

Trigonometric Substitution

Trigonometric Identities:

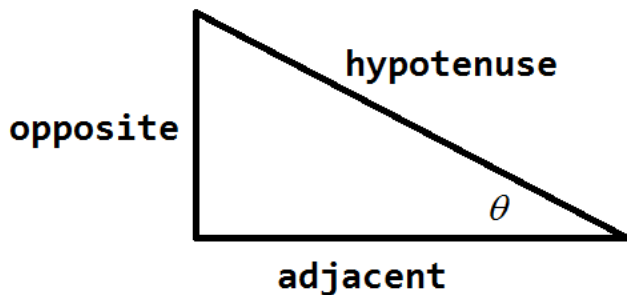
$$\sin^2 x + \cos^2 x = 1; \quad \sin^2 x = 1 - \cos^2 x; \quad \cos^2 x = 1 - \sin^2 x$$

$$\sec^2 x = 1 + \tan^2 x \Rightarrow \tan^2 x = \sec^2 x - 1$$

For integrals containing $\sqrt{a^2 - u^2}$, let $u = a \sin \theta$

For integrals containing $\sqrt{a^2 + u^2}$, let $u = a \tan \theta$

For integrals containing $\sqrt{u^2 - a^2}$, let $u = a \sec \theta$



$$(\text{hypotenuse})^2 = (\text{opposite})^2 + (\text{adjacent})^2$$

$$\text{hypotenuse} = \sqrt{(\text{opposite})^2 + (\text{adjacent})^2}$$

$$(\text{opposite})^2 = (\text{hypotenuse})^2 - (\text{adjacent})^2$$

$$\text{opposite} = \sqrt{(\text{hypotenuse})^2 - (\text{adjacent})^2}$$

$$(\text{adjacent})^2 = (\text{hypotenuse})^2 - (\text{opposite})^2$$

$$\text{adjacent} = \sqrt{(\text{hypotenuse})^2 - (\text{opposite})^2}$$

$$(hypotenuse)^2 = (opposite)^2 + (adjacent)^2$$

$$hypotenuse = \sqrt{(opposite)^2 + (adjacent)^2}$$

$$(opposite)^2 = (hypotenuse)^2 - (adjacent)^2$$

$$opposite = \sqrt{(hypotenuse)^2 - (adjacent)^2}$$

$$(adjacent)^2 = (hypotenuse)^2 - (opposite)^2$$

$$adjacent = \sqrt{(hypotenuse)^2 - (opposite)^2}$$

Example1: Find the indefinite integral $\int \frac{\sqrt{x^2 - 25}}{x} dx$.

Note: Integral contains Form $u^2 - a^2$.

$$\text{Let } u^2 = x^2 \quad \text{and} \quad a^2 = 25$$

$$\Rightarrow u = x \quad \text{and} \quad a = 5$$

If we let $u = a \sec \theta$, then $x = 5 \sec \theta$

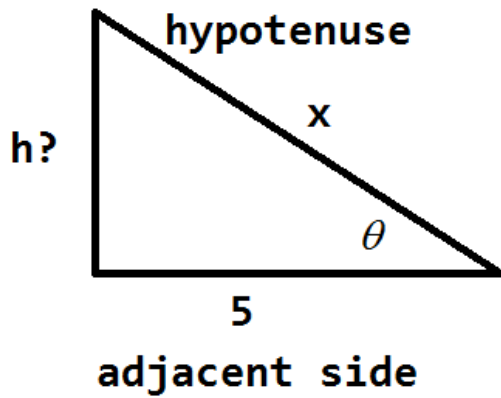
$$\text{Hence, } \frac{dx}{d\theta} = 5 \sec \theta \tan \theta \Rightarrow dx = 5 \sec \theta \tan \theta d\theta$$

$$\begin{aligned} \sqrt{x^2 - 25} &= \sqrt{(5 \sec \theta)^2 - 25} = \sqrt{25(\sec \theta)^2 - 25} \\ &= \sqrt{25[(\sec \theta)^2 - 1]} = \sqrt{25 \tan^2 \theta} = 5 \tan \theta \end{aligned}$$

$$\begin{aligned} \int \frac{\sqrt{x^2 - 25}}{x} dx &= \int \frac{5 \tan \theta}{5 \sec \theta} (5 \sec \theta \tan \theta d\theta) = \int (5 \tan \theta)(\tan \theta d\theta) \\ &= 5 \int \tan^2 \theta d\theta \\ &= 5[-\theta + \tan \theta] \quad \text{Using Formula \#63} \end{aligned}$$

Note : $x = 5 \sec \theta$

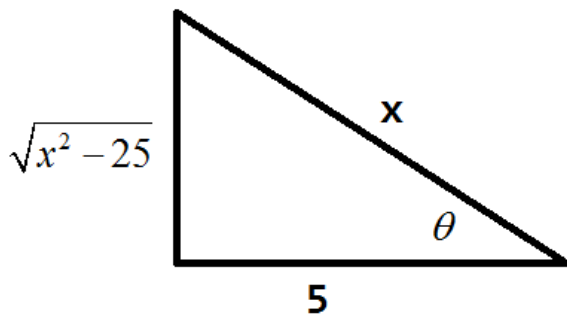
$$\sec \theta = \frac{x}{5} = \frac{\text{hypotenuse}}{\text{adjacent}}$$



$$\text{Note: } h^2 + 5^2 = x^2$$

$$h^2 = x^2 - 25$$

$$h = \sqrt{x^2 - 25}$$



$$\text{Note: } \left(\sqrt{x^2 - 25}\right)^2 + 5^2 = x^2$$

From triangle:

$$\sec \theta = \frac{\text{hypotenuse}}{\text{adjacent}} = \frac{x}{5} \Rightarrow \theta = \sec^{-1} \left(\frac{x}{5} \right)$$

$$\tan \theta = \frac{\text{hypotenuse}}{\text{opposite}} = \frac{\sqrt{x^2 - 25}}{5}$$

$$\int \frac{\sqrt{x^2 - 25}}{x} dx = \int \frac{5 \tan \theta}{5 \sec \theta} (5 \sec \theta \tan \theta d\theta) = \int (5 \tan \theta) (\tan \theta d\theta)$$

$$= 5 \int \tan^2 \theta d\theta$$

$$= 5[-\theta + \tan \theta] \quad \text{Using Formula \#63}$$

$$= 5 \left[-\sec^{-1} \frac{x}{5} + \frac{\sqrt{x^2 - 25}}{x} \right] + C$$

Example 2: Find the indefinite integral $\int \frac{x^3}{\sqrt{x^2 - 25}} dx$.

Note: Integral contains Form $u^2 - a^2$.

$$\text{Let } u^2 = x^2 \quad \text{and} \quad a^2 = 25$$

$$\Rightarrow u = x \quad \text{and} \quad a = 5$$

If we let $u = a \sec \theta$, then $x = 5 \sec \theta$

$$\text{Hence, } \frac{dx}{d\theta} = 5 \sec \theta \tan \theta \Rightarrow dx = 5 \sec \theta \tan \theta d\theta$$

$$\begin{aligned} \sqrt{x^2 - 25} &= \sqrt{(5 \sec \theta)^2 - 25} = \sqrt{25(\sec \theta)^2 - 25} \\ &= \sqrt{25[(\sec \theta)^2 - 1]} = \sqrt{25 \tan^2 \theta} = 5 \tan \theta \end{aligned}$$

$$\begin{aligned} \int \frac{x^3}{\sqrt{x^2 - 25}} dx &= \int \frac{(5 \sec \theta)^3}{5 \tan \theta} (5 \sec \theta \tan \theta d\theta) \\ &= \int \frac{125(\sec \theta)^3}{5 \tan \theta} (5 \sec \theta \tan \theta d\theta) \\ &= \int 125(\sec \theta)^3 (\sec \theta d\theta) \\ &= 125 \int (\sec \theta)^4 d\theta \end{aligned}$$

$$\int \frac{x^3}{\sqrt{x^2 - 25}} dx = 125 \int (\sec \theta)^4 d\theta$$

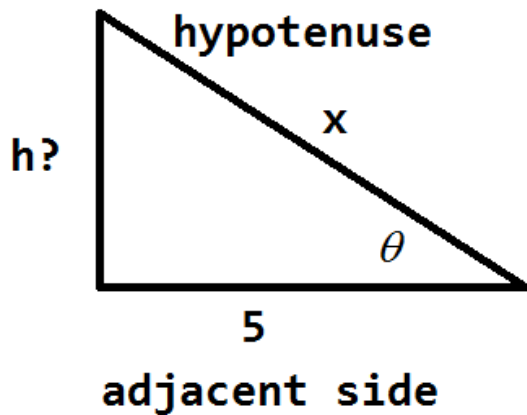
Using Formula #69 with $n = 4$:

$$\begin{aligned} \int (\sec \theta)^4 d\theta &= \frac{\sec^2 \theta \tan \theta}{3} + \frac{2}{3} \int \sec^2 \theta d\theta \\ &= \frac{\sec^2 \theta \tan \theta}{3} + \frac{2}{3} [\tan \theta] \quad \text{Using Formula \#65} \end{aligned}$$

$$\begin{aligned} \int \frac{x^3}{\sqrt{x^2 - 25}} dx &= 125 \int (\sec \theta)^4 d\theta \\ &= 125 \left[\frac{\sec^2 \theta \tan \theta}{3} + \frac{2}{3} [\tan \theta] \right] + C \end{aligned}$$

Note: $x = 5 \sec \theta$

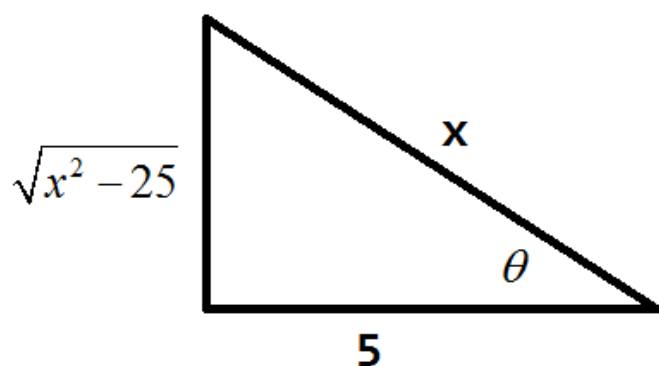
$$\sec \theta = \frac{x}{5} = \frac{\text{hypotenuse}}{\text{adjacent}}$$



$$\text{Note: } h^2 + 5^2 = x^2$$

$$h^2 = x^2 - 25$$

$$h = \sqrt{x^2 - 25}$$



$$\text{Note: } \left(\sqrt{x^2 - 25}\right)^2 + 5^2 = x^2$$

From triangle:

$$\sec \theta = \frac{\text{hypotenuse}}{\text{adjacent}} = \frac{x}{5} \Rightarrow \theta = \sec^{-1}\left(\frac{x}{5}\right)$$

$$\tan \theta = \frac{\text{hypotenuse}}{\text{opposite}} = \frac{\sqrt{x^2 - 25}}{5}$$

$$\int \frac{x^3}{\sqrt{x^2 - 25}} dx = 125 \left[\frac{\sec^2 \theta \tan \theta}{3} + \frac{2}{3} [\tan \theta] \right] + C$$

$$= 125 \left[\frac{\left(\frac{x}{5}\right)^2 \left(\frac{\sqrt{x^2 - 25}}{5}\right)}{3} + \frac{2}{3} \left[\frac{\sqrt{x^2 - 25}}{5} \right] \right] + C$$

Example 3: Find the indefinite integral $\int \sqrt{4+x^2} dx$.

Note: Integral contains Form $a^2 + u^2$.

$$\text{Let } u^2 = x^2 \quad \text{and} \quad a^2 = 4$$

$$\Rightarrow u = x \quad \text{and} \quad a = 2$$

If we let $u = a \tan \theta$, then $x = 2 \tan \theta$

$$\text{Hence, } \frac{dx}{d\theta} = 2 \sec^2 \theta \Rightarrow dx = 2 \sec^2 \theta d\theta$$

$$\begin{aligned} \sqrt{4+x^2} &= \sqrt{4+(2 \tan \theta)^2} = \sqrt{4+4(\tan \theta)^2} \\ &= \sqrt{4[1+(\tan \theta)^2]} = 2\sqrt{1+(\tan \theta)^2} \\ &= 2\sqrt{(\sec \theta)^2} = 2 \sec \theta \end{aligned}$$

$$\int \sqrt{4+x^2} dx = \int 2 \sec \theta (2 \sec^2 \theta d\theta) = 4 \int \sec^3 \theta d\theta$$

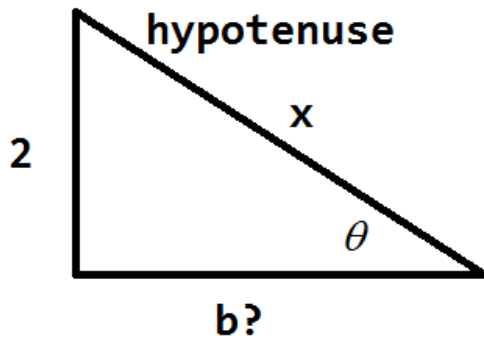
Using Formula #69 with $n = 3$:

$$\begin{aligned} \int \sec^3 \theta d\theta &= \frac{\sec \theta \tan \theta}{2} + \frac{1}{2} \int \sec \theta d\theta \\ &= \frac{\sec \theta \tan \theta}{2} + \frac{1}{2} [\ln |\sec \theta + \tan \theta|] \quad \text{Using Formula \#61} \end{aligned}$$

$$\begin{aligned} \int \sqrt{4+x^2} dx &= 4 \int \sec^3 \theta d\theta \\ &= 4 \left[\frac{\sec \theta \tan \theta}{2} + \frac{1}{2} [\ln |\sec \theta + \tan \theta|] \right] \end{aligned}$$

Note : $x = 2 \tan \theta$

$$\tan \theta = \frac{x}{2} = \frac{\text{hypotenuse}}{\text{opposite}}$$



$$b^2 + 2^2 = x^2$$

$$b^2 + 4 = x^2$$

$$b^2 = x^2 - 4$$

$$b = \sqrt{x^2 - 4}$$

$$\sec \theta = \frac{\text{hypotenuse}}{\text{adjacent}} = \frac{x}{\sqrt{x^2 - 4}}$$

$$\int \sqrt{4 + x^2} dx = 4 \left[\frac{\sec \theta \tan \theta}{2} + \frac{1}{2} \left[\ln |\sec \theta + \tan \theta| \right] \right]$$

$$= 4 \left[\frac{\left(\frac{x}{\sqrt{x^2 - 4}} \right) \left(\frac{x}{2} \right)}{2} + \frac{1}{2} \left[\ln \left| \frac{x}{\sqrt{x^2 - 4}} + \frac{x}{2} \right| \right] \right] + C$$

Example 4: Find the indefinite integral $\int \sqrt{25x^2 - 1} dx$.

Note: Integral contains Form $u^2 - a^2$.

$$\text{Let } u^2 = 25x^2 \quad \text{and} \quad a^2 = 1$$

$$\Rightarrow u = 5x \quad \text{and} \quad a = 1$$

$$\frac{du}{dx} = 5 \Rightarrow du = 5dx \Rightarrow \frac{1}{5} du = dx$$

Using Formula 26:

$$\int \sqrt{25x^2 - 1} dx = \int \sqrt{u^2 - a^2} \left(\frac{1}{5} du \right) = \frac{1}{5} \int \sqrt{u^2 - a^2} du$$

$$= \frac{1}{5} \left[\frac{1}{2} \left[u \sqrt{u^2 - a^2} + a^2 \ln \left| u + \sqrt{u^2 - a^2} \right| \right] \right] + C$$

$$= \frac{1}{10} \left[\left[5x \sqrt{(5x)^2 - 1} + \ln \left| 5x + \sqrt{(5x)^2 - 1} \right| \right] \right] + C$$

Example 5: Find the indefinite integral $\int \frac{x^2}{\sqrt{36-x^2}} dx$.

Note: Integral contains Form $a^2 - u^2$.

$$\text{Let } u^2 = x^2 \quad \text{and} \quad a^2 = 36$$

$$\Rightarrow u = x \quad \text{and} \quad a = 6$$

$$\text{Let } u = a \sin \theta \Rightarrow x = a \sin \theta = 6 \sin \theta$$

$$\frac{dx}{d\theta} = 6 \cos \theta \Rightarrow dx = 6 \cos \theta d\theta$$

$$\begin{aligned} \sqrt{36-x^2} &= \sqrt{36-(6 \sin \theta)^2} = \sqrt{36-36(\sin \theta)^2} \\ &= \sqrt{36[1-(\sin \theta)^2]} = \sqrt{36 \cos^2 \theta} = 6 \cos \theta \end{aligned}$$

$$\int \frac{x^2}{\sqrt{36-x^2}} dx = \int \frac{(6 \sin \theta)^2}{6 \cos \theta} 6 \cos \theta d\theta = 36 \int (\sin \theta)^2 d\theta$$

$$\int \frac{x^2}{\sqrt{36-x^2}} dx = \int \frac{(6 \sin \theta)^2}{6 \cos \theta} 6 \cos \theta d\theta = 36 \int (\sin \theta)^2 d\theta$$

$$\text{Using Formula 48: } \int (\sin \theta)^2 d\theta = \frac{1}{2} [\theta - \sin \theta \cos \theta]$$

$$\begin{aligned} \int \frac{x^2}{\sqrt{36-x^2}} dx &= \int \frac{(6 \sin \theta)^2}{6 \cos \theta} 6 \cos \theta d\theta = 36 \int (\sin \theta)^2 d\theta \\ &= 36 \left[\frac{1}{2} [\theta - \sin \theta \cos \theta] \right] = 18 [\theta - \sin \theta \cos \theta] \end{aligned}$$

Note: $x = 6 \sin \theta$

$$\sin \theta = \frac{x}{6} = \frac{\textit{opposite}}{\textit{hypotenuse}}; \quad \text{and } \theta = \sin^{-1} \left(\frac{x}{6} \right)$$

$$\textit{adjacent} = \sqrt{(\textit{hypotenuse})^2 - (\textit{opposite})^2} = \sqrt{36 - x^2}$$

$$\cos \theta = \frac{\textit{adjacent}}{\textit{hypotenuse}} = \frac{\sqrt{36 - x^2}}{6}$$

$$\int \frac{x^2}{\sqrt{36 - x^2}} dx = 18 [\theta - \sin \theta \cos \theta]$$

$$= 18 \left[\sin^{-1} \left(\frac{x}{6} \right) - \left(\frac{x}{6} \right) \left(\frac{\sqrt{36 - x^2}}{6} \right) \right] + C$$

Example 6: Find the indefinite integral $\int \frac{\sqrt{25x^2 + 4}}{x^4} dx$.

Note: Integral contains Form $a^2 + u^2$.

$$\text{Let } u^2 = 25x^2 \quad \text{and} \quad a^2 = 4$$

$$\Rightarrow u = 5x \quad \text{and} \quad a = 2$$

$$\text{Let } u = a \tan \theta \Rightarrow 5x = a \tan \theta \Rightarrow x = \frac{2 \tan \theta}{5}$$

$$\frac{dx}{d\theta} = \frac{2}{5} \sec^2 \theta \Rightarrow dx = \frac{2}{5} \sec^2 \theta d\theta$$

$$\begin{aligned} \sqrt{25x^2 + 4} &= \sqrt{25 \left(\frac{2 \tan \theta}{5} \right)^2 + 4} = \sqrt{25 \cdot \frac{4}{25} (\tan \theta)^2 + 4} \\ &= \sqrt{4(\tan \theta)^2 + 4} = \sqrt{4[(\tan \theta)^2 + 1]} = \sqrt{4[(\sec \theta)^2]} = 2 \sec \theta \end{aligned}$$

$$\int \frac{\sqrt{25x^2 + 4}}{x^4} dx = \int \frac{2 \sec \theta}{\left(\frac{2 \tan \theta}{5} \right)^4} \cdot \frac{2}{5} \sec^2 \theta d\theta = \frac{2 \cdot \frac{2}{5}}{\left(\frac{2}{5} \right)^4} \int \frac{\sec^3 \theta}{(\tan \theta)^4} d\theta$$

$$= 31.25 \int \frac{\left(\frac{1}{\cos^3 \theta} \right)}{\left(\frac{\sin \theta}{\cos \theta} \right)^4} d\theta = 31.25 \cos \theta (\sin \theta)^{-4} d\theta$$

$$\text{Note: } \frac{\left(\frac{1}{\cos^3 \theta} \right)}{\left(\frac{\sin \theta}{\cos \theta} \right)^4} \cdot \frac{\left(\frac{1}{\cos^3 \theta} \right)}{\left(\frac{\sin \theta}{\cos \theta} \right)^4} = \left(\frac{1}{\cos^3 \theta} \right) \frac{(\cos \theta)^4}{(\sin \theta)^4} = \frac{\cos \theta}{(\sin \theta)^4} = \cos \theta (\sin \theta)^{-4}$$

$$\int \frac{\sqrt{25x^2 + 4}}{x^4} dx = 31.25 \int \cos \theta (\sin \theta)^{-4} d\theta$$

Let $u = \sin \theta$; $\frac{du}{d\theta} = \cos \theta$; $du = \cos \theta d\theta$

$$\begin{aligned} \int \frac{\sqrt{25x^2 + 4}}{x^4} dx &= 31.25 \int \cos \theta (\sin \theta)^{-4} d\theta \\ &= 31.25 \int (\sin \theta)^{-4} \cos \theta d\theta = 31.25 \int (u)^{-4} du \end{aligned}$$

$$= 31.25 \left[\frac{u^{-3}}{-3} \right] = 31.25 \left[\frac{(\sin \theta)^{-3}}{-3} \right]$$

$$= 31.25 \left[\frac{u^{-3}}{-3} \right] = 31.25 \left[\frac{(\sin \theta)^{-3}}{-3} \right]$$

Note : $x = \frac{2 \tan \theta}{5} \Rightarrow \frac{5}{2} x = \frac{5}{2} \cdot \frac{2 \tan \theta}{5}$

$$\Rightarrow \tan \theta = \frac{5x}{2} = \frac{\text{opposite}}{\text{adjacent}}$$

Note :

$$(\text{hypotenuse})^2 = (\text{opposite})^2 + (\text{adjacent})^2$$

$$(\text{hypotenuse})^2 = (5x)^2 + (2)^2 = 25x^2 + 4$$

$$\text{hypotenuse} = \sqrt{25x^2 + 4}$$

$$\sin \theta = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{5x}{\sqrt{25x^2 + 4}}$$

$$\cos \theta = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{2}{\sqrt{25x^2 + 4}}$$

$$\int \frac{\sqrt{25x^2 + 4}}{x^4} dx = 31.25 \left[\frac{(\sin \theta)^{-3}}{-3} \right]$$

$$= 31.25 \left[\frac{\left(\frac{5x}{\sqrt{25x^2 + 4}} \right)^{-3}}{-3} \right] + C$$

Example 7: Find the indefinite integral $\int \frac{1}{x\sqrt{9x^2+1}}dx$.

Formula 32: $\int \frac{1}{u\sqrt{u^2+a^2}}du$.

Let $u^2 = 9x^2$ and $a^2 = 1$

$\Rightarrow u = 3x$ and $a = 1$

Also, $\frac{du}{dx} = 3$; $du = 3dx$; $\frac{1}{3}du = dx$

and $u = 3x \Rightarrow x = \frac{1}{3}u$

$$\int \frac{1}{x\sqrt{9x^2+1}}dx = \int \frac{1}{\frac{1}{3}u\sqrt{u^2+1}}\left(\frac{1}{3}du\right) = \int \frac{1}{u\sqrt{u^2+1}}(du)$$

$$\int \frac{1}{x\sqrt{9x^2+1}}dx = \int \frac{1}{\frac{1}{3}u\sqrt{u^2+1}}\left(\frac{1}{3}du\right)$$

$$= \int \frac{1}{u\sqrt{u^2+1}}(du) = \frac{-1}{a} \ln \left| \frac{a + \sqrt{u^2 + a^2}}{u} \right| + C \quad \text{Formula \#32}$$

$$= \frac{-1}{1} \ln \left| \frac{1 + \sqrt{9x^2 + 1}}{3x} \right| + C$$

$$= -\ln \left| \frac{1 + \sqrt{9x^2 + 1}}{3x} \right| + C$$

Example 8: Find the indefinite integral $\int \frac{x^2}{(1+x^2)^2} dx$.

Integral contains Form $a^2 + u^2$.

Let $u^2 = x^2$ and $a^2 = 1$

$$\Rightarrow u = x; \quad a = 1$$

Also, let $u = a \tan \theta \Rightarrow x = a \tan \theta \Rightarrow x = \tan \theta$

$$\frac{dx}{d\theta} = (\sec \theta)^2 \Rightarrow dx = (\sec \theta)^2 d\theta$$

$$(1+x^2)^2 = (1+(\tan \theta)^2)^2 = ((\sec \theta)^2)^2 = (\sec \theta)^4$$

$$\int \frac{x^2}{(1+x^2)^2} dx = \int \frac{(\tan \theta)^2}{(\sec \theta)^4} (\sec \theta)^2 d\theta$$

$$\int \frac{x^2}{(1+x^2)^2} dx = \int \frac{(\tan \theta)^2}{(\sec \theta)^4} (\sec \theta)^2 d\theta = \int \frac{(\tan \theta)^2}{(\sec \theta)^2} d\theta$$

$$= \int \frac{\left(\frac{\sin^2 \theta}{\cos^2 \theta} \right)}{\frac{1}{\cos^2 \theta}} d\theta = \int \sin^2 \theta d\theta$$

$$= \frac{1}{2} [\theta - \sin \theta \cos \theta] \quad \text{Formula \#48}$$

Note : $x = \tan \theta$

$$\tan \theta = \frac{x}{1} = \frac{\textit{opposite}}{\textit{adjacent}}; \quad \text{and} \quad \theta = \tan^{-1} x$$

$$\textit{hypotenuse} = \sqrt{(\textit{opposite})^2 + (\textit{adjacent})^2}$$

$$\textit{hypotenuse} = \sqrt{(x)^2 + (1)^2} = \sqrt{x^2 + 1}$$

$$\sin \theta = \frac{\textit{opposite}}{\textit{hypotenuse}} = \frac{x}{\sqrt{x^2 + 1}}$$

$$\cos \theta = \frac{\textit{adjacent}}{\textit{hypotenuse}} = \frac{1}{\sqrt{x^2 + 1}}$$

$$\begin{aligned} \int \frac{x^2}{(1+x^2)^2} dx &= \frac{1}{2} [\theta - \sin \theta \cos \theta] \\ &= \frac{1}{2} \left[\tan^{-1} x - \frac{x}{\sqrt{x^2 + 1}} \frac{1}{\sqrt{x^2 + 1}} \right] + C \end{aligned}$$