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Why Risk Is Not Variance: An Expository Note

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Variance (or standard deviation) of return is widely used as a measure of risk in financial investment risk analysis applications, where mean-variance analysis is applied to calculate efficient frontiers and undominated portfolios. Why, then, do health, safety, and environmental (HS&E) and reliability engineering risk analysts insist on defining risk more flexibly, as being determined by probabilities and consequences, rather than simply by variances? This note suggests an answer by providing a simple proof that mean-variance decision making violates the principle that a rational decisionmaker should prefer higher to lower probabilities of receiving a fixed gain, all else being equal. Indeed, simply hypothesizing a continuous increasing indifference curve for mean-variance combinations at the origin is enough to imply that a decisionmaker must find unacceptable some prospects that offer a positive probability of gain and zero probability of loss. Unlike some previous analyses of limitations of variance as a risk metric, this expository note uses only simple mathematics and does not require the additional framework of von Neumann Morgenstern utility theory.

KEY WORDS: Expected utility; mean-variance analysis; risk measure; stochastic dominance; variance

1. INTRODUCTION

Two plausible principles for managing financial investment risks are the following:

1. *Rule 1: Make dominating choices:* Other things being held equal, given a choice between a smaller probability of gain and a larger probability of gain, a decisionmaker should always choose the larger probability of gain. For example, given a choice between winning \$100 with probability 0.1 and winning \$100 with probability 0.2, rational decisionmakers who prefer more dollars to less should choose the option that gives a 0.2 probability of winning the \$100.
2. *Rule 2: Seek mean-variance efficiency (higher variance requires higher mean return):* Given a choice among risky prospects, an investor should require more expected return to accept a prospect with more variance than to accept a prospect with less variance. For example, a 0.2 chance of winning \$100 (else nothing) has a higher variance than a 0.1 chance of winning \$100, but it also has a higher mean.

Rule 1 is implied by the decision-analytic principle of first-order stochastic dominance (Sheldon & Sproule, 1997): prospects that give higher probabilities of preferred outcomes (and lower probabilities of less preferred outcomes) should be preferred. Rule 2 provides the basis for many current efficient portfolio and mathematical optimization (e.g., quadratic programming) approaches to optimal investment (http://en.wikipedia.org/wiki/Modern_portfolio_theory). Although theorists have noted that some risk-averse decisionmakers may prefer some mean-preserving increases in variance (Sheldon & Sproule, 1997), the idea that volatility in returns, as measured by variance or standard deviation, is generally undesirable to risk-averse investors, and that it should be avoided

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or compensated by higher expected returns, is still widely taught and practiced.

This note emphasizes that *Rules 1 and 2 are incompatible* in general. Simply hypothesizing that a decisionmaker has continuous upward-sloping indifference curves for mean-variance combinations (so that increasing the variance in the random return from an investment prospect or portfolio requires increasing its mean return in order to leave the investor equally well off) violates Rule 1 for some simple prospects, as demonstrated next.

2. FRAMEWORK AND TERMINOLOGY: CERTAINTY EQUIVALENTS AND ACCEPTABLE RISKS IN MEAN-VARIANCE SPACE

Following the literature on mean-variance decision making, suppose that a decisionmaker has positively sloped continuous indifference curves in mean-variance space (e.g., Wong, 2006). To any mean-variance pair (m, v) (a point in the mean-variance space) there corresponds a *certainty equivalent*, namely, the point at which the indifference curve through (m, v) reaches the horizontal (mean) axis. The indifference curve through the origin (0, 0) separates *acceptable risks* (those with positive certainty equivalents, lying below and to the right of the curve, if return is desirable) from *unacceptable risks* (those with negative certainty equivalents, lying above and to the left of it). To make an unacceptable risk acceptable in this framework, one must increase its mean return or reduce its variance. (A risk-neutral decisionmaker, who cares only about means and not about variances, would have vertical indifference curves, but we will focus on the case, implied by Rule 2, of positively sloped indifference curves.)

3. RESULTS

The hypothesis that upward-sloping mean-variance indifference curves exist has some surprising consequences.

THEOREM 1: *If the indifference curve through the origin slopes upward, then the decisionmaker finds unacceptable some prospects with positive expected values and no possibility of loss.*

Proof: We offer a constructive proof. Let the slope of the indifference curve through the origin at the origin be s. By hypothesis, $0 < s < \infty$. Now, consider a Bernoulli random variable $X(p)$ that gives a positive return of \$2s with probability p (the "win probability") and gives no return (\$0) with probability ($1-p$). For a given value of p between 0 and 1 inclusive, $X(p)$ has mean $2ps$ and variance $4s^2p(1-p)$ (since it is a scaled version of a Bernoulli random variable). Therefore, as p ranges from 0 to 1, $X(p)$ traces out a parabola in mean-variance space, with variance = 0 at $p = 0$ and at $p = 1$ and with a positive maximum variance of s^2 at $p = 0.5$ (see Fig. 1). A line from (0, 0) to the point on this parabola corresponding to a particular value of p has slope $4s^2p(1-p)/2ps = 2s(1-p)$. As p approaches 0, this slope approaches $2s$. Hence,

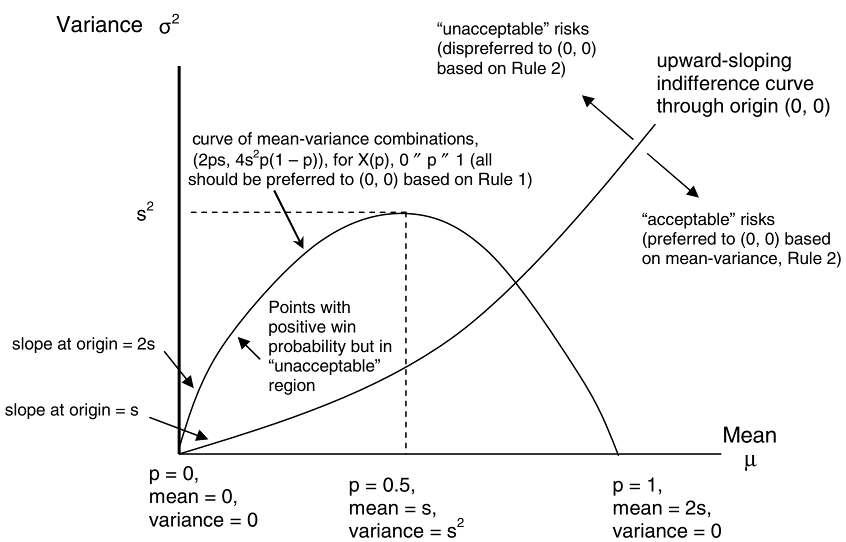

Fig. 1. Geometry of inconsistency between Rules 1 and 2.

the parabola traced out by $X(p)$ as p ranges from 0 to 1 starts above and to the left of the indifference curve through the origin (since it is constructed to have twice the slope of the indifference curve at the origin), but it ends below and to the right of the indifference curve (since it is constructed to pass through the point ($2s$, 0) when $p = 1$.) Therefore, the parabola must intersect the indifference curve somewhere above and to the right of the origin (since it starts above it and ends below it). Let p^* denote the value of the win probability for this intersection point. Then the decisionmaker prefers (0, 0) to all prospects $X(p)$ with $p < p^*$, since by construction, these are unacceptable (i.e., above and to the left of the indifference curve through the origin). Hence, the decisionmaker finds unacceptable all such prospects giving probability p of \$$2s$ (else \$0) for $p < p^*$, even though they have positive win probabilities and even though none offers the possibility of a loss. QED

The proof of Theorem 1 implies that, if the indifference curve through the origin has positive slope, then the decisionmaker prefers some prospects that give 0 probability of winning a positive amount (namely, $2s$) to other prospects that give positive probability of winning the positive amount (and otherwise nothing). Such a decisionmaker prefers the *status quo* or "nothing ventured, nothing gained" point (0, 0) to the possibility of winning a positive amount without any possibility of a loss, violating Rule 1. In this sense, Rules 1 and 2 are incompatible.

More generally, other parabolas can easily be constructed that intersect indifference curves twice, once for the ascending (positively sloped) portion of the parabola and once for its descending (negatively sloped) portion (Borch, 1969). In any such construction, the right-most intersection represents a stochastically dominant prospect (which should be preferred, by Rule 1) compared to the left-most intersection. That both points lie on the same indifference curve violates Rule 1.

4. DISCUSSION AND CONCLUSIONS

Students of finance are often taught that "risk" is measured by the variance or standard deviation of returns around an expected value. On the other hand, students of health, safety, and environmental (HS&E) and reliability engineering risk analysis are often taught that "risk" is determined by the probabilities of different consequences. This note has suggested a simple construction to show why the second approach (considering different specific consequences, such as \$0 and \$$2s$, and their probabilities) can be preferable to considering only means and variances.

The finding that variance is problematic as a measure of risk is not new. The most common critique in the theoretical decision analysis and financial economics literatures is that mean-variance analysis is compatible with von Neumann-Morgenstern expected utility theory only under very restrictive conditions (e.g., if all risky prospects have normal or location-scale distributions and utility functions are quadratic, implying that less money is preferred to more, for some amounts) (Markowitz, 1959; Baron, 1977). Mean-variance dominance and stochastic dominance relations for location-scale distributions do not coincide in general (Wong, 2006). Indeed, expected utility theory is inconsistent with all possible moment-based preference models (in which preferences are determined by mean, variance, skewness, kurtosis, etc.) for many utility functions (Brockett & Kahane, 1992). Variance is also inconsistent with proposed normative axioms for financial risk measures (non-negativity, homogeneity, and subadditivity, which together imply that deterministic outcomes have zero risk; and shift-invariance, which implies that adding a constant to a random variable does not change its risk) (Pedersen & Satchell, 1999). Empirical studies since the 1960s have demonstrated that real decisionmakers pay attention to more than mean and variance in their choices among risky prospects (Jia *et al*., 1999).

Thus, Theorem 1 is consistent with a long line of previous research. However, in contrast to much previous work, it demonstrates a conflict between Rules 1 and 2 making only minimal assumptions (in particular, not requiring the framework of von Neumann-Morgenstern expected utility theory or other sets of normative axioms for risk measures) and using only elementary mathematics. It may, therefore, be useful to students of risk analysis who wish to understand why knowing the variances and expected returns from alternative investment choices (or other actions) does not necessarily suffice to identify the choice with the most desirable probability distribution of consequences.

REFERENCES

Baron, D. P. (1977). On the utility-theoretic foundation of mean-variance analysis. *Journal of Finance*, *32*(5), 1683–1697.

Borch, K. H. (1969). A note on uncertainty and indifference curves. *Review of Economic Studies*, *36*(1), 1–14.

Brockett, P. L., & Kahane, Y. (1992). Risk, return, skewness and preference. *Management Science*, *38*(6), 851–866.

Jia, J., Dyer, J. S., & Butler, J. C. (1999). Measures of perceived risk. *Management Science*, *45*(4), 519–532. Available at http://fisher.osu.edu/~butler_267/DAPapers/WP970005.pdf.

Markowitz, H. M. (1959). *Portfolio Selection: Efficient Diversification of Investments*. New York: John Wiley & Sons.

Pedersen, C. S., & Satchell, S. E. (1999). *Choosing the Right Risk Measure: A Survey*. Available at http://citeseer.ist.psu.edu/52(0923).html.

Sheldon, A., & Sproule, R. (1997). Variance as a proxy for risk: The case of the binomial distribution. *Statistician*, *46*(3), 317–322.

Wong, W.-K. (2006). Stochastic dominance theory for location-scale family. *Journal of Applied Mathematics and Decision Sciences Volume 2006*, Article ID 82049, 1–19. www.emis.de/journals/HOA/JAMDS/f84d.html?doi=10.1155/JAMDS/2006/82049&e=ref.