对于两点边值问题：

$$\begin{cases} -\frac{d^2u}{dx^2} = x^2, & 0 < x < 1 \\ u(0) = 0, \quad u(1) = 0 \end{cases}$$

其Ritz变分问题为：

$$\begin{cases} u \in K, \quad s.t. \\ J(u) \leq J(w), \quad \forall w \in K \end{cases}$$

其中$J(u) = \frac{1}{2}D(u,u) - F(u), \quad D(u,v) = \int_0^1 \frac{du}{dx}\frac{du}{dx}dx, \quad F(u) = \int_0^1 x^2 v dx.$

采用Ritz方法求解，并取$K = C_0^1[0,1]$，取基函数$\varphi_1, \varphi_2, ...\varphi_N$,其中$\varphi_i = x^i(1-x)$。则该问题化简为：求解$Dc^0 = F$，
其中$D \in R^{n*n}, D_{ij} = D(\varphi_i, \varphi_j)$，$c^0 \in R^n$，$F \in R^n, F_i = F(\varphi_i)$.
则近似解$u_N = \sum_{i=1}^N c_i^0 \varphi_i$.

1.求解D与F，解方程组$Dc^0 = F$：

$$\begin{aligned} D_{ij} &= D(x^i(1-x), x^j(1-x)) \\ &= \int_0^1 [ix^{i-1}(1-x) - x^i][jx^{i-1}(1-x) - x^i]dx \\ &= \frac{(i+1)(j+1)}{i+j+1} - \frac{i(j+1) + j(i+1)}{i+j} + \frac{ij}{i+j-1} \\ &= \frac{ij}{(i+j+1)(i+j)(i+j-1)} \end{aligned}$$

,

$$\begin{aligned} F_i &= F(x^i(1-x)) \\ &= \int_0^1 x^{2+i}(1-x)dx \\ &= \frac{1}{3+i} - \frac{1}{4+i} \\ &= \frac{1}{(3+i)(4+i)} \end{aligned}$$

解方程组$Dc^0 = F$，$u_N = \sum_{i=1}^N c_i^0 \varphi_i$.

2.估计误差：

已知方程精确解$u = \frac{x}{12}(1-x^3)$，令相对误差$e = \frac{\int_0^1 (u-u_N)^2 dx}{\int_0^1 u^2 dx} = \frac{\int_0^1 [\frac{x}{12}(1-x^3) - \sum_{i=1}^N c_i^0 x^i (1-x)]^2 dx}{\frac{1}{108}}$，由于N较大时，直接求解该积分较麻烦，因此采用数值方法：

这里我采用Romberg算法，令T_m^k为二分k次，加速m次得到的梯形值，令$f = (u-u_N)^2$：
$T_0^0 = \frac{f(0)+f(1)}{2}$

$h = \frac{1}{2^k}$
$T_0^k = \frac{h}{2}[f(a) + 2\sum_{i=1}^{2^k-1} f(x_i) + f(b)] = \frac{h}{2}[f(a) + 2\sum_{i=1}^{2^k-1} f(\frac{i}{2^k}) + f(b)]$
$T_0^{k+1} = \frac{1}{2}T_0^k + \frac{h}{2}\sum_{i=0}^{2^k-1} f(x_{i+\frac{1}{2}}) = \frac{1}{2}T_0^k + \frac{h}{2}\sum_{i=0}^{2^k-1} f(hi + \frac{h}{2})$
$T_1^k = \frac{4}{3}T_0^{k+1} - \frac{1}{3}T_0^k$
$e = T_1^k$
将N从1开始增加至10，得到如下误差曲线：

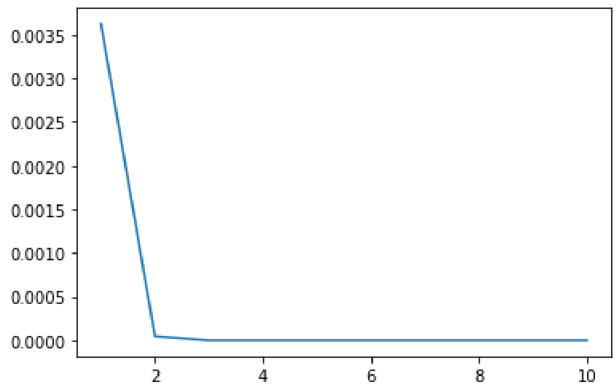

图 1: $\epsilon = 10^{-4}$误差图

故N取3时误差最小，约为$7 * 10^{-32}$，此时向量c^0的各项均为$\frac{0.25}{3}$，故近似解$u_N = \frac{x(1-x^3)}{12}$，与精确解相同