(***Manufacturing Costs***) A furniture manufacturer makes chairs and tables, each of which must go through an assembly process and a finishing process. The times required for these processes are given (in hours) by the matrix

$$\begin{array}{cc} & \begin{array}{cc} \text{Assembly process} & \text{Finishing process} \end{array} \\ A = & \left[\begin{array}{cc} 2 & 2 \\ 3 & 4 \end{array}\right] \begin{array}{l} \text{Chair} \\ \text{Table} \end{array} \end{array}.$$

The manufacturer has a plant in Salt Lake City and another in Chicago. The hourly rates for each of the processes are given (in dollars) by the matrix

$$\begin{array}{cc} & \begin{array}{cc} \text{Salt Lake City} & \text{Chicago} \end{array} \\ B = & \left[\begin{array}{cc} 9 & 10 \\ 10 & 12 \end{array}\right] \begin{array}{l} \text{Assembly process} \\ \text{Finishing process} \end{array} \end{array}.$$

What do the entries in the matrix product $AB$ tell the manufacturer?

Find the total amount to manufacture the chairs and tables in each plant.

**Solution:** Given that

$$A = \begin{bmatrix} 2 & 2 \\ 3 & 4 \end{bmatrix} \text{ and } B = \begin{bmatrix} 9 & 10 \\ 10 & 12 \end{bmatrix}.$$

Therefore,

$$AB = \begin{bmatrix} 2 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 9 & 10 \\ 10 & 12 \end{bmatrix}$$

$$= \begin{bmatrix} 2\times9+2\times10 & 2\times10+2\times12 \\ 3\times9+4\times10 & 3\times10+4\times12 \end{bmatrix}$$

$$= \begin{bmatrix} 38 & 44 \\ 67 & 78 \end{bmatrix}$$

(***Medicine***) A diet research project includes adults and children of both sexes. The composition of the participants in the project is given by the matrix

$$A = \begin{array}{c} \\ \\ \end{array}\begin{array}{cc} \text{Adults} & \text{Children} \\ \left[\begin{array}{cc} 80 & 120 \\ 100 & 200 \end{array}\right] \end{array}\begin{array}{l} \\ \text{Male} \\ \text{Female} \end{array}.$$

The number of daily grams of protein, fat, and carbohydrate consumed by each child and adult is given by the matrix

$$B = \begin{array}{ccc} \text{Protein} & \text{Fat} & \text{Carbo-hydrate} \end{array}$$
$$B = \left[\begin{array}{ccc} 20 & 20 & 20 \\ 10 & 20 & 30 \end{array}\right]\begin{array}{l} \text{Adult} \\ \text{Child} \end{array}.$$

**(a)** How many grams of protein are consumed daily by the males in the project?

**(b)** How many grams of fat are consumed daily by the females in the project?

**Solution:** Given that

$A=\begin{bmatrix} 80 & 120 \\ 100 & 200 \end{bmatrix}$ and $B=\begin{bmatrix} 20 & 20 & 20 \\ 10 & 20 & 30 \end{bmatrix}$.

(a) The number of daily grams of protein by the males is $80\times20+120\times10=1600+1200=2800$

(b) The number of daily grams of fat by the females is $100\times20+200\times20=2000+4000=6000$

Given that,

$$A=\begin{bmatrix} 1 & 2 \\ 3 & -4 \end{bmatrix}$$

and $f(x)=2x^2-3x+5$

$\therefore f(A)=2A^2-3A+5I$ , where $I$ is an identity matrix.

Now,

$$A^2=AA=\begin{bmatrix} 1 & 2 \\ 3 & -4 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & -4 \end{bmatrix}=\begin{bmatrix} 1\times1+2\times3 & 1\times2+2\times(-4) \\ 3\times1+(-4)\times3 & 3\times2+(-4)\times(-4) \end{bmatrix}$$

$$\therefore A^2=\begin{bmatrix} 7 & -6 \\ -9 & 22 \end{bmatrix}$$

$$\therefore f(A) = 2\begin{bmatrix} 7 & -6 \\ -9 & 22 \end{bmatrix} - 3\begin{bmatrix} 1 & 2 \\ 3 & -4 \end{bmatrix} + 5\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 14 & -12 \\ -18 & 44 \end{bmatrix} - \begin{bmatrix} 3 & 6 \\ 9 & -12 \end{bmatrix} + \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 14-3+5 & -12-6+0 \\ -18-9+0 & 44+12+5 \end{bmatrix}$$

$$= \begin{bmatrix} 16 & -18 \\ -27 & 61 \end{bmatrix}$$

Hermitian Matrix: The conjugate transpose of a complex matrix is called the Hermitian matrix. Symbolically, if $A$ is a complex matrix then $A^H = \left(\bar{A}\right)^T$

Here

$$A = \begin{bmatrix} 3-5i & 2+4i \\ 6+7i & 1+8i \end{bmatrix}$$

$$\bar{A} = \begin{bmatrix} 3+5i & 2-4i \\ 6-7i & 1-8i \end{bmatrix}$$

Therefore, $A^H = \left(\bar{A}\right)^T = \begin{bmatrix} 3+5i & 2-4i \\ 6-7i & 1-8i \end{bmatrix}^T = \begin{bmatrix} 3+5i & 6-7i \\ 2-4i & 1-8i \end{bmatrix}$

Here

$$A = \begin{bmatrix} 2-3i & 5+8i \\ -4 & 3-7i \\ -6-i & 5i \end{bmatrix}$$

$$\bar{A} = \begin{bmatrix} 2+3i & 5-8i \\ -4 & 3+7i \\ -6+i & -5i \end{bmatrix}$$

Therefore, $A^H = \left(\bar{A}\right)^T = \begin{bmatrix} 2+3i & 5-8i \\ -4 & 3+7i \\ -6+i & -5i \end{bmatrix}^T = \begin{bmatrix} 2+3i & -4 & -6+i \\ 5-8i & 3+7i & -5i \end{bmatrix}$

**Adjoint of a matrix:** Let $A=[a_{ij}]$ be an $n \times n$ matrix over a field $K$ and let $A_{ij}$ denote the cofactor of $a_{ij}$. The classical adjoint of $A$, denoted by adj $A$, is the transpose of the matrix of cofactors of $A$. Namely,

$$\text{adj } A = \left[A_{ij}\right]^T$$

**Inverse of a matrix:** $A^{-1}=\dfrac{1}{D}\text{adj }A=\dfrac{1}{|A|}\text{adj }A$

Rule of signs: $A=\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$

**Problem:** Find the inverse of the matrix $A=\begin{bmatrix} 2 & 3 & -4 \\ 0 & -4 & 2 \\ 1 & -1 & 5 \end{bmatrix}$.

**Solution:** Given the matrix

$$A=\begin{bmatrix} 2 & 3 & -4 \\ 0 & -4 & 2 \\ 1 & -1 & 5 \end{bmatrix}$$

Determinant of A, $|A|=2\times(-20+2)-3\times(0-2)-4\times(0+4)=-36+6-16=-46\neq 0$

The cofactors of A are

$$A_{11}=\begin{vmatrix} -4 & 2 \\ -1 & 5 \end{vmatrix}=-20+2=-18,\quad A_{12}=-\begin{vmatrix} 0 & 2 \\ 1 & 5 \end{vmatrix}=-(0-2)=2,\quad A_{13}=\begin{vmatrix} 0 & -4 \\ 1 & -1 \end{vmatrix}=0+4=4$$

$$A_{21}=-\begin{vmatrix} 3 & -4 \\ -1 & 5 \end{vmatrix}=-(15-4)=-11,\quad A_{22}=\begin{vmatrix} 2 & -4 \\ 1 & 5 \end{vmatrix}=10+4=14,\quad A_{23}=-\begin{vmatrix} 2 & 3 \\ 1 & -1 \end{vmatrix}=-(-2-3)=5$$

$$A_{31}=\begin{vmatrix} 3 & -4 \\ -4 & 2 \end{vmatrix}=6-16=-10,\quad A_{32}=-\begin{vmatrix} 2 & -4 \\ 0 & 2 \end{vmatrix}=-(4-0)=-4,\quad A_{33}=\begin{vmatrix} 2 & 3 \\ 0 & -4 \end{vmatrix}=-8+0=-8$$

So,

$$\text{adj }A=\begin{bmatrix} -18 & 2 & 4 \\ -11 & 14 & 5 \\ -10 & -4 & -8 \end{bmatrix}^T=\begin{bmatrix} -18 & -11 & -10 \\ 2 & 14 & -4 \\ 4 & 5 & -8 \end{bmatrix}$$

Therefore,

$$A^{-1}=\frac{1}{|A|}\text{adj }A=-\frac{1}{46}\begin{bmatrix} -18 & -11 & -10 \\ 2 & 14 & -4 \\ 4 & 5 & -8 \end{bmatrix}=\begin{bmatrix} \frac{9}{23} & \frac{11}{46} & \frac{5}{23} \\ -\frac{1}{23} & -\frac{7}{23} & \frac{2}{23} \\ -\frac{2}{23} & -\frac{5}{46} & \frac{4}{23} \end{bmatrix}$$

**Problem:** Find the inverse of the matrix $A=\begin{bmatrix} 2 & -1 & -1 \\ 1 & -2 & 1 \\ 1 & -1 & 2 \end{bmatrix}$.

**Solution:** Given the matrix

$$A=\begin{bmatrix} 2 & -1 & -1 \\ 1 & -2 & 1 \\ 1 & -1 & 2 \end{bmatrix}$$

Determinant of A, $|A|=2\times(-4+1)+1\times(2-1)-1\times(-1+2)=-6+1-1=-6\neq 0$

The cofactors of A are

$$A_{11}=\begin{vmatrix} -2 & 1 \\ -1 & 2 \end{vmatrix}=-4+1=-3,\quad A_{12}=-\begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix}=-(2-1)=-1,\quad A_{13}=\begin{vmatrix} 1 & -2 \\ 1 & -1 \end{vmatrix}=-1+2=1$$

$$A_{21}=-\begin{vmatrix} -1 & -1 \\ -1 & 2 \end{vmatrix}=-(-2-1)=3,\quad A_{22}=\begin{vmatrix} 2 & -1 \\ 1 & 2 \end{vmatrix}=4+1=5,\quad A_{23}=-\begin{vmatrix} 2 & -1 \\ 1 & -1 \end{vmatrix}=-(-2+1)=1$$

$$A_{31}=\begin{vmatrix} -1 & -1 \\ -2 & 1 \end{vmatrix}=-1-2=-3,\quad A_{32}=-\begin{vmatrix} 2 & -1 \\ 1 & 1 \end{vmatrix}=-(2+1)=-3,\quad A_{33}=\begin{vmatrix} 2 & -1 \\ 1 & -2 \end{vmatrix}=-4+1=-3$$

So,

$$\text{adj } A=\begin{bmatrix} -3 & -1 & 1 \\ 3 & 5 & 1 \\ -3 & -3 & -3 \end{bmatrix}^T=\begin{bmatrix} -3 & 3 & -3 \\ -1 & 5 & -3 \\ 1 & 1 & -3 \end{bmatrix}$$

Therefore,

$$A^{-1}=\frac{1}{|A|}\text{adj } A=-\frac{1}{6}\begin{bmatrix} -3 & 3 & -3 \\ -1 & 5 & -3 \\ 1 & 1 & -3 \end{bmatrix}=\begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{6} & -\frac{5}{6} & \frac{1}{2} \\ -\frac{1}{6} & -\frac{1}{6} & \frac{1}{2} \end{bmatrix}$$

$x+2y=3$

$2x+4y=6$

$2(3-2y)+4y=6\Rightarrow 6-4y+4y=6\Rightarrow 0=0$

$\Rightarrow x+2y=3$

$x=1,$

$1+2y=3$

$\therefore y=1$

Variables=2

Equation=1

Number of Variables-Number of Equations=2-1=1

Therefore, number of free variable is 1.

$x+2y=3$

$2x+4y=2$

$\Rightarrow x+2y=3$

$1=3$

$x+2y=3$

$4y=2$

$\Rightarrow y=1/2$

**System of linear equations**

$AX=B$

Multiplying both sides by $A^{-1}$ then we have

$A^{-1}AX=A^{-1}B$

$\Rightarrow IX=A^{-1}B$

$\therefore X=A^{-1}B$

$$A=\begin{bmatrix} 2 & -1 & -1 \\ 1 & -2 & 1 \\ 1 & -1 & 2 \end{bmatrix}$$

Example 10 (page 25):

**Solution:** Given the system of equations

$$\left.\begin{array}{l} x+y-z=1 \\ 2x+3y+\lambda z=3 \\ x+\lambda y+3z=2 \end{array}\right\} \quad \begin{array}{c} L_2 \to -2L_1+L_2 \\ L_3 \to -L_1+L_3 \end{array} \quad \left.\begin{array}{l} x+y-z=1 \\ y+(\lambda+2)z=1 \\ (\lambda-1)y+4z=1 \end{array}\right\}$$

$$L_3 \to -(\lambda-1)L_2+L_3 \quad \left.\begin{array}{l} x+y-z=1 \\ y+(\lambda+2)z=1 \\ \{-(\lambda-1)(\lambda+2)+4\}z=-\lambda+1+1 \end{array}\right\}$$

$$\left.\begin{array}{l} x+y-z=1 \\ y+(\lambda+2)z=1 \\ (\lambda+3)(2-\lambda)z=2-\lambda \end{array}\right\}$$

(i) The above system has a unique solution if $(\lambda+3)(2-\lambda)\neq 0$,

=> $\lambda+3\neq 0$ and $2-\lambda\neq 0$ => $\lambda\neq -3$ and $\lambda\neq 2$

(ii) The above system has more than one solution if $2-\lambda=0$ => $\lambda=2$

(iii) The above system has no solution if $\lambda+3=0$ => $\lambda=-3$

**Exercise 30 (Page 36):** Given the system of equations

$$\left.\begin{aligned} &x-3z=-3\\ &2x+\lambda y-z=-2\\ &x+2y+\lambda z=1 \end{aligned}\right\} \quad L_1 \leftrightarrow L_3$$

$$\left.\begin{aligned} &x+2y+\lambda z=1\\ &2x+\lambda y-z=-2\\ &x-3z=-3 \end{aligned}\right\} \quad \begin{aligned} &L_2 \longrightarrow -2L_1+L_2\\ &L_3 \longrightarrow -L_1+L_3 \end{aligned} \quad \left.\begin{aligned} &x+2y+\lambda z=1\\ &(\lambda+4)y+(2\lambda-2)z=0\\ &-2y-(\lambda+3)z=-4 \end{aligned}\right\}$$

$$L_3 \longrightarrow 2L_2+(\lambda+4)L_3 \quad \left.\begin{aligned} &x+2y+\lambda z=1\\ &(\lambda+4)y+(2\lambda-1)z=0\\ &\{(4\lambda-4)-(\lambda+4)(\lambda+3)\}z=-4(\lambda+4) \end{aligned}\right\}$$

$$\left.\begin{aligned} &x+2y+\lambda z=1\\ &(\lambda+4)y+(2\lambda-1)z=0\\ &(\lambda^2+7\lambda+12-4\lambda+4)z=4(\lambda+4) \end{aligned}\right\}$$

$$\text{Or, } \left.\begin{aligned} &x+2y+\lambda z=1\\ &(\lambda+4)y+(2\lambda-1)z=0\\ &(\lambda^2+3\lambda+16)z=4(\lambda+4) \end{aligned}\right\}$$

$$\text{Or, } \left.\begin{aligned} &x+2y+\lambda z=1\\ &(\lambda+4)y+(2\lambda-1)z=0\\ &(\lambda+5)(\lambda-2)z=4(\lambda-2) \end{aligned}\right\}$$

(i) The above system has a unique solution if $(\lambda+5)(\lambda-2)\neq 0$,

=> $\lambda+5\neq 0$ and $\lambda-2\neq 0$ => $\lambda\neq -5$ and $\lambda\neq 2$

(ii) The above system has more than one solution if $\lambda-2=0$ => $\lambda=2$

(iii) The above system has no solution if $\lambda+5=0$ => $\lambda=-5$

() For S is a subspace,

(i) S is nonempty

(ii) $u,v\in S$, $\alpha u+\beta v\in S$

$R_1 \leftrightarrow R_3$

$L_3 \rightarrow L_3/2$