Chapter 2

Centre of Mass

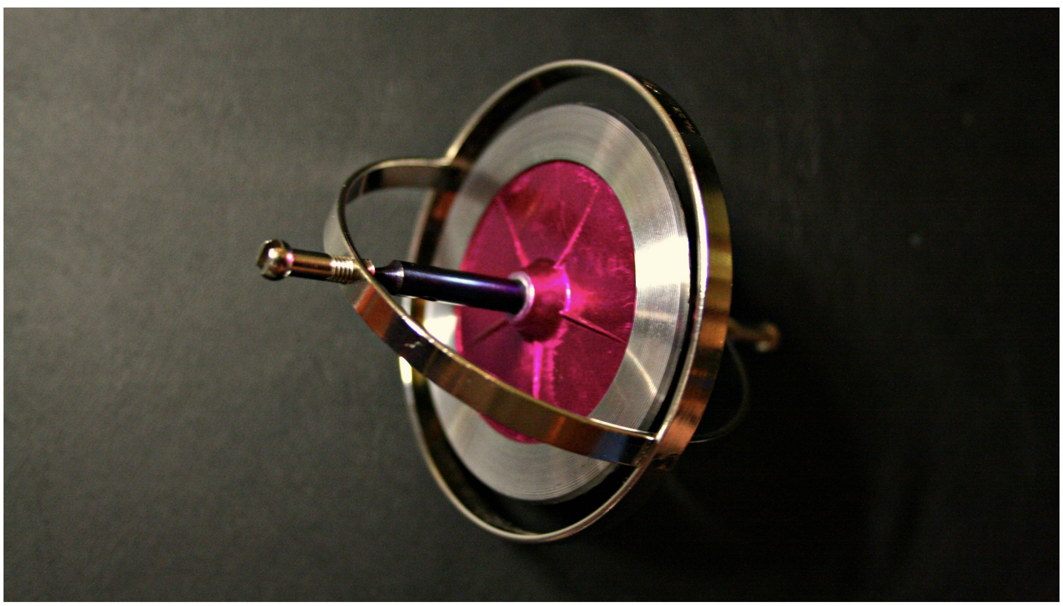


Figure 2.1: Gyroscope, non-trivially rotating body that seems to defy gravity – due to rotations about its centre of mass.

Assignment 3: Centre of Mass

Fri, 6 Dec 19:00

2.1 The Centre of Mass

2.1.1 Definition and Simple Examples

Suppose we have N point objects with masses $m_1, m_2, \ldots, m_N$ with the respective positions $\vec{r}_1, \vec{r}_2, \ldots, \vec{r}_N$. The centre of mass is the point whose position is defined by:

Definition 7 (Centre of mass).

$$\vec{r}_{\text{c.m.}} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2 + \cdots + m_N\vec{r}_N}{m_1 + m_2 + \cdots + m_N}.$$

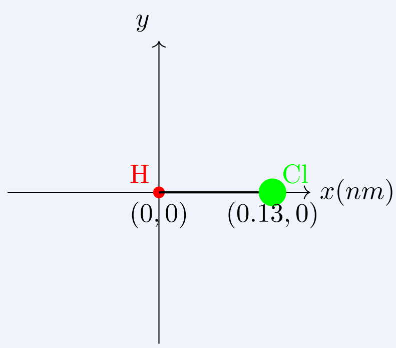
This can be written using summation notation:

$$\vec{r}_{\text{c.m.}} = \frac{\sum_{i=1}^{N} m_i \vec{r}_i}{\sum_{i=1}^{N} m_i}.$$

The vectors can be projected onto any axis to find the x, y, or z-coordinate of the centre of mass. For example, the x-coordinate is given by:

$$x_{\text{c.m.}} = \frac{\sum_{i=1}^{N} m_i x_i}{\sum_{i=1}^{N} m_i}.$$

Example (Centre of Mass for molecule of Hydrochloric Acid)**.** We can make use of the formula for the centre of mass to deduce the centre of mass for Hydrochloric Acid, HCl. We note that Hydrogen has a mass of 1 a.m.u (atomic mass unit, equivalent to the mass of a proton). Chlorine has a mass of 35.5 a.m.u. The length of the ionic bond separating Hydrogen and Chlorine is 1.3×10^{-10} m. We set up the following diagram for HCl, where we centre the molecule such that the x-axis goes through the symmetry axis of the molecule:



$$x_{\text{cm}} = \frac{(m_H \times 0) + (m_{Cl} \times 0.13)}{m_H + m_{Cl}} = \frac{35.5}{36.5} \times 0.15 \approx 0.126nm$$

2.1.2 Motion of the Centre of Mass

The motion of the centre of mass (CoM) can be described using Newton's second law – here's how. If we differentiate the position of the centre of mass with respect to time – we find the acceleration of the CoM is:

$$\vec{a}_{\text{c.m.}} = \frac{\sum_i m_i \vec{a}_i}{\sum_i m_i}.$$

Multiplying through by the total mass $m_{\text{tot}} = \sum m_i$, we get:

$$m_{\text{tot}} \vec{a}_{\text{c.m.}} = \sum_i m_i \vec{a}_i.$$

Using Newton's Second Law:

$$\sum_i m_i \vec{a}_i = \sum_i \vec{F}_{\text{int},i} + \sum_i \vec{F}_{\text{ext},i}.$$

From Newton's Third Law, $\sum_i \vec{F}_{\text{int},i} = 0$, so:

$$m_{\text{tot}} \vec{a}_{\text{c.m.}} = \sum_i \vec{F}_{\text{ext},i}.$$

From this, we arrive at an important result in context of the centre of mass:

Definition 8 (External forces and the centre of mass)**.** The centre of mass of a system moves as if all the external forces were acting on a single point object of mass m_{tot}.

Internal forces cancel each other out according to Newton's laws of motion, and so we are left with the external forces – which are the only forces that impact the motion of the centre of mass.

2.2 Finding Centre of Mass of irregular shapes using the Geometric Centre

For irregular shapes, it is convenient to split the shape up into i regular ones. The formula for the geometric centre is given below, it is dependent on the area of the i regular shapes, as well as their geometric centre.

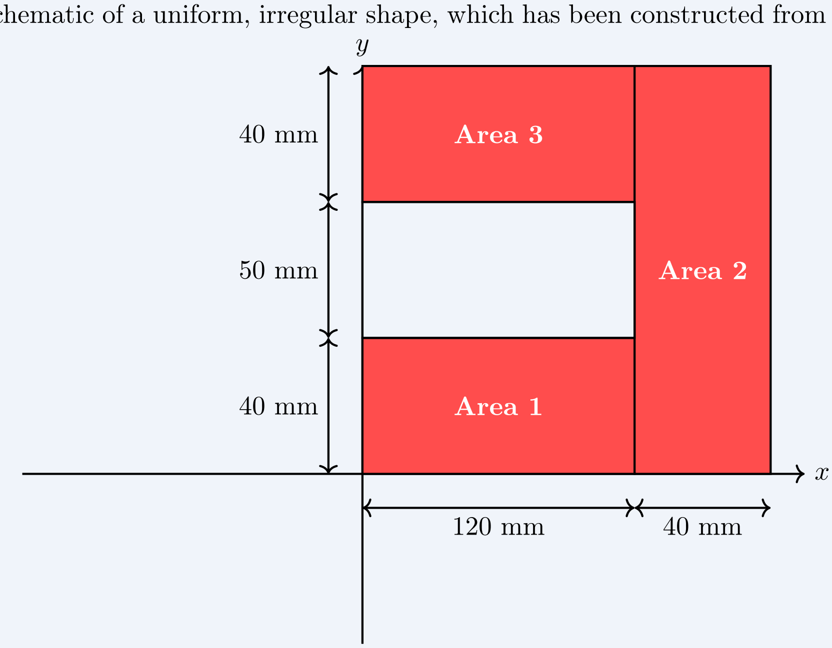
Definition 9 (Geocentric centre)**.** The cooridnate q, C_q, representing the geocentric centre of an irregular shape is defined as:

$$C_q = \frac{\sum_i C_{iq} A_i}{\sum_i A_i}$$

where C_{iq} denotes the geocentric centre of the ith shape, A_i is the area of the ith shape.

Corollary. The geocentric centre and the COM will coincide when the mass density is uniform. And so, calculating the geocentric centre is one method of calculating an object's centre of mass, iff its mass distribution is uniform.

Example (Using the geometric centre method for an irrelgular shape.)**.** Shown below is a schematic of a uniform, irregular shape, which has been constructed from 3 rectangles.



The geocentric centre of this irregular shape can be found by calculating:

$$C_x = \frac{\sum_i C_{ix} A_i}{\sum_i A_i}$$

$$C_y = \frac{\sum_i C_{iy} A_i}{\sum_i A_i}$$

The geocentric centre is then given by the point (C_x, C_y). Performing the calculation first for the x-cooridnate of the geocentric centre, we find:

$$C_x = \frac{(60 \times 4800) + (140 \times 5200) + (60 \times 4800)}{4800 + 5200 + 4800} = 88.11 \text{ mm}$$

$$C_y = \frac{(20 \times 4800) + (65 \times 5200) + (110 \times 4800)}{4800 + 5200 + 4800} = 65\text{mm}$$

Note, the irregular shape enjoys symmetry at the line y = 65, and so the centre of mass given it is a uniform shape should reside here. This calculation is therefore a nice sanity check, as we do expect the y cooridnate of the centre of mass to be at 65mm – given its symmetry. Hence, the centre of mass of this irregular shape resides at the cooridnate $(88.11, 65)$.

2.3 Finding Centre of Mass using Integration

The formula in the summand form may be treated with infinitesimal masses. Treating the sum with infinitesimal elements converts the sum into an intergal. Here is an example of where infinitesimal mass elements would come in handy in calculating the centre of mass – calculating the centre of mass of hills.

Figure 2.2: The Chocolate Hills in the Philippines. They enjoy a nice circular symmetry, as they are modelled as a cone with a height h of 50 m and a radius r.

Example (Centre of Mass of the Chocolate Hills). The Chocolate Hills are a geological formation in the Philippines. They are covered in green grass that turns into a chocolate-like brown during the dry season, hence the name. Let us assume it is conical in shape, as shown in the figure, with a uniform density and a height of $h = 50\,\mathrm{m}$. We aim to find the height of its centre of mass (y_{cm}).

Note, due to circular symmetry and uniform mass density, the x coordinate of the centre of mass will be zero, mass elements on left cancel the mass elements on the right of the cone. As such, we only need to consider the y-direction. The centre of mass is calculated as:

$$y_{\mathrm{cm}} = \frac{\int_0^h y\, dm}{\int_0^h dm}.$$

The mass element (dm) of the cone can be expressed as:

$$dm = \rho\,(\pi r^2)\,dy = \rho\,\pi y^2 \tan^2\theta\,dy,$$

where ρ is the density, $r = y\tan\theta$, and θ is the angle of the cone's slope. Substituting dm into the expression for y_{cm}:

$$y_{\mathrm{cm}} = \frac{\int_0^h y\,\rho\,\pi y^2 \tan^2\theta\,dy}{\int_0^h \rho\,\pi y^2 \tan^2\theta\,dy}.$$

Since ρ, π, and $\tan^2\theta$ are constants, they cancel out:

$$y_{\mathrm{cm}} = \frac{\int_0^h y^3\,dy}{\int_0^h y^2\,dy}.$$

The numerator and denominator are solved as follows:

$$\int_0^h y^3\,dy = \left[\frac{y^4}{4}\right]_0^h = \frac{h^4}{4},$$

$$\int_0^h y^2\,dy = \left[\frac{y^3}{3}\right]_0^h = \frac{h^3}{3}.$$

Substituting these results into the expression for y_{cm}:

$$y_{\mathrm{cm}} = \frac{\frac{h^4}{4}}{\frac{h^3}{3}} = \frac{3h}{4}.$$

Thus, the centre of mass is located at:

$$y_{\mathrm{cm}} = \frac{3h}{4} \quad \text{(measured from the top of the cone).}$$

For $h = 50\,\mathrm{m}$, the height of the centre of mass from the top of the cone is 37.5m, meaning that from the base:

$$y_{\mathrm{cm}} = 12.5\,\mathrm{m}.$$

2.4 A note on the Line of Action: relating stability to the COM

Do review Q10 on Physics Assignment 3 which is a direct application of the line of action! The line of action is purely a geometric representation of how a force is applied to a rotating system. It is simply a straight line through the point at which the force is applied, and is in the same direction as the Force applied. It is a useful aid in visualising perpendicular distances, e.g. say we wish to calculate the torque, τ as a result of the second mass. The torque is equal to the cross-product of the distance from the pivot (in this example, the table is the point of contact) and the force causing the rotation about the pivot:

$$\tau = \mathbf{r_2} \times m_2\mathbf{g} = m_2 g\{|\mathbf{r_2}|\sin\theta\} = m_2 g \times x_2$$

where the x_2 is the perpendicular distance between the line of action of the force and the pivot and θ is the tilt of the hemisphere, and is also the angle separating the forces and the position vectors from the mass to the pivot. This may be repeated for the COM. The rest of the analysis is something you had to do in Q10 of Physics Assignment 3; with the main punchline of the problem being the construction of lines of action to see what the perpendicular distance from the pivot is, which are needed order to calculate the torque of the COM and the second mass.