Polar Equations of Conics:

$$r = \frac{ed}{1 \pm e\cos\theta} \quad \text{or} \quad r = \frac{ed}{1 \pm e\sin\theta}$$

where e $\left(\text{eccentricity}\right) > 0$ and

$\left|d\right| =$ distance between the focus at the pole and directrix.

Classification of Conics by Eccentricity

Let $e =$ eccentricity.

a) The conic is a ellipse for $0 < e < 1$.

b) The conic is a parabola for $e = 1$.

c) The conic is a hyperbola for $e > 1$.

Four types of equations based on location and type of directrix:

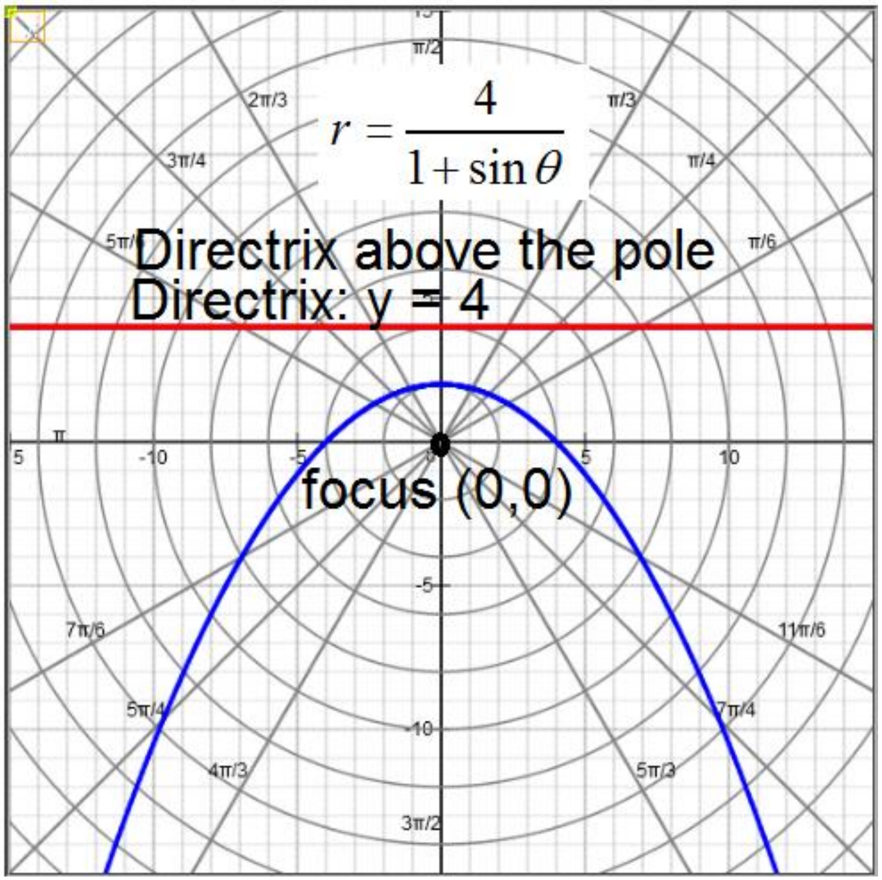
1) Horizontal directrix above the pole: $r = \dfrac{ed}{1+e\sin\theta}$

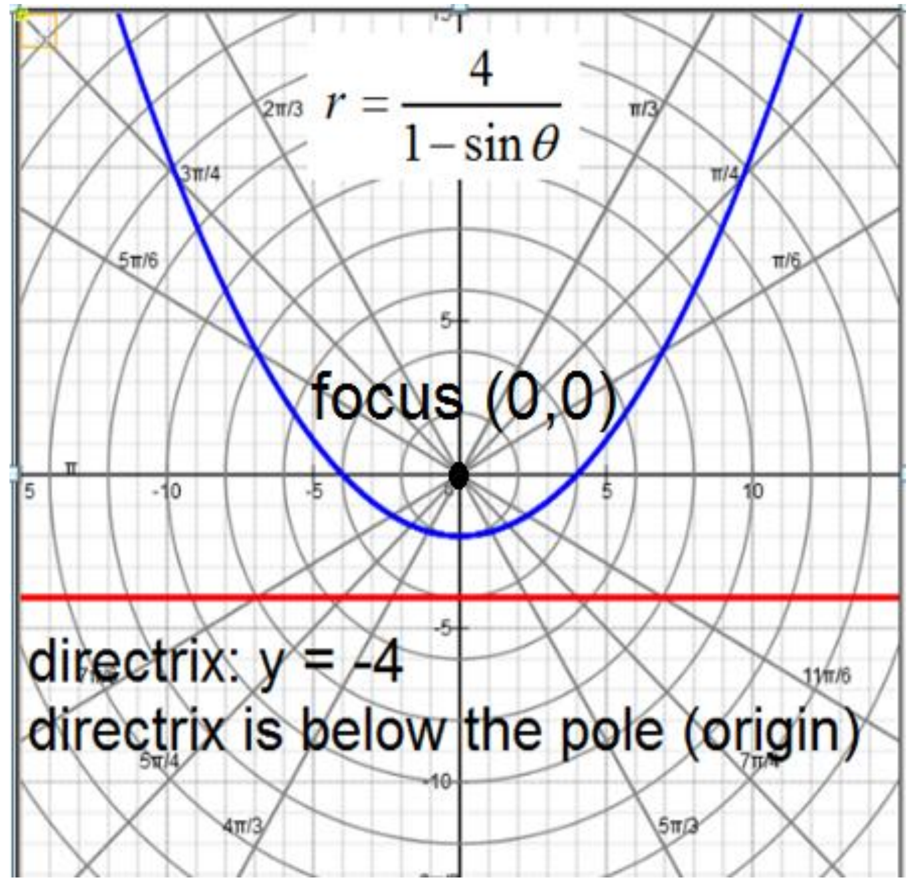
2) Horizontal directrix below the pole: $r = \dfrac{ed}{1-e\sin\theta}$

3) Vertical directrix to the right of the pole: $r = \dfrac{ed}{1+e\cos\theta}$

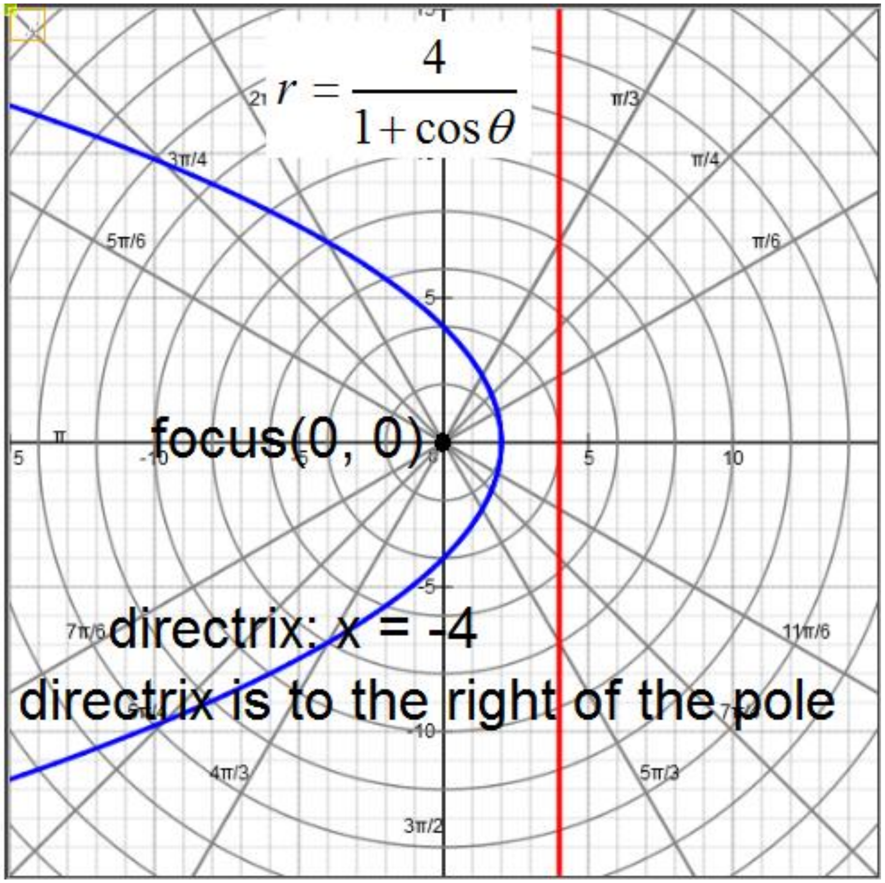
4) Vertical directrix to the left of the pole: $r = \dfrac{ed}{1-e\cos\theta}$

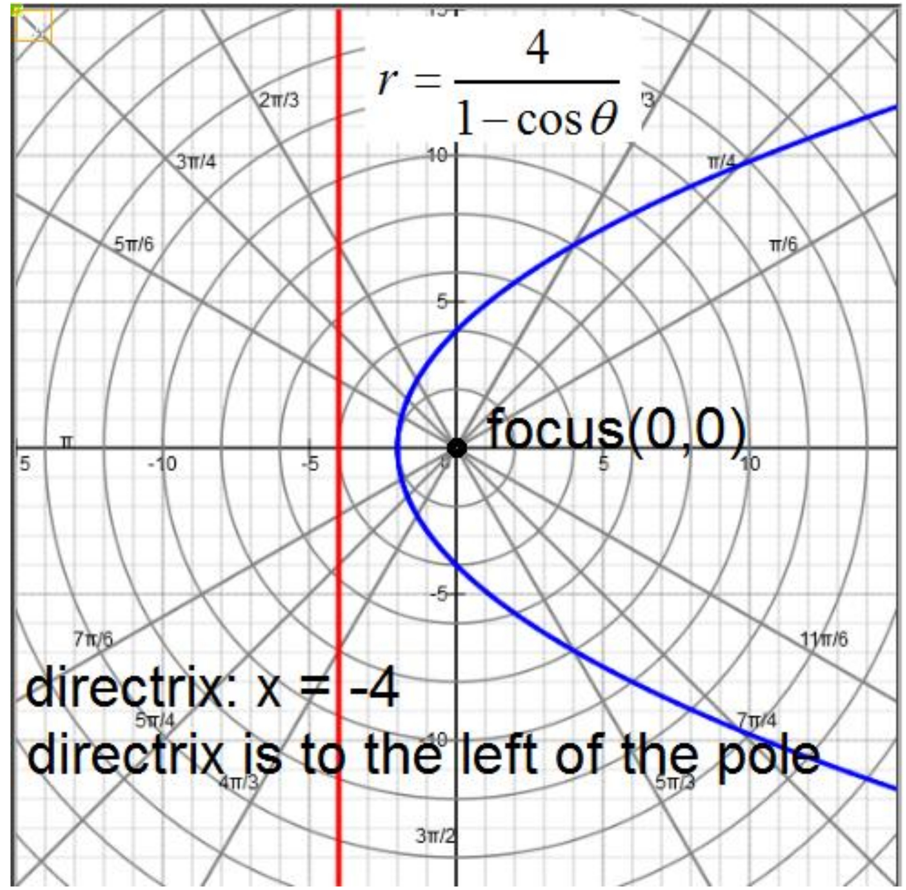
Examples of Parabola and Location of Directrix:



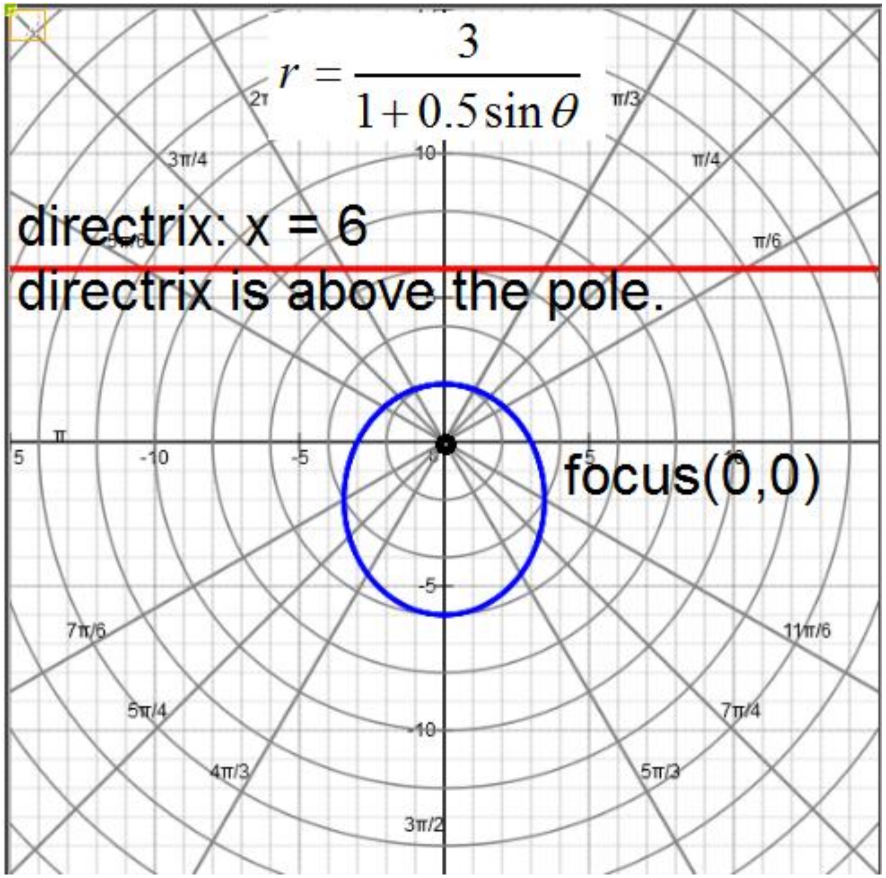


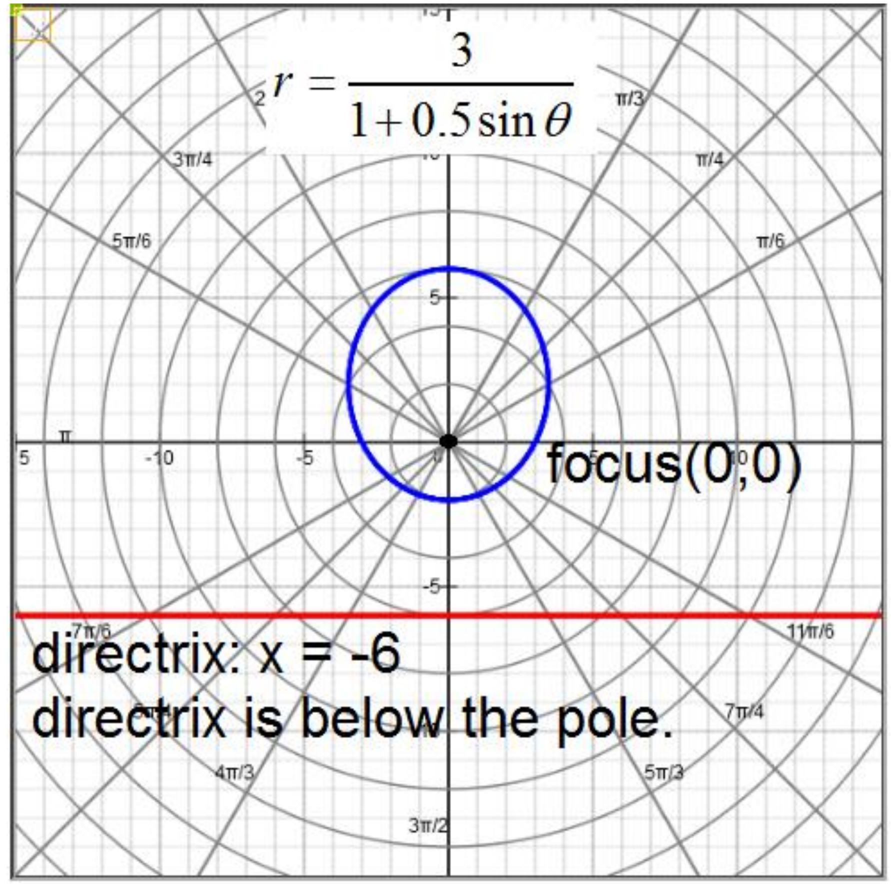
Examples of Parabola and Location of Directrix:



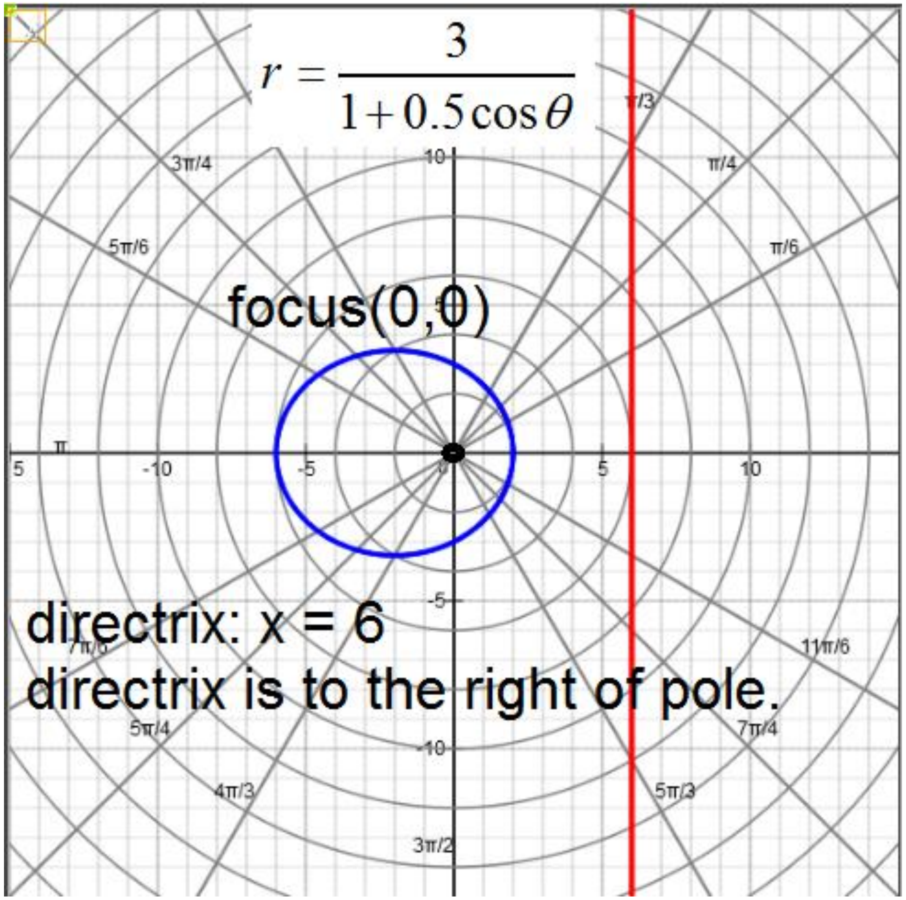


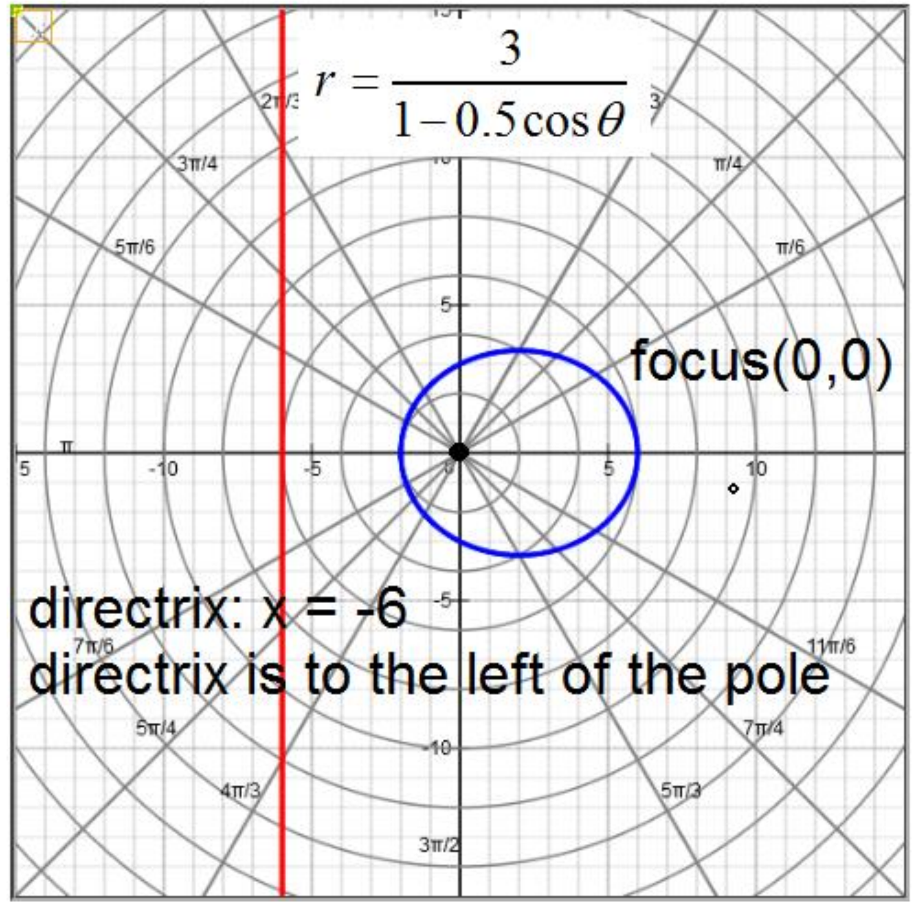
Examples of Ellipse and Location of Directrix:



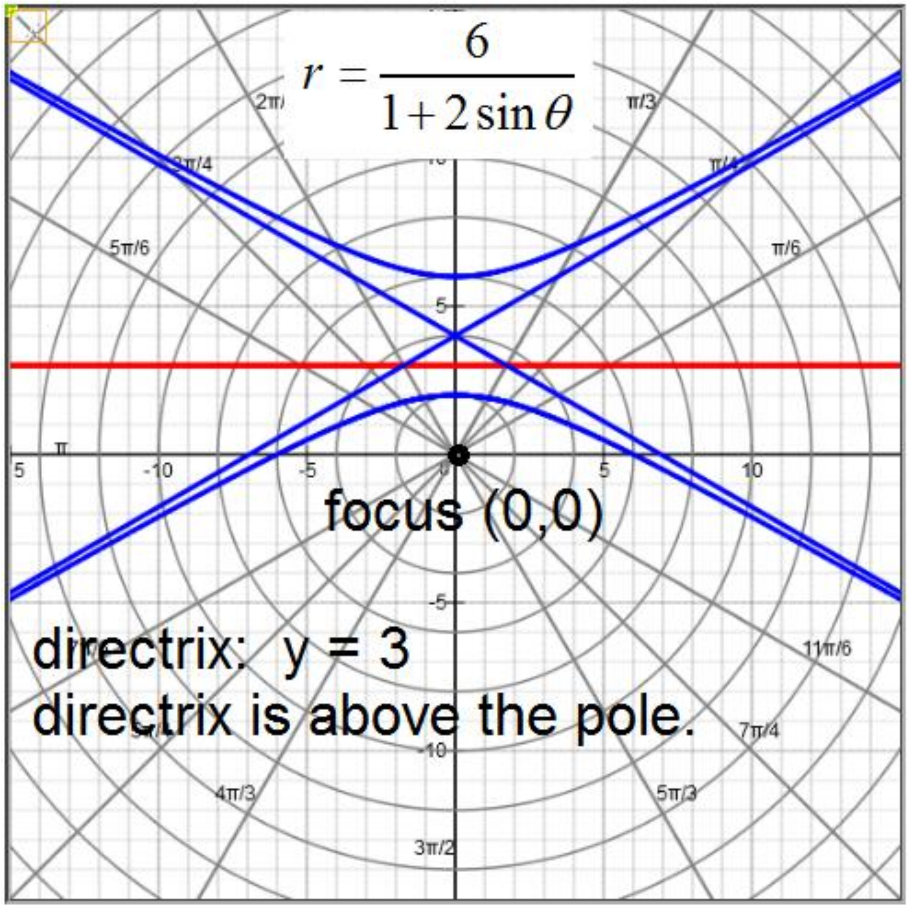


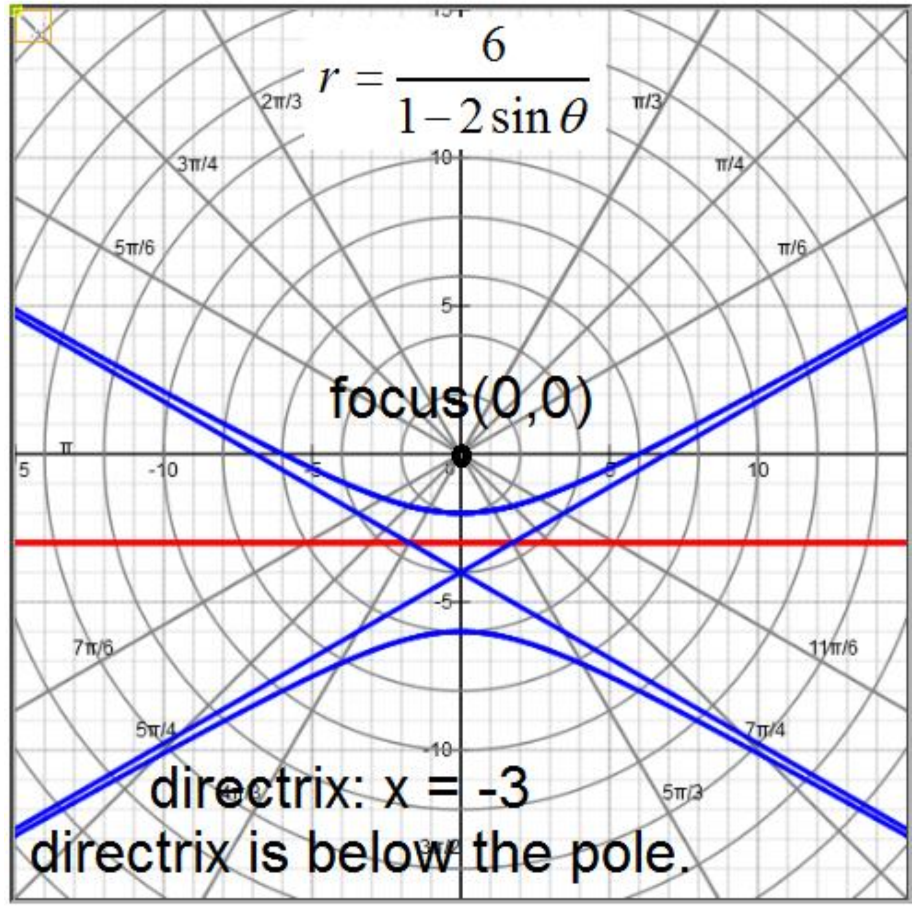
Examples of Ellipse and Location of Directrix:



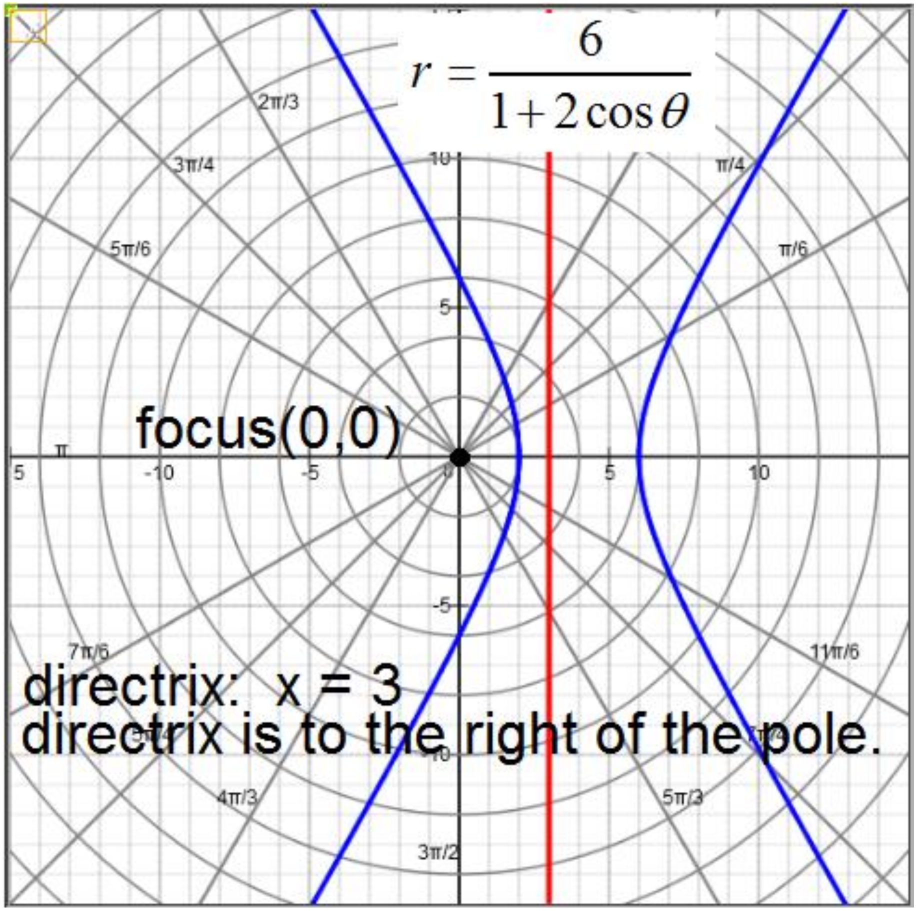


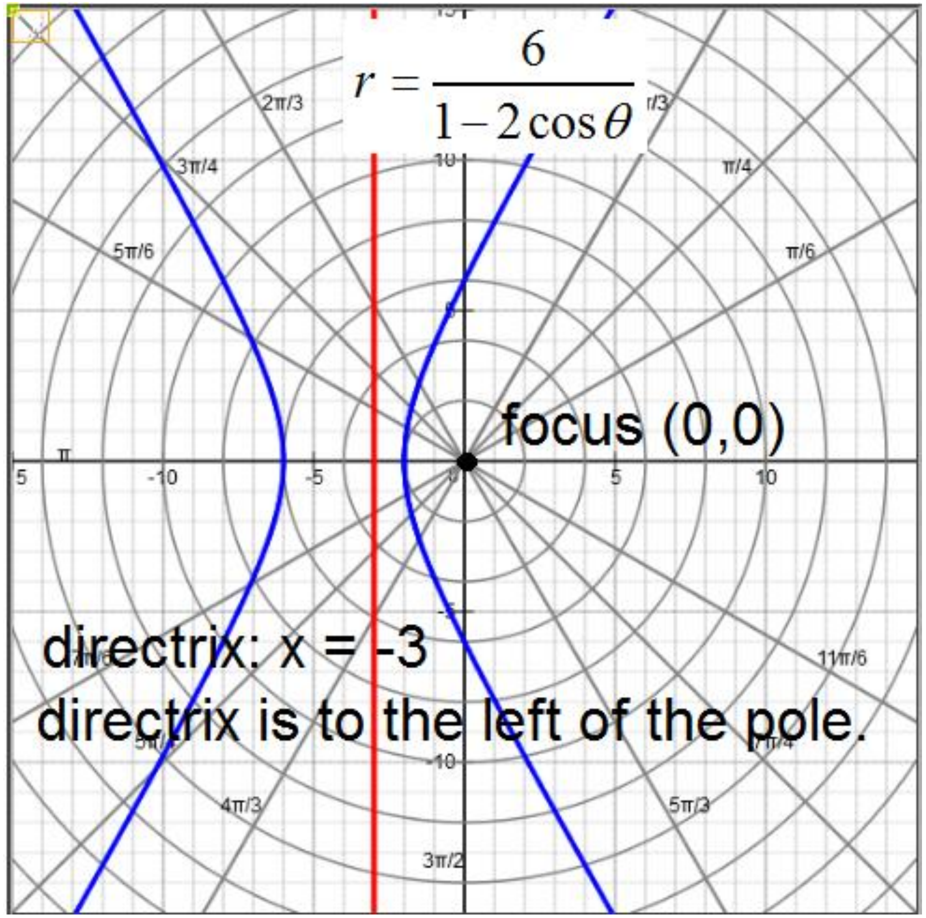
Examples of Hyperbola and Location of Directrix:





Examples of Hyperbola and Location of Directrix:





Example 1: Let $r = \dfrac{6}{1 - \cos\theta}$

Corresponding Equation: $r = \dfrac{ed}{1 - e\cos\theta}$

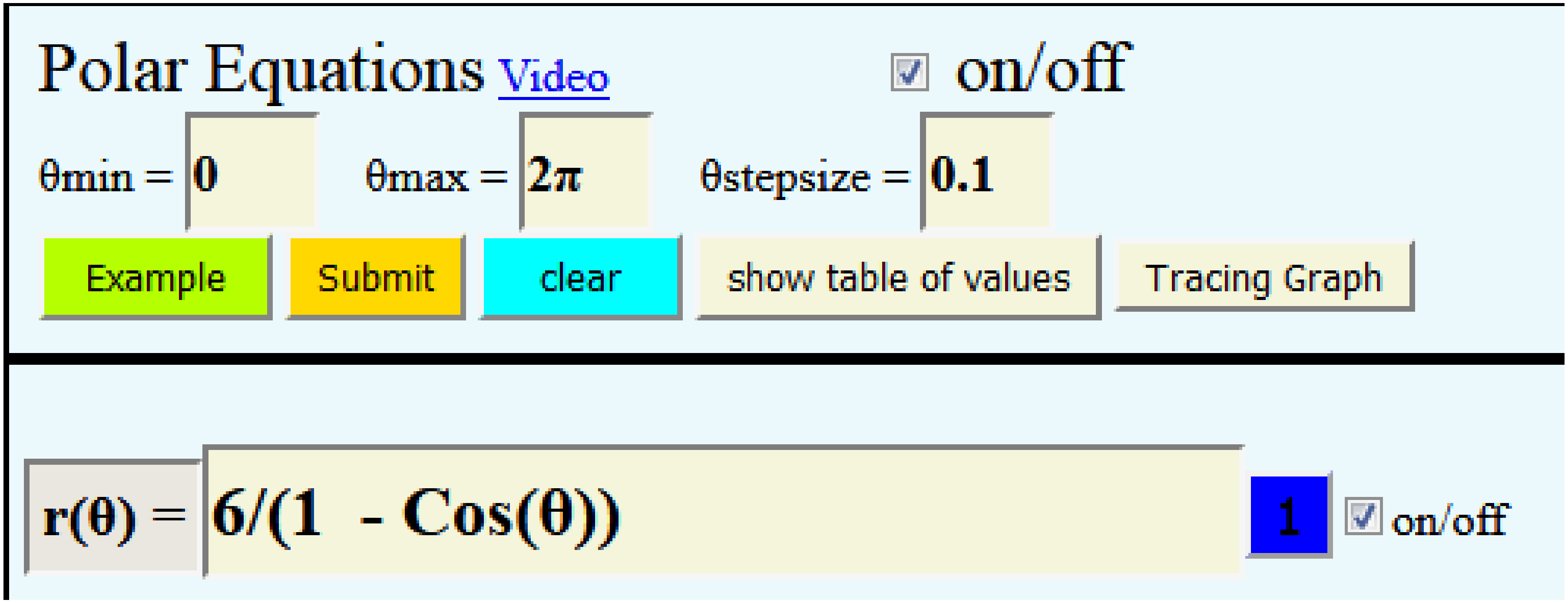
$$r = \frac{6}{1 - \cos\theta} = r = \frac{6(1)}{1 - 1\cos\theta}$$

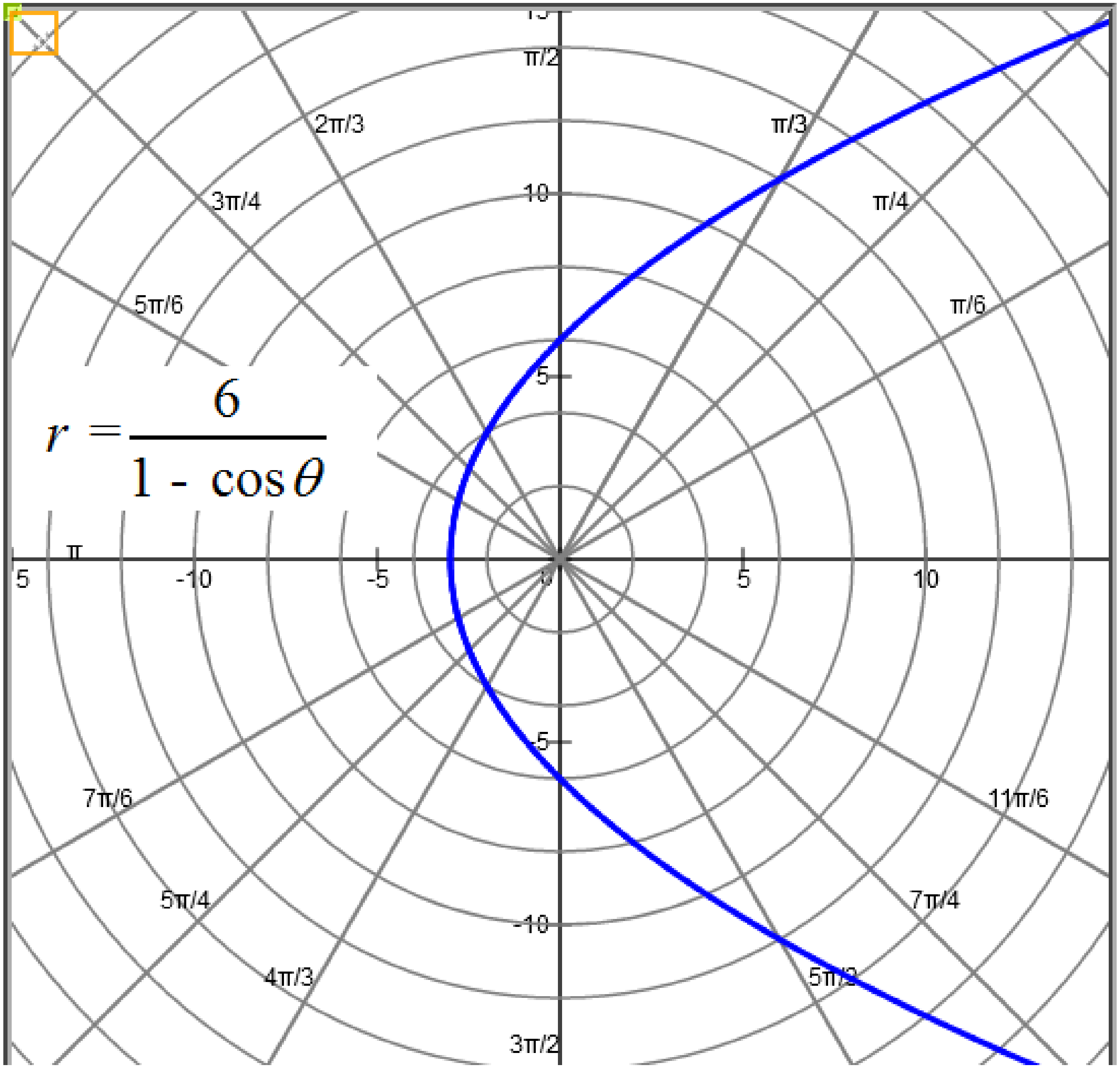
a) $e =$ eccentricity = 1 $\Rightarrow$ conic is a parabola

b) $d =$ distance from focus at the pole to directrix = 6

c) Graph $r = \dfrac{6}{1 - \cos\theta}$

Graphing $r = \dfrac{6}{1 - \cos\theta}$




$r = \frac{6}{1 - \cos\theta}$
π/2
2π/3
π/3
3π/4
π/4
5π/6
π/6
π
7π/6
11π/6
5π/4
7π/4
4π/3
5π/3
3π/2
10
5
-5
-10
-10
-5
5
10

Example 2: Let $r = \dfrac{3}{-1 + 2\cos\theta}$

Note: $r = \dfrac{3}{-1 + 2\cos\theta} = \dfrac{3/(-1)}{-1/(-1) + 2\cos\theta/(-1)}$

$$= \frac{-3}{1 - 2\cos\theta} = \frac{-3}{1 - 2\cos\theta}$$

Corresponding Equation: $r = \dfrac{ed}{1 - e\cos\theta}$

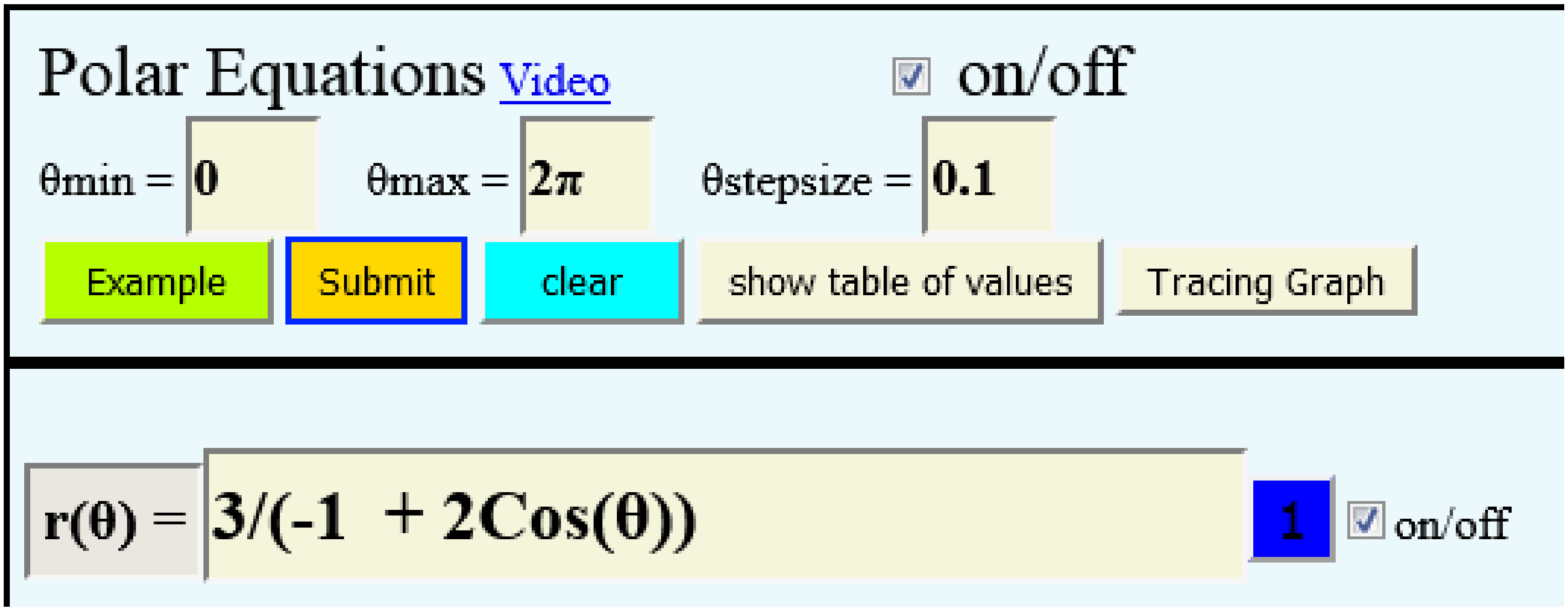
$e = 2; \quad ed = -3 \quad \Leftrightarrow \quad 2d = -3 \quad \Leftrightarrow \quad d = -3/2$

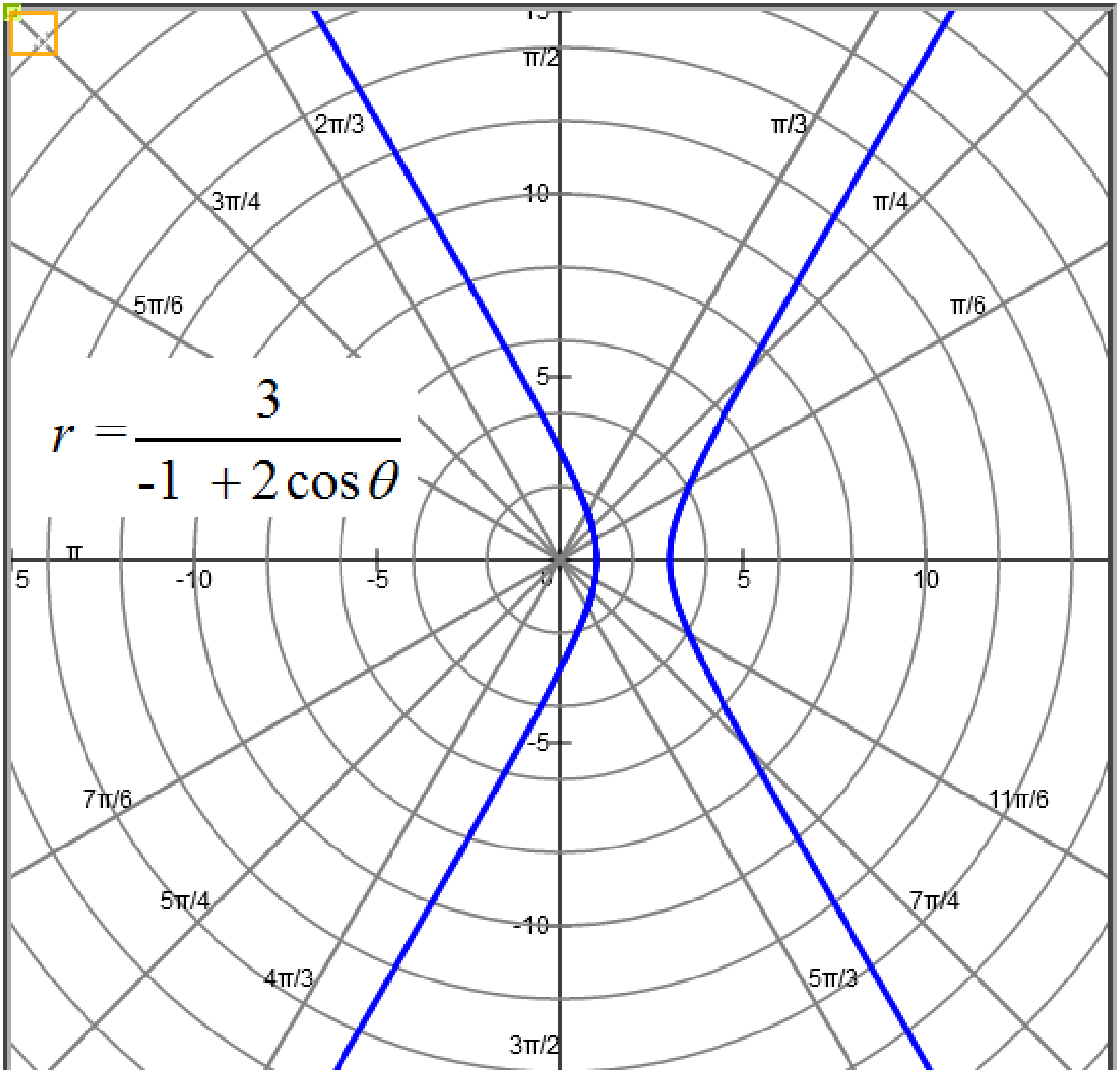
a) $e =$ eccentricity = 2

b) $|d| =$ distance from focus at the pole to directrix = $|-3/2| = 3/2$

c) Graph $r = \dfrac{3}{-1 + 2\cos\theta}$

Graphing $r = \dfrac{3}{-1 + 2\cos\theta}$




$r = \dfrac{3}{-1 + 2\cos\theta}$
π/2
2π/3
π/3
3π/4
π/4
5π/6
π/6
π
7π/6
11π/6
5π/4
7π/4
4π/3
5π/3
3π/2
10
5
-5
-10
0

Example 3:

Find a polar equation for the parabola with its focus at the pole.

Parabola has: $e =$ eccentricity = 1; Directrix: $x = -2$

a) Find distance from focus at the pole to directrix = $|d| = 2$

b) Polar equation for parabola:

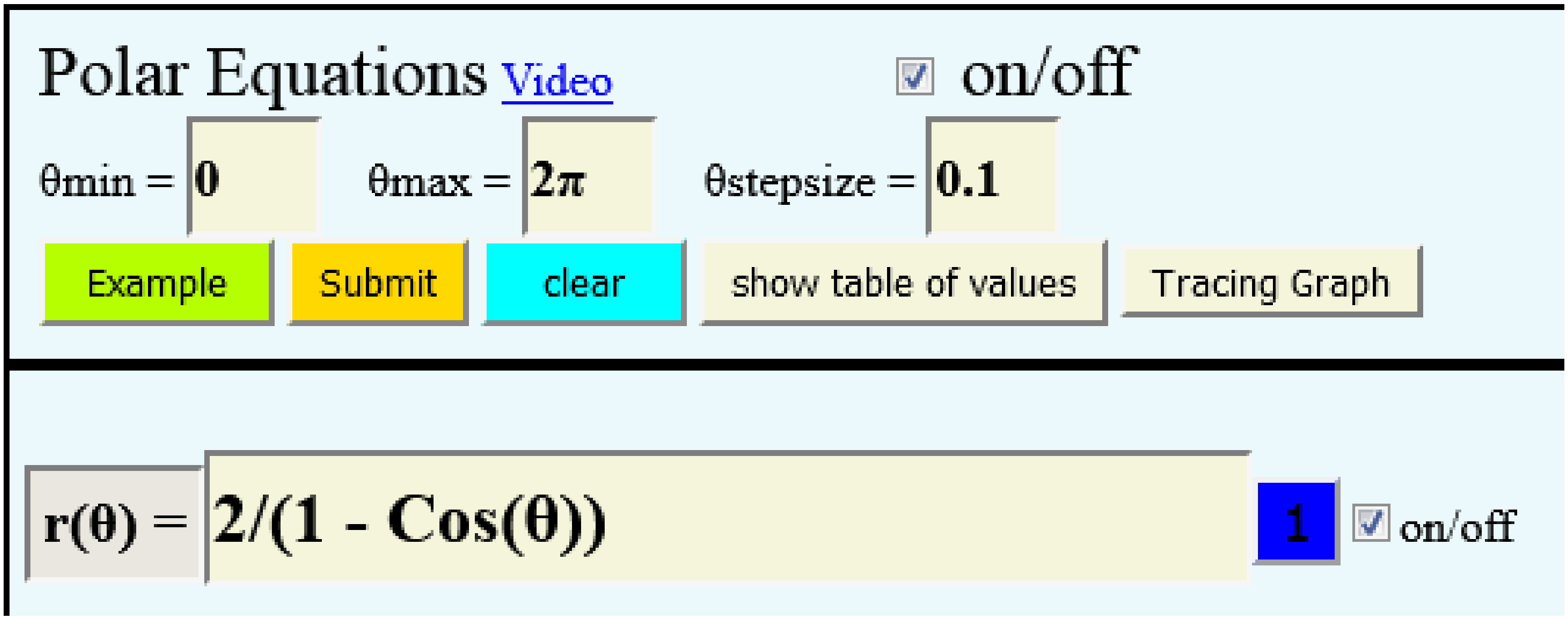
Note: The equation $x = -2$ is a vertical line to the left of the origin.

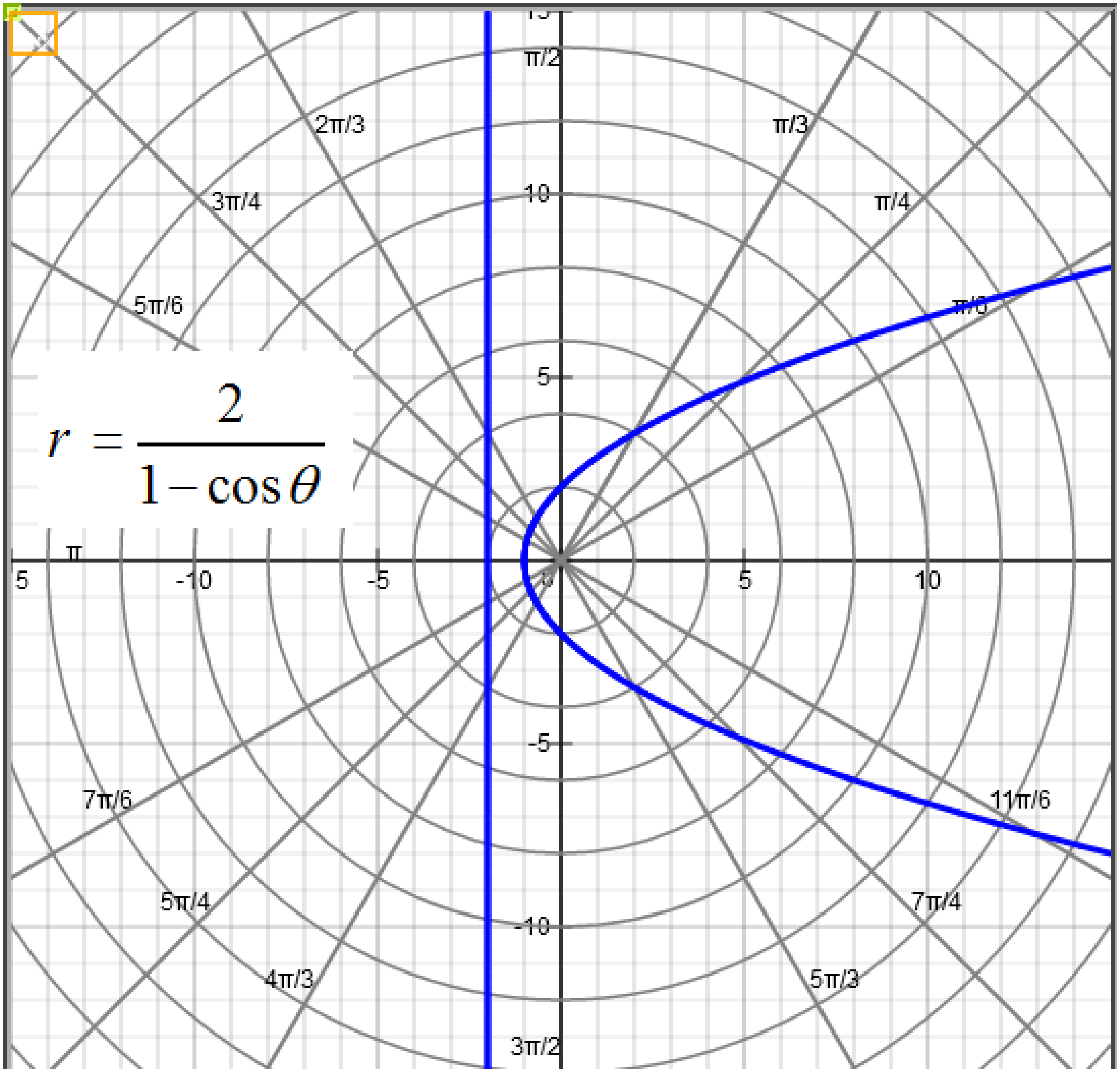
Hence, directrix is vertical and is to the left of the pole (origin).

Therefore, corresponding equation is $r = \dfrac{ed}{1 - e\cos\theta}$.

$$r = \frac{ed}{1 - e\cos\theta} = \frac{(1)(2)}{1 - 1\cos\theta} = \frac{2}{1 - \cos\theta}$$

Graphing $r = \dfrac{2}{1-\cos\theta}$




π/2
2π/3
π/3
3π/4
π/4
5π/6
π/6
π
7π/6
11π/6
5π/4
7π/4
4π/3
5π/3
3π/2
10
5
-5
-10
$r = \frac{2}{1 - \cos\theta}$

Example 4:

Find a polar equation for the parabola with its focus at the pole.

Parabola has vertex at $\left(1,\ \pi/2\right)$.

Note: Focus is at the pole or (0, 0) for polar equation of conics;

Polar Coordinates $\left(1,\ \pi/2\right) \Rightarrow$ Rectangular Coordinates $\left(0,1\right)$.

Parabola opens down because Focus is $\left(0,0\right)$ and Vertex is $\left(0,1\right)$;

Directrix is $y = 2$; Directrix is above the pole and horizontal.

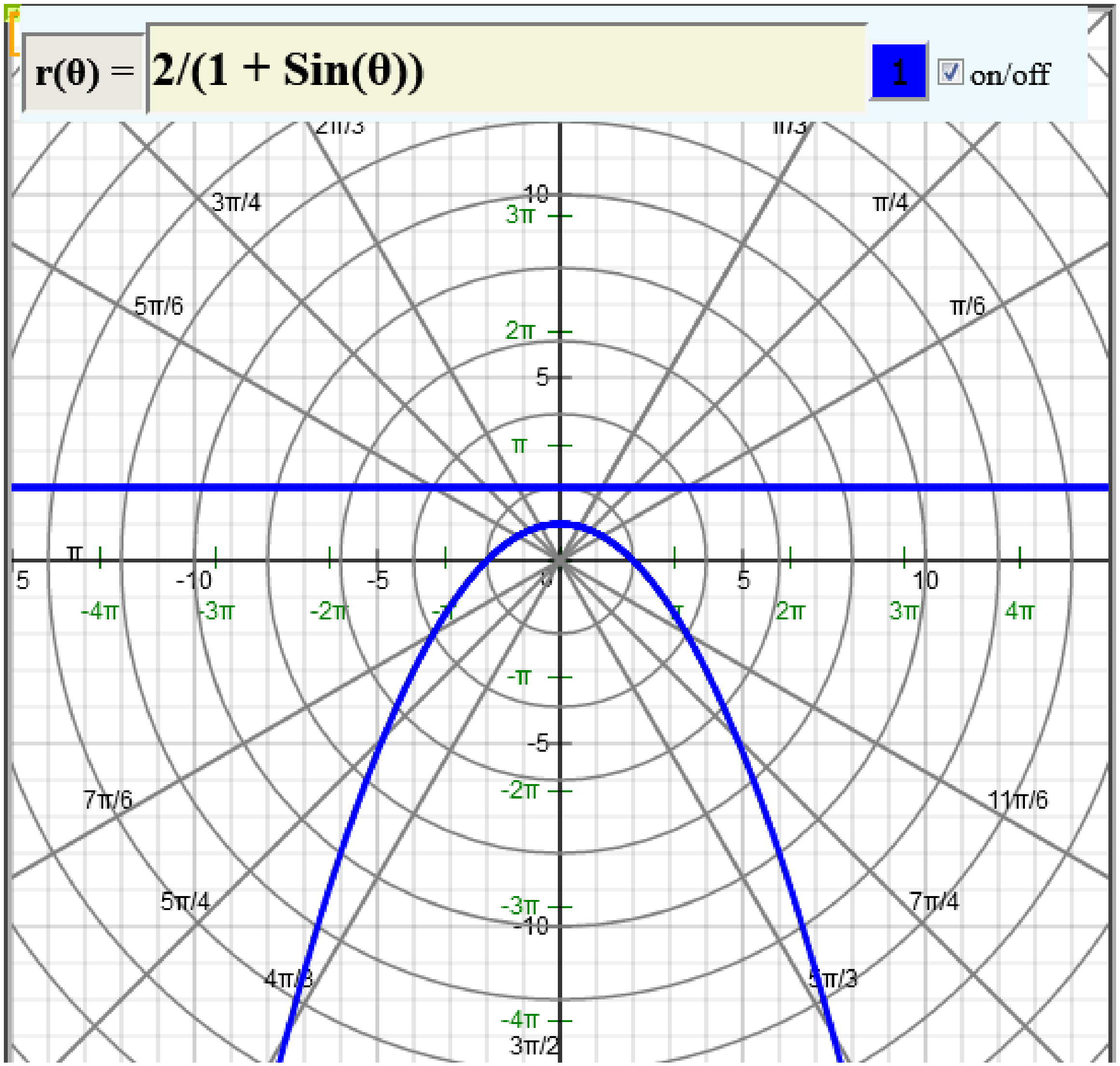
$d =$ distance between the focus at the pole and directrix $= 2$

$e = 1$ because conic is parabola.

Polar equation for parabola with horizontal directrix above the pole:

$$r = \frac{ed}{1 + e\sin\theta} = \frac{\left(1\right)\left(2\right)}{1 + \left(1\right)\sin\theta} = \frac{2}{1 + \sin\theta}$$

Graphing $r = \dfrac{2}{1+\sin\theta}$



Example 5:

Find a polar equation for the ellipse with its focus at the pole.

Ellipse has: $e =$ eccentricity = 1/4; Directrix: $y = 2$

a) distance from focus at the pole to directrix = $d = 2$

b) Polar equation for ellipse:

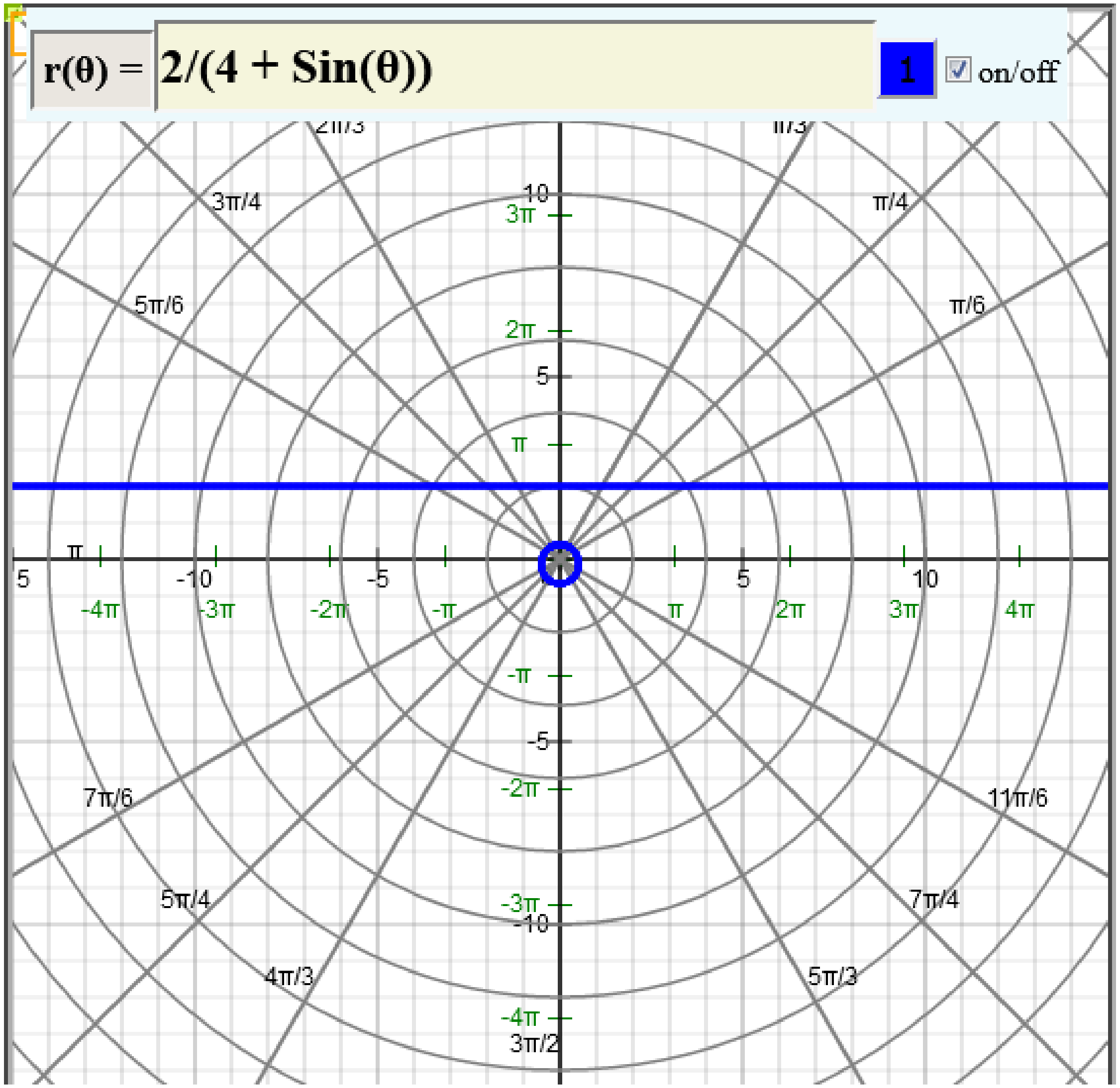
The equation $y = 2$ is a horizontal line above the origin.

Hence, directrix is horizontal and above the pole.

Corresponding polar equation for horizontal directrix above the pole:

$$r = \frac{ed}{1 + e\sin\theta} = \frac{(1/4)(2)}{1 + (1/4)\sin\theta} = \frac{1/2}{1 + (1/4)\sin\theta} = \frac{2}{4 + \sin\theta}.$$

Graphing $r = \dfrac{2}{4 + \sin\theta}$



Example 6:

Find a polar equation for the ellipse with its focus at the pole.

Ellipse has vertices at (1, 0) and (4, π).

Note: Polar Coordinates (1, 0) $\Rightarrow$ Rectangular Coordinates (1, 0)

Polar Coordinates (4, π) $\Rightarrow$ Rectangular Coordinates (-4, 0)

Center of ellipse is (-1.5, 0)

$a =$ distance from center to vertex = 2.5

$c =$ distance from center to focus = 1.5

$$e = \text{ eccentricity} = \frac{c}{a} = \frac{1.5}{2.5} = \frac{3}{5}$$

Note that ellipse elongates horizontally and has vertical directrix to the right of the pole. To find $d,$ we can use the point $(r,\theta) = (1, 0)$ and the equation $r = \dfrac{ed}{1+e\cos\theta}$.

Example 6 (con't):

$$r=\frac{ed}{1+e\cos\theta}=\frac{(3/5)d}{1+(3/5)\cos\theta}$$

Using $(r,\theta)=(1,0)$:

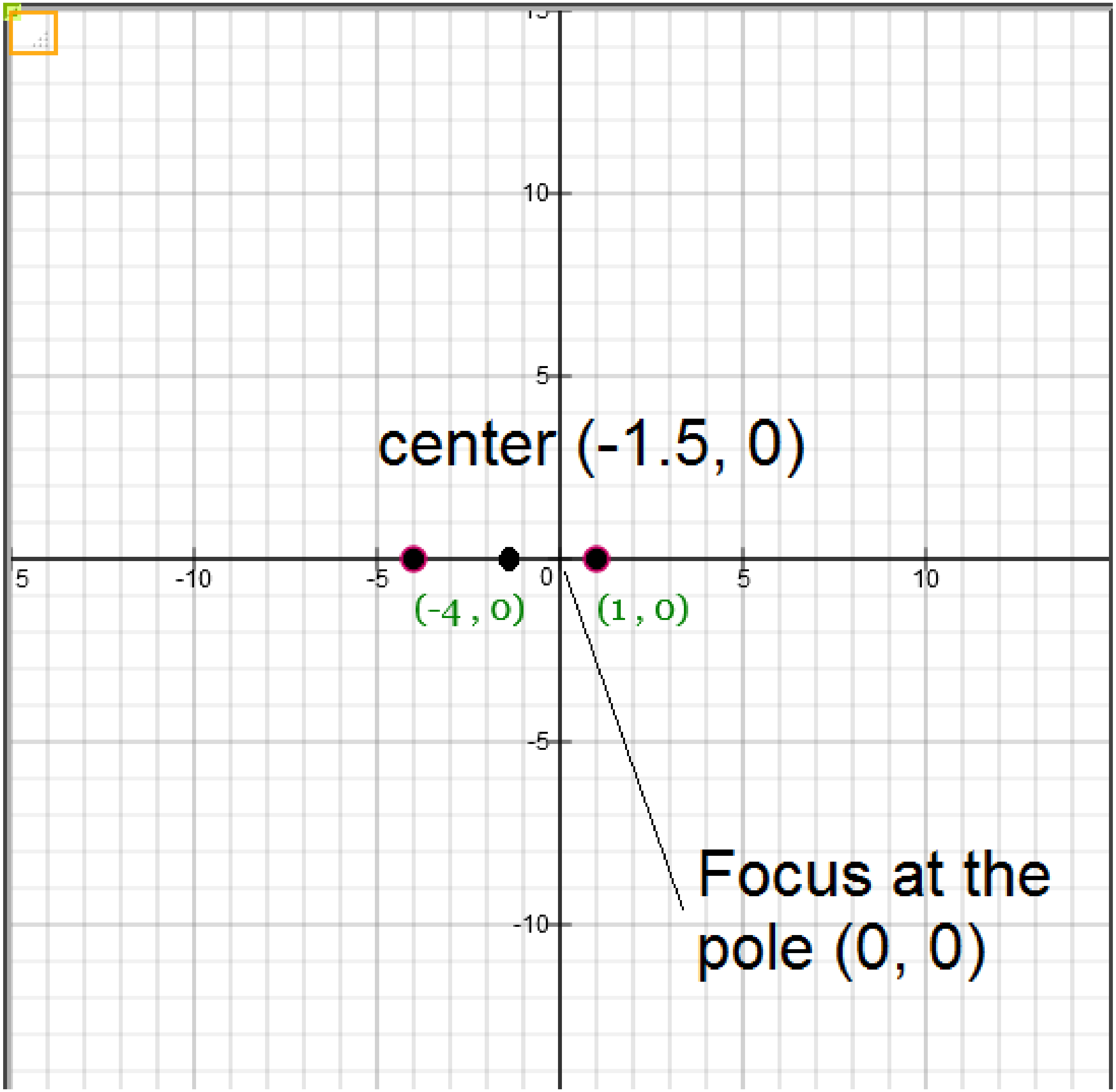
$$r=\frac{(3/5)d}{1+(3/5)\cos\theta}=\frac{3d}{5+3\cos\theta}\quad\Leftrightarrow\quad 1=\frac{3d}{5+3\cos(0)}$$

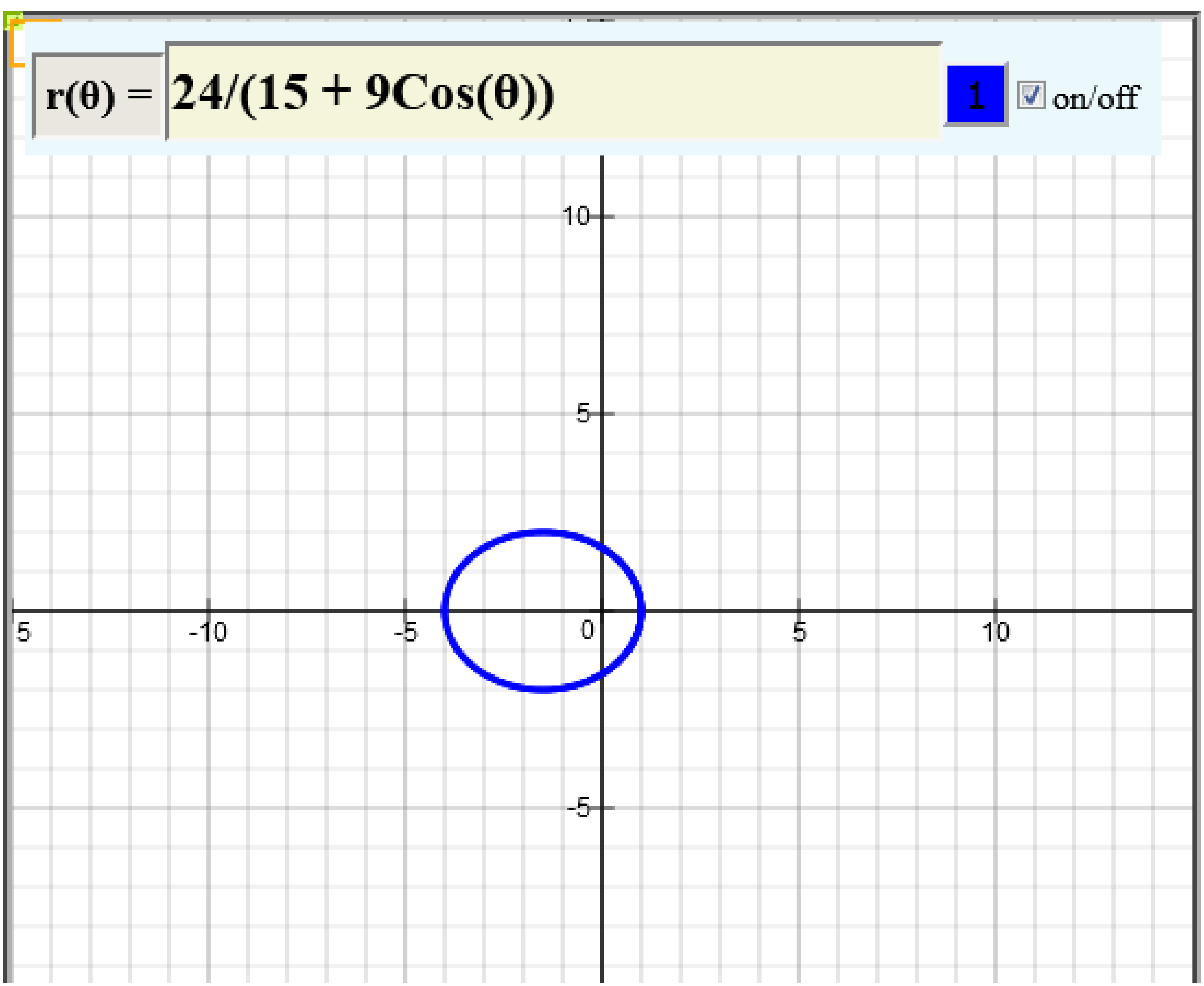
$$\Leftrightarrow\quad 1=\frac{3d}{8}\quad\Leftrightarrow\quad 8=3d\quad\Leftrightarrow\quad d=8/3$$

Hence, distance from focus at the pole to directrix = $d=\dfrac{8}{3}$

Polar equation for ellipse:

$$r=\frac{ed}{1+e\cos\theta}=\frac{(3/5)(8/3)}{1+(3/5)\cos\theta}=\frac{24/15}{1+(3/5)\cos\theta}=\frac{24}{15+9\cos\theta}$$


center (-1.5, 0)
(-4 , 0)
(1 , 0)
Focus at the pole (0, 0)
10
5
-5
-10
-10
-5
0
5
10


r(θ) = 24/(15 + 9Cos(θ))
1
on/off
10
5
-5
5
-10
-5
0
5
10

Example 7:

Find a polar equation for the hyperbola with its focus at the pole.

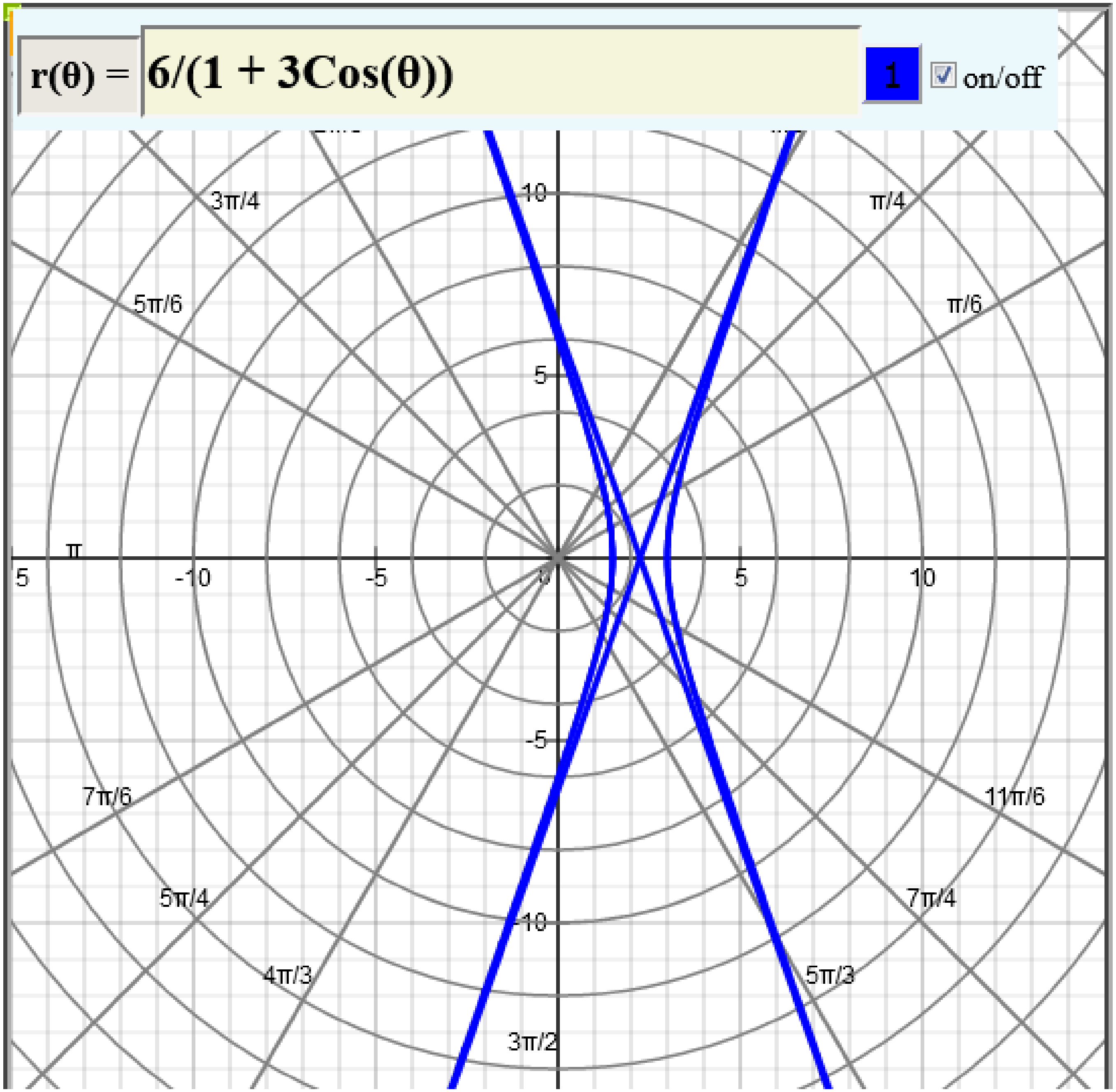
Hyperbola has: $e =$ eccentricity = 3; Directrix: $x = 2$

a) distance from focus at the pole to directrix = $d = 2$

b) Polar equation for hyperbola:

polar equation for vertical directrix to the right of the pole:

$$r = \frac{ed}{1 + e\cos\theta} = \frac{(3)(2)}{1 + (3)\cos\theta} = \frac{6}{1 + 3\cos\theta}$$



r(θ) = 6/(1 + 3Cos(θ))
1
on/off
3π/4
π/4
5π/6
π/6
10
5
π
-10
-5
0
5
10
-5
-10
7π/6
11π/6
5π/4
7π/4
4π/3
5π/3
3π/2

Example 8:

Find a polar equation for the hyperbola with its focus at the pole.

Hyperbola has vertices at $\left(2,\ 3\pi/2\right)$ and $\left(10,\ 3\pi/2\right)$.

Note:

Polar Coordinates $\left(2,\ 3\pi/2\right) \Rightarrow$ Rectangular Coordinates (0,-2)

Polar Coordinates $\left(10,\ 3\pi/2\right) \Rightarrow$ Rectangular Coordinates (0,-10)

Center of ellipse is (0,-6)

$a =$ distance from center to vertex = 4

$c =$ distance from center to focus = 6

$$e = \text{ eccentricity} = \frac{c}{a} = \frac{6}{4} = \frac{3}{2}$$

Note that hyperbola has horizontal directrix below the pole.

Example 8 (con't):

To find d, we can use the point $(r,\theta)=\left(2,\ 3\pi/2\right)$ and the equation $r=\dfrac{ed}{1-e\sin\theta}$.

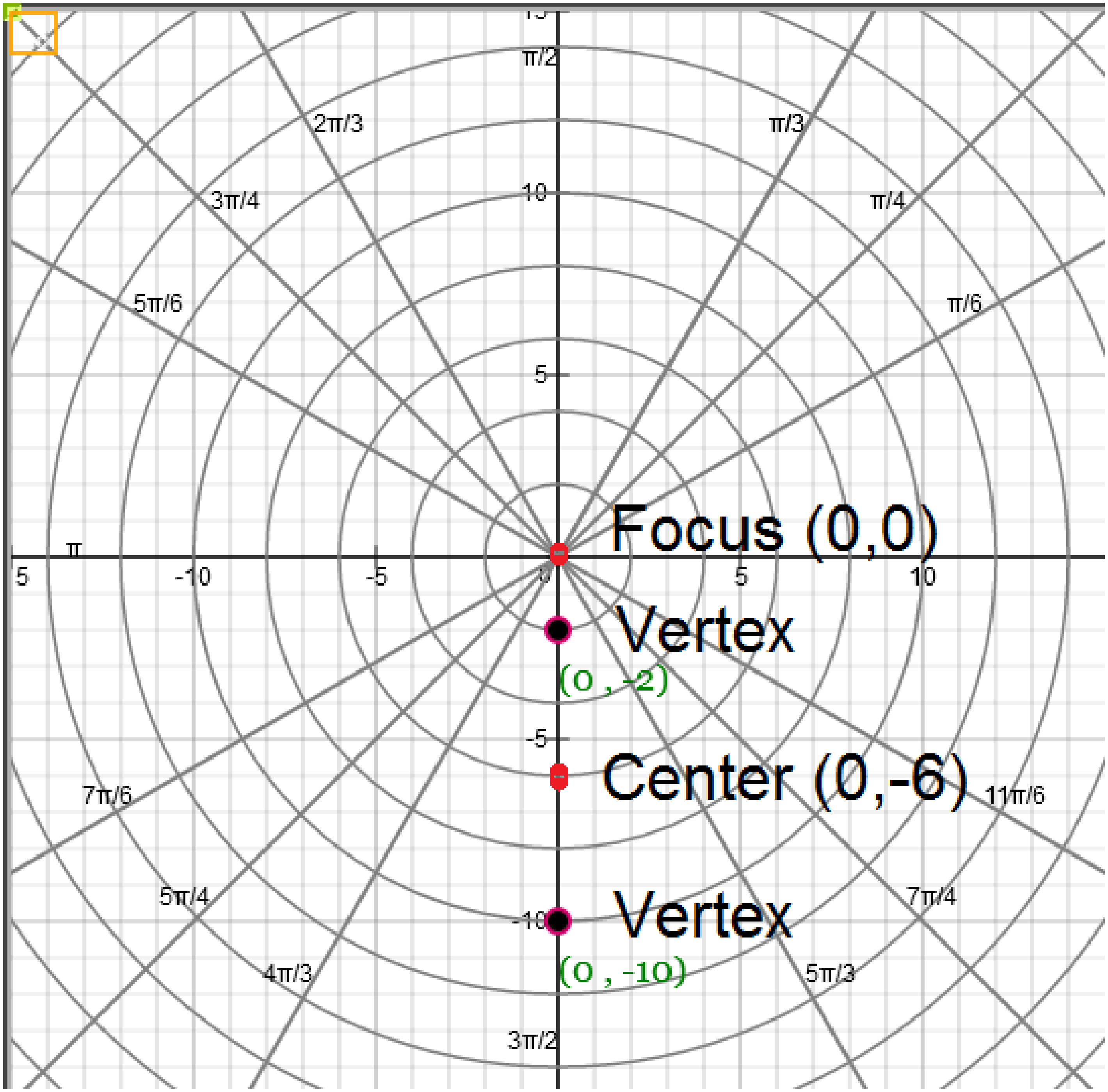
$$r=\frac{ed}{1-e\sin\theta}$$

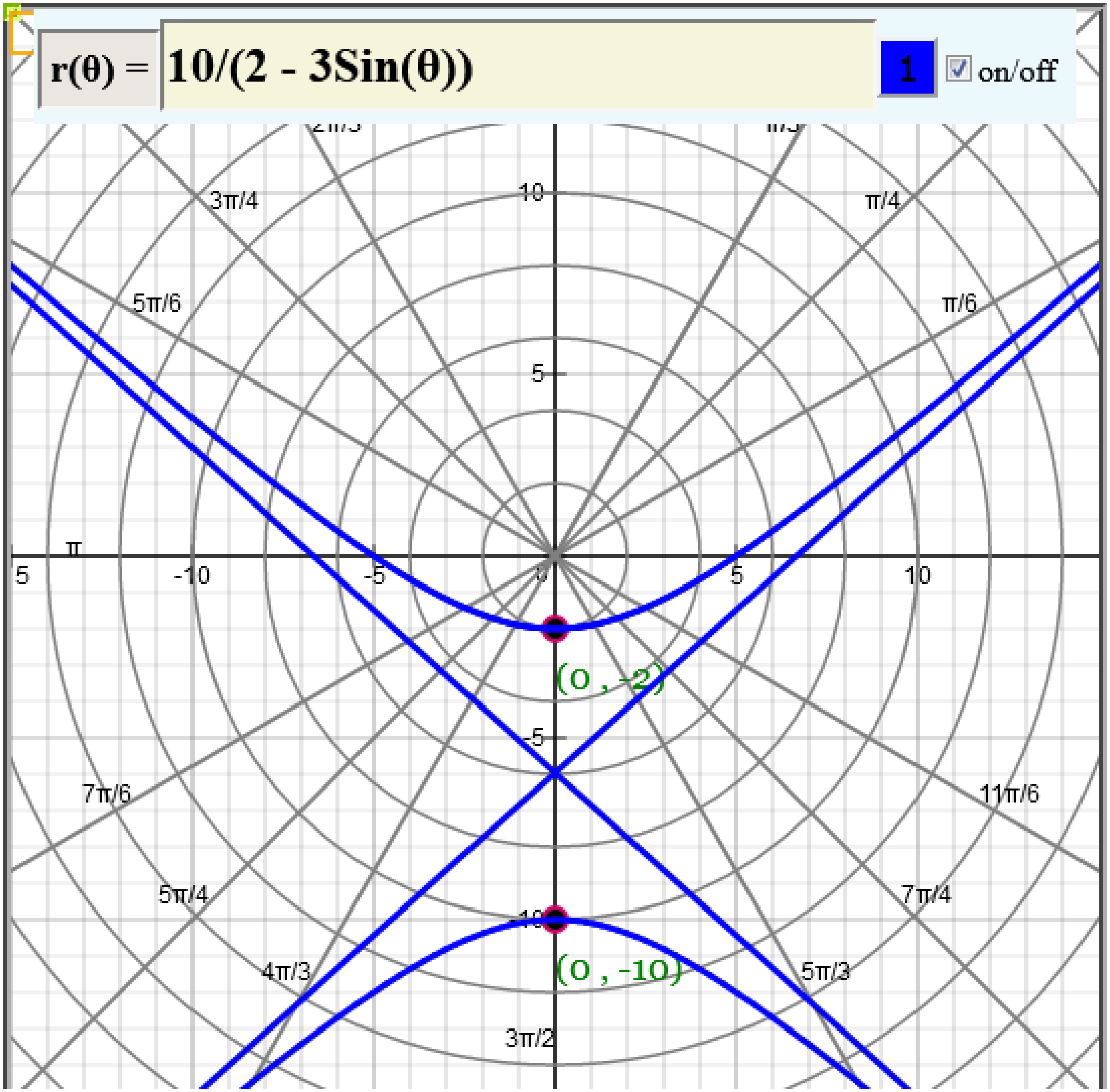
$$2=\frac{(3/2)d}{1-(3/2)\sin(3\pi/2)}=\frac{(3/2)d}{1-3/2(-1)}=\frac{(3/2)d}{5/2}=\frac{3}{5}d$$

$$d=(5/3)(2)=10/3$$

Therefore, equation of hyperbola is:

$$r=\frac{ed}{1-e\sin\theta}=\frac{(3/2)(10/3)}{1-(3/2)\sin\theta}=\frac{5}{1-(3/2)\sin\theta}=\frac{10}{2-3\sin\theta}$$


π/2
2π/3
π/3
3π/4
π/4
5π/6
π/6
π
7π/6
11π/6
5π/4
7π/4
4π/3
5π/3
3π/2
10
5
-5
-10
Focus (0,0)
Vertex
(0 , -2)
Center (0,-6)
Vertex
(0 , -10)



r(θ) =
10/(2 - 3Sin(θ))
1
on/off
3π/4
π/4
5π/6
π/6
π
10
5
-5
-10
0
7π/6
11π/6
5π/4
7π/4
4π/3
5π/3
3π/2
(0 , -2)
(0 , -10)