Research Report

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June 23, 2025

An informal definition of a fusion ring: Let $(R, \times, +)$ be a unital ring (has a multiplicative identity), let $I \subset R$ be a finite subset of R called the basis. What this means is that every element of R can be written as a $\mathbb{Z}$-linear combination of the basis elements in I. Say $I = \{a, b, c, d\}$, $x \in R$ can be written as $x = n_a a + n_b b + ... + n_z z$ where all of the coefficients are integers.

$(R, \times, +)$ becomes a fusion ring if $a \times b = \sum_{c \in I} N_c^{ab} c$. The collection of all of those expressions are called the fusion rules, they explain everything about the ring. If you know how to multiply the basis elements of the ring, you can multiply any arbitrary elements in the ring. The term N_c^{ab} is called the "fusion coefficient" or the "structure constant of the fusion ring." The fusion coefficient and the basis element together ($N_c^{ab} c$) are called the "fusion channel of a and b." There also must exist an involution on I, $* : I \to I$ such that $1* = 1$ and $\forall a \in I,\ N_1^{ab} = \begin{cases} 1 & \text{if } b = a* \\ 0 & \text{else} \end{cases}$ This means that in the product of a and b, 1 will show up only if a is the dual of b.

An involution is a function that is its own inverse, applying an involution twice will always produce the original value inputted. The value outputted by an involution is called the "dual" of the value inputted. Given a group, there are always two inherent involutions, one that maps an element to its inverse, and the identity involution that maps an element to itself.

Example: **Ising fusion ring** has a basis set $I = \{1, \sigma, \psi\}$

Fusion Rules:
$\sigma \times \psi = \sigma$
$\psi \times \sigma = \sigma$
$\psi \times \psi = 1$
$\sigma \times \sigma = 1 + \psi$

Fusion coefficients:
$N_1^{\sigma\psi} = 0,\ N_\sigma^{\sigma\psi} = 1,\ N_\psi^{\sigma\psi} = 0$
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Fusion table:

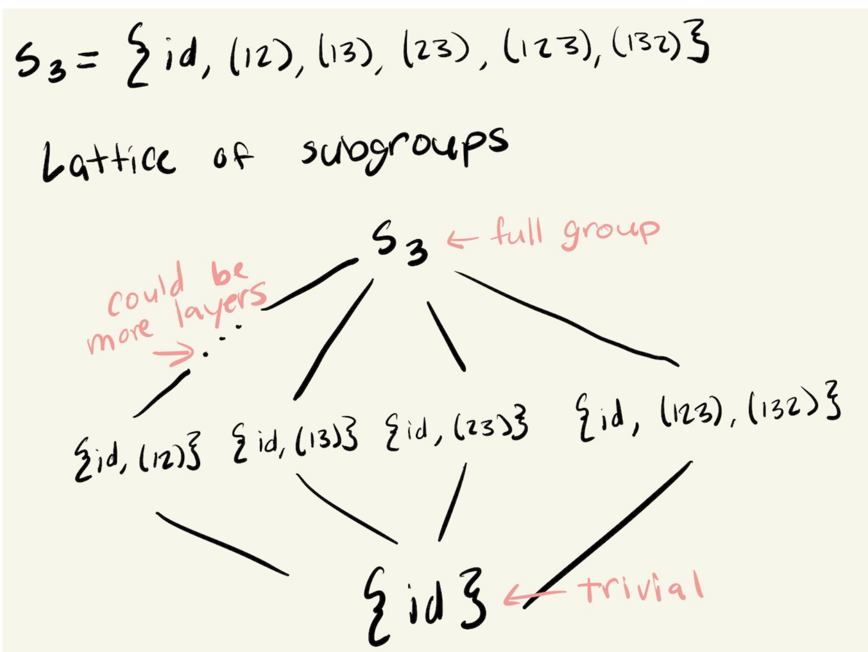
$\times$	1	σ	ψ
1	1	σ	ψ
σ	σ	$1+\psi$	σ
ψ	ψ	σ	1

Every finite group can lead to a fusion ring. Let G be a finite group, you can define a "group ring" of G over any field k, denoted by $k[G]$. this would look like $(k[G], \times, +)$ where the multiplication is induced by multiplication in G, and the sum is a formal sum, (more like a combination of terms rather than a traditional sum that can be evaluated/simplified). Essentially the group becomes the basis elements, and the ring is represented by any linear combination of these elements.

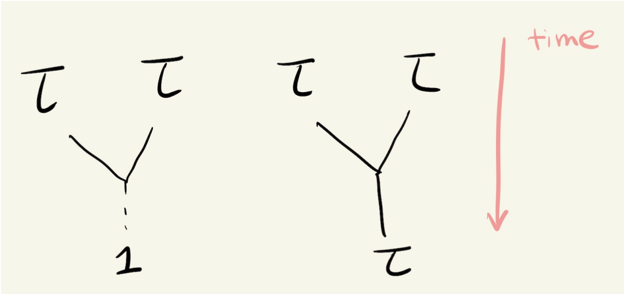
Cayley's theorem states that every finite group is isomorphic to a subgroup of a symmetric group. Given a group G with $|G| = n$, there exists an injective group homomorphism from $G \to S_n$.

If $(G, \times)$ is a group, a subset H of G is a subgroup if H is also a group. This means that while it takes its elements from G, it must also satisfy the axioms of a group: H must be closed, contain an identity, contain inverses, and be associative. A subgroup H is written like $H \leq G$. Proper subgroups are the set of subgroups that do not include $H = G$, these would be written like $H < G$.

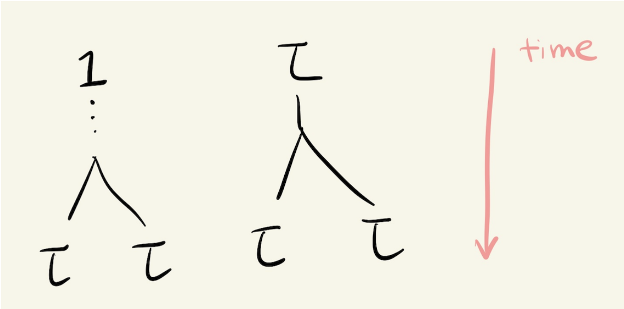
A graph that shows the relationship between groups and subgroups is called a lattice of subgroups of a group. This lattice starts at the full group, then goes down in layers with each layer being a subgroup of the layer above, until you reach the trivial group.



The "fusion" of anyons are modeled by a commutative fusion ring. Every anyon is represented by a basis element in the fusion ring, and their fusion properties come from the fusion rules. In the Fibonacci fusion ring, the identity 1 represents the vacuum anyon, and τ represents the Fibonacci anyon. The rule $\tau \times \tau = 1 + \tau$ can be depicted with "fusion trees", this is another case where the plus doesn't necessarily represent the actual sum, but instead the possible states after fusing τ with τ:



Another way of depicting this is with "splitting" trees:



Considering the fact that there are multiple possible results for the fusion of anyons, there must exist multiple trees given a set of initial anyons, an ordering for the fusion of these anyons (the shape of the tree), and a final value (called the specified total charge or a reference anyon, essentially what all of the anyons fuse down to). Each of these possible trees is a state, and the general state of the entire system is a $\mathbb{C}$-linear combination, or a superposition, of these states. More broadly, for a set of n anyons, there is a vector space of quantum states associated to the anyons, the basis of this space is given by admissably-labeled fusion trees connecting those n anyons and the specified total charge.