

(Manufacturing Costs) A furniture manufacturer makes chairs and tables, each of which must go through an assembly process and a finishing process. The times required for these processes are given (in hours) by the matrix

$$A = \begin{array}{cc} & \begin{array}{c} \text{Assembly} \\ \text{process} \end{array} & \begin{array}{c} \text{Finishing} \\ \text{process} \end{array} \\ \begin{bmatrix} 2 \\ 3 \end{bmatrix} & \begin{bmatrix} 2 \\ 4 \end{bmatrix} & \begin{array}{l} \text{Chair} \\ \text{Table} \end{array} \end{array}$$

The manufacturer has a plant in Salt Lake City and another in Chicago. The hourly rates for each of the processes are given (in dollars) by the matrix

$$B = \begin{array}{cc} & \begin{array}{c} \text{Salt Lake} \\ \text{City} \end{array} & \text{Chicago} \\ \begin{bmatrix} 9 \\ 10 \end{bmatrix} & \begin{bmatrix} 10 \\ 12 \end{bmatrix} & \begin{array}{l} \text{Assembly process} \\ \text{Finishing process} \end{array} \end{array}$$

What do the entries in the matrix product AB tell the manufacturer?

Find the total amount to manufacture the chairs and tables in each plant.

Solution: Given that

$$A = \begin{bmatrix} 2 & 2 \\ 3 & 4 \end{bmatrix} \text{ and } B = \begin{bmatrix} 9 & 10 \\ 10 & 12 \end{bmatrix}.$$

Therefore,

$$\begin{aligned} AB &= \begin{bmatrix} 2 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 9 & 10 \\ 10 & 12 \end{bmatrix} \\ &= \begin{bmatrix} 2 \times 9 + 2 \times 10 & 2 \times 10 + 2 \times 12 \\ 3 \times 9 + 4 \times 10 & 3 \times 10 + 4 \times 12 \end{bmatrix} \\ &= \begin{bmatrix} 38 & 44 \\ 67 & 78 \end{bmatrix} \end{aligned}$$

(Medicine) A diet research project includes adults and children of both sexes. The composition of the participants in the project is given by the matrix

$$A = \begin{array}{cc} & \begin{array}{cc} \text{Adults} & \text{Children} \end{array} \\ \begin{bmatrix} 80 & 120 \\ 100 & 200 \end{bmatrix} & \begin{array}{c} \text{Male} \\ \text{Female} \end{array} \end{array}$$

The number of daily grams of protein, fat, and carbohydrate consumed by each child and adult is given by the matrix

$$B = \begin{array}{ccc} & \begin{array}{cc} \text{Protein} & \text{Fat} \end{array} & \begin{array}{c} \text{Carbo-} \\ \text{hydrate} \end{array} \\ \begin{bmatrix} 20 & 20 \\ 10 & 20 \end{bmatrix} & \begin{array}{c} \text{Adult} \\ \text{Child} \end{array} \end{array}$$

- How many grams of protein are consumed daily by the males in the project?
- How many grams of fat are consumed daily by the females in the project?

Solution: Given that

$$A = \begin{bmatrix} 80 & 120 \\ 100 & 200 \end{bmatrix} \text{ and } B = \begin{bmatrix} 20 & 20 & 20 \\ 10 & 20 & 30 \end{bmatrix}.$$

- The number of daily grams of protein by the males is $80 \times 20 + 120 \times 10 = 1600 + 1200 = 2800$
- The number of daily grams of fat by the females is $100 \times 20 + 200 \times 20 = 2000 + 4000 = 6000$

Page 34 (Example 2.9) Schaum's Outline series
Given that,

$$A = \begin{bmatrix} 1 & 2 \\ 3 & -4 \end{bmatrix}$$

$$\text{and } f(x) = 2x^2 - 3x + 5$$

$$\therefore f(A) = 2A^2 - 3A + 5I, \text{ where } I \text{ is an identity matrix.}$$

Now,

$$A^2 = AA = \begin{bmatrix} 1 & 2 \\ 3 & -4 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & -4 \end{bmatrix} = \begin{bmatrix} 1 \times 1 + 2 \times 3 & 1 \times 2 + 2 \times (-4) \\ 3 \times 1 + (-4) \times 3 & 3 \times 2 + (-4) \times (-4) \end{bmatrix}$$

$$\therefore A^2 = \begin{bmatrix} 7 & -6 \\ -9 & 22 \end{bmatrix}$$

$$\begin{aligned}
\therefore f(A) &= 2 \begin{bmatrix} 7 & -6 \\ -9 & 22 \end{bmatrix} - 3 \begin{bmatrix} 1 & 2 \\ 3 & -4 \end{bmatrix} + 5 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 14 & -12 \\ -18 & 44 \end{bmatrix} - \begin{bmatrix} 3 & 6 \\ 9 & -12 \end{bmatrix} + \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix} \\
&= \begin{bmatrix} 14-3+5 & -12-6+0 \\ -18-9+0 & 44+12+5 \end{bmatrix} \\
&= \begin{bmatrix} 16 & -18 \\ -27 & 61 \end{bmatrix}
\end{aligned}$$

Hermitian Matrix: The conjugate transpose of a complex matrix is called the Hermitian matrix. Symbolically, if A is a complex matrix then $A^H = (\bar{A})^T$

Here

$$\begin{aligned}
A &= \begin{bmatrix} 3-5i & 2+4i \\ 6+7i & 1+8i \end{bmatrix} \\
\bar{A} &= \begin{bmatrix} 3+5i & 2-4i \\ 6-7i & 1-8i \end{bmatrix}
\end{aligned}$$

$$\text{Therefore, } A^H = (\bar{A})^T = \begin{bmatrix} 3+5i & 2-4i \\ 6-7i & 1-8i \end{bmatrix}^T = \begin{bmatrix} 3+5i & 6-7i \\ 2-4i & 1-8i \end{bmatrix}$$

Here

$$\begin{aligned}
A &= \begin{bmatrix} 2-3i & 5+8i \\ -4 & 3-7i \\ -6-i & 5i \end{bmatrix} \\
\bar{A} &= \begin{bmatrix} 2+3i & 5-8i \\ -4 & 3+7i \\ -6+i & -5i \end{bmatrix}
\end{aligned}$$

$$\text{Therefore, } A^H = (\bar{A})^T = \begin{bmatrix} 2+3i & 5-8i \\ -4 & 3+7i \\ -6+i & -5i \end{bmatrix}^T = \begin{bmatrix} 2+3i & -4 & -6+i \\ 5-8i & 3+7i & -5i \end{bmatrix}$$

Adjoint of a matrix: Let $A=[a_{ij}]$ be an $n \times n$ matrix over a field K and let A_{ij} denote the cofactor of a_{ij} . The classical adjoint of A , denoted by $\text{adj } A$, is the transpose of the matrix of cofactors of A . Namely,

$$\text{adj } A = [A_{ij}]^T$$

Inverse of a matrix: $A^{-1} = \frac{1}{D} \text{adj } A = \frac{1}{|A|} \text{adj } A$

Rule of signs: $A = \begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$

Problem: Find the inverse of the matrix $A = \begin{bmatrix} 2 & 3 & -4 \\ 0 & -4 & 2 \\ 1 & -1 & 5 \end{bmatrix}$.

Solution: Given the matrix

$$A = \begin{bmatrix} 2 & 3 & -4 \\ 0 & -4 & 2 \\ 1 & -1 & 5 \end{bmatrix}$$

Determinant of A, $|A| = 2 \times (-20 + 2) - 3 \times (0 - 2) - 4 \times (0 + 4) = -36 + 6 - 16 = -46 \neq 0$

The cofactors of A are

$$A_{11} = \begin{vmatrix} -4 & 2 \\ -1 & 5 \end{vmatrix} = -20 + 2 = -18, \quad A_{12} = -\begin{vmatrix} 0 & 2 \\ 1 & 5 \end{vmatrix} = -(0 - 2) = 2, \quad A_{13} = \begin{vmatrix} 0 & -4 \\ 1 & -1 \end{vmatrix} = 0 + 4 = 4$$

$$A_{21} = -\begin{vmatrix} 3 & -4 \\ -1 & 5 \end{vmatrix} = -(15 - 4) = -11, \quad A_{22} = \begin{vmatrix} 2 & -4 \\ 1 & 5 \end{vmatrix} = 10 + 4 = 14, \quad A_{23} = -\begin{vmatrix} 2 & 3 \\ 1 & -1 \end{vmatrix} = -(-2 - 3) = 5$$

$$A_{31} = \begin{vmatrix} 3 & -4 \\ -4 & 2 \end{vmatrix} = 6 - 16 = -10, \quad A_{32} = -\begin{vmatrix} 2 & -4 \\ 0 & 2 \end{vmatrix} = -(4 - 0) = -4, \quad A_{33} = \begin{vmatrix} 2 & 3 \\ 0 & -4 \end{vmatrix} = -8 + 0 = -8$$

So,

$$\text{adj } A = \begin{bmatrix} -18 & 2 & 4 \\ -11 & 14 & 5 \\ -10 & -4 & -8 \end{bmatrix}^T = \begin{bmatrix} -18 & -11 & -10 \\ 2 & 14 & -4 \\ 4 & 5 & -8 \end{bmatrix}$$

Therefore,

$$A^{-1} = \frac{1}{|A|} \text{adj } A = -\frac{1}{46} \begin{bmatrix} -18 & -11 & -10 \\ 2 & 14 & -4 \\ 4 & 5 & -8 \end{bmatrix} = \begin{bmatrix} \frac{9}{23} & \frac{11}{46} & \frac{5}{23} \\ -\frac{1}{23} & -\frac{7}{23} & \frac{2}{23} \\ -\frac{2}{23} & -\frac{5}{46} & \frac{4}{23} \end{bmatrix}$$

Problem: Find the inverse of the matrix $A = \begin{bmatrix} 2 & -1 & -1 \\ 1 & -2 & 1 \\ 1 & -1 & 2 \end{bmatrix}$.

Solution: Given the matrix

$$A = \begin{bmatrix} 2 & -1 & -1 \\ 1 & -2 & 1 \\ 1 & -1 & 2 \end{bmatrix}$$

Determinant of A, $|A| = 2 \times (-4 + 1) + 1 \times (2 - 1) - 1 \times (-1 + 2) = -6 + 1 - 1 = -6 \neq 0$

The cofactors of A are

$$A_{11} = \begin{vmatrix} -2 & 1 \\ -1 & 2 \end{vmatrix} = -4 + 1 = -3, \quad A_{12} = -\begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} = -(2 - 1) = -1, \quad A_{13} = \begin{vmatrix} 1 & -2 \\ 1 & -1 \end{vmatrix} = -1 + 2 = 1$$

$$A_{21} = -\begin{vmatrix} -1 & -1 \\ -1 & 2 \end{vmatrix} = -(-2 - 1) = 3, \quad A_{22} = \begin{vmatrix} 2 & -1 \\ 1 & 2 \end{vmatrix} = 4 + 1 = 5, \quad A_{23} = -\begin{vmatrix} 2 & -1 \\ 1 & -1 \end{vmatrix} = -(-2 + 1) = 1$$

$$A_{31} = \begin{vmatrix} -1 & -1 \\ -2 & 1 \end{vmatrix} = -1 - 2 = -3, \quad A_{32} = -\begin{vmatrix} 2 & -1 \\ 1 & 1 \end{vmatrix} = -(2 + 1) = -3, \quad A_{33} = \begin{vmatrix} 2 & -1 \\ 1 & -2 \end{vmatrix} = -4 + 1 = -3$$

So,

$$\text{adj } A = \begin{bmatrix} -3 & -1 & 1 \\ 3 & 5 & 1 \\ -3 & -3 & -3 \end{bmatrix}^T = \begin{bmatrix} -3 & 3 & -3 \\ -1 & 5 & -3 \\ 1 & 1 & -3 \end{bmatrix}$$

Therefore,

$$A^{-1} = \frac{1}{|A|} \text{adj } A = -\frac{1}{6} \begin{bmatrix} -3 & 3 & -3 \\ -1 & 5 & -3 \\ 1 & 1 & -3 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{6} & -\frac{5}{6} & \frac{1}{2} \\ -\frac{1}{6} & -\frac{1}{6} & \frac{1}{2} \end{bmatrix}$$

$$x + 2y = 3$$

$$2x + 4y = 6$$

$$2(3 - 2y) + 4y = 6 \Rightarrow 6 - 4y + 4y = 6 \Rightarrow 0 = 0$$

$$\Rightarrow x + 2y = 3$$

$$x = 1,$$

$$1 + 2y = 3$$

$$\therefore y = 1$$

Variables=2

Equation=1

Number of Variables-Number of Equations=2-1=1

Therefore, number of free variable is 1.

$$x + 2y = 3$$

$$2x + 4y = 2$$

$$\Rightarrow x + 2y = 3$$

$$1 = 3$$

$$x + 2y = 3$$

$$4y = 2$$

$$\Rightarrow y = 1/2$$

System of linear equations

$$AX = B$$

Multiplying both sides by A^{-1} then we have

$$A^{-1}AX = A^{-1}B$$

$$\Rightarrow IX = A^{-1}B$$

$$\therefore X = A^{-1}B$$

$$A = \begin{bmatrix} 2 & -1 & -1 \\ 1 & -2 & 1 \\ 1 & -1 & 2 \end{bmatrix}$$

Book-Abdur Rahman Example 10 (page 25):

Solution: Given the system of equations

$$\left. \begin{array}{l} x + y - z = 1 \\ 2x + 3y + \lambda z = 3 \\ x + \lambda y + 3z = 2 \end{array} \right\} \begin{array}{l} L_2 \rightarrow -2L_1 + L_2 \\ L_3 \rightarrow -L_1 + L_3 \end{array} \left. \begin{array}{l} x + y - z = 1 \\ y + (\lambda + 2)z = 1 \\ (\lambda - 1)y + 4z = 1 \end{array} \right\}$$

$$\left. \begin{array}{l} L_3 \rightarrow -(\lambda - 1)L_2 + L_3 \\ x + y - z = 1 \\ y + (\lambda + 2)z = 1 \\ \{-(\lambda - 1)(\lambda + 2) + 4\}z = -\lambda + 1 + 1 \end{array} \right\}$$

$$\left. \begin{array}{l} x + y - z = 1 \\ y + (\lambda + 2)z = 1 \\ (\lambda + 3)(2 - \lambda)z = 2 - \lambda \end{array} \right\}$$

(i) The above system has a unique solution if $(\lambda + 3)(2 - \lambda) \neq 0$,

$$\Rightarrow \lambda + 3 \neq 0 \text{ and } 2 - \lambda \neq 0 \Rightarrow \lambda \neq -3 \text{ and } \lambda \neq 2$$

(ii) The above system has more than one solution if $2 - \lambda = 0 \Rightarrow \lambda = 2$

(iii) The above system has no solution if $\lambda + 3 = 0 \Rightarrow \lambda = -3$

Exercise 30 (Page 36): Given the system of equations

$$\left. \begin{array}{l} x-3z=-3 \\ 2x+\lambda y-z=-2 \\ x+2y+\lambda z=1 \end{array} \right\} \quad L_1 \leftrightarrow L_3$$

$$\left. \begin{array}{l} x+2y+\lambda z=1 \\ 2x+\lambda y-z=-2 \\ x-3z=-3 \end{array} \right\} \quad \begin{array}{l} L_2 \rightarrow -2L_1 + L_2 \\ L_3 \rightarrow -L_1 + L_3 \end{array} \quad \left. \begin{array}{l} x+2y+\lambda z=1 \\ (\lambda-4)y-(2\lambda+1)z=-4 \\ -2y-(\lambda+3)z=-4 \end{array} \right\}$$

$$\left. \begin{array}{l} L_3 \rightarrow 2L_2 + (\lambda-4)L_3 \\ x+2y+\lambda z=1 \\ (\lambda-4)y-(2\lambda+1)z=0 \\ \{-(4\lambda+2)-(\lambda-4)(\lambda+3)\}z=-4(\lambda-4)-8 \end{array} \right\}$$

$$\left. \begin{array}{l} x+2y+\lambda z=1 \\ (\lambda+4)y+(2\lambda-1)z=0 \\ -(\lambda^2-\lambda-12+4\lambda+2)z=-4(\lambda-2) \end{array} \right\}$$

$$\left. \begin{array}{l} x+2y+\lambda z=1 \\ \text{Or, } (\lambda+4)y+(2\lambda-1)z=0 \\ (\lambda^2+3\lambda-10)z=4(\lambda-2) \end{array} \right\}$$

$$\left. \begin{array}{l} x+2y+\lambda z=1 \\ \text{Or, } (\lambda+4)y+(2\lambda-1)z=0 \\ (\lambda+5)(\lambda-2)z=4(\lambda-2) \end{array} \right\}$$

(i) The above system has a unique solution if $(\lambda+5)(\lambda-2) \neq 0$,

$$\Rightarrow \lambda+5 \neq 0 \text{ and } \lambda-2 \neq 0 \Rightarrow \lambda \neq -5 \text{ and } \lambda \neq 2$$

(ii) The above system has more than one solution if $\lambda-2=0 \Rightarrow \lambda=2$

(iii) The above system has no solution if $\lambda+5=0 \Rightarrow \lambda=-5$

() For S is a subspace,

(i) S is nonempty

(ii) $u, v \in S, \alpha u + \beta v \in S$

Let,

$$v = \alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3, \text{ where } \alpha_1, \alpha_2, \alpha_3 \in F$$

$$R_1 \leftrightarrow R_3$$

$$L_3 \rightarrow L_3/2$$

Solution: We form the following matrix using the given vectors:

$$\begin{bmatrix} 2 & -1 & 4 \\ 3 & 6 & 2 \\ 2 & 10 & -4 \end{bmatrix} \xrightarrow[\substack{L_2 \rightarrow 3L_1 - 2L_2 \\ L_3 \rightarrow L_1 - L_3}]{\substack{L_2 \rightarrow L_2/(-15) \\ L_3 \rightarrow L_3/(-11)}} \begin{bmatrix} 2 & -1 & 4 \\ 0 & -15 & 8 \\ 0 & -11 & 8 \end{bmatrix} \xrightarrow[\substack{L_2 \rightarrow L_2/(-15) \\ L_3 \rightarrow L_3/(-11)}]{L_3 \rightarrow L_2 - L_3} \begin{bmatrix} 2 & -1 & 4 \\ 0 & 1 & -\frac{8}{15} \\ 0 & 1 & -\frac{8}{11} \end{bmatrix}$$

$$\xrightarrow{L_3 \rightarrow L_2 - L_3} \begin{bmatrix} 2 & -1 & 4 \\ 0 & 1 & -\frac{8}{15} \\ 0 & 0 & \frac{32}{165} \end{bmatrix}$$

The matrix is in echelon form and it has no zero row. Hence the given vectors are linearly independent.

We set

(i) $y=1, z=0$.

Then $x-3y+z=0 \Rightarrow x-3.1+0=0 \Rightarrow x=3$

(ii) $y=0, z=1$.

Then $x-3y+z=0 \Rightarrow x-3.0+1=0 \Rightarrow x=-1$

Thus we obtain the solutions $(3, 1, 0)$ and $(-1, 0, 1)$.

Hence the set $\{(3, 1, 0), (-1, 0, 1)\}$ is a basis of the solution space.

$x=1, y=0, z=0$

$x=0, y=1, z=0$

$x=0, y=0, z=1$

$u=(u_1, u_2, u_3)$

$v=(v_1, v_2, v_3)$

Inner product, $(u, v)=u_1*v_1+u_2*v_2+u_3*v_3$

$\|u\|=\text{Sqrt}(u, u)=\text{Sqrt}(u_1*u_1+u_2*u_2+u_3*u_3)$

$=\text{Sqrt}(u_1^2+u_2^2+u_3^2)$

Lu Factorization

3.39. Find the LU factorization of (a) $A = \begin{bmatrix} 1 & -3 & 5 \\ 2 & -4 & 7 \\ -1 & -2 & 1 \end{bmatrix}$, (b) $B = \begin{bmatrix} 1 & 4 & -3 \\ 2 & 8 & 1 \\ -5 & -9 & 7 \end{bmatrix}$.

(a) Reduce A to triangular form by the following operations:

“Replace R_2 by $-2R_1 + R_2$,” “Replace R_3 by $R_1 + R_3$,” and then
“Replace R_3 by $\frac{5}{2}R_2 + R_3$ ”

These operations yield the following, where the triangular form is U :

$$A \sim \begin{bmatrix} 1 & -3 & 5 \\ 0 & 2 & -3 \\ 0 & -5 & 6 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & 5 \\ 0 & 2 & -3 \\ 0 & 0 & -\frac{3}{2} \end{bmatrix} = U \quad \text{and} \quad L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & -\frac{5}{2} & 1 \end{bmatrix}$$

The entries $2, -1, -\frac{5}{2}$ in L are the negatives of the multipliers $-2, 1, \frac{5}{2}$ in the above row operations. (As a check, multiply L and U to verify $A = LU$.)

(b) Reduce B to triangular form by first applying the operations “Replace R_2 by $-2R_1 + R_2$ ” and “Replace R_3 by $5R_1 + R_3$.” These operations yield

$$B \sim \begin{bmatrix} 1 & 4 & -3 \\ 0 & 0 & 7 \\ 0 & 11 & -8 \end{bmatrix}.$$

Observe that the second diagonal entry is 0. Thus, B cannot be brought into triangular form without row interchange operations. Accordingly, B is not LU -factorable. (There does exist a PLU factorization of such a matrix B , where P is a permutation matrix, but such a factorization lies beyond the scope of this text.)

3.41. Find the LU factorization of the matrix $A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 3 & 3 \\ -3 & -10 & 2 \end{bmatrix}$.

Reduce A to triangular form by the following operations:

(1) “Replace R_2 by $-2R_1 + R_2$,” (2) “Replace R_3 by $3R_1 + R_3$,” (3) “Replace R_3 by $-4R_2 + R_3$ ”

These operations yield the following, where the triangular form is U :

$$A \sim \begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & 1 \\ 0 & -4 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = U \quad \text{and} \quad L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -3 & 4 & 1 \end{bmatrix}$$

The entries $2, -3, 4$ in L are the negatives of the multipliers $-2, 3, -4$ in the above row operations. (As a check, multiply L and U to verify $A = LU$.)

Lu Factorization

3.69. Find the LU factorization of each of the following matrices:

$$(a) \begin{bmatrix} 1 & -1 & -1 \\ 3 & -4 & -2 \\ 2 & -3 & -2 \end{bmatrix}, (b) \begin{bmatrix} 1 & 3 & -1 \\ 2 & 5 & 1 \\ 3 & 4 & 2 \end{bmatrix}, (c) \begin{bmatrix} 2 & 3 & 6 \\ 4 & 7 & 9 \\ 3 & 5 & 4 \end{bmatrix}, (d) \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 7 \\ 3 & 7 & 10 \end{bmatrix}$$

Least Squares Problem Given a linear system $A\mathbf{x} = \mathbf{b}$ of m equations in n unknowns, find a vector \mathbf{x} , if possible, that minimizes $\|A\mathbf{x} - \mathbf{b}\|$ with respect to the Euclidean inner product on \mathbb{R}^m . Such a vector is called a **least squares solution** of $A\mathbf{x} = \mathbf{b}$.

For any linear system $A\mathbf{x} = \mathbf{b}$, the associated normal system

$$A^T A\mathbf{x} = A^T \mathbf{b}$$

is consistent, and all solutions of the normal system are least squares solutions of $A\mathbf{x} = \mathbf{b}$. Moreover, if W is the column space of A , and \mathbf{x} is any least squares solution of $A\mathbf{x} = \mathbf{b}$, then the orthogonal projection of \mathbf{b} on W is

$$\text{proj}_W \mathbf{b} = A\mathbf{x}$$

Find the least squares solution of the linear system $A\mathbf{x} = \mathbf{b}$ given by

$$x_1 - x_2 = 4$$

$$3x_1 + 2x_2 = 1$$

$$-2x_1 + 4x_2 = 3$$

and find the orthogonal projection of \mathbf{b} on the column space of A .

Solution

Here

$$A = \begin{bmatrix} 1 & -1 \\ 3 & 2 \\ -2 & 4 \end{bmatrix} \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} 4 \\ 1 \\ 3 \end{bmatrix}$$

Observe that A has linearly independent column vectors, so we know in advance that there is a unique least squares solution. We have

$$A^T A = \begin{bmatrix} 1 & 3 & -2 \\ -1 & 2 & 4 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 3 & 2 \\ -2 & 4 \end{bmatrix} = \begin{bmatrix} 14 & -3 \\ -3 & 21 \end{bmatrix}$$

$$A^T \mathbf{b} = \begin{bmatrix} 1 & 3 & -2 \\ -1 & 2 & 4 \end{bmatrix} \begin{bmatrix} 4 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 10 \end{bmatrix}$$

so the normal system $A^T A\mathbf{x} = A^T \mathbf{b}$ in this case is

$$\begin{bmatrix} 14 & -3 \\ -3 & 21 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 10 \end{bmatrix}$$

Solving this system yields the least squares solution

$$x_1 = \frac{17}{95}, \quad x_2 = \frac{143}{285}$$

From Formula 5, the orthogonal projection of \mathbf{b} on the column space of A is

$$A\mathbf{x} = \begin{bmatrix} 1 & -1 \\ 3 & 2 \\ -2 & 4 \end{bmatrix} \begin{bmatrix} \frac{17}{95} \\ \frac{143}{285} \end{bmatrix} = \begin{bmatrix} -\frac{92}{285} \\ \frac{439}{285} \\ \frac{94}{57} \end{bmatrix}$$

Find the least squares solution of the linear system $A\mathbf{x} = \mathbf{b}$, and find the orthogonal projection of \mathbf{b} onto the column space of A .

$$(a) \quad A = \begin{bmatrix} 1 & 1 & 1 \\ -1 & 1 & 1 \\ -1 & 2 & 2 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 7 \\ 0 \\ -7 \end{bmatrix}$$

$$(b) \quad A = \begin{bmatrix} 2 & -2 \\ 1 & 1 \\ 3 & 1 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$$

$$(c) \quad A = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & -2 \\ 1 & 1 & 0 \\ 1 & 1 & -1 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 6 \\ 0 \\ 9 \\ 3 \end{bmatrix}$$

Solution:

The associated normal system is $A^T A\mathbf{x} = A^T \mathbf{b}$, or

$$\begin{bmatrix} 1 & -1 & -1 \\ 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & 1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 & -1 & -1 \\ 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} 7 \\ 0 \\ -7 \end{bmatrix}$$

or

$$\begin{bmatrix} 3 & -2 \\ -2 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 14 \\ -7 \end{bmatrix}$$

This system has solution $x_1 = 5$, $x_2 = 1/2$, which is the least squares solution of $A\mathbf{x} = \mathbf{b}$.

The orthogonal projection of \mathbf{b} on the column space of A is $A\mathbf{x}$, or

$$\begin{bmatrix} 1 & 1 \\ -1 & 1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 5 \\ 1/2 \end{bmatrix} = \begin{bmatrix} 11/2 \\ -9/2 \\ -4 \end{bmatrix}$$

(c) The associated normal system is

$$\begin{bmatrix} 1 & 2 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ -1 & -2 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & -2 \\ 1 & 1 & 0 \\ 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \\ = \begin{bmatrix} 1 & 2 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ -1 & -2 & 0 & -1 \end{bmatrix} \begin{bmatrix} 6 \\ 0 \\ 9 \\ 3 \end{bmatrix}$$

or

$$\begin{bmatrix} 7 & 4 & -6 \\ 4 & 3 & -3 \\ -6 & -3 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 18 \\ 12 \\ -9 \end{bmatrix}$$

This system has solution $x_1 = 12$, $x_2 = -3$, $x_3 = 9$, which is the least squares solution of $A\mathbf{x} = \mathbf{b}$.

The orthogonal projection of \mathbf{b} on the column space of A is $A\mathbf{x}$, or

$$\begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & -2 \\ 1 & 1 & 0 \\ 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} 12 \\ -3 \\ 9 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ 9 \\ 0 \end{bmatrix}$$

which can be written as $(3, 3, 9, 0)$.