Chapter 1

Trigonometry

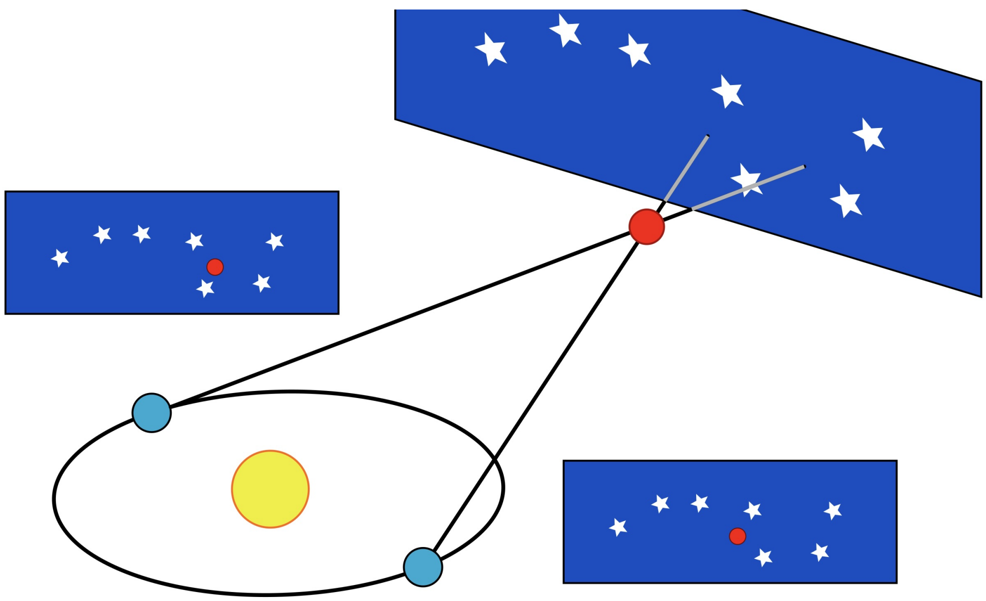


Figure 1.1: Example of a parallax. Shown in the figure is Earth in January, vs Earth in July. There is a relative difference in the apparent position of the start in January versus July, given by the Parallax Angle.

Assignment 2: Trigonometry

Fri, 15 Nov 19:00

1.1 Trigonometric Functions

The coordinates of any point on the unit circle ($r = 1$) are defined by the following trigonometric functions.

Definition 11 (Sine)**.** The sine of an angle θ is the ratio of the length of the opposite side to the length of the hypotenuse:

$$\sin(\theta) = \frac{\text{opposite}}{\text{hypotenuse}}$$

Domain: $\theta \in \mathbb{R}$ (all real numbers)
Range: $\sin(\theta) \in [-1, 1]$

Definition 12 (Cosine)**.** The cosine of an angle θ is the ratio of the length of the adjacent side to the length of the hypotenuse:

$$\cos(\theta) = \frac{\text{adjacent}}{\text{hypotenuse}}$$

Domain: $\theta \in \mathbb{R}$
Range: $\cos(\theta) \in [-1, 1]$

Definition 13 (Tangent)**.** The tangent of an angle θ is the ratio of the length of the opposite side to the length of the adjacent side:

$$\tan(\theta) = \frac{\text{opposite}}{\text{adjacent}} \equiv \frac{\sin(\theta)}{\cos(\theta)}$$

Domain: $\theta \in \mathbb{R} \setminus \left\{\frac{\pi}{2} + n\pi \mid n \in \mathbb{Z}\right\}$ (all real numbers except odd multiples of $\frac{\pi}{2}$)
Range: $\tan(\theta) \in \mathbb{R}$ (all real numbers)

The Unit Circle

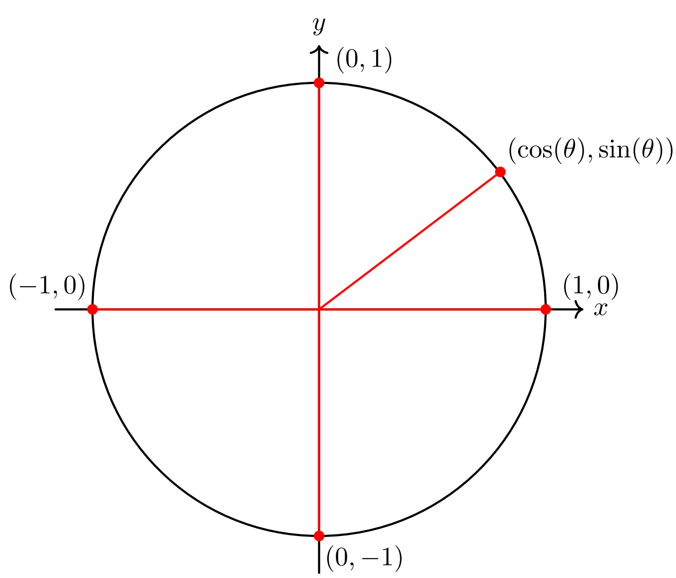


Figure 1.2: The Unit Circle, circle on the xy plane with radius, $r = 1$.

Example (Unit Circle).

- The unit circle is represented by the equation:

$$x^2 + y^2 = 1$$

where x and y are the coordinates of any point on the circle.

- Any point on the unit circle can be described by the coordinates $(\cos(\theta), \sin(\theta))$, where θ is the angle formed by the radius vector and the positive x-axis.
- As θ varies from 0 to 2π, the point on the unit circle traces out a full circle, with $\cos(\theta)$ representing the x-coordinate and $\sin(\theta)$ representing the y-coordinate of the point.
- Key points on the unit circle include:
  - At $\theta = 0°$ (or 0 radians), the point is $(1, 0)$.
  - At $\theta = 90°$ (or $\frac{\pi}{2}$ radians), the point is $(0, 1)$.
  - At $\theta = 180°$ (or π radians), the point is $(-1, 0)$.
  - At $\theta = 270°$ (or $\frac{3\pi}{2}$ radians), the point is $(0, -1)$.

1.2 Reciprocal Trigonometric Functions

The reciprocal trigonometric functions are defined as the inverses of sine, cosine, and tangent.

Definition 14 (Cosecant). The cosecant of θ is the reciprocal of the sine function:

$$\csc(\theta) := \frac{1}{\sin(\theta)}$$

Domain: $\theta \in \mathbb{R} \setminus n\pi$ (all real numbers except integer multiples of π)
Range: $\csc(\theta) \in (-\infty, -1] \cup [1, \infty)$

Definition 15 (Secant). The secant of θ is the reciprocal of the cosine function:

$$\sec(\theta) := \frac{1}{\cos(\theta)}$$

Domain: $\theta \in \mathbb{R} \setminus \frac{\pi}{2} + n\pi$ (all real numbers except odd multiples of $\frac{\pi}{2}$)
Range: $\sec(\theta) \in (-\infty, -1] \cup [1, \infty)$

Definition 16 (Cotangent). The cotangent of θ is the reciprocal of the tangent function:

$$\cot(\theta) := \frac{1}{\tan(\theta)}$$

Domain: $\theta \in \mathbb{R} \setminus n\pi$ (all real numbers except integer multiples of π)
Range: $\cot(\theta) \in \mathbb{R}$ (all real numbers)

1.3 Inverse Trigonometric Functions

Inverse trigonometric functions allow us to determine the angle when given a trigonometric ratio. Here, we discuss arcsine (arcsin) and arccosine (arccos).

Definition 17 (Arcsine (arcsin)). The arcsine function $\arcsin(x)$ is the inverse of the sine function. It returns the angle θ such that:

$$\sin(\theta) = x \Rightarrow \theta = \arcsin(x) \quad \text{and} \quad \theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right].$$

Domain: $x \in [-1, 1]$
Range: $\theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

Definition 18 (Arccosine (arccos)). The arccosine function $\arccos(x)$ is the inverse of the cosine function. It returns the angle θ such that:

$$\cos(\theta) = x \Rightarrow \theta = \arccos(x) \quad \text{and} \quad \theta \in [0, \pi].$$

Domain: $x \in [-1, 1]$
Range: $\theta \in [0, \pi]$

1.3.1 Relationship to the Unit Circle

The arcsine and arccosine functions correspond to angles on the unit circle.

- For $\arcsin(x)$, the angle θ is measured counterclockwise from the positive x-axis and is restricted to the first and fourth quadrants ($-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$).
- For $\arccos(x)$, the angle θ is measured counterclockwise from the positive x-axis and is restricted to the first and second quadrants ($0 \leq \theta \leq \pi$).

1.4 Pythagorean Identities

A fundamental identity in trigonometry.

Definition 19 (Pythagorean Identity). The Pythagorean identity states that for any angle θ:

$$\sin^2(\theta) + \cos^2(\theta) = 1$$

1.5 Sum and Difference Formulas

The sum and difference formulas allow us to compute the trigonometric functions of sums or differences of angles.

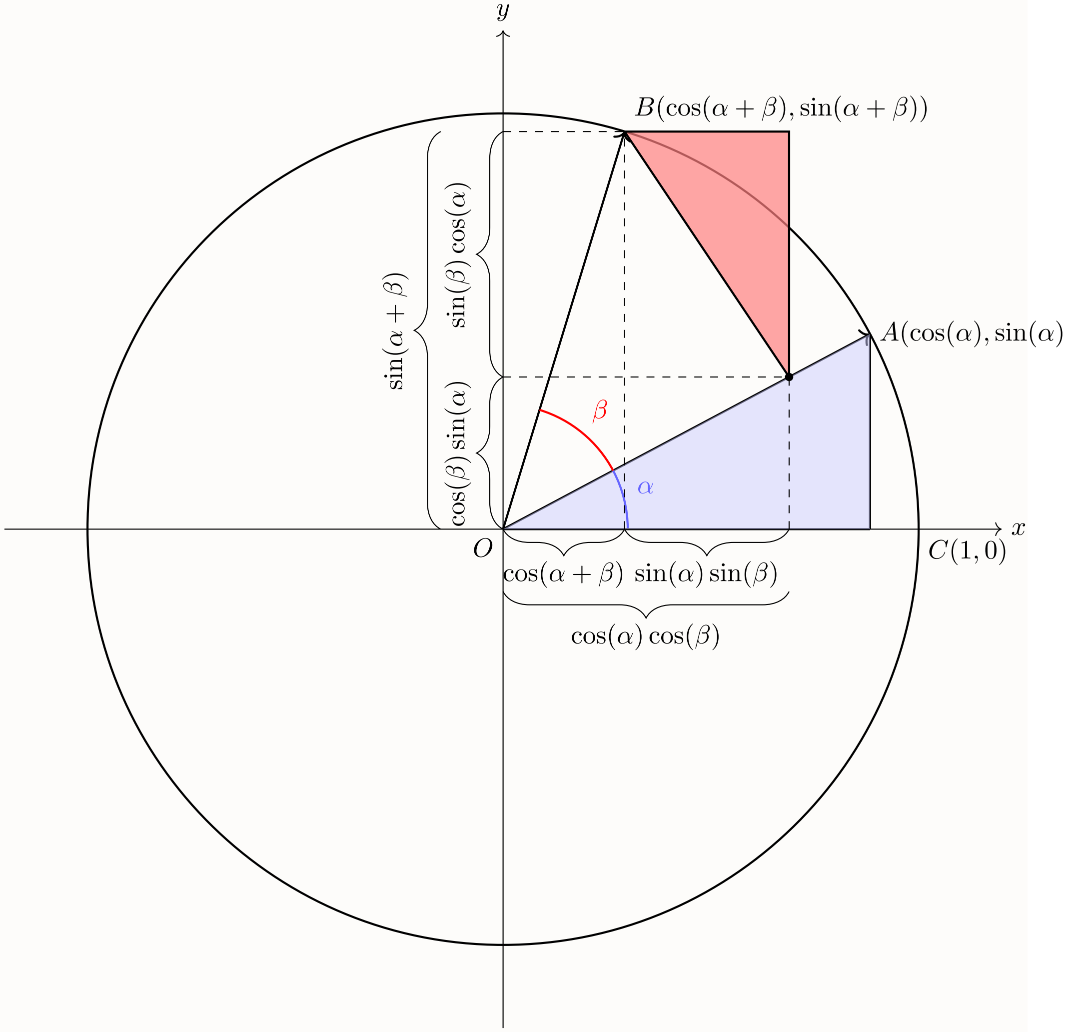
Definition 20 (Sum of Angles for Sine). The sine of the sum of two angles is given by:

$$\sin(A + B) = \sin(A)\cos(B) + \cos(A)\sin(B)$$

Definition 21 (Sum of Angles for Cosine)**.** The cosine of the sum of two angles is given by:

$$\cos(A+B) = \cos(A)\cos(B) - \sin(A)\sin(B)$$

Proof. To prove, we can construct a unit circle, and draw two angles, α and β. Note, at A, this point has an angle of α while B has an angle of $\alpha+\beta$. The key to this problem is the creation of a perpendicular bisector emanating from the point B, and hits the triangle with an angle α. What this creates is a new triangle, with angle β, which is but a triangle with angle β rotated by angle α from the origin. Performing trigonometry on all triangles discussed, we find by matching horizontal and vertical distances highlighted by the underbraces in the figure below: $\sin(\alpha+\beta) = \sin(\alpha)\cos(\beta) + \sin(\beta)\cos(\alpha)$ and $\cos(\alpha)\cos(\beta) = \cos(\alpha+\beta) + \sin(\alpha)\sin(\beta) \Rightarrow \cos(\alpha+\beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)$, completing the proof of definitions 15 and 16.



$\square$

Corollary. Noting symmetry of $\sin(-x) = -\sin(x)$ and $\cos(-x) = \cos(x)$, we find as a corollary the following generalised addition formulae:

$$\sin(A \pm B) = \sin(A)\cos(B) \pm \cos(A)\sin(B)$$

$$\cos(A + B) = \cos(A)\cos(B) \mp \sin(A)\sin(B)$$

Definition 22 (Sum of Angles for Tangent)**.** The tangent of the sum of two angles is given by:

$$\tan(A + B) = \frac{\tan(A) + \tan(B)}{1 - \tan(A)\tan(B)}$$

1.6 Trigonometric Equations and Identities

Trigonometric equations and identities are useful tools in simplifying and solving problems in maths.

Definition 23 (Double Angle Formula for Sine)**.** The sine of double an angle is given by:

$$\sin(2\theta) = 2\sin(\theta)\cos(\theta)$$

Definition 24 (Double Angle Formula for Cosine)**.** The cosine of double an angle is given by:

$$\cos(2\theta) = \cos^2(\theta) - \sin^2(\theta)$$

1.7 Applications of Trigonometry

Definition 25 (Law of Sines)**.** In any triangle, the ratio of the length of a side to the sine of its opposite angle is constant:

$$\frac{a}{\sin(A)} = \frac{b}{\sin(B)} = \frac{c}{\sin(C)}$$

Definition 26 (Law of Cosines)**.** In any triangle, the cosine of an angle is related to the lengths of the sides as:

$$c^2 = a^2 + b^2 - 2ab\cos(C)$$