Section 9.4

Basic Comparison Test and Limit Comparison Test

Recall:

$\sum_{n=1}^{\infty} \frac{1}{n}$ diverges because it a p-series with p = 1

$\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges because it a p-series with p = 2

$\sum_{n=1}^{\infty} \frac{1}{n^3}$ converges because it a p-series with p = 3

$\sum_{n=1}^{\infty} \frac{1}{n^4}$ converges because it a p-series with p = 4

Recall:

$\sum_{n=1}^{\infty} \frac{\ln}{n}$ diverges by Integral Test.

$\sum_{n=1}^{\infty} \frac{\ln n}{n^2}$ converges by Integral Test.

$\sum_{n=1}^{\infty} \frac{\ln n}{n^3}$ converges by Integral Test.

$\sum_{n=1}^{\infty} \frac{\ln n}{n^4}$ converges by Integral Test.

Basic Comparison Test:

If $\sum_{n=1}^{\infty} a_n \leq \sum_{n=1}^{\infty} b_n$ and $\sum_{n=1}^{\infty} b_n$ converges then the series $\sum_{n=1}^{\infty} a_n$ also coverges.

If $\sum_{n=1}^{\infty} b_n \leq \sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ diverges then the series $\sum_{n=1}^{\infty} a_n$ also diverges.

Use the Direct Comparison Test (DCT) to show that the

series $\sum_{n=1}^{\infty} \frac{1}{5+2^n}$ converges.

Note: $5 + 2^n > 2^n$

$$\Rightarrow \frac{1}{5+2^n} < \frac{1}{2^n}$$

$$\Rightarrow \sum_{n=1}^{\infty} \frac{1}{5+2^n} < \sum_{n=1}^{\infty} \frac{1}{2^n}$$

The series $\sum_{n=1}^{\infty} \frac{1}{2^n} = \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n$ is a Geometric Series with $r = \frac{1}{2}$.

Hence, $\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n$ converges by Geometric Series Test.

Therefore, the series $\sum_{n=1}^{\infty} \frac{1}{5+2^n}$ converges by Direct Comparison Test

Use the Direct Comparison Test (DCT) to show that the series $\sum_{n=1}^{\infty} \frac{1}{5+\sqrt{n^3}}$ converges.

Note: $5+\sqrt{n^3} > \sqrt{n^3}$

$$\Rightarrow \frac{1}{5+\sqrt{n^3}} < \frac{1}{\sqrt{n^3}}$$

$$\Rightarrow \sum_{n=1}^{\infty} \frac{1}{5+\sqrt{n^3}} < \sum_{n=1}^{\infty} \frac{1}{\sqrt{n^3}}$$

The series $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^3}} = \sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ is a $p-$series with $p = 3/2$.

Hence, $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ converges by p-series Test.

Therefore, the series $\sum_{n=1}^{\infty} \frac{1}{5+\sqrt{n^3}}$ converges by Direct Comparison Test

Use the Direct Comparison Test (DCT) to show that the series $\sum_{n=1}^{\infty} \frac{4^n}{5^n + 7}$ converges.

Note: $\frac{4^n}{5^n + 7} < \frac{4^n}{5^n}$

$$\Rightarrow \sum_{n=1}^{\infty} \frac{4^n}{5^n + 7} < \sum_{n=1}^{\infty} \frac{4^n}{5^n}$$

The series $\sum_{n=1}^{\infty} \frac{4^n}{5^n} = \sum_{n=1}^{\infty} \left(\frac{4}{5}\right)^n$ is a Geometric Series with $r = \frac{4}{5}$.

Hence, $\sum_{n=1}^{\infty} \left(\frac{4}{5}\right)^n$ converges by Geometric Series Test.

Therefore, the series $\sum_{n=1}^{\infty} \frac{4^n}{5^n + 7}$ converges by Direct Comparison Test

Use the Direct Comparison Test (DCT) to show that the

series $\sum_{n=1}^{\infty} \frac{4}{3^n + 1}$ converges.

Note: $\frac{4}{3^n + 1} < \frac{4}{3^n}$

$$\Rightarrow \sum_{n=1}^{\infty} \frac{4}{3^n + 1} < \sum_{n=1}^{\infty} \frac{4}{3^n}$$

The series $\sum_{n=1}^{\infty} \frac{4}{3^n} = \sum_{n=1}^{\infty} 4\left(\frac{1}{3}\right)^n = 4\sum_{n=1}^{\infty}\left(\frac{1}{3}\right)^n$.

$\sum_{n=1}^{\infty}\left(\frac{1}{3}\right)^n$ is a Geometric Series with $r = \frac{1}{3}$.

Hence, $\sum_{n=1}^{\infty}\left(\frac{1}{3}\right)^n$ converges by Geometric Series Test.

Which means $4\sum_{n=1}^{\infty}\left(\frac{1}{3}\right)^n$ also converges.

Therefore, the series $\sum_{n=1}^{\infty} \frac{4}{3^n + 1}$ converges by Direct Comparison Test

Limit Comparison Test:

If $\lim_{n\to\infty} \frac{a_n}{b_n} = L > 0$, then either both series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ converge; or both series diverge.

Use the Limit Comparison Test (LCT) to show that the series $\sum_{n=1}^{\infty} \frac{1}{n^2(n+3)}$ converges.

Let $a_n = \frac{1}{n^2(n+3)}$; Let $b_n = \frac{1}{n^2(n)} = \frac{1}{n^3}$

Recall: $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{n^3}$ converges.

$$\lim_{n\to\infty}\left(\frac{a_n}{b_n}\right) = \lim_{n\to\infty}\left(\frac{\frac{1}{n^2(n+3)}}{\frac{1}{n^3}}\right) = \lim_{n\to\infty}\left(\frac{n^3}{n^2(n+3)}\right) = \lim_{n\to\infty}\left(\frac{n^3}{n^3+3n^2}\right)$$

$= 1$ Using L'Hopital's Rule

Therefore, $\sum_{n=1}^{\infty} \frac{1}{n^2(n+3)}$ converges because $\sum_{n=1}^{\infty} \frac{1}{n^3}$ converges.

Use the Limit Comparison Test (LCT) to show that the series $\sum_{n=1}^{\infty} \frac{\ln n}{n^2+1}$ converges.

Let $a_n = \frac{\ln n}{n^2+1}$; Let $b_n = \frac{\ln n}{n^2}$

Recall: $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{\ln n}{n^2}$ converges.

$$\lim_{n\to\infty}\left(\frac{a_n}{b_n}\right) = \lim_{n\to\infty}\left(\frac{\frac{\ln n}{n^2+1}}{\frac{\ln n}{n^2}}\right) = \lim_{n\to\infty}\left(\frac{n^2}{n^2+1}\right) = 1$$

Using L'Hopital's Rule

Therefore, the series $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{\ln n}{n^2+1}$ converges because $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{\ln n}{n^2}$ converges.