Topology I - Exercises and Solutions

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1 Metric spaces

1. Let p be a prime number, and $d\colon \mathbb{Z} \times \mathbb{Z} \to [0, +\infty)$ be a function defined by

$$d_p(x, y) = p^{-\max\{m \in \mathbb{N} \,:\, p^m | x-y\}}.$$

Prove that d_p is a metric on $\mathbb{Z}$ and that $d_p(x, y) \leq \max\{d_p(x, z), d_p(z, y)\}$ for every $x, y, z \in \mathbb{Z}$.

Proof. The symmetry condition is trivial. By $p > 0$ and $m \in \mathbb{N}$, we have $p^{-m} > 0$ for every $m \in \mathbb{N}$ with 0 as the limit. Indeed, $d(x, y)$ can only be zero, if $p^m | x - y$ for all natural numbers m. This can only happen if $x - y = 0$, thus $d(x, y) = 0 \Leftrightarrow x = y$. For the triangle inequality condition, assume wlog that $m_{x,z} := -\log_p(d(x, z))$ is not larger than $m_{y,z}$ (i.e. $d(x, z) \geq d(y, z)$). Then $p^{m_{x,z}}$ divides both $x - z$ and $z - y$ and therefore also $x - z + z - y = x - y$, which means that $m_{x,y} \geq m_{x,z}$ or equivalently $d(x, y) \leq d(x, z) = \max\{d(x, z), d(y, z)\}$, so in particular $d(x, y) \leq d(x, z) + d(y, z)$ holds. □

2. Let $X \neq \emptyset$ be a set and $\mathcal{F}$ the family of all finite subsets of X. Prove that the function $d\colon \mathcal{F} \times \mathcal{F} \to [0, +\infty)$ defined by

$$d(A, B) = \#((A \backslash B) \cup (B \backslash A))$$

is a metric on X.

Proof. $d(A, B) = 0$ implies that both $A \backslash B$ and $B \backslash A$ must be empty, thus the sets A and B must coincide. Conversely $d(A, A) = 0$. Looking at the Venn diagram for sets A, B, C, it can be seen that $A \Delta C \cup B \Delta C$ contains $A \Delta B$ (where Δ denotes the symmetric difference). □

3. Let (X, d) be a metric space and $F \subseteq X$ be a finite subset. Prove that F is closed in X.

Proof. Let $x \in X \setminus F$ and define $r := \min\{d(x, y) :\ y \in F\}$. As F is finite the minimum exists. The open ball $B(x, r)$ around x does not contain any point of F, thus $X \setminus F$ is open and F closed. □

4. Let (X, d) be a metric space and $\emptyset \neq Y \subseteq X$ be a subset. The *distance* of a point $x \in X$ from the subset Y is a function $X \to [0, +\infty)$ defined by:

$$d(x, Y) = \inf\{d(x, y) : y \in Y\}.$$

Verify that the distance function is well defined. Prove that

$$\overline{Y} = \{x \in X : d(x, Y) = 0\}.$$

Proof. The fact that d is bounded from below ensures that for each $x \in X$ the infimum exists. If the infimum exists, it is unique, so the function is well-defined. The claim directly follows from Lemma 1.5 in the lecture: Take an $x \notin \overline{Y}$ and $r > 0$ such that $B_r(x) \cap Y = \emptyset$. Then all points outside the ball, in particular the points of Y, must have distance at least r, so $x \notin \{x \in X : d(x, Y) = 0\}$. Now let $x \notin \{x \in X : d(x, Y) = 0\}$. Then by the triangle inequality any ball $B(x, r)$ with $0 < r < d(x, Y)$ will be an open neighbourhood of x that doesn't intersect $\{x \in X : d(x, Y) = 0\}$. Hence it is closed and because it contains Y, it is a superset of $\overline{Y}$. □

5. Let X be the set of all sequences $x\colon \mathbb{N} \to \mathbb{R}$ converging to zero. Prove that the function $d\colon X \times X \to [0, +\infty)$ defined by
$$d(x_n, y_n) = \sup_{n\in\mathbb{N}} |x_n - y_n|$$
is a metric on X. Show that the metric space (X, d) is separable.

Proof. Symmetry and coincidence with zero are obvious. For the triangle inequality in this metric, use the triangle inequality for the Euclidean metric. For each $n \in \mathbb{N}$, we have
$$|x_n - y_n| = |x_n - z_n + z_n - y_n| \le |x_n - z_n| + |z_n - y_n| \le d(x, z) + d(y, z).$$
Then by the definition of the supremum, this also holds for $d(x, y)$.
To prove that X is separable we need to find a countable set Q such that $\overline{Q} = X$. Let Q be the set of sequences $r : \mathbb{N} \to \mathbb{Q}$ converging to zero. Cantors diagonal argument shows that Q is countable and for every $x \in X$, $x(i)$ can be approximated by a sequence of rationals $r_n(i)$, so $r_n \to x$. $\square$

6. Let (X, d) be a metric space and $Y \subseteq X$ be a subspace. Then $U \subset Y$ is open (closed) in Y if and only if there exist $V \subseteq X$ open (closed) in X such that $U = V \cap Y$.

Proof. By definition a set $U \subset Y$ is open in Y if for all points $x \in U$ there exists an $r_x > 0$ such that $B_{d'}(x, r_x)$ lies in U, where $d' := d \restriction_{Y\times Y}$. Thus U can be expressed as $\bigcup_{x\in U} B_{d'}(x, r_x)$. One direction of the claim can be shown by setting $V := \bigcup_{x\in U} B_d(x, r_x)$. Intersecting each ball $B_d(x, r_x)$ with Y, we obtain the corresponding ball with respect to the induced metric d' and deduce $Y \cap V = U$. The converse direction is straight forward. If there is an open V with the above mentioned properties, it must contain an open ball $B_d(z, r_z)$ for every point $z \in V$ which gives us balls which are open in Y and which all lie in U. $\square$

7. Let (X, d) be a metric space, and $\emptyset \ne Y \subseteq X$. Prove that the function $X \to \mathbb{R}$ defined by $x \mapsto d(x, Y)$ is a continuous function.

Proof. It suffices to show that the preimage of any open set $U \subseteq \mathbb{R}$ is open in X. Take an arbitrary $x \in f^{-1}(U)$. U is open, so there exists an $\varepsilon > 0$ with the property that $(z - \varepsilon, z + \varepsilon) \subseteq U$ where $z := d(x, Y)$. Every point $x' \in B(x, \varepsilon)$ satisfies $d(x', Y) \le d(x', x) + d(x, Y) < \varepsilon + z$ by the triangle inequality. Conversely, we also have $z = d(x, Y) \le d(x, x') + d(x', Y) < \varepsilon + d(x', Y)$which rearranges to $z - \varepsilon < d(x', Y)$. We deduce that $x' \in f^{-1}(U)$ which yields the claim.

$\square$

8. Let (X, d) be a metric space and $U \ne X$ be an open subset in X. Consider a function $d_U : U \times U \to [0, +\infty)$ defined by
$$d_U(x, y) = d(x, y) + \left|\tfrac{1}{d(x,X\setminus U)} - \tfrac{1}{d(y,X\setminus U)}\right|.$$
Prove that d_U is a metric on U and that it is equivalent to the induced metric $d|_{U\times U}$.

Proof. Again, symmetry and coincidence with zero are obvious. Using the triangle inequality first for the metric d and then for the absolute value function, we obtain
$$\begin{aligned} d_U(x, y) &= d(x, y) + \left|\tfrac{1}{d(x,X\setminus U)} - \tfrac{1}{d(y,X\setminus U)}\right| \\ &\le d(x, z) + d(y, z) + \left|\tfrac{1}{d(x,X\setminus U)} - \tfrac{1}{d(z,X\setminus U)} + \tfrac{1}{d(z,X\setminus U)} - \tfrac{1}{d(y,X\setminus U)}\right| \\ &\le d(x, z) + d(y, z) + \left|\tfrac{1}{d(x,X\setminus U)} - \tfrac{1}{d(z,X\setminus U)}\right| + \left|\tfrac{1}{d(z,X\setminus U)} - \tfrac{1}{d(y,X\setminus U)}\right| \\ &= d_U(x, z) + d_U(y, z). \end{aligned}$$

To show the equivalence of the metrics, take a set $V \subseteq U$ that is open with respect to d. This set contains an open r_x-ball contained in V for every element $x \in V$. The open ball with radius r_x with respect to d_U is a subset of $B_d(x, r_x)$ and therefore lies in V.
Conversely, take a set W that is open with respect to metric d_U and a point $x \in W$. There exists a radius R_x such that $B_{d_U}(x, R_x) \subseteq W$. For every point y in $B_d(x, r_x)$ for some r_x that is yet to

determine, the distance $d(y, X \setminus U)$ differs from $D_x := d(x, X \setminus U)$ by at most r_x. Without loss of generality, assume that $D_x > r_x$, otherwise scale r_x accordingly. Now

$$\left|\tfrac{1}{d(x,X\setminus U)} - \tfrac{1}{d(y,X\setminus U)}\right| = \left|\tfrac{d(y,X\setminus U)-d(x,X\setminus U)}{d(x,X\setminus U)\cdot d(y,X\setminus U)}\right| \leq \tfrac{r_x}{D_x\cdot(D_x-r_x)} =: N_x$$

Thus $d_U(x,y) \leq d(x,y) + N_x < r_x + N_x$. Setting r_x such that $r_x + N_x < R_x$, ensures that $B_d(x,r_x) \subset B_{d_U}(x,R_x)$ and therefore lies in W. □

9. Let $S \neq \emptyset$ be a set and (X,d) a complete metric space. Prove that the metric space of all bounded functions $B(S,X)$ equipped with the metric $D(f,g) = \sup_{s\in S} d(f(s),g(s))$ is also complete.

   Proof. Take a Cauchy sequence $(f_n) \subseteq B(S,X)$ in the sup-metric. For every $\varepsilon > 0$ there exists an $n_\varepsilon \in \mathbb{N}$ so that $d(f_n(s), f_m(s)) \leq D(f_n, f_m) < \frac{\varepsilon}{3}$ for all $n, m \geq n_\varepsilon$. From the definition of D we know that $(f_n(s))$ is a Cauchy sequence in X for every $s \in S$. Since X is complete, for every $s \in S$ there exists a $f(s) \in X$ so that $f_n(s) \to f(s)$. Define the function $f : S \to X$ so that $s \mapsto f(s)$ (well defined due to the uniqueness of the limit). To check that (f_n) converges to f, fix $s \in S$. Now there exists $n_s \in \mathbb{N}$ so that $d(f_n(s), f(s)) < \frac{\varepsilon}{3}$ for all $n \geq n_s$. Define $N := \max\{n_\varepsilon, n_s\}$. Then for every $n \geq n_\varepsilon$ we have

   $$d(f_n(s), f(s)) \leq d(f_n(s), f_N(s)) + d(f_N(s), f(s)) < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \frac{2\varepsilon}{3}.$$

   In other words, for every $s \in S$ the difference $d(f_n(s), f(s))$ is bounded by $\frac{2\varepsilon}{3}$. Hence by taking supremum over all $s \in S$, we have $D(f_n, f) \leq \frac{2\varepsilon}{3} < \varepsilon$ for all $n \geq n_\varepsilon$. To show that f is bounded, i.e. $\sup_{r,s\in S} d(f(s), f(r)) < \infty$, use the triangle inequality twice and the fact that f_N is bounded.

   $$d(f(s), f(r)) \leq d(f(s), f_N(s)) + d(f_N(s), f_N(r)) + d(f_N(r), f(r)) \leq \frac{2\varepsilon}{3} + d(f_N(s), f_N(r)) < \infty$$

   □

10. Prove that a discrete metric space is compact if and only if its underlying set is finite.

    Proof. One direction is obvious, as each subset of a finite set is finite. For the other direction, take a compact space (X,d) with the discrete metric, suppose the underlying set X were infinite and look at the open cover $\mathcal{C} = \{\{x\} : x \in X\}$. As $\mathcal{C}$ is infinite, any finite subcover $\mathcal{C}'$ would yield an element $y \in X$ that is not covered by $\mathcal{C}'$ which contradicts the compactness of (X,d). □

11. Let (X,d) be a metric space and $\lim x_n = x_0$ in X. Show that the set $K = \{x_0, x_1, \dots\}$ is compact.

    Proof. As x_0 lies in K, there must be one open set U_0 in any open covering $\mathcal{C}$ that contains it. This U_0 must contain an open ball $B(x_0, \varepsilon)$. As the sequence (x_n) converges to x_0, there is a n_0 such that for all $n \geq n_0$, we have $d(x_n, x_0) < \varepsilon$ which leaves only finitely many points outside the ball. The (finite) subset of those, that aren't contained in U_0 can be covered by finitely many open sets from $\mathcal{C}$. □

12. Let (X,d) be a metric space, and let $F \subseteq X$ be a closed subset and $K \subseteq X$ a compact subset of X. Prove that

    $$F \cap K \neq \emptyset \iff \inf\{d(x,y) : x \in F, y \in K\} = 0.$$

    Proof. The one direction is straight forward. A point $x \in F \cap K$ satisfies $d(x,x) = 0$. For the other direction, take sequences $(x_n) \in F$, $(y_n) \in K$ that satisfy $\inf_{n\in\mathbb{N}} d(x_n, y_n) = 0$. For all $\varepsilon > 0$ we have an $n_1 \in \mathbb{N}$ s.t. for all $n \geq n_1$, $d(x_n, y_n) < \frac{\varepsilon}{2}$. K is compact, so (y_n) has a converging subsequence (y_{n_i}) with limit $y \in K$. In particular, we find an $n_2 \in \mathbb{N}$ such that for all $n \geq n_2$, $d(y_{n_i}, y) < \frac{\varepsilon}{2}$. With the triangle inequality, we get $d(y, x_{n_i}) \leq d(y, y_{n_i}) + d(y_{n_i}, x_{n_i}) < \varepsilon$. Thus y is a limit point of F and since F is closed, ist must be contained in F. □

13. Let $(X_1, d_1), \dots, (X_n, d_n)$ be compact metric spaces, and let $X := X_1 \times \dots \times X_n$ be equipped with either the D_1 or D_∞ metric. Prove that X is also compact.

Proof. Take a sequence (x^k) in X and look at the sequences $(x_1^k), \dots, (x_n^k)$ in $X_1, \dots, X_n$. Since (X_1, d_1) is compact, it has a convergent subsequence $(x_1^{k_l})$ with index set I_1. The induced sequence $(x_2^{k_l})$ in X_2 has a converging subsequence $(x_2^{k_{l_m}})$ for an index set $I_2 \subseteq I_1$. Iterating gives us an index set I_n such that the sequences x_j^K for $K \in I_n$ converge for all $j = 1, \dots, n$. Thus, with both metrics, the corresponding subsequence of (x^k) converges. From the lecture we know that this implies compactness (Chapter 1.9, Lemma). $\square$

14. Let E be a normed space and A and B two compact subspaces. Then the Minkowski sum

$$A + B := \{a + b : a \in A, b \in B\}$$

is also compact.

Proof. This can again be shown using sequential compactness. Another approach is the following. Let $\mathcal{C}$ be an open cover of $A+B$. Then the families $\mathcal{C}_\mathcal{A} := \{U - B : U \in \mathcal{C}\}$ and $\mathcal{C}_\mathcal{B} := \{U - A : U \in \mathcal{C}\}$ are covers of A and B respectively. Those sets are compact, so there exist finite subcovers. Both families (translated by A or B) together yield a finite subcover of $A + B$. $\square$

15. Let (X, d) be a metric space. Define a function $d' \colon X \times X \to [0, +\infty)$ by

$$d'(x, y) := \min\{d(x, y), 1\}$$

for every $(x, y) \in X \times X$. Show that d' is a metric on X, and moreover is equivalent to d, i.e., induces the same topology as d.

Proof. Checking the metric properties is straight forward. To see that the topologies coincide, use that for $\varepsilon < 1$, $B_d(x, \varepsilon) = B_{d'}(x, \varepsilon)$. $\square$

2 Topological spaces and Continuous Functions

1. Let X be a set and $\mathcal{T} = \{U \subseteq X : U = \emptyset \text{ or } X \backslash U \text{ is countable}\}$. Prove that $\mathcal{T}$ is a topology on X.

Proof. Obviously $\emptyset$ and X are contained in $\mathcal{T}$. For the second criterion, look at a subfamily $\{U_\alpha : \alpha \in A\}$ of $\mathcal{T}$. Without loss of generality, assume that all U_α have a countable complement. Then $X \setminus \bigcup_{\alpha \in A} U_\alpha = \bigcap_{\alpha \in A} X \setminus U_\alpha$ is the intersection of countable sets and therefore countable. Also, the finite union of countable sets is countable which yields the third criterion. $\square$

2. Let X be a set and $\mathcal{B}$ be a basis for a topology on X. Prove that the topology generated by $\mathcal{B}$ is indeed a topology.

Proof. Denote the topology generated by $\mathcal{B}$ with $\mathcal{T}$. The empty set is trivially contained in it. For the whole space X, the first property of a basis ensures that $C \in \mathcal{T}$. The second criterion is obvious, as any point x in the union of open sets must lie in one of them. Thus we find a basis set B containing x and also contained in that set and therefore also in the union. If we take a finite subfamily $\{U_1, \dots, U_n\}$, then any $x \in \bigcap_{i=1,\dots,n} U_i$ has basis sets B_i such that $x \in B_i \subseteq U_i$. The second base property reformulates as $\forall n : x \in \bigcap_{i=1,\dots,n} B_i \Rightarrow \exists B \in \mathcal{B} : x \in B \subseteq \bigcap_{i=1,\dots,n} B_i$ (by repeatedly applying the property). Thus there exists a basis set B such that $x \in B \subseteq \bigcap_{i=1,\dots,n} U_i$. $\square$

3. Let $X = \mathbb{R}$ and $K = \{\frac{1}{n} : n \in \mathbb{N}\}$. Prove that the following families are bases for a topology on X:

$$\begin{aligned} \mathcal{B} &= \{(a, b) : a, b \in \mathbb{R},\ a < b\}, \\ \mathcal{B}' &= \{[a, b) : a, b \in \mathbb{R},\ a < b\}, \\ \mathcal{B}'' &= \{(a, b) : a, b \in \mathbb{R},\ a < b\} \cup \{(a, b) \backslash K : a, b \in \mathbb{R},\ a < b\}. \end{aligned}$$

Furthermore, let $\mathcal{T}$, $\mathcal{T}'$ and $\mathcal{T}''$ denote the topologies on X generated by $\mathcal{B}$, $\mathcal{B}'$ and $\mathcal{B}''$, respectively. Prove that $\mathcal{T}'$ and $\mathcal{T}''$ are finer than $\mathcal{T}$, and that $\mathcal{T}'$ and $\mathcal{T}''$ are not comparable.

Proof. Remark: $\mathcal{T}'$ is called the lower limit topology or right half-open interval topology; the topological space is called Sorgenfrey line. $\mathcal{T}''$ is called K-topology.
First, check the two basis properties for the three families. This is straightforward. The only thing one has to watch out for is the second property for $\mathcal{B}''$. There one needs to do a (simple) case analysis.
$\mathcal{T} \subseteq \mathcal{T}'$: Let $U \in \mathcal{T}$. Then $\forall x \in U \exists (a_x, b_x) \subseteq U : x \in (a_x, b_x)$. We can rewrite any interval (a, b) as $\bigcup_{a<c<b} [c, b)$, so $(a_x, b_x) \in \mathcal{T}'$, thus for all $y \in (a_x, b_x)$, so in particular for the previous x, there exist $B_y \in \mathcal{B}' : y \in B_y \subseteq (a_x, b_x)$. Thus $U \in \mathcal{T}'$.
$\mathcal{T} \subseteq \mathcal{T}''$: Obvious.
$\mathcal{T}'' \not\subseteq \mathcal{T}'$: Use $\mathbb{R} \setminus K$. It is open in $\mathcal{T}''$ but has nonempty intersection with every open set in $\mathcal{T}''$ that also contains zero.
$\mathcal{T}' \not\subseteq \mathcal{T}''$: Take for instance $[2, \infty)$. It is open in $\mathcal{T}''$ but any basis set in $\mathcal{B}''$ that contains 2 will also contain numbers smaller than 2. $\square$

4. Let $X = \mathbb{R}$ and $\mathcal{B} = \{(a, b) : a, b \in \mathbb{Q},\ a < b\}$. Prove that $\mathcal{B}$ is a basis for a topology on X, and that it generates the standard topology on X.

   Proof. Let $x \in \mathbb{R}$. Let n be some integer larger than x. Then $-n < x < n$. Now let (a, b) and (c, d) be basis sets that contain x. Then $(\max(a, c), \min(b, d))$ lies in $\mathcal{B}$ because $\max(a, c) < x$ and $x < \min(b, d)$. Finally we need to check that the generated topology coincides with the standard topology. The one inclusion is obvious. For the other one, take a set U that is open in the standard topology. Then there are real numbers $a < b$ such that $x \in (a, b)$ for all $x \in U$. There exists an N such that $N > \frac{1}{x-a}$. which can be rearranged to $1 < xN - aN$, so there must be another integer M between aN and xN. This again can be rearranged to $a < \frac{M}{N} < x$. Doing the same thing for x and b, we get $x \in (\frac{M}{N}, \frac{M'}{N'}) \subseteq (a, b)$. Hence U is open in the topology generated by $\mathcal{B}$. $\square$

5. Let A be a subspace of X and let B be a subspace of Y. We equip A and B with the subspace topologies. Prove that the product topology on $A \times B$ is the same as the topology $A \times B$ inherits as a subspace of $X \times Y$.

   Proof. Denote with $\mathcal{T}_{A\times B,prod}$ the product topology on $A \times B$ and with $\mathcal{T}_{A\times B,sub}$ the subspace topology induced by $X \times Y$.
   $\mathcal{T}_{A\times B,prod} \subseteq \mathcal{T}_{A\times B,sub}$: Let $U_A \times U_B \in \mathcal{T}_A \times \mathcal{T}_B$. The open sets in A and B respectively give us open sets V_A and V_B in the spaces X and Y by the definition of the subspace topology and if we intersect $V_A \times V_B$ with $A \times B$, we obtain $U_A \times U_B$. Thus the basis of $\mathcal{T}_{A\times B,prod}$ is contained in $\mathcal{T}_{A\times B,sub}$, and the same holds for the topology generated by it.
   $\mathcal{T}_{A\times B,sub} \subseteq \mathcal{T}_{A\times B,prod}$: Let $U \in \mathcal{T}_{A\times B,sub}$. Then there exists a set $V \in \mathcal{T}_{X\times Y}$ such that $V \cap A \times B = U$. By the construction of the product topology, for all $v \in V$ we have a basis set $V_X \times V_Y \in \mathcal{T}_X \times \mathcal{T}_Y$ with $v \in V_X \times V_Y \subseteq V$. In particular this holds for all $v \in U$ with $v \in U_A \times U_B := V_X \times V_Y \cap A \times B \subseteq U$, so $U \in \mathcal{T}_{A\times B,prod}$. $\square$

6. Let Y be a subspace of X. Prove that if A is closed in Y, and Y is closed in X, then A is closed in X.

   Proof. Denote by $\mathcal{U}_Z(x)$ the family of sets which contain x and are open in Z and by $\overline{A}_Z \subseteq Z$ the closure of A in Z. Recall the following criterion from the lecture (a theorem from Chapter 2.3):

   $$x \in Z \text{ lies in } \overline{A}_Z \Leftrightarrow \forall U \in \mathcal{U}_Z(x) : U \cap A \neq \emptyset.$$

   Let $x \in \overline{A}_X$, then every element of $\mathcal{U}_X(x)$ must have non-empty intersection with Y, because $A \subseteq Y$. This means that $x \in \overline{Y}_X$ which equals Y because Y is closed in X. Also, all sets in $\mathcal{U}_Y(x)$ originate from a set in $\mathcal{U}_X(x)$, so they all have non-empty intersection with A. Thus $x \in \overline{A}_Y$ which coincides with A because A is closed in Y. $A \subseteq \overline{A}_X$ is trivial, so we deduce $A = \overline{A}_X$. $\square$

7. Let X be a Hausdorff space. Prove that a sequence in X converges to at most one point in X.

   Proof. Denote by $\mathcal{U}(x)$ the family of open sets which contain x. Assume we have a converging sequence (x_n) with distinct limit points x and y. Then by definition for all $U \in \mathcal{U}(x), V \in \mathcal{U}(y)$ there exist $n_U, n_V \in \mathbb{N}$ such that for all $n \geq n_U, n_V$, we have $x_n \in U$ and $x_n \in V$ respectively.

X is Hausdorff, so there exist disjoint neighbourhoods $U' \in \mathcal{U}(x)$ and $V' \in \mathcal{U}(y)$ and for $n \geq \max\{n_{U'}, n_{V'}\}$ we have $x_n \in U' \cap V' = \emptyset$. Contradiction. $\square$

8. Show that if A is closed in X, and B is closed in Y, then $A \times B$ is closed in $X \times Y$.

   Proof. $X \setminus A$ and $Y \setminus B$ are open and so is $(X \setminus A) \times Y$ and $X \times (Y \setminus B)$. Thus $(X \times Y) \setminus (A \times B) = ((X \setminus A) \times Y) \cup (X \times (Y \setminus B))$ is also open. Alternatively, use $\overline{A \times B} = \overline{A} \times \overline{B}$ from exercise 10. $\square$

9. Show that if U is open in X and A is closed in X, then $U \setminus A$ is open in X, and $A \setminus U$ is closed in X.

   Proof. $U \setminus A = U \cap (X \setminus U)$ and $X \setminus (A \setminus U) = (X \setminus A) \cap U$ are the finite intersection resp. union of open sets and therefore open. $\square$

10. Let $A \subseteq X$ and $B \subseteq Y$. Show that $\overline{A \times B} = \overline{A} \times \overline{B}$ in $X \times Y$.

    Proof. Wlog only look at the limit points (for isolated points there is no problem).
    - $\subseteq$ Let (a_n, b_n) be a convergent sequence in $A \times B$ with limit point (a, b). Let $V \in \mathcal{T}_X$ and $W \in \mathcal{T}_Y$ with $a \in V$ and $b \in W$, then $V \times W \in \mathcal{T}_{X \times Y}$ and $(a, b) \in V \times W$. Thus there exists an $n_{V,W} \in \mathbb{N}$ such that all subsequent elements (a_n, b_n) lie in $V \times W$. In particular $a_n \in V, b_n \in W$ for all $n \geq n_{V,W}$ which makes a and b limit points. A and B are closed, therefore $a \in A$ and $b \in B$.
    - $\supseteq$ Let $a_n \to a$ in A and $b_n \to b$ in B. Take an arbitrary $U \in \mathcal{T}_{X \times Y}$ with $(a, b) \in U$. Then for all $u \in U$ there is a basis set $V \times W \in \mathcal{T}_{X \times Y}$ such that $u \in V \times W \subseteq U$. The open sets V and W give us numbers $n_V, n_W \in \mathbb{N}$ such that $a_n \in V, b_n \in W$ for all subsequent n. Define $n_U := \max(n_v, n_W)$, then $\forall n \geq n_U : (a_n.b_n) \in U$. We deduce that (a, b) is a limit point.

    $\square$

11. Show that the product of two Hausdorff spaces is Hausdorff.

    Proof. Let $(a, b) \neq (c, d)$ be points in $X \times Y$, where X, Y are Hausdorff spaces. Then $a \neq c$ or $b \neq d$. Wlog assume $a \neq c$. Then there exist disjoint open neighbourhoods A and C for a and c in X. This gives us disjoint open neighbourhoods $A \times Y$ and $C \times Y$ of (a, b) and (c, d). $\square$

12. Show that X is Hausdorff if and only if the *diagonal* $\Delta := \{(x, x) : x \in X\}$ is closed in $X \times X$.

    Proof. X is Hausdorff. $\Leftrightarrow$ For every pair $a, b \in X$ with $a \neq b$ there there exist disjoint open neighbourhoods $A \ni a$ and $B \ni b$. $\Leftrightarrow$ Every element (a, b) of $X \setminus \Delta$ has a basis set $A \times B$ with $(a, b) \in A \times B \subseteq X \setminus \Delta$. $\Leftrightarrow$ $X \setminus \Delta$ is open. $\Leftrightarrow$ Δ is closed. $\square$

13. If $A \subseteq X$, let us define the *boundary* of A by the equation:

    $$\partial A := \overline{A} \cap \overline{X \setminus A}.$$

    Show that:
    (a) $\mathring{A} \cap \partial A = \emptyset$ and $\overline{A} = \mathring{A} \cup \partial A$;
    (b) $\partial A = \emptyset$ if and only if A is both open and closed;
    (c) A is open if and only if $\partial A = \overline{A} \setminus A$.

    Proof.
    (a) $x \in \mathring{A}$. $\Leftrightarrow$ There is an open neighbourhood $U \subset A$ of x. $\Leftrightarrow$ There is an open neighbourhood U of x with $U \cap X \setminus A = \emptyset$. $\Leftrightarrow$ $x \notin \overline{X \setminus A}$. This means $\mathring{A} = X \setminus \overline{X \setminus A}$. From this we deduce two things: Firstly, $\mathring{A} \cap \overline{X \setminus A} = \emptyset$ and therefore $\mathring{A} \cap \partial A = \emptyset$ and secondly $\partial A = \overline{A} \cap (X \setminus \mathring{A})$ and therefore $\partial A \cup \mathring{A} = \overline{A} \cap (X \setminus \mathring{A}) \cup \mathring{A} = \overline{A}$.
    (b) A is open and closed if and only if $\mathring{A} = A = \overline{A}$ and using (a), this corresponds to $\partial A = \emptyset$.
    (c) A is open if and only if $\mathring{A} = A$.
    - $\Rightarrow$ Subtracting $\mathring{A}$ from the second equation (and using the first statement) of (a), we get that $\partial A = \overline{A} \setminus \mathring{A}$. Inserting $\mathring{A} = A$ yields the claim.
    - $\Leftarrow$ $\overline{A} = \mathring{A} \cup (\overline{A} \setminus A) = \overline{A} \setminus (A \setminus \mathring{A})$, so $A \setminus \mathring{A} = \emptyset$.

    $\square$

14. Let $f\colon X \to Y$ be a continuous map, $A \subset X$ and A' denotes the set of all limit points of A. If $x \in A'$ is it true that $f(x) \in f(A)'$, i.e., is $f(x)$ a limit point of $f(A)$?

    Proof. No, in general this is not true. Isolated points can cause a problem. Consider for instance the constant map $f : \mathbb{R} \to [0,1] \cup \{2\}$ given by $f(x) = 2$. It is obviously continuous but 2 is no limit point. $\square$

15. Let $f\colon A \to B$ and $g\colon C \to D$ be continuous functions. Consider the map $f \times g\colon A \times C \to B \times D$ defined by
$$(f \times g)(a, c) := (f(a), g(c))$$
for every $(a, c) \in A \times C$. Prove that $f \times g$ is also continuous.

    Proof. Let $U \in \mathcal{T}_{B\times D}$. Then for all $u \in U$ there is a set $B_u \times D_u \in \mathcal{T}_B \times \mathcal{T}_D$ with $u \in B_u \times D_u \subseteq U$. Define $A_u := f^{-1}(B_u)$ and $C_u := g^{-1}(D_u)$. $A_u \times C_u$ is an open neighbourhood for any $v \in (f \times g)^{-1}(u)$ due to the fact that f and g are continuous. Thus for every $v \in (f \times g)^{-1}(U)$ there is a set $A_{(f\times g)(v)} \times A_{(f\times g)(v)} \in \mathcal{T}_A \times \mathcal{T}_C$ with $v \in A_{(f\times g)(v)} \times C_{(f\times g)(v)} \subseteq (f \times g)^{-1}(U)$ which proves our claim. $\square$

16. Let X and Y be topological spaces with Y being Hausdorff. Let $A \subseteq X$ and $f\colon A \to Y$ be a continuous map. Prove that if f can be extended to a continuous function $g\colon \overline{A} \to Y$, then g is uniquely determined by f.

    Proof. From exercise 2.14 we know that a convergent sequence a_n in A with limit point a yields a convergent sequence $g(a_n)$ with limit point $g(a)$. In exercise 2.7 we learned that the limit of a convergent sequence is unique in Hausdorff spaces, so $g(a) = \lim_{n\to\infty} f(a_n)$ is uniquely defined. $\square$

17. Let $\{X_\alpha\}_{\alpha\in J}$ be an indexed family of topological spaces. Assume that the topology on each X_α is given by a basis $\mathcal{B}_\alpha$. Prove that the family
$$\Big\{ \prod_{\alpha\in J} B_\alpha : B_\alpha \in \mathcal{B}_\alpha \Big\}$$
is a basis for the box topology on $\prod_{\alpha\in J} X_\alpha$, while
$$\Big\{ \prod_{\alpha\in J} B_\alpha : B_\alpha \in \mathcal{B}_\alpha \text{ and } B_\alpha = X_\alpha \text{ for all but finitely many } \alpha \in J \Big\}$$
is a basis for the product topology on $\prod_{\alpha\in J} X_\alpha$.

    Proof. Let $U \in \mathcal{T}_{box}$, then for all $u \in U$ there exists a set $\prod_{\alpha\in J} U_\alpha$ with the property that $u \in \prod_{\alpha\in J} U_\alpha \subseteq U$. For all α, we have $u_\alpha \in U_\alpha$ and a basis set B_α such that $u_\alpha \in B_\alpha \subseteq U_\alpha$, so $u \in \prod_{\alpha\in J} B_\alpha \subseteq \prod_{\alpha\in J} U_\alpha \subseteq U$. The same line of reasoning works for the product topology. $\square$

18. Let $\{X_\alpha\}_{\alpha\in J}$ be an indexed family of Hausdorff spaces. Prove that $\prod_{\alpha\in J} X_\alpha$ is a Hausdorff space in both the box and product topology.

    Proof. If two points (x_α) and (y_α) are different, then there must be an index $\alpha \in J$ such that $x_\alpha \neq y_\alpha$. Use the same argument as in 11. to prove that the claim (for both topologies). $\square$

19. Let $p\colon X \to Y$ be a continuous map. Show that if there is a continuous map $f\colon Y \to X$ such that $p \circ f = \mathrm{id}_Y$, then p is a quotient map.

    Proof. For every $y \in Y$ the element $x := f(y)$ is in $p^{-1}(y)$, so p is surjective. Now let $p^{-1}(U) \in \mathcal{T}_X$. Then $U = (p \circ f)^{-1}(U) = f^{-1} \circ p^{-1}(U)$ is open because f is continuous. Together with the fact that p is continuous, this yields the claim. $\square$

20. Let X be a topological space and $A \subseteq X$ a subspace. A continuous map $r\colon X \to A$ is a *retraction* if $r|_A = \mathrm{id}_A$, i.e., $r(a) = a$ for every $a \in A$. Show that a retraction is a quotient map.

    Proof. Use the previous exercise with $p := r$ and $f : A \hookrightarrow X$, the embedding of A into X. $\square$

3 Connectedness and Compactness

1. Let $\{C_n\}$ be a sequence of connected subspaces of X with the property that $C_n \cap C_{n+1} \neq \emptyset$ for every $n \in \mathbb{N}$. Show that $\bigcup_{n\in\mathbb{N}} C_n$ is connected.

   Proof. Assume there exist non-empty open sets U, V such that $\bigcup_{n\in\mathbb{N}} C_n$ is the disjoint union of the two. For every $n \in \mathbb{N}$ we know that either $U \cap C_n$ or $V \cap C_n$ must be empty because C_n is connected. So we have that $C_n \subseteq U$ or $C_n \subseteq V$ (This is also a lemma in Chapter 3). Using induction and the fact that for all $n \in \mathbb{N}$ the subspaces C_n and C_{n+1} have non-empty intersection, we deduce that all C_n must lie in the same set U or V which contradicts our assumption. □

2. Let $\{C_\alpha\}$ be a sequence of connected subspaces of X. Let moreover C be a connected subspace of X with the property that $C \cap C_\alpha \neq \emptyset$ for every α. Show that $C \cup \left(\bigcup_\alpha C_\alpha\right)$ is connected.

   Proof. Assume U and V form a separation of $C \cup (\bigcup_{\alpha\in\mathbb{N}} C_n)$. Very much like in the previous exercise, we deduce that C_α and C lie either in U or in V. Wlog say $C \subseteq U$, then $U \cap C_\alpha \subseteq U$ and therefore $C_\alpha \subseteq U$ for all α. Contradiction. □

3. Let X be an infinite set. Show that if X is equipped with the finite complement topology then X is connected.

   Proof. For every pair of sets U, V with the property that $X = U \sqcup V$, i.e. $U = X \setminus V$, we have the following. If one of them, say U, is open, then its complement V must be finite by the definition of the topology. But this means that U itself must be infinite (as X is infinite) and therefore V cannot be open. □

4. Let $A \subset X$. Show that if C is a connected subspace of X that intersects both A and $X \backslash A$, then C intersects ∂A.

   Proof. We can write X as the disjoint union of $\mathring{A}, \partial A$ and $(X \overset{\circ}{\setminus} A)$. If $C \cap \partial A$ is empty, then $C = C \cap (\mathring{A} \sqcup \partial A \sqcup (X \overset{\circ}{\setminus} A)) = (C \cap \mathring{A}) \sqcup (C \cap (X \overset{\circ}{\setminus} A))$. These two sets are open in the subspace topology which contradicts the connectedness assumption. □

5. Let $A \subsetneq X$ and $B \subsetneq Y$. If X and Y are connected, show that $(X \times Y) \backslash (A \times B)$ is also connected.

   Proof. A and B don't coincide with X and Y, so we find points $x \in X \setminus A$ and $y \in Y \setminus B$ and can construct sets $\{x\} \times Y$ and $X \times \{y\}$ that are disjoint from $A \times B$. The union $(\{x\} \times Y) \cup (X \times \{y\})$ is connected as it is the union of connected sets with a point (x, y) in common (see Exercise 3.1). □

6. Let $p\colon X \to Y$ be a quotient map, $p^{-1}(\{y\})$ be connected for every $y \in Y$, and Y be connected. Show that X is connected.

   Proof. Assume U and V form a separation of X. We know that $p(U) \cup p(U) = Y$ because p is surjective. Assume there exists a $y \in p(U) \cap p(V)$. Denote the fibre $p^{-1}(\{y\})$ with C. C is connected and the sets $C \cap U$ and $C \cap V$ are open in C and $C = (C \cap U) \sqcup (C \cap V)$ which gives us a contradiction. This means that $p(U)$ and $p(V)$ are disjoint but also that $p^{-1}(p(U)) = U$. p is a quotient map, so if $p^{-1}(p(U))$ is open, then so is $p(U)$. The same holds for $p(V)$. Thus $p(U)$ and $p(V)$ form a separation of Y, which cannot be true. □

7. Let Y be a subspace of X, and both connected. Show that if A and B form a separation of $X \backslash Y$, then $Y \cup A$ and $Y \cup B$ are connected

   Proof. Let us work with the definition of connectedness using limit points, which was introduced in Lemma 1 of the third chapter. Assume that $Y \cup A$ is not connected. Then we find non-empty U, V with the property that $U \cap V' = \emptyset$, $U' \cap V = \emptyset$ and $Y \cup A = U \cup V$. Y is connected, so by the second lemma of Chapter 3 either $Y \subseteq U$ or $Y \subseteq V$. Say $Y \subseteq U$. Then $V \subseteq A \subseteq A'$, hence $V \cap B' = \emptyset$ and $V' \cap B = \emptyset$. If we now look at the union $U \cup B$, we see that $V \cap (U \cup B)' = V \cap (U' \cup B') = \emptyset$ and $V' \cap (U \cup B) = \emptyset$. So V and $U \cup B$ form a separation of $X = Y \cup A \cup B = V \cup U \cup B$. Contradiction. The same line of reasoning proves that $Y \cup B$ must be connected. □

8. Show that $\mathbb{R}^d$ and $\mathbb{R}$ are not homeomorphic if $d > 1$.

Proof. Assume there is a homeomorphism $f : \mathbb{R}^n \to \mathbb{R}$. Define $\tilde{f} : \mathbb{R}^n \setminus \{f^{-1}(\mathbf{0})\} \to \mathbb{R} \setminus \{\mathbf{0}\}$ to be the restriction of f. It is easy to see that this is again a homeomorphism. But $\mathbb{R}^n \setminus \{h^{-1}(\mathbf{0})\}$ is connected (consider $\mathbf{0}$ as $\{0\} \times \cdots \times \{0\}$ and see Exercise 3.5) and $\mathbb{R} \setminus \{\mathbf{0}\}$ is not. The theorem which states that the image of a connected space under a continuous map is connected gives us the desired contradiction. □

9. Let $f\colon S^1 \to \mathbb{R}$ be a continuous map. Show that there exists a point $x \in S^1$ such that $f(x) = f(-x)$.

   Proof. Assume there is no such point. Then we can construct the (well-defined) map $g : S^1 \to S^0$ given by $g(x) := \frac{f(x)-f(-x)}{|f(x)-f(-x)|}$. It is continuous and therefore its image should be connected, because S^1 is connected. $g(x)$ and $g(-x)$ differ in sign for any $x \in S^1$, so $\{-1, 1\}$ is the image of g and disconnected which gives us a contradiction.
   Remark: This statement generalizes to the Borsuk-Ulam theorem. □

10. Let $f\colon [0, 1] \to [0, 1]$ be a continuous map. Show that there exists a point $x \in [0, 1]$ such that $f(x) = x$.

    Proof. Analogously to the previous exercise, assume there is no such point and define $g : [0, 1] \to \{-1, 1\}$ given by $g(x) := \frac{f(x)-x}{|f(x)-x|}$. This map is well-defined and continuous. It is also surjective because $f(1) < 1$ and $f(0) > 0$ and hence $g(1) = -1$ and $g(0) = 1$, which – as before – gives us a contradiction.
    Remark: This is a special case of the Brouwer fixed-point theorem. □

11. Show that the finite union of compact subspaces of X is compact.

    Proof. Let $\mathcal{U}$ be an open covering of $X = \bigcup_{i=1}^n X_i$. It is also an open covering of the X_i. Compactness gives us a finite subcollection $\tilde{\mathcal{U}_\rangle}$ for every i. Then the union of these families is a finite subcover of X. □

12. Show that every compact subspace Y of a metric space X is bounded with respect to the metric and is closed.

    Proof. Fix an $y \in Y$. Then $\{B(y, \varepsilon) \subseteq X : \varepsilon > 0\}$ is an open cover of Y. From the compactness property we know that there exists a finite subcover with say N sets and ε_{max} the radius of the largest ball. Then the diameter of Y is at most $2\varepsilon_{max}$. The fact that Y is also closed was proved in Chapter 1.9. □

13. Let A and B be disjoint compact subspaces of the Hausdorff space X. Show that there exist disjoint open sets U and V in X such that $A \subseteq U$ and $B \subseteq V$.

    Proof. Recall that in the lecture we extracted the following property from the proof of the claim that every compact subspace of a Hausdorff space is closed: If Y is a compact subspace of a Hausdorff space X and $x_0 \notin Y$, then there exist disjoint open sets U and V of X containing Y and x_0, respectively. Apply this to the compact set A and all points b in B. Then $\{V_b\}$ is an open cover of B with finite subcover $\{V_{b_1}, \dots, V_{b_n}\}$. The sets $\bigcap_{i=1}^n U_{b_i}$ and $\bigcup_{i=1}^n V_{b_i}$ are disjoint and open with $A \subseteq \bigcap_{i=1}^n U_{b_i}$ and $b \subseteq \bigcup_{i=1}^n V_{b_i}$ □

14. Let X be compact and Y be a Hausdorff space. If $f\colon X \to Y$ is a continuous map then f is a closed map.

    Proof. Let Z be a closed subset of X. From the lecture we know that it must be compact too. Images of compact spaces under continuous maps are again compact, so $f(Z)$ is compact. But that means it is also closed, because Y is Hausdorff. □

15. Show that if Y is compact, then the projection $\pi_1\colon X \times Y \to X$ is a closed map.

    Proof. Let $Z \subseteq X \times Y$ be closed and $x \in X \setminus \pi(Z)$. For every $y \in Y$, we have that $(x, y) \notin Z$. The set $X \setminus Z$ is open, so we find basis sets $U_y \times V_y \subseteq X \times Y \setminus Z$ that contain (x, y). The sets V_y cover the compact space Y, so there is a finite subfamily, say $V_{y_1}, ..., V_{y_n}$ such that $Y = \bigcup_{i=1}^n V_{y_i}$. Every set $U_{y_i} \times V_{y_i}$ lies in $X \times Y \setminus Z$, thus $U_{y_i} \cap \pi(Z) = \emptyset$. Hence the intersection $\bigcap_{i=1}^n U_{y_i}$ is an open neighbourhood of x that does not intersect $\pi(Z)$, so $\pi(Z)$ is closed.

□

16. Let $f\colon X \to Y$ be a continuous map with Y compact Hausdorff. Show that f is continuous if and only if the *graph* of f
$$G_f := \{(x, f(x)) : x \in X\}$$
is a closed subset of $X \times Y$.

Proof.

⇒ Let $(x, y) \in (X \times Y) \setminus G_f$. Then $y \neq f(x)$ and therefore we can find disjoint open neighbourhoods $U \ni y, V \ni f(x)$. Define $W := f^{-1}(V)$. Then $f(W) \subseteq V \subseteq Y \setminus U$ and thus $W \times U \cap G_f = \emptyset$. $W \times U$ is open, so we found an open neighbourhood of (x, y) in $(X \times Y) \setminus G_f$.

⇐ Now let G_f be closed and let V be an open set in Y. Then the intersection $G_f \cap X \times (Y \setminus V)$ is also closed. If we apply Exercise 3.15, we can deduce that $\pi_1(G_f \cap X \times (Y \setminus V)) = \{x \in X : f(x) \notin V\} = X \setminus f^{-1}(V)$ is closed and equivalently, that $f^{-1}(V)$ is open.

□