Part I

Physics

Chapter 1

Circular Motion

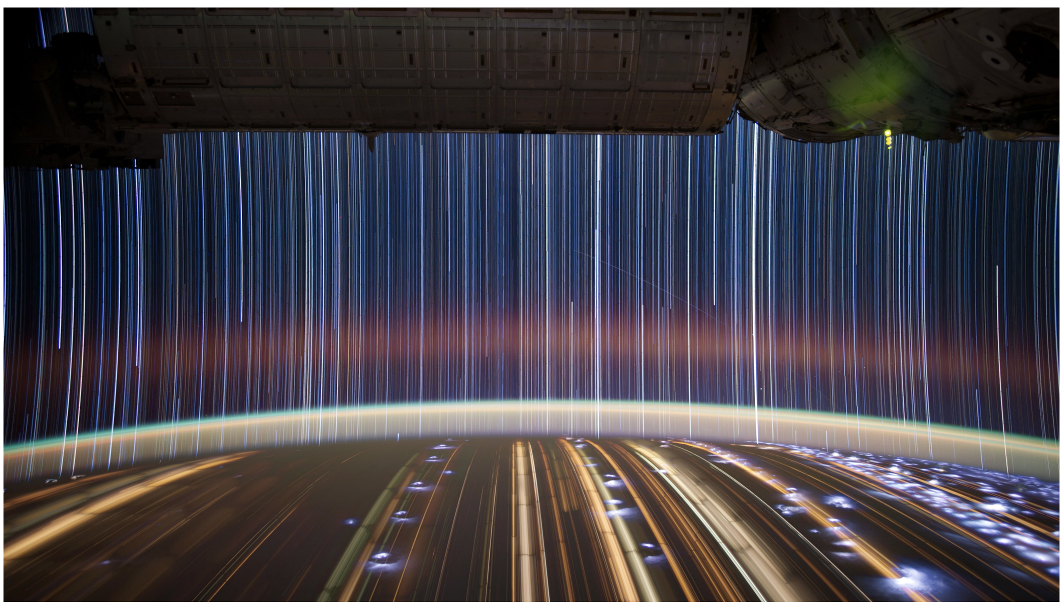


Figure 1.1: Star trails and light-on-Earth trails composite image created by International Space Station Expedition 30 crew member Don Pettit. The ISS is one of 11,000 active satellites as of June 2024 currently in orbit with the Earth.

Assignment 2: Circular Motion

Fri, 8 Nov 19:00

1.1 Angular Motion Kinematics

When an object moves along a circular path, its motion is defined by a few key relationships and variables. This section introduces the core concepts of angular motion.

> **Definition 1** (Angular Displacement). For an object moving in a circle of radius R, the distance it travels along the curve is noted as $x(t)$. The angle swept out by this path,

called the angular displacement, is:

$$\theta(t) = \frac{x(t)}{R}$$

Definition 2 (Angular Velocity and Acceleration)**.** By taking derivatives of angular displacement with respect to time, we get the angular velocity $\omega(t)$ and angular acceleration $\alpha(t)$:

$$\omega(t) = \frac{v(t)}{R}, \quad \alpha(t) = \frac{a_\tau(t)}{R}$$

For **uniform circular motion**, both $v(t)$ and $\omega(t)$ stay constant, giving us $x(t) = vt$ and $\theta(t) = \omega t$.

1.2 Centripetal Acceleration

As an object moves at a constant speed along a circular path, its direction changes continuously, creating a special kind of acceleration toward the circle's center, called centripetal acceleration.

Definition 3 (Centripetal Acceleration)**.** The acceleration that always points toward the center of the circle is known as centripetal acceleration and is given by:

$$a_c = \frac{v^2}{r} = \omega^2 r$$

Proof. Consider velocity vectors $\vec{v}_1$ and $\vec{v}_2$ at points A and B tangent to a circular trajectory and form an angle $\Delta\theta$ with each other. The change in velocity $\Delta\vec{v}$ can be approximated by the vector difference between $\vec{v}_1$ and $\vec{v}_2$. Since $|\vec{v}_1| = |\vec{v}_2| = v$, we find the magnitude of $\Delta\vec{v}$ as follows:

$$|\Delta\vec{v}| = 2v\sin\frac{\Delta\theta}{2}$$

For very small angles, $\sin\frac{\Delta\theta}{2} \approx \frac{\Delta\theta}{2}$, so we can write:

$$|\Delta\vec{v}| \approx v\Delta\theta$$

Now, dividing by the time interval Δt gives the average acceleration:

$$a = \frac{|\Delta\vec{v}|}{\Delta t} = \frac{v\Delta\theta}{\Delta t}$$

Since the angular velocity ω is defined as $\omega = \frac{\Delta\theta}{\Delta t}$, we have:

$$a = v\omega$$

Using the relationship $v = r\omega$, we substitute for ω:

$$a = \frac{v^2}{r}$$

Thus, the centripetal acceleration, directed toward the center of the circular path, is given by:

$$a_c = \frac{v^2}{r}$$

Corollary. Centripetal acceleration works at a right angle to the object's velocity, affecting only its direction and not its speed.

1.3 Uniform Circular Motion Dynamics

When an object moves in a circle, it needs a force to keep it on that path. This force, called centripetal force, keeps it from flying off in a straight line.

Definition 4 (Centripetal Force)**.** The force that pulls an object toward the center of its circular path is the centripetal force:

$$F_c = \frac{mv^2}{r} = m\omega^2 r$$

Centripetal force can be provided by different interactions, such as gravity for a satellite or friction for a car taking a turn. **Period** T and **frequency** f relate to motion around a circle by:

$$T = \frac{1}{f}, \quad \omega = 2\pi f$$

1.4 Non-uniform Circular Motion Dynamics

If an object speeds up or slows down as it moves in a circle, there's more going on. Here, we see both centripetal and tangential accelerations, affecting both direction and speed.

Definition 5 (Total Acceleration)**.** In non-uniform circular motion, total acceleration is the sum of tangential a_τ and centripetal a_c components:

$$a = a_\tau + a_c$$

To analyze this, we typically rely on **Newton's second law** and the principles of energy conservation.

1.5 Centrifugal Force (Apparent Force)

When you're in a rotating reference frame, an outward force, known as centrifugal force, seems to push you away from the center of rotation. This force is an illusion caused by the frame's motion, but it's useful in calculations.

Definition 6 (Centrifugal Force)**.** The centrifugal force appears as an outward pull in a rotating reference frame:

$$F_{\text{centrifugal}} = -ma_c$$

Although it feels like a real force, centrifugal force results from the rotating frame's acceleration and doesn't actually act on the object.

Chapter 2

Centre of Mass

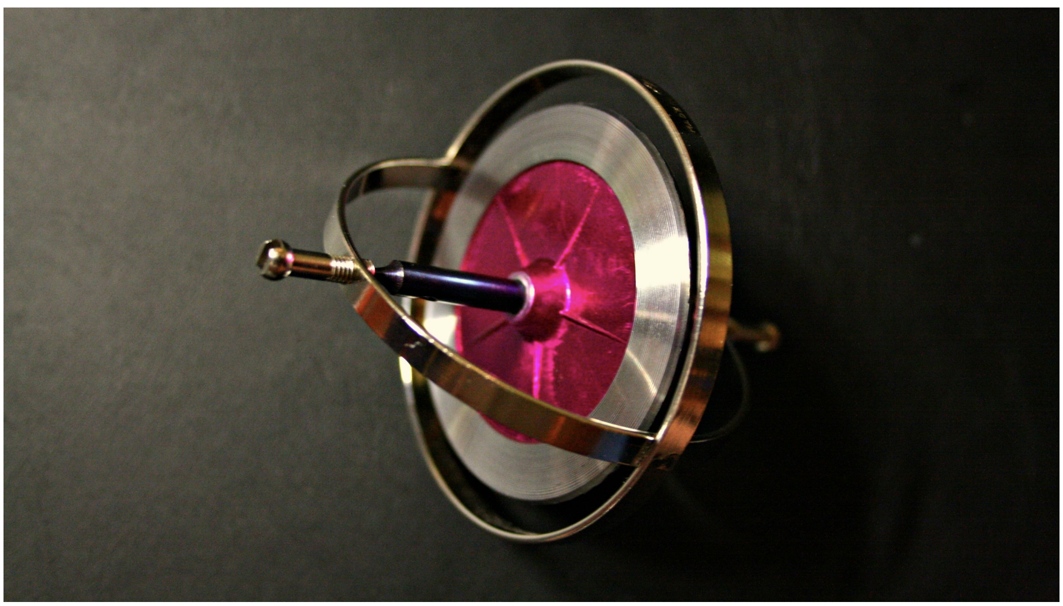


Figure 2.1: Gyroscope, non-trivially rotating body that seems to defy gravity – due to rotations about its centre of mass.

Assignment 3: Centre of Mass

Fri, 6 Dec 19:00

2.1 The Centre of Mass

2.1.1 Definition and Simple Examples

Suppose we have N point objects with masses $m_1, m_2, \ldots, m_N$ with the respective positions $\vec{r}_1, \vec{r}_2, \ldots, \vec{r}_N$. The centre of mass is the point whose position is defined by:

Definition 7 (Centre of mass).

$$\vec{r}_{\text{c.m.}} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2 + \cdots + m_N\vec{r}_N}{m_1 + m_2 + \cdots + m_N}.$$

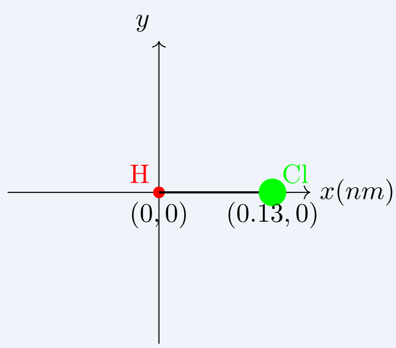
This can be written using summation notation:

$$\vec{r}_{\text{c.m.}} = \frac{\sum_{i=1}^{N} m_i \vec{r}_i}{\sum_{i=1}^{N} m_i}.$$

The vectors can be projected onto any axis to find the x, y, or z-coordinate of the centre of mass. For example, the x-coordinate is given by:

$$x_{\text{c.m.}} = \frac{\sum_{i=1}^{N} m_i x_i}{\sum_{i=1}^{N} m_i}.$$

Example (Centre of Mass for molecule of Hydrochloric Acid)**.** We can make use of the formula for the centre of mass to deduce the centre of mass for Hydrochloric Acid, HCl. We note that Hydrogen has a mass of 1 a.m.u (atomic mass unit, equivalent to the mass of a proton). Chlorine has a mass of 35.5 a.m.u. The length of the ionic bond separating Hydrogen and Chlorine is 1.3×10^{-10} m. We set up the following diagram for HCl, where we centre the molecule such that the x-axis goes through the symmetry axis of the molecule:



$$x_{\text{cm}} = \frac{(m_H \times 0) + (m_{Cl} \times 0.13)}{m_H + m_{Cl}} = \frac{35.5}{36.5} \times 0.15 \approx 0.126nm$$

2.1.2 Motion of the Centre of Mass

The motion of the centre of mass (CoM) can be described using Newton's second law – here's how. If we differentiate the position of the centre of mass with respect to time – we find the acceleration of the CoM is:

$$\vec{a}_{\text{c.m.}} = \frac{\sum_i m_i \vec{a}_i}{\sum_i m_i}.$$

Multiplying through by the total mass $m_{\text{tot}} = \sum m_i$, we get:

$$m_{\text{tot}} \vec{a}_{\text{c.m.}} = \sum_i m_i \vec{a}_i.$$

Using Newton's Second Law:

$$\sum_i m_i \vec{a}_i = \sum_i \vec{F}_{\text{int},i} + \sum_i \vec{F}_{\text{ext},i}.$$

From Newton's Third Law, $\sum_i \vec{F}_{\text{int},i} = 0$, so:

$$m_{\text{tot}} \vec{a}_{\text{c.m.}} = \sum_i \vec{F}_{\text{ext},i}.$$

From this, we arrive at an important result in context of the centre of mass:

Definition 8 (External forces and the centre of mass)**.** The centre of mass of a system moves as if all the external forces were acting on a single point object of mass m_{tot}.

Internal forces cancel each other out according to Newton's laws of motion, and so we are left with the external forces – which are the only forces that impact the motion of the centre of mass.

2.2 Finding Centre of Mass of irregular shapes using the Geometric Centre

For irregular shapes, it is convenient to split the shape up into i regular ones. The formula for the geometric centre is given below, it is dependent on the area of the i regular shapes, as well as their geometric centre.

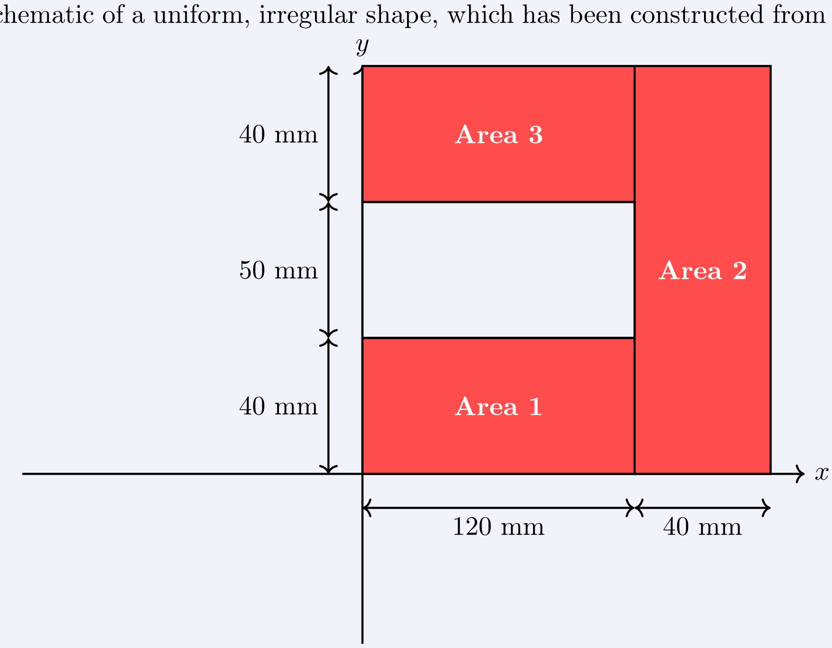
Definition 9 (Geocentric centre)**.** The cooridnate q, C_q, representing the geocentric centre of an irregular shape is defined as:

$$C_q = \frac{\sum_i C_{iq} A_i}{\sum_i A_i}$$

where C_{iq} denotes the geocentric centre of the ith shape, A_i is the area of the ith shape.

Corollary. The geocentric centre and the COM will coincide when the mass density is uniform. And so, calculating the geocentric centre is one method of calculating an object's centre of mass, iff its mass distribution is uniform.

Example (Using the geometric centre method for an irrelgular shape.)**.** Shown below is a schematic of a uniform, irregular shape, which has been constructed from 3 rectangles.



The geocentric centre of this irregular shape can be found by calculating:

$$C_x = \frac{\sum_i C_{ix} A_i}{\sum_i A_i}$$

$$C_y = \frac{\sum_i C_{iy} A_i}{\sum_i A_i}$$

The geocentric centre is then given by the point (C_x, C_y). Performing the calculation first for the x-cooridnate of the geocentric centre, we find:

$$C_x = \frac{(60 \times 4800) + (140 \times 5200) + (60 \times 4800)}{4800 + 5200 + 4800} = 88.11 \text{ mm}$$

$$C_y = \frac{(20 \times 4800) + (65 \times 5200) + (110 \times 4800)}{4800 + 5200 + 4800} = 65\text{mm}$$

Note, the irregular shape enjoys symmetry at the line y = 65, and so the centre of mass given it is a uniform shape should reside here. This calculation is therefore a nice sanity check, as we do expect the y cooridnate of the centre of mass to be at 65mm – given its symmetry. Hence, the centre of mass of this irregular shape resides at the cooridnate $(88.11, 65)$.

2.3 Finding Centre of Mass using Integration

The formula in the summand form may be treated with infinitesimal masses. Treating the sum with infinitesimal elements converts the sum into an intergal. Here is an example of where infinitesimal mass elements would come in handy in calculating the centre of mass – calculating the centre of mass of hills.

Figure 2.2: The Chocolate Hills in the Philippines. They enjoy a nice circular symmetry, as they are modelled as a cone with a height h of 50 m and a radius r.

Example (Centre of Mass of the Chocolate Hills). The Chocolate Hills are a geological formation in the Philippines. They are covered in green grass that turns into a chocolate-like brown during the dry season, hence the name. Let us assume it is conical in shape, as shown in the figure, with a uniform density and a height of $h = 50\,\text{m}$. We aim to find the height of its centre of mass (y_{cm}).

Note, due to circular symmetry and uniform mass density, the x coordinate of the centre of mass will be zero, mass elements on left cancel the mass elements on the right of the cone. As such, we only need to consider the y-direction. The centre of mass is calculated as:

$$y_{\text{cm}} = \frac{\int_0^h y\,dm}{\int_0^h dm}.$$

The mass element (dm) of the cone can be expressed as:

$$dm = \rho\,(\pi r^2)\,dy = \rho\,\pi y^2 \tan^2\theta\,dy,$$

where ρ is the density, $r = y\tan\theta$, and θ is the angle of the cone's slope. Substituting dm into the expression for y_{cm}:

$$y_{\text{cm}} = \frac{\int_0^h y\,\rho\,\pi y^2 \tan^2\theta\,dy}{\int_0^h \rho\,\pi y^2 \tan^2\theta\,dy}.$$

Since ρ, π, and $\tan^2\theta$ are constants, they cancel out:

$$y_{\text{cm}} = \frac{\int_0^h y^3\,dy}{\int_0^h y^2\,dy}.$$

The numerator and denominator are solved as follows:

$$\int_0^h y^3\,dy = \left[\frac{y^4}{4}\right]_0^h = \frac{h^4}{4},$$

$$\int_0^h y^2\,dy = \left[\frac{y^3}{3}\right]_0^h = \frac{h^3}{3}.$$

Substituting these results into the expression for y_{cm}:

$$y_{\text{cm}} = \frac{\frac{h^4}{4}}{\frac{h^3}{3}} = \frac{3h}{4}.$$

Thus, the centre of mass is located at:

$$y_{\text{cm}} = \frac{3h}{4} \quad \text{(measured from the top of the cone).}$$

For $h = 50\,\text{m}$, the height of the centre of mass from the top of the cone is 37.5m, meaning that from the base:

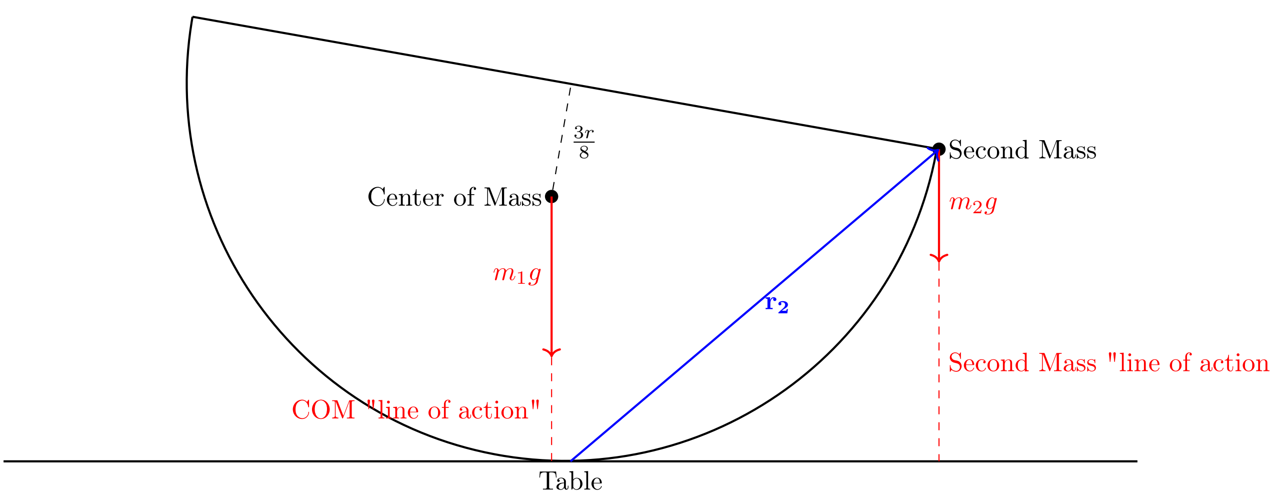
$$y_{\text{cm}} = 12.5\,\text{m}.$$

2.4 A note on the Line of Action: relating stability to the COM

Do review Q10 on Physics Assignment 3 which is a direct application of the line of action! The line of action is purely a geometric representation of how a force is applied to a rotating system. It is simply a straight line through the point at which the force is applied, and is in the same direction as the Force applied. It is a useful aid in visualising perpendicular distances, e.g. say we wish to calculate the torque, τ as a result of the second mass. The torque is equal to the cross-product of the distance from the pivot (in this example, the table is the point of contact) and the force causing the rotation about the pivot:

$$\tau = \mathbf{r_2} \times m_2\mathbf{g} = m_2 g\{|\mathbf{r_2}|\sin\theta\} = m_2 g \times x_2$$

where the x_2 is the perpendicular distance between the line of action of the force and the pivot and θ is the tilt of the hemisphere, and is also the angle separating the forces and the position vectors from the mass to the pivot. This may be repeated for the COM. The rest of the analysis is something you had to do in Q10 of Physics Assignment 3; with the main punchline of the problem being the construction of lines of action to see what the perpendicular distance from the pivot is, which are needed order to calculate the torque of the COM and the second mass.



Chapter 3

Momentum

Figure 3.1: Launch of ESA's Ariane 5 rocket, carrying rather important payload to space – the James Webb Space Telescope – on Christmas Day in 2021. Rockets use momentum, in principle, to get into space.

Assignment 4: Momentum

Fri, 10 Jan 19:00

3.1 Momentum and Impulse

Definition 10 (Momentum and Impulse).

Linear Momentum:

$$\mathbf{p} = m \cdot \mathbf{v}$$

Here, $\mathbf{p}$ is the linear momentum, m is the mass, and $\mathbf{v}$ is the velocity.

Conservation of Linear Momentum:

$$\sum m_i \cdot \mathbf{v_i} = \sum m_f \cdot \mathbf{v_f}$$

This equation expresses the conservation of linear momentum in a system, where m_i and $\mathbf{v_i}$ are the initial mass and velocity, and m_f and $\mathbf{v_f}$ are the final mass and velocity of the system.

Impulse-Momentum Theorem:

$$\mathbf{F} \cdot \Delta t = \Delta \mathbf{p}$$

This theorem relates the impulse ($\mathbf{F} \cdot \Delta t$) applied to an object to the change in its momentum ($\Delta \mathbf{p}$). This is true for a constant force over time. We can generalise this theorem for a variable force, one that changes with time, by considering an infinitesimal time period, such that: $\Delta t \to \delta t$. One can write the macroscopic momentum, $\Delta \mathbf{p}$ as:

$$\Delta \mathbf{p} = \int_{p_1}^{p_2} \delta \mathbf{p}(t) = \underbrace{\int_{t_1}^{t_2} \mathbf{F}(t)\,\mathrm{d}t}_{\text{via. Impulse-Momentum Theorem}}$$

This is a more general result of the Impulse-Momentum Theorem, it accounts for the Force potentially being time dependent.

Angular Momentum:

$$\mathbf{L} = I \cdot \boldsymbol{\omega}$$

Angular momentum ($\mathbf{L}$) is the product of moment of inertia (I) and angular velocity ($\boldsymbol{\omega}$).

3.2 Collisions

Via the principle of the conservation of linear momentum: the momentum describing the state of a system before a collision must equal the momentum describing the state of the system after the collision. Kinetic energy, though, needn't be conserved. If it is conserved after a collision event, it is an elastic collision. If kinetic energy is not conserved after a collision event, it is an inelastic collision.

Example. In an elastic collision between two equal masses, assuming they both collide, and one of the masses is at rest, the angle of separation after the collision is 90 degrees.

Proof. Some interesting situations arise when the two colliding objects have equal mass and the collision is elastic. An elastic collision conserves internal kinetic energy.

$$\frac{1}{2}mv_1^2 = \frac{1}{2}mv_1'^2 + \frac{1}{2}mv_2'^2.$$

Because the masses are equal ($m_1 = m_2 = m$), algebraic manipulation of conservation of momentum in the x- and y-directions can show that

$$\frac{1}{2}mv_2^2 = \frac{1}{2}mv_1'^2 + \frac{1}{2}mv_2'^2 + mv_1'v_2'\cos(\theta_1 - \theta_2).$$

(Remember that θ_2 is negative here.) The two preceding equations can both be true only if

$$mv_1'v_2'\cos(\theta_1 - \theta_2) = 0.$$

There are three ways that this term can be zero. They are

- $v_1' = 0$: head-on collision; incoming ball stops
- $v_2' = 0$: no collision; incoming ball continues unaffected
- $\cos(\theta_1 - \theta_2) = 0$: angle of separation ($\theta_1 - \theta_2$) is $90°$ after the collision. Thus, as we are interested solely in the last case, the angle of separation between the two balls will be 90 degrees.