Section 9.3   Integral Test and p-series Test

Integral Test

Use the Integral Test to show that the series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \ \dots. \qquad \text{Harmonic Series}$$

Corresponding continuous function $f(x) = \frac{1}{x} \qquad x \geq 1$

$$\int_{1}^{\infty} \frac{1}{x} dx = \lim_{b \to \infty} \int_{1}^{b} \frac{1}{x} dx = \lim_{b \to \infty} \left(\ln|x| \Big|_{1}^{b} \right) = \lim_{b \to \infty} \left(\ln|b| - \ln 1 \right) = \infty = diverges$$

The Integral Test states that if the corresponding integral

$\int_{1}^{\infty} \frac{1}{x} dx$ diverges, then the series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

Integral Test

Use the Integral Test to show that the series $\sum\limits_{n=1}^{\infty} \dfrac{1}{n^2}$ converges.

Corresponding continuous function $f(x) = \dfrac{1}{x^2}$ $x \geq 1$

$$\int_{1}^{\infty} \frac{1}{x^2}\,dx = \lim_{b\to\infty} \int_{1}^{b} x^{-2}\,dx = \lim_{b\to\infty}\left(-x^{-1}\Big|_{1}^{b}\right) = \lim_{b\to\infty}\left(\frac{-1}{x}\Big|_{1}^{b}\right)$$

$$= \lim_{b\to\infty}\left(\frac{-1}{b} - \frac{-1}{1}\right) = 1 = \text{ converges}$$

The Integral Test states that if the corresponding integral

$\int_{1}^{\infty} \dfrac{1}{x^2}\,dx$ converges, then the series $\sum\limits_{n=1}^{\infty} \dfrac{1}{n^2}$ converges.

Use the Integral Test to show that the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^3}$ converges.

Corresponding continuous function $f(x) = \dfrac{1}{x^3}$ $x \geq 1$

$$\int_{1}^{\infty} \frac{1}{x^3}\,dx = \lim_{b\to\infty}\int_{1}^{b} x^{-3}\,dx = \lim_{b\to\infty}\left(-2x^{-2}\Big|_{1}^{b}\right) = \lim_{b\to\infty}\left(\frac{-2}{x^2}\Big|_{1}^{b}\right)$$

$$= \lim_{b\to\infty}\left(\frac{-2}{b^2} - \frac{-2}{1}\right) = 2 = \text{converges}$$

The Integral Test states that if the corresponding integral

$\displaystyle\int_{1}^{\infty} \frac{1}{x^3}\,dx$ converges, then the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^3}$ converges.

Use the Integral Test to show that the series $\displaystyle\sum_{n=1}^{\infty}\frac{2}{3n+5}$ diverges.

Corresponding continuous function $f(x)=\dfrac{2}{3x+5}$ $x\geq 1$

$$\int_{1}^{\infty}\frac{2}{3x+5}\,dx=\lim_{b\to\infty}\int_{1}^{b}\frac{2}{3x+5}\,dx$$

$$Note:\int_{1}^{b}\frac{2}{3x+5}\,dx=2\int_{1}^{b}\frac{1}{3x+5}\,dx=2\left[\frac{1}{3}\ln|3x+5|\right]_{1}^{b}=2\left[\frac{1}{3}\ln|3b+5|\right]-2\left[\frac{1}{3}\ln|8|\right]$$

$$\int_{1}^{\infty}\frac{2}{3x+5}\,dx=\lim_{b\to\infty}\int_{1}^{b}\frac{2}{3x+5}\,dx=\lim_{b\to\infty}\left[2\left[\frac{1}{3}\ln|3b+5|\right]-2\left[\frac{1}{3}\ln|8|\right]\right]$$

$$=\infty-2\left[\frac{1}{3}\ln|8|\right]=\infty=diverges$$

The Integral Test states that if the corresponding integral

$\displaystyle\int_{1}^{\infty}\frac{2}{3x+5}\,dx$ diverges, then the series $\displaystyle\sum_{n=1}^{\infty}\frac{2}{3n+5}$ diverges.

Use the Integral Test to show that the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2 + 25}$ converges.

Corresponding continuous function $f(x) = \dfrac{1}{x^2 + 25}$ $x \geq 1$

$$\int_1^{\infty} \frac{1}{x^2 + 25} dx = \lim_{b \to \infty} \int_1^b \frac{1}{x^2 + 25} dx$$

$$Note : \int_1^b \frac{1}{x^2 + 25} dx = \left[\frac{1}{5} \tan^{-1}\left(\frac{x}{5}\right) \right]_1^b = \left[\frac{1}{5} \tan^{-1}\left(\frac{b}{5}\right) \right] - \left[\frac{1}{5} \tan^{-1}\left(\frac{1}{5}\right) \right]$$

$$\int_1^{\infty} \frac{2}{3x + 5} dx = \lim_{b \to \infty} \left[\left[\frac{1}{5} \tan^{-1}\left(\frac{b}{5}\right) \right] - \left[\frac{1}{5} \tan^{-1}\left(\frac{1}{5}\right) \right] \right]$$

$$= \left[\frac{1}{5} \tan^{-1}(\infty) \right] - \left[\frac{1}{5} \tan^{-1}\left(\frac{1}{5}\right) \right]$$

$$= \left[\frac{1}{5} \cdot \frac{\pi}{2} \right] - \left[\frac{1}{5} \tan^{-1}\left(\frac{1}{5}\right) \right] = \text{finite} = \text{converges}   \text{Note: } \tan^{-1}(\infty) = \frac{\pi}{2}$$

The Integral Test states that if the corresponding integral

$$\int_1^{\infty} \frac{1}{x^2 + 25} dx \text{ converges, then the series } \sum_{n=1}^{\infty} \frac{1}{n^2 + 25} \text{ diverges.}$$

Use the Integral Test to show that the series $\sum\limits_{n=1}^{\infty}\left(\dfrac{5}{2}\right)^n$ diverges.

Corresponding continuous function $f(x) = \left(\dfrac{5}{2}\right)^x \qquad x \geq 1$

$$\int_{1}^{\infty}\left(\frac{5}{2}\right)^x dx = \lim_{b\to\infty}\int_{1}^{b}\left(\frac{5}{2}\right)^x dx$$

$Note : \displaystyle\int\left(\frac{5}{2}\right)^x dx = \frac{1}{\ln(5/2)}\left(\frac{5}{2}\right)^x$

$$\int_{1}^{\infty}\left(\frac{5}{2}\right)^x dx = \lim_{b\to\infty}\int_{1}^{b}\left(\frac{5}{2}\right)^x dx = \lim_{b\to\infty}\left[\frac{1}{\ln(5/2)}\left(\frac{5}{2}\right)^b - \frac{1}{\ln(5/2)}\left(\frac{5}{2}\right)^1\right]$$

$$= \left[\frac{1}{\ln(5/2)}\left(\frac{5}{2}\right)^{\infty} - \frac{1}{\ln(5/2)}\left(\frac{5}{2}\right)^1\right] = \infty - \frac{1}{\ln(5/2)}\left(\frac{5}{2}\right) = \infty$$

The Integral Test states that if the corresponding integral

$\displaystyle\int_{1}^{\infty}\left(\frac{5}{2}\right)^x dx$ diverges, then the series $\sum\limits_{n=1}^{\infty}\left(\dfrac{5}{2}\right)^n$ diverges.

Use the Integral Test to show that the series $\displaystyle\sum_{n=1}^{\infty}\frac{\ln n}{n^3}$ converges.

Corresponding continuous function $f(x) = \dfrac{\ln x}{x^3}$ $x \geq 1$

$$\int_{1}^{\infty}\frac{\ln x}{x^3}\,dx = \lim_{b\to\infty}\int_{1}^{b}\frac{\ln x}{x^3}\,dx$$

$$Note: \int\frac{\ln x}{x^3}\,dx = \int x^{-3}\ln x\,dx = \frac{x^{-2}}{4}\left[-1+(-2)\ln x\right]$$

$$= \frac{1}{4x^2}\left[-1-2\ln x\right] = \frac{-1-2\ln x}{4x^2} \qquad \text{Using Formula 89}$$

$$\int_{1}^{\infty}\frac{\ln x}{x^3}\,dx = \lim_{b\to\infty}\int_{1}^{b}\frac{\ln x}{x^3}\,dx = \lim_{b\to\infty}\left[\frac{-1-2\ln b}{4b^2} - \frac{-1-2\ln 1}{4(1)^2}\right]$$

$$= \lim_{b\to\infty}\left[\frac{-1-2\ln b}{4b^2} - \frac{-1-2\ln 1}{4(1)^2}\right] = -\infty - \frac{-1-2\ln 1}{4(1)^2} = -\infty$$

The Integral Test states that if the corresponding integral

$\displaystyle\int_{1}^{\infty}\frac{\ln x}{x^3}\,dx$ diverges, then the series $\displaystyle\sum_{n=1}^{\infty}\frac{\ln n}{n^3}$ diverges.

Use the Integral Test to show that the series $\displaystyle\sum_{n=1}^{\infty}\left(\frac{5}{2}\right)^{n}$ diverges.

Corresponding continuous function $f(x) = \left(\dfrac{5}{2}\right)^{x}$ $x \geq 1$

$$\int_{1}^{\infty}\left(\frac{5}{2}\right)^{x} dx = \lim_{b\to\infty}\left(-x^{-1}\Big|_{1}^{b}\right) = \lim_{b\to\infty}\left(\frac{-1}{x}\Big|_{1}^{b}\right) = \lim_{b\to\infty}\left(\frac{-1}{b}-\frac{-1}{1}\right) = 1 = \text{ converges}$$

The Integral Test states that if the corresponding integral $\displaystyle\int_{1}^{\infty}\left(\frac{5}{2}\right)^{x} dx$

converges then the series $\displaystyle\sum_{n=1}^{\infty}\left(\frac{5}{2}\right)^{n}$ diverges.

p-series Test

We have already seen:

The series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ diverges by the Integral Test.

The series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges by the Integral Test.

The series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^3}$ converges by the Integral Test.

p-series Test:

The series $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^p}$ diverges if $p \le 1$.

The series $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^p}$ converges if $p > 1$.

Use p-series Test to show that the series $\displaystyle\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}$ diverges.

$\displaystyle\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}} = \sum_{n=1}^{\infty}\frac{1}{n^{1/2}}$ is a p-series with $p = 1/2$; therefore, $\displaystyle\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}$ diverges.

Use p-series Test to show that the series $\sum_{n=1}^{\infty} \dfrac{1}{\sqrt[4]{x^3}}$ diverges.

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt[4]{x^3}} = \sum_{n=1}^{\infty} \frac{1}{n^{3/4}} \text{ is a p-series with } p = 3/4; \text{ therefore, } \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \text{ diverges.}$$

Use p-series Test to show that the series $\sum_{n=1}^{\infty} \dfrac{1}{\sqrt[3]{x^5}}$ converges.

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{x^5}} = \sum_{n=1}^{\infty} \frac{1}{n^{5/3}} \text{ is a p-series with } p = 5/3; \text{ therefore, } \sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{x^5}} \text{ converges.}$$

Use p-series Test to show that the series $\displaystyle\sum_{n=1}^{\infty}\left(\frac{1}{\sqrt[3]{n^4}}+\frac{1}{n^3}\right)$ converges.

$$\sum_{n=1}^{\infty}\left(\frac{1}{\sqrt[3]{n^4}}+\frac{1}{n^3}\right)=\sum_{n=1}^{\infty}\frac{1}{\sqrt[3]{n^4}}+\sum_{n=1}^{\infty}\frac{1}{n^3}.$$

$\displaystyle\sum_{n=1}^{\infty}\frac{1}{\sqrt[3]{n^4}}=\sum_{n=1}^{\infty}\frac{1}{n^{4/3}}$ is a p-series with $p=4/3$; therefore, $\displaystyle\sum_{n=1}^{\infty}\frac{1}{\sqrt[3]{n^4}}$ converges.

$\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^3}$ is a p-series with $p=3$; therefore, $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^3}$ converges.

In summary: $\displaystyle\sum_{n=1}^{\infty}\left(\frac{1}{\sqrt[3]{n^4}}+\frac{1}{n^3}\right)$ converges.

Use p-series Test to show that the series $\displaystyle\sum_{n=1}^{\infty}\left(\frac{1}{\sqrt[3]{n^7}}+\frac{1}{n}\right)$ diverges.

$$\sum_{n=1}^{\infty}\left(\frac{1}{\sqrt[3]{n^7}}+\frac{1}{n}\right)=\sum_{n=1}^{\infty}\frac{1}{\sqrt[3]{n^7}}+\sum_{n=1}^{\infty}\frac{1}{n}.$$

$\displaystyle\sum_{n=1}^{\infty}\frac{1}{\sqrt[3]{n^7}}=\sum_{n=1}^{\infty}\frac{1}{n^{7/3}}$ is a p-series with $p=7/3$; therefore, $\displaystyle\sum_{n=1}^{\infty}\frac{1}{\sqrt[7]{n^4}}$ converges.

$\displaystyle\sum_{n=1}^{\infty}\frac{1}{n}$ is a p-series with $p=1$; therefore, $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n}$ divverges to ∞.

In summary: $\displaystyle\sum_{n=1}^{\infty}\left(\frac{1}{\sqrt[3]{n^7}}+\frac{1}{n}\right)=\sum_{n=1}^{\infty}\frac{1}{\sqrt[3]{n^7}}+\sum_{n=1}^{\infty}\frac{1}{n}=\infty;\ \ \sum_{n=1}^{\infty}\left(\frac{1}{\sqrt[3]{n^7}}+\frac{1}{n}\right)$ diverges.