Trigonometric Integrals

Trigonometric Identities:

$$\sin^2 x + \cos^2 x = 1; \quad \sin^2 x = 1 - \cos^2 x; \quad \cos^2 x = 1 - \sin^2 x$$

$$\sin^2 x = \frac{1 - \cos 2x}{2} = \frac{1}{2}(1 - \cos 2x)$$

$$\cos^2 x = \frac{1 + \cos 2x}{2} = \frac{1}{2}(1 + \cos 2x)$$

Example1: Find the indefinite integral $\int \cos^3 x \sin^4 x dx$.

Note: Power of the cosine is *odd*.

$$\int \cos^3 x \sin^4 x dx = \int \cos^2 x \cdot \cos x \sin^4 x dx$$

$$= \int \cos^2 x \cdot \sin^4 x \cdot \cos x dx = \int (1\text{-}\sin^2 x) \cdot \sin^4 x \cdot \cos x dx$$

Now let $u = \sin x \Rightarrow \dfrac{du}{dx} = \cos x \Rightarrow du = \cos x dx$

$$\int \cos^3 x \sin^4 x dx = \int \cos^2 x \cdot \cos x \sin^4 x dx$$

$$= \int \cos^2 x \cdot \sin^4 x \cdot \cos x dx$$

$$= \int (1\text{-}\sin^2 x) \cdot \sin^4 x \cdot \cos x dx$$

$$= \int (1\text{-}u^2) \cdot u^4 \cdot du$$

$$= \int (u^4\text{-}u^6) \cdot du$$

$$= \frac{u^5}{5} - \frac{u^7}{7} + C$$

$$= \frac{(\sin x)^5}{5} - \frac{(\sin x)^7}{7} + C$$

Example 2: Find the indefinite integral $\int \sin^3 3x dx$.

Note: Power of the sine is *odd*.

$$\int \sin^3 3x dx = \int \sin^2 3x \cdot \sin 3x dx = \int \left(1-\cos^2 3x\right)\cdot \sin 3x dx$$

Now let $u = \cos 3x \Rightarrow \dfrac{du}{dx} = -\sin 3x \cdot (3) = -3 \cdot \sin 3x$

$$\Rightarrow du = -3\sin 3x dx$$

$$\Rightarrow -\frac{1}{3} du = \sin 3x dx$$

$$\int \sin^3 3x dx = \int \sin^2 3x \cdot \sin 3x dx = \int \left(1-\cos^2 3x\right)\cdot \sin 3x dx$$

$$= \int \left(1-u^2\right)\cdot\left(-\frac{1}{3}du\right) = -\frac{1}{3}\int \left(1-u^2\right)\cdot du$$

$$= -\frac{1}{3}\left(u - \frac{u^3}{3}\right) + C$$

$$= -\frac{1}{3}\left(\cos 3x - \frac{\left(\cos 3x\right)^3}{3}\right) + C$$

Example 3: Find the indefinite integral $\int \cos^3 \frac{x}{3} dx.$

Note: Power of the cosine is *odd*.

$$\int \cos^3 \frac{x}{3} dx = \int \cos^2 \frac{x}{3} \cdot \cos \frac{x}{3} dx = \int \left(1 - \sin^2 \frac{x}{3}\right) \cdot \cos \frac{x}{3} dx$$

Now let $u = \sin \frac{x}{3} \Rightarrow \frac{du}{dx} = \cos \frac{x}{3} \cdot D_x \left(\frac{x}{3}\right) = \cos \frac{x}{3} \cdot \left(\frac{1}{3}\right)$

$$\Rightarrow du = \left(\frac{1}{3}\right) \cdot \cos \frac{x}{3} dx$$

$$\Rightarrow 3du = \cos \frac{x}{3} dx$$

$$\int \cos^3 \frac{x}{3} dx = \int \cos^2 \frac{x}{3} \cdot \cos \frac{x}{3} dx = \int \left(1 - \sin^2 \frac{x}{3}\right) \cdot \cos \frac{x}{3} dx$$

$$= \int \left(1 - u^2\right) \cdot \left(3du\right) = 3\int \left(1 - u^2\right) \cdot du$$

$$= 3\left(u - \frac{u^3}{3}\right) + C$$

$$= 3\left(\sin \frac{x}{3} - \frac{\left(\sin \frac{x}{3}\right)^3}{3}\right) + C$$

Example 4: Find the indefinite integral $\int \frac{\cos^5 x}{\sqrt{\sin x}} dx$.

Note: Power of the cosine is *odd*.

$$\int \frac{\cos^5 x}{\sqrt{\sin x}} dx = \int \cos^5 x \frac{1}{\sqrt{\sin x}} dx = \int \cos^2 x \cdot \cos^2 x \cdot \cos x \frac{1}{\sqrt{\sin x}} dx$$

$$= \int \left(1-\sin^2 x\right)\left(1-\sin^2 x\right) \cdot \cos x \frac{1}{\left(\sin x\right)^{1/2}} dx$$

$$= \int \left(1-\sin^2 x\right)\left(1-\sin^2 x\right)\left(\sin x\right)^{-1/2} \cdot \cos x dx$$

Now let $u = \sin x \Rightarrow \frac{du}{dx} = \cos x \Rightarrow du = \cos x dx$

$$\int \frac{\cos^5 x}{\sqrt{\sin x}} dx = \int \left(1-\sin^2 x\right)\left(1-\sin^2 x\right)\left(\sin x\right)^{-1/2} \cdot \cos x dx$$

$$= \int \left(1-u^2\right)\left(1-u^2\right)\left(u\right)^{-1/2} \cdot \left(du\right)$$

$$= \int \left(1-2u^2+u^4\right)\left(u\right)^{-1/2} \cdot \left(du\right)$$

$$= \int \left(u^{-1/2}-2u^{3/2}+u^{7/2}\right) \cdot \left(du\right)$$

$$= \frac{u^{1/2}}{1/2} - 2\frac{u^{5/2}}{5/2} + \frac{u^{9/2}}{9/2} + C$$

$$= 2u^{1/2} - \frac{4}{5}u^{5/2} + \frac{2}{9}u^{9/2} + C$$

$$= 2\left(\sin x\right)^{1/2} - \frac{4}{5}\left(\sin x\right)^{5/2} + \frac{2}{9}\left(\sin x\right)^{9/2} + C$$

Example 5: Find the indefinite integral $\int \sin^4 x dx$.

Recall: $\sin^2 x = \frac{1}{2}(1-\cos 2x)$

$$\int \sin^4 x dx = \int \sin^2 x \sin^2 x dx$$

$$= \int \left[\frac{1}{2}(1-\cos 2x)\right]\left[\frac{1}{2}(1-\cos 2x)\right] dx$$

$$= \frac{1}{4}\int (1-\cos 2x)(1-\cos 2x) dx$$

$$= \frac{1}{4}\int \left(1-2\cos 2x + (\cos 2x)^2\right) dx$$

$$Note: \int (\cos 2x)^2 dx = \int (\cos^2 2x) dx$$

Let $u = 2x \Rightarrow \frac{du}{dx} = 2 \Rightarrow du = 2dx \Rightarrow \frac{1}{2} du = dx$

$$\int (\cos 2x)^2 dx = \int (\cos^2 2x) dx = \int \cos^2 u \cdot \left(\frac{1}{2} du\right)$$

$$= \frac{1}{2}\int \cos^2 u \cdot du = \frac{1}{2}\left[\frac{1}{2}(u + \sin u \cos u)\right] \qquad \text{Formula \#49}$$

$$= \frac{1}{4}(u + \sin u \cos u) = \frac{1}{4}(2x + \sin 2x \cos 2x)$$

$For \ \int (\cos 2x) dx:$

$$\text{Let } u = 2x \Rightarrow \frac{du}{dx} = 2 \Rightarrow du = 2dx \Rightarrow \frac{1}{2} du = dx$$

$$\int (\cos 2x) dx = \int (\cos u) \left(\frac{1}{2} du \right) = \frac{1}{2} \int (\cos u) du$$

$$= \frac{1}{2} [\sin u] = \frac{1}{2} [\sin 2x]$$

$Hence,$

$$\int \sin^4 x dx = \int \sin^2 x \sin^2 x dx$$

$$= \frac{1}{4} \int \left(1 - 2\cos 2x + (\cos 2x)^2 \right) dx$$

$$= \frac{1}{4} \left[x - 2\left[\frac{1}{2} [\sin 2x] \right] + \left[\frac{1}{4} (2x + \sin 2x \cos 2x) \right] \right] + C$$

Example 6: Find the indefinite integral $\int x^2 \sin^2 x dx$.

Recall: $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$

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$$\int x^2 \sin^2 x dx = \int x^2 \left[\frac{1}{2}(1 - \cos 2x) \right] dx$$

$$= \frac{1}{2} \int (x^2 - x^2 \cos 2x) dx$$

Note:

Formula #55 with $n = 2$: $\int u^2 \cos u du = u^2 \sin u - 2\int u \sin u du$

Formula 52: $\int u \sin u du = \sin u - u \cos u$

Hence, $\int u^2 \cos u du = u^2 \sin u - 2[\sin u - u \cos u]$

Hence, $\int u^2 \cos u du = u^2 \sin u - 2[\sin u - u \cos u]$

For $\int (x^2 \cos 2x) dx$:

Let $u = 2x \Rightarrow \frac{du}{dx} = 2 \Rightarrow du = 2dx \Rightarrow \frac{1}{2} du = dx$

Also, $u = 2x \Rightarrow \frac{u}{2} = x \Rightarrow \frac{u^2}{4} = x^2 \Rightarrow \frac{1}{4} u^2 = x^2$

$$\int (x^2 \cos 2x) dx = \int \left(\frac{1}{4} u^2 \right) (\cos u) \left(\frac{1}{2} du \right)$$

$$= \frac{1}{8} \int (u^2 \cos u) du$$

$$= \frac{1}{8} \left[u^2 \sin u - 2[\sin u - u \cos u] \right] + C$$

$$= \frac{1}{8} \left[(2x)^2 \sin 2x - 2[\sin 2x - 2x \cos 2x] \right] + C$$

Therefore, $\int x^2 \sin^2 x dx = \int x^2 \left[\frac{1}{2} (1 - \cos 2x) \right] dx$

$$= \frac{1}{2} \int (x^2 - x^2 \cos 2x) dx$$

$$= \frac{1}{2} \left[\frac{x^3}{3} - \left[\frac{1}{8} \left[(2x)^2 \sin 2x - 2[\sin 2x - 2x \cos 2x] \right] \right] \right] + C$$

Example 7: $\int_{0}^{\pi/2} \cos^9 x dx$ Note: $n = 9$

Using Wallis's Formula on p. 526.

$$\int_{0}^{\pi/2} \cos^9 x dx = \left(\frac{2}{3}\right)\left(\frac{4}{5}\right)\left(\frac{6}{7}\right)\left(\frac{8}{9}\right) = \frac{384}{945}$$

Example 8:

$\int \sec^4 2x dx$ Note: power of the secant is even

Trigonometric Identity: $1+\tan^2 x = \sec^2 x$

$$\int \sec^4 2x dx = \int \left(\sec^2 2x\right)\left(\sec^2 2x\right) dx$$

$$= \int \left(1+\tan^2 2x\right)\left(\sec^2 2x\right) dx$$

Now Let $u = \tan 2x \Rightarrow \dfrac{du}{dx} = \sec^2 2x \cdot D_x\left(2x\right) = \sec^2 2x \cdot \left(2\right)$

$$\Rightarrow du = 2\sec^2 2x dx$$

$$\Rightarrow \frac{1}{2} du = \sec^2 2x dx$$

$$\int \sec^4 2x dx = \int \left(1+\tan^2 2x\right)\left(\sec^2 2x\right) dx$$

$$= \int \left(1+u^2\right)\left(\frac{1}{2} du\right) = \frac{1}{2}\int \left(1+u^2\right) du$$

$$= \frac{1}{2}\left[u + \frac{u^3}{3}\right]$$

$$= \frac{1}{2}\left[\tan 2x + \frac{\left(\tan 2x\right)^3}{3}\right] + C$$

Example 9:

$\int \tan^6 3x dx$ Note: power of the tangent is even

Trigonometric Identity: $1+\tan^2 x = \sec^2 x \Rightarrow \tan^2 x = \sec^2 x - 1$

$$\int \tan^6 3x dx = \int \left(\tan^4 3x\right)\left(\tan^2 3x\right) dx$$

$$= \int \left(\tan^4 3x\right)\left(\sec^2 3x - 1\right) dx$$

$$= \int \left(\tan^4 3x \sec^2 3x - \tan^4 3x\right) dx$$

$$= \int \left(\tan^4 3x \sec^2 3x\right) dx - \int \left(\tan^4 3x\right) dx$$

For $\int \left(\tan^4 3x \sec^2 3x\right) dx$:

Let $u = \tan 3x \Rightarrow \dfrac{du}{dx} = \left(\sec^2 3x\right) D_x(3x) = \left(\sec^2 3x\right) \cdot 3$

$$\Rightarrow du = 3\left(\sec^2 3x\right) dx$$

$$\Rightarrow \frac{1}{3} du = \left(\sec^2 3x\right) dx$$

$$\int \left(\tan^4 3x \sec^2 3x\right) dx = \int \left(\tan^4 3x\right)\left(\sec^2 3x\right) dx$$

$$= \int u^4 \left(\frac{1}{3} du\right) = \frac{1}{3} \int u^4 du$$

$$= \frac{1}{3}\left[\frac{u^5}{5}\right] = \frac{1}{15}\left[u^5\right]$$

$$= \frac{1}{15}\left[\left(\tan 3x\right)^5\right]$$

For $\int\left(\tan^4 3x\right)dx,$ use Formula #67 with $n=4$

Let $u=3x \Rightarrow \dfrac{du}{dx}=3 \Rightarrow du=3dx \Rightarrow \dfrac{1}{3}du=dx$

$$\int\left(\tan^4 3x\right)dx=\int\left(\tan^4 u\right)\left(\frac{1}{3}du\right)=\frac{1}{3}\int\left(\tan^4 u\right)du$$

$$=\frac{1}{3}\left[\frac{\tan^3 u}{3}-\int \tan^2 u du\right]$$

$$=\frac{1}{3}\left[\frac{\tan^3 u}{3}-\left[-u+\tan u\right]\right] \quad \text{Using Formula \#63}$$

$$=\frac{1}{3}\left[\frac{\tan^3 3x}{3}-\left[-3x+\tan 3x\right]\right]$$

Therefore:

$$\int \tan^6 3x dx=\int\left(\tan^4 3x \sec^2 3x\right)dx-\int\left(\tan^4 3x\right)dx$$

$$=\left[\frac{1}{15}\left[\left(\tan 3x\right)^5\right]\right]-\left[\frac{1}{3}\left[\frac{\tan^3 3x}{3}-\left[-3x+\tan 3x\right]\right]\right]+C$$

Example 10: $\int\left(\tan^3 3x\right)dx$

For $\int\left(\tan^3 3x\right)dx$, use Formula #67 with $n=3$

Let $u=3x \Rightarrow \dfrac{du}{dx}=3 \Rightarrow du=3dx \Rightarrow \dfrac{1}{3}du=dx$

$$\int\left(\tan^3 3x\right)dx=\int\left(\tan^3 u\right)\left(\frac{1}{3}du\right)=\frac{1}{3}\int\left(\tan^3 u\right)du$$

$$=\frac{1}{3}\left[\frac{\tan^3 u}{2}-\int \tan u\,du\right]$$

$$=\frac{1}{3}\left[\frac{\tan^3 u}{2}-\left[\ln\left|\cos u\right|\right]\right] \quad \text{Using Formula #59}$$

$$=\frac{1}{3}\left[\frac{\tan^3 3x}{2}-\left[\ln\left|\cos 3x\right|\right]\right]+C$$