ The pole $z = a$ lies inside the contour C which is a simple pole.

Residue at $z = a$ is $\lim\limits_{z\to a} (z-a)\cdot f(z)$

$$= \lim_{z\to a}(z-a)\cdot\frac{1}{az^2-(1+a^2)z+a}$$

$$= \lim_{z\to a}(z-a)\cdot\frac{1}{a(z-a)\left(z-\frac{1}{a}\right)}$$

$$= \lim_{z\to a}\frac{1}{a\left(z-\frac{1}{a}\right)} = \frac{1}{a\left(a-\frac{1}{a}\right)} = \frac{1}{a^2-1}$$

By Cauchy's residue theorem we have from (1)

$$\int_0^{2\pi}\frac{d\theta}{1+a^2-2a\cos\theta} = \frac{-1}{i}\cdot 2\pi i\cdot(\text{Residue at } z = a)$$

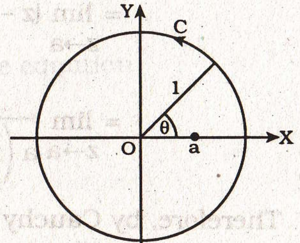
$$= -2\pi\cdot\frac{1}{a^2-1} = \frac{2\pi}{1-a^2} \quad \text{(Ans)}$$

Solution–16. (i) & (ii) : Let us consider the unit circle $|z| = 1$ as the contour C.

Then $z = e^{i\theta} \Rightarrow dz = ie^{i\theta}\,d\theta = iz\,d\theta \Rightarrow d\theta = \frac{1}{iz}dz$, where $0 \le \theta \le 2\pi$

$$\cos\theta = \frac{1}{2}(e^{i\theta}+e^{-i\theta})$$

$$= \frac{1}{2}\left(z+\frac{1}{z}\right) = \frac{z^2+1}{2z}$$



$$\therefore \int_0^{2\pi}\frac{\cos 2\theta}{1-2a\cos\theta+a^2}d\theta$$

$$= \text{Real part of } \int_0^{2\pi}\frac{\cos 2\theta + i\sin 2\theta}{1-2a\cos\theta+a^2}d\theta$$

$$= \text{R. P.} \int_0^{2\pi}\frac{e^{i2\theta}}{1-2a\cos\theta+a^2}d\theta, \text{ where R. P. means real part of}$$

$$= \text{R. P.} \oint_C \frac{z^2}{1-2a\cdot\frac{z^2+1}{2z}+a^2}\cdot\frac{1}{iz}dz$$

$$= \text{R. P.} \frac{1}{i}\oint_C \frac{z^2}{z-az^2-a+a^2z}dz$$

$$= \text{R. P.} \frac{-1}{i}\oint_C f(z)\,dz, \text{ say } \cdots\cdots (1)$$

where $f(z) = \dfrac{z^2}{az^2-z-a^2z+a}$

The poles of f(z) are obtained by solving the equation

$$az^2 - z - a^2z + a = 0$$

$$\Rightarrow z(az-1) - a(az-1) = 0$$

$$\Rightarrow (z-a)(az-1) = 0$$

$$\Rightarrow z = a, \frac{1}{a}$$

Now by Cauchy's residue theorem we have

$$\oint_C f(z)\,dz = 2\pi i \text{ (sum of the residues)}$$

$$\Rightarrow \int_{-R}^{R} f(z)\,dz + \int_\Gamma f(z)\,dz = 2\pi i\left(\frac{9i-12}{100}+\frac{3-4i}{25}\right) \;\ldots\ldots (1)$$

When $R \to \infty$ then $\int_{-R}^{R} f(z)\,dz = \int_{-\infty}^{\infty} f(x)\,dx$

$$= \int_{-\infty}^{\infty} \frac{x^2}{(x^2+1)^2\,(x^2+2x+2)}\,dx$$

Also, $\lim\limits_{z\to\infty} z\,f(z) = \lim\limits_{z\to\infty} \dfrac{z^3}{(z^2+1)^2\,(z^2+2z+2)}$

$$= \lim_{z\to\infty} \frac{1}{z^3\left(1+\frac{1}{z^2}\right)^2\left(1+\frac{2}{z}+\frac{2}{z^2}\right)} = \frac{1}{\infty} = 0$$

$$\therefore \lim_{R\to\infty} \int_\Gamma f(z)\,dz = 0.$$

Now taking the limits $R \to \infty$ in (1) and putting these values we get,

$$\int_{-\infty}^{\infty} \frac{x^2}{(x^2+1)^2\,(x^2+2x+2)}\,dx + 0 = 2\pi i \times \frac{9i-12+12-16i}{100}$$

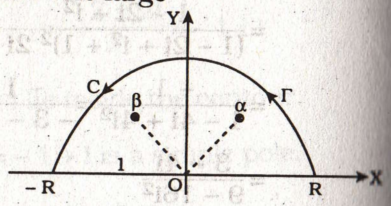
$$\Rightarrow \int_{-\infty}^{\infty} \frac{x^2}{(x^2+1)^2\,(x^2+2x+2)}\,dx = \frac{-14i^2\pi}{100}$$

$$= \frac{7\pi}{50} \quad \text{(Ans)}$$

Solution–28. Consider $\oint_C f(z)\,dz$, where $f(z) = \dfrac{1}{z^4+1}$ and C is the contour consisting of

(i) the x-axis from $-R$ to R, where R is large

(ii) the upper semi-circle Γ of the circle $|z| = R$, which lies above the x-axis.



The poles of $f(z) = \dfrac{1}{z^4+1}$ will be obtained by solving the equation

$$z^4 + 1 = 0$$

$$\Rightarrow z^4 = -1 = \cos\pi + i\sin\pi$$

$$= \cos(2n\pi+\pi) + i\sin(2n\pi+\pi)$$

$$= \cos(2n+1)\pi + i\sin(2n+1)\pi$$

$$\Rightarrow z = \cos\frac{2n+1}{4} + i\sin\left(\frac{2n+1}{4}\right)\pi$$

$$= e^{i(2n+1)\pi/4}, \text{ where } n = 0, 1, 2, 3.$$

The poles are $e^{i\pi/4}$, $e^{i3\pi/4}$, $e^{i5\pi/4}$, $e^{i7\pi/4}$

The amplitudes of the first two poles lies between 0 and π.

Therefore, the poles, $z = e^{i\pi/4}$ and $z = e^{i3\pi/4}$ lie inside the contour, both of them are simple poles. Let $z = \alpha = e^{i\pi/4}$ and $z = \beta = e^{i3\pi/4}$.

Residue at $z = \alpha$ is $\lim\limits_{z\to\alpha} (z-\alpha)\,f(z)$

$$= \lim_{z\to\alpha} \frac{z-\alpha}{z^4+1}; \frac{0}{0} \text{ form} \qquad z^4 = -1$$

$$\Rightarrow \alpha^4 = -1$$

$$= \lim_{z\to\alpha} \frac{1}{4z^3}; \text{ by L. Hospital rule} \qquad \Rightarrow \alpha^4 + 1 = 0$$

$$= \frac{1}{4\alpha^3}$$

Similarly, Residue at $z = \beta$ is $\dfrac{1}{4\beta^3}$

Now by Cauchy's residue theorem we have

$$\oint_C f(z)\,dz = 2\pi i \text{ [Sum of the residues]}$$

$$\Rightarrow \int_{-R}^{R} f(z)\,dz + \int_\Gamma f(z)\,dz = 2\pi i\left(\frac{1}{4\alpha^3}+\frac{1}{4\beta^3}\right) \;\ldots\ldots (1)$$

When $R \to \infty$ then $\int_{-R}^{R} f(z)\,dz = \int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{\infty} \dfrac{dx}{x^4+1}$

$$= 2\int_0^\infty \frac{dx}{x^4+1}, \text{ since } f(x) \text{ is an even function.}$$

Also, $\lim\limits_{z\to\infty} zf(z) = \lim\limits_{z\to\infty} \dfrac{z}{z^4+1}$; $\dfrac{\infty}{\infty}$ form

$$= \lim_{z\to\infty} \frac{1}{4z^3}, \text{ by L. Hospital rule}$$

$$= \frac{1}{\infty} = 0$$

$$\therefore \lim_{R\to\infty} \int_{\Gamma} f(z)\,dz = 0$$

$$\frac{1}{4\alpha^3} + \frac{1}{4\beta^3} = \frac{1}{4}\left(\frac{1}{\alpha^3} + \frac{1}{\beta^3}\right)$$

$$= \frac{1}{4}\left(\frac{\alpha}{\alpha^4} + \frac{\beta}{\beta^4}\right)$$

$$= \frac{1}{4}(-\alpha - \beta); \quad \because z^4 = -1 \Rightarrow \alpha^4 = -1 \text{ and } \beta^4 = -1$$

$$= -\frac{1}{4}\left(e^{i\pi/4} + e^{i3\pi/4}\right)$$

$$= -\frac{1}{4}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4} + \cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)$$

$$= -\frac{1}{4}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4} - \cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$$

$$= \frac{-2}{4}\, i\sin\frac{\pi}{4} = \frac{-i}{2\sqrt{2}}$$

Now taking limit $R \to \infty$ in (1) and then putting above all results we get

$$2\int_0^{\infty} \frac{dx}{x^4+1} + 0 = 2\pi i\left(\frac{-i}{2\sqrt{2}}\right)$$

$$\Rightarrow \int_0^{\infty} \frac{dx}{x^4+1} = \frac{\pi}{2\sqrt{2}} = \frac{\sqrt{2}}{4}\pi \quad \text{(Ans)}$$

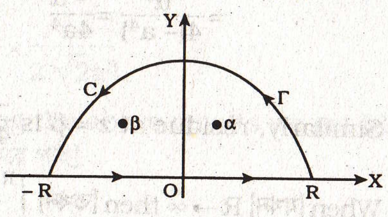
Solution–29. Consider $\oint_C f(z)\,dz$, where $f(z) = \dfrac{1}{z^4 + a^4}$ and C is the contour consisting of

(i) the x-axis from $-R$ to R, where R is large

(ii) the upper semi-circle Γ of the circle $|z| = R$, which lies above the x-axis.

The poles of $f(z) = \dfrac{1}{z^4 + a^4}$ will be obtained from the equation

$$z^4 + a^4 = 0$$



[$\oint_C f(z)\,dz$ বিবেচনা করি, যেখানে $f(z) = \dfrac{1}{z^4 + a^4}$ এবং C কন্টুরটি গঠিত হচ্ছে

(i) x অক্ষ, $-R$ হতে R, যেখানে R অনেক বৃহৎ।

(ii) $|z| = R$ বৃত্তের উর্ধ্ব অর্ধবৃত্ত Γ, যাহা x অক্ষের উর্ধ্বে অবিস্থত।

$f(z) = \dfrac{1}{z^4 + a^4}$ এর পোল পাওয়া যাবে $z^4 + a^4 = 0$ সমীকরণ হতে।]

$$\Rightarrow z^4 = -a^4$$

$$\Rightarrow z = a(-1)^{1/4}$$

$$= a(\cos\pi + i\sin\pi)^{1/4}$$

$$= a[\cos(2n\pi + \pi) + i\sin(2n\pi + \pi)]^{1/4}$$

$$= a\left[\cos\frac{(2n+1)\pi}{4} + i\sin\frac{(2n+1)\pi}{4}\right]$$

$$= ae^{i(2n+1)\pi/4}, \text{ where } n = 0, 1, 2, 3.$$

The poles are [পোলগুলি হলো] $ae^{i\pi/4}$, $ae^{i3\pi/4}$, $ae^{i5\pi/4}$, $ae^{i7\pi/4}$.

The amplitudes of the first two poles are lie between 0 and π. So they are inside the contour and each of them are simple pole. [প্রথম দুইটি পোলের কোণাঙ্ক 0 ও π এর মধ্যে অবস্থিত। সুতরাং তারা কন্টুরের ভিতরে অবস্থিত এবং প্রত্যেকে সরল পোল।]

Let [ধরি] $z = ae^{i\pi/4} = \alpha$ and [এবং] $z = ae^{i3\pi/4} = \beta$.

Residue at $z = \alpha$ is [$z = \alpha$ এ অবশেষ] $\lim\limits_{z\to\alpha} (z - \alpha) \cdot f(z)$

$$= \lim_{z\to\alpha} \frac{z-\alpha}{z^4 + a^4}; \ \frac{\infty}{\infty} \text{ form}$$

$$= \lim_{z\to\alpha} \frac{1}{4z^3}, \text{ by L. Hospital rule}$$

$$= \frac{1}{4\alpha^3} = \frac{\alpha}{4\alpha^4}$$

$$= \frac{\alpha}{4(-a^4)} = \frac{-\alpha}{4a^4} \qquad \because z^4 = -a^4 \Rightarrow \alpha^4 = -a^4$$

Similarly, residue at $z = \beta$ is $\frac{-\beta}{4a^4}$ [একইভাবে $z = \beta$ এ অবশেষ $\frac{-\beta}{4a^4}$]

When [যখন] $R \to \infty$ then [তখন] $\int_{-R}^{R} f(z)\, dz = \int_{-\infty}^{\infty} f(x)\, dx$

$$= \int_{-\infty}^{\infty} \frac{dx}{x^4 + a^4}$$

$$= 2\int_{0}^{\infty} \frac{dx}{x^4 + a^4}$$

Also [আরো] $\lim\limits_{z\to\infty} zf(z) = \lim\limits_{z\to\infty} \frac{z}{z^4 + a^4}$; $\frac{\infty}{\infty}$ form

$$= \lim_{z\to\infty} \frac{1}{4z^3}\text{ ; by L. Hospital rule}$$

$$= \frac{1}{\infty} = 0$$

Sum of the residues [অবশেষগুলির যোগফল]

$$= \frac{-\alpha}{4a^4} - \frac{\beta}{4a^4}$$

$$= \frac{-1}{4a^4}(\alpha + \beta)$$

$$= \frac{-a}{4a^4}(e^{i\pi/4} + e^{i3\pi/4})$$

$$= \frac{-1}{4a^3}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4} + \cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)$$

$$= \frac{-1}{4a^3}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4} - \cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$$

$$= \frac{-1}{4a^3}\left(2i\sin\frac{\pi}{4}\right)$$

$$= \frac{-i}{2\sqrt{2}a^3}$$

By Cauchy's residue theorem we have [কচির অবশেষ উপপাদ্য দ্বারা পাই]

$$\oint_C f(z)\, dz = 2\pi i \text{ (sum of the residues)}$$

$$\Rightarrow \int_{-R}^{R} f(z)\, dz + \int_{\Gamma} f(z)\, dz = 2\pi i \times \frac{-i}{2\sqrt{2}a^3}$$

Taking limit $R \to \infty$ and using the above results we get [$R \to \infty$ লিমিট নিয়ে এবং উপরের ফলগুলি ব্যবহার করে পাই]

$$\int_{-\infty}^{\infty} f(x)\, dx + 0 = \frac{\pi}{\sqrt{2}a^3}$$

$$\Rightarrow 2\int_{0}^{\infty} \frac{dx}{x^4 + a^4} = \frac{\pi}{\sqrt{2}a^3}$$

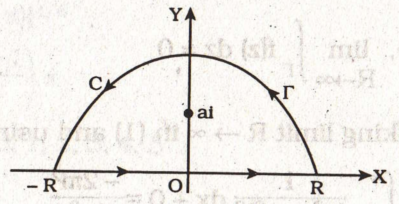
$$\Rightarrow \int_{0}^{\infty} \frac{dx}{x^4 + a^4} = \frac{\pi}{2\sqrt{2}a^3} = \frac{\sqrt{2}\pi}{4a^3}. \quad \text{(Ans)}$$

Solution–30. Consider the integral $\oint_C f(z)\, dz$, where $f(z) = \frac{1}{(z^2 + a^2)^2}$ and C is the closed contour consisting of

(i) the x-axis from $-R$ to R, where R is large

(ii) the upper semi-circle Γ of the circle $|z| = R$, which lies above the x-axis.

The poles of $f(z) = \frac{1}{(z^2 + a^2)^2}$ are obtained from the equation



$$(z^2 + a^2)^2 = 0$$

$$\Rightarrow z^2 + a^2 = 0$$

$$\Rightarrow z^2 = -a^2$$

$$\Rightarrow z = \pm ai$$

Only the pole $z = ai$ lies inside the contour which is a double pole.

Residue at $z = ai$ is $\lim\limits_{z\to ai} \frac{1}{1!}\frac{d}{dz}\{(z - ai)^2 \cdot f(z)\}$

$$= \lim_{z\to ai} \frac{d}{dz}\left\{(z - ai)^2 \cdot \frac{1}{(z^2 + a^2)^2}\right\}$$