Modeling and Simulation

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Level: 2nd SC Class
Date:February 16, 2025

Lecture 4:

Modeling Dynamical Systems

General Introduction to Dynamical Systems

What is a dynamical system?

- A **dynamical system** is any system that varies through time.
- We used to describe such a system by state $x(t)$
- The state x can be multivariate (a vector)

$$x(t) = (x_1(t), ..., x_n(t))^T$$

Some examples of dynamical systems:

- Ecological models (e.g., changes in population size)
- Chemistry (change in number of molecules during chemical reactions)
- Climate science (e.g., change in temperature through time)
- Economics models (e.g., change in stock prices)
- ...

Modeling Dynamical Systems

Types of models for Dynamical Systems

Two classes of description for dynamical systems:

- **Discrete-time dynamical system:** Discrete-time models assume that time proceeds in a step-wise manner.These dynamical systems are described by recurrence relations e.g.:

$$x_t = f(x_{t-1}, t)$$

- **Continuous-time dynamical system:** Continuous-time models assume that time is a real number (technically: "infinitesimally" small). Theses systems are described by differential equations e.g.:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = f(x, t)$$

In either case, x_t or x is the state variable of the system at time t. f is a function that determines the rules by which the system changes its state over time

Modeling Dynamical Systems

Simulating Discrete-Time Systems

- A good starting point to understand discrete-time dynamical models are sequences of numbers that follow certain rules.
- **Example:**

x_0	x_1	x_2	x_3	x_4	x_5
1	-2	4	-8	16	-32

  Can you guess the underlying rule for these sequences (The recursion equation f that describes the relationship between x_t and x_{i-1}) ?

$$x_i = f(x_{i-1})$$

  Solution:

$$x_i = -2x_{i-1}, \text{with } x_0 = 1$$

- The example above belongs to a specific type of Discrete-Time Dynamical systems called **First-order Linear system**.

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Simulating Discrete-Time First-order Systems

- **First-order system:** A difference equation whose rules involve state variables of the immediate past time (at time $t-1$).

Theorem 1.

The general solution of the first-order linear difference equation:

$$x_t = a.x_{t-1} + b, t = 0, 1, 2, 3, ...,$$

is given by:

$$x_t = a^t.x_0 + \begin{cases} \frac{a^n-1}{a-1}.b, \text{if } a \neq 1 \\ n.b, \text{if } a = 1 \end{cases}$$

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Simulating Discrete-Time First-order Systems

Example:

- Now we want to simulate a simple exponential population growth model governed by the following difference equation model:

$$N_t = a.N_{t-1}$$

  Where $a = 1.1$ is the rate of population generation and N_t is the number of individuals in generation at time t. The initial generation $N_0 = 1$.

- Our objective is to find out what kind of behavior this model will show through computer simulation.
- The general solution of the above example is:

$$N_t = a^t.N_0 + \cancelto{0}{\begin{cases} \frac{a^n-1}{a-1}.b, \text{if } a \neq 1 \\ n.b, \text{if } a = 1 \end{cases}} = a^t.N_0$$

Simulating Discrete-Time First-order Systems - Mathematical Solution

We use the `sympy` python library to solve this system (https://www.sympy.org/)

```python
from sympy import Function,rsolve
from sympy.abc import n

#rate of population generation
a=1.1
#Functions Declarations
N = Function('N')
f = N(t+1) - a * N(t)
#Solving the equation
sol = rsolve(f, N(t), {N(0):1})
#Printing the solution
print('N_t = {}'.format(sol))
N_5 = round(sol.subs(t, 5), 2)
print('N(5) = {:,}'.format(N_5))
```

```
N_t = 1.1**t
N(5) = 1.61
```

Modeling Dynamical Systems

Simulating Discrete-Time First-order Systems - Numerical Solution

```python
import matplotlib.pyplot as plt
import numpy as np

N_0 = 1.0 #initial gen.
a   = 1.1 #rate of  pop. gen.
N_t = 59  #Max Time steps
#generate time steps
t = np.linspace(0, (N_t+1), (N_t+2))
#Initialize N with 0
N = np.zeros(N_t+2)
#Set N[0] to N_0
N[0] = N_0
#Exact Solution
E=N_0*np.power(a,t)
#Solve Numerically
for n in range(N_t+1):
    N[n+1] =  N[n] * a
#Plotting(solution)
fig, axs = plt.subplots(2)
axs[0].scatter(x=t, y=N,  c='r', s=1, label='numerical')
axs[0].plot(t,E , 'b',label='exact')
axs[0].legend(loc='upper left')
axs[0].set(xlabel='t', ylabel='N(t)')
axs[1].plot(t,(E-N)**2,label='error')
axs[1].legend(loc='upper left')
axs[1].set(xlabel='t', ylabel='Err(t)')
plt.show()
```

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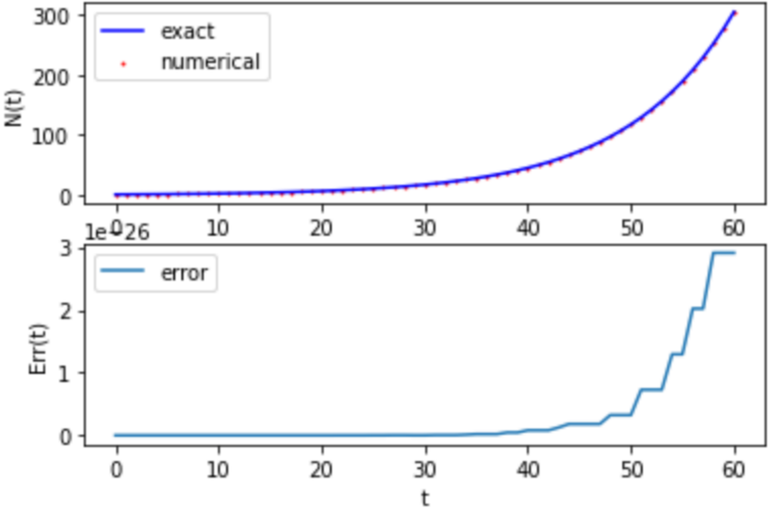
Simulating Discrete-Time First-order Systems - Numerical Solution



Modeling Dynamical Systems

Simulating Continuous Time dynamical Systems

- Continuous Time Dynamical systems are usually described using differential equation models, specifically ordinary differential equations (ODE) and Partially Differential Equations (PDE)
- Probably more mainstream in science and engineering, and studied more extensively, than discrete-time models
- Can be used to model/simulate various natural phenomena (e.g., motion of objects, flow of electric current).
- In this course, we are interested in systems that are modeled using ODE.

Modeling Dynamical Systems

Ordinary differential equations (ODEs)

ODEs are a type of differential equations that have:

- One independent variable, t
- May be several dependent variables, x_i :

$$x_i = \{x_1(t), x_2(t), ...\},$$

- and a set of functions f_i relating x_i and its derivatives,

$$f_i(x_i, \dot{x}_i, \ddot{x}_i, ...; t) = 0$$

where:

$$\dot{x}_i = \frac{\mathrm{d}x_i}{\mathrm{d}t}, \ddot{x}_i = \frac{\mathrm{d}^2 x_i}{\mathrm{d}t^2}, .., etc$$

Features

- Order: refers to the highest derivative: kth order $\Leftrightarrow \frac{\mathrm{d}^k x}{\mathrm{d}t^k}$
- Dimension: refers to the number of dependent variables $x = [x_1 ... x_d]$, and the number of independent equations.
- Autonomous: $\Leftrightarrow f_i$ have no explicit dependence on t.
- Linear: if f_i has only linear dependence on $x_i, \dot{x}_i$,...and their combinations. Otherwise it is non-linear.

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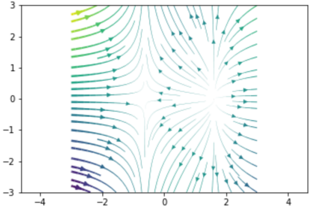
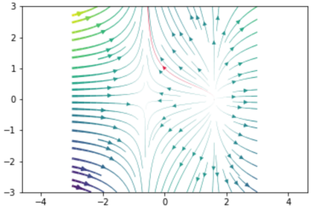
Ordinary differential equations (ODEs)- Example

Consider a system of the following ODE:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = x^2 - x - 1,$$





This system is:

- First-order, as $\frac{\mathrm{d}x}{\mathrm{d}t}$ is the highest derivative.
- One-dimensional, as x is the only dependent variable.
- Autonomous, as the rate of change $\frac{\mathrm{d}x}{\mathrm{d}t}$ does not depend on the independent variable t.
- Non-linear, because of the non-linear term x^2 on the right-hand side.

To find one solution among many, we specify an initial condition, e.g. $x(0) = 1$

Ordinary differential equations (ODEs) - 1D Autonomous Equation

Consider the 1st-order autonomous case:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = f(x),$$

- A solution is typically found by separation of variables.
- Divide by $f(x)$ and integrate.

$$\int \frac{dx}{f(x)} = t + c$$

- Some cases can be solved exactly, e.g,

$$f(x) = x \Rightarrow ln(x) = t + c \Rightarrow x(t) = C_1 e^t$$

- In some cases integral can't be found analytically $\Rightarrow$ Solve Numerically.

Modeling Dynamical Systems

ODEs - 1D Autonomous Equation - Example 1

Consider the simple exponential population growth system modeled by the 1st-order autonomous model:

$$\frac{dx}{dt} = x,$$

- A general mathematical solution for this system can be found using `sympy` python library:

$$x(t) = C_1 e^t$$

- To specify a specific solution, we have to specify an initial condition.

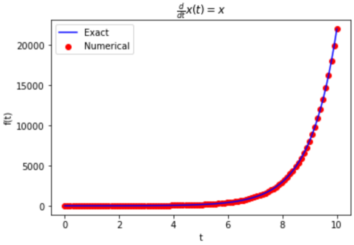
Mathematical Solution for f(x)=x

```
import sympy as sy
sy.init_printing()
# declare math. symbols
 used in Eq
t = sy.symbols('t')
f = sy.Function('x')
# Solve the equation f'(t)
  = x(t)
diffEq = f(t).diff(t) - f(
 t)
sy.dsolve(diffEq, f(t))
```

$$x(t) = C_1 e^t$$

ODEs - 1D Autonomous Equation - Example 1 (Numerical Solution)

```python
import matplotlib.pyplot as plt
from scipy.integrate import odeint
def PopGrowth(x, t):
    #Returns the gradient dx/dt for the exponential equation
    return x
ts = np.linspace(0.0, 10.0, 100) # values of independent variable
x0 = 1 # an initial condition, x(0) = x0
y = odeint(PopGrowth, x0, ts)
# odeint returns an array of x values, at the times in ts.
plt.xlabel('t')
plt.ylabel('f(t)')
plt.scatter(ts,y, label='Numerical', color='r')
plt.legend()
```



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ODEs - 1D Autonomous Equation - Example 2

Consider the logistic population growth system modeled by the 1st-order autonomous model:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = x(1-x),$$

- The general mathematical solution for this system using `sympy` python library is:

$$x(t) = \frac{1}{C_1 e^{-t} + 1}$$

- To specify a specific solution, we have to specify an initial condition.

Mathematical Solution for f(x)=x(1−x)

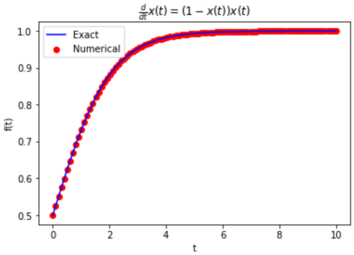
```python
import sympy as sy
sy.init_printing()
# declare math. symbols used in Eq
t = sy.symbols('t')
f = sy.Function('x')
# Solve the equation f'(t) = f(t)(1-f(t)).
diffEq = f(t).diff(t) - f(t)*(1-f(t))
sy.dsolve(diffEq, f(t))
```

$$x(t) = \frac{1}{C_1 e^{-t} + 1}$$

Modeling Dynamical Systems

ODEs - 1D Autonomous Equation - Example 2 (Numerical Solution (1))

```python
import matplotlib.pyplot as plt
from scipy.integrate import odeint
def LogGrowth(x, t):
    #Returns the gradient dx/dt for the Logistic equation
    return x*(1-x)
ts = np.linspace(0.0, 10.0, 100) # values of independent variable
x0 = 0.5 # an initial condition, x(0) = x0
y = odeint(LogGrowth, x0, ts)
# odeint returns an array of x values, at the times in ts.
plt.xlabel('t')
plt.ylabel('f(t)')
plt.scatter(ts,y, label='Numerical', color='r')
plt.legend()
```



- Here $x_0 = 0.5$.

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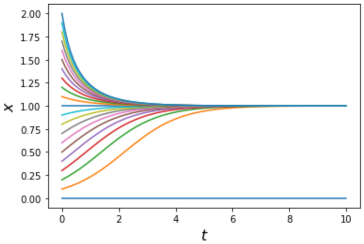
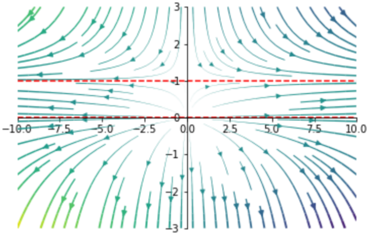
ODEs - 1D Autonomous Equation - Example 2 (Numerical Solution (2))

- Let's plot curves for several initial conditions

```python
# Plot curves for several initial conditions
ics = np.linspace(0.0, 2.0, 21) # a list of initial conditions
for x0 in ics:
    xs = odeint(LogGrowth, x0, ts)
    plt.plot(ts, xs)
plt.xlabel('$t$', fontsize=16);
plt.ylabel('$x$', fontsize=16)
```





- Two equilibrium positions: $x = 0$ and $x = 1$.
- $x = 0$ is an unstable equilibrium.
- $x = 1$ is a stable equilibrium.

Modeling Dynamical Systems

Ordinary differential equations (ODEs) - 2D autonomous equations

Now consider a first order system with two dependent variables, x and y,

$$\begin{cases} \frac{\mathrm{d}x}{\mathrm{d}t} = f(x, y; t), \\ \frac{\mathrm{d}y}{\mathrm{d}t} = g(x, y; t), \end{cases}$$

- System is autonomous if f and g do not depend on t.
- **Example:** Modelling the populations of rabbits and foxes.

ODEs - 2D Autonomous Equations - Example

Consider the Predator-prey equations example which is also known as Lotka-Volterra equations.

- the predator-prey equations are a pair of coupled first-order non-linear ordinary differential equations.
- They represent a simplified model of the change in populations of two species which interact via predation. For example, foxes (predators) and rabbits (prey).
- Let x and y represent rabbit and fox populations, respectively. Then:

$$\begin{cases} \frac{\mathrm{d}x}{\mathrm{d}t} = ax - bxy, \\ \frac{\mathrm{d}y}{\mathrm{d}t} = -cy + dxy, \end{cases}$$

Where a, b, c and d are parameters, which are assumed to be positive.

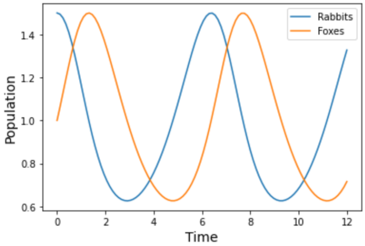
ODEs - 2D Autonomous Equations - Predator-Prey Equations (Numerical Solution

```python
import numpy as np
import matplotlib.pyplot as plt
from scipy.integrate import odeint

def predprey(Z, t, a=1, b=1, c=1, d=1): # a,b,c,d
    optional arguments.
    x, y = Z[0], Z[1]
    dxdt, dydt = x*(a - b*y), -y*(c - d*x)
    return [dxdt, dydt]
ts = np.linspace(0, 12, 100)
Z0 = [1.5, 1.0] # initial conditions for x and y
Sol = odeint(predprey, Z0, ts, args=(1,1,1,1))
# use optional argument 'args' to pass parameters to
    dZ_dt
prey = Sol[:,0] # first column
predators = Sol[:,1] # second column
```

ODEs - 2D Autonomous Equations - Predator-Prey Equations (Numerical Solution

```
#Let's plot 'rabbit' and 'fox' populations as a function
    of time
plt.plot(ts, prey, "+", label="Rabbits")
plt.plot(ts, predators, "x", label="Foxes")
plt.xlabel("Time", fontsize=14)
plt.ylabel("Population", fontsize=14)
plt.legend();
```



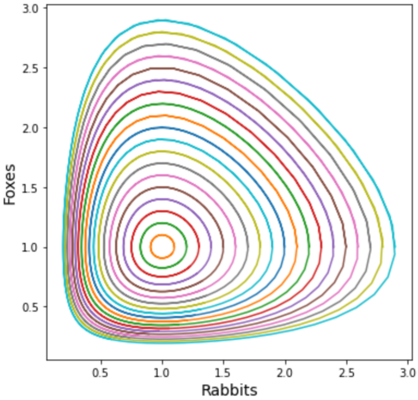
$$\begin{cases} \frac{\mathrm{d}x}{\mathrm{d}t} = ax - bxy, \\ \frac{\mathrm{d}y}{\mathrm{d}t} = -cy + dxy, \end{cases}$$

ODEs - 2D Autonomous Equations - Predator-prey equations: Phase plot

- The ODEs are autonomous: no explicit dependence on t
- Phase portrait: Plot x vs y (instead of x,y vs t).
- One curve for each initial condition
- Curves will not cross (typically) for an autonomous system.

```
fig = plt.figure()
fig.set_size_inches(6,6) # Square plot, 1:1 aspect ratio
ics = np.arange(1.0, 3.0, 0.1) # initial conditions
for r in ics:
    Z0 = [r, 1.0]
    Sol = odeint(predprey, Z0, ts)
    plt.plot(Sol[:,0], Sol[:,1])
plt.xlabel("Rabbits", fontsize=14)
plt.ylabel("Foxes", fontsize=14)
```

ODEs - 2D Autonomous Equations - Predator-prey equations: Phase plot



- Curves do not cross
- Closed curves: Periodic solutions
- Equilibrium at:
- $x = y = 1 \Longrightarrow \dot{x} = \dot{y} = 0$

ODEs - Second-Order Systems - Example: The Van der Pol oscillator

- In dynamics, the Van der Pol oscillator is a non-conservative oscillator with non-linear damping. It evolves in time according to the following ODE:

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} - a\left(1 - x^2(t)\right)\frac{\mathrm{d}x}{\mathrm{d}t} + x(t) = 0$$

  or

$$\ddot{x} - a(1 - x^2)\dot{x} + x = 0$$

- This is a 2nd-order ODE with one parameter, a
  - $|x| > 1$: loses energy
  - $|x| < 1$: absorbs energy
- Originally, used as a model for an electric circuit with a vacuum tube.
- Used to model biological processes such as heart beat, circadian rhythms, biochemical oscillators, and pacemaker neuron

ODEs - Second-Order Systems - Example: The Van der Pol oscillator

$$\ddot{x} - a(1 - x^2)\dot{x} + x = 0$$

First-order reduction:

Any second-order equation can be written as two coupled first-order equations, by introducing a new variable.

- Let $y = \frac{\mathrm{d}x}{\mathrm{d}t}$. Then

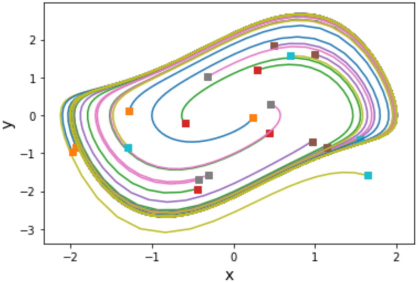
$$\dot{x} = y$$
$$\dot{y} = a(1 - x^2)y - x$$

Modeling Dynamical Systems

ODEs - Second-Order Systems - Example: The Van der Pol oscillator

```
def Van_der_Pol (Z, t, a = 1.0):
    x, y = Z[0], Z[1]
    dxdt = y
    dydt = a*(1-x**2)*y - x
    return [dxdt, dydt]
def random_ic(scalefac=2.0): # stochastic initial condition
    return scalefac*(2.0*np.random.rand(2) - 1.0)

ts = np.linspace(0.0, 40.0, 400)
nlines = 20
for ic in [random_ic() for i in range(nlines)]:
    Zs = odeint(Van_der_Pol, ic, ts)
    plt.plot(Zs[:,0], Zs[:,1])
    plt.plot([Zs[0,0]],[Zs[0,1]], 's') # plot the first point
```



- All curves tend towards a limit cycle

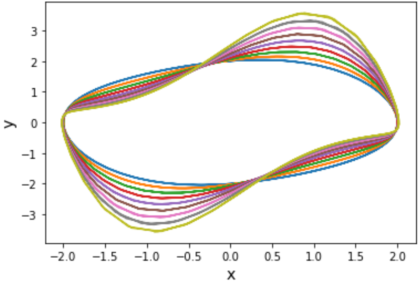
Modeling Dynamical Systems

ODEs - Second-Order Systems - Example: The Van der Pol oscillator

- Investigate how the limit cycle varies with the parameter a:

```
avals = np.arange(0.2, 2.0, 0.2) # parameters
minpt = int(len(ts) / 2) # look at late-time behaviour

for a in avals:
    Zs = odeint(Van_der_Pol, random_ic(), ts, args=(a,))
    plt.plot(Zs[minpt:,0], Zs[minpt:,1])
plt.xlabel("x", fontsize =14)
plt.ylabel("y", fontsize =14)
```



End of Lecture 4:

Modeling Dynamical Systems