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Conditional Value-at-Risk, spectral risk measures and (non-)diversification in portfolio selection problems – A comparison with mean–variance analysis

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ABSTRACT

We study portfolio selection under Conditional Value-at-Risk and, as its natural extension, spectral risk measures, and compare it with traditional mean–variance analysis. Unlike the previous literature that considers an investor's mean-spectral risk preferences for the choice of optimal portfolios only implicitly, we explicitly model these preferences in the form of a so-called spectral utility function. Within this more general framework, spectral risk measures tend towards corner solutions. If a risk free asset exists, diversification is never optimal. Similarly, without a risk free asset, only limited diversification is obtained. The reason is that spectral risk measures are based on a regulatory concept of diversification that differs fundamentally from the reward-risk tradeoff underlying the mean–variance framework.

1. Introduction

Quantile-based risk measures and utility functions, respectively, have a long tradition in finance, insurance theory, and decision analysis. Originally, these functionals have been proposed in connection with non-expected utility models by the dual theory of choice (Yaari, 1987). Moreover, distortion measures have been introduced for the pricing of insurance contracts (Wang et al., 1997). From both approaches it is well-known that they tend towards corner solutions in portfolio selection problems (e.g., Yaari, 1987). Especially, for a risk free and a risky asset, diversification is never optimal, but the exclusive investment in one of the assets.

In the last decade, quantile-based risk measures have been "re-discovered" in the context of axiomatic approaches to risk measurement, initiated by the seminal work of Artzner et al., 1999, on coherent risk measures. Since then, spectral risk measures as the most widespread subclass, and Conditional Value-at-Risk as the most prominent representative, have been applied to portfolio selection problems extensively, replacing the traditional variance. Surprisingly, and unlike one would expect from the earlier results on corner solutions, these studies regularly find diversification (e.g., Adam et al., 2008; Bassett et al., 2004; Benati, 2003; Bertsimas et al., 2004; Giorgi, 2002; Krokhmal et al., 2002; Rockafellar and Uryasev, 2000).

In this paper, we address these contradicting results on diversification under Conditional Value-at-Risk and spectral risk measures. Our contribution is twofold. First and most importantly, we show that the tendency towards corner solutions also prevails under spectral risk measures. More specifically, we find non-diversification if the risk free asset exists, and only limited diversification without a risk free asset. As diversification is a key issue in portfolio selection, its lack is a major drawback one should be aware of when replacing the traditional variance by spectral risk measures.

Our results differ from the aforementioned studies, as we apply a less restrictive approach for the choice of optimal portfolios. This literature finds optimal portfolios by fixing a certain level of expected return, $\bar{\mu}$, and choosing the corresponding portfolio that minimizes a spectral risk measure, ρ_ϕ, over a set of alternatives $\mathcal{X}$:

$$\min\ \rho_\phi(X),\ \text{s.t.}\ X \in \mathcal{X},\ E(X) = \bar{\mu}. \tag{1}$$

Many applications in practice are based on such *limited analysis*. For example, it might be that the tradeoff between reward and

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risk is made by, say, a higher management hierarchy by fixing $\bar{\mu}$, while the risk analyst should only assess the risk side. However, as $\bar{\mu}$ is given exogenously, the tradeoff between reward and risk is not subject to considerations within the choice of optimal portfolios. Note, in particular, that by fixing an "interior" $\bar{\mu}$, the limited analysis by definition shows diversification: Assume, for example, a risk free asset with a return of 2% and a risky asset with an expected return of 6%, and fix $\bar{\mu} = 4\%$. Then a 50/50-mix is "optimal". However, it has not yet been proved that the 50/50-mix with $\bar{\mu} = 4\%$ actually is an optimal portfolio under spectral risk measures. By contrast, it will turn out that investors with (μ,ρ_ϕ)-preferences always prefer the corner positions $\bar{\mu} = 2\%$ or $\bar{\mu} = 6\%$ to any diversified portfolio.

In order to disclose this tendency towards corner solutions under spectral risk measures, we have to consider the tradeoff between reward and risk made by the higher hierarchy instead of restricting to a limited analysis only. We do so by applying a *tradeoff analysis*, which explicitly models an investor's (μ,ρ_ϕ)-preferences in the form of a (μ,ρ_ϕ)-utility function, $\pi = \pi(\mu,\rho_\phi)$. To this end, we reproduce a portfolio selection approach that is well-established in the mean–variance framework for decades, and obtain the *spectral utility function*

$$\pi_\phi(X) = (1-\lambda)\cdot E(X) - \lambda \cdot \rho_\phi(X), \quad \lambda \in [0,1]. \tag{2}$$

This utility function naturally arises from two different perspectives. From a decision theoretic perspective, the two components $-E(X)$ and $\rho_\phi(X)$ satisfy the properties of spectral risk measures, and by forming a (negative) convex combination, the spectral utility function π_ϕ itself satisfies these properties as well. As the use of spectral risk measures instead of the variance in the literature is mainly motivated by referring to their axiomatic framework, the spectral utility function (2) establishes one consistent axiomatic framework that simultaneously covers the determination of efficient frontiers and the choice of optimal portfolios. From an optimization perspective, the spectral utility function (2) is the tradeoff version of the (μ,ρ_ϕ)-efficient frontier program (1), and thus also establishes a consistent framework.

As will be shown below, the limited analysis and the tradeoff analysis are equivalent approaches for finding optimal portfolios in the traditional mean–variance framework only, but they fundamentally differ under spectral risk measures. As a consequence, the prevalent limited analysis (1) is not covered by the more general tradeoff analysis (2) and leads to (non-optimal) diversified portfolios, while actually non-diversification and limited diversification, respectively, are optimal.

As a second contribution, we provide an explanation for the tendency towards corner solutions under spectral risk measures. We argue that the corner solutions arise from mixing two different contexts, each with its own concept of diversification: Spectral risk measures originally have been introduced for the assessment of solvency capital in bank regulation ("risk" context), where they add to bank's objective (or utility) function π as a regulatory constraint:

$$\max \pi(X), \text{ s.t. } X \in \mathcal{X}, \ \rho_\phi(X) \leqslant \bar{\rho}. \tag{3}$$

Hence, spectral risk measures are not used to find optimal portfolios, but only restrict the set of alternatives $\mathcal{X}$ to the so-called acceptance set $\mathcal{A} = \{X \in \mathcal{X} | \rho_\phi(X) \leqslant \bar{\rho}\}$. Especially, the regulatory framework does not impose any assumption on the bank's utility function π.

By contrast, the portfolio selection approaches (1) and (2) ("decision" context) implicitly and explicitly, respectively, aim to model this utility function π, as the problem of finding an optimal portfolio is given by

$$\max \pi(\mu(X), \rho_\phi(X)), \text{ s.t. } X \in \mathcal{X}. \tag{4}$$

Now, spectral risk measures constitute the risk part within an investor's (μ,ρ_ϕ)-utility function, $\pi = \pi(\mu,\rho_\phi)$.

We show that the axiomatic framework underlying spectral risk measures and the concept of diversification induced thereby originally have been proposed for regulatory purposes ("risk" context), where diversification is based on the dependence structure between assets. By contrast, in standard portfolio selection approaches ("decision" context), such as the mean–variance framework, the tradeoff between reward and risk is relevant for diversification. Applying spectral risk measures instead of the variance for portfolio selection means adopting their regulatory concept of diversification, and leads to corner solutions.

The paper proceeds as follows. Section 2 reviews the axiomatic framework and characterizes the regulatory concept of diversification underlying spectral risk measures. Section 3 derives the (μ,ρ_ϕ)-efficient frontiers. Section 4 analyzes the choice of optimal portfolios by using spectral utility functions. In both Sections 3 and 4, we confront these results with those from the traditional (μ,σ^2)-framework, which serves as a well-established benchmark. Section 5 discusses the economic implications of the findings. Section 6 concludes.

2. Theoretical framework

2.1. Spectral risk measures and the regulatory concept of diversification

In order to prepare for the regulatory concept of diversification underlying spectral risk measures ("risk" context), we first introduce the notion of comonotonicity (e.g., Dhaene et al., 2002).

Definition 2.1. Two random variables $X_1, X_2 \in \mathcal{X}$ are called comonotonic if

$$(X_1(\omega_i) - X_1(\omega_j)) \cdot (X_2(\omega_i) - X_2(\omega_j)) \geqslant 0,$$
$$\text{for all } \omega_i, \omega_j \in \Omega, P(\Omega) = 1. \tag{5}$$

Two random variables are comonotonic if they increase and decrease simultaneously in their state-dependent realizations. Comonotonicity thus denotes perfect positive dependence between two random variables. In particular, comonotonicity is more general than perfect positive correlation. As $corr(X_1,X_2) = 1$ if and only if $X_2 = a \cdot X_1 + b, a > 0, b \in \mathbb{R}$, perfect positive correlation implies comonotonicity, but the converse is not true.

We proceed with the definition of spectral risk measures (Acerbi, 2002, 2004).[1]

Definition 2.2. A mapping $\rho_\phi : \mathcal{X} \to \mathbb{R}$ is called spectral risk measure if it satisfies

- Monotonicity with respect to first order stochastic dominance: For $X_1, X_2 \in \mathcal{X}$ with $F_{X_1}(t) \geqslant F_{X_2}(t)$ and $t \in \mathbb{R}$, $\rho_\phi(X_1) \geqslant \rho_\phi(X_2)$.
- Translation invariance: For $X \in \mathcal{X}$ and $c \in \mathbb{R}, \rho_\phi(X + c) = \rho_\phi(X) - c$.
- Subadditivity: For $X_1, X_2 \in \mathcal{X}, \rho_\phi(X_1 + X_2) \leqslant \rho_\phi(X_1) + \rho_\phi(X_2)$.
- Comonotonic Additivity: For comonotonic $X_1, X_2 \in \mathcal{X}$, $\rho_\phi(X_1 + X_2) = \rho_\phi(X_1) + \rho_\phi(X_2)$.

[1] The given properties differ slightly from those by Acerbi (2004) in that they do not explicitly consider law invariance and positive homogeneity. Law invariance is implied by monotonicity with respect to first order stochastic dominance. Further, monotonicity and comonotonic additivity imply positive homogeneity, $\rho_\phi(\lambda \cdot X) = \lambda \cdot \rho_\phi(X)$, $\lambda \geqslant 0$ (Song and Yan, 2009, Section 5.1).

As to the above properties, spectral risk measures originally have been introduced for the assessment of solvency capital in bank regulation ("risk" context). Monotonicity and translation invariance are straightforward requirements for measuring risk in monetary terms. Especially, due to $\rho_\phi(X+\rho_\phi(X)) = \rho_\phi(X) - \rho_\phi(X) = 0$, translation invariance allows for the interpretation of $\rho_\phi(X)$ as required solvency capital.

The regulatory concept of diversification underlying spectral risk measures is captured jointly by the properties of subadditivity and comonotonic additivity, and relates exclusively to the dependence structure between assets. If two assets X_1 and X_2 within a portfolio are non-comonotonic, a "high" realization in one state of the world of asset X_1 (partially) compensates for a "low" realization of asset X_2 in the same state of the world (and vice versa). Subadditivity implements this idea by demanding that a portfolio of two assets regularly requires less solvency capital than the two single assets do; the diversification benefit is positive. If two assets X_1 and X_2 within a portfolio are comonotonic, i.e., "high" and "low" realizations coincide in all states of the world, a compensation does not exist. This is captured by the additivity of spectral risk measures for comonotonic assets; the diversification benefit is zero. We summarize the above argument in the following proposition (see also Cherny, 2006, Theorem 5.1).

Proposition 2.3. *Let ρ_ϕ be a spectral risk measure and $X_1, X_2 \in \mathcal{X}$. Non-comonotonicity between X_1 and X_2 is a necessary condition for a positive diversification benefit, $\lambda \cdot \rho_\phi(X_1) + (1-\lambda) \cdot \rho_\phi(X_2) - \rho_\phi(\lambda \cdot X_1 + (1-\lambda) \cdot X_2) > 0$, $\lambda \in [0,1]$.*

The following stylized example illustrates the regulatory concept of diversification underlying spectral risk measures.

Example 2.4. Let $X_0 = x_0$ be a risk free asset and X_1, X_2 and X_3 be risky assets with state-dependent returns

$$X_1 = \begin{cases} -2 & P(\omega_1) = 1/3 \\ 0 & P(\omega_2) = 1/3 \\ 3 & P(\omega_3) = 1/3 \end{cases}, \quad X_2 = \begin{cases} -3 & P(\omega_1) = 1/3 \\ 1 & P(\omega_2) = 1/3 \\ 4 & P(\omega_3) = 1/3 \end{cases}, \quad X_3 = \begin{cases} 4 & P(\omega_1) = 1/3 \\ 1 & P(\omega_2) = 1/3 \\ -3 & P(\omega_3) = 1/3 \end{cases}.$$

For the non-comonotonic assets X_1 and X_3, we observe a compensation between "high" and "low" realizations within a portfolio: While the single assets suffer losses in state of the world ω_1 and ω_3, respectively, the portfolios $\gamma \cdot X_1 + (1-\gamma) \cdot X_3$ for $\gamma \in [0.5, 0.67]$ have only non-negative state-dependent returns. Consequently, subadditivity regularly yields reduced solvency capital requirements and a positive diversification benefit:

$$\rho_\phi(\gamma \cdot X_1 + (1-\gamma) \cdot X_3) < \gamma \cdot \rho_\phi(X_1) + (1-\gamma) \cdot \rho_\phi(X_3), \gamma \in (0,1).$$

For the comonotonic assets X_1 and X_2, a compensation between "high" and "low" realizations within the portfolio does not prevail. Hence, comonotonic additivity ensures that the solvency capital requirements are not reduced and the diversification benefit is zero:

$$\rho_\phi(\gamma \cdot X_1 + (1-\gamma) \cdot X_2) = \gamma \cdot \rho_\phi(X_1) + (1-\gamma) \cdot \rho_\phi(X_2), \gamma \in [0,1].$$

As another example, building a portfolio of a risky asset X_i, $i = 1, \ldots, 3$ and the risk free asset X_0 yields

$$\rho_\phi(\beta \cdot X_i + (1-\beta) \cdot X_0) = \beta \cdot \rho_\phi(X_i) - (1-\beta) \cdot X_0, \quad \beta \in [0,1].$$

In this case, the solvency capital requirements decrease linearly in the proportion $(1-\beta)$. Beyond, there is no (additional) diversification benefit from the dependence structure. Note that this result is not only driven by the comonotonicity between X_i and X_0, but is also required by the property of translation invariance. □

We finally come to the representation of spectral risk measures as weighted sum of quantiles.

Proposition 2.5. *Any spectral risk measure ρ_ϕ of a random variable X is of the form*

$$\rho_\phi(X) = -\int_0^1 F_X^*(p) \cdot \phi(p) \mathrm{d}p, \tag{6}$$

where $F_X^*(p) = \sup\{x \in \mathbb{R} | F_X(x) < p\}, p \in (0,1]$ is the quantile function of X, and the risk spectrum $\phi : [0,1] \to \mathbb{R}$ is a non-increasing density function.

For the proof see Acerbi, 2002, Theorem 4.1. Spectral risk measures are characterized by a risk spectrum, ϕ, which assigns subjective weights to the p-quantiles, with smaller quantiles receiving greater weights to ensure the subadditivity property. Further properties of spectral risk measures are given by Dhaene et al., 2006.

Currently, the most widely discussed spectral risk measure is Conditional Value-at-Risk (e.g., Acerbi and Tasche, 2002b; Rockafellar and Uryasev, 2002). Its risk spectrum is given by

$$\phi(p) = \begin{cases} \alpha^{-1} & \text{for } 0 < p \leqslant \alpha \\ 0 & \text{for } \alpha < p \leqslant 1 \end{cases}. \tag{7}$$

Conversely, spectral risk measures can be seen as a natural extension of Conditional Value-at-Risk, as any convex combination of Conditional Value-at-Risks yields a spectral risk measure (Acerbi, 2002, Proposition 2.2).

Also, the (negative) mean, $\rho_\phi(X) = -E(X)$, is a spectral risk measure with $\phi(p) = 1$, $p \in [0,1]$. By contrast, the variance, $Var(X) = \sigma^2$, is not a spectral risk measure, as it satisfies none of the required properties.

2.2. Portfolio selection problems

We now change the context from "risk" to "decision" and introduce two standard portfolio selection problems, in which spectral risk measures will be applied: An investor can split his initial wealth W_0 between different assets. The return from this investment (i.e., the final wealth) is given by a random variable $X \in \mathcal{X}$ that stems from one of the following settings:

- *Setting 1:* There are two risky assets X_1 and X_2, i.e., $\mathcal{X} = \{\gamma \cdot X_1 + (1-\gamma) \cdot X_2 | \gamma \in \mathbb{R}\}$.[2] We assume the risky assets to be (μ, ρ)-efficient,[3] i.e., $E(X_1) < E(X_2) \wedge \rho(X_1) < \rho(X_2)$.
- *Setting 2:* There are two risky assets X_1 and X_2, and a risk free asset X_0, i.e., $\mathcal{X} = \{\beta \cdot (\gamma \cdot X_1 + (1-\gamma) \cdot X_2) + (1-\beta) \cdot X_0 | \beta \geqslant 0, \gamma \in \mathbb{R}\}$. Again, we assume the risky assets to be (μ, ρ)-efficient. Moreover, we restrict the correlation coefficient to *corr*-$(X_1, X_2) \in (-1, 1)$ to ensure that one cannot construct an additional risk free asset from the risky assets. Further assumptions about the return of the risk free asset are made in the relevant sections.

So far, the theoretical literature mostly assumes normally distributed returns to easily capture the complete dependence structure by the correlation coefficient (e.g., Alexander and Baptista, 2002, 2004; Giorgi, 2002; Deng et al., 2009). We refrain from this assumption and apply a *state space approach* instead, which does not require any assumption on the distribution of the returns, and characterizes the assets $X : \Omega \to \mathbb{R}$ via their state-dependent realizations $X = (X(\omega_1), \ldots, X(\omega_n))' = (x_1, \ldots, x_n)'$ and the corresponding vector of the probabilities of the states of the world $P = (P(\omega_1), \ldots, P(\omega_n))' = (p_1, \ldots, p_n)'$, i.e., any alternative is given

[2] $X_1 := W_0 \cdot (1 + R_1)$ and $X_2 := W_0 \cdot (1 + R_2)$ denote the returns from investing the initial wealth W_0 in assets 1 and 2.

[3] We use ρ as a placeholder for the variance, σ^2, and spectral risk measures, ρ_ϕ. The term "(μ, ρ)-efficient", for example, stands for (μ, σ^2)-efficient and (μ, ρ_ϕ)-efficient.

by the pair (X,P). This approach captures the dependence structure completely by the vectors X, and both the variance and spectral risk measures can be calculated directly from (X,P). For the ease of demonstration, the analysis remains mostly restricted to a finite state space, as certain portfolio structures "get lost" in the case of infinitely many states. However, we also refer to the case of normally distributed returns.

Also for the ease of demonstration, we first restrict the analysis to $m = 2$ risky assets; later we show that the results hold for more general cases as well.

The (μ,ρ)-boundary and the (μ,ρ)-efficient frontier are defined as follows.

Definition 2.6. A portfolio $X \in \mathcal{X}$ belongs to the (μ,ρ)-boundary if for some expected return $\bar{\mu} \in \mathbb{R}$ it has minimum risk ρ.

Definition 2.7. A portfolio $X \in \mathcal{X}$ belongs to the (μ,ρ)-efficient frontier if there is no portfolio $\overline{X} \in \mathcal{X}$ with $E(\overline{X}) \geqslant E(X)$ and $\rho(\overline{X}) \leqslant \rho(X)$, where at least one of the inequalities is strict.

We use the subscript $i = 1, 2$ to indicate that the $(\mu,\rho)_i$-boundaries and the $(\mu,\rho)_i$-efficient frontiers refer to settings 1 and 2, respectively. As is common in portfolio selection, we illustrate the $(\mu,\rho)_i$-efficient frontiers in the respective (ρ,μ)-planes.[4] Extending the previous literature, we are not only interested in the (μ,ρ)-efficient frontiers themselves, but especially in their (e.g., (piecewise) linear or (strictly) concave) shape.

3. (μ,σ^2)-efficient frontiers versus (μ,ρ_ϕ)-efficient frontiers

3.1. Comonotonic subsets of alternatives

As spectral risk measures are comonotonic additive, comonotonic subsets of alternatives become an essential part of the analysis.

Definition 3.1. A subset $\mathcal{S} \subseteq \mathcal{X}$ of a set of alternatives $\mathcal{X}$ is called comonotonic subset of alternatives if all its elements are pairwise comonotonic, i.e., for all $X_1, X_2 \in \mathcal{S}$ it holds that X_1, X_2 are comonotonic random variables in the sense of Definition 2.1.

The state space approach allows to make the comonotonic subsets of alternatives explicit. In setting 1 with two risky assets, the set of alternatives is given by

$$\mathcal{X} = \left\{ X_\gamma = \gamma \cdot X_1 + (1-\gamma) \cdot X_2 = \begin{pmatrix} \gamma \cdot x_{1_1} + (1-\gamma) \cdot x_{2_1} \\ \vdots \\ \gamma \cdot x_{1_n} + (1-\gamma) \cdot x_{2_n} \end{pmatrix} \middle| \gamma \in \mathbb{R} \right\}. \tag{8}$$

Equalizing any two portfolio realizations and solving for γ yields the following proportions

$$\gamma_{ij} := \frac{x_{2_i} - x_{2_j}}{(x_{2_i} - x_{2_j}) - (x_{1_i} - x_{1_j})}, \quad i = 1, \ldots, n-1, \ j = 2, \ldots, n, \ i < j, \tag{9}$$

which indicate a switch in the ranking of the realizations, and thus give the boundaries of the comonotonic subsets of alternatives. Rearranging the proportions with respect to size yields the following $k + 1$ comonotonic subsets of alternatives:

$$\mathcal{S}_1 = \{X_\gamma | \gamma \in (-\infty, \gamma_{ij,1:k}]\}, \ldots, \mathcal{S}_{k+1} = \{X_\gamma | \gamma \in (\gamma_{ij,k:k}, \infty)\}. \tag{10}$$

Their number depends mainly on the number of states of the world. For $n \to \infty$, k may (but does not necessarily need to) tend to infinity.

In setting 2 with one risk free and one risky asset, the complete set of alternatives

$$\mathcal{X} = \{X_\beta = \beta \cdot X_{\bar{\gamma}} + (1-\beta) \cdot X_0 | \beta \geqslant 0\} \tag{11}$$

is comonotonic itself.

3.2. Two risky assets

We start portfolio selection with analyzing the (μ,ρ)-boundaries and the (μ,ρ)-efficient frontiers. As we restrict the analysis to two risky assets, the complete set of alternatives $X_\gamma = \gamma \cdot X_1 + (1-\gamma) \cdot X_2, \gamma \in \mathbb{R}$ belongs to the $(\mu,\rho)_1$-boundaries.

First, we briefly recall the traditional (μ,σ^2)-framework. The $(\mu,\sigma^2)_1$-boundary is obtained by solving the portfolio's expected return for the proportion γ and plugging it into its variance:

$$\begin{aligned} Var(X_\gamma) &= \left(\frac{E(X_\gamma) - E(X_2)}{E(X_1) - E(X_2)}\right)^2 \cdot a + 2 \cdot \frac{E(X_\gamma) - E(X_2)}{E(X_1) - E(X_2)} \cdot b + c, \\ a &= Var(X_1) + Var(X_2) - 2 \cdot \sqrt{Var(X_1)} \\ &\quad \cdot \sqrt{Var(X_2)} \cdot corr(X_1, X_2), \\ b &= \sqrt{Var(X_1)} \cdot \sqrt{Var(X_2)} \cdot corr(X_1, X_2) - Var(X_2), \\ c &= Var(X_2). \end{aligned} \tag{12}$$

The $(\mu,\sigma^2)_1$-boundary is a parabola that opens to the right (see Fig. 1). The $(\mu,\sigma^2)_1$-efficient frontier lies on the upper branch of the parabola starting from the minimum-variance portfolio (MVP).

Proposition 3.2. *Let $\mathcal{X}$ be as in setting 1. Then*

1. *the minimum-variance portfolio is given by $\gamma_{MVP} = -\frac{b}{a}$;*
2. *the $(\mu,\sigma^2)_1$-efficient frontier contains all portfolios $\gamma \in (-\infty, \gamma_{MVP}]$ and is a strictly concave curve for any correlation coefficient $corr(X_1,X_2) \in [-1,1]$.*

The proof is straightforward and therefore omitted. Essentially, the strict concavity of the $(\mu,\sigma^2)_1$-efficient frontier follows from the strict convexity of the variance on $\mathcal{X}$ for any correlation coefficient $corr(X_1,X_2) \in [-1,1]$.

We now consider (μ,ρ_ϕ)-preferences. The $(\mu,\rho_\phi)_1$-boundary is obtained by writing the portfolio's expected return as a function of its spectral risk.

As a first step, we analyze a comonotonic subset of alternatives X_γ, $\gamma \in [\gamma_d, \gamma_u]$ as given in (10). For any $\delta := \frac{\gamma - \gamma_d}{\gamma_u - \gamma_d} \in [0,1]$, the spectral risk of portfolio X_γ

$$\begin{aligned} \rho_\phi(X_\gamma) &= \rho_\phi(\delta \cdot X_{\gamma_d} + (1-\delta) \cdot X_{\gamma_u}) \\ &= \delta \cdot \rho_\phi(X_{\gamma_d}) + (1-\delta) \cdot \rho_\phi(X_{\gamma_u}) \end{aligned} \tag{13}$$

can be solved for

$$\delta = \frac{\rho_\phi(X_\gamma) - \rho_\phi(X_{\gamma_u})}{\rho_\phi(X_{\gamma_d}) - \rho_\phi(X_{\gamma_u})} \tag{14}$$

and inserted into the portfolio's expected return to give the linear risk-return schedule

$$\begin{aligned} E(X_\gamma) &= \delta \cdot E(X_{\gamma_d}) + (1-\delta) \cdot E(X_{\gamma_u}) \\ &= \frac{E(X_{\gamma_d}) - E(X_{\gamma_u})}{\rho_\phi(X_{\gamma_d}) - \rho_\phi(X_{\gamma_u})} \cdot \rho_\phi(X_\gamma) - \frac{E(X_{\gamma_d}) - E(X_{\gamma_u})}{\rho_\phi(X_{\gamma_d}) - \rho_\phi(X_{\gamma_u})} \\ &\quad \cdot \rho_\phi(X_{\gamma_u}) + E(X_{\gamma_u}). \end{aligned} \tag{15}$$

[4] For the variance we use the (σ^2,μ)-plane instead of the more standard (σ,μ)-plane. The reason is that for a more straightforward comparison of the choice of optimal portfolios in Section 4 we want both the induced indifference curves of the mean–variance utility function, $\pi(X) = E(X) - \frac{\lambda}{2} \cdot Var(X)$, and spectral utility functions, $\pi_\phi(X) = (1-\lambda) \cdot E(X) - \lambda \cdot \rho_\phi(X)$, to be linear. This in turn requires having the variance instead of the standard deviation on the abscissa. None of the results on the choice of optimal portfolios would change if we were to use the (σ,μ)-plane instead.

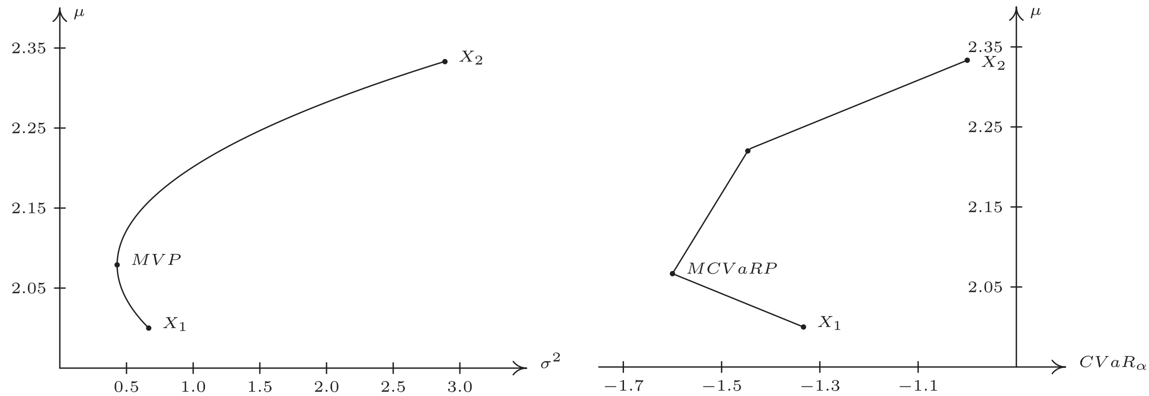

Fig. 1. $(\mu,\sigma^2)_1$-versus $(\mu, CVaR_\alpha)_1$-boundary with two risky assets.

If the comonotonic subset of alternatives is (μ, ρ_ϕ)-efficient, i.e., $E(X_{\gamma_d}) \leqslant E(X_{\gamma_u}) \wedge \rho_\phi(X_{\gamma_d}) \leqslant \rho_\phi(X_{\gamma_u})$, (15) is linearly increasing, and linearly decreasing otherwise.

Regarding the complete set of alternatives $X_\gamma, \gamma \in \mathbb{R}$, the portfolio's spectral risk is convex on $\mathcal{X}$ due to subadditivity and positive homogeneity. As the portfolio's expected return $E(X_\gamma)$ and the proportion γ are linearly related, the portfolio's spectral risk is also a convex function of its expected return that, according to (15), is piecewise linear for comonotonic subsets of alternatives. The $(\mu,\rho_\phi)_1$-boundary thus is a piecewise linear and overall convex curve that opens to the right (see Fig. 1). The $(\mu,\rho_\phi)_1$-efficient frontier lies on the upper branch of the $(\mu,\rho_\phi)_1$-boundary starting from the minimum-spectral risk portfolio (MSP); its existence and the non-emptiness of the $(\mu,\rho_\phi)_1$-efficient frontier are guaranteed by the assumption of (μ,ρ_ϕ)-efficient basic assets. We summarize the results in the following proposition.

Proposition 3.3. *Let $\mathcal{X}$ be as in setting 1. Then*

1. *the minimum-spectral risk portfolio γ_{MSP} lies in the set $\{\gamma_{ij,1:k}, \ldots, \gamma_{ij,k:k}\}$;*
2. *the $(\mu,\rho_\phi)_1$-efficient frontier contains all portfolios $\gamma \in (-\infty, \gamma_{MSP}]$ and is a concave curve that is piecewise linear for comonotonic subsets of alternatives as given in (10).*

We give the following stylized example for numerical illustration.

Example 3.4. An investor can split his initial wealth between two risky assets with state-dependent returns

$$X_1 = \begin{cases} 1 & P(\omega_1) = 1/3 \\ 2 & P(\omega_2) = 1/3 \\ 3 & P(\omega_3) = 1/3 \end{cases} \quad \text{and} \quad X_2 = \begin{cases} 4 & P(\omega_1) = 1/3 \\ 0 & P(\omega_2) = 1/3 \\ 3 & P(\omega_3) = 1/3 \end{cases}.$$

As risk measures, the investor applies the variance and Conditional Value-at-Risk at the confidence level $\alpha = 0.5$.

Fig. 1 shows the $(\mu,\sigma^2)_1$-boundary. The minimum-variance portfolio is given by $X_{0.76} = (1.53, 1.71, 3)'$, and the $(\mu,\sigma^2)_1$-efficient frontier contains all portfolios X_γ, $\gamma \in (-\infty, 0.76]$.

Further, Fig. 1 shows the $(\mu, CVaR_\alpha)_1$-boundary. The $(\mu, CVaR_\alpha)_1$-efficient frontier contains all portfolios X_γ, $\gamma \in (-\infty, 0.8]$, with $X_{0.8} = (1.6, 1.6, 3)'$ as the minimum-Conditional Value-at-Risk portfolio. The linear segments correspond to the portfolios $\gamma \in (-\infty, 0.34]$ $(x_2 \leqslant x_3 \leqslant x_1)$, $\gamma \in (0.34, 0.8]$ $(x_2 \leqslant x_1 \leqslant x_3)$, $\gamma \in (0.8, 1.5]$ $(x_1 \leqslant x_2 \leqslant x_3)$, and $\gamma \in (1.5, \infty]$ $(x_1 \leqslant x_3 \leqslant x_2)$. The corner positions $X_{0.34} = (3, 0.67, 3)'$, $X_{0.8} = (1.6, 1.6, 3)'$, and $X_{1.5} = (-0.5, 3, 3)'$ are characterized by having at least two identical state-dependent realizations. □

3.3. One risk free and two risky assets

We continue portfolio selection by introducing a risk free asset. For (μ,σ^2)-preferences, we stay in line with the literature and assume $X_0 < E(X_{MVP})$ to ensure that the risk free asset lies below the intersection of the asymptote of the $(\mu,\sigma^2)_1$-efficient frontier with the ordinate. In the (μ,ρ_ϕ)-framework, the risk free asset satisfies $X_0 = E(X_0) = -\rho_\phi(X_0)$ and lies on the bisector of the second quadrant. Therefore, we assume $-\rho_\phi(X_{MSP}) < X_0 < E(X_{MSP})$ (see Fig. 2).

As the first step, the $(\mu,\rho)_2$-efficient frontiers for one risk free asset X_0 and only one risky asset $X_{\bar{\gamma}}$ are analyzed, i.e., the set of alternatives is $X_{\beta,\bar{\gamma}} = \beta \cdot X_{\bar{\gamma}} + (1-\beta) \cdot X_0, \beta \geqslant 0$. Afterwards, we interpret the risky asset $X_{\bar{\gamma}}$ as a $(\mu,\rho)_1$-efficient portfolio that is composed of the risky basic assets.

Again, we briefly recall the traditional (μ,σ^2)-framework. The derivation of the $(\mu,\sigma^2)_2$-efficient frontier with respect to the set of alternatives $X_{\beta,\bar{\gamma}}, \beta \geqslant 0$ requires solving the portfolio's variance for the proportion β and plugging it into its expected return, which gives the well-known strictly concave (square root) function

$$E(X_{\beta,\bar{\gamma}}) = \frac{E(X_{\bar{\gamma}}) - X_0}{\sqrt{Var(X_{\bar{\gamma}})}} \cdot \sqrt{Var(X_{\beta,\bar{\gamma}})} + X_0. \tag{16}$$

Generally, any $(\mu,\sigma^2)_1$-efficient portfolio can serve as a risky asset $X_{\bar{\gamma}}$ in the above considerations. $(\mu,\sigma^2)_2$-efficient mean–variance combinations consist of the risk free asset X_0 and the $(\mu,\sigma^2)_1$-efficient portfolio X_{T,σ^2} that touches the parabola (12) at only one point, and thus is called *tangency portfolio* (see Fig. 3).

Proposition 3.5. *Let $\mathcal{X}$ be as in Setting 2 and $X_0 < E(X_{MVP})$. Then*

1. *the $(\mu,\sigma^2)_2$-efficient frontier is a strictly concave curve through the risk free asset and the tangency portfolio;*
2. *the tangency portfolio is given by*

$$\gamma_{T,\sigma^2} = \frac{(E(X_2) - X_0) \cdot b - (E(X_1) - E(X_2)) \cdot c}{(E(X_1) - E(X_2)) \cdot b - (E(X_2) - X_0) \cdot a} \in (-\infty, \gamma_{MVP}). \tag{17}$$

The proof is straightforward and therefore omitted. Proposition 3.5 provides the well-known Tobin separation (Tobin, 1958): Any $(\mu,\sigma^2)_2$-efficient portfolio is a linear combination of the risk free

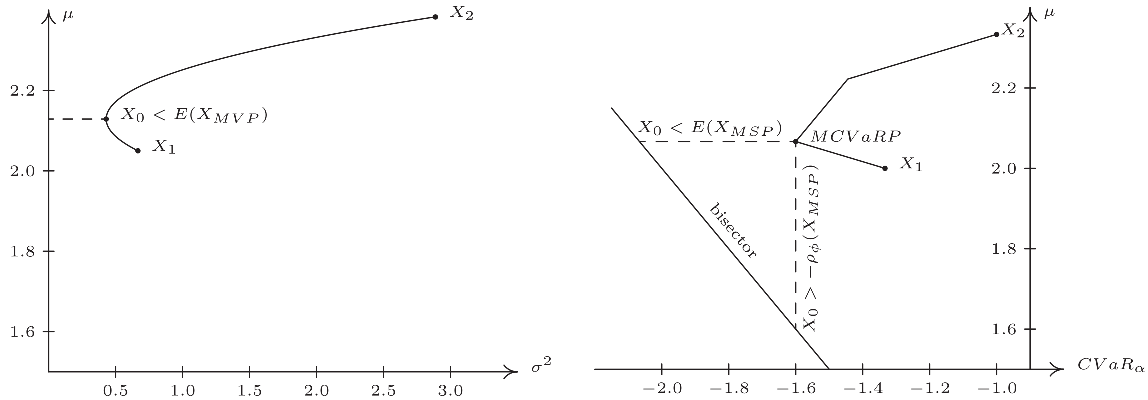

Fig. 2. Locus of the risk free asset.

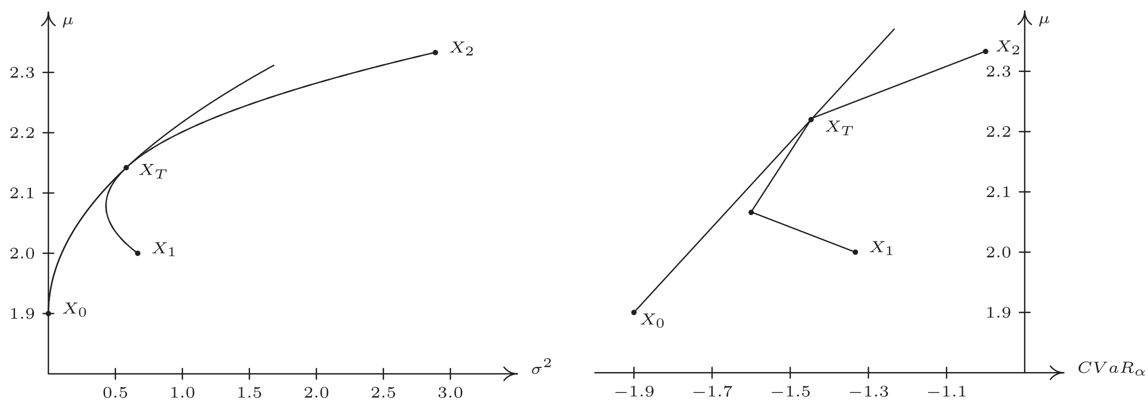

Fig. 3. $(\mu,\sigma^2)_2$-versus $(\mu,CVaR_\alpha)_2$-boundary with a risk free and two risky assets.

asset and the tangency portfolio. An investor's individual risk aversion only affects the proportions of the initial wealth that are invested in these assets.

A similar argument applies to the $(\mu,\rho_\phi)_2$-efficient frontier. As the set of alternatives $X_{\beta,\bar{\gamma}}, \beta \geqslant 0$ is comonotonic and spectral risk measures are comonotonic additive and translation invariant, we can solve the portfolio's spectral risk for the proportion β as

$$\begin{aligned}\rho_\phi(X_{\beta,\bar{\gamma}}) &= \rho_\phi(\beta\cdot X_{\bar{\gamma}} + (1-\beta)\cdot X_0)\\ &= \beta\cdot\rho_\phi(X_{\bar{\gamma}}) + (1-\beta)\cdot\rho_\phi(X_0) \;\Rightarrow\; \beta\\ &= \frac{\rho_\phi(X_{\beta,\bar{\gamma}}) - \rho_\phi(X_0)}{\rho_\phi(X_{\bar{\gamma}}) - \rho_\phi(X_0)}\end{aligned} \tag{18}$$

and substitute β into its expected return:

$$\begin{aligned}E(X_{\beta,\bar{\gamma}}) &= \beta\cdot E(X_{\bar{\gamma}}) + (1-\beta)\cdot X_0\\ &= \frac{E(X_{\bar{\gamma}}) - X_0}{\rho_\phi(X_{\bar{\gamma}}) - \rho_\phi(X_0)}\cdot\rho_\phi(X_{\beta,\bar{\gamma}}) - \frac{E(X_{\bar{\gamma}}) - X_0}{\rho_\phi(X_{\bar{\gamma}}) - \rho_\phi(X_0)}\\ &\quad\cdot\rho_\phi(X_0) + X_0.\end{aligned} \tag{19}$$

The portfolio's expected return is linearly increasing in its spectral risk.

Again, any $(\mu,\rho_\phi)_1$-efficient portfolio $X_{\bar{\gamma}}$ can serve as the risky asset. However, the only $(\mu,\rho_\phi)_2$-efficient combination consists of the risk free asset X_0 and the $(\mu,\rho_\phi)_1$-efficient portfolio, X_{T,ρ_ϕ}, where (19) is a tangent to the $(\mu,\rho_\phi)_1$-efficient frontier (tangency portfolio) (see Fig. 3); Tobin separation still holds. Note that this result crucially depends on the assumption of (μ,ρ_ϕ)-efficient risky basic assets; without this assumption, the $(\mu,\rho_\phi)_1$-efficient frontier might be empty and Tobin separation does not hold. The results are summarized in the following proposition.

Proposition 3.6. *Let $\mathcal{X}$ as in setting 2 and $-\rho_\phi(X_{MSP}) < X_0 < E(X_{MSP})$. Then*

1. *the $(\mu,\rho_\phi)_2$-efficient frontier is a straight line through the risk free asset and the tangency portfolio;*
2. *the tangency portfolio γ_{T,ρ_ϕ} lies in the set $\{\gamma_{ij,1:k},\ldots,\gamma_{ij,k:k}\}$.*

We continue the stylized Example 3.4 by adding a risk free asset.

Example 3.7. Let $X_0 = 1.9$ be the return of the risk free asset, which is added to the risky assets X_1 and X_2.

Fig. 3 shows the $(\mu,\sigma^2)_2$-efficient frontier, which is a strictly concave curve through X_0 and X_{T,σ^2}. The tangency portfolio $\gamma_{T,\sigma^2} = 0.57$ is characterized by the state-dependent returns $X_T = (2.28, 1.15, 3)'$.

Further, Fig. 3 shows the $(\mu,CVaR_\alpha)_2$-efficient frontier as a straight line through X_0 and $X_{T,CVaR_\alpha}$. The tangency portfolio $\gamma_{T,CVaR_\alpha} = 0.8$ with $X_{T,CVaR_\alpha} = (1.6, 1.6, 3)'$ is characterized by having at least two identical state-dependent realizations. □

3.4. Extensions

In order to keep the analysis simple, so far we have imposed two assumptions: (i) $m = 2$ risky assets, and (ii) (μ,ρ)-efficiency of these risky basic assets. We now show that relaxing the assumptions does not change the shape of the (μ,ρ)-efficient frontiers.

For the (μ,σ^2)-framework, it is well-known that the strict concavity of the (μ,σ^2)-efficient frontiers as well as Tobin separation hold in the absence of the above restrictions (see the Appendix A for another proof).

We proceed with the (μ,ρ_ϕ)-framework, and start by omitting the (μ,ρ_ϕ)-efficiency of the risky basic assets, but still restrict their number to $m = 2$. The efficiency assumption ensures that the minimum-spectral risk portfolio exists and the $(\mu,\rho_\phi)_1$-efficient frontier is non-empty. The following counter-example shows that this result no longer holds when relaxing the assumption.

Example 3.8. Let the risky assets be given as in Example 3.4. As spectral risk measures, the investor applies Conditional Value-at-Risk at the confidence level $\alpha_1 = 0.8$ and $\alpha_2 = 0.9$. In both cases, X_1 is not $(\mu, CVaR_\alpha)$-efficient.

For $\alpha_1 = 0.8$, the minimum-spectral risk portfolio is given by $X_{0.34} = (3, 0.67, 3)'$, and the $(\mu, CVaR_{\alpha_1})_1$-efficient frontier contains all portfolios $\gamma \in (-\infty, 0.34]$. For $\alpha_2 = 0.9$, the minimum-spectral risk portfolio does not exist. Instead, the $(\mu, CVaR_{\alpha_2})_1$-boundary is strictly decreasing. Fig. 4 shows the corresponding $(\mu,\ CVaR_\alpha)_1$-boundaries. □

The non-existence of the minimum-spectral risk portfolio is closely related to the property of translation invariance; due to

$$\rho_\phi(X) = \rho_\phi(X + E(X) - E(X)) = -E(X) + \rho_\phi(X - E(X)), \tag{20}$$

two separate effects can be identified when moving upward along the $(\mu,\rho_\phi)_1$-boundary. The first effect is captured by $-E(X) < 0$ (mean effect), and leads to a decrease in spectral risk. The second effect (deviation effect) refers to $\rho_\phi(X - E(X)) > 0$, and has been introduced as *deviation measure* by Rockafellar et al., 2006. The deviation effect leads to an increase in spectral risk. Depending on whether the mean effect outweighs the deviation effect, the $(\mu,\rho_\phi)_1$-efficient frontier is empty or non-empty.

These effects show that spectral risk measures exhibit both properties of location measures and deviation measures simultaneously. Especially, the former is a reasonable requirement for monetary and regulatory risk measurement ("risk" context), as an increase in an asset's mean should result in reduced solvency capital requirements. At the same time, the location property may lead to the non-existence of the minimum-spectral risk portfolio when applied to portfolio selection ("decision" context). We summarize as follows.

Theorem 3.9. *For $m = 2$ risky assets, the assumption of (μ,ρ_ϕ)-efficiency is a sufficient, although not necessary, condition for the existence of the minimum-spectral risk portfolio and the non-emptiness of the $(\mu,\rho_\phi)_1$-efficient frontier. For a risk free asset and $m = 2$ risky assets with $-\rho_\phi(X_{MSP}) < X_0 < E(X_{MSP})$, the assumption of (μ,ρ_ϕ)-efficient risky basic assets is a sufficient, although not necessary, condition for Tobin separation.*

The sufficiency immediately follows from the relations $E(X_1) < E(X_2) \wedge \rho_\phi(X_1) < \rho_\phi(X_2)$ in connection with the convexity of spectral risk measures; Example 3.8 shows the non-necessity.

Considering $m \geqslant 2$ risky assets and omitting the efficiency restriction does not change the results from the two-asset framework either.

Theorem 3.10. *For $m \geqslant 2$ risky assets and in the absence of efficiency restrictions,*

1. *if the $(\mu,\rho_\phi)_1$-efficient frontier is non-empty, it is a concave curve that is piecewise linear for comonotonic subsets of alternatives*;
2. *if the $(\mu,\rho_\phi)_1$-efficient frontier is non-empty and a risk free asset with $-\rho_\phi(X_{MSP}) < X_0 < E(X_{MSP})$ exists, the $(\mu,\rho_\phi)_2$-efficient frontier is a straight line through the risk free asset and the tangency portfolio.*

For the proof, see Benati, 2003, Theorem 2, in connection with (14) and (19) for the linearity property.

Finally note that the linearity of the $(\mu,\rho_\phi)_2$-efficient frontier in Extension 3.10, Part 2, is valid under arbitrary distributions, as the derivation in (18) and (19) only relies on the properties of the spectral risk measure, but does not impose any assumption on the distribution of the asset returns X. Especially, the linearity preserves under the assumption of normally distributed returns, which is common in the theoretical literature on portfolio selection with Conditional Value-at-Risk and spectral risk measures as yet. For this case, Giorgi, 2002, Section 5.1, has proved the following proposition, which is a special case of the more general Extension 3.10.

Proposition 3.11. *For one risk free asset and $m \geqslant 2$ risky assets with multivariate normally distributed returns, $-\rho_\phi(X_{MSP}) < X_0 < E(X_{MSP})$, and in the absence of efficiency restrictions*,

1. *if the $(\mu,\rho_\phi)_1$-efficient frontier is non-empty, the $(\mu,\rho_\phi)_2$-efficient frontier is a straight line through the risk free asset and the tangency portfolio*;
2. *if the $(\mu,\rho_\phi)_1$-efficient frontier is non-empty, the tangency portfolios in the (μ,σ^2)-framework and the (μ,ρ_ϕ)-framework coincide.*

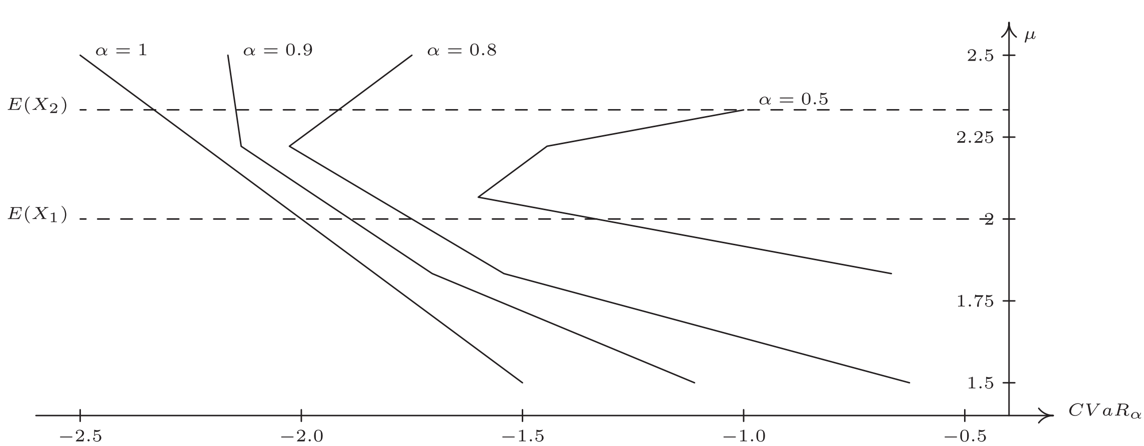

Fig. 4. $(\mu, CVaR_\alpha)_1$-boundary without efficiency restriction.

Under the assumption of normally distributed returns, the portfolios on the $(\mu,\sigma^2)_2$- and the $(\mu,\rho_\phi)_2$-efficient frontiers coincide, but are still different in shape: While the $(\mu,\sigma^2)_2$-efficient frontier is a strictly concave square root function, the $(\mu,\rho_\phi)_2$-efficient frontier is a straight line. As a consequence, the choice of optimal portfolios also differs fundamentally under the variance and spectral risk measures.

4. Optimal portfolios

4.1. Determination of optimal portfolios

Based on the (μ,ρ)-efficient frontiers, we now turn to the choice of optimal portfolios ("decision" context), which is the key issue of this paper.

In the prevailing literature, optimal portfolios are chosen by fixing a certain level of expected return, $\bar{\mu}$, and finding the corresponding portfolio on the (μ,ρ)-efficient frontier (e.g., Adam et al., 2008; Bassett et al., 2004; Benati, 2003; Bertsimas et al., 2004; Giorgi, 2002; Krokhmal et al., 2002; Rockafellar and Uryasev, 2000). Unlike this limited analysis, we apply a more general tradeoff analysis, which explicitly models an investors (μ,ρ)-preferences in the form of a (μ,ρ)-utility function, and finds the optimal portfolio where the induced indifference curves are tangent to the (μ,ρ)-efficient frontiers.

In the (μ,σ^2)-framework, the limited analysis and the tradeoff analysis are equivalent approaches. Searching for the $(\mu,\sigma^2)_2$-efficient frontier requires solving

$$\begin{aligned}&\min_{\beta\geqslant 0,\gamma\in\mathbb{R}} \frac{1}{2}\cdot Var(X_{\beta,\gamma})\\ &\quad \text{s.t. } E(X_{\beta,\gamma})=\bar{\mu}\geqslant E(X_{MVP}),\end{aligned} \tag{21}$$

which can be written in the tradeoff form

$$\max_{\beta\geqslant 0,\gamma\in\mathbb{R}} E(X_{\beta,\gamma})-\frac{\bar{\lambda}}{2}\cdot Var(X_{\beta,\gamma}). \tag{22}$$

We refer to (22) as the *mean–variance utility function*. The two problems (21) and (22) with parameters $\bar{\mu}$ and $\bar{\lambda}$, respectively, are equivalent if and only if $\bar{\mu}=X_0+\frac{(E(X_{T,\sigma^2})-X_0)^2}{\bar{\lambda}\cdot Var(X_{T,\sigma^2})}$ (e.g., Steinbach, 2001, Theorem 1.9). Due to this one-to-one-correspondence between $\bar{\mu}$ and $\bar{\lambda}$, one can choose between two equivalent approaches for optimal portfolio selection. As a first approach (limited analysis), an investor can fix a certain level of expected return $\bar{\mu}$. The optimal portfolio then is given by the corresponding portfolio on the $(\mu,\sigma^2)_2$-efficient frontier. As a second approach (tradeoff analysis), the same optimal portfolio obtains if the investor applies the indifference curve induced by the mean–variance utility function (22) with risk aversion $\bar{\lambda}=\frac{(E(X_{T,\sigma^2})-X_0)^2}{(\bar{\mu}-X_0)\cdot Var(X_{T,\sigma^2})}$ to the $(\mu,\sigma^2)_2$-efficient frontier.

From a decision theoretic perspective, the mean–variance utility function (22) follows from the assumptions of an expected utility maximizer with constant absolute risk aversion $\lambda=\bar{\lambda}$ and normally distributed returns (e.g., Bamberg, 1986, p. 20). These assumptions and the mean–variance utility function are well-established in portfolio selection due to their striking analytical advantages (e.g., Alexander and Baptista, 2002; Lintner, 1969; Sentana, 2003).

Next, we reproduce the above argument in the (μ,ρ_ϕ)-framework, and show that the situation becomes fundamentally different. Searching for the $(\mu,\rho_\phi)_2$-efficient frontier requires solving

$$\begin{aligned}&\min_{\beta\geqslant 0,\gamma\in\mathbb{R}} \rho_\phi(X_{\beta,\gamma})\\ &\quad \text{s.t. } E(X_{\beta,\gamma})=\bar{\mu}\geqslant E(X_{MSP}),\end{aligned} \tag{23}$$

which has a tradeoff version of the form

$$\max_{\beta\geqslant 0,\gamma\in\mathbb{R}} (1-\bar{\lambda})\cdot E(X_{\beta,\gamma})-\bar{\lambda}\cdot\rho_\phi(X_{\beta,\gamma}),\quad \bar{\lambda}\in[0,1]. \tag{24}$$

For reasons stated below, we refer to (24) as *spectral utility function*. While the two problems (23) and (24) induce the same $(\mu,\rho_\phi)_2$-efficient frontier (e.g., Krokhmal et al., 2002, Theorem 3; Acerbi and Simonetti, 2002, Proposition 4.2), we no longer observe a one-to-one correspondence between $\bar{\mu}$ and $\bar{\lambda}$. As Tobin separation holds in setting 2, the relevant first-order condition of (24) is given by

$$\frac{\mathrm{d}(\cdot)}{\mathrm{d}\beta}=(E(X_{T,\rho_\phi})-X_0)-\bar{\lambda}\cdot(E(X_{T,\rho_\phi})+\rho_\phi(X_{T,\rho_\phi})), \tag{25}$$

and has

$$\bar{\lambda}=\frac{E(X_{T,\rho_\phi})-X_0}{E(X_{T,\rho_\phi})+\rho_\phi(X_{T,\rho_\phi})}\geqslant 0 \tag{26}$$

as a unique solution. As $\bar{\lambda}$ does not depend on $\bar{\mu}$, an investor's specific (μ,ρ_ϕ)-preferences in the form of a spectral utility function with risk aversion $\bar{\lambda}$ do not imply a unique level of expected return $\bar{\mu}$. Hence, for the choice of optimal portfolios it is no longer sufficient to fix a certain level of expected return $\bar{\mu}$ and to find the corresponding portfolio on the $(\mu,\rho_\phi)_2$-efficient frontier as is done by the limited analysis. Rather, this approach neglects that certain levels of expected return are not under an investor's consideration if he maximizes a spectral utility function. It instead becomes necessary to apply the tradeoff analysis, which finds the optimal portfolio at the tangential point between the indifference curves induced by the spectral utility function (24) and the $(\mu,\rho_\phi)_2$-efficient frontier.

The spectral utility function (24) receives strong support also from a decision theoretic perspective. Besides the two components (negative) mean $-E(X)$ and the spectral risk measure $\rho_\phi(X)$, the negative convex combination $\pi_\phi(X):=-\rho_\phi(X)=(1-\lambda)\cdot E(X)-\lambda\cdot\rho_\phi(X)$, $\lambda\in[0,1]$ satisfies (up to the algebraic sign) the properties of spectral risk measures as well (Acerbi, 2002, Proposition 2.2). Therefore, the determination of the (μ,ρ_ϕ)-efficient frontiers and the consequent choice of optimal portfolios by maximizing a spectral utility function π_ϕ are based on one consistent axiomatic framework (e.g., Acerbi and Simonetti, 2002). However, this framework is still based on the underlying regulatory concept of diversification, which relates exclusively to the dependence structure.

Based on the above argument we give the following definition, which implements the tradeoff analysis.

Definition 4.1. A portfolio is called optimal if it maximizes a utility function π over a set of alternatives $\mathcal{X}$.

The optimal portfolios are located where the indifference curves induced by the mean–variance utility function (22) and the spectral utility function (24) are tangent to the (μ,ρ)-efficient frontiers. As the analysis so far has been based on the (μ,σ^2)-plane and the (μ,ρ_ϕ)-plane, the induced indifference curves both are linearly increasing with slope $\frac{\mathrm{d}E}{\mathrm{d}Var}=\frac{\lambda}{2}\geqslant 0$ and $\frac{\mathrm{d}E}{\mathrm{d}\rho_\phi}=\frac{\lambda}{1-\lambda}\geqslant 0$, respectively. For both the mean–variance utility function and the spectral utility function, an investor's risk aversion increases with increasing λ, as the corresponding certainty equivalents, $\pi(X)$ and $\pi_\phi(X)$, decrease.

4.2. The mean–variance utility function and full diversification

As has been argued in Section 3, the $(\mu,\sigma^2)_2$-efficient frontier is a strictly concave curve. The marginal rate of transformation according to (16) is

$$\frac{\mathrm{d}E}{\mathrm{d}Var}=\frac{E(X_{T,\sigma^2})-X_0}{2\cdot\beta\cdot Var(X_{T,\sigma^2})}\in(0,\infty)\quad\text{for } \beta\in(0,\infty). \tag{27}$$

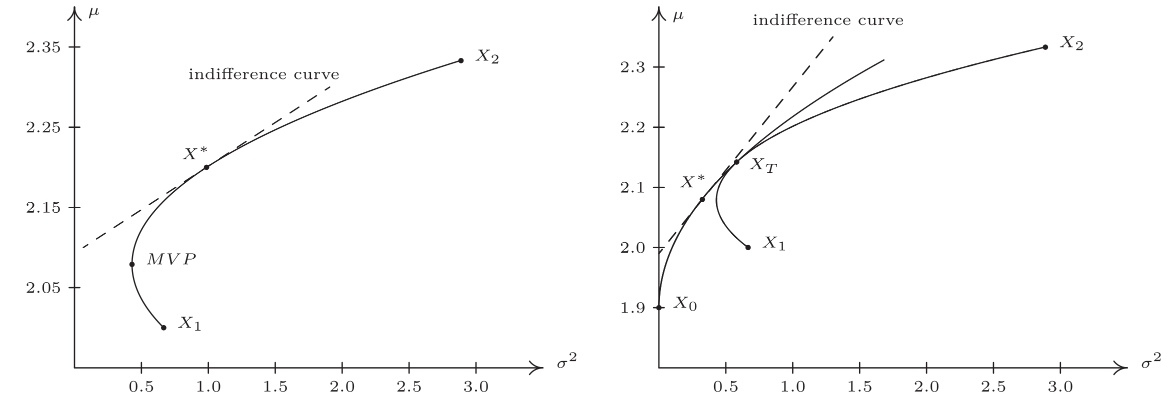

Fig. 5. Choice of optimal portfolios with the mean–variance utility function.

Similarly, the $(\mu,\sigma^2)_1$-efficient frontier corresponds to the strictly concave upper branch of a parabola. Its marginal rate of transformation according to (12) is given by

$$\frac{\mathrm{d}E}{\mathrm{d}Var} = \frac{E(X_1) - E(X_2)}{2 \cdot (\gamma \cdot a + b)} \in (0,\infty) \quad \text{for } \gamma \in (-\infty, \gamma_{MVP}). \tag{28}$$

Taking into account that the indifference curves of the mean–variance utility function (22) are linear, i.e., the marginal rate of substitution $\frac{\mathrm{d}E}{\mathrm{d}Var} = \frac{\lambda}{2}$ is constant, we immediately get the following proposition (see Fig. 5).

Proposition 4.2. *Suppose an investor maximizes the mean–variance utility function (22) with respect to β and γ in settings 1 and 2, respectively. Then*

$$\beta^* = \frac{E(X_{T,\sigma^2}) - X_0}{\lambda \cdot Var(X_{T,\sigma^2})}, \tag{29}$$

$$\gamma^* = \gamma_{MVP} - \frac{E(X_2) - E(X_1)}{\lambda \cdot a}. \tag{30}$$

We are now prepared to characterize the concept of diversification underlying the traditional (μ,σ^2)-framework. First, consider the case with a risk free asset. The set of the $(\mu,\sigma^2)_2$-efficient portfolios is given by $\beta \in [0,\infty)$, where $\beta = 0$ describes the risk free asset and $\beta = 1$ corresponds to the tangency portfolio. We obtain the following results, which we refer to as *full diversification*:

- *Efficient versus optimal portfolios:* Any $(\mu,\sigma^2)_2$-efficient portfolio $\beta \in [0,\infty)$ can be optimal if the risk aversion is chosen adequately as

$$\lambda(\beta^*) = \frac{E(X_{T,\sigma^2}) - X_0}{\beta^* \cdot Var(X_{T,\sigma^2})} \geqslant 0; \tag{31}$$

i.e., the set of $(\mu,\sigma^2)_2$-efficient portfolios and the set of optimal portfolios coincide.

- *Comparative risk aversion:* The investment in the risk free asset is continuously increasing in the risk aversion λ, with $\lim_{\lambda\to 0}\beta^* = \infty$ and $\lim_{\lambda\to\infty}\beta^* = 0$.

Now consider the case without the risk free asset. The set of the $(\mu,\sigma^2)_1$-efficient portfolios is given by $\gamma \in (-\infty,\gamma_{MVP}]$, where $\gamma = 0$ corresponds to the risky asset X_2. Again, we observe full diversification:

- *Efficient versus optimal portfolios:* Any $(\mu,\sigma^2)_1$-efficient portfolio $\gamma \in (-\infty,\gamma_{MVP}]$ can be optimal if the risk aversion is adequately chosen as

$$\lambda(\gamma^*) = \frac{E(X_2) - E(X_1)}{(\gamma_{MVP} - \gamma^*) \cdot a} \geqslant 0; \tag{32}$$

i.e., the set of $(\mu,\sigma^2)_1$-efficient portfolios and the set of optimal portfolios, for any correlation coefficient $corr(X_1,X_2) \in [-1,1]$, coincide.

- *Comparative risk aversion:* The investment towards the minimum-variance portfolio is continuously increasing in the risk aversion λ, with $\lim_{\lambda\to 0}\gamma^* = \infty$ and $\lim_{\lambda\to\infty}\gamma^* = \gamma_{MVP}$.

The optimal proportions β^* and γ^*, respectively, depend on (i) reward and risk, given by the expected returns and the variances of the assets, and (ii) the dependence structure, given by $corr(X_1,X_2)$. Diversification in the (μ,σ^2)-framework, and in particular full diversification, is based on the tradeoff between reward and risk, while the dependence structure has an "indirect" impact only: If the risk free asset exists, a positive risk premium, $E(X_{T,\sigma^2}) - X_0 > 0$, is a necessary and sufficient condition for full diversification. This result is independent of the dependence structure, as it holds for any $corr(X_1,X_2) \in (-1,1)$ by choosing λ according to (31). Without a risk free asset, a positive excess return, $E(X_2) - E(X_1) > 0$, is a necessary and sufficient condition for full diversification. Again, this result obtains for any $corr(X_1,X_2) \in [-1,1]$. Note that full diversification also holds for $corr(X_1,X_2) = 1$. Even if the risky assets are linearly dependent, still any portfolio $\gamma^* \in (-\infty,\gamma_{MVP})$ can be optimal by choosing λ according to (32).

Full diversification under comonotonicity as, for example, between the risk free asset and the tangency portfolio, and especially between two risky assets with $corr(X_1,X_2) = 1$, may appear counterintuitive initially, but consistently reflects diversification based on the tradeoff between reward and risk: Even if the dependence structure does not provide an additional diversification benefit in reducing a portfolio's variance, an investor may still prefer a (μ,σ^2)-efficient reward-risk profile that lies in the interior of the comonotonic or perfectly positive correlated assets' reward-risk profiles.

4.3. Spectral utility functions and non-diversification

We take the case of a risk free asset as a starting point, as it brings us to our main result. The $(\mu,\rho_\phi)_2$-efficient frontier is a straight line through the risk free asset and the tangency portfolio. Its constant marginal rate of transformation according to (19) is given by

$$\frac{\mathrm{d}E}{\mathrm{d}\rho_\phi} = \frac{E(X_{T,\rho_\phi}) - X_0}{\rho_\phi(X_{T,\rho_\phi}) - \rho_\phi(X_0)} \geqslant 0. \tag{33}$$

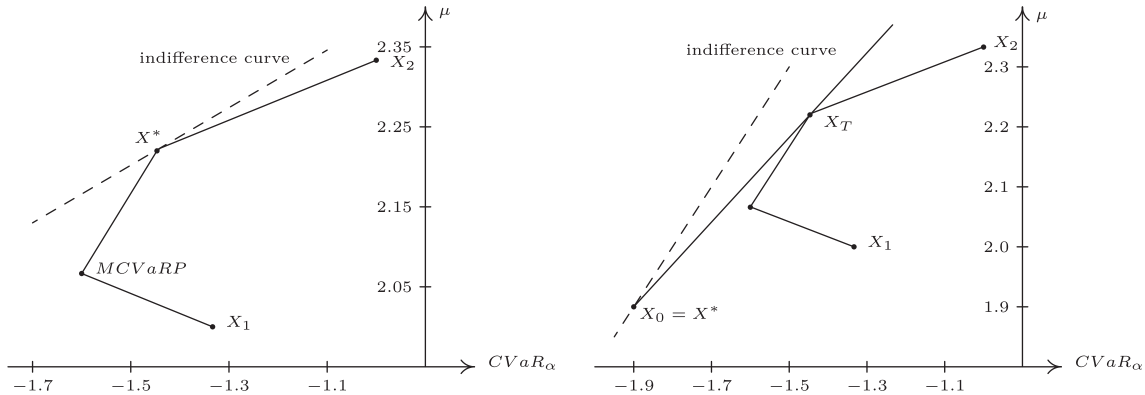

Fig. 6. Choice of optimal portfolios with the spectral utility function.

As the indifference curves of the spectral utility function are linear as well, and the marginal rate of substitution $\frac{dE}{d\rho_\phi} = \frac{\lambda}{1-\lambda}$ is constant, we immediately obtain the following Proposition (see Fig. 6).[5]

Proposition 4.3. *Suppose an investor maximizes a spectral utility function (24) with respect to β in Setting 2. Then*

$$\beta^* = \begin{cases} 0 & \text{if } \frac{E(X_{T,\rho_\phi})-X_0}{\rho_\phi(X_{T,\rho_\phi})-\rho_\phi(X_0)} \geqslant \frac{\lambda}{1-\lambda} \\ \infty & \text{else} \end{cases}. \tag{34}$$

In order to provide a more real-world interpretation, we impose the short-sale constraint $\beta \in [0,1]$ and get $\beta^* \in \{0,1\}$, i.e., the investor either invests in the risk free asset ($\beta = 0$) or the tangency portfolio ($\beta = 1$). We refer to this portfolio structure as *non-diversification*, which is characterized as follows:

- *Efficient versus optimal portfolios:* The set of $(\mu,\rho_\phi)_2$-efficient portfolios, $\beta \in [0,1]$, does not coincide with the set of optimal portfolios, $\beta^* \in \{0,1\}$, which only contains the corner positions.
- *Comparative risk aversion:* The optimal proportion β^* is not continuous in the risk aversion λ. Up to a certain degree of risk aversion, the exclusive investment in the tangency portfolio is optimal, and subsequently the optimum jumps towards the exclusive risk free investment.

Recall from Section 2.1 and Proposition 2.3 that spectral risk measures and spectral utility functions, respectively, are based on a regulatory concept of diversification. Within that concept, and unlike under the variance, it is no longer the tradeoff between reward and risk that induces diversification. Instead, non-comonotonicity is a necessary condition for a positive diversification benefit. As the risk free asset and the tangency portfolio are comonotonic, we do not find diversification. Note that, under non-diversification, a risk averse investor may decide for the exclusive investment in the tangency portfolio though a risk free asset is available. On the other hand, a risk averse investor may also decide for the exclusive investment in the risk free asset though the tangency portfolio offers a positive risk premium. Both portfolio selection decisions contradict the tradeoff between reward and risk, and thus appear as rather counter-intuitive when compared to the traditional (μ,σ^2)-framework.

From a decision theoretic perspective, non-diversification is not a convincing result, as there is no reason for excluding some $(\mu,\rho_\phi)_2$-efficient portfolios from being optimal, merely because they belong to a comonotonic set of alternatives. In this case, both expected return and spectral risk increase linearly. However, it might well be possible, and is economically plausible, that an investor prefers a $(\mu,\rho_\phi)_2$-efficient reward-risk profile somewhere in the interior of a comonotonic set of alternatives.

Further, Extension 3.10 shows that the all-or-nothing decision holds in settings that are far more general than our setting 2, as the $(\mu,\rho_\phi)_2$-efficient frontier is a straight line through the risk free asset and the tangency portfolio also for $m \geqslant 2$ risky assets, and for arbitrary return distributions. Therefore, the all-or-nothing decision also obtains under the assumption of normally distributed returns, and provides an interesting result: Although the efficient frontiers in the (μ,σ^2)-framework and the (μ,ρ_ϕ)-framework coincide, the choice of optimal portfolios differs fundamentally between full diversification and non-diversification.

The case without the risk free asset yields similar results. The $(\mu,\rho_\phi)_1$-efficient frontier is a concave curve that for comonotonic subsets of alternatives is piecewise linear. For a comonotonic subset of alternatives X_γ, $\gamma \in [\gamma_d, \gamma_u]$, the constant marginal rate of transformation is given by (see (15)):

$$\frac{dE}{d\rho_\phi} = \frac{E(X_{\gamma_d}) - E(X_{\gamma_u})}{\rho_\phi(X_{\gamma_d}) - \rho_\phi(X_{\gamma_u})}. \tag{35}$$

Together with the linear indifference curves of the spectral utility function, i.e., the marginal rate of substitution $\frac{dE}{d\rho_\phi} = \frac{\lambda}{1-\lambda}$ is constant, we immediately get the following result (see Fig. 6).

Proposition 4.4. *Suppose an investor maximizes a spectral utility function (24) with respect to γ in setting 1. Then*

$$\gamma^* \in \{-\infty, \gamma_{ij,1:k}, \ldots, \gamma_{ij,k:k}\}. \tag{36}$$

Diversification under spectral risk measures and risky assets thus provides similar results to the case with a risk free asset, and we refer to this portfolio structure as *limited diversification*:

- *Efficient versus optimal portfolios:* The set of optimal portfolios is restricted to the $(\mu,\rho_\phi)_1$-efficient boundaries of the comonotonic subsets of alternatives, whereas the set of $(\mu,\rho_\phi)_1$-efficient portfolios also contains the interior portfolios. Therefore, the set of efficient portfolios and the set of optimal portfolios do not coincide.
- *Comparative risk aversion:* The optimal proportion γ^* is not continuous in the risk aversion λ. With risk aversion increasing, the same proportion remains optimal until the portfolio jumps to the next corner position.

[5] Without loss of generality, we assume that if the marginal rate of transformation and the marginal rate of substitution coincide, the investor decides for a corner position.

5. Discussion

The spectral utility function (24) obviously is crucial for our results on non-diversification and limited diversification, respectively. Note, however, that this utility function is not an ad hoc choice. Rather, we obtain the spectral utility function by reproducing a portfolio selection approach that is well-established in the (μ, σ^2)-framework already for decades, in order to consistently compare the (μ, σ^2)-framework and the (μ, ρ_ϕ)-framework, and to highlight major differences and drawbacks that have been overlooked so far. In doing so, we contradict the recent literature that regularly states that "the model is a straightforward extension of the classic Markovitz mean variance approach" (Benati, 2003, p. 573). Instead, we show that earlier results on non-diversification under quantile-based risk measures and utility functions, respectively, can be recovered. For example, already Yaari, 1987, Section 6, notes that the dual theory of choice tends towards all-or-nothing decisions instead of diversification.

In order to obtain diversification under spectral risk measures, one would need to determine (μ, ρ_ϕ)-efficient frontiers first, and afterwards apply another type of (μ, ρ_ϕ)-utility function, which is nonlinear; such utility function then is not covered by the axiomatic framework of spectral risk measures anymore. However, using spectral risk measures instead of the variance for portfolio selection in the literature is mainly motivated by referring to their axiomatic framework. For example, Acerbi and Tasche, 2002a, p. 380, state that "in our opinion speaking of non-coherent [by this, they basically mean what we call in the present paper "non-spectral"] measures of risk is (..) useless and dangerous. In our language, the adjective coherent is simply redundant." When following this argument, it appears rather inconsistent to apply the axiomatic framework for the determination of (μ, ρ_ϕ)-efficient frontiers, but to refrain from it for the consequent choice optimal portfolios. This argument is further supported by numerous studies that maximize spectral utility functions and minimize spectral risk measures, respectively, in order to obtain optimal portfolios (e.g., Acerbi and Simonetti, 2002; Dhaene et al., 2005), optimal insurance contracts (e.g., Cai and Tan, 2007), or optimal order quantities (e.g., Jammernegg and Kischka, 2007). In these studies, the use of spectral risk measures and spectral utility functions, respectively, as an objective function ("decision" context) is also motivated by referring to their axiomatic framework.

When following the above argument, our results raise doubts on the theoretical foundation of empirical findings on portfolio selection with Conditional Value-at-Risk and spectral risk measures. The relevant literature compares optimal portfolios in the (μ, σ^2)-framework and the (μ, ρ_ϕ)-framework based on a fixed level of expected return $\bar{\mu} \in [X_0, E(X_{T,\rho_\phi})]$. This approach attempts to model and compare investors with "similar" preferences in the two frameworks. Beyond, many applications in practice are based on such limited analysis, where the tradeoff between reward and risk is made by a higher hierarchy by fixing $\bar{\mu}$, while the risk analyst should only assess the risk side afterwards. However, from a decision theoretic point of view, comparing portfolios based on a fixed level of expected return, i.e., applying a limited analysis, is admissible only when is has been proved that this level of expected return can actually be optimal in the respective frameworks. While the proof succeeds in the (μ, σ^2)-framework, it fails in the (μ, ρ_ϕ)-framework. Therefore, empirical studies so far neglect that (μ, ρ_ϕ)-investors would not choose expected returns in the interval $\bar{\mu} \in (X_0, E(X_{T,\rho_\phi}))$, but always prefer one of the corner positions $\bar{\mu} = X_0$ or $\bar{\mu} = E(X_{T,\rho_\phi})$ to any diversified portfolio.

Moreover, experimental evidence is mixed. A first strand of literature shows that investors diversify between a risk free and a risky asset (e.g., Benartzi and Thaler, 1999). In another strand of literature underdiversification is documented (e.g., Mitton and Vorkink, 2007). However, underdiversification and limited diversification, as in the present paper, are different concepts. Whereas underdiversification denotes the investment in only a few assets, limited diversification refers to the fact that only a few (μ, ρ_ϕ)-efficient portfolios are actually chosen by an investor; whether these portfolios consist of a few or a large number of assets is not subject to considerations here. On the other hand, the (μ, ρ_ϕ)-framework might be a possible explanation for the stock market participation puzzle. For example, Mankiw and Zeldes, 1991, find that only 25% of investors hold stocks, which can easily be reconciled with a (μ, ρ_ϕ)-framework, but not with a (μ, σ^2)-framework.

Therefore, the use of spectral risk measures for portfolio selection faces some serious drawbacks from a decision theoretic, an empirical and also, in parts, an experimental perspective. In a sense, already Markowitz, 1952, raised serious doubts in this respect by stating that "diversification is both observed and sensible; a rule of behavior which does not imply the superiority of diversification must be rejected both as a hypothesis and as a maxim" (p. 77).

6. Conclusions

In this paper, we applied Conditional Value-at-Risk and, as its natural extension, spectral risk measures to some standard portfolio selection problems. Our approach differs from the previous literature in two respects. First, we use a finite state space approach to show that the efficient frontiers are piecewise linear without the risk free asset, and linear if the risk free asset exists. By contrast, in the traditional mean–variance framework the efficient frontiers are strictly concave in any case. Second, we show that choosing optimal portfolios by fixing a certain level of expected return and finding the corresponding portfolio on the efficient frontier (limited analysis), as is done in the relevant literature so far, provides misleading results. By applying the indifference curves induced by a spectral utility function (tradeoff analysis), which naturally arises both from a decision theoretic and an optimization perspective, we obtain fundamentally different optimal portfolios. If the risk free asset is available, diversification is never optimal. Likewise, without the risk free asset, we find only limited diversification. By contrast, the mean–variance framework shows full diversification in any case.

The reason is that spectral risk measures originally have been introduced for the assessment of solvency capital ("risk" context). The underlying regulatory concept of diversification regards the dependence structure between the assets as the only source for a positive diversification benefit. For the special case of perfect positive dependence (comonotonicity) the diversification benefit is zero, which is an adequate requirement for the assessment of solvency capital. Traditional mean–variance portfolio selection ("decision" context), by contrast, is based on a fundamentally different concept of diversification that refers to the tradeoff between reward and risk, and that is only indirectly affected by the dependence structure between the assets. The incompatibility of these conflicting concepts of diversification produces non-diversification and limited diversification, respectively.

In formal terms, the concept of diversification underlying spectral risk measures (and spectral utility functions) is determined jointly by the properties of subadditivity (superadditivity) and comonotonic additivity. The relevant literature regularly is focused on subadditivity only and omits considering comonotonic additivity. We indeed agree that subadditivity is an essential property for the assessment of solvency capital and for portfolio selection. However, we have shown that comonotonic additivity is essential only for the assessment of solvency capital ("risk" context), but leads to paradoxical results if applied to portfolio selection ("decision" context). As a more appropriate alternative to spectral risk measures, and as an agenda for future research, we would instead propose convex risk measures, which do not require comonotonic additivity (e.g., Föllmer and Schied, 2002).

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Appendix A. Extensions of the (μ,σ^2)-framework

Considering $m \geqslant 2$ risky assets and omitting the efficiency restriction yields the well-known mutual fund theorem (Merton, 1972, Section 3): The $(\mu,\sigma^2)_1$-boundary can be generated by any two distinct $(\mu,\sigma^2)_1$-boundary portfolios. The $(\mu,\sigma^2)_1$-efficient frontier lies on the upper branch of the $(\mu,\sigma^2)_1$-boundary starting from the minimum-variance portfolio. For a rigorous analytical treatment see, for example, Giorgi, 2002, Section 4.1. Recall from Proposition 3.2 that the $(\mu,\sigma^2)_1$-efficient frontier for two risky assets is a strictly concave curve for any correlation coefficient $corr(X_1,X_2) \in [-1,1]$. As the case of $m \geqslant 2$ risky assets is formally equivalent to the case of two distinct $(\mu,\sigma^2)_1$-boundary portfolios, we obtain the following extension.

Theorem A.1. *For $m \geqslant 2$ risky assets and in the absence of the efficiency restriction, the $(\mu,\sigma^2)_1$-efficient frontier is a strictly concave curve that starts from the minimum-variance portfolio.*

Also, it is well-known that by additionally considering a risk free asset, Tobin separation from Proposition 3.5 still holds. For a full analytical treatment see again Giorgi, 2002, Section 5.1. We summarize as follows.

Theorem A.2. *For a risk free asset and $m \geqslant 2$ risky assets with $E(X_0) < E(X_{MVP})$ and in the absence of the efficiency restriction, the $(\mu,\sigma^2)_2$-efficient frontier is a strictly concave curve through the risk free asset and the tangency portfolio.*

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