A Note on *Z*-numbers

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ABSTRACT

Decisions are based on information. To be useful, information must be reliable. Basically, the concept of a *Z*-number relates to the issue of reliability of information. A *Z*-number, *Z*, has two components, *Z* = (*A*,*B*). The first component, *A*, is a restriction (constraint) on the values which a real-valued uncertain variable, *X*, is allowed to take. The second component, *B*, is a measure of reliability (certainty) of the first component. Typically, *A* and *B* are described in a natural language. Example: (about 45 min, very sure). An important issue relates to computation with *Z*-numbers. Examples: What is the sum of (about 45 min, very sure) and (about 30 min, sure)? What is the square root of (approximately 100, likely)? Computation with *Z*-numbers falls within the province of Computing with Words (CW or CWW). In this note, the concept of a *Z*-number is introduced and methods of computation with *Z*-numbers are outlined. The concept of a *Z*-number has a potential for many applications, especially in the realms of economics, decision analysis, risk assessment, prediction, anticipation and rule-based characterization of imprecise functions and relations.

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1. Introduction

In the real world, uncertainty is a pervasive phenomenon. Much of the information on which decisions are based is uncertain. Humans have a remarkable capability to make rational decisions based on information which is uncertain, imprecise and/or incomplete. Formalization of this capability, at least to some degree, is a challenge that is hard to meet. It is this challenge that motivates the concepts and ideas outlined in this note.

A *Z*-number is an ordered pair of fuzzy numbers, (*A*,*B*). For simplicity, *A* and *B* are assumed to be trapezoidal fuzzy numbers. A *Z*-number is associated with a real-valued uncertain variable, *X*, with the first component, A, playing the role of a fuzzy restriction, *R*(*X*), on the values which *X* can take, written as *X* is A, where A is a fuzzy set. What should be noted is that, strictly speaking, the concept of a restriction has greater generality than the concept of a constraint. A probability distribution is a restriction but is not a constraint [10]. A restriction may be viewed as a generalized constraint [16]. In this note, the terms restriction and constraint are used interchangeably.

The restriction

$$R(X) : X \text{ is } A,$$

is referred to as a possibilistic restriction (constraint), with A playing the role of the possibility distribution of *X*. More specifically,

$$R(X) : X \text{ is } A \rightarrow Poss(X = u) = \mu_A(u)$$

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where μ_A is the membership function of A and u is a generic value of X. μ_A may be viewed as a constraint which is associated with $R(X)$, meaning that $\mu_A(u)$ is the degree to which u satisfies the constraint.

When X is a random variable, the probability distribution of X plays the role of a probabilistic restriction on X. A probabilistic restriction is expressed as:

$R(X)$: X is$p$ p

where p is the probability density function of X. In this case,

$R(X)$: X is$p$ $p \rightarrow \text{Prob}(u \leq X \leq u + du) = p(u)du$

Note. Generally, the term "restriction" applies to X is R. Occasionally, "restriction" applies to R. Context serves to disambiguate the meaning of "restriction."

The ordered triple (X,A,B) is referred to as a Z-valuation. A Z-valuation is equivalent to an assignment statement, X is (A,B). X is an uncertain variable if A is not a singleton. In a related way, uncertain computation is a system of computation in which the objects of computation are not values of variables but restrictions on values of variables. In this note, unless stated to the contrary, X is assumed to be a random variable. For convenience, A is referred to as a value of X, with the understanding that, strictly speaking, A is not a value of X but a restriction on the values which X can take. The second component, B, is referred to as certainty. Closely related to certainty are the concepts of sureness, confidence, reliability, strength of belief, probability, possibility, etc. When X is a random variable, certainty may be equated to probability. Informally, B may be interpreted as a response to the question: How sure are you that X is A? Typically, A and B are perception-based and are described in a natural language. Example: (about 45 min, usually.) A collection of Z-valuations is referred to as Z-information. It should be noted that much of everyday reasoning and decision-making is based, in effect, on Z-information. For purposes of computation, when A and B are described in a natural language, the meaning of A and B is precisiated (graduated) through association with membership functions, μ_A and μ_B, respectively, Figure 1. The membership function of A, μ_A, may be elicited by asking a succession of questions of the form: To what degree does the number, a, fit your perception of A? Example: To what degree does 50 min fit your perception of about 45 min? The same applies to B. The fuzzy set, A, may be interpreted as the possibility distribution of X. The concept of a Z-number may be generalized in various ways. In particular, X may be assumed to take values in R^n, in which case A is a Cartesian product of fuzzy numbers. Simple examples of Z-valuations are:

(anticipated budget deficit, close to 2 million dollars, very likely)
(population of Spain, about 45 million, quite sure)
(degree of Robert's honesty, very high, absolutely)
(degree of Robert's honesty, high, not sure)
(travel time by car from Berkeley to San Francisco, about 30 min, usually)
(price of oil in the near future, significantly over 100 dollars/barrel, very likely)
It is important to note that many propositions in a natural language are expressible as Z-valuations. Example: The proposition, p,
p: Usually, it takes Robert about 1 h to get home from work, is expressible as a Z-valuation:
(Robert's travel time from office to home, about 1 h, usually)
If X is a random variable, then X is A represents a fuzzy event in R, the real line. The probability of this event, p, may be expressed as: [9]

$$p = \int_R \mu_A(u)p_X(u)du,$$

where p_X is the underlying (hidden) probability density of X. In effect, the Z-valuation (X,A,B) may be viewed as a restriction (generalized constraint) on X defined by:

$\text{Prob}(X \text{ is } A)$ is B.

What should be underscored is that in a Z-number, (A,B), the underlying probability distribution, p_X, is not known. What is known is a restriction on p_X which may be expressed as:
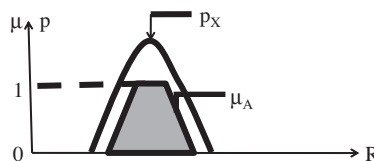
$$\int_R \mu_A(u)p_X(u)du \text{ is } B$$

Fig. 1. Membership function of A and probability density function of X.

A subtle point is that B is a restriction on the probability measure of A rather than on the probability of A. Conversely, if B is a restriction on the probability of A rather than on the probability measure of A, then (A, B) is not a Z-number.

Note. In this note, the term "probability distribution" is not used in its strict technical sense.

In effect, a Z-number may be viewed as a summary of p_X. It is important to note that in everyday decision-making most decisions are based on summaries of information. Viewing a Z-number as a summary is consistent with this reality. In applications to decision analysis, a basic problem which arises relates to ranking of Z-numbers. Example: Is (approximately 100, likely) greater than (approximately 90, very likely)? Is this a meaningful question?

An immediate consequence of the relation between p_X and B is the following. If $Z = (A, B)$ then $Z' = (A', 1 - B)$, where A' is the complement of A and Z' plays the role of the complement of Z. $1 - B$ is the antonym of B [6]. Example: The complement of $Z = (A, \text{likely})$ is $Z' = (\text{not } A, \text{unlikely})$.

An important qualitative attribute of a Z-number is informativeness. Generally, but not always, a Z-number is informative if its value has high specificity, that is, is tightly constrained [7], and its certainty is high. Informativeness is a desideratum when a Z-number is a basis for a decision. A basic question is: When is the informativeness of a Z-number sufficient to serve as a basis for an intelligent decision?

The concept of a Z-number is based on the concept of a fuzzy granule [12,13,16]. It should be noted that the concept of a Z-number is much more general than the concept of confidence interval in probability theory. There are some links between the concept of a Z-number, the concept of a fuzzy random number and the concept of a fuzzy random variable [4,2,5]. An alternative interpretation of the concept of a Z-number may be based on the concept of a fuzzy-set-valued random variable — a concept which is discussed in [12]. This interpretation is not considered in the following.

A concept which is closely related to the concept of a Z-number is the concept of a Z^+-number. Basically, a Z^+-number, Z^+, is a combination of a fuzzy number, A, and a random number, R, written as an ordered pair $Z^+ = (A, R)$. In this pair, A plays the same role as it does in a Z-number, and R is the probability distribution of a random number. Equivalently, R may be viewed as the underlying probability distribution of X in the Z-valuation (X, A, B). Alternatively, a Z^+-number may be expressed as (A, p_X) or (μ_A, p_X), where μ_A is the membership function of A. A Z^+-valuation is expressed as (X, A, p_X) or, equivalently, as (X, μ_A, p_X), where p_X is the probability distribution (density) of X. A Z^+-number is associated with what is referred to as a bimodal distribution, that is, a distribution which combines the possibility and probability distributions of X. Informally, these distributions are compatible if the centroids of μ_A and p_X are coincident, that is,

$$\int_R u p_X(u) du = \frac{\int_R u \mu_A(u) du}{\int_R \mu_A(u) du}$$

The scalar product of μ_A and p_X, $\mu_A \cdot p_X$, is the probability measure, P_A, of A. More concretely,

$$\mu_A \cdot p_X = P_A = \int_R \mu_A(u) p_X(u) du$$

It is this relation that links the concept of a Z-number to that of a Z^+-number. More concretely,

$$Z(A, B) = Z^+(A, \mu_A \cdot p_X \text{ is } B)$$

What should be underscored is that in the case of a Z-number what is known is not p_X but a restriction on p_X expressed as: $\mu_A \cdot p_X$ is B. By definition, a Z^+-number carries more information than a Z-number. This is the reason why it is labeled a Z^+-number. As will be seen in the sequel, computation with Z^+-numbers is a portal to computation with Z-numbers.

The concept of a bimodal distribution is of interest in its own right. Let X be a real-valued variable taking values in U. For our purposes, it will be convenient to assume that U is a finite set, $U = \{u_1, \ldots, u_n\}$. We can associate with X a possibility distribution, μ, and a probability distribution, p, expressed as:

$$\mu = \mu_1/u_1 + \cdots + \mu_n/u_n$$

$$p = p_1 \setminus u_1 + \cdots + p_n \setminus u_n$$

in which μ_i/u_i means that μ_i, $i = 1, \ldots, n$, is the possibility that $X = u_i$. Similarly, $p_i \setminus u_i$ means that p_i is the probability that $X = u_i$.

The possibility distribution, μ, may be combined with the probability distribution, p, through what is referred to as confluence. More concretely,

$$\mu : p = (\mu_1, p_1)/u_1 + \cdots + (\mu_n, p_n)/u_n$$

As was noted earlier, the scalar product, expressed as $\mu \cdot p$, is the probability measure of A. In terms of the bimodal distribution, the Z^+-valuation and the Z-valuation associated with X may be expressed as:

$$(X, A, p_X)$$

$$(X, A, B), \quad \mu_A \cdot p_X \text{ is } B,$$

respectively, with the understanding that B is a possibilistic restriction on $\mu_A \cdot p_X$.

Both Z and Z^+ may be viewed as restrictions on the values which X may take, written as: X is Z and X is Z^+, respectively. Viewing Z and Z^+ as restrictions on X adds important concepts to representation of information and characterization of dependencies. In this connection, what should be noted is that the concept of a fuzzy if-then rule plays a pivotal role in most applications of fuzzy logic. What follows is a very brief discussion of what are referred to as Z-rules – if-then rules in which the antecedents and/or consequents involve Z-numbers or Z^+-numbers.

A basic fuzzy if-then rule may be expressed as: if X is A then Y is B, where A and B are fuzzy numbers. The meaning of such a rule is defined as:

$$\text{if } X \text{ is } A \text{ then } Y \text{ is } B \rightarrow (X, Y) \text{ is } A \times B$$

where $A \times B$ is the Cartesian product of A and B [14]. It is convenient to express a generalization of the basic if-then rule to Z-numbers in terms of Z-valuations. More concretely,

$$\text{if } (X, A_X, B_X) \text{ then } (Y, A_Y, B_Y)$$

Examples:

> if (anticipated budget deficit, about two million dollars, very likely) then (reduction in staff, about ten percent, very likely)
> if (degree of Robert's honesty, high, not sure) then (offer a position, not, sure)
> if (X, small) then (Y, large, usually).

An important question relates to the meaning of Z-rules and Z^+-rules. The meaning of a Z^+-rule may be expressed as:

$$\text{if } (X, A_X, p_X) \text{ then } (Y, A_Y, p_Y) \rightarrow (X, Y) \text{ is } (A_X \times A_Y, p_X p_Y)$$

where $A_X \times A_Y$ is the Cartesian product of A_X and A_Y

The meaning of Z-rules is more complex and will not be considered in this note. Z-rules have the potential for important applications in decision analysis and modeling of complex systems, especially in the realm of economics.

A problem which plays a key role in many applications of fuzzy logic, especially in the realm of fuzzy control, is that of interpolation. More concretely, the problem of interpolation may be formulated as follows. Consider a collection of fuzzy if-then rules of the form:

$$\text{if } X \text{ is } A_i \text{ then } Y \text{ is } B_i, \quad i = 1, \ldots, n$$

where the A_i and B_i are fuzzy sets with specified membership functions. If X is A, where A is not one of the A_i, then what is the restriction on Y?

The problem of interpolation may be generalized in various ways. A generalization to Z-numbers may be described as follows. Consider a collection of Z-rules of the form:

$$\text{if } X \text{ is } A_i \text{ then usually } (Y \text{ is } B_i), \quad i = 1, \ldots, n$$

where the A_i and B_i are fuzzy sets. Let A be a fuzzy set which is not one of the A_i. What is the restriction on Y expressed as a Z-number? An answer to this question would add a useful formalism to the analysis of complex systems and decision processes.

Representation of Z-numbers is facilitated through the use of what is called a Z-mouse [17]. Basically, a Z-mouse is a visual means of entry and retrieval of fuzzy data. A different system of visual entry and retrieval of fuzzy data was employed by Buisson for balancing meals [3].

The cursor of a Z-mouse is a circular fuzzy mark, called an f-mark, with a trapezoidal distribution of light intensity. This distribution is interpreted as a trapezoidal membership function of a fuzzy set. The parameters of the trapezoid are controlled by the user. A fuzzy number such as "approximately 3" is represented as an f-mark on a scale, with 3 being the centroid of the f-mark (Fig. 2a). The size of the f-mark is a measure of the user's uncertainty about the value of the number. As
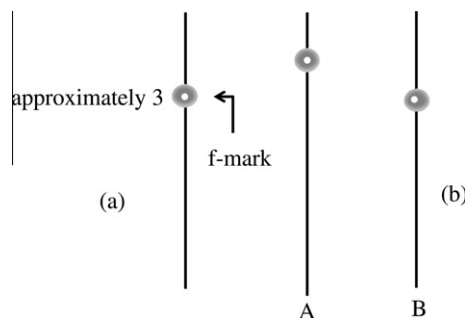

Fig. 2. (a) f-mark of approximately 3; (b) f-marks of a Z-number.

was noted already, the Z-mouse interprets an f-mark as the membership function of a trapezoidal fuzzy set. This membership function serves as an object of computation. A Z-mouse can be used to draw curves and plot functions.

A key idea which underlies the concept of a Z-mouse is that visual interpretation of uncertainty is much more natural than its description in natural language or as a membership function of a fuzzy set. This idea is closely related to the remarkable human capability to precisiate (graduate) perceptions, that is, to associate perceptions with degrees. As an illustration, if I am asked "What is the probability that Obama will be reelected?" I would find it easy to put an f-mark on a scale from 0 to l. Similarly, I could put an f-mark on a scale from 0 to l if I were asked to indicate the degree to which I like my job. It is of interest to note that a Z-mouse could be used as an informative means of polling, making it possible to indicate one's strength of feeling about an issue. Conventional polling techniques do not assess strength of feeling.

Using a Z-mouse, a Z-number is represented as two f-marks on two different scales (Fig. 2b). The trapezoidal fuzzy sets which are associated with the f-marks serve as objects of computation.

2. Computation with *Z*-numbers

What is meant by computation with Z-numbers? Here is a simple example. Suppose that I intend to drive from Berkeley to San Jose via Palo Alto. The perception-based information which I have may be expressed as Z-valuations: (travel time from Berkeley to Palo Alto, about 1 h, usually) and (travel time from Palo Alto to San Jose, about 25 min, usually.) How long will it take me to drive from Berkeley to San Jose? In this case, we are dealing with the sum of two Z-numbers (about 1 h, usually) and (about 25 min, usually.) Problems involving computation with Z-numbers are easy to state but far from easy to solve. Example: What is the square root of (A, B)? Computation with Z-numbers falls within the province of Computing with Words (CW or CWW) [15]. In large measure, computation with Z-numbers is a move into an uncharted territory. What is described in the following is merely a preliminary step toward exploration of this territory.

Computation with Z^+-numbers is much simpler than computation with Z-numbers. Assume that $*$ is a binary operation whose operands are Z^+-numbers, $Z_X^+ = (A_X, R_X)$ and $Z_Y^+ = (A_Y, R_Y)$ By definition,

$$Z_X^+ * Z_Y^+ = (A_X * A_Y, R_X * R_Y)$$

with the understanding that the meaning of $*$ in $R_X * R_Y$ is not the same as the meaning of $*$ in $A_X * A_Y$. In this expression, the operands of $*$ in $A_X * A_Y$ are fuzzy numbers; the operands of $*$ in $R_X * R_Y$ are probability distributions.

Example: Assume that $*$ is sum. In this case, $A_X + A_Y$ is defined by:

$$\mu_{(A_X+A_Y)}(v) = \sup_u(\mu_{A_X}(u) \wedge \mu_{A_Y}(v-u)), \quad \wedge = \min$$

Similarly, assuming that R_X and R_Y are independent, the probability density function of $R_X * R_Y$ is the convolution, $\circ$, of the probability density functions of R_X and R_Y. Denoting these probability density functions as p_{R_X} and p_{R_Y}, respectively, we have:

$$p_{R_X+R_Y}(v) = \int_R p_{R_X}(u)p_{R_Y}(v-u)du$$

Thus,

$$Z_X^+ + Z_Y^+ = (A_X + A_Y, p_{R_X} \circ p_{R_Y})$$

It should be noted that the assumption that R_X and R_Y are independent implies worst case analysis.

More generally, to compute $Z_X * Z_Y$ what is needed is the extension principle of fuzzy logic [7,9]. Basically, the extension principle is a rule for evaluating a function when what are known are not the values of arguments but restrictions on the values of arguments. In other words, the rule involves evaluation of the value of a function under less than complete information about the values of arguments.

Note. Originally, the term "extension principle" was employed to describe a rule which serves to extend the domain of definition of a function from numbers to fuzzy numbers. In this note, the term "extension principle" has a more general meaning which is stated in terms of restrictions. What should be noted is that, more generally, incompleteness of information about the values of arguments applies also to incompleteness of information about functions, in particular, about functions which are described as collections of if-then rules.

There are many versions of the extension principle. A basic version was given in [8]. In this version, the extension principle may be described as:

$$Y = f(X)$$

$$\frac{R(X): X \text{ is } A \text{ (constraint on } u \text{ is } \mu_A(u))}{R(Y): \quad \mu_Y(v) = \sup_u \mu_A(u) \quad (f(A) = R(Y))}$$

subject to

$$v = f(u)$$

where A is a fuzzy set, μ_A is the membership function of A, μ_Y is the membership function of Y, and u and v are generic values of X and Y, respectively.

A discrete version of this rule is:

$$Y = f(X)$$
$$\frac{R(X): X \text{ is } (\mu_1/u_1 + \cdots + \mu_n/u_n)}{R(Y): \mu_Y(v) = \sup_{u_1,\ldots,u_n} \mu_i}$$

subject to

$$v = f(u_i)$$

A more general version was described in [11]. In this version, we have

$$Y = f(X)$$
$$\frac{R(X): g(X) \text{ is } A \text{ (constraint on } u \text{ is } \mu_A(g(u)))}{R(Y): \mu_Y(v) = \sup_u \mu_A(g(u))}$$

subject to

$$v = f(u)$$

For a function with two arguments, the extension principle reads:

$$Z = f(X,Y)$$
$$R(X): \ g(X) \text{ is } A \text{ (constraint on } v \text{ is } \mu_A(g(u)))$$
$$\frac{R(Y): \ h(Y) \text{ is } B \text{ (constraint on } v \text{ is } v_B(h(u)))}{R(Z): \ \mu_Z(w) = \sup_{u,v}(\mu_X(g(u)) \wedge \mu_Y(h(v))), \ \ \wedge = \min}$$

subject to

$$w = f(u,v)$$

In application to probabilistic restrictions, the extension principle leads to results which coincide with standard results which relate to functions of probability distributions [1]. Specifically, for discrete probability distributions, we have:

$$Y = f(X)$$
$$\frac{R(X): X \text{ is } p \ p, \ p = p_1 \setminus u_1 + \cdots p_n \setminus u_n}{R(Y): p_Y(v) = \Sigma_i p_i \quad (f(p) = R(Y))}$$

subject to

$$v = f(u_i)$$

For functions with two arguments, we have:

$$Z = f(X,Y)$$
$$R(X): \ X \text{ is} p \ p, \ p = p_1 \setminus u_1 + \cdots p_m \setminus u_m$$
$$\frac{R(Y): Y \text{ is} p \ q, \ \ q = q_1 \setminus v_1 + \cdots q_n \setminus v_n}{R(Z): p_z(w) = \Sigma_{i,j} p_i q_j \quad (f(p,q) = R(Z))}$$

subject to

$$w = f(u_i, v_j)$$

For the case where the restrictions are Z^+-numbers, the extension principle reads:

$$Z = f(X,Y)$$
$$R(X): \ X \text{ is } (A_X, p_X)$$
$$\frac{R(Y): \ Y \text{ is } (A_Y, p_Y)}{R(Z) = (f(A_X, A_Y) \quad f(p_X, p_Y))}$$

It is this version of the extension principle that is the basis for computation with Z-numbers. Question: Is $f(p_X, p_Y)$ compatible with $f(A_X, A_Y)$?

Turning to computation with Z-numbers, assume for simplicity that $* = \text{sum}$. Assume that $Z_X = (A_X, B_X)$ and $Z_Y = (A_Y, B_Y)$. Our problem is to compute the sum $Z = X + Y$. Assume that the associated Z-valuations are (X, A_X, B_X), (Y, A_Y, B_Y) and (Z, A_Z, B_Z).

The first step involves computation of p_Z. To begin with, let us assume that p_X and p_Y are known, and let us proceed as we did in computing the sum of Z^+-numbers. Then

$$p_Z = p_X \circ p_Y$$

or more concretely,

$$p_Z(v) = \int_R p_X(u)p_Y(v-u)du$$

In the case of Z-numbers what we know are not p_X and p_Y but restrictions on p_X and p_Y

$$\int_R \mu_{A_X}(u)p_X(u)du \text{ is } B_X$$

$$\int_R \mu_{A_Y}(u)p_Y(u)du \text{ is } B_Y$$

In terms of the membership functions of B_X and B_Y, these restrictions may be expressed as:

$$\mu_{B_X}\left(\int_R \mu_{A_X}(u)p_X(u)du\right)$$

$$\mu_{B_Y}\left(\int_R \mu_{A_Y}(u)p_Y(u)du\right)$$

Additional restrictions on p_X and p_Y are:

$$\int_R p_X(u)du = 1$$

$$\int_R p_Y(u)du = 1$$

$$\int_R up_X(u)du = \frac{\int_R u\mu_{A_X}(u)du}{\int_R \mu_{A_X}(u)du} \quad \text{(compatibility)}$$

$$\int_R up_Y(u)du = \frac{\int_R u\mu_{A_Y}(u)du}{\int_R \mu_{A_Y}(u)du} \quad \text{(compatibility)}$$

Applying the extension principle, the membership function of p_Z may be expressed as:

$$\mu_{p_Z}(p_Z) = \sup_{p_X,p_Y}\left(\mu_{B_X}\left(\int_R \mu_{A_X}(u)p_X(u)du\right) \wedge \mu_{B_Y}\left(\int_R \mu_{A_Y}(u)p_Y(u)du\right)\right)$$

subject to

$$p_Z = p_X \circ p_Y$$

$$\int_R p_X(u)du = 1$$

$$\int_R p_Y(u)du = 1$$

$$\int_R up_X(u)du = \frac{\int_R u\mu_{A_X}(u)du}{\int_R \mu_{A_X}(u)du}$$

$$\int_R up_Y(u)du = \frac{\int_R u\mu_{A_Y}(u)du}{\int_R \mu_{A_Y}(u)du}$$

In this case, the combined restriction on the arguments is expressed as a conjunction of their restrictions, with $\wedge$ interpreted as min. In effect, application of the extension principle reduces computation of p_Z to a problem in functional optimization. What is important to note is that the solution is not a value of p_Z but a restriction on the values of p_Z, consistent with the restrictions on p_X and p_Y.

At this point it is helpful to pause and summarize where we stand. Proceeding as if we are dealing with Z^+-numbers, we arrive at an expression for p_Z as a function of p_X and p_Y. Using this expression and applying the extension principle we can compute the restriction on p_Z which is induced by the restrictions on p_X and p_Y. The allowed values of p_Z consist of those values of p_Z which are consistent with the given information, with the understanding that consistency is a matter of degree.

The second step involves computation of the probability of the fuzzy event, Z is A_Z, given p_Z. As was noted earlier, in fuzzy logic the probability measure of the fuzzy event X is A, where A is a fuzzy set and X is a random variable with probability density p_X, is defined as:

$$\int_R \mu_A(u)p_X(u)du$$

Using this expression, the probability measure of A_Z may be expressed as:

$$B_Z = \int_R \mu_{A_Z}(u)p_Z(u)du,$$

where

$$\mu_{A_Z}(u) = \sup_v(\mu_{A_X}(v) \wedge \mu_{A_Y}(u-v))$$

It should be noted that B_Z is a number when p_Z is a known probability density function. Since what we know about p_Z is its possibility distribution, $\mu_{pZ}(p_Z)$, B_Z is a fuzzy set with membership function μ_{BZ}. Applying the extension principle, we arrive at an expression for μ_{BZ}. More specifically,

$$\mu_{B_Z}(w) = \sup_{p_Z} \mu_{p_Z}(p_Z)$$

subject to

$$w = \int_R \mu_{A_Z}(u)p_Z(u)du$$

where $\mu_{p_Z}(p_Z)$ is the result of the first step. In principle, this completes computation of the sum of Z-numbers, Z_X and Z_Y.

In a similar way, we can compute various functions of Z-numbers. The basic idea which underlies these computations may be summarized as follows. Suppose that our problem is that of computing $f(Z_X,Z_Y)$, where Z_X and Z_Y are Z-numbers, $Z_X = (A_X,B_X)$ and $Z_Y = (A_Y,B_Y)$, respectively, and $f(Z_X,Z_Y) = (A_Z,B_Z)$. We begin by assuming that the underlying probability distributions p_X and p_Y are known. This assumption reduces the computation of $f(Z_X,Z_Y)$ to computation of $f(Z_X^+,Z_Y^+)$, which can be carried out through the use of the version of the extension principle which applies to restrictions which are Z^+-numbers. At this point, we recognize that what we know are not p_X and p_Y but restrictions on p_X and p_Y. Applying the version of the extension principle which relates to probabilistic restrictions, we are led to $f(Z_X,Z_Y)$. We can compute the restriction, B_Z, on the scalar product of $f(A_X,A_Y)$ and $f(p_X,p_Y)$. Since $A_Z = f(A_X,A_Y)$, computation of B_Z completes the computation of $f(Z_X,Z_Y)$.

It is helpful to express the summary as a version of the extension principle. More concretely, we can write:

$$Z = f(X,Y)$$

$$X \text{ is } (A_X,B_X) \quad \text{(restriction on } X)$$

$$Y \text{ is } (A_Y,B_Y) \quad \text{(restriction on } Y)$$

$$\frac{Z \text{ is } (A_Z,B_Z) \quad \text{(induced restriction on } Z)}{A_Z = f(A_X,A_Y) \quad \text{(application of extension principle for fuzzy numbers)}}$$

$$B_Z = \mu_{A_Z} \cdot f(p_X,p_Y)$$

where p_X and p_Y are constrained by:

$$\int_R \mu_{A_X}(u)p_X(u)du \text{ is } B_X$$

$$\int_R \mu_{A_Y}(u)p_Y(u)du \text{ is } B_Y$$

In terms of the membership functions of B_X and B_Y, these restrictions may be expressed as:

$$\mu_{B_X}\left(\int_R \mu_{A_X}(u)p_X(u)du\right)$$

$$\mu_{B_Y}\left(\int_R \mu_{A_Y}(u)p_Y(u)du\right)$$

Additional restrictions on p_X and p_Y are:

$$\int_R p_X(u)du = 1$$

$$\int_R p_Y(u)du = 1$$

$$\int_R up_X(u)du = \frac{\int_R u\mu_{A_X}(u)du}{\int_R \mu_{A_X}(u)du} \quad \text{(compatibility)}$$

$$\int_R up_Y(u)du = \frac{\int_R u\mu_{A_Y}(u)du}{\int_R \mu_{A_Y}(u)du} \quad \text{(compatibility)}$$

Consequently, in agreement with earlier results we can write:

$$\mu_{p_Z}(p_Z) = \sup_{p_X,p_Y}\left(\mu_{B_X}\left(\int_R \mu_{A_X}(u)p_X(u)du\right) \wedge \mu_{B_Y}\left(\int_R \mu_{A_Y}(u)p_Y(u)du\right)\right)$$

subject to

$$p_Z = p_X \circ p_Y$$

$$\int_R p_X(u)du = 1$$

$$\int_R p_Y(u)du = 1$$

$$\int_R up_X(u)du = \frac{\int_R u\mu_{A_X}(u)du}{\int_R \mu_{A_X}(u)du}$$

$$\int_R up_Y(u)du = \frac{\int_R u\mu_{A_Y}(u)du}{\int_R \mu_{A_Y}(u)du}$$

What is important to keep in mind is that A and B are, for the most part, perception-based and hence intrinsically imprecise. Imprecision of A and B may be exploited by making simplifying assumptions about A and B – assumptions that are aimed at reduction of complexity of computation with Z-numbers and increasing the informativeness of results of computation. Two examples of such assumptions are sketched in the following.

Briefly, a realistic simplifying assumption is that p_X and p_Y are parametric distributions, in particular, Gaussian distributions with parameters m_X, σ_X^2 and m_Y, σ_Y^2, respectively. Compatibility conditions fix the values of m_X and m_Y. Consequently, if b_X and b_Y are numerical measures of certainty, then b_X and b_Y determine p_X and p_Y, respectively. Thus, the assumption that we know b_X and b_Y is equivalent to the assumption that we know p_X and p_Y. Employing the rules governing computation of functions of Z^+-numbers, we can compute B_Z as a function of b_X and b_Y. At this point, we recognize that B_X and B_Y are restrictions on b_X and b_Y, respectively. Employment of a general version of the extension principle leads to B_Z and completes the process of computation. This may well be a very effective way of computing with Z-numbers.

Another effective way of exploiting the imprecision of A and B involves approximation of the trapezoidal membership function of A by an interval-valued membership function, A^b, where A^b is the bandwidth of A (Fig. 3). Since A is a crisp set, we can write:

$$\left(A_X^b, B_X\right) * (A_Y^b, B_Y) = \left(A_X^b * A_Y^b, B_X \times B_Y\right)$$

where $B_X \times B_Y$ is the product of the fuzzy numbers B_X and B_Y. Validity of this expression depends on how well an interval-valued membership function approximates to a trapezoidal membership function.
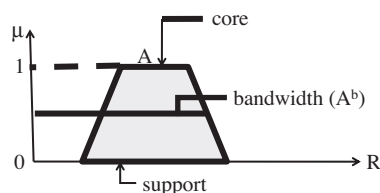

Fig. 3. Interval-valued approximation to a trapezoidal fuzzy set.

3. Concluding remark

Clearly, the issue of reliability of information is of pivotal importance in planning, decision-making, formulation of algorithms and management of information. The issue of reliability is intrinsically complex – an issue that does not lend itself to rigorous formal analysis. The concept of a *Z*-number which is introduced in this note merely opens the door to an uncharted territory and should be viewed as a first step toward development of methods of computation with *Z*-numbers. It is possible that some of the assumptions regarding *Z*-numbers may have to be revised in the light of experience. There are many important directions which remain to be explored, especially in the realm of calculi of *Z*-rules and their application to decision analysis and modeling of complex systems.

Computation with *Z*-numbers may be viewed as a generalization of computation with numbers, intervals, fuzzy numbers and random numbers. More concretely, the levels of generality are: computation with numbers (ground level 1); computation with intervals (level 1); computation with fuzzy numbers (level 2); computation with random numbers (level 2); and computation with *Z*-numbers (level 3). The higher the level of generality, the greater is the capability to construct realistic models of real-world systems, especially in the realms of economics, decision analysis, risk assessment, planning and analysis of causality.

It should be noted that many numbers, especially in fields such as economics and decision analysis are in reality *Z*-numbers, but they are not treated as such because it is much simpler to compute with numbers than with *Z*-numbers. Basically, the concept of a *Z*-number is a step toward formalization of the remarkable human capability to make rational decisions in an environment of imprecision and uncertainty.

There are many questions which are touched upon but not answered in this note. A basic question relates to informativeness. Are *Z*-numbers and functions of *Z*-numbers sufficiently informative to serve a particular purpose? What assumptions need to be made to insure informativeness?

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