Seminar 5

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7. An internet service provider has two connection lines for its customers. Eighty percent of customers are connected through Line I, the rest through Line II. Line I has a $Gamma(3, \frac{1}{2})$ connection time (in minutes), while Line II has a $U(20, 50)$ connection time (in seconds). Find the probability that it takes a randomly selected customer more than 30 seconds to connect to the internet.

Proof. Firstly, we want to compute the probability that it takes a randomly selected customer from Line I more than 30 seconds to connect to the internet. We know that the connection time is $Gamma(3, \frac{1}{2})$, so we can compute

$$\text{pdf } f(x) = 4x^2e^{-2x}$$

$$F(\frac{1}{2}) = P(X \leq \frac{1}{2}) = \int_{-\infty}^{\frac{1}{2}} f(x)\, dx = \int_{0}^{\frac{1}{2}} f(x)\, dx$$

since one cannot wait < 0 minutes.

$$\int 4x^2e^{-2x}\, dx = -\left(2x^2 + 2x + 1\right)\mathrm{e}^{-2x} + C$$

$$\text{Thus, } F(\frac{1}{2}) \approx 0.08030$$

$$\begin{aligned} P(T > \frac{1}{2}\text{min}) &= 1 - P(T \leq \frac{1}{2}\text{min}) \\ &= 1 - F(\frac{1}{2}) \\ &= 0.91970 = P1 \end{aligned}$$

or

$$\begin{aligned} P(T > \frac{1}{2}\text{min}) &= 1 - P(T \leq \frac{1}{2}\text{min}) \\ &= 1 - gamcdf(\frac{1}{2}, 3, \frac{1}{2}) \textit{ (in matlab)} \\ &= 0.91970 = P1 \end{aligned} \tag{1}$$

Secondly, we want to compute the probability that it takes a randomly selected customer from Line II more than 30 seconds to connect to the internet. We know that the connection time is $U(20, 50)$, so we can compute

$$\text{pdf } f(x) = \frac{1}{30},\ x \in [20, 50]$$

$$F(30) = P(X \leq 30) = \int_{-\infty}^{30} f(x)\,dx = \int_{20}^{30} f(x)\,dx$$

since f is defined on $[20, 50]$

$$\int \frac{1}{30}\,dx = \frac{x}{30} + C$$

$$\text{Thus, } F(30) = \frac{30}{30} - \frac{20}{30} = \frac{1}{3} = 0.3333$$

$$\begin{aligned} P(T > 30\text{sec}) &= 1 - P(T \leq 30\text{sec}) \\ &= 1 - F(30) \\ &= 0.66667 = P2 \end{aligned}$$

or

$$\begin{aligned} P(T > 30\text{sec}) &= 1 - P(T \leq 30\text{sec}) \\ &= 1 - \mathit{unifcdf}(30, 20, 50)\ \textit{(in matlab)} \\ &= 0.66667 = P2 \end{aligned} \tag{2}$$

Finally, we need to compute the probability that it takes a randomly selected customer more than 30 seconds to connect to the internet. We have that

$$\begin{aligned} P(T > 30) &= 80\%P1 + 20\%P2 \\ &= 80\% \cdot 0.91970 + 20\% \cdot 0.66667 \\ &= 0.73576 + 0.133334 \\ &= 0.869094 \end{aligned}$$

□

8. Let $X, Y \in N(0, 1)$ be independent random variables. Let D_r be the disk centered at the origin with radius r. Find r such that $P((X, Y) \in D_r) = 0.3$.

Proof. The pdf of X is $f_X(x) = \frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}}$. The pdf of Y is $f_Y(y) = \frac{e^{-\frac{y^2}{2}}}{\sqrt{2\pi}}$. Since X and Y are independent, $f_{X,Y}(x, y) = f_X(x) \cdot f_Y(y) = \frac{e^{-\frac{x^2+y^2}{2}}}{2\pi}$

We know that $P((X, Y) \in D_r) = \int_{D_r} \int f(x, y)\,dx\,dy$.

A disk D_r is defined with the formula

$$x^2 + y^2 \leq r^2$$

Thus,

$$P(D_r) = 0, \text{ if } x^2 + y^2 > r^2$$

For $x^2 + y^2 \leq r^2$, we have

$$\begin{aligned}
P(D_r) &= \int_{-r}^{r} \int_{-\sqrt{r^2-x^2}}^{\sqrt{r^2-x^2}} \frac{e^{-\frac{x^2+y^2}{2}}}{2\pi} \, dy \, dx \\
&= \int_0^r \int_0^{2\pi} \frac{e^{\frac{-R^2}{2}}}{2\pi} R \, d\theta \, dR \\
&= \int_0^r \frac{e^{\frac{-R^2}{2}}}{2\pi} 2\pi R \, dR \\
&= 1 - e^{-\frac{r^2}{2}}
\end{aligned}$$

(by writing to polar coordinates) the probabilty of $(X, Y) \in D_r$. This way, we need to compute

$$\begin{aligned}
&P(R = r) = 0.3 \\
&1 - e^{-\frac{r^2}{2}} = 0.3 \\
&e^{-\frac{r^2}{2}} = 0.7 \\
&-r^2 = 2ln(0.7) \\
&r \approx 0.8446
\end{aligned}$$

$\square$