Research Report

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Let G and H be groups. The direct product of these groups is represented by the set $G \times H$, which is the Cartesian product of the sets of G and H. Elements in $G \times H$ are pairs of elements, $(g, h), g \in G, h \in G$. Multiplication between pairs is given by the multiplication within respective groups. For example: $(g_1, h_1) \cdot (g_2, h_2) = (g_1 \cdot g_2, h_1 \cdot h_2)$. The identity of a product of groups is also given by the pair of each groups respective identity: (e_g, e_h). And inverses by each groups respective inverse: $(g, h)^{-1} = (g^{-1}, h^{-1})$. The order of a direct product of groups is the product of the order of the two groups: $|G \times H| = |G| \times |H|$.

A cyclic group is a group that can be generated by a single element. The presentation on generators and relations for a cyclic group G would look like: $\langle g \mid g^N = e \rangle$ where $N = |G|$. One example of a cyclic group is the modulo group $\mathbb{Z}/_{N\mathbb{Z}}$. In fact, every cyclic group is isomorphic to the modulo group of the same order.

Fundamental theorem of finite abelian groups: If A is an abelian group, then $A \cong \mathbb{Z}/_{N_1\mathbb{Z}} \times \mathbb{Z}/_{N_2\mathbb{Z}} \times ... \times \mathbb{Z}/_{N_m\mathbb{Z}}$ for some m, and some positive $N_1, N_2, ..., N_m \leq |A|$. This means that any abelian group A, (besides those with a prime order) is isomorphic to the direct product of cyclic groups with orders corresponding to factors of $|A|$. For example: An abelian group A, with $|A| = 8$ means that $A \cong \mathbb{Z}/_{N_8\mathbb{Z}} \wedge A \cong \mathbb{Z}/_{N_2\mathbb{Z}} \times \mathbb{Z}/_{N_4\mathbb{Z}} \wedge A \cong \mathbb{Z}/_{N_2\mathbb{Z}} \times \mathbb{Z}/_{N_2\mathbb{Z}} \times \mathbb{Z}/_{N_2\mathbb{Z}}$

Deligne (direct) product of fusion rings: Given fusion rings F and $\tilde{F}$ with bases $I = \{1, a, b, c, ...\}$ and $\tilde{I} = \{\tilde{1}, \tilde{a}, \tilde{b}, \tilde{c}, ...\}$, and fusion rules $a \times b = \sum_{c \in I} N_c^{ab} c$ and $\tilde{a} \times \tilde{b} = \sum_{\tilde{c} \in \tilde{I}} \tilde{N}_{\tilde{c}}^{\tilde{a}\tilde{b}} \tilde{c}$ respectively, there exists a fusion ring $F \boxtimes \tilde{F}$ with basis $I \times \tilde{I}$. Its fusion rules are given by $(a \boxtimes \tilde{a}) \times (b \boxtimes \tilde{b}) = (a \times b) \boxtimes (\tilde{a} \times \tilde{b}) = \sum_{c, \tilde{c}} N_c^{ab} c \boxtimes N_{\tilde{c}}^{\tilde{a}\tilde{b}} \tilde{c}$.

Example: Fib $\boxtimes$ Ising

Fibonacci has basis elements $\{1, \tau\}$, Ising has basis elements $\{1, \sigma, \psi\}$. Their Deligne product has basis elements: $\{1 \boxtimes 1, 1 \boxtimes \sigma, 1 \boxtimes \psi, \tau \boxtimes 1, \tau \boxtimes \sigma, \tau \boxtimes \psi\}$. It is apparent here that the order of the Deligne product of two fusion rings follows the same rule as the direct product of groups, as $|\text{Fib} \boxtimes \text{Ising}| = |\text{Fib}| \cdot |\text{Ising}|$. Additionally, multiplication in the new fusion ring follows the same pattern as direct products:

$$
\begin{aligned}
&(\tau \boxtimes \sigma) \times (\tau \boxtimes \sigma) \\
&\quad = (\tau \times \tau) \boxtimes (\sigma \times \sigma) \\
&\quad = (1 + \tau) \boxtimes (1 + \psi) \\
&\quad = 1 \boxtimes 1 + \tau \boxtimes 1 + 1 \boxtimes \psi + \tau \boxtimes \psi
\end{aligned}
$$

Group actions on sets: Let G be a group and X be a set, a group action of G on X is a map $\sigma_g : G \times X \longrightarrow X$, $(g, x) \mapsto g \cdot x$ such that

1. $g_1 \cdot (g_2 \cdot x) = (g_1 g_2) \cdot x \ \forall x \in X, g_1, g_2 \in G$
2. $e \cdot x = x \ \forall x \in X$

For every $g \in G$, the mapping σ_g is a permutation on the elements of X. Also, the map $G \longrightarrow S_x,\ g \mapsto \sigma_g$ is a homomorphism. For example, take S_2 acting on a bilayer Fibonacci anyon: $X = \{1\boxtimes 1, 1\boxtimes\tau, \tau\boxtimes 1, \tau\boxtimes\tau\}$. A group action, or a symmetry of S_2 on X could be $(12)\cdot 1\boxtimes\tau = \tau\boxtimes 1$.

The permutation defect fusion ring: Permutation defects X^σ in the context of a multilayer anyon theory derive from an n-fold Deligne product of fusion rings: $F^{\boxtimes n}$ (equivalent to $F \boxtimes F \boxtimes ... \boxtimes F$ n times). There are n spatially separated layers of anyons with fusion rules given by F. A permutation defect "couples" these layers together, and can also teleport anyons between layers, this is where the permutations arise. The multilayer anyons and the defects together form a larger fusion ring of the form $(F^{\boxtimes n})^X_{S_n}$

Example: Three layers and the Fibonacci fusion ring

Fibonacci has a basis, $I = \{1, \tau\}$ and the fusion rule $\tau \times \tau = 1 + \tau$. Given three decoupled layers of these anyons, there exists a trilayer anyon fusion ring (the Deligne product) $F^{\boxtimes 3}$, with basis:

$$I^{\boxtimes 3} = \{1 \boxtimes 1 \boxtimes 1,\ 1 \boxtimes 1 \boxtimes \tau,\ 1 \boxtimes \tau \boxtimes 1,\ 1 \boxtimes \tau \boxtimes \tau,\ \tau \boxtimes 1 \boxtimes 1,\ \tau \boxtimes 1 \boxtimes \tau,\ \tau \boxtimes \tau \boxtimes 1,\ \tau \boxtimes \tau \boxtimes \tau\}$$

These anyons also represent trivial permutation defects, since they are all fixed under the identity mapping. For example: $1 \boxtimes 1 \boxtimes 1 \cong X^{\mathrm{id}}_{1\boxtimes 1\boxtimes 1}$.

$F^{\boxtimes 3}$ lives inside the trilayer permutation defect fusion ring: $(F^{\boxtimes 3})^X_{S_3}$. More specifically, the permutation defect fusion ring is divided into sectors, one for each permutation in S_3, and $F^{\boxtimes 3}$ holds the permutation defects for the identity. The remaining sectors would look like this (with the box-times notation suppressed in the subscript):

$$I_{(12)} = \{X^{(12)}_{111}, X^{(12)}_{11\tau}, X^{(12)}_{\tau\tau 1}, X^{(12)}_{\tau\tau\tau}\}$$

$$I_{(13)} = \{X^{(13)}_{111}, X^{(13)}_{1\tau 1}, X^{(13)}_{\tau 1\tau}, X^{(13)}_{\tau\tau\tau}\}$$

$$I_{(23)} = \{X^{(23)}_{111}, X^{(23)}_{\tau 11}, X^{(23)}_{1\tau\tau}, X^{(23)}_{\tau\tau\tau}\}$$

$$I_{(123)} = \{X^{(123)}_{111}, X^{(123)}_{\tau\tau\tau}\}$$

$$I_{(132)} = \{X^{(132)}_{111}, X^{(132)}_{\tau\tau\tau}\}$$

The key to these permutation defects is that the anyons are fixed under the "group action" applied to them by S_3. This means that every sector only has anyons which are unchanged or **fixed** after the given permutation is applied to them.

Fusion of anyons and permutation defects: The fusion of anyons and permutation defects holds onto the idea that a permutation defect needs is a multi-layer anyon that is fixed under the permutation. It is best visualized through an example.

Example: Three layers and the Fibonacci fusion ring

This fusion ring has already been shown, so the product of an anyon and permutation defect should result in one (or the sum of multiples) of the permutation defects defined previously in $(F^{\boxtimes 3})^X_{S_3}$.

$\tau \boxtimes 1 \boxtimes 1 \;\otimes\; X^{(13)}_{111} = X^{(13)}_{\tau\otimes 1,1,\tau\otimes 1} = X^{(13)}_{\tau 1\tau}$

Here, τ has been "smeared" across the layers affected by the permutation, creating a fixed point under (13) with the given trilayer anyon.

$\tau \boxtimes 1 \boxtimes \tau \;\otimes\; X^{(13)}_{111} = X^{(13)}_{\tau\otimes\tau,1,\tau\otimes\tau} = X^{(13)}_{1+\tau,1,1+\tau} = X^{(13)}_{111} \oplus X^{(13)}_{\tau 1\tau}$

Here, the fusion rule $\tau \times \tau = 1 + \tau$ comes into play. The result is the superposition for the possible values of $\tau \times \tau$. There are other possible combinations such as $X^{(13)}_{11\tau}$ and $X^{(13)}_{\tau 11}$, but since they are not fixed under (13) they do not count as valid permutation defects.

$\tau \boxtimes 1 \boxtimes 1 \;\otimes\; X^{(13)}_{\tau 11} = X^{(13)}_{\tau\otimes\tau,1,\tau\otimes 1} = X^{(13)}_{1+\tau,1,\tau} = X^{(13)}_{\tau 1\tau}$

Here, the defect is no longer a **bare** permutation defect. The smearing process is the same and there is only one fixed outcome.