Integration Rules

Example 1: Find the indefinite Integral

$$\int \frac{3}{(x+4)^4}\,dx$$

Let $u = x + 4$

$$\frac{du}{dx} = 1$$

$$du = dx$$

Rewriting integral in terms of u and du:

$$\int \frac{3}{(x+4)^4}\,dx = 3\int \frac{1}{(u)^4}\,du = 3\int u^{-4}\,du = 3\left[\frac{u^{-3}}{-3}\right] = -1u^{-3} = -u^{-3} + C$$

Writing answer in terms of x:

$$\int \frac{3}{(x+4)^4}\,dx = -u^{-3} + C = -(x+4)^{-3} + C$$

Example 2: Find the indefinite Integral

$$\int x^2\sqrt{5+x^3}\,dx$$

Let $u = 5 + x^3$

$$\frac{du}{dx} = 3x^2$$

$$du = 3x^2\,dx$$

$$\frac{1}{3}du = \frac{1}{3}\cdot 3x^2\,dx$$

$$\frac{1}{3}du = x^2\,dx$$

Rewriting integral in terms of u and du:

$$\int x^2\sqrt{5+x^3}\,dx = \int \sqrt{5+x^3}\,x^2\,dx = \int \sqrt{u}\left(\frac{1}{3}du\right) = \frac{1}{3}\int u^{1/2}\,du$$

$$= \frac{1}{3}\left[\frac{u^{3/2}}{3/2}\right] + C$$

$$= \frac{1}{3}\cdot\frac{2}{3}\left[u^{3/2}\right] + C$$ dividing by 3/2 is same multiplying by 2/3

$$= \frac{2}{9}\left[u^{3/2}\right] + C$$

Writing answer in terms of x:

$$\int x^2\sqrt{5+x^3}\,dx = \frac{2}{9}\left[u^{3/2}\right] + C = \frac{2}{9}\left[\left(5+x^3\right)^{3/2}\right] + C$$

Example 3: Find the indefinite Integral

$$\int \left[2x - \frac{4}{(2x+5)^3} \right] dx$$

Note: $\int \left[2x - \frac{4}{(2x+5)^3} \right] dx = \int [2x]\, dx \ - \ \int \left[\frac{4}{(2x+5)^3} \right] dx$

$$\int [2x]\, dx = \frac{2x^2}{2} + C = x^2 + C$$

Fore $\int \left[\frac{4}{(2x+5)^3} \right] dx$:

Let $u = 2x + 5$

$$\frac{du}{dx} = 2$$

$$du = 2dx$$

$$\frac{1}{2} du = \frac{1}{2} \cdot 2dx$$

$$\frac{1}{2} du = dx$$

Rewriting integral in terms of u and du:

$$\int \left[\frac{4}{(2x+5)^3} \right] dx = \int \left[\frac{4}{(u)^3} \right] \cdot \frac{1}{2} du = \int \left[\frac{2}{(u)^3} \right] du = 2\int \left[u^{-3} \right] du$$

$$= \frac{2u^{-2}}{-2} + C = -1(2x+5)^{-2} + C$$

Writing answer in terms of x:

$$\int \left[2x - \frac{4}{(2x+5)^3} \right] dx = \int [2x]\, dx \ - \ \int \left[\frac{4}{(2x+5)^3} \right] dx$$

$$= \frac{2x^2}{2} - 1(2x+5)^{-2} + C$$

$$= x^2 - (2x+5)^{-2} + C$$

Example 4: Find the indefinite Integral

$$\int\left[\frac{4x}{x+2}\right]dx$$

Note: Degree of numerator is 1; degree of denominator is 1;

When degree of numerator is equal or greater than that of denominator

try to divide expression using long division.

$$\frac{4x}{x+2} \quad \Leftrightarrow \quad x+2\overline{)4x}$$

Hence, $\dfrac{4x}{x+2} = 4 + \dfrac{-8}{x+2}$

$$\int\left[\frac{4x}{x+2}\right]dx = \int\left[4+\frac{-8}{x+2}\right]dx = \int[4]dx \ + \ \int\left[\frac{-8}{x+2}\right]dx$$

Note: $\int[4]dx = 4x$

Fore $\int\left[\dfrac{-8}{x+2}\right]dx$:

$$\int\left[\frac{-8}{x+2}\right]dx = -8\int\left[\frac{1}{x+2}\right]dx$$

Recall: $\int\left[\dfrac{1}{x}\right]dx = \ln|x|$; $\qquad \int\left[\dfrac{1}{x+a}\right]dx = \ln|x+a|$

$$\int\left[\frac{-8}{x+2}\right]dx = -8\int\left[\frac{1}{x+2}\right]dx = -8\ln|x+2|+C$$

Writing Answer:

$$\int\left[\frac{4x}{x+2}\right]dx = \int\left[4+\frac{-8}{x+2}\right]dx = \int[4]dx \ + \ \int\left[\frac{-8}{x+2}\right]dx$$

$$= 4x - 8\ln|x+2|+C$$

Example 5: Find the indefinite Integral

$$\int \left[\frac{1}{3x+2} - \frac{4}{5x+2} \right] dx$$

Note: $\int \left[\frac{1}{3x+2} - \frac{4}{5x+2} \right] dx = \int \left[\frac{1}{3x+2} \right] dx - \int \left[\frac{4}{5x+2} \right] dx$

Recall: $\int \left[\frac{1}{x} \right] dx = \ln |\, x \,|$; $\qquad \int \left[\frac{1}{x+a} \right] dx = \ln |\, x+a \,|$

$$\int \left[\frac{1}{bx+a} \right] dx = \frac{1}{b} \ln |\, bx+a \,|$$

$$\int \left[\frac{1}{3x+2} - \frac{4}{5x+2} \right] dx = \int \left[\frac{1}{3x+2} \right] dx - 4 \int \left[\frac{1}{5x+2} \right] dx$$

$$= \frac{1}{3} \ln |\, 3x+2 \,| \; - \; 4 \left[\frac{1}{5} \ln |\, 5x+2 \,| \right] \; + \; C$$

$$= \frac{1}{3} \ln |\, 3x+2 \,| \; - \; \frac{4}{5} \ln |\, 5x+2 \,| \; + \; C$$

Example 6: Find the indefinite Integral

$$\int [\csc \pi x \cot \pi x]\, dx$$

Recall: $\int [\csc u \cot u]\, du = -\csc u + C$

Let $u = \pi x$

$$\frac{du}{dx} = \pi$$

$$du = \pi\, dx$$

$$\frac{1}{\pi} du = \frac{1}{\pi} \cdot \pi\, dx$$

$$\frac{1}{\pi} du = dx$$

Rewriting integral in terms of u and du :

$$\int [\csc \pi x \cot \pi x]\, dx = \int [\csc u \cot u] \frac{1}{\pi} du$$

$$= \frac{1}{\pi} \int [\csc u \cot u]\, du$$

$$= \frac{1}{\pi}[-\csc u] + \mathrm{C}$$

$$= \frac{1}{\pi}[-\csc \pi x] + \mathrm{C}$$ Writing answer in terms of x

Example 7: Find the indefinite Integral

$$\int \left[\csc^2 x e^{\cot x} \right] dx$$

Let $u = \cot x$

$$\frac{du}{dx} = -\csc^2 x$$

$$du = -\csc^2 x dx$$

$$-1du = (-1)\left(-\csc^2 x dx\right)$$

$$-1du = \csc^2 x dx$$

Rewriting integral in terms of u and du:

$$\int \left[\csc^2 x e^{\cot x} \right] dx = \int \left[e^{\cot x} \right] \csc^2 x dx$$

$$= \int \left[e^{u} \right] (-1) du$$

$$= -1 \int \left[e^{u} \right] du$$

$$= -1 e^{u} + C$$ Note: $\int \left[e^{u} \right] du = e^{u} + C$

$$= -1 e^{\cot x} + C$$ Writing answer in terms of x

Example 8: Find the indefinite Integral

$$\int [\tan x][\ln(\cos x)]\,dx$$

Recall: $\int [\tan u]\,du = -\ln|\cos u| + C$

Let $u = \ln(\cos x)$

$$\frac{du}{dx} = \frac{1}{\cos x}(-\sin x)$$

Note: $D_x(\ln u) = \frac{1}{u} \cdot u'$

$$du = -\frac{\sin x}{\cos x}dx$$

$$dx = -\tan x dx$$

$$-1du = (-1)(-\tan x dx)$$

$$-1du = \tan x dx$$

Rewriting integral in terms of u and du :

$$\int [\tan x][\ln(\cos x)]\,dx = \int [\ln(\cos x)] \tan x dx$$

$$= \int [u](-1)du$$

$$= -1\int [u]\,du$$

$$= -1\left[\frac{u^2}{2}\right] + C$$

$$= \frac{-1}{2}[\ln(\cos x)]^2 + C$$

Writing answer in terms of x

Example 9: Find the definite Integral

$$\int_0^4 \frac{1}{\sqrt{36-x^2}}\,dx$$

Recall: $\int \frac{1}{\sqrt{a^2-u^2}}\,du = \arcsin\frac{u}{a} + C$

Let $a^2 = 36 \quad \Rightarrow \quad a = 6$

Let $u^2 = x^2 \quad \Rightarrow \quad u = x$

$$\frac{du}{dx} = 1$$

$$du = dx$$

Rewriting integral in terms of u and du :

$$\int_0^4 \frac{1}{\sqrt{36-x^2}}\,dx = \int_0^4 \frac{1}{\sqrt{a^2-u^2}}\,du$$

$$= \arcsin\frac{u}{a}$$

$$= \arcsin\frac{x}{6}\Big]_0^4$$

$$= \arcsin\frac{4}{6} - \arcsin\frac{0}{6}$$

= 0.7297276562269663 - 0 = 0.7297276562269663

Example 10: Find the definite integral

$$\int_0^4 \frac{1}{9+16x^2}\,dx$$

Recall: $\int \frac{1}{a^2+u^2}\,du = \frac{1}{a}\arctan\frac{u}{a} + C$

Let $a^2 = 9 \quad \Rightarrow \quad a = 3$

Let $u^2 = 16x^2 \quad \Rightarrow \quad u = 4x$

$$\frac{du}{dx} = 4$$

$$du = 4dx$$

$$\frac{1}{4}du = dx$$

Rewriting integral in terms of u and du:

$$\int_0^4 \frac{1}{9+16x^2}\,dx = \int \frac{1}{a^2+u^2}\frac{1}{4}\,du = \frac{1}{4}\int \frac{1}{a^2+u^2}\,du$$

$$= \frac{1}{4}\left[\frac{1}{a}\arctan\frac{u}{a}\right]$$

$$= \frac{1}{4}\left[\frac{1}{3}\arctan\frac{4x}{3}\right]_0^4$$

$$= \frac{1}{4}\left[\frac{1}{3}\arctan\frac{16}{3} - \frac{1}{3}\arctan\frac{0}{3}\right]$$

$$= 0.11545403139993349$$

Example 11: Find the definite integral

$$\int \frac{4x}{\sqrt{x^2-9}}\,dx$$

Let $u = x^2 - 9$

$$\frac{du}{dx} = 2x$$

$$du = 2x\,dx$$

$$2du = 2 \cdot 2x\,dx$$

$$2du = 4x\,dx$$

Rewriting integral in terms of u and du:

$$\int \frac{4x}{\sqrt{x^2-9}}\,dx = \int \frac{1}{\sqrt{x^2-9}}\,4x\,dx = \int \frac{1}{\sqrt{u}}\,2dx$$

$$= 2\int \frac{1}{u^{1/2}}\,dx$$

$$= 2\int u^{-1/2}\,dx$$

$$= 2\frac{u^{1/2}}{1/2} \quad = 2 \cdot 2u^{1/2}$$ Dividing by 1/2 is same as multiplying by 2

$$= 4u^{1/2} + \mathrm{C}$$

$$= 4\left(x^2-9\right)^{1/2} + \mathrm{C}$$ Writing answer in terms of x