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Source: *Management Science*, Vol. 37, No. 5 (May, 1991), pp. 519-531
Published by: INFORMS
Stable URL: http://www.jstor.org/stable/2632458

MANAGEMENT SCIENCE
Vol. 37, No. 5, May 1991
Printed in U.S.A.

MEAN-ABSOLUTE DEVIATION PORTFOLIO OPTIMIZATION MODEL AND ITS APPLICATIONS TO TOKYO STOCK MARKET*

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The purpose of this paper is to demonstrate that a portfolio optimization model using the L_1 risk (mean absolute deviation risk) function can remove most of the difficulties associated with the classical Markowitz's model while maintaining its advantages over equilibrium models. In particular, the L_1 risk model leads to a linear program instead of a quadratic program, so that a large-scale optimization problem consisting of more than 1,000 stocks may be solved on a real time basis. Numerical experiments using the historical data of NIKKEI 225 stocks show that the L_1 risk model generates a portfolio quite similar to that of the Markowitz's model within a fraction of time required to solve the latter.
(PORTFOLIO OPTIMIZATION; L_1 RISK FUNCTION; LINEAR PROGRAMMING; MARKOWITZ'S MODEL; SINGLE-FACTOR MODEL)

1. Introduction

Markowitz's portfolio optimization model, contrary to its theoretical reputation, has not been used extensively in its original form to construct a large-scale portfolio. One of the most significant reasons behind this is the computational difficulty associated with solving a large-scale quadratic programming problem with a dense covariance matrix.

Several authors tried to alleviate this difficulty by using various approximation schemes (Sharpe 1967, 1971, Stone 1973) in the early years of the history. Also, the use of the index model enables one to reduce the amount of computation by introducing the notion of "factors" influencing the stock prices (Perold 1984, Sharpe 1963). Yet, these efforts are largely discounted because of the popularity of equilibrium models such as CAPM and APT which are computationally less demanding.

True that the idea of Markowitz is still being used indirectly because CAPM is based upon this model. But we must not forget that there is a fundamental difference between these two models. In particular, equilibrium models have to impose several unrealistic assumptions to derive a simple relation between the rate of return of individual assets and the market portfolio (see Elton and Gruber 1987, Sharpe 1964 for the details of CAPM). Unfortunately, however, the recent studies of data collected in Tokyo Stock Market show that this relation is very unstable. In fact, calculated "Beta" of stocks behave so erratically that the information provided by CAPM can at best serve as a first order approximation.

These observations motivated the authors' efforts to improve Markowitz's model both computationally and theoretically. In a series of papers (Konno 1988, 1989), we proposed a new portfolio optimization model using piecewise linear risk functions. In Konno (1989), we showed that our model can achieve the intention of Markowitz by solving a linear program instead of a "not so easy" quadratic program. Also, we demonstrated several nice properties of our model through preliminary numerical experiments using the historical data of 50 stocks included in OSAKA 50.

The main purpose of this paper is to demonstrate further that the L_1 risk model, a

* Accepted by W. T. Ziemba; received December 1989. This paper has been with the authors 2 months for 2 revisions.

0025-1909/91/3705/0519$01.25

special case of the piecewise linear risk model, can remove most of the difficulties of Markowitz's model while maintaining its advantages over equilibrium models.

In §2, we will discuss the reasons why Markowitz's original model could not get more popularity among practitioners. In §3, we review some of the more important results of the L_1 risk model. In §4, we will compare the performance of Markowitz's L_2 risk (standard deviation) model, L_1 risk (absolute deviation) model and Sharpe's (1963) single-factor QP model using the historical data of 224 stocks included in NIKKEI 225 index.

2. Review of Markowitz's Model

Let R_j be a random variable representing the rate of return (per period) of the asset $S_j, j = 1, \ldots, n$. Also let x_j be the amount of money to be invested in S_j out of the total fund M_0.

The expected return (per period) of this investment is given by

$$r(x_1, \ldots, x_n) = E\left[\sum_{j=1}^{n} R_j x_j\right] = \sum_{j=1}^{n} E[R_j]x_j \tag{2.1}$$

where $E[\cdot]$ represents the expected value of the random variable in the bracket. An investor prefers to have $r(x_1, \ldots, x_n)$ as large as possible. At the same time, he wants to make the risk as small as possible.

Harry Markowitz, in his seminal work (1959), employed the standard deviation of the (per period) return

$$\sigma(x_1, \ldots, x_n) = \sqrt{E\left[\left\{\sum_{j=1}^{n} R_j x_j - E\left[\sum_{j=1}^{n} R_j x_j\right]\right\}^2\right]} \tag{2.2}$$

as the measure of risk and formulated the portfolio optimization problem as a parametric quadratic programming problem:

$$\left|\begin{array}{ll} \text{minimize} & \displaystyle\sum_{i=1}^{n}\sum_{j=1}^{n} \sigma_{ij} x_i x_j \\ \text{subject to} & \displaystyle\sum_{j=1}^{n} r_j x_j \geq \rho M_0, \\ & \displaystyle\sum_{j=1}^{n} x_j = M_0, \\ & 0 \leq x_j \leq u_j, \qquad j = 1, \ldots, n, \end{array}\right. \tag{2.3}$$

where $r_j = E[R_j]$ and $\sigma_{ij} = E[(R_i - r_i)(R_j - r_j)]$ and ρ is a parameter representing the minimal rate of return required by an investor. Also, u_j is the maximum amount of money which can be invested into S_j.

This model is known to be valid if (i) R_j's are multivariate normally distributed and/or (ii) an investor is risk averse in the sense that he prefers less standard deviation of the portfolio to more. Also this model is now widely recognized as the starting point of the modern portfolio theory. In fact, many important theories in financial economics are based upon this model. It seems, however, that Markowitz's model itself was not used extensively by practitioners as a tool for optimizing a large-scale portfolio. According to a fund manager of a leading security company of Japan, the problems containing more than 200 variables are rarely solved in practice. Let us list some of the reasons why it is so.

(a) *Computational burden.* To build a model, we have to calculate $n(n+1)/2$ constants σ_{ij}'s through historical data or through some future projection.[1] We would not be surprised if practitioners felt that this computation is quite tedious. Furthermore, solving a large-scale dense quadratic programming problem (2.3) where almost all σ_{ij}'s are nonzero is very difficult if n is over, say 500. This computational difficulty can be substantially alleviated through the use of the factor (index) models (Sharpe 1963, Perold 1984) and sparse matrix techniques (Pang 1980, Perold 1984), but it is still not easy to obtain an optimal solution of a large-scale quadratic programming problems on a real time basis.

(b) *Investors' perception against risk and distribution of stock prices.* Many practitioners were not fully convinced of the validity of the standard deviation as a measure of risk (Kroll et al. 1984). They are certainly unhappy to have small or negative profit, but they usually feel happy to have larger profit. This means that the investors' perception against risk is not symmetric around the mean. Unfortunately, however, recent studies of stock prices in Tokyo Stock Market (Kariya et al. 1989) revealed that most of R_j's are not normally nor even symmetrically distributed. Thus we need to consider the third moment of the distribution in addition to the first and second (i.e. mean and variance). In other words, Markowitz's model should be viewed as an approximation to the more complicated optimization problem facing an investor.

(c) *Transaction/Management cost and cut-off effect.* An optimal solution $x^* = (x_1^*, \ldots, x_n^*)$ of a large-scale quadratic programming problem (2.3) usually contains many nonzero elements. In fact, at least 100–200 components of x^* are expected to be positive when n is over 1000. This means that an investor has to purchase many different stocks, most of which by a fraction of one percent of the total fund. This is very inconvenient in practice, since we have to pay significant amount of transaction costs to buy many different stocks by a small amount. Also, we may not be able to purchase small amounts of stock below minimum transaction units. Thus we have to round the numbers to the integer multiples of this minimal unit or else we have to solve an integer quadratic programming problem which is intractable if n is larger than, say, 20. Moreover, managing a portfolio consisting of over 100 stocks is tedious if not impossible. Thus we are forced to eliminate stocks with smaller weight to get around this difficulty. But this cut-off process may distort the portfolio to an extent that the resulting standard deviation is considerably larger than the one obtained through an exact model.

Perold (1984) proposed an approximation scheme which can partially take care of the transaction costs and the constraints associated with minimal transaction units. Also, he demonstrated that this scheme is very effective for the model consisting of up to 500 stocks. The success of his scheme owes critically to the data reduction due to the employment of a "factor" model. It should be pointed out, however, that the factor model need not lead to a portfolio close enough to the one derived through the Markowitz's model as will be demonstrated in §4.

In summary, large-scale portfolio optimization using Markowitz's (full covariance matrix) model has been considered impractical not only because of its computational difficulty but also because of the complications associated with the implementation of its solution.

3. L_1 Risk Model

In Konno (1988), we introduced the L_1 risk (absolute deviation) function

$$w(x) = E\left[\left|\sum_{j=1}^{n} R_j x_j - E\left[\sum_{j=1}^{n} R_j x_j\right]\right|\right]$$

[1] Let us note that Markowitz did not (1959, pp. 96–100) and does not (1990) suggest the use of historical covariance matrix to estimate the covariance structure of the portfolio. Instead, he recommends the use of factor models such as the ones employed by Sharpe (1963) and Perold (1984).

instead of the L_2 risk (standard deviation) function of the (per period) earning out of the portfolio. These two measures are essentially the same if $(R_1, \ldots, R_n)$ are multivariate normally distributed.

THEOREM 3.1. *If $(R_1, \ldots, R_n)$ are multivariate normally distributed, then*

$$w(x) = \sqrt{\frac{2}{\pi}}\,\sigma(x).$$

PROOF. Let $(\mu_1, \ldots, \mu_n)$ be the mean of $(R_1, \ldots, R_n)$. Also let $(\sigma_{ij}) \in R^{n\times n}$ be the covariance matrix of $(R_1, \ldots, R_n)$. Then $\sum R_j x_j$ is normally distributed (Rao 1965) with mean $\sum_{j=1}^{n} \mu_j x_j$ and standard deviation

$$\sigma(x) = \sqrt{\sum_{i=1}^{n} \sum_{j=1}^{n} \sigma_{ij} x_i x_j}.$$

Therefore,

$$w(x) = \frac{1}{\sqrt{2\pi}\,\sigma(x)} \int_{-\infty}^{\infty} |u| \exp - \frac{u^2}{2\sigma^2(x)}\, du = \sqrt{\frac{2}{\pi}}\,\sigma(x). \qquad \square$$

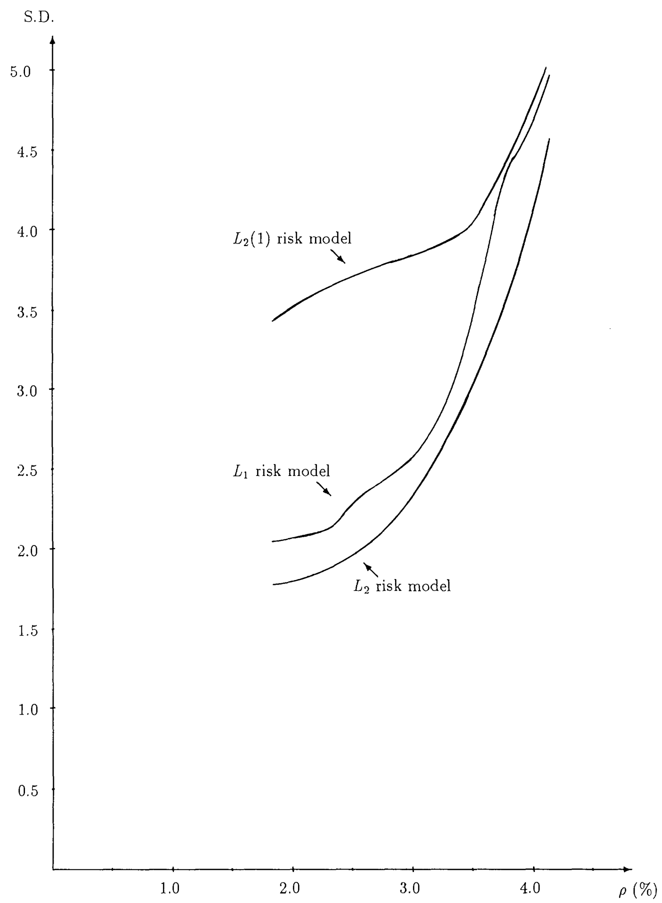


FIGURE 4.1

This theorem implies that minimizing $w(x)$ is equivalent to minimizing $\sigma(x)$ if $(R_1, \dots, R_n)$ is multivariate normally distributed. Thus we are led to an alternative L_1 risk minimization problem.

$$\left|\begin{array}{ll} \text{minimize} & w(x) E\left[\left|\sum_{j=1}^{n} R_j x_j - E\left[\sum_{j=1}^{n} R_j x_j\right]\right|\right] \\ \text{subject to} & \sum_{j=1}^{n} E[R_j] x_j \geq \rho M_0, \\ & \sum_{j=1}^{n} x_j = M_0, \\ & 0 \leq x_j \leq u_j, \qquad j = 1, \dots, n. \end{array}\right. \tag{3.1}$$

Let r_{jt} be the realization of random variable R_j during period t ($t = 1, \dots, T$) which we assume to be available through the historical data or from some future projection. We also assume that the expected value of the random variable can be approximated by the average derived from these data.

In particular, let

$$r_j = E[R_j] = \sum_{t=1}^{T} r_{jt}/T. \tag{3.2}$$

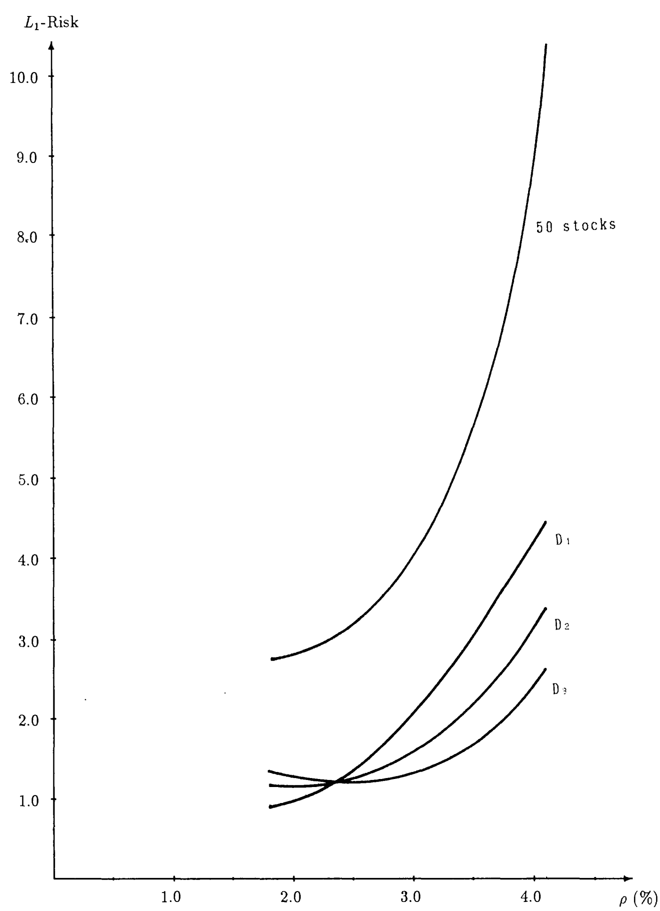


FIGURE 4.2

Then $w(x)$ can be approximated as follows:

$$E\left[\left|\sum_{j=1}^{n} R_j x_j - E\left[\sum_{j=1}^{n} R_j x_j\right]\right|\right] = \frac{1}{T}\sum_{t=1}^{T}\left|\sum_{j=1}^{n}(r_{jt} - r_j)x_j\right|. \tag{3.3}$$

Let us denote

$$a_{jt} = r_{jt} - r_j, \qquad j = 1, \ldots, n; \qquad t = 1, \ldots, T. \tag{3.4}$$

Then (3.1) leads to the following minimization problem:

$$\begin{array}{|ll} \text{minimize} & \sum_{t=1}^{T}\left|\sum_{j=1}^{n} a_{jt}x_j\right| \Big/ T \\ \text{subject to} & \sum_{j=1}^{n} r_j x_j \geq \rho M_0, \\ & \sum_{j=1}^{n} x_j = M_0, \\ & 0 \leq x_j \leq u_j, \qquad j = 1, \ldots, n, \end{array} \tag{3.5}$$

which is equivalent to the linear program

$$\begin{array}{|ll} \text{minimize} & \sum_{t=1}^{T} y_t \Big/ T \\ \text{subject to} & y_t + \sum_{j=1}^{n} a_{jt}x_j \geq 0, \qquad t = 1, \ldots, T, \\ & y_t - \sum_{j=1}^{n} a_{jt}x_j \geq 0, \qquad t = 1, \ldots, T, \\ & \sum_{j=1}^{n} r_j x_j \geq \rho M_0, \\ & \sum_{j=1}^{n} x_j = M_0, \\ & 0 \leq x_j \leq u_j, \qquad j = 1, \ldots, n. \end{array} \tag{3.6}$$

Let us list some of the advantages of this formulation over the L_2 risk model:

(a) We don't have to calculate the covariance matrix to set up the model. Also, it is very easy to update the model when new data are added.

(b) Solving a linear program (3.6) is much easier than solving its counterpart (2.3). Note that the number of functional constraints remains constant (i.e. $2T + 2$) regardless the number of stocks included in the model, so that we may be able to solve a problem with more than one thousand assets on a real time basis.

(c) An optimal solution $x^* = (x_1^*, \ldots, x_n^*)$ of (3.6) contains at most $2T + 2$ positive components if $u_j = \infty$, $j = 1, \ldots, n$ (see Chvátal 1983, Dantzig 1963). This means that an optimal portfolio consists of at most $2T + 2$ assets regardless of the size of n. An optimal portfolio derived from L_2 risk model, on the other hand, may contain as many as n assets. The difference can be very substantial when n is over 1,000.

(d) We can use T as a control variable when we want to restrict the number of assets in the portfolio.

(e) Values of at least $n - (2T + 2)$ components of x_j^*'s are either 0 or u_j when u_j's are finite. Thus by choosing u_j as an integer multiple of minimum transaction unit of S_j, values of at least $n - (2T + 2)$ components of x_j^*'s are integer multiple of the minimum transaction unit. Also, it is relatively easy to obtain a genuine integer solution by using appropriate integer linear programming techniques.

Another feature of the L_1 risk model is that we can derive an equilibrium relation between the "market portfolio" and an individual asset analogous to CAPM model. In fact, it was shown in Konno (1989) that the following relation holds in equilibrium under a certain nondegeneracy assumption:

$$r_i = r_F + \theta_i(r_M - r_F), \qquad i = 1, \ldots, n, \tag{3.7}$$

where r_i is the average rate of return of S_i, r_M is the average rate of return of the market portfolio, and r_F is the interest rate of risk free asset. Also θ_i is a constant so called "theta" of S_i. (See Konno 1989 for details.)

4. Comparison of the L_1 Risk Model with Classical Models

In this section, we will compare the performance of L_1 risk model with that of Markowitz's L_2 risk model using the historical data of 224 stocks included in NIKKEI 225 index. We also compare the performance of the above two models with Sharpe's single-factor L_2 risk model (Sharpe 1971). This model assumes the relation

$$R_i = \alpha_i + \beta_i F + \epsilon_i, \qquad i = 1, \ldots, n, \tag{4.1}$$

between the rate of return R_i and a common factor F, where ϵ_i's are independently distributed random variables with mean zero and variance σ_i^2. Also F is a random variable with mean f and variance σ_0^2. The advantage of this model is that the associated variance of the return of the portfolio has a diagonal expression, i.e.,

$$\sigma^2(x_1, \ldots, x_n) = \sum_{i=1}^{n} \sigma_i^2 x_i + \sigma_0^2 x_{n+1}^2 \qquad \text{where} \tag{4.2}$$

$$x_{n+1} = \sum_{i=1}^{n} \beta_i x_i, \tag{4.3}$$

so that the amount of computation can be substantially reduced. Also, Sharpe claims that this model leads to a portfolio similar to the one obtained by solving an exact

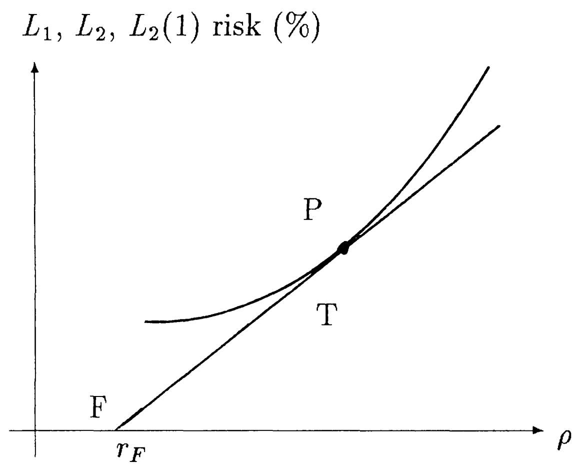


FIGURE 4.3

quadratic program for n up to 20. We employed R_M, the average return of the TOPIX index, as the factor F and applied the standard least square technique to estimate β_i's.

We prepared three sets of data D_1, D_2, D_3 all of which consist of 60 data representing the rate of return of 224 stocks for 60 months. D_1 covers five years from 1981 to 1985, while D_2 and D_3 cover 60 months from 1982 to 1986, 1983 to 1987, respectively. Throughout these computations, u_j's were set at ∞.

Figure 4.1 shows the efficient frontiers generated by the L_1 risk model, the L_2 risk model and a single-factor L_2 risk model (which we refer to as the $L_2(1)$ risk model in the sequel). We observe that the difference of the standard deviation of the optimal portfolio generated by L_2 and L_1 risk models is at most 10% for whatever value of ρ. Note that these two lines should coincide if the R_j's are multivariate normally distributed (Theorem 3.1). Thus this difference can be largely attributed to the nonnormality of the data.

On the other hand, the efficient frontier associated with the $L_2(1)$ model is radically different from that of the L_2 risk model contrary to the results of Sharpe (1963). In fact, the standard deviation of the former is almost twice as much as that of the latter. Solving a linear program (3.6) for a fixed value of ρ required 3.3 minutes on the average on a

TABLE 4.1

Data Set	D_1			D_2			D_3		
Portfolio	P_1	P_2	$P_2(1)$	P_1	P_2	$P_2(1)$	P_1	P_2	$P_2(1)$
r_M: average rate of return/ mo. (%)	2.03	2.16	1.90	2.77	2.85	2.63	3.15	3.18	2.88
σ_M: standard deviation of the rate of return/mo. (%)	1.81	1.61	3.24	2.45	2.22	3.75	2.66	2.34	3.74
$(r_M - r_F)/\sigma_M$: Sharpe ratio	0.97	1.17	0.50	1.02	1.16	0.63	1.08	1.24	0.70
Number of stocks in the portfolio	28	25	88	26	25	67	31	29	71
Stock with maximum share (%)	13.4	14.9	4.5	10.7	10.3	5.3	10.6	10.2	4.9
Stock with minimum share (%)	0.01	0.48	0.004	0.45	0.004	0.007	0.15	0.054	0.02
Sum of weights of the top 5 stocks	45.1	48.6	17.5	46.5	43.3	23.4	43.2	40.6	22.5
Sum of weights of the top 10 stocks	67.5	72.7	29.7	70.8	70.2	40.1	71.3	63.1	38.3
Number of stocks with positive weight less than 3%	14	13	84	14	11	59	18	14	63
Number of stocks with positive weight less than 1%	5	2	47	3	4	29	10	5	32

D_1: Jan. '81–Dec. '85 P_1: L_1 Risk Market Portfolio
D_2: Jan. '82–Dec. '86 P_2: L_2 Risk Market Portfolio
D_3: Jan. '83–Dec. '87 $P_2(1)$: $L_2(1)$ Risk Market Portfolio

SUN IV computer. We used the MINOS IV Code for 20 different values of ρ, so it took about an hour to generate an efficient frontier. Computation time would have been 4–5 minutes if a parametric right-hand side simplex routine were available. Solving a dense quadratic program associated with Markowitz's L_2 risk model took about 28 minutes on the average for a fixed value of ρ. We used a QP code developed by Mathematical Systems, Inc., which uses active set strategies and a rudimentary sparse matrix technique. It is not equipped with a parametric right-hand side routine, so that it took about 10 hours to generate an efficient frontier by solving 20 QP's for 20 different values of ρ. We guess that generating an efficient frontier for the L_2 risk model requires at least an hour even if we use a parametric quadratic programming code. Solving a sparse quadratic program associated with $L_2(1)$ model took 4.8 minutes using the same code as above. Computation time could be reduced if we use a more elaborate QP code. For large n, the amount of computation to solve L_1 model would be $O(n)$. The amount of computation associated with $L_2(1)$ and L_2 risk models on the other hand are expected to be at least $O(n^2)$ and $O(n^3)$, respectively (see Chvátal 1983, Dantzig 1963).

Figure 4.2 shows the efficient frontiers of L_1 risk model for three data sets D_1, D_2, D_3. It also shows the efficient frontier associated with 5-years' data of 50 stocks included in OSAKA 50. Forty-seven stocks out of these 50 stocks are included in NIKKEI 225. L_1 risk of the NIKKEI 225 portfolio is approximately one third of the L_1 risk of the OSAKA

TABLE 4.2

Name and Rank in the Portfolio Associated with Data Set D_2

Name	*Rank in P_1*	*Rank in P_2*	*Rank in $P_2(1)$*	*Name*	*Rank in P_1*	*Rank in P_2*	*Rank in $P_2(1)$*
Kyokuyou		23		Nichi-densou	23		
Teikei	25	13		Yuasa-denchi	5	1	1
Satou	13	12	19	Toyota		17	
Tobishima	1	4	3	Nikon	6	9	
Fujita	24	25		Ricoh	3	5	
Tekken	17	14		Sichizun	26		
Morinaga	21		16	Yamaha	16	18	
Goudou	8	7	9	Matuzakaya	22	16	2
Yamanouchi	14	15	18	Ichikan		11	
Dainichi-yaku	9	8		Tougin	12	20	
Tounen	19			ANA	4	6	7
Gaisi	20			Tougasu	2	2	
Nichi-denkou		24	22	Shouchiku	7	3	4
Mitui-kin		10	30	Touhou	18	21	
Sumitomo-kou	15	22		Nikkatu	10		
NEC	11	19					

50 portfolio. We expect from this that the risk can be substantially reduced if we include all 1,100 stocks being transacted in the Tokyo Stock Market.

We next calculated the market portfolios P_1, P_2, $P_2(1)$ associated with L_1, L_2, $L_2(1)$ risk models. These portfolios are associated with the supporting point T_1, T_2, $T_2(1)$ of the lines emanating from point F corresponding to the interest rate r_F of the risk-free asset with the efficient frontier. (See Figure 4.3.)

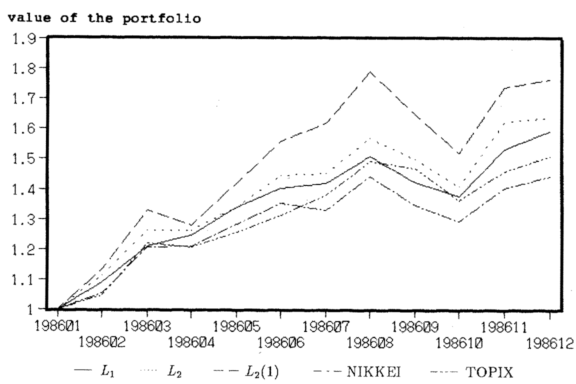
Table 4.1 shows the statistics of these portfolios. Let us note the striking similarities of P_1 and P_2 and a surprising difference between P_2 and $P_2(1)$ portfolios.

Table 4.2 shows the names of stocks in P_1, P_2, $P_2(1)$ together with their rankings. We find that 20 stocks are included in both P_1 and P_2. Also, 9 out of the top 10 stocks included in P_1 and P_2 are same (the order is different, of course). $P_2(1)$, on the other hand, is quite different. In particular, the $P_2(1)$ portfolio contains about three times as many stocks. Similar results have been obtained for the data sets D_1 and D_3, the details of which will be omitted due to the space restrictions.

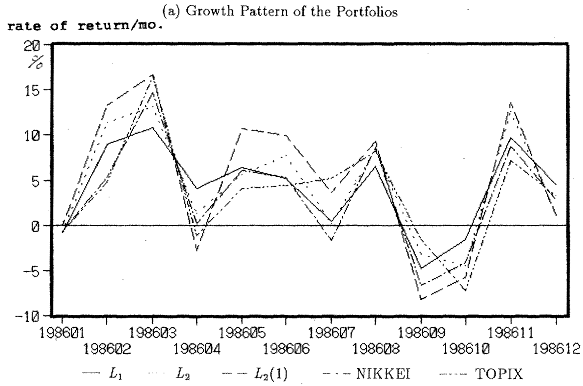
We calculated the monthly rate of return of each portfolio for twelve months succeeding the periods D_1, D_2 and D_3. Figures 4.4–4.6 show the performance of the portfolios P_1, P_2, $P_2(1)$ associated with the data sets D_1–D_3 and $\rho = 2.0\%$/mouth. Also these figures show the performance of NIKKEI 225 Index and TOPIX Index (average of all the stocks of the Tokyo Stock Market).

Table 4.3 summarizes the statistics of the ex-post performance. We calculated the average rate of return r (per month), its standard deviation σ and the Sharpe ratio $(r - r_F)/\sigma$. We observe from Table 4.3 and Figures 4.4–4.6 that

(i) P_1 always outperforms NIKKEI 225 and TOPIX, and appears to be somewhat better than P_2.

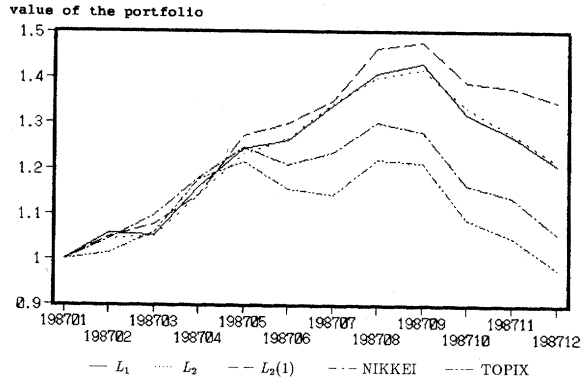


(a) Growth Pattern of the Portfolios

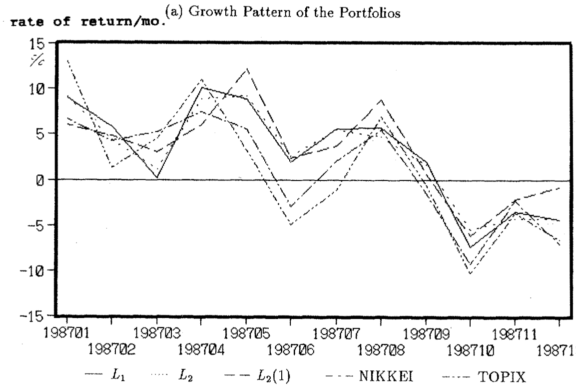


(b) Monthly Rate of Return of the Portfolios

FIGURE 4.4. Ex-Post Performance of Portfolios (Data Set D_1, Test Period 1986, $\rho = 2.0\%$/month).



(a) Growth Pattern of the Portfolios



(b) Monthly Rate of Return of the Portfolios

FIGURE 4.5. Ex-Post Performance of Portfolios (Data Set D_2, Test Period 1987, $\rho = 2.0\%/\text{month}$).

(ii) P_2 is better than NIKKEI and TOPIX most of the time.

(iii) $P_2(1)$ is better than NIKKEI 225 and TOPIX for D_2 and D_3, while all three of them are comparable for D_1.

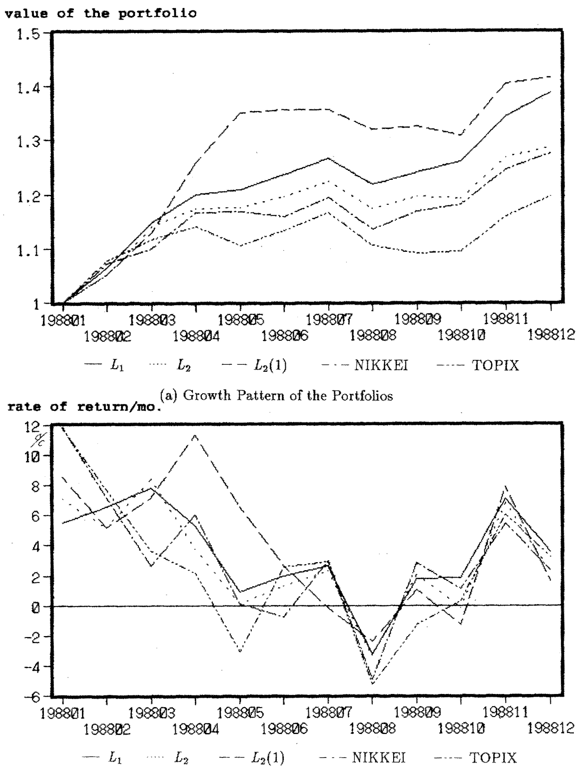
(iv) $P_1(1)$ usually leads to larger r and σ than P_1 and P_2.

(v) The largest Sharpe ratio is attained by P_1 for D_1 and D_3, while it is attained by $P_2(1)$ for D_2.

5. Concluding Remarks and Future Direction of Research

We showed in this paper that the L_1 risk model can be used as an alternative to Markowitz's L_2 risk model. In fact, calculated optimal portfolios and their performance are quite similar to each other. We believe that these portfolios will not be very much different for the model with larger n. We are planning a test using the data of 500 stocks included in NIKKEI 500 and all 1,100 stocks being transacted in the Tokyo Stock Market. We don't know, however, whether a quadratic programming problem can be solved within a reasonable amount of time if n is over 1,000. Note that we have to handle 1,000 × 1,000 dense covariance matrix. The L_1 risk model, on the other hand, could be solved in less than 10 minutes since the computational time for solving a linear program is expected to increase at most linearly in n. By extending the universe, we would be able to obtain a portfolio with substantially smaller risk. (Note the remarkable reduction of the risk when we increased n from 50 to 225. See Figure 4.2.)

It turned out that the $L_2(1)$ risk model leads to a portfolio very different from that of the L_2 risk model contrary to the assertion of Sharpe (1971). Thus we need to introduce



(a) Growth Pattern of the Portfolios

(b) Monthly Rate of Return of the Portfolios

FIGURE 4.6. Ex-Post Performance of Portfolios (Data Set D_3, Test Period 1988, ρ = 2.0%/month).

at least several factors to generate a portfolio close enough to that of the L_2 risk model, at the expense of the increase in the amount of computation. Relative advantage of multi-factor L_2 risk model (Perold 1984) and the L_1 risk model is an open question.

Our next step is to compute an optimal portfolio using the more general piecewise linear risk model proposed in Konno (1989). This model can better represent the risk perception of an individual investor. Also the associated optimization problem can be formulated as a linear program as long as the risk function is convex. We are planning a comparison of the L_1 risk model and the order-made piecewise linear risk model by

TABLE 4.3(a)

Performance of Portfolios (ρ = 2.0%/*month*)

Data	*Test Period*		P_1	P_2	$P_2(1)$	*NIKKEI* 225	*TOPIX*
D_1	Jan. '86–	r	4.12%	4.41%	5.12%	3.21%	3.57%
	Dec. '86	σ	4.87%	5.42%	8.25%	6.10%	5.95%
		$(r - r_F)/\sigma$	0.79	0.76	0.59	0.48	0.55
D_2	Jan. '87–	r	2.82%	2.77%	3.20%	1.15%	1.08%
	Dec. '87	σ	5.71%	5.27%	4.96%	5.61%	7.04%
		$(r - r_F)/\sigma$	0.44	0.47	0.59	0.16	0.11
D_3	Jan. '88–	r	3.47%	2.97%	4.00%	3.08%	2.55%
	Dec. '88	σ	3.15%	3.43%	4.38%	4.26%	4.66%
		$(r - r_F)/\sigma$	1.01	0.78	0.85	0.66	0.49

TABLE 4.3(b)
Performance of Portfolios (ρ = 2.5%/month)

Data	Test Period		P_1	P_2	$P_2(1)$	NIKKEI 225	TOPIX
D_1	Jan. '86–	r	4.69%	4.11%	4.67%	3.21%	3.57%
	Dec. '86	σ	4.47%	5.96%	8.06%	6.10%	5.95%
		$(r - r_F)/\sigma$	0.99	0.64	0.54	0.48	0.55
D_2	Jan. '87–	r	2.42%	2.53%	2.65%	1.15%	1.08%
	Dec. '87	σ	5.01%	5.05%	4.96%	5.61%	7.04%
		$(r - r_F)/\sigma$	0.43	0.45	0.48	0.16	0.11
D_3	Jan. '88–	r	3.21%	2.86%	3.76%	3.08%	2.55%
	Dec. '88	σ	3.57%	3.38%	4.25%	4.26%	4.66%
		$(r - r_F)/\sigma$	0.82	0.76	0.82	0.66	0.49

first measuring the risk function of a fund manager by using a technique developed in decision analysis and then comparing his degree of satisfaction. The results of all these tests will be reported subsequently.[2]

[2] The authors are greatly indebted to Mr. M. Mizuno of the Mathematical Systems, Inc. for his assistance during the course of computation.

This research was partly supported by Grant-in-Aid for Scientific Research of the Ministry of Education, Science and Culture, Grant No. 63490010.

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