Seminar 4

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8. The independent variables X, Y have binomial distributions with parameters m, p and n, p, respectively. Find the pdf of $X + Y$. What distribution does $X + Y$ have?

Proof. a) Denote $Z = X + Y$. We have to compute $P(Z = z) = P(X + Y = z)$. We can fix the first variable, so we have $P(X = x, Y = z - x)$, where $x \in [0, z]$. Thus, we have

$$
\begin{aligned}
P(Z = z) &= \sum_{x=0}^{z} P(X = x, Y = z - x) \\
&= \sum_{x=0}^{z} P(X = x)P(Y = z - x) \\
&= \sum_{x=0}^{z} C_n^x p^x (1 - p)^{n-x} \cdot C_m^{z-x} p^{z-x} (1 - p)^{m-z+x} \\
&= \sum_{x=0}^{z} C_n^x p^{x+z-x} (1 - p)^{n-x+m-z+x} C_m^{z-x} \qquad (1) \\
&= \sum_{x=0}^{z} C_n^x C_m^{z-x} p^z (1 - p)^{n+m-z} \\
&= p^z (1 - p)^{n+m-z} \sum_{x=0}^{z} C_n^x C_m^{z-x} \\
&= p^z (1 - p)^{n+m-z} C_{n+m}^{z-x+x} \\
&= p^z (1 - p)^{n+m-z} C_{n+m}^{z}
\end{aligned}
$$

from X and Y being independent and by using the properties of combinations.

Thus, we have that the pdf of $X + Y$ is

$$X + Y \begin{pmatrix} k \\ C_{n+m}^k p(k)(1 - p)^{n+m-k} \end{pmatrix} \qquad (2)$$

which is a variable of **binomial distribution**. □

9. Two dice are rolled. Let X be the smaller number of points and Y the larger number of points. If both dice show the same number, say z, then X = Y = z. a) Find the joint pdf of (X, Y); b) Are X and Y independent? Explain; c) If the smaller number shown is 2, what is the probability that the larger one will be 5?

Proof. Let D_1 denote the value of the first die, and D_2 the value of the second die. Then $X = min(D_1, D_2)$ and $Y = max(D_1, D_2)$. Note that D_1 and D_2 are independent.

For $x = y$, we have

$$P(X = x, Y = y) = P(D_1 = x, D_2 = x) = P(D_1 = x)P(D_2 = x) \tag{3}$$

For $x \neq y$, we have

$$\begin{aligned} P(X = x, Y = y) &= P(D_1 = x, D_2 = y) + P(D_1 = y, D_2 = x) \\ &= 2P(D_1 = x)P(D_2 = y) \end{aligned} \tag{4}$$

since we don't know which value between D_1 and D_2 is greater, so both (X, Y) and (Y, X) are possible outcomes.

The **joint pdf** of (X, Y) is

$$p_{ij} = P(X = x_i, Y = y_j) \tag{5}$$

We know that

$$\begin{aligned} pdf(D_1) = D_1 \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \end{pmatrix} \\ pdf(D_2) = D_2 \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \end{pmatrix} \end{aligned} \tag{6}$$

Thus, we have the joint pdf represented in the table below

$X \backslash Y$	1	2	3	4	5	6
1	$\frac{1}{36}$	$\frac{1}{18}$	$\frac{1}{18}$	$\frac{1}{18}$	$\frac{1}{18}$	$\frac{1}{18}$
2	0	$\frac{1}{36}$	$\frac{1}{18}$	$\frac{1}{18}$	$\frac{1}{18}$	$\frac{1}{18}$
3	0	0	$\frac{1}{36}$	$\frac{1}{18}$	$\frac{1}{18}$	$\frac{1}{18}$
4	0	0	0	$\frac{1}{36}$	$\frac{1}{18}$	$\frac{1}{18}$
5	0	0	0	0	$\frac{1}{36}$	$\frac{1}{18}$
6	0	0	0	0	0	$\frac{1}{36}$

having values of 0 in cells with column ¿ row, as X denotes the smaller value.

b)

$$X \text{ and } Y \text{ are independent} \iff P(X = x, Y = y) = P(X = x)P(Y = y)$$

For $X = 1$ and $Y = 2 \Rightarrow$

$$\begin{aligned} &P(X = 1, Y = 2) = P(X = 1)P(Y = 2) \iff \\ &\frac{1}{18} = \frac{1}{36} \textit{(false)} \end{aligned} \tag{7}$$

c) The probability of having the minimum value 2 is the sum on the 2nd row of the joint pdf. Thus, we have $P(X = 2) = \frac{1}{4}$.

The probability of having the maximum value 5 is the sum on the 5th column. Thus, we have $P(Y = 5) = \frac{1}{4}$.

Finally,

$$\begin{aligned} P(Y = 5|X = 2) &= \frac{P(X = 2, Y = 5)}{P(X = 2)} \\ &= \frac{\frac{1}{18}}{\frac{1}{4}} \\ &= \frac{4}{18} \\ &= 0.222 \end{aligned} \tag{8}$$

□