Q1. If z is a complex number, prove that $|z|^2 = z\bar{z}$.

Proof: Let $z = x + iy$.

Then

$$|z|^2 = \left(\sqrt{x^2+y^2}\right)^2 = x^2 + y^2$$

and

$$z\bar{z} = (x+iy)(x-iy) = x^2 - i^2y^2 = x^2 + y^2 \quad \left[\therefore\, i^2 = -1\right]$$

Thus we have

$$|z|^2 = z\bar{z}$$

Q1. If z is a complex number, prove that $\bar{\bar{z}} = z$.

Proof: Let $z = x + iy$.

By definition of complex conjugate, we get

$$\bar{z} = x - iy$$

Again

$$\begin{aligned} \bar{\bar{z}} &= \overline{x - iy} \\ &= x + iy \\ &= z \end{aligned}$$

$$\therefore\ \bar{\bar{z}} = z$$

Q2. If z_1 and z_2 are complex numbers, prove that $\overline{z_1z_2} = \bar{z}_1\bar{z}_2$.

Proof: Let $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$.

Then we get

$$\begin{aligned} z_1z_2 &= (x_1+iy_1)(x_2+iy_2) \\ &= x_1x_2 + ix_1y_2 + ix_2y_1 + i^2y_1y_2 \\ &= x_1x_2 + i(x_1y_2 + x_2y_1) + (-1)\,y_1y_2 \\ &= x_1x_2 - y_1y_2 + i(x_1y_2 + x_2y_1) \end{aligned}$$

$$\therefore\ \overline{z_1z_2} = x_1x_2 - y_1y_2 - i(x_1y_2 + x_2y_1) \qquad (1)$$

Now by definition of complex conjugate, we have

$\bar{z}_1 = x_1 - iy_1$ and $\bar{z}_2 = x_2 - iy_2$

Therefore

$$\begin{aligned} \bar{z}_1\bar{z}_2 &= (x_1-iy_1)(x_2-iy_2) \\ &= x_1x_2 - ix_1y_2 - ix_2y_1 + i^2y_1y_2 \\ &= x_1x_2 - y_1y_2 - i(x_1y_2 + x_2y_1) \end{aligned}$$

$$\therefore\ \bar{z}_1\bar{z}_2 = x_1x_2 - y_1y_2 - i(x_1y_2 + x_2y_1) \qquad (2)$$

From (1) and (2), we can write

$\overline{z_1 z_2} = \bar{z}_1 \bar{z}_2$

Q2. If z_1 and z_2 are complex numbers, prove that

(i) $\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$ (ii) $|z_1 z_2| = |z_1||z_2|$ (iii) $|z_1 + z_2| \le |z_1| + |z_2|$

Proof: Let $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$.

By definition of complex conjugate, we have

$$\bar{z}_1 = x_1 - iy_1 \text{ and } \bar{z}_2 = x_2 - iy_2 \tag{3}$$

(i) Now $z_1 + z_2 = x_1 + iy_1 + x_2 + iy_2$

$$= x_1 + x_2 + i(y_1 + y_2)$$

$\therefore \overline{z_1 + z_2} = x_1 + x_2 - i(y_1 + y_2)$

$$= x_1 - iy_1 + x_2 - iy_2$$

$$= \bar{z}_1 + \bar{z}_2 \quad \text{[Using (3)]}$$

$$\therefore \overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2 \tag{4}$$

(ii) We have

$$|z_1 z_2|^2 = (z_1 z_2)\overline{(z_1 z_2)}$$

$$= (z_1 z_2)(\bar{z}_1 \bar{z}_2) \quad \left[\text{since } \overline{z_1 z_2} = \bar{z}_1 \bar{z}_2\right]$$

$$= (z_1 \bar{z}_1)(z_2 \bar{z}_2)$$

$$= |z_1|^2 |z_2|^2$$

$$\therefore |z_1 z_2| = |z_1||z_2|$$

Note: $z = x + iy$ and $\bar{z} = x - iy$

$$z + \bar{z} = x + iy + x - iy = 2x = 2\operatorname{Re}(z)$$

$$|z| = |x + iy| = \sqrt{x^2 + y^2} \ge \sqrt{x^2} = x = \operatorname{Re}(z)$$

$$\therefore \operatorname{Re}(z) \le |z|$$

$$|\bar{z}| = |x - iy| = \sqrt{x^2 + (-y)^2} = \sqrt{x^2 + y^2} = |z|$$

(iii) We can write

$$|z_1 + z_2|^2 = (z_1 + z_2)\left(\overline{z_1 + z_2}\right)$$

$$= (z_1 + z_2)(\bar{z}_1 + \bar{z}_2)$$

$$= z_1\bar{z}_1 + z_1\bar{z}_2 + \bar{z}_1 z_2 + z_2\bar{z}_2$$

$$= z_1\bar{z}_1 + z_1\bar{z}_2 + \overline{z_1\bar{z}_2} + z_2\bar{z}_2 \quad [\because \bar{\bar{z}} = z]$$

$$= z_1\bar{z}_1 + z_3 + \bar{z}_3 + z_2\bar{z}_2 \quad [\because z_3 = z_1\bar{z}_2]$$

$$= z_1\bar{z}_1 + 2\operatorname{Re}(z_1\bar{z}_2) + z_2\bar{z}_2 \quad [\because z + \bar{z} = 2\operatorname{Re}(z)]$$

$$\therefore\ |z_1 + z_2|^2 \le |z_1|^2 + 2|z_1\bar{z}_2| + |z_2|^2 \qquad [\because\ \mathrm{Re}(z) \le |z|]$$

$$\Rightarrow |z_1 + z_2|^2 \le |z_1|^2 + 2|z_1||\bar{z}_2| + |z_2|^2 \qquad [\because\ |z_1 z_2| = |z_1||z_2|]$$

$$\Rightarrow |z_1 + z_2|^2 \le |z_1|^2 + 2|z_1||z_2| + |z_2|^2$$

$$\Rightarrow |z_1 + z_2|^2 \le (|z_1| + |z_2|)^2$$

$$\therefore\ |z_1 + z_2| \le |z_1| + |z_2|$$

3.1 Derivatives

If $f(z)$ is single-valued in some region $\mathcal{R}$ of the z plane, the *derivative* of $f(z)$ is defined as

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z} \tag{3.1}$$

provided that the limit exists independent of the manner in which $\Delta z \to 0$. In such a case, we say that $f(z)$ is *differentiable* at z. In the definition (3.1), we sometimes use h instead of Δz. Although differentiability implies continuity, the reverse is not true (see Problem 3.4).

3.2 Analytic Functions

If the derivative $f'(z)$ exists at all points z of a region $\mathcal{R}$, then $f(z)$ is said to be *analytic in* $\mathcal{R}$ and is referred to as an *analytic function in* $\mathcal{R}$ or a function *analytic in* $\mathcal{R}$. The terms *regular* and *holomorphic* are sometimes used as synonyms for analytic.

A function $f(z)$ is said to be *analytic at a point* z_0 if there exists a neighborhood $|z - z_0| < \delta$ at all points of which $f'(z)$ exists.

3.3 Cauchy–Riemann Equations

A necessary condition that $w = f(z) = u(x, y) + iv(x, y)$ be analytic in a region $\mathcal{R}$ is that, in $\mathcal{R}$, u and v satisfy the *Cauchy–Riemann equations*

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \tag{3.2}$$

If the partial derivatives in (3.2) are continuous in $\mathcal{R}$, then the Cauchy–Riemann equations are sufficient conditions that $f(z)$ be analytic in $\mathcal{R}$. See Problem 3.5.

The functions $u(x, y)$ and $v(x, y)$ are sometimes called *conjugate functions*. Given u having continuous first partials on a simply connected region $\mathcal{R}$ (see Section 4.6), we can find v (within an arbitrary additive constant) so that $u + iv = f(z)$ is analytic (see Problems 3.7 and 3.8).

3.4 Harmonic Functions

If the second partial derivatives of u and v with respect to x and y exist and are continuous in a region $\mathcal{R}$, then we find from (3.2) that (see Problem 3.6)

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0, \qquad \frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}=0 \tag{3.3}$$

It follows that under these conditions, the real and imaginary parts of an analytic function satisfy *Laplace's equation* denoted by

$$\frac{\partial^2 \Psi}{\partial^2 x}+\frac{\partial^2 \Psi}{\partial^2 y}=0 \quad \text{or} \quad \nabla^2\Psi=0 \quad \text{where } \nabla^2 \equiv \frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2} \tag{3.4}$$

The operator ∇^2 is often called the *Laplacian*.

Functions such as $u(x, y)$ and $v(x, y)$ which satisfy Laplace's equation in a region $\mathcal{R}$ are called *harmonic functions* and are said to be *harmonic in* $\mathcal{R}$.

3.2. Show that $(d/dz)\bar{z}$ does not exist anywhere, i.e., $f(z)=\bar{z}$ is non-analytic anywhere.

Solution

By definition,

$$\frac{d}{dz}f(z)=\lim_{\Delta z\to 0}\frac{f(z+\Delta z)-f(z)}{\Delta z}$$

if this limit exists independent of the manner in which $\Delta z=\Delta x+i\Delta y$ approaches zero.

Then

$$\frac{d}{dz}\bar{z}=\lim_{\Delta z\to 0}\frac{\overline{z+\Delta z}-\bar{z}}{\Delta z}=\lim_{\substack{\Delta x\to 0\\ \Delta y\to 0}}\frac{\overline{x+iy+\Delta x+i\Delta y}-\overline{x+iy}}{\Delta x+i\Delta y}$$

$$=\lim_{\substack{\Delta x\to 0\\ \Delta y\to 0}}\frac{x-iy+\Delta x-i\Delta y-(x-iy)}{\Delta x+i\Delta y}=\lim_{\substack{\Delta x\to 0\\ \Delta y\to 0}}\frac{\Delta x-i\Delta y}{\Delta x+i\Delta y}$$

If $\Delta y=0$, the required limit is

$$\lim_{\Delta x\to 0}\frac{\Delta x}{\Delta x}=1$$

If $\Delta x=0$, the required limit is

$$\lim_{\Delta y\to 0}\frac{-i\Delta y}{i\Delta y}=-1$$

Then, since the limit depends on the manner in which $\Delta z\to 0$, the derivative does not exist, i.e., $f(z)=\bar{z}$ is *non-analytic* anywhere.

$$\int y\cos y\,dy = y\int \cos y\,dy-\int\left\{\frac{d}{dy}(y)\int\cos y\,dy\right\}dy$$

$$= y\sin y-\int \sin y\,dy$$

$$= y\sin y-(-\cos y)$$

$$= y\sin y+\cos y$$

3.7. (a) Prove that $u = e^{-x}(x \sin y - y \cos y)$ is harmonic.

(b) Find v such that $f(z) = u + iv$ is analytic.

Solution

(a) $\dfrac{\partial u}{\partial x} = (e^{-x})(\sin y) + (-e^{-x})(x \sin y - y \cos y) = e^{-x} \sin y - xe^{-x} \sin y + ye^{-x} \cos y$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}(e^{-x} \sin y - xe^{-x} \sin y + ye^{-x} \cos y) = -2e^{-x} \sin y + xe^{-x} \sin y - ye^{-x} \cos y \tag{1}$$

$$\frac{\partial u}{\partial y} = e^{-x}(x \cos y + y \sin y - \cos y) = xe^{-x} \cos y + ye^{-x} \sin y - e^{-x} \cos y$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y}(xe^{-x} \cos y + ye^{-x} \sin y - e^{-x} \cos y) = -xe^{-x} \sin y + 2e^{-x} \sin y + ye^{-x} \cos y \tag{2}$$

Adding (1) and (2) yields $(\partial^2 u/\partial x^2) + (\partial^2 u/\partial y^2) = 0$ and u is harmonic.

(b) From the Cauchy–Riemann equations,

$$\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x} = e^{-x} \sin y - xe^{-x} \sin y + ye^{-x} \cos y \tag{3}$$

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} = e^{-x} \cos y - xe^{-x} \cos y - ye^{-z} \sin y \tag{4}$$

Integrate (3) with respect to y, keeping x constant. Then

$$\begin{aligned} v &= -e^{-x} \cos y + xe^{-x} \cos y + e^{-x}(y \sin y + \cos y) + F(x) \\ &= ye^{-x} \sin y + xe^{-x} \cos y + F(x) \end{aligned} \tag{5}$$

where $F(x)$ is an arbitrary real function of x.

Substitute (5) into (4) and obtain

$$-ye^{-x} \sin y - xe^{-x} \cos y + e^{-x} \cos y + F'(x) = -ye^{-x} \sin y - xe^{-x} \cos y - ye^{-x} \sin y$$

or $F'(x) = 0$ and $F(x) = c$, a constant. Then, from (5),

$$v = e^{-x}(y \sin y + x \cos y) + c$$

3.8. Find $f(z)$ in Problem 3.7.

Solution

Method 1

We have $\qquad f(z) = f(x + iy) = u(x, y) + iv(x, y).$

Putting $y = 0$ $\qquad f(x) = u(x, 0) + iv(x, 0)$.

Replacing x by z, $\qquad f(z) = u(z, 0) + iv(z, 0)$.

Then, from Problem 3.7, $u(z, 0) = 0$, $v(z, 0) = ze^{-z}$ and so $f(z) = u(z, 0) + iv(z, 0) = ize^{-z}$, apart from an arbitrary additive constant.

Method 2

Apart from an arbitrary additive constant, we have from the results of Problem 3.7,

$$\begin{aligned} f(z) = u + iv &= e^{-x}(x \sin y - y \cos y) + ie^{-x}(y \sin y + x \cos y) \\ &= e^{-x}\left\{x\left(\frac{e^{iy} - e^{-iy}}{2i}\right) - y\left(\frac{e^{iy} + e^{-iy}}{2}\right)\right\} + ie^{-x}\left\{y\left(\frac{e^{iy} - e^{-iy}}{2i}\right) + x\left(\frac{e^{iy} + e^{-iy}}{2}\right)\right\} \\ &= i(x + iy)e^{-(x+iy)} = ize^{-z} \end{aligned}$$

Q. Prove that $\lim\limits_{z\to 0} \frac{\bar{z}}{z}$ does not exist.

Proof: We have

$$\begin{aligned} \lim_{z\to 0} \frac{\bar{z}}{z} &= \lim_{\substack{x\to 0 \\ y\to 0}} \frac{\overline{x+iy}}{x+iy} \\ &= \lim_{\substack{x\to 0 \\ y\to 0}} \frac{x-iy}{x+iy} \end{aligned}$$

Now along the real axis, $y = 0$, $x\to 0$ and we obtain

$$\begin{aligned} \lim_{z\to 0} \frac{\bar{z}}{z} &= \lim_{x\to 0} \frac{x - i.0}{x + i.0} \\ &= \lim_{x\to 0} \frac{x}{x} \\ &= 1 \end{aligned}$$

But, along the imaginary axis, $x = 0$, $y\to 0$ and we get

$$\begin{aligned} \lim_{z\to 0} \frac{\bar{z}}{z} &= \lim_{y\to 0} \frac{0 - iy}{0 + iy} \\ &= \lim_{y\to 0} \frac{-iy}{iy} \\ &= -1 \end{aligned}$$

Since the limits vary on the manner in which $z\to 0$, hence $\lim\limits_{z\to 0} \frac{\bar{z}}{z}$ does not exist.

Q. For the function, $f(z)$ defined by

$$f(z) = \begin{cases} \frac{(\bar{z})^2}{z}, & z \neq 0 \\ 0, & z = 0 \end{cases}$$

show that the C-R equations are satisfied at $z = 0$ but the function is not differentiable at $z = 0$.

Proof: We assume that

$$f(z) = u(x,y) + i\upsilon(x,y)$$

Then at z = 0, $\quad f(0) = u(0,0) + i\upsilon(0,0)$

$$\Rightarrow 0 = u(0,0) + i\upsilon(0,0)$$

Therefore, $u(0,0) = 0,\ \upsilon(0,0) = 0$

Now for $z \neq 0$,

$$f(z) = \frac{(\bar{z})^2}{z}$$

$$= \frac{(\bar{z})^3}{z\bar{z}}$$

$$= \frac{(x-iy)^3}{(x+iy)(x-iy)}$$

$$= \frac{x^3 - 3x^2.iy + 3x(iy)^2 - (iy)^3}{(x^2 - i^2y^2)}$$

$$= \frac{x^3 - 3ix^2y - 3xy^2 + iy^3}{(x^2 + y^2)}$$

$$= \frac{x^3 - 3xy^2}{x^2 + y^2} + i\,\frac{y^3 - 3x^2y}{x^2 + y^2}$$

So $\quad u(x,y) = \dfrac{x^3 - 3xy^2}{x^2 + y^2},\ \upsilon(x,y) = \dfrac{y^3 - 3x^2y}{x^2 + y^2}$

We know

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z}$$

$$= \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{f(x + \Delta x, y + \Delta y) - f(x,y)}{\Delta x + i\Delta y}$$

$$f'(0) = \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{f(\Delta x, \Delta y) - f(0,0)}{\Delta x + i\Delta y}$$

At z = 0, we find

$$\frac{\partial u}{\partial x} = \lim_{\Delta x \to 0} \frac{u(\Delta x, 0) - u(0,0)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\dfrac{(\Delta x)^3 - 3\Delta x.0^2}{(\Delta x)^2 + 0^2} - 0}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\frac{(\Delta x)^3}{(\Delta x)^2}}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\Delta x}{\Delta x}$$

$$= 1$$

$$\frac{\partial u}{\partial y} = \lim_{\Delta y \to 0} \frac{u(0, \Delta y) - u(0,0)}{\Delta y}$$

$$= \lim_{\Delta y \to 0} \frac{\frac{0^3 - 3.0.\Delta y^2}{0^2 + (\Delta y)^2} - 0}{\Delta y}$$

$$= \lim_{\Delta y \to 0} \frac{0}{\Delta y}$$

$$= 0$$

$$\frac{\partial \upsilon}{\partial x} = \lim_{\Delta x \to 0} \frac{\upsilon(\Delta x, 0) - \upsilon(0,0)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\frac{0^3 - 3\Delta x.0^2}{(\Delta x)^2 + 0^2} - 0}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{0}{\Delta x}$$

$$= 0$$

$$\frac{\partial \upsilon}{\partial y} = \lim_{\Delta y \to 0} \frac{\upsilon(0, \Delta y) - \upsilon(0,0)}{\Delta y}$$

$$= \lim_{\Delta y \to 0} \frac{\frac{(\Delta y)^3 - 3\Delta x^2.0}{0^2 + (\Delta y)^2} - 0}{\Delta y}$$

$$= \lim_{\Delta y \to 0} \frac{\frac{(\Delta y)^3}{(\Delta y)^2}}{\Delta y}$$

$$= \lim_{\Delta y \to 0} \frac{\Delta y}{\Delta y}$$

$$= 1$$

Therefore, we have $\frac{\partial u}{\partial x} = \frac{\partial \upsilon}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial \upsilon}{\partial x}$ at $z = 0$.

Hence Cauchy-Riemann equations are satisfied at $z = 0$.

Differentiability: By definition, we have

$$f'(0) = \lim_{z \to 0} \frac{f(z) - f(0)}{z - 0}$$

$$= \lim_{z \to 0} \frac{\frac{(\bar{z})^2}{z} - 0}{z - 0}$$

$$= \lim_{z \to 0} \frac{(\bar{z})^2}{z^2}$$

$$= \lim_{\substack{x \to 0 \\ y \to 0}} \left(\frac{x - iy}{x + iy} \right)^2$$

Along the real axis $y = 0$, $x \to 0$ and we get

$$f'(0) = \lim_{x \to 0} \left(\frac{x}{x} \right)^2 = 1$$

But along the line $y = x$, we take $x \to 0$. Therefore,

$$f'(0) = \lim_{x \to 0} \left(\frac{x - ix}{x + ix} \right)^2$$

$$= \lim_{x \to 0} \left(\frac{1 - i}{1 + i} \right)^2$$

$$= \frac{1 - 2i + i^2}{1 + 2i + i^2}$$

$$= \frac{1 - 2i - 1}{1 + 2i - 1}$$

$$= \frac{-2i}{2i}$$

$$= -1$$

Since $f'(0)$ depends on the variation of manner in which $z \to 0$, hence the function is not differentiable at $z = 0$.

{Mid-2}

Simply and multiply connected regions

A region R is called simply-connected if any simple closed curve which lies in R can be shrunk to a point without leaving R. A region R which is not simply-connected is called multiply-connected.

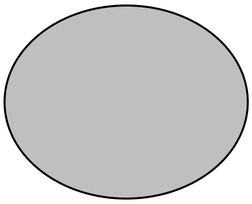


Fig. 4-1 Simply-connected

For example, suppose $\mathcal{R}$ is the region defined by $|z| < 2$ shown shaded in Fig. 4-2. If Γ is any simple closed curve lying in $\mathcal{R}$ [i.e., whose points are in $\mathcal{R}$], we see that it can be shrunk to a point that lies in $\mathcal{R}$, and thus does not leave $\mathcal{R}$, so that $\mathcal{R}$ is simply-connected. On the other hand, if $\mathcal{R}$ is the region defined by $1 < |z| < 2$, shown shaded in Fig. 4-3, then there is a simple closed curve Γ lying in $\mathcal{R}$ that cannot possibly be shrunk to a point without leaving $\mathcal{R}$, so that $\mathcal{R}$ is multiply-connected.

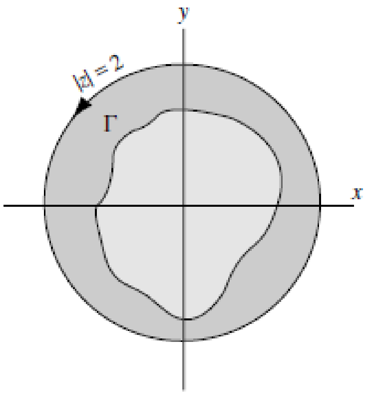


Fig. 4-2

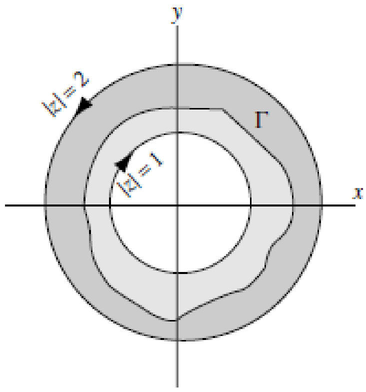


Fig. 4-3

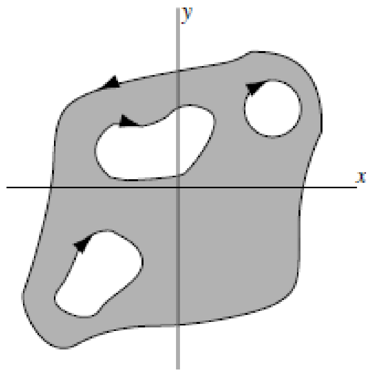


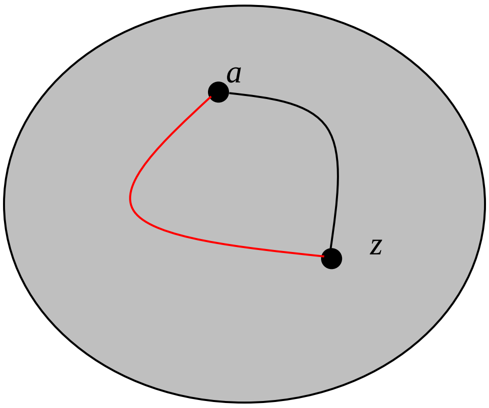
Fig. 4-4

Intuitively, a simply-connected region is one that does not have any "holes" in it, while a multiply-connected region is one that does. The multiply-connected regions of Figs. 4-3 and 4-4 have, respectively, one and three holes in them.

Theorem 1. Let $f(z)$ be analytic in a simply-connected region R. If a and z are any two points in R, then

$$\int_a^z f(z)\,dz$$

is independent of the path in R joining a and z.



Theorem 4. Let $f(z)$ be analytic in a region bounded by two simple closed curves C and C_1 [where C_1 lies inside C as in Fig. 4-5 below] and on these curves. Then

$$\oint_C f(z)\,dz = \oint_{C_1} f(z)\,dz \tag{16}$$

where C and C_1 are both traversed in the positive sense relative to their interiors [counter-clockwise in Fig. 4-5].

The result shows that if we wish to integrate $f(z)$ along curve C we can equivalently replace C by any curve C_1 so long as $f(z)$ is analytic in the region between C and C_1.

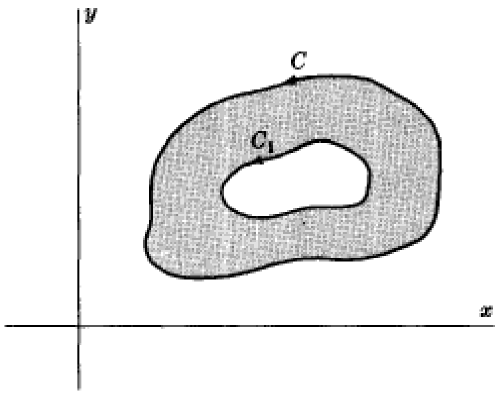


Fig. 4-5

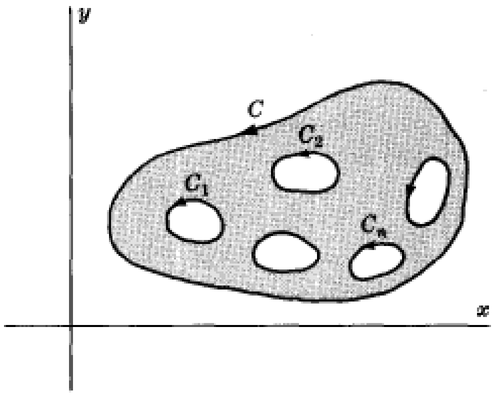
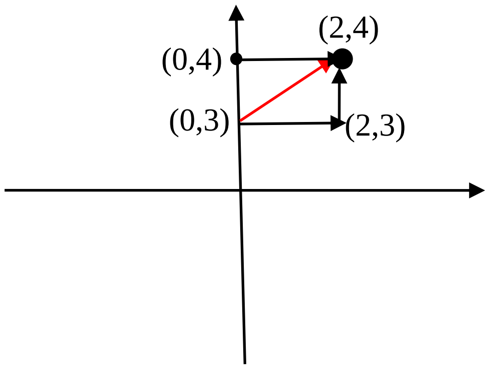


Fig. 4-6

Theorem 5. Let $f(z)$ be analytic in a region bounded by the non-overlapping simple closed curves $C, C_1, C_2, C_3, \ldots, C_n$ [where $C_1, C_2, \ldots, C_n$ are inside C as in Fig. 4-6 above] and on these curves. Then

$$\oint_C f(z)\,dz = \oint_{C_1} f(z)\,dz + \oint_{C_2} f(z)\,dz + \cdots + \oint_{C_n} f(z)\,dz \tag{17}$$

This is a generalization of Theorem 4.



LINE INTEGRALS

1. Evaluate $\int_{(0,3)}^{(2,4)} (2y + x^2)\,dx + (3x - y)\,dy$ along: (*a*) the parabola $x = 2t$, $y = t^2 + 3$; (*b*) straight lines from $(0,3)$ to $(2,3)$ and then from $(2,3)$ to $(2,4)$; (*c*) a straight line from $(0,3)$ to $(2,4)$.

(*a*) The points $(0,3)$ and $(2,4)$ on the parabola correspond to $t = 0$ and $t = 1$ respectively. Then the given integral equals

$$\int_{t=0}^{1} \{2(t^2+3) + (2t)^2\}\,2\,dt + \{3(2t) - (t^2+3)\}\,2t\,dt = \int_0^1 (24t^2 + 12 - 2t^3 - 6t)\,dt = 33/2$$

(*b*) Along the straight line from $(0,3)$ to $(2,3)$, $y = 3$, $dy = 0$ and the line integral equals

$$\int_{x=0}^{2} (6 + x^2)\,dx + (3x-3)0 = \int_{x=0}^{2} (6+x^2)\,dx = 44/3$$

Along the straight line from $(2,3)$ to $(2,4)$, $x = 2$, $dx = 0$ and the line integral equals

$$\int_{y=3}^{4} (2y+4)0 + (6-y)\,dy = \int_{y=3}^{4} (6-y)\,dy = 5/2$$

Then the required value $= 44/3 + 5/2 = 103/6$.

(*c*) An equation for the line joining $(0,3)$ and $(2,4)$ is $2y - x = 6$. Solving for x, we have $x = 2y - 6$. Then the line integral equals

$$\int_{y=3}^{4} \{2y + (2y-6)^2\}\,2\,dy + \{3(2y-6) - y\}\,dy = \int_3^4 (8y^2 - 39y + 54)\,dy = 97/6$$

The result can also be obtained by using $y = \frac{1}{2}(x+6)$.

2. Evaluate $\int_C \bar{z}\,dz$ from $z = 0$ to $z = 4 + 2i$ along the curve C given by (*a*) $z = t^2 + it$, (*b*) the line from $z = 0$ to $z = 2i$ and then the line from $z = 2i$ to $z = 4 + 2i$.

(*a*) The points $z = 0$ and $z = 4 + 2i$ on C correspond to $t = 0$ and $t = 2$ respectively. Then the line integral equals

$$\int_{t=0}^{2} (\overline{t^2 + it})\,d(t^2 + it) = \int_0^2 (t^2 - it)(2t + i)\,dt = \int_0^2 (2t^3 - it^2 + t)\,dt = 10 - 8i/3$$

1(a) $\int_{t=0}^{1}\left(24t^2 + 12 - 2t^3 - 6t\right)dt$

$$= \left[24\times\frac{t^3}{3} + 12t - 2\times\frac{t^4}{4} - 6\times\frac{t^2}{2}\right]_0^1$$

$$= 8\left(1^3 - 0^3\right) + 12(1-0) - \frac{1}{2}\left(1^4 - 0^4\right) - 3\left(1^2 - 0^2\right)$$

$$= 8 + 12 - \frac{1}{2} - 3 = \frac{33}{2}$$

1(c) Equation of a straight line passing through (0,3) and (2,4) is

$$y-3=\frac{4-3}{2-0}(x-0)$$

$$\Rightarrow y-3=\frac{1}{2}x$$

$$\Rightarrow 2y-6=x$$

$$\therefore x=2y-6$$

$$\therefore dx=2dy$$

2(a) $\int_{t=0}^{2}\left(2t^3-it^2+t\right)dt=\left[2\times\frac{t^4}{4}-i\frac{t^3}{3}+\frac{t^2}{2}\right]_0^2=\frac{2^4}{2}-i\frac{2^3}{3}+\frac{2^2}{2}=8-\frac{8i}{3}+2=10-\frac{8i}{3}$

(b) The given line integral equals

$$\int_C (x-iy)(dx+i\,dy) \quad = \quad \int_C x\,dx+y\,dy \quad + \quad i\int_C x\,dy-y\,dx$$

The line from $z=0$ to $z=2i$ is the same as the line from $(0,0)$ to $(0,2)$ for which $x=0$, $dx=0$ and the line integral equals

$$\int_{y=0}^{2}(0)(0)+y\,dy \quad + \quad i\int_{y=0}^{2}(0)(dy)-y(0) \quad = \quad \int_{y=0}^{2}y\,dy \quad = \quad 2$$

The line from $z=2i$ to $z=4+2i$ is the same as the line from $(0,2)$ to $(4,2)$ for which $y=2$, $dy=0$ and the line integral equals

$$\int_{x=0}^{4}x\,dx+2\cdot 0 \quad + \quad i\int_{x=0}^{4}x\cdot 0-2\,dx \quad = \quad \int_0^4 x\,dx \quad + \quad i\int_0^4 -2\,dx \quad = \quad 8-8i$$

Then the required value $= 2+(8-8i) = 10-8i$.

23. If C is the curve $y = x^3-3x^2+4x-1$ joining points $(1,1)$ and $(2,3)$, find the value of

$$\int_C (12z^2-4iz)\,dz$$

Method 1. By Problem 17, the integral is independent of the path joining $(1,1)$ and $(2,3)$. Hence any path can be chosen. In particular let us choose the straight line paths from $(1,1)$ to $(2,1)$ and then from $(2,1)$ to $(2,3)$.

Case 1. Along the path from $(1,1)$ to $(2,1)$, $y=1$, $dy=0$ so that $z = x+iy = x+i$, $dz = dx$. Then the integral equals

$$\int_{x=1}^{2}\{12(x+i)^2-4i(x+i)\}\,dx \quad = \quad \{4(x+i)^3-2i(x+i)^2\}\Big|_1^2 \quad = \quad 20+30i$$

Case 2. Along the path from $(2,1)$ to $(2,3)$, $x=2$, $dx=0$ so that $z = x+iy = 2+iy$, $dz = i\,dy$. Then the integral equals

$$\int_{y=1}^{3}\{12(2+iy)^2-4i(2+iy)\}i\,dy \quad = \quad \{4(2+iy)^3-2i(2+iy)^2\}\Big|_1^3 \quad = \quad -176+8i$$

Then adding, the required value $= (20+30i)+(-176+8i) = -156+38i$.

Method 2. The given integral equals

$$\int_{1+i}^{2+3i}(12z^2-4iz)\,dz \quad = \quad (4z^3-2iz^2)\Big|_{1+i}^{2+3i} \quad = \quad -156+38i$$

It is clear that Method 2 is easier.

LINE INTEGRALS

32. Evaluate $\int_{(0,1)}^{(2,5)} (3x+y)\,dx + (2y-x)\,dy$ along (*a*) the curve $y = x^2+1$, (*b*) the straight line joining $(0,1)$ and $(2,5)$, (*c*) the straight lines from $(0,1)$ to $(0,5)$ and then from $(0,5)$ to $(2,5)$, (*d*) the straight lines from $(0,1)$ to $(2,1)$ and then from $(2,1)$ to $(2,5)$.
Ans. (*a*) 88/3, (*b*) 32, (*c*) 40, (*d*) 24

33. (*a*) Evaluate $\oint_C (x+2y)\,dx + (y-2x)\,dy$ around the ellipse C defined by $x = 4\cos\theta$, $y = 3\sin\theta$, $0 \leq \theta < 2\pi$ if C is described in a counterclockwise direction. (*b*) What is the answer to (*a*) if C is described in a clockwise direction? *Ans.* (*a*) -48π, (*b*) 48π

34. Evaluate $\int_C (x^2 - iy^2)\,dz$ along (*a*) the parabola $y = 2x^2$ from $(1,2)$ to $(2,8)$, (*b*) the straight lines from $(1,1)$ to $(1,8)$ and then from $(1,8)$ to $(2,8)$, (*c*) the straight line from $(1,1)$ to $(2,8)$.
Ans. (*a*) $\frac{511}{3} - \frac{49}{5}i$, (*b*) $\frac{518}{3} - 57i$, (*c*) $\frac{518}{3} - 8i$

35. Evaluate $\oint_C |z|^2\,dz$ around the square with vertices at $(0,0)$, $(1,0)$, $(1,1)$, $(0,1)$. *Ans.* $-1+i$

38. Evaluate $\int_i^{2-i} (3xy + iy^2)\,dz$ (*a*) along the straight line joining $z = i$ and $z = 2-i$, (*b*) along the curve $x = 2t-2$, $y = 1+t-t^2$. *Ans.* (*a*) $-\frac{4}{3} + \frac{8}{3}i$, (*b*) $-\frac{1}{3} + \frac{79}{30}i$

39. Evaluate $\oint_C \bar{z}^2\,dz$ around the circles (*a*) $|z| = 1$, (*b*) $|z-1| = 1$. *Ans.* (*a*) 0, (*b*) $4\pi i$

40. Evaluate $\oint_C (5z^4 - z^3 + 2)\,dz$ around (*a*) the circle $|z| = 1$, (*b*) the square with vertices at $(0,0)$, $(1,0)$, $(1,1)$ and $(0,1)$, (*c*) the curve consisting of the parabolas $y = x^2$ from $(0,0)$ to $(1,1)$ and $y^2 = x$ from $(1,1)$ to $(0,0)$. *Ans.* 0 in all cases

41. Evaluate $\int_C (z^2+1)^2\,dz$ along the arc of the cycloid $x = a(\theta - \sin\theta)$, $y = a(1-\cos\theta)$ from the point where $\theta = 0$ to the point where $\theta = 2\pi$. *Ans.* $(96\pi^5a^5 + 80\pi^3a^3 + 30\pi a)/15$

42. Evaluate $\int_C \bar{z}^2\,dz + z^2\,d\bar{z}$ along the curve C defined by $z^2 + 2z\bar{z} + \bar{z}^2 = (2-2i)z + (2+2i)\bar{z}$ from the point $z = 1$ to $z = 2+2i$. *Ans.* 248/15

43. Evaluate $\oint_C \frac{dz}{z-2}$ around (*a*) the circle $|z-2| = 4$, (*b*) the circle $|z-1| = 5$, (*c*) the square with vertices at $3 \pm 3i$, $-3 \pm 3i$. *Ans.* $2\pi i$ in all cases

44. Evaluate $\oint_C (x^2 + iy^2)\,ds$ around the circle $|z| = 2$ where s is the arc length. *Ans.* $8\pi(1+i)$

Cauchy's integral formulas

1. If $f(z)$ is analytic inside and on the boundary C of a simply-connected region R, then

$$f(a) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{z-a}\,dz$$

2. If $f(z)$ is analytic inside and on the boundary C of a simply-connected region R, then

$$f'(a) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{(z-a)^2}\,dz$$

3. If $f(z)$ is analytic inside and on the boundary C of a simply-connected region R, then

$$f^{(n)}(a) = \frac{n!}{2\pi i}\oint_C \frac{f(z)}{(z-a)^{n+1}}\,dz, \qquad n = 0,1,2,\dots$$

Q. Evaluate (a) $\oint_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)}\,dz$ (b) $\oint_C \frac{e^{2z}}{(z+1)^4}\,dz$ where C is the circle $|z| = 3$.

Solution: Since $\frac{1}{(z-1)(z-2)} = \frac{1}{z-2} - \frac{1}{z-1}$, we have

$$\oint_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)}\,dz = \oint_C \frac{\sin \pi z^2 + \cos \pi z^2}{z-2}\,dz - \oint_C \frac{\sin \pi z^2 + \cos \pi z^2}{z-1}\,dz \qquad (1)$$

Let, $f(z) = \sin \pi z^2 + \cos \pi z^2$.

Here a=1 and 2.

Then by Cauchy's integral formula, we get

$$f(2) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{z-2}\,dz$$

$$\Rightarrow \oint_C \frac{\sin \pi z^2 + \cos \pi z^2}{z-2}\,dz = 2\pi i f(2)$$

$$= 2\pi i\left\{\sin\left(\pi.2^2\right) + \cos\left(\pi.2^2\right)\right\}$$

$$= 2\pi i\left\{\sin(4\pi) + \cos(4\pi)\right\}$$

$$= 2\pi i(0+1)$$

$$\therefore \oint_C \frac{\sin \pi z^2 + \cos \pi z^2}{z-2}\,dz = 2\pi i \qquad (2)$$

and

$$f(1) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{z-1}\,dz$$

$$\Rightarrow \oint_C \frac{\sin \pi z^2 + \cos \pi z^2}{z-1}\,dz = 2\pi i f(1)$$

$$= 2\pi i\left\{\sin\left(\pi.1^2\right) + \cos\left(\pi.1^2\right)\right\}$$

$$= 2\pi i\left\{\sin(\pi) + \cos(\pi)\right\}$$

$$= 2\pi i(0-1)$$

$$\therefore \oint_C \frac{\sin \pi z^2 + \cos \pi z^2}{z-1}\,dz = -2\pi i \qquad (3)$$

Using (2) and (3) into (1), we find

$$\oint_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)}\,dz = 2\pi i - (-2\pi i) = 4\pi i.$$

$$\therefore \oint_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)}\,dz = 4\pi i.$$

$$\oint_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)}\,dz = \oint_C \frac{\sin \pi z^2 + \cos \pi z^2}{z-2}\,dz - \oint_C \frac{\sin \pi z^2 + \cos \pi z^2}{z-1}\,dz$$

By Cauchy's integral formula with $a = 2$ and $a = 1$ respectively, we have

$$\oint_C \frac{\sin \pi z^2 + \cos \pi z^2}{z-2}\,dz = 2\pi i\{\sin \pi(2)^2 + \cos \pi(2)^2\} = 2\pi i$$

$$\oint_C \frac{\sin \pi z^2 + \cos \pi z^2}{z-1}\,dz = 2\pi i\{\sin \pi(1)^2 + \cos \pi(1)^2\} = -2\pi i$$

since $z = 1$ and $z = 2$ are inside C and $\sin \pi z^2 + \cos \pi z^2$ is analytic inside C. Then the required integral has the value $2\pi i - (-2\pi i) = 4\pi i$.

(b) Let $f(z) = e^{2z}$.

Here $a=-1$ and $n=3$.

Then by Cauchy's integral formula, we get

$$f'''(-1) = \frac{3!}{2\pi i}\oint_C \frac{f(z)}{(z+1)^4}\,dz$$

$$\Rightarrow \oint_C \frac{f(z)}{(z+1)^4}\,dz = \frac{2\pi i}{6}\,f'''(-1)$$

$$\therefore \oint_C \frac{e^{2z}}{(z+1)^4}\,dz = \frac{\pi i}{3}\,f'''(-1) \qquad (1)$$

Now,

$$f'(z) = 2e^{2z},\ \ f''(z) = 4e^{2z},\ \ f'''(z) = 8e^{2z}$$

Substituting this value in (1), we have

$$\oint_C \frac{e^{2z}}{(z+1)^4}\,dz = \frac{\pi i}{3}8e^{2.(-1)}$$

$$\therefore \oint_C \frac{e^{2z}}{(z+1)^4}\,dz = \frac{8\pi i}{3}e^{-2}.$$

(*b*) Let $f(z) = e^{2z}$ and $a = -1$ in the Cauchy integral formula

$$f^{(n)}(a) = \frac{n!}{2\pi i}\oint_C \frac{f(z)}{(z-a)^{n+1}}\,dz \qquad (1)$$

If $n = 3$, then $f'''(z) = 8e^{2z}$ and $f'''(-1) = 8e^{-2}$. Hence (*1*) becomes

$$8e^{-2} = \frac{3!}{2\pi i}\oint_C \frac{e^{2z}}{(z+1)^4}\,dz$$

from which we see that the required integral has the value $8\pi i e^{-2}/3$.

Poles. If $f(z)$ has the form (6.8) in which the principal part has only a finite number of terms given by

$$\frac{a_{-1}}{z-a} + \frac{a_{-2}}{(z-a)^2} + \cdots + \frac{a_{-n}}{(z-a)^n}$$

where $a_{-n} \neq 0$, then $z = a$ is called a *pole of order n*. If $n = 1$, it is called a *simple pole*.

If $f(z)$ has a pole at $z = a$, then $\lim_{z\to a} f(z) = \infty$ [see Problem 6.32].

Theorem: An isolated singular point z_0 of a function *f* is a pole of order *m* if and only if *f*(*z*) can be written in the form

$$f(z) = \frac{\varphi(z)}{(z-z_0)^m},$$

where $\varphi(z)$ is analytic and nonzero at z_0. Moreover,

$$\operatorname*{Res}_{z\to z_0} f(z) = \varphi(z_0) \quad if \quad m = 1$$

and

$$\operatorname*{Res}_{z\to z_0} f(z) = \frac{\varphi^{(m-1)}(z_0)}{(m-1)!} = \lim_{z\to z_0}\frac{1}{(m-1)!}\frac{d^{m-1}}{dz^{m-1}}\left\{(z-z_0)^m f(z)\right\} \qquad if \quad m \geq 1$$

(*a*) If $F(z)$ is analytic inside and on a simple closed curve *C* except for a pole of order *m* at $z = a$ inside *C*, prove that

$$\frac{1}{2\pi i}\oint_C F(z)\,dz = \lim_{z\to a}\frac{1}{(m-1)!}\frac{d^{m-1}}{dz^{m-1}}\{(z-a)^m F(z)\}$$

(*b*) How would you modify the result in (*a*) if more than one pole were inside *C*?

(*a*) If $F(z)$ has a pole of order m at $z = a$, then $F(z) = f(z)/(z-a)^m$ where $f(z)$ is analytic inside and on C, and $f(a) \neq 0$. Then by Cauchy's integral formula,

$$\frac{1}{2\pi i}\oint_C F(z)\,dz = \frac{1}{2\pi i}\oint_C \frac{f(z)}{(z-a)^m}\,dz = \frac{f^{(m-1)}(a)}{(m-1)!}$$

$$= \lim_{z\to a}\frac{1}{(m-1)!}\frac{d^{m-1}}{dz^{m-1}}\{(z-a)^m F(z)\}$$

(b) Suppose there are two poles at $z = a_1$ and $z = a_2$ inside C, of orders m_1 and m_2 respectively. Let Γ_1 and Γ_2 be circles inside C having radii ϵ_1 and ϵ_2 and centres at a_1 and a_2 respectively. Then

$$\frac{1}{2\pi i}\oint_C F(z)\,dz = \frac{1}{2\pi i}\oint_{\Gamma_1} F(z)\,dz + \frac{1}{2\pi i}\oint_{\Gamma_2} F(z)\,dz \qquad (1)$$

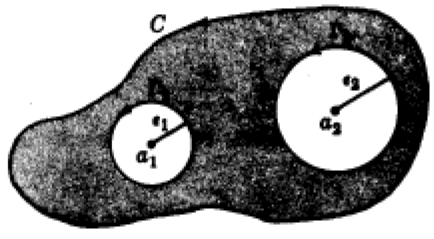


Fig. 5-12

If $F(z)$ has a pole of order m_1 at $z = a_1$, then

$$F(z) = \frac{f_1(z)}{(z-a_1)^{m_1}} \qquad \text{where } f_1(z) \text{ is analytic and } f_1(a_1) \neq 0$$

If $F(z)$ has a pole of order m_2 at $z = a_2$, then

$$F(z) = \frac{f_2(z)}{(z-a_2)^{m_2}} \qquad \text{where } f_2(z) \text{ is analytic and } f_2(a_2) \neq 0$$

Then by (1) and part (a),

$$\begin{aligned}\frac{1}{2\pi i}\oint_C F(z)\,dz &= \frac{1}{2\pi i}\oint_{\Gamma_1} \frac{f_1(z)}{(z-a_1)^{m_1}}\,dz + \frac{1}{2\pi i}\oint_{\Gamma_2} \frac{f_2(z)}{(z-a_2)^{m_2}}\,dz \\ &= \lim_{z\to a_1} \frac{1}{(m_1-1)!}\frac{d^{m_1-1}}{dz^{m_1-1}}\{(z-a_1)^{m_1}F(z)\} \\ &\qquad + \lim_{z\to a_2} \frac{1}{(m_2-1)!}\frac{d^{m_2-1}}{dz^{m_2-1}}\{(z-a_2)^{m_2}F(z)\}\end{aligned}$$

If the limits on the right are denoted by R_1 and R_2, we can write

$$\oint_C F(z)\,dz = 2\pi i(R_1 + R_2)$$

where R_1 and R_2 are called the *residues* of $F(z)$ at the poles $z = a_1$ and $z = a_2$.

In general if $F(z)$ has a number of poles inside C with residues $R_1, R_2, \ldots$, then $\oint_C F(z)\,dz =$ $2\pi i$ times the sum of the residues. This result is called the *residue theorem*. Applications of this theorem together with generalization to singularities other than poles, are treated in Chap. 7.

Evaluate $\oint_C \frac{e^z}{(z^2+\pi^2)^2}dz$ where C is the circle $|z| = 4$.

The poles of $\frac{e^z}{(z^2+\pi^2)^2} = \frac{e^z}{(z-\pi i)^2(z+\pi i)^2}$ are at $z = \pm\pi i$ inside C and are both of order two.

Residue at $z = \pi i$ is $\quad \lim_{z\to\pi i} \frac{1}{1!}\frac{d}{dz}\left\{(z-\pi i)^2 \frac{e^z}{(z-\pi i)^2(z+\pi i)^2}\right\} = \frac{\pi+i}{4\pi^3}$.

Residue at $z = -\pi i$ is $\quad \lim_{z\to-\pi i} \frac{1}{1!}\frac{d}{dz}\left\{(z+\pi i)^2 \frac{e^z}{(z-\pi i)^2(z+\pi i)^2}\right\} = \frac{\pi-i}{4\pi^3}$.

Then $\quad \oint_C \frac{e^z}{(z^2+\pi^2)^2}\,dz = 2\pi i(\text{sum of residues}) = 2\pi i\left(\frac{\pi+i}{4\pi^3} + \frac{\pi-i}{4\pi^3}\right) = \frac{i}{\pi}$.

CAUCHY'S INTEGRAL FORMULAE

30. Evaluate $\frac{1}{2\pi i}\oint_C \frac{e^z}{z-2}\,dz$ if C is (a) the circle $|z|=3$, (b) the circle $|z|=1$. *Ans.* (a) e^2, (b) 0

31. Evaluate $\oint_C \frac{\sin 3z}{z+\pi/2}\,dz$ if C is the circle $|z|=5$. *Ans.* $2\pi i$

32. Evaluate $\oint_C \frac{e^{3z}}{z-\pi i}\,dz$ if C is (a) the circle $|z-1|=4$, (b) the ellipse $|z-2|+|z+2|=6$.
Ans. (a) $-2\pi i$, (b) 0

33. Evaluate $\frac{1}{2\pi i}\oint_C \frac{\cos \pi z}{z^2-1}\,dz$ around a rectangle with vertices at: (a) $2\pm i$, $-2\pm i$; (b) $-i$, $2-i$, $2+i$, i.
Ans. (a) 0, (b) $-\frac{1}{2}$

34. Show that $\frac{1}{2\pi i}\oint_C \frac{e^{zt}}{z^2+1}\,dz = \sin t$ if $t>0$ and C is the circle $|z|=3$.

35. Evaluate $\oint_C \frac{e^{iz}}{z^3}\,dz$ where C is the circle $|z|=2$. *Ans.* $-\pi i$

36. Prove that $f'''(a) = \frac{3!}{2\pi i}\oint_C \frac{f(z)\,dz}{(z-a)^4}$ if C is a simple closed curve enclosing $z=a$ and $f(z)$ is analytic inside and on C.

37. Prove Cauchy's integral formulae for all positive integral values of n. [*Hint*: Use mathematical induction.]

38. Find the value of (a) $\oint_C \frac{\sin^6 z}{z-\pi/6}\,dz$, (b) $\oint_C \frac{\sin^6 z}{(z-\pi/6)^3}\,dz$ if C is the circle $|z|=1$.
Ans. (a) $\pi i/32$, (b) $21\pi i/16$

39. Evaluate $\frac{1}{2\pi i}\oint_C \frac{e^{zt}}{(z^2+1)^2}\,dz$ if $t>0$ and C is the circle $|z|=3$. *Ans.* $\frac{1}{2}(\sin t - t\cos t)$

Exercise 39: Let $f(z)=\dfrac{e^{zt}}{\left(z^2+1\right)^2}$.

Now, $\left(z^2+1\right)^2=0 \Rightarrow \left(z^2-i^2\right)^2=0 \Rightarrow (z+i)^2(z-i)^2=0 \Rightarrow (z+i)(z-i)=0$

$\therefore z=\pm i$

So, the poles of *f*(*z*) are at $z=\pm i$ within *C* and are of order 2.

Residue at $z=i$ is

$$\lim_{z\to i}\frac{1}{(2-1)!}\frac{d}{dz}\left\{(z-i)^2\frac{e^{zt}}{(z-i)^2(z+i)^2}\right\}=\lim_{z\to i}\frac{d}{dz}\left\{\frac{e^{zt}}{(z+i)^2}\right\}$$

$$=\lim_{z\to i}\left\{\frac{(z+i)^2te^{zt}-2(z+i)e^{zt}}{(z+i)^4}\right\}$$

$$=\lim_{z\to i}\left\{\frac{t(z+i)e^{zt}-2e^{zt}}{(z+i)^3}\right\}$$

$$=\frac{t(i+i)e^{it}-2e^{it}}{(i+i)^3}$$

$$=\frac{2e^{it}\left(it-1\right)}{8i^3}$$

$$=-\frac{e^{it}\left(it-1\right)}{4i}$$

$$=\frac{e^{it}\left(1-it\right)}{4i}$$

and residue at $z=-i$ is

$$\lim_{z\to -i}\frac{1}{\left(2-1\right)!}\frac{d}{dz}\left\{\left(z+i\right)^2\frac{e^{zt}}{\left(z-i\right)^2\left(z+i\right)^2}\right\}=\lim_{z\to i}\frac{d}{dz}\left\{\frac{e^{zt}}{\left(z-i\right)^2}\right\}$$

$$=\lim_{z\to -i}\left\{\frac{\left(z-i\right)^2te^{zt}-2\left(z-i\right)e^{zt}}{\left(z-i\right)^4}\right\}$$

$$=\lim_{z\to -i}\left\{\frac{t\left(z-i\right)e^{zt}-2e^{zt}}{\left(z-i\right)^3}\right\}$$

$$=\frac{t\left(-i-i\right)e^{-it}-2e^{-it}}{\left(-i-i\right)^3}$$

$$=\frac{-2e^{-it}\left(it+1\right)}{-8i^3}$$

$$=-\frac{e^{-it}\left(1+it\right)}{4i}$$

By Residue theorem, we therefore obtain

$$\frac{1}{2\pi i}\oint_C\frac{e^{zt}}{\left(z^2+1\right)^2}dz=\text{Sum of residues}$$

$$\Rightarrow\frac{1}{2\pi i}\oint_C\frac{e^{zt}}{\left(z^2+1\right)^2}dz=\frac{e^{it}\left(1-it\right)}{4i}-\frac{e^{-it}\left(1+it\right)}{4i}$$

$$=\frac{e^{it}-e^{-it}}{4i}-\frac{it\left(e^{it}+e^{-it}\right)}{4i}$$

$$=\frac{\cos t+i\sin t-\cos t+i\sin t}{4i}-\frac{t\left(\cos t+i\sin t+\cos t-i\sin t\right)}{4}$$

$$=\frac{2i\sin t}{4i}-\frac{2t\cos t}{4}$$

$$=\frac{\sin t}{2}-\frac{t\cos t}{2}$$

$$\therefore\frac{1}{2\pi i}\oint_C\frac{e^{zt}}{\left(z^2+1\right)^2}dz=\frac{1}{2}\left(\sin t-t\cos t\right)\ \text{(Proved).}$$

6.7 Taylor's Theorem

Let $f(z)$ be analytic inside and on a simple closed curve C. Let a and $a+h$ be two points inside C. Then

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!}f''(a) + \cdots + \frac{h^n}{n!}f^{(n)}(a) + \cdots \tag{6.3}$$

or writing $z = a + h$, $h = z - a$,

$$f(z) = f(a) + f'(a)(z-a) + \frac{f''(a)}{2!}(z-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(z-a)^n + \cdots \tag{6.4}$$

This is called *Taylor's theorem* and the series (6.3) or (6.4) is called a *Taylor series* or *expansion* for $f(a+h)$ or $f(z)$.

6.9 Laurent's Theorem

Let C_1 and C_2 be concentric circles of radii R_1 and R_2, respectively, and center at a [Fig. 6-1]. Suppose that $f(z)$ is single-valued and analytic on C_1 and C_2 and, in the ring-shaped region $\mathcal{R}$ [also called the *annulus* or *annular region*] between C_1 and C_2, is shown shaded in Fig. 6-1. Let $a+h$ be any point in $\mathcal{R}$. Then we have

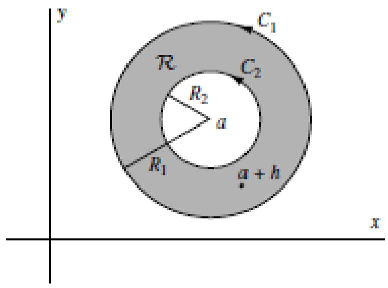


Fig. 6-1

$$f(a+h) = a_0 + a_1 h + a_2 h^2 + \cdots + \frac{a_{-1}}{h} + \frac{a_{-2}}{h^2} + \frac{a_{-3}}{h^3} + \cdots \tag{6.5}$$

where

$$\begin{aligned} a_n &= \frac{1}{2\pi i}\oint_{C_1} \frac{f(z)}{(z-a)^{n+1}}\,dz \qquad n = 0, 1, 2, \ldots \\ a_{-n} &= \frac{1}{2\pi i}\oint_{C_1} (z-a)^{n-1} f(z)\,dz \quad n = 1, 2, 3, \ldots \end{aligned} \tag{6.6}$$

C_1 and C_2 being traversed in the positive direction with respect to their interiors.

In the above integrations, we can replace C_1 and C_2 by any concentric circle C between C_1 and C_2 [see Problem 6.100]. Then, the coefficients (6.6) can be written in a single formula,

$$a_n = \frac{1}{2\pi i}\oint_C \frac{f(z)}{(z-a)^{n+1}}\,dz \qquad n = 0, \pm 1, \pm 2, \ldots \tag{6.7}$$

With an appropriate change of notation, we can write the above as

$$f(z) = a_0 + a_1(z-a) + a_2(z-a)^2 + \cdots + \frac{a_{-1}}{z-a} + \frac{a_{-2}}{(z-a)^2} + \cdots \tag{6.8}$$

where

$$a_n = \frac{1}{2\pi i}\oint_C \frac{f(\zeta)}{(\zeta-a)^{n+1}}\,d\zeta \qquad n = 0, \pm 1, \pm 2, \ldots \tag{6.9}$$

This is called *Laurent's theorem* and (6.5) or (6.8) with coefficients (6.6), (6.7), or (6.9) is called a *Laurent series* or *expansion*.

6.23. Let $f(z) = \ln(1+z)$, where we consider the branch that has the zero value when $z = 0$. (a) Expand $f(z)$ in a Taylor series about $z = 0$. (b) Determine the region of convergence for the series in (a). (c) Expand $\ln(1+z/1-z)$ in a Taylor series about $z = 0$.

Solution

(a)
$$\begin{aligned}
f(z) &= \ln(1+z), & f(0) &= 0\\
f'(z) &= \frac{1}{1+z} = (1+z)^{-1}, & f'(0) &= 1\\
f''(z) &= -(1+z)^{-2}, & f''(0) &= -1\\
f'''(z) &= (-1)(-2)(1+z)^{-3}, & f'''(0) &= 2!\\
&\vdots & &\vdots\\
f^{(n+1)}(z) &= (-1)^n n!(1+z)^{-(n+1)}, & f^{(n+1)}(0) &= (-1)^n n!
\end{aligned}$$

Then

$$\begin{aligned}
f(z) = \ln(1+z) &= f(0) + f'(0)z + \frac{f''(0)}{2!}z^2 + \frac{f'''(0)}{3!}z^3 + \cdots\\
&= z - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4} + \cdots
\end{aligned}$$

(c) From the result in (a) we have, on replacing z by $-z$,

$$\ln(1+z) = z - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4} + \cdots$$

$$\ln(1-z) = -z - \frac{z^2}{2} - \frac{z^3}{3} - \frac{z^4}{4} - \cdots$$

both series convergent for $|z| < 1$. By subtraction, we have

$$\ln\left(\frac{1+z}{1-z}\right) = 2\left(z + \frac{z^3}{3} + \frac{z^5}{5} + \cdots\right) = \sum_{n=0}^{\infty} \frac{2z^{2n+1}}{2n+1}$$

(c) We have,

$$f(z) = \ln\left(\frac{1+z}{1-z}\right) = \ln(1+z) - \ln(1-z)$$

Therefore,

$$f'(z) = \frac{1}{1+z}.1 - \frac{1}{1-z}.(-1) = \frac{1}{1+z} + \frac{1}{1-z} = (1+z)^{-1} + (1-z)^{-1}$$

$$f''(z) = (-1)\frac{1}{(1+z)^2}.1 + (-1).\frac{1}{(1-z)^2}.(-1) = -\frac{1}{(1+z)^2} + \frac{1}{(1-z)^2}$$

$$f'''(z) = -(-2)\frac{1}{(1+z)^3}.1 + (-2).\frac{1}{(1-z)^3}.(-1) = \frac{2}{(1+z)^3} + \frac{2}{(1-z)^3}$$

$$f^{iv}(z) = (-3)\frac{2}{(1+z)^4}.1 + (-3).\frac{2}{(1-z)^4}.(-1) = -\frac{6}{(1+z)^4} + \frac{6}{(1-z)^4}$$

$$f^{v}(z) = -(-4)\frac{6}{(1+z)^5}.1 + (-4).\frac{6}{(1-z)^5}.(-1) = \frac{24}{(1+z)^5} + \frac{24}{(1-z)^5}$$

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So,

$$f(0)=\ln\left(\frac{1+0}{1-0}\right)=\ln(1)=0$$

$$f'(0)=\frac{1}{1+0}+\frac{1}{1-0}=1+1=2$$

$$f''(0)=-\frac{1}{(1+0)^2}+\frac{1}{(1-0)^2}=-1+1=0$$

$$f'''(0)=\frac{2}{(1+0)^3}+\frac{2}{(1-0)^3}=2+2=4$$

$$f^{iv}(0)=-\frac{6}{(1+0)^4}+\frac{6}{(1-0)^4}=-6+6=0$$

$$f^{v}(0)=\frac{24}{(1+0)^4}+\frac{24}{(1-0)^4}=24+24=48$$

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By Taylor's theorem about z=0, we have

$$f(z)=f(0)+zf'(0)+\frac{z^2}{2!}f''(0)+\frac{z^3}{3!}f'''(0)+\frac{z^4}{4!}f^{iv}(0)+\frac{z^5}{5!}f^{v}(0)+\cdots$$

$$\Rightarrow \ln\left(\frac{1+z}{1-z}\right)=0+z.2+\frac{z^2}{2!}.0+\frac{z^3}{3!}.4+\frac{z^4}{4!}.0+\frac{z^5}{5!}.48+\cdots$$

$$=2z+\frac{z^3}{3.2.1}.4+\frac{z^5}{5.4.3.2.1}.48+\cdots$$

$$=2z+\frac{2z^3}{3}+\frac{2z^5}{5}+\cdots$$

$$=2\left(z+\frac{z^3}{3}+\frac{z^5}{5}+\cdots\right)$$

$$=\sum_{n=0}^{\infty}\frac{2z^{2n+1}}{2n+1}$$

6.24. (a) Expand $f(z) = \sin z$ in a Taylor series about $z = \pi/4$
(b) Determine the region of convergence of this series.

Solution

(a) $f(z) = \sin z,\ f'(z) = \cos z,\ f''(z) = -\sin z,\ f'''(z) = -\cos z,\ f^{\text{IV}}(z) = \sin z, \ldots$

$f(\pi/4) = \sqrt{2}/2,\ f'(\pi/4) = \sqrt{2}/2,\ f''(\pi/4) = -\sqrt{2}/2,\ f'''(\pi/4) = -\sqrt{2}/2,\ f^{\text{IV}}(\pi/4) = \sqrt{2}/2, \ldots$

Then, since $a = \pi/4$,

$$\begin{aligned} f(z) &= f(a) + f'(a)(z-a) + \frac{f''(a)(z-a)^2}{2!} + \frac{f'''(a)(z-a)^3}{3!} + \cdots \\ &= \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}(z-\pi/4) - \frac{\sqrt{2}}{2\cdot 2!}(z-\pi/4)^2 - \frac{\sqrt{2}}{2\cdot 3!}(z-\pi/4)^3 + \cdots \\ &= \frac{\sqrt{2}}{2}\left\{1 + (z-\pi/4) - \frac{(z-\pi/4)^2}{2!} - \frac{(z-\pi/4)^3}{3!} + \cdots\right\} \end{aligned}$$

Another Method. Let $u = z - \pi/4$ or $z = u + \pi/4$. Then, we have,

$$\begin{aligned} \sin z &= \sin(u+\pi/4) = \sin u \cos(\pi/4) + \cos u \sin(\pi/4) \\ &= \frac{\sqrt{2}}{2}(\sin u + \cos u) \\ &= \frac{\sqrt{2}}{2}\left\{\left(u - \frac{u^3}{3!} + \frac{u^5}{5!} - \cdots\right) + \left(1 - \frac{u^2}{2!} + \frac{u^4}{4!} - \cdots\right)\right\} \\ &= \frac{\sqrt{2}}{2}\left\{1 + u - \frac{u^2}{2!} - \frac{u^3}{3!} + \frac{u^4}{4!} + \cdots\right\} \\ &= \frac{\sqrt{2}}{2}\left\{1 + (z-\pi/4) - \frac{(z-\pi/4)^2}{2!} - \frac{(z-\pi/4)^3}{3!} + \cdots\right\} \end{aligned}$$

6.26. Find Laurent series about the indicated singularity for each of the following functions:

(a) $\dfrac{e^{2z}}{(z-1)^3}$; $z = 1$. (c) $\dfrac{z - \sin z}{z^3}$; $z = 0$. (e) $\dfrac{1}{z^2(z-3)^2}$; $z = 3$.

(b) $(z-3)\sin\dfrac{1}{z+2}$; $z = -2$. (d) $\dfrac{z}{(z+1)(z+2)}$; $z = -2$.

Name the singularity in each case and give the region of convergence of each series.

Solution

(a) Let $z - 1 = u$. Then $z = 1 + u$ and

$$\frac{e^{2z}}{(z-1)^3} = \frac{e^{2+2u}}{u^3} = \frac{e^2}{u^3}\cdot e^{2u} = \frac{e^2}{u^3}\left\{1 + 2u + \frac{(2u)^2}{2!} + \frac{(2u)^3}{3!} + \frac{(2u)^4}{4!} + \cdots\right\}$$

$$= \frac{e^2}{(z-1)^3} + \frac{2e^2}{(z-1)^2} + \frac{2e^2}{z-1} + \frac{4e^2}{3} + \frac{2e^2}{3}(z-1) + \cdots$$

$z = 1$ is a *pole of order* 3, or *triple pole*.

The series converges for all values of $z \neq 1$.

(b) Let $z + 2 = u$ or $z = u - 2$. Then

$$(z-3)\sin\frac{1}{z+2} = (u-5)\sin\frac{1}{u} = (u-5)\left\{\frac{1}{u} - \frac{1}{3!\,u^3} + \frac{1}{5!\,u^5} - \cdots\right\}$$

$$= 1 - \frac{5}{u} - \frac{1}{3!u^2} + \frac{5}{3!\,u^3} + \frac{1}{5!\,u^4} - \cdots$$

$$= 1 - \frac{5}{z+2} - \frac{1}{6(z+2)^2} + \frac{5}{6(z+2)^3} + \frac{1}{120(z+2)^4} - \cdots$$

$z = -2$ is an *essential singularity.*

The series converges for all values of $z \neq -2$.

(c)
$$\frac{z - \sin z}{z^3} = \frac{1}{z^3}\left\{z - \left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \cdots\right)\right\}$$

$$= \frac{1}{z^3}\left\{\frac{z^3}{3!} - \frac{z^5}{5!} + \frac{z^7}{7!} - \cdots\right\} = \frac{1}{3!} - \frac{z^2}{5!} + \frac{z^4}{7!} - \cdots$$

Then the required Laurent expansion valid for both $|z| > 1$ and $|z| > 3$, i.e., $|z| > 3$, is by subtraction

$$\frac{1}{z^2} - \frac{4}{z^3} + \frac{13}{z^4} - \frac{40}{z^5} + \cdots$$

(c) Let $z + 1 = u$. Then

$$\frac{1}{(z+1)(z+3)} = \frac{1}{u(u+2)} = \frac{1}{2u(1+u/2)} = \frac{1}{2u}\left(1 - \frac{u}{2} + \frac{u^2}{4} - \frac{u^3}{8} + \cdots\right)$$

$$= \frac{1}{2(z+1)} - \frac{1}{4} + \frac{1}{8}(z+1) - \frac{1}{16}(z+1)^2 + \cdots$$

valid for $|u| < 2$, $u \neq 0$ or $0 < |z+1| < 2$.

(d) If $|z| < 1$,

$$\frac{1}{2(z+1)} = \frac{1}{2(1+z)} = \frac{1}{2}(1 - z + z^2 - z^3 + \cdots) = \frac{1}{2} - \frac{1}{2}z + \frac{1}{2}z^2 - \frac{1}{2}z^3 + \cdots$$

If $|z| < 3$, we have by part (a),

$$\frac{1}{2(z+3)} = \frac{1}{6} - \frac{z}{18} + \frac{z^2}{54} - \frac{z^3}{162} + \cdots$$

Then the required Laurent expansion, valid for both $|z| < 1$ and $|z| < 3$, i.e., $|z| < 1$, is by subtraction

$$\frac{1}{3} - \frac{4}{9}z + \frac{13}{27}z^2 - \frac{40}{81}z^3 + \cdots$$

We have

$$\frac{z}{(z+1)(z+3)} = \frac{1}{2(z+1)} - \frac{1}{2(z+3)} \qquad (1)$$

$$\frac{1}{2(z+3)} = \frac{1}{6\left(1+\frac{z}{3}\right)} = \frac{1}{6}\left(1+\frac{z}{3}\right)^{-1} = \frac{1}{6}\left(1-\frac{z}{3}+\frac{z^2}{9}-\frac{z^3}{27}+\cdots\right)$$