Seminar 3

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8. Three contestants participate in a trivia game show. Their probabilities of answering a question correctly are 0.8, 0.9 and 0.75, respectively. If 10 questions are asked and every contestant answers every question, find the probability that one contestant (any one) answers exactly 7 questions correctly, while the other two give any other number of correct answers. (event A)?

Proof. Denote by A_1, A_2 and A_3 the probability that the contestant no 1, 2, respectively 3 answers exactly 7 questions right. It can be seen that the 3 probabilities are computed using a **Binomial model**, since every one of them has equal probability for every iteration. We have

$$P(A_1) = C_n^k \cdot 0.8^7 \cdot 0.2^3 = 0.2$$

$$P(A_2) = C_n^k \cdot 0.9^7 \cdot 0.1^3 = 0.05$$

$$P(A_3) = C_n^k \cdot 0.75^7 \cdot 0.25^3 = 0.25$$

In order to compute P(A), we need exactly one value from the 3 probabilities A_1, A_2 and A_3, thus using a **Poisson Model**.

$P(A) =$ the coefficient of x^1 in the polynomial expansion of $(P(A_1) \cdot x + 1 - P(A_1)) \cdot (P(A_2) \cdot x + 1 - P(A_2)) \cdot (P(A_3) \cdot x + 1 - P(A_3))$

$$(0.2 \cdot x + 0.8) \cdot (0.05 \cdot x + 0.95) \cdot (0.25 \cdot x + 0.75) = 0.0025 \cdot x^3 + 0.065 \cdot x^2 + 0.3625 \cdot x + 0.57 \tag{1}$$

We can see that the coefficient of x^1 is $0.3625 = P(A)$, which is our final probability. □

9. In a department store at the mall, black and brown gloves are on sale. There are N (identical) pairs of black gloves and N (identical) pairs of brown gloves. If N customers come in, one at a time and randomly choose and buy 2 pairs each, find the probability of event A: each customer buys 2 pairs of different colors (one black and one brown).

Proof. There are N trials, where we need N successes, and every trial has a different probability. Thus, we need a **Poisson Model**. Denote by $P(A_K)$ the probability that the kth person chooses gloves of different colours, knowing that the first the first k - 1 people took gloves of different colours.

$$P(A_1) = \frac{N}{2 \cdot N - 1}, P(A_2) = \frac{N-1}{2 \cdot N - 3} \ ... \ P(A_i) = \frac{N-i+1}{2 \cdot N - 2 \cdot i + 1} \ ... \ P(A_N) = \frac{1}{1}$$

Thus, we observe that the probability of N successes in N trials is the coefficient of x^N in the polynomial expansion of $\prod_{i=1}^{N} (\frac{N-i+1}{2 \cdot N - 2 \cdot i + 1} \cdot x + \frac{N-i}{2 \cdot N - 2 \cdot i + 1})$. We can observe that the coefficient of x^N is the product of the coefficients of x in the above product, so

$$P(A) = \prod_{i=1}^{N} \frac{N-i+1}{2 \cdot N - 2 \cdot i + 1} = \frac{N!}{(2 \cdot N - 1) \cdot (2 \cdot N - 3) \cdot \ ... \ \cdot 3 \cdot 1}$$

Denote $Q = 2 \cdot 4 \cdot 6 \cdots 2n$. $Q = (1 \cdot 2)(2 \cdot 2)(3 \cdot 2) \cdots (n \cdot 2) = n! \cdot 2^n$. Also, $(2n)! = (1 \cdot 3 \cdot 5 \cdots (2n-1)) \cdot (2 \cdot 4 \cdot 6 \cdots 2n) = (1 \cdot 3 \cdot 5 \cdots (2n-1)) \cdot Q$. So $(1 \cdot 3 \cdot 5 \cdots (2n-1)) = \frac{(2n)!}{2^n \cdot n!}$ In conclusion,

$$P(A) = \frac{N!^2 \cdot 2^N}{(2N)!}$$

$\square$