Seminar 6

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8. Two independent customers are scheduled to arrive in the afternoon. Their arrival times are uniformly distributed between 2 pm and 8 pm. What is the expected time of
a) the first (earlier) arrival;
b) the last (later) arrival?

Proof. We need to compute the expectations of min and max functions. Let $Z = max(X, Y)$ and $U = min(X, Y)$

The cdf of max is

$$F_Z(z) = P(Z \leq z) = P(X \leq z, Y \leq z) = P(X \leq z)P(Y \leq z) = (\frac{z-2}{8-2})^2$$

Thus, the pdf is $f_Z(z) = \frac{z-2}{18}$. The cdf of min is

$$\begin{aligned} F_U(u) = P(U \leq u) &= 1 - P(U \geq u) \\ &= P(X \geq u)P(Y \geq u) \\ &= 1 - (1 - \frac{u-2}{6})^2 \\ &= 1 - (\frac{8-u}{6})^2 \end{aligned}$$

The pdf is $f_U(u) = \frac{8-u}{18}$.

Finally, we need to compute $E(Z)$ and $E(U)$.

$$E(U) = \int_2^8 u\frac{8-u}{18}\,du = 4pm, \text{earliest arrival}$$

$$E(Z) = \int_2^8 z\frac{z-2}{18}\,dz = 6pm, \text{latest arrival}$$

□

9. In an office n different letters are placed randomly into n addressed envelopes. Let Z_n denote the number of correct mailings. For each $k \in \{1, ..., n\}$., let X_k be the random variable defined by

$$X_k = \begin{cases} 1, \text{ if the k-th letter is placed correctly} \\ 0, , \text{ otherwise.} \end{cases}$$

a) Find $E(X_k)$ and $V(X_k)$ for each $k \in \{1, ..., n\}$.
b) Find $E(Z_n)$ and $V(Z_n)$.
c) How many correct mailings are to be expected?

Proof. a, b) We know that $X = X_1 + X_2 + X_3 + ... + X_n$, thus $E(X) = E(X_1) + E(X_2) + ... + E(X_n)$

We need to compute $E(X_k)$, for $k \in \{1, ..., n\}$. Since all possibilities of mailing are equally possible, $E(X_k) = E(X_1)$. Since X_1 is a variable with values 0 or 1, then $E(X_1) = P(X_1 = 1) = \frac{1}{n}$

$$E(Z_n) = E(X_1) + E(X_2) + ... + E(X_n) = n \cdot \frac{1}{n} = 1$$

$$Z_n^2 = \sum_{i=1}^{n} X_i^2 + \sum_{(i,j)=(1,1), i \neq j}^{(n,n)} X_i X_j$$

We need to compute $E(X_k^2) = E(X_1^2) = E(X_1) = \frac{1}{n}$, since the pdf of X_1^2 is

$$\begin{pmatrix} 0 & 1 \\ \frac{n-1}{n} & \frac{1}{n} \end{pmatrix}$$

$$\begin{aligned} E(X_k X_p) &= E(X_1 X_2) = P(X_1 X_2 = 1) \\ &= P(X_1 = 1, X_2 = 1) = P(X_1 = 1) P(X_2 = 1 | X_1 = 1) = \frac{1}{n} \cdot \frac{1}{n-1} \end{aligned}$$

$$\begin{aligned} E(Z_n^2) &= \sum_{i=1}^{n} E(X_i^2) + \sum_{(i,j)=(1,1), i \neq j}^{(n,n)} E(X_i X_j) \\ &= n \cdot \frac{1}{n} + n \cdot (n-1) \cdot \frac{1}{n} \cdot \frac{1}{n-1} \\ &= 2 \end{aligned}$$

$$V(X_k) = E(X_k^2) - (E(X_k))^2 = \frac{1}{n} - \frac{1}{n^2} = \frac{n-1}{n^2}$$

$$V(Z_n) = E(Z_n^2) - (E(Z_n))^2 = 2 - 1 = 1$$

c) There are to be expected $E(Z_n) = 1$ correct mailing. □