Research Report

Tayden White

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Let k be the number of overlapping supports between two permutation defects. Each step of the algorithm for the fusion of permutation defects can be defined by one of three cases, depending on the value of k.

1. $k \geq 2$: In this case, a transposition containing the overlapping supports from the two permutations should be factored out. The result will be a sum of multi-layer anyons that can be pushed to the side because fusion between anyons and permutation defects is commutative.
   $X^{(ijk)} \otimes X^{(ijl)} = [\sum_{a \in I} \cdots \boxtimes a_i \boxtimes \cdots \boxtimes a_j^* \boxtimes \cdots] \otimes X^{(ik)} \otimes X^{(jl)}$

2. $k = 1$: In this case, the associated permutations should be multiplied.
   $X^{(ijk)} \otimes X^{(klm)} = X^{(ijklm)}$

3. $k = 0$: In this case, the associated permutations should be multiplied.
   $X^{(ijk)} \otimes X^{(lmn)} = X^{(ijk)(lmn)}$

Upon factoring out a transposition like described in rule 1, there may be a case where the remaining permutation is actually the product of two disjoint cycles. These disjoint cycles can and should be represented as two separate permutation defects that can commute, but only with each other.

Example: $X^{(123456)(78)} \otimes X^{(124378596)}$

1. $X_1^{(123456)(78)} \otimes X_2^{(124378596)}$
   Rule 3: Treat disjoint cycles as separate commutable defects $\rightarrow$

2. $X_1^{(78)} \otimes X_2^{(123456)} \otimes X_3^{(124378596)}$
   Defects 2 & 3 — $k = 6$, Rule 1: Factor out transposition (12) $\rightarrow$

3. $[\sum_{a \in I} a_1 \boxtimes a_2^* \boxtimes \cdots] \otimes X_1^{(78)} \otimes X_2^{(13456)} \otimes X_3^{(24378596)}$
   Defects 2 & 3 — $k = 4$, Rule 1: Factor out transposition (34) $\rightarrow$

4. $[\sum_{a \in I} a_1 \boxtimes a_2^* \boxtimes \cdots] \otimes [\sum_{a \in I} \cdots \boxtimes a_3 \boxtimes a_4^* \boxtimes \cdots] \otimes X_1^{(78)} \otimes X_2^{(1356)} \otimes X_3^{(2378596)}$
   Defects 2 & 3 — $k = 3$, Rule 1: Factor out transposition (35) $\rightarrow$

5. $[\sum_{a \in I} a_1 \boxtimes a_2^* \boxtimes \cdots] \otimes [\sum_{a \in I} \cdots \boxtimes a_3 \boxtimes a_4^* \boxtimes \cdots] \otimes [\sum_{a \in I} \cdots \boxtimes a_3 \boxtimes 1_4 \boxtimes a_5^* \boxtimes \cdots] \otimes$
   $X_1^{(78)} \otimes X_2^{(136)} \otimes X_3^{(2596)(378)}$
   Rule 3: Treat disjoint cycles as separate defects $\rightarrow$

6. $[\sum_{a\in I} a_1 \boxtimes a_2^* \boxtimes \cdots] \otimes [\sum_{a\in I} \cdots \boxtimes a_3 \boxtimes a_4^* \boxtimes \cdots] \otimes [\sum_{a\in I} \cdots \boxtimes a_3 \boxtimes 1_4 \boxtimes a_5^* \boxtimes \cdots] \otimes X_1^{(78)} \otimes X_2^{(136)} \otimes X_3^{(2596)} \otimes X_4^{(378)}$
   Commute disjoint defects $\rightarrow$

7. $[\sum_{a\in I} a_1 \boxtimes a_2^* \boxtimes \cdots] \otimes [\sum_{a\in I} \cdots \boxtimes a_3 \boxtimes a_4^* \boxtimes \cdots] \otimes [\sum_{a\in I} \cdots \boxtimes a_3 \boxtimes 1_4 \boxtimes a_5^* \boxtimes \cdots] \otimes X_2^{(136)} \otimes X_1^{(78)} \otimes X_4^{(378)} \otimes X_3^{(2596)}$
   Defects 1 & 4 — $k = 2$, Rule 1: Factor out transposition (78) $\rightarrow$

8. $[\sum_{a\in I} a_1 \boxtimes a_2^* \boxtimes \cdots] \otimes [\sum_{a\in I} \cdots \boxtimes a_3 \boxtimes a_4^* \boxtimes \cdots] \otimes [\sum_{a\in I} \cdots \boxtimes a_3 \boxtimes 1_4 \boxtimes a_5^* \boxtimes \cdots] \otimes [\sum_{a\in I} \cdots \boxtimes a_7 \boxtimes a_8^* \boxtimes \cdots] \otimes X_2^{(136)} \otimes X_4^{(38)} \otimes X_3^{(2596)}$
   Defects 2 & 4 — $k = 1$, Rule 2: Multiply permutations $\rightarrow$

9. $[\sum_{a\in I} a_1 \boxtimes a_2^* \boxtimes \cdots] \otimes [\sum_{a\in I} \cdots \boxtimes a_3 \boxtimes a_4^* \boxtimes \cdots] \otimes [\sum_{a\in I} \cdots \boxtimes a_3 \boxtimes 1_4 \boxtimes a_5^* \boxtimes \cdots] \otimes [\sum_{a\in I} \cdots \boxtimes a_7 \boxtimes a_8^* \boxtimes \cdots] \otimes X_2^{(1386)} \otimes X_3^{(2596)}$
   Defects 2 & 3 — $k = 1$, Rule 2: Multiply permutations $\rightarrow$

10. $[\sum_{a\in I} a_1 \boxtimes a_2^* \boxtimes \cdots] \otimes [\sum_{a\in I} \cdots \boxtimes a_3 \boxtimes a_4^* \boxtimes \cdots] \otimes [\sum_{a\in I} \cdots \boxtimes a_3 \boxtimes 1_4 \boxtimes a_5^* \boxtimes \cdots] \otimes [\sum_{a\in I} \cdots \boxtimes a_7 \boxtimes a_8^* \boxtimes \cdots] \otimes X_2^{(1386259)}$