

Objective: Write power series representation of function $f(x)$.

Recall: Geometric Series $\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}$ for $-1 < r < 1$.

Examples:

$$\sum_{n=0}^{\infty} 2 \left(\frac{1}{3} \right)^n = 2 + 2 \left(\frac{1}{3} \right) + 2 \left(\frac{1}{3} \right)^2 + 2 \left(\frac{1}{3} \right)^3 + \dots = \frac{a}{1-r} = \frac{2}{1-\frac{1}{3}} = \frac{2}{2/3} = 3$$

$$\sum_{n=0}^{\infty} 4 \left(\frac{1}{5} \right)^n = 4 + 4 \left(\frac{1}{5} \right) + 4 \left(\frac{1}{5} \right)^2 + 4 \left(\frac{1}{5} \right)^3 + \dots = \frac{a}{1-r} = \frac{4}{1-\frac{1}{5}} = \frac{4}{4/5} = 5$$

$$\sum_{n=0}^{\infty} \left(\frac{1}{6} \right)^n = 1 + \left(\frac{1}{6} \right) + \left(\frac{1}{6} \right)^2 + \left(\frac{1}{6} \right)^3 + \dots = \frac{a}{1-r} = \frac{1}{1-\frac{1}{6}} = \frac{1}{5/6} = 6/5$$

Example 1: Find a geometric power series for the function

$$f(x) = \frac{1}{7 - x} \text{ centered at } 0.$$

Recall: Geometric Series $\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}$ for $-1 < r < 1$.

$$\text{Note: } \frac{1}{7 - x} = \frac{1/7}{(7 - x)/7} = \frac{1/7}{1 - x/7} = \frac{a}{1-r}$$

$$a = 1/7 ; r = x/7$$

Power Series Representation for $f(x)$:

$$f(x) = \frac{1}{7 - x} = \sum_{n=0}^{\infty} ar^n = \sum_{n=0}^{\infty} \left(\frac{1}{7}\right) \left(\frac{x}{7}\right)^n$$

Comparing $f(x) = \frac{1}{7-x}$ and $\sum_{n=0}^{\infty} \left(\frac{1}{7}\right) \left(\frac{x}{7}\right)^n$

For $x = 2$,

$$f(x) = \frac{1}{7-x} = \frac{1}{7-2} = \frac{1}{5}$$

Power Series: $\sum_{n=0}^{\infty} \left(\frac{1}{7}\right) \left(\frac{x}{7}\right)^n = \sum_{n=0}^{\infty} \left(\frac{1}{7}\right) \left(\frac{2}{7}\right)^n = \frac{a}{1-r} = \frac{1/7}{1-2/7} = \frac{1/7}{5/7} = 1/5$

Note: The power series $\sum_{n=0}^{\infty} (1/7) (x/7)^n$ is good when x is around $c = 0$.

Example 2: Find a geometric power series for the

function $f(x) = \frac{5}{2 - x}$ centered at 0.

Note: $\frac{5}{2 - x} = \frac{(5)/2}{(2 - x)/2} = \frac{5/2}{1 - x/2} = \frac{a}{1 - r}$

$$a = 5/2$$

$$r = x/2$$

Power Series Representation for $f(x)$:

$$f(x) = \frac{5}{2 - x} = \sum_{n=0}^{\infty} ar^n = \sum_{n=0}^{\infty} \left(\frac{5}{2}\right) \left(\frac{x}{2}\right)^n$$

Comparing $f(x) = \frac{5}{2-x}$ and $\sum_{n=0}^{\infty} ar^n = \sum_{n=0}^{\infty} (5/2)(x/2)^n$

For $x=1$,

$$f(x) = \frac{5}{2-x} = \frac{5}{2-1} = 5$$

Power Series: $\sum_{n=0}^{\infty} (5/2)(x/2)^n = \sum_{n=0}^{\infty} (5/2)(1/2)^n = \frac{a}{1-r} = \frac{5/2}{1-1/2} = \frac{5/2}{1/2} = 5$

Note: The power series $\sum_{n=0}^{\infty} (5/2)(x/2)^n$ is good when x is around $c=0$.

Example 3: Find a geometric power series for the

$$\text{function } f(x) = \frac{2}{5 - x} \text{ centered at } c = 1.$$

First, rewrite $5 - x$ as $5 - (x - 1) - 1$ since we want the power series centered at $x = 1$.

$$\text{Note: } 5 - (x - 1) - 1 = 5 - x$$

$$\begin{aligned} \text{Hence: } \frac{2}{5 - x} &= \frac{2}{5 - (x - 1) - 1} = \frac{2}{4 - (x - 1)} \\ &= \frac{2/4}{4/4 - (x - 1)/4} = \frac{1/2}{1 - (x - 1)/4} = \frac{a}{1 - r} \end{aligned}$$

$$\text{So we have: } a = \frac{1}{2} ; \quad r = \frac{x - 1}{4}$$

Power Series Representation for $f(x)$:

$$f(x) = \frac{2}{5 - x} = \sum_{n=0}^{\infty} ar^n = \sum_{n=0}^{\infty} \left(\frac{1}{2} \right) \left[\frac{x - 1}{4} \right]^n$$

Comparing $f(x) = \frac{2}{5-x}$ and $\sum_{n=0}^{\infty} \left(\frac{1}{2}\right) \left[\frac{x-1}{4}\right]^n$

Note that $\sum_{n=0}^{\infty} \left(\frac{1}{2}\right) \left[\frac{x-1}{4}\right]^n$ converges if $r = \frac{x-1}{4}$ is between -1 and 1.

Hence, $-1 < \frac{x-1}{4} < 1 \Leftrightarrow -4 < x-1 < 4 \Leftrightarrow -3 < x < 5$

Or $\sum_{n=0}^{\infty} \left(\frac{1}{2}\right) \left[\frac{x-1}{4}\right]^n$ converges if $-3 < x < 5$ (this interval is centered at 1).

For $x = 1.5$,

$$f(x) = \frac{5}{2-x} = \frac{5}{2-1.5} = 10$$

$$\begin{aligned} \text{Power Series: } \sum_{n=0}^{\infty} \left(\frac{1}{2}\right) \left[\frac{x-1}{4}\right]^n &= \sum_{n=0}^{\infty} \left(\frac{1}{2}\right) \left[\frac{1.5-1}{4}\right]^n = \sum_{n=0}^{\infty} \left(\frac{1}{2}\right) \left[\frac{0.5}{4}\right]^n \\ &= \frac{1/2}{1-0.5/4} = 10 \end{aligned}$$

Example 4: Find a geometric power series for the

$$\text{function } f(x) = \frac{4}{3x - 6} \text{ centered at } c = -2.$$

$$\text{Recall: Geometric Series } \sum_{n=0}^{\infty} ar^n = \frac{a}{1-r} \quad \text{for } -1 < r < 1.$$

Since we want a power series centered at $x = -2$,

rewrite $3x - 6$ as $3(x + 2) - 6 - 6$.

$$\text{Note: } 3(x + 2) - 6 - 6 = 3x - 6$$

$$\begin{aligned} \text{Note: } \frac{4}{3x - 6} &= \frac{4}{3(x + 2) - 6 - 6} = \frac{4}{-12 + 3(x + 2)} = \frac{4/(-12)}{-12/(-12) + 3(x + 2)/(-12)} \\ &= \frac{\frac{-1}{3}}{1 - \frac{x + 2}{4}} = \frac{a}{1-r} \end{aligned}$$

$$\text{Hence, } a = \frac{-1}{3} \quad \text{and} \quad r = \frac{x + 2}{4}$$

Hence, $a = -1/3$; $r = \frac{(x + 2)}{4}$

Power Series Representation for $f(x)$: $f(x) = \frac{4}{3x - 6} = \sum_{n=0}^{\infty} \left(\frac{-1}{3} \right) \left[\frac{(x + 2)}{4} \right]^n$

Comparing $f(x) = \frac{4}{3x - 6}$ and $\sum_{n=0}^{\infty} \left(\frac{-1}{3}\right) \left[\frac{(x+2)}{4}\right]^n$

Let $x = -1$.

$$f(x) = \frac{4}{3x - 6} = \frac{4}{3(-1) - 6} = -\frac{4}{9}$$

$$\begin{aligned} \sum_{n=0}^{\infty} \left(\frac{-1}{3}\right) \left[\frac{(x+2)}{4}\right]^n &= \sum_{n=0}^{\infty} \left(\frac{-1}{3}\right) \left[\frac{(-1+2)}{4}\right]^n = \sum_{n=0}^{\infty} \left(\frac{-1}{3}\right) \left[\frac{1}{4}\right]^n \\ &= \frac{a}{1-r} = \frac{-1/3}{1-1/4} = \frac{-1/3}{3/4} = -4/9 \end{aligned}$$

Note: The power series $\left(\frac{-1}{3}\right) \left[\frac{(x+2)}{4}\right]^n$ is good when x is around $c = -2$.

Example 5: Find a power series for the function

$$f(x) = \frac{4x}{x^2 + 5x + 4} \text{ centered at } c = 0:$$

$$\text{Decomposing (see below): } \frac{4x}{x^2 + 5x + 4} = \frac{16/3}{x + 4} + \frac{-4/3}{x + 1}$$

$$\frac{16/3}{x + 4} = \frac{16/3}{4 + x} = \frac{(16/3)/4}{4/4 + x/4} = \frac{4/3}{1 - (-x/4)} = \frac{a}{1 - r}$$

$$\frac{-4/3}{x + 1} = \frac{-4/3}{1 + x} = \frac{-4/3}{1 - (-x)} = \frac{a}{1 - r}$$

Power Series Representation for $f(x)$:

$$\begin{aligned} f(x) &= \frac{4x}{x^2 + 5x + 4} = \sum_{n=0}^{\infty} (4/3)(-x/4)^n + \sum_{n=0}^{\infty} (-4/3)(-x)^n \\ &= \sum_{n=0}^{\infty} \left[(4/3)(-x/4)^n + (-4/3)(-x)^n \right] \end{aligned}$$

Using Partial Fraction To Decompose $\frac{4x}{x^2 + 5x + 4}$:

$$\frac{4x}{x^2 + 5x + 4} = \frac{4x}{(x+4)(x+1)} = \frac{A}{x+4} + \frac{B}{x+1}$$

$$\text{Hence, } \frac{4x}{(x+4)(x+1)} = \frac{A(x+1) + B(x+4)}{(x+4)(x+1)}$$

$$4x = A(x+1) + B(x+4)$$

$$\text{Let } x = -1 \Rightarrow 4(-1) = A(-1+1) + B(-1+4)$$

$$-4 = 3B \Rightarrow B = -4/3$$

$$\text{Let } x = -4 \Rightarrow 4(-4) = A(-4+1) + B(-4+4)$$

$$-16 = -3A \Rightarrow A = 16/3$$