Research Report

Tayden White

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Enumerating all of the cases for the fusion products of bare defects for $n = 4$. Let α and β be the respective permutations describing each permutation defect in a given product. Let i, j, k, l each describe a different layer of the 4-layer defects. There should be $23 \times 23 = 529$ cases describing the different products that can be made up from these cycles:

2-cycles: $(12), (13), (14), (23), (24), (34)$

3-cycles: $(123), (132), (124), (142), (134), (143), (234), (243)$

4-cycles: $(1234), (1243), (1324), (1342), (1423), (1432)$

2 2-cycles: $(12)(34), (13)(24), (14)(23)$

Cases:

1. $2 \otimes 2$ – 36 cases
   (a) $k = 0$ (6 cases): $X^{(ij)} \otimes X^{(kl)} = X^{(ij)(kl)}$
   (b) $k = 1$ (24 cases): $X^{(ij)} \otimes X^{(jk)} = X^{(ijk)}$
   (c) $k = 2$ (6 cases): $X^{(ij)} \otimes X^{(ij)} = \sum_{a \in I} a_i \boxtimes a_j^* \boxtimes 1_k \boxtimes 1_l$
2. $2 \otimes 3$ – 48 cases
   (a) $k = 1$ (24 cases): $X^{(ij)} \otimes X^{(jkl)} = X^{(ijkl)}$
   (b) $k = 2$ (24 cases): $X^{(ij)} \otimes X^{(ijk)} = [\sum_{a \in I} a_i \boxtimes a_j^* \boxtimes 1_k \boxtimes 1_l] \otimes X^{(jk)}$
3. $2 \otimes 4$ – 36 cases
   (a) Elements in α are adjacent in β (24 cases):
   $X^{(ij)} \otimes X^{(ijkl)} = [\sum_{a \in I} a_i \boxtimes a_j^* \boxtimes 1_k \boxtimes 1_l] \otimes X^{(jkl)}$
   (b) Elements in α are non-adjacent in β (12 cases):
   $X^{(ij)} \otimes X^{(ikjl)} = [\sum_{a \in I} a_i \boxtimes a_j^* \boxtimes 1_k \boxtimes 1_l] \otimes X^{(ik)(jl)}$
4. $2 \otimes 22$ – 18 cases
   (a) Elements in α are in the same transposition in β (6 cases):
   $X^{(ij)} \otimes X^{(ij)(kl)} = [\sum_{a \in I} a_i \boxtimes a_j^* \boxtimes 1_k \boxtimes 1_l] \otimes X^{(kl)}$
   (b) Elements in α are in different transpositions in β (12 cases):
   $X^{(ij)} \otimes X^{(ik)(jl)} = X^{(ikjl)}$

5. $3 \otimes 2$ – 48 cases
    (a) $k = 1$ (24 cases): $X^{(ijk)} \otimes X^{(kl)} = X^{(ijkl)}$
    (b) $k = 2$ (24 cases): $X^{(ijk)} \otimes X^{(ij)} = [\sum_{a \in I} a_i \boxtimes a_j^* \boxtimes 1_k \boxtimes 1_l] \otimes X^{(ik)}$
6. $3 \otimes 3$ – 64 cases
    (a) $k = 2$, overlapping supports are in the same order (24 cases):
    $X^{(ijk)} \otimes X^{(ijl)} = [\sum_{a \in I} a_i \boxtimes a_i^* \boxtimes 1_j \boxtimes 1_k] \otimes X^{(ik)(jl)}$
    (b) $k = 2$, overlapping supports are in a different order (24 cases):
    $X^{(ijk)} \otimes X^{(jil)} = [\sum_{a \in I} a_i \boxtimes a_j^* \boxtimes 1_k \boxtimes 1_l] \otimes X^{(ilk)}$
    (c) $k = 3$, α is the same as β (8 cases):
    $X^{(ijk)} \otimes X^{(ijk)} = [\sum_{a \in I} a_i \boxtimes a_j^* \boxtimes 1_k \boxtimes 1_l] \otimes X^{(ikj)}$
    (d) $k = 3$, α is the inverse of β (8 cases):
    $X^{(ijk)} \otimes X^{(ikj)} = [\sum_{a \in I} a_i \boxtimes a_j^* \boxtimes 1_k \boxtimes 1_l] \otimes [\sum_{a \in I} a_i \boxtimes 1_j \boxtimes a_k^* \boxtimes 1_l]$
7. $3 \otimes 4$: Let ψ be the 3-cycle made by excluding the non-overlapping element in β – 48 cases:
    (a) α is the same as ψ (24 cases):
    $X^{(ijk)} \otimes X^{(ijkl)} = [\sum_{a \in I} a_i \boxtimes a_j^* \boxtimes 1_k \boxtimes 1_l] \otimes X^{(iklj)}$
    (b) α is the inverse of ψ (24 cases):
    $X^{(ijk)} \otimes X^{(ikjl)} = [\sum_{a \in I} a_i \boxtimes a_j^* \boxtimes 1_k \boxtimes 1_l] \otimes [\sum_{a \in I} a_i \boxtimes 1_j \boxtimes a_k^* \boxtimes 1_l] \otimes X^{(jl)}$
8. $3 \otimes 22$ – 24 cases
    (a) $X^{(ijk)} \otimes X^{(ij)(kl)} = [\sum_{a \in I} a_i \boxtimes a_j^* \boxtimes 1_k \boxtimes 1_l] \otimes X^{(ikl)}$
9. $4 \otimes 2$ – 36 cases
    (a) Elements in β are adjacent in α (24 cases):
    $X^{(ijkl)} \otimes X^{(ij)} = [\sum_{a \in I} a_i \boxtimes a_j^* \boxtimes 1_k \boxtimes 1_l] \otimes X^{(ikl)}$
    (b) Elements in β are non-adjacent in α (12 cases):
    $X^{(ijkl)} \otimes X^{(ik)} = [\sum_{a \in I} a_i \boxtimes 1_j \boxtimes a_k^* \boxtimes 1_l] \otimes X^{(il)(jk)}$
10. $4 \otimes 3$: Let ψ be the 3-cycle made by excluding the non-overlapping element in α – 48 cases:
    (a) α is the same as ψ (24 cases):
    $X^{(ijkl)} \otimes X^{(ijk)} = [\sum_{a \in I} a_i \boxtimes a_j^* \boxtimes 1_k \boxtimes 1_l] \otimes X^{(ikjl)}$
    (b) α is the inverse of ψ (24 cases):
    $X^{(ijkl)} \otimes X^{(ikj)} = [\sum_{a \in I} a_i \boxtimes a_j^* \boxtimes 1_k \boxtimes 1_l] \otimes [\sum_{a \in I} a_i \boxtimes 1_j \boxtimes a_k^* \boxtimes 1_l] \otimes X^{(il)}$
11. $4 \otimes 4$ – 36 cases
    (a) α is the same as β (6 cases):
    $X^{(ijkl)} \otimes X^{(ijkl)} = [\sum_{a \in I} a_i \boxtimes a_j^* \boxtimes 1_k \boxtimes 1_l] \otimes [\sum_{a \in I} 1_i \boxtimes 1_j \boxtimes a_k \boxtimes a_l^*] \otimes X^{(ik)(jl)}$
    (b) α is the inverse of β (6 cases):
    $X^{(ijkl)} \otimes X^{(ilkj)} = [\sum_{a \in I} a_i \boxtimes a_j^* \boxtimes 1_k \boxtimes 1_l] \otimes [\sum_{a \in I} a_i \boxtimes 1_j \boxtimes a_k^* \boxtimes 1_l] \otimes [\sum_{a \in I} 1_i \boxtimes 1_j \boxtimes a_k \boxtimes a_l^*]$
    (c) Otherwise, one pair of same ordered adjacent supports will exist (24 cases):
    $X^{(ijkl)} \otimes X^{(ijlk)} = [\sum_{a \in I} a_i \boxtimes a_j^* \boxtimes 1_k \boxtimes 1_l] \otimes [\sum_{a \in I} 1_i \boxtimes 1_j \boxtimes a_k \boxtimes a_l^*] \otimes X^{(ikj)}$

12. $4 \otimes 22$ – 18 cases
    (a) Transpositions in β are adjacent in α (12 cases):
    $X^{(ijkl)} \otimes X^{(ij)(kl)} = [\sum_{a\in I} a_i \boxtimes a_j^* \boxtimes 1_k \boxtimes 1_l] \otimes [\sum_{a\in I} 1_i \boxtimes 1_j \boxtimes a_k \boxtimes a_l^*] \otimes X^{(ik)}$
    (b) Transpositions in β are non-adjacent in α (6 cases):
    $X^{(ijkl)} \otimes X^{(ik)(jl)} = [\sum_{a\in I} a_i \boxtimes 1_j \boxtimes a_k^* \boxtimes 1_l] \otimes X^{(ilkj)}$

13. $22 \otimes 2$ – 18 cases
    (a) Elements in β are in the same transposition in α (6 cases):
    $X^{(ij)(kl)} \otimes X^{(ij)} = [\sum_{a\in I} a_i \boxtimes a_j^* \boxtimes 1_k \boxtimes 1_l] \otimes X^{(kl)}$
    (b) Elements in β are in different transpositions in α (12 cases):
    $X^{(ij)(kl)} \otimes X^{(ik)} = X^{(ilkj)}$

14. $22 \otimes 3$ – 24 cases
    (a) $X^{(ij)(kl)} \otimes X^{(ijk)} = [\sum_{a\in I} a_i \boxtimes a_j^* \boxtimes 1_k \boxtimes 1_l] \otimes X^{(jlk)}$

15. $22 \otimes 4$ – 18 cases
    (a) Transpositions in α are adjacent in β (12 cases):
    $X^{(ij)(kl)} \otimes X^{(ijkl)} = [\sum_{a\in I} a_i \boxtimes a_j^* \boxtimes 1_k \boxtimes 1_l] \otimes [\sum_{a\in I} 1_i \boxtimes 1_j \boxtimes a_k \boxtimes a_l^*] \otimes X^{(jl)}$
    (b) Transpositions in α are non-adjacent in β (6 cases):
    $X^{(ij)(kl)} \otimes X^{(ikjl)} = [\sum_{a\in I} a_i \boxtimes a_j^* \boxtimes 1_k \boxtimes 1_l] \otimes X^{(iljk)}$

16. $22 \otimes 22$ – 9 cases
    (a) $\alpha = \beta$ (3 cases): $X^{(ij)(kl)} \otimes X^{(ij)(kl)} = [\sum_{a\in I} a_i \boxtimes a_j^* \boxtimes 1_k \boxtimes 1_l] \otimes [\sum_{a\in I} 1_i \boxtimes 1_j \boxtimes a_k \boxtimes a_l^*]$
    (b) $\alpha \neq \beta$ (6 cases): $X^{(ij)(kl)} \otimes X^{(ik)(jl)} = [\sum_{a\in I} a_i \boxtimes a_j^* \boxtimes 1_k \boxtimes 1_l] \otimes X^{(il)(jk)}$