COMPOS - Maths and Physics Handbook

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February 6, 2025

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Part I

Physics

Chapter 1

Circular Motion

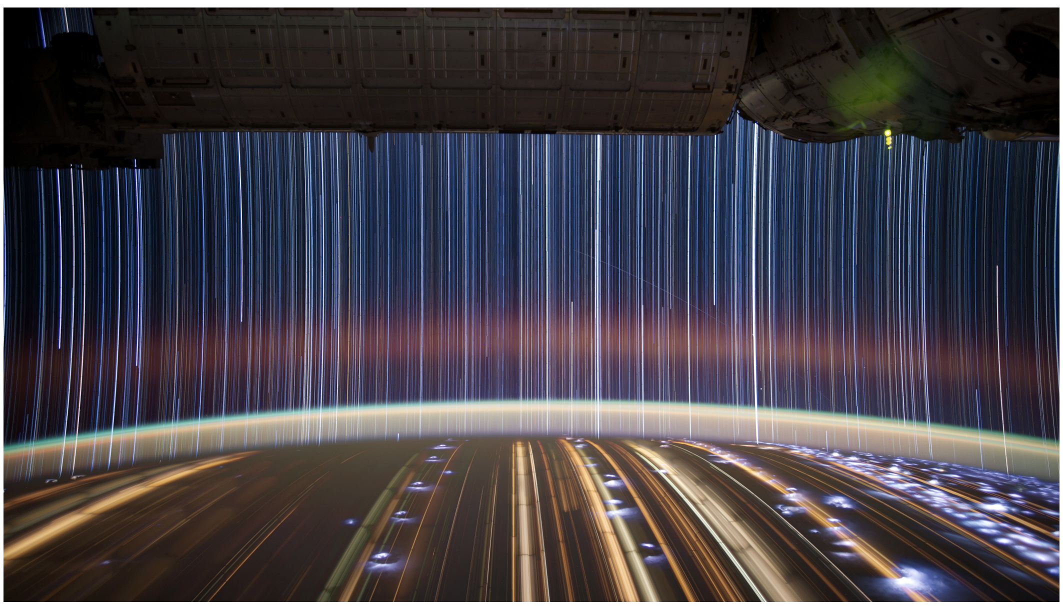


Figure 1.1: Star trails and light-on-Earth trails composite image created by International Space Station Expedition 30 crew member Don Pettit. The ISS is one of 11,000 active satellites as of June 2024 currently in orbit with the Earth.

Assignment 2: Circular Motion

Fri, 8 Nov 19:00

1.1 Angular Motion Kinematics

When an object moves along a circular path, its motion is defined by a few key relationships and variables. This section introduces the core concepts of angular motion.

Definition 1 (Angular Displacement). For an object moving in a circle of radius R, the distance it travels along the curve is noted as $x(t)$. The angle swept out by this path, called the angular displacement, is:

$$\theta(t) = \frac{x(t)}{R}$$

Definition 2 (Angular Velocity and Acceleration)**.** By taking derivatives of angular displacement with respect to time, we get the angular velocity $\omega(t)$ and angular acceleration $\alpha(t)$:

$$\omega(t) = \frac{v(t)}{R}, \quad \alpha(t) = \frac{a_\tau(t)}{R}$$

For **uniform circular motion**, both $v(t)$ and $\omega(t)$ stay constant, giving us $x(t) = vt$ and $\theta(t) = \omega t$.

1.2 Centripetal Acceleration

As an object moves at a constant speed along a circular path, its direction changes continuously, creating a special kind of acceleration toward the circle's center, called centripetal acceleration.

Definition 3 (Centripetal Acceleration)**.** The acceleration that always points toward the center of the circle is known as centripetal acceleration and is given by:

$$a_c = \frac{v^2}{r} = \omega^2 r$$

Proof. Consider velocity vectors $\vec{v}_1$ and $\vec{v}_2$ at points A and B tangent to a circular trajectory and form an angle $\Delta\theta$ with each other. The change in velocity $\Delta\vec{v}$ can be approximated by the vector difference between $\vec{v}_1$ and $\vec{v}_2$. Since $|\vec{v}_1| = |\vec{v}_2| = v$, we find the magnitude of $\Delta\vec{v}$ as follows:

$$|\Delta\vec{v}| = 2v \sin\frac{\Delta\theta}{2}$$

For very small angles, $\sin\frac{\Delta\theta}{2} \approx \frac{\Delta\theta}{2}$, so we can write:

$$|\Delta\vec{v}| \approx v\Delta\theta$$

Now, dividing by the time interval Δt gives the average acceleration:

$$a = \frac{|\Delta\vec{v}|}{\Delta t} = \frac{v\Delta\theta}{\Delta t}$$

Since the angular velocity ω is defined as $\omega = \frac{\Delta\theta}{\Delta t}$, we have:

$$a = v\omega$$

Using the relationship $v = r\omega$, we substitute for ω:

$$a = \frac{v^2}{r}$$

Thus, the centripetal acceleration, directed toward the center of the circular path, is given by:

$$a_c = \frac{v^2}{r}$$

Corollary. Centripetal acceleration works at a right angle to the object's velocity, affecting only its direction and not its speed.

1.3 Uniform Circular Motion Dynamics

When an object moves in a circle, it needs a force to keep it on that path. This force, called centripetal force, keeps it from flying off in a straight line.

Definition 4 (Centripetal Force)**.** The force that pulls an object toward the center of its circular path is the centripetal force:

$$F_c = \frac{mv^2}{r} = m\omega^2 r$$

Centripetal force can be provided by different interactions, such as gravity for a satellite or friction for a car taking a turn. **Period** T and **frequency** f relate to motion around a circle by:

$$T = \frac{1}{f}, \quad \omega = 2\pi f$$

1.4 Non-uniform Circular Motion Dynamics

If an object speeds up or slows down as it moves in a circle, there's more going on. Here, we see both centripetal and tangential accelerations, affecting both direction and speed.

Definition 5 (Total Acceleration)**.** In non-uniform circular motion, total acceleration is the sum of tangential a_τ and centripetal a_c components:

$$a = a_\tau + a_c$$

To analyze this, we typically rely on **Newton's second law** and the principles of energy conservation.

1.5 Centrifugal Force (Apparent Force)

When you're in a rotating reference frame, an outward force, known as centrifugal force, seems to push you away from the center of rotation. This force is an illusion caused by the frame's motion, but it's useful in calculations.

Definition 6 (Centrifugal Force)**.** The centrifugal force appears as an outward pull in a rotating reference frame:

$$F_{\text{centrifugal}} = -ma_c$$

Although it feels like a real force, centrifugal force results from the rotating frame's acceleration and doesn't actually act on the object.

Chapter 2

Centre of Mass

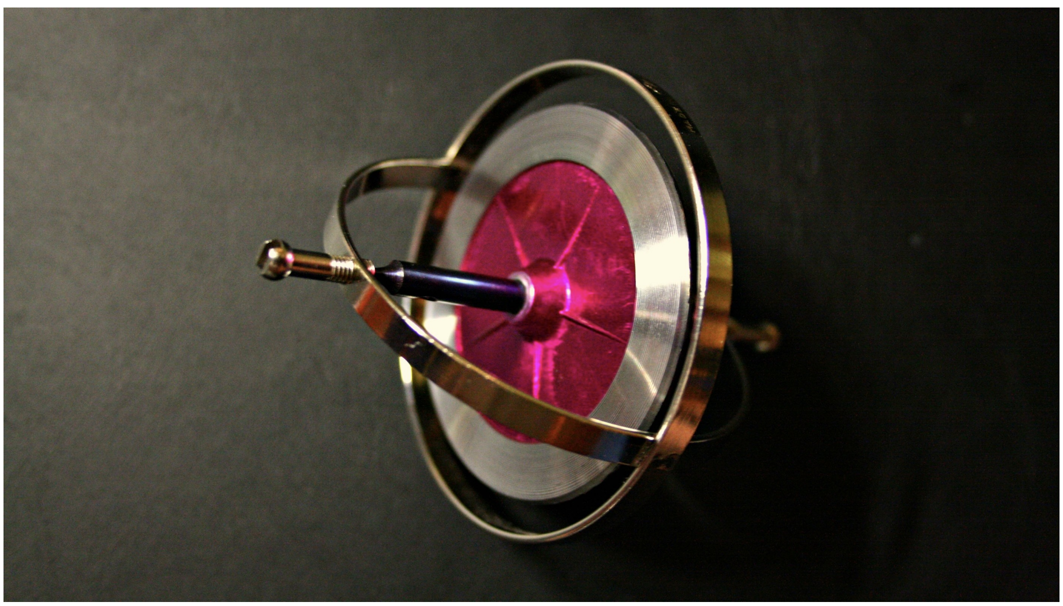


Figure 2.1: Gyroscope, non-trivially rotating body that seems to defy gravity – due to rotations about its centre of mass.

Assignment 3: Centre of Mass

Fri, 6 Dec 19:00

2.1 The Centre of Mass

2.1.1 Definition and Simple Examples

Suppose we have N point objects with masses $m_1, m_2, \ldots, m_N$ with the respective positions $\vec{r}_1, \vec{r}_2, \ldots, \vec{r}_N$. The centre of mass is the point whose position is defined by:

Definition 7 (Centre of mass).

$$\vec{r}_{\text{c.m.}} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2 + \cdots + m_N\vec{r}_N}{m_1 + m_2 + \cdots + m_N}.$$

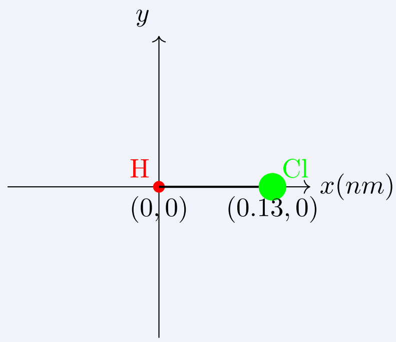
This can be written using summation notation:

$$\vec{r}_{\text{c.m.}} = \frac{\sum_{i=1}^{N} m_i \vec{r}_i}{\sum_{i=1}^{N} m_i}.$$

The vectors can be projected onto any axis to find the x, y, or z-coordinate of the centre of mass. For example, the x-coordinate is given by:

$$x_{\text{c.m.}} = \frac{\sum_{i=1}^{N} m_i x_i}{\sum_{i=1}^{N} m_i}.$$

Example (Centre of Mass for molecule of Hydrochloric Acid)**.** We can make use of the formula for the centre of mass to deduce the centre of mass for Hydrochloric Acid, HCl. We note that Hydrogen has a mass of 1 a.m.u (atomic mass unit, equivalent to the mass of a proton). Chlorine has a mass of 35.5 a.m.u. The length of the ionic bond separating Hydrogen and Chlorine is 1.3×10^{-10} m. We set up the following diagram for HCl, where we centre the molecule such that the x-axis goes through the symmetry axis of the molecule:



$$x_{\text{cm}} = \frac{(m_H \times 0) + (m_{Cl} \times 0.13)}{m_H + m_{Cl}} = \frac{35.5}{36.5} \times 0.15 \approx 0.126nm$$

2.1.2 Motion of the Centre of Mass

The motion of the centre of mass (CoM) can be described using Newton's second law – here's how. If we differentiate the position of the centre of mass with respect to time – we find the acceleration of the CoM is:

$$\vec{a}_{\text{c.m.}} = \frac{\sum_i m_i \vec{a}_i}{\sum_i m_i}.$$

Multiplying through by the total mass $m_{\text{tot}} = \sum m_i$, we get:

$$m_{\text{tot}} \vec{a}_{\text{c.m.}} = \sum_i m_i \vec{a}_i.$$

Using Newton's Second Law:

$$\sum_i m_i \vec{a}_i = \sum_i \vec{F}_{\text{int},i} + \sum_i \vec{F}_{\text{ext},i}.$$

From Newton's Third Law, $\sum_i \vec{F}_{\text{int},i} = 0$, so:

$$m_{\text{tot}} \vec{a}_{\text{c.m.}} = \sum_i \vec{F}_{\text{ext},i}.$$

From this, we arrive at an important result in context of the centre of mass:

Definition 8 (External forces and the centre of mass). The centre of mass of a system moves as if all the external forces were acting on a single point object of mass m_{tot}.

Internal forces cancel each other out according to Newton's laws of motion, and so we are left with the external forces – which are the only forces that impact the motion of the centre of mass.

2.2 Finding Centre of Mass of irregular shapes using the Geometric Centre

For irregular shapes, it is convenient to split the shape up into i regular ones. The formula for the geometric centre is given below, it is dependent on the area of the i regular shapes, as well as their geometric centre.

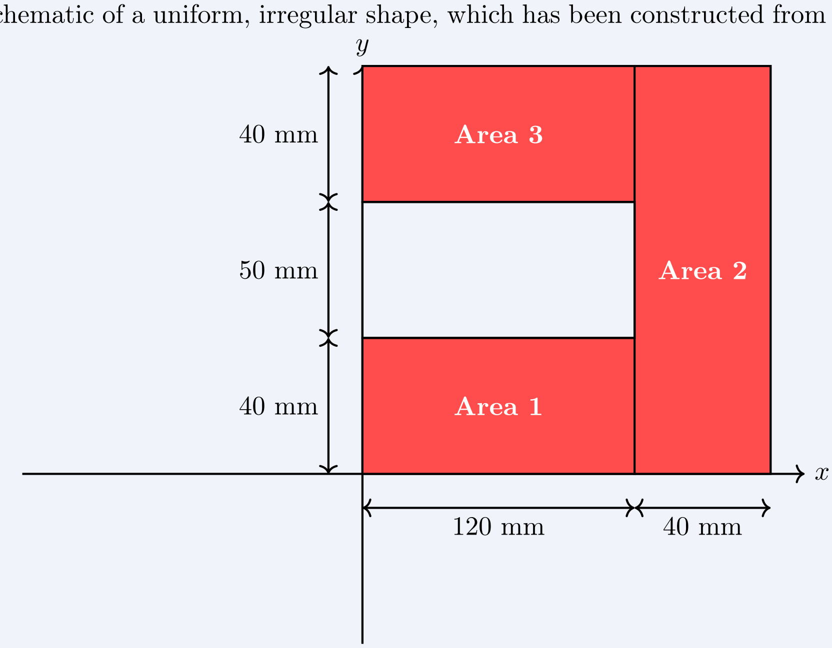
Definition 9 (Geocentric centre). The cooridnate q, C_q, representing the geocentric centre of an irregular shape is defined as:

$$C_q = \frac{\sum_i C_{iq} A_i}{\sum_i A_i}$$

where C_{iq} denotes the geocentric centre of the ith shape, A_i is the area of the ith shape.

Corollary. The geocentric centre and the COM will coincide when the mass density is uniform. And so, calculating the geocentric centre is one method of calculating an object's centre of mass, iff its mass distribution is uniform.

Example (Using the geometric centre method for an irrelgular shape.). Shown below is a schematic of a uniform, irregular shape, which has been constructed from 3 rectangles.



The geocentric centre of this irregular shape can be found by calculating:

$$C_x = \frac{\sum_i C_{ix} A_i}{\sum_i A_i}$$

$$C_y = \frac{\sum_i C_{iy} A_i}{\sum_i A_i}$$

The geocentric centre is then given by the point (C_x, C_y). Performing the calculation first for the x-cooridnate of the geocentric centre, we find:

$$C_x = \frac{(60 \times 4800) + (140 \times 5200) + (60 \times 4800)}{4800 + 5200 + 4800} = 88.11 \text{ mm}$$

$$C_y = \frac{(20 \times 4800) + (65 \times 5200) + (110 \times 4800)}{4800 + 5200 + 4800} = 65\text{mm}$$

Note, the irregular shape enjoys symmetry at the line y = 65, and so the centre of mass given it is a uniform shape should reside here. This calculation is therefore a nice sanity check, as we do expect the y cooridnate of the centre of mass to be at 65mm – given its symmetry. Hence, the centre of mass of this irregular shape resides at the cooridnate $(88.11, 65)$.

2.3 Finding Centre of Mass using Integration

The formula in the summand form may be treated with infinitesimal masses. Treating the sum with infinitesimal elements converts the sum into an intergal. Here is an example of where infinitesimal mass elements would come in handy in calculating the centre of mass – calculating the centre of mass of hills.

Figure 2.2: The Chocolate Hills in the Philippines. They enjoy a nice circular symmetry, as they are modelled as a cone with a height h of 50 m and a radius r.

Example (Centre of Mass of the Chocolate Hills). The Chocolate Hills are a geological formation in the Philippines. They are covered in green grass that turns into a chocolate-like brown during the dry season, hence the name. Let us assume it is conical in shape, as shown in the figure, with a uniform density and a height of $h = 50\,\text{m}$. We aim to find the height of its centre of mass (y_{cm}).

Note, due to circular symmetry and uniform mass density, the x coordinate of the centre of mass will be zero, mass elements on left cancel the mass elements on the right of the cone. As such, we only need to consider the y-direction. The centre of mass is calculated as:

$$y_{\text{cm}} = \frac{\int_0^h y\,dm}{\int_0^h dm}.$$

The mass element (dm) of the cone can be expressed as:

$$dm = \rho\,(\pi r^2)\,dy = \rho\,\pi y^2 \tan^2\theta\,dy,$$

where ρ is the density, $r = y\tan\theta$, and θ is the angle of the cone's slope. Substituting dm into the expression for y_{cm}:

$$y_{\text{cm}} = \frac{\int_0^h y\,\rho\,\pi y^2 \tan^2\theta\,dy}{\int_0^h \rho\,\pi y^2 \tan^2\theta\,dy}.$$

Since ρ, π, and $\tan^2\theta$ are constants, they cancel out:

$$y_{\text{cm}} = \frac{\int_0^h y^3\,dy}{\int_0^h y^2\,dy}.$$

The numerator and denominator are solved as follows:

$$\int_0^h y^3\,dy = \left[\frac{y^4}{4}\right]_0^h = \frac{h^4}{4},$$

$$\int_0^h y^2\,dy = \left[\frac{y^3}{3}\right]_0^h = \frac{h^3}{3}.$$

Substituting these results into the expression for y_{cm}:

$$y_{\text{cm}} = \frac{\frac{h^4}{4}}{\frac{h^3}{3}} = \frac{3h}{4}.$$

Thus, the centre of mass is located at:

$$y_{\text{cm}} = \frac{3h}{4} \quad \text{(measured from the top of the cone).}$$

For $h = 50\,\text{m}$, the height of the centre of mass from the top of the cone is 37.5m, meaning that from the base:

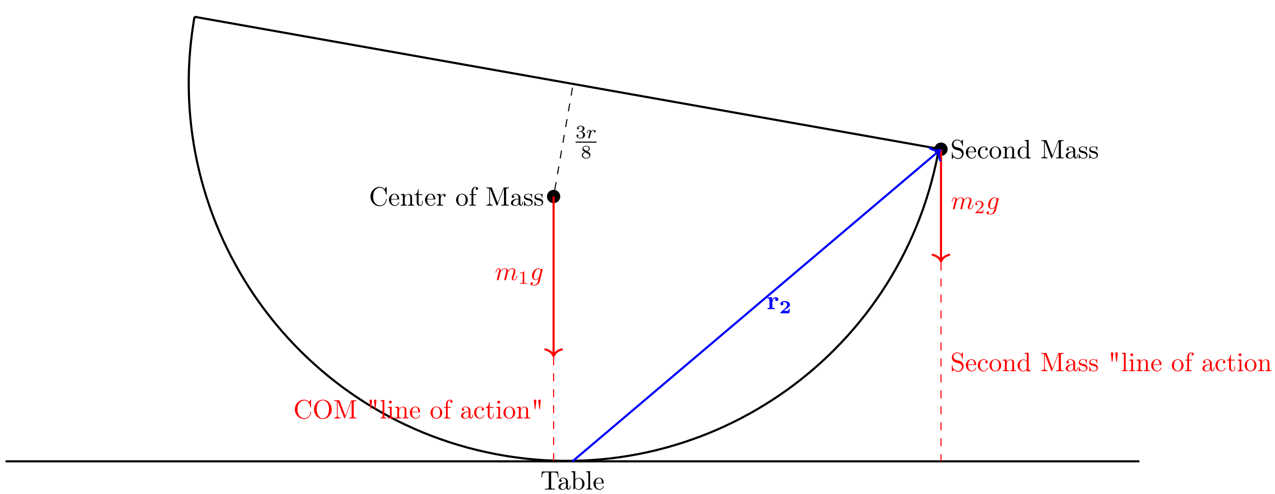
$$y_{\text{cm}} = 12.5\,\text{m}.$$

2.4 A note on the Line of Action: relating stability to the COM

Do review Q10 on Physics Assignment 3 which is a direct application of the line of action! The line of action is purely a geometric representation of how a force is applied to a rotating system. It is simply a straight line through the point at which the force is applied, and is in the same direction as the Force applied. It is a useful aid in visualising perpendicular distances, e.g. say we wish to calculate the torque, τ as a result of the second mass. The torque is equal to the cross-product of the distance from the pivot (in this example, the table is the point of contact) and the force causing the rotation about the pivot:

$$\tau = \mathbf{r_2} \times m_2\mathbf{g} = m_2 g\{|\mathbf{r_2}|\sin\theta\} = m_2 g \times x_2$$

where the x_2 is the perpendicular distance between the line of action of the force and the pivot and θ is the tilt of the hemisphere, and is also the angle separating the forces and the position vectors from the mass to the pivot. This may be repeated for the COM. The rest of the analysis is something you had to do in Q10 of Physics Assignment 3; with the main punchline of the problem being the construction of lines of action to see what the perpendicular distance from the pivot is, which are needed order to calculate the torque of the COM and the second mass.



Chapter 3

Momentum

Figure 3.1: Launch of ESA's Ariane 5 rocket, carrying rather important payload to space – the James Webb Space Telescope – on Christmas Day in 2021. Rockets use momentum, in principle, to get into space.

Assignment 4: Momentum

Fri, 10 Jan 19:00

3.1 Momentum and Impulse

Definition 10 (Momentum and Impulse).

Linear Momentum:

$$\mathbf{p} = m \cdot \mathbf{v}$$

Here, $\mathbf{p}$ is the linear momentum, m is the mass, and $\mathbf{v}$ is the velocity.

Conservation of Linear Momentum:

$$\sum m_i \cdot \mathbf{v_i} = \sum m_f \cdot \mathbf{v_f}$$

This equation expresses the conservation of linear momentum in a system, where m_i and $\mathbf{v_i}$ are the initial mass and velocity, and m_f and $\mathbf{v_f}$ are the final mass and velocity of the system.

Impulse-Momentum Theorem:

$$\mathbf{F} \cdot \Delta t = \Delta \mathbf{p}$$

This theorem relates the impulse ($\mathbf{F} \cdot \Delta t$) applied to an object to the change in its momentum ($\Delta \mathbf{p}$). This is true for a constant force over time. We can generalise this theorem for a variable force, one that changes with time, by considering an infinitesimal time period, such that: $\Delta t \to \delta t$. One can write the macroscopic momentum, $\Delta \mathbf{p}$ as:

$$\Delta \mathbf{p} = \int_{p_1}^{p_2} \delta \mathbf{p}(t) = \underbrace{\int_{t_1}^{t_2} \mathbf{F}(t)\, \mathrm{d}t}_{\text{via. Impulse-Momentum Theorem}}$$

This is a more general result of the Impulse-Momentum Theorem, it accounts for the Force potentially being time dependent.

Angular Momentum:

$$\mathbf{L} = I \cdot \boldsymbol{\omega}$$

Angular momentum ($\mathbf{L}$) is the product of moment of inertia (I) and angular velocity ($\boldsymbol{\omega}$).

3.2 Collisions

Via the principle of the conservation of linear momentum: the momentum describing the state of a system before a collision must equal the momentum describing the state of the system after the collision. Kinetic energy, though, needn't be conserved. If it is conserved after a collision event, it is an elastic collision. If kinetic energy is not conserved after a collision event, it is an inelastic collision.

Example. In an elastic collision between two equal masses, assuming they both collide, and one of the masses is at rest, the angle of separation after the collision is 90 degrees.

Proof. Some interesting situations arise when the two colliding objects have equal mass and the collision is elastic. An elastic collision conserves internal kinetic energy.

$$\frac{1}{2}mv_1^2 = \frac{1}{2}mv_1'^2 + \frac{1}{2}mv_2'^2.$$

Because the masses are equal ($m_1 = m_2 = m$), algebraic manipulation of conservation of momentum in the x- and y-directions can show that

$$\frac{1}{2}mv_2^2 = \frac{1}{2}mv_1'^2 + \frac{1}{2}mv_2'^2 + mv_1'v_2' \cos(\theta_1 - \theta_2).$$

(Remember that θ_2 is negative here.) The two preceding equations can both be true only if

$$mv_1'v_2' \cos(\theta_1 - \theta_2) = 0.$$

There are three ways that this term can be zero. They are

- $v_1' = 0$: head-on collision; incoming ball stops
- $v_2' = 0$: no collision; incoming ball continues unaffected
- $\cos(\theta_1 - \theta_2) = 0$: angle of separation ($\theta_1 - \theta_2$) is $90°$ after the collision. Thus, as we are interested solely in the last case, the angle of separation between the two balls will be 90 degrees.

Part II

Maths

Chapter 1

Trigonometry

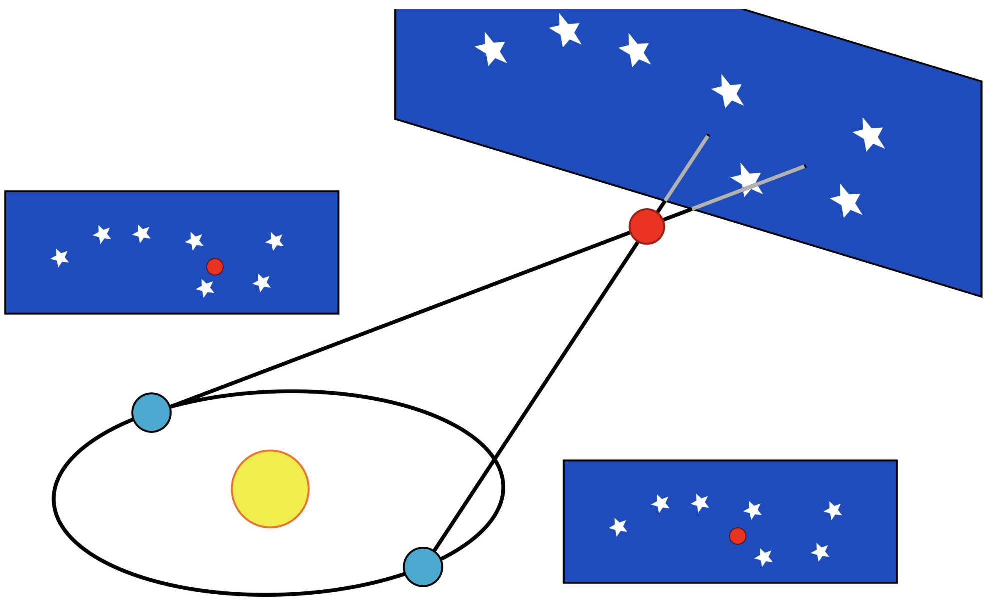


Figure 1.1: Example of a parallax. Shown in the figure is Earth in January, vs Earth in July. There is a relative difference in the apparent position of the start in January versus July, given by the Parallax Angle.

Assignment 2: Trigonometry

Fri, 15 Nov 19:00

1.1 Trigonometric Functions

The coordinates of any point on the unit circle ($r = 1$) are defined by the following trigonometric functions.

Definition 11 (Sine). The sine of an angle θ is the ratio of the length of the opposite side to the length of the hypotenuse:

$$\sin(\theta) = \frac{\text{opposite}}{\text{hypotenuse}}$$

Domain: $\theta \in \mathbb{R}$ (all real numbers)
Range: $\sin(\theta) \in [-1, 1]$

Definition 12 (Cosine). The cosine of an angle θ is the ratio of the length of the adjacent side to the length of the hypotenuse:

$$\cos(\theta) = \frac{\text{adjacent}}{\text{hypotenuse}}$$

Domain: $\theta \in \mathbb{R}$
Range: $\cos(\theta) \in [-1, 1]$

Definition 13 (Tangent). The tangent of an angle θ is the ratio of the length of the opposite side to the length of the adjacent side:

$$\tan(\theta) = \frac{\text{opposite}}{\text{adjacent}} \equiv \frac{\sin(\theta)}{\cos(\theta)}$$

Domain: $\theta \in \mathbb{R} \setminus \left\{\frac{\pi}{2} + n\pi \mid n \in \mathbb{Z}\right\}$ (all real numbers except odd multiples of $\frac{\pi}{2}$)
Range: $\tan(\theta) \in \mathbb{R}$ (all real numbers)

The Unit Circle

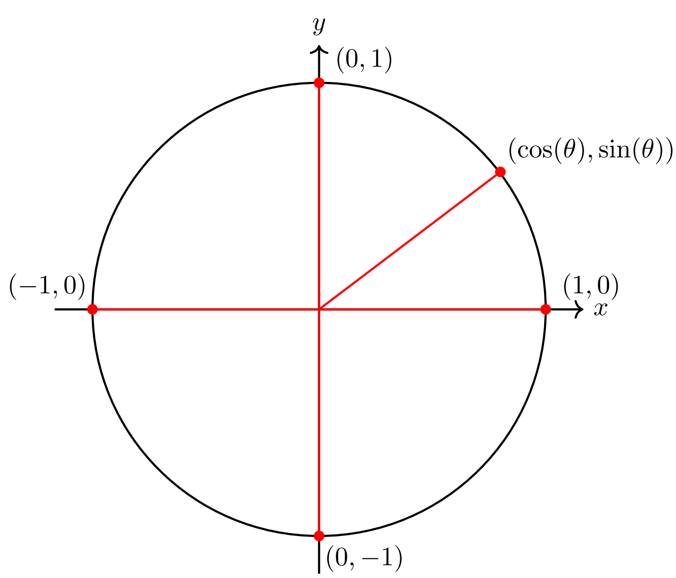


Figure 1.2: The Unit Circle, circle on the xy plane with radius, $r = 1$.

Example (Unit Circle).

- The unit circle is represented by the equation:

$$x^2 + y^2 = 1$$

  where x and y are the coordinates of any point on the circle.

- Any point on the unit circle can be described by the coordinates $(\cos(\theta), \sin(\theta))$, where θ is the angle formed by the radius vector and the positive x-axis.
- As θ varies from 0 to 2π, the point on the unit circle traces out a full circle, with $\cos(\theta)$ representing the x-coordinate and $\sin(\theta)$ representing the y-coordinate of the point.
- Key points on the unit circle include:
  - At $\theta = 0°$ (or 0 radians), the point is $(1, 0)$.
  - At $\theta = 90°$ (or $\frac{\pi}{2}$ radians), the point is $(0, 1)$.
  - At $\theta = 180°$ (or π radians), the point is $(-1, 0)$.
  - At $\theta = 270°$ (or $\frac{3\pi}{2}$ radians), the point is $(0, -1)$.

1.2 Reciprocal Trigonometric Functions

The reciprocal trigonometric functions are defined as the inverses of sine, cosine, and tangent.

Definition 14 (Cosecant). The cosecant of θ is the reciprocal of the sine function:

$$\csc(\theta) := \frac{1}{\sin(\theta)}$$

Domain: $\theta \in \mathbb{R} \setminus n\pi$ (all real numbers except integer multiples of π)
Range: $\csc(\theta) \in (-\infty, -1] \cup [1, \infty)$

Definition 15 (Secant). The secant of θ is the reciprocal of the cosine function:

$$\sec(\theta) := \frac{1}{\cos(\theta)}$$

Domain: $\theta \in \mathbb{R} \setminus \frac{\pi}{2} + n\pi$ (all real numbers except odd multiples of $\frac{\pi}{2}$)
Range: $\sec(\theta) \in (-\infty, -1] \cup [1, \infty)$

Definition 16 (Cotangent). The cotangent of θ is the reciprocal of the tangent function:

$$\cot(\theta) := \frac{1}{\tan(\theta)}$$

Domain: $\theta \in \mathbb{R} \setminus n\pi$ (all real numbers except integer multiples of π)
Range: $\cot(\theta) \in \mathbb{R}$ (all real numbers)

1.3 Inverse Trigonometric Functions

Inverse trigonometric functions allow us to determine the angle when given a trigonometric ratio. Here, we discuss arcsine (arcsin) and arccosine (arccos).

Definition 17 (Arcsine (arcsin)). The arcsine function $\arcsin(x)$ is the inverse of the sine function. It returns the angle θ such that:

$$\sin(\theta) = x \Rightarrow \theta = \arcsin(x) \quad \text{and} \quad \theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right].$$

Domain: $x \in [-1, 1]$
Range: $\theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

Definition 18 (Arccosine (arccos)). The arccosine function $\arccos(x)$ is the inverse of the cosine function. It returns the angle θ such that:

$$\cos(\theta) = x \Rightarrow \theta = \arccos(x) \quad \text{and} \quad \theta \in [0, \pi].$$

Domain: $x \in [-1, 1]$
Range: $\theta \in [0, \pi]$

1.3.1 Relationship to the Unit Circle

The arcsine and arccosine functions correspond to angles on the unit circle.

- For $\arcsin(x)$, the angle θ is measured counterclockwise from the positive x-axis and is restricted to the first and fourth quadrants ($-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$).
- For $\arccos(x)$, the angle θ is measured counterclockwise from the positive x-axis and is restricted to the first and second quadrants ($0 \leq \theta \leq \pi$).

1.4 Pythagorean Identities

A fundamental identity in trigonometry.

Definition 19 (Pythagorean Identity). The Pythagorean identity states that for any angle θ:

$$\sin^2(\theta) + \cos^2(\theta) = 1$$

1.5 Sum and Difference Formulas

The sum and difference formulas allow us to compute the trigonometric functions of sums or differences of angles.

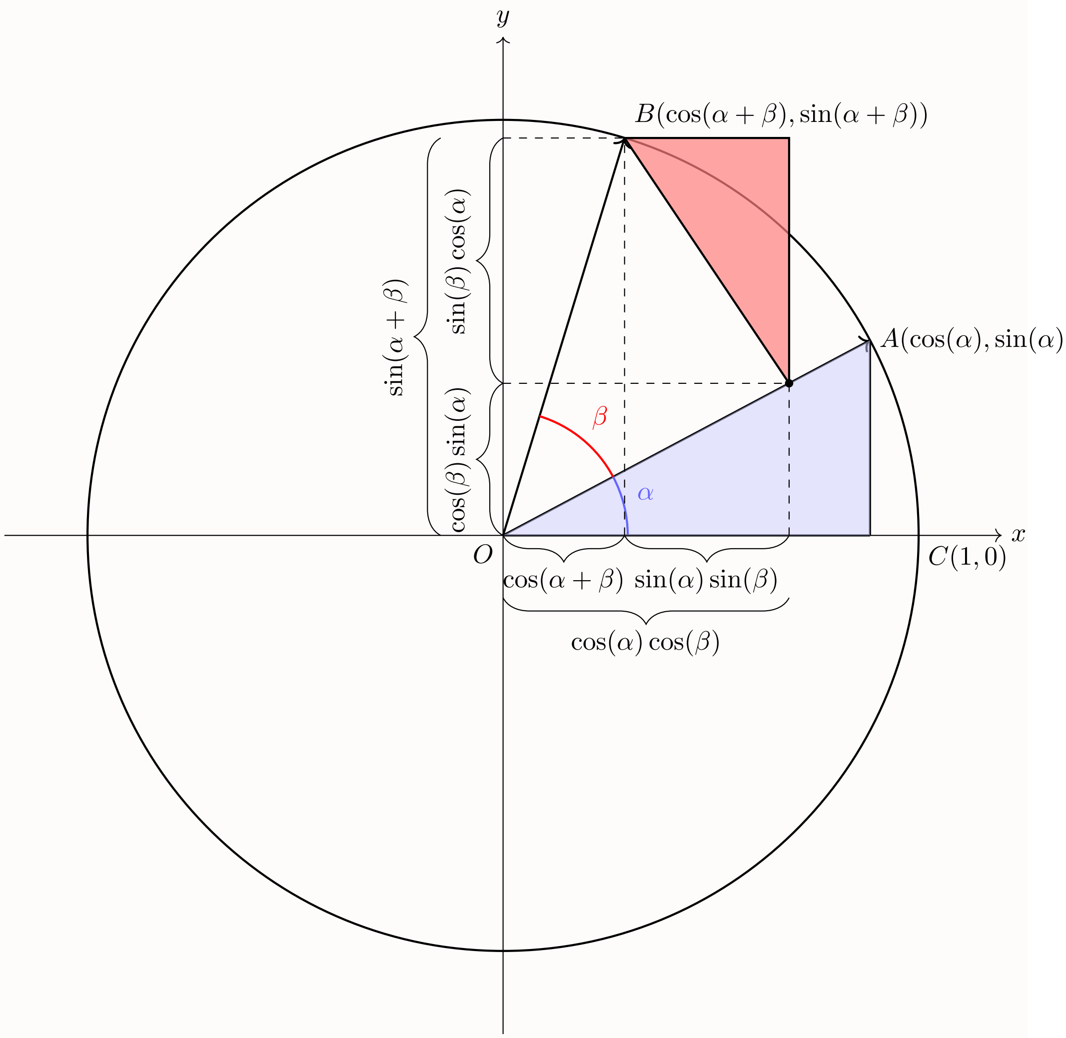
Definition 20 (Sum of Angles for Sine). The sine of the sum of two angles is given by:

$$\sin(A + B) = \sin(A)\cos(B) + \cos(A)\sin(B)$$

Definition 21 (Sum of Angles for Cosine)**.** The cosine of the sum of two angles is given by:

$$\cos(A+B) = \cos(A)\cos(B) - \sin(A)\sin(B)$$

Proof. To prove, we can construct a unit circle, and draw two angles, α and β. Note, at A, this point has an angle of α while B has an angle of $\alpha+\beta$. The key to this problem is the creation of a perpendicular bisector emanating from the point B, and hits the triangle with an angle α. What this creates is a new triangle, with angle β, which is but a triangle with angle β rotated by angle α from the origin. Performing trigonometry on all triangles discussed, we find by matching horizontal and vertical distances highlighted by the underbraces in the figure below: $\sin(\alpha+\beta) = \sin(\alpha)\cos(\beta) + \sin(\beta)\cos(\alpha)$ and $\cos(\alpha)\cos(\beta) = \cos(\alpha+\beta) + \sin(\alpha)\sin(\beta) \Rightarrow \cos(\alpha+\beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)$, completing the proof of definitions 15 and 16.



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Corollary. Noting symmetry of $\sin(-x) = -\sin(x)$ and $\cos(-x) = \cos(x)$, we find as a corollary the following generalised addition formulae:

$$\sin(A \pm B) = \sin(A)\cos(B) \pm \cos(A)\sin(B)$$

$$\cos(A + B) = \cos(A)\cos(B) \mp \sin(A)\sin(B)$$

Definition 22 (Sum of Angles for Tangent). The tangent of the sum of two angles is given by:

$$\tan(A + B) = \frac{\tan(A) + \tan(B)}{1 - \tan(A)\tan(B)}$$

1.6 Trigonometric Equations and Identities

Trigonometric equations and identities are useful tools in simplifying and solving problems in maths.

Definition 23 (Double Angle Formula for Sine). The sine of double an angle is given by:

$$\sin(2\theta) = 2\sin(\theta)\cos(\theta)$$

Definition 24 (Double Angle Formula for Cosine). The cosine of double an angle is given by:

$$\cos(2\theta) = \cos^2(\theta) - \sin^2(\theta)$$

1.7 Applications of Trigonometry

Definition 25 (Law of Sines). In any triangle, the ratio of the length of a side to the sine of its opposite angle is constant:

$$\frac{a}{\sin(A)} = \frac{b}{\sin(B)} = \frac{c}{\sin(C)}$$

Definition 26 (Law of Cosines). In any triangle, the cosine of an angle is related to the lengths of the sides as:

$$c^2 = a^2 + b^2 - 2ab\cos(C)$$

Chapter 2

Calculus I – Differentiation

Figure 2.1: Gottfried Wilhelm Leibniz, Joseph-Louis Lagrange, Sir Isaac Newton, Leonhard Euler (pictured from left to right). Each introduced their notation for the differential operator acting on a function f: Leibniz – $\frac{df}{dx}$, Lagrange – f', Newton – $\dot{f}$, Euler – $D_x f$

Assignment 3: Calculus I

Fri, 13 Dec 19:00

2.1 Introducing the differential operator

Differentiation is the process of finding the instantaneous rate of change of a function $f(x)$, denoted as $f'(x) \equiv \frac{d}{dx} f(x)$. Below, we can write some foundational rules and definitions:

2.1.1 Power Rule

For the power function $f(x) = x^n$, the derivative is:

Definition 27 (Power Rule).

$$\frac{d}{dx}x^n = nx^{n-1}.$$

2.1.2 Linearity of Differentiation

Differentiation is represented by the operator $\frac{d}{dx}$. It is a linear operator, meaning:

$$\frac{d}{dx}\left[af(x) + bg(x) + ch(x) + \dots\right] = a\frac{d}{dx}f(x) + b\frac{d}{dx}g(x) + c\frac{d}{dx}h(x) + \dots$$

2.1.3 First Principles

The derivative from first principles is defined as:

Definition 28 (Differentiation from first principles).

$$f'(x) = \lim_{h\to 0}\frac{f(x+h) - f(x)}{h}.$$

2.2 Derivatives of Sine and Cosine

For trigonometric functions:

$$\frac{d}{dx}\sin x = \cos x, \quad \frac{d}{dx}\cos x = -\sin x.$$

For a constant a:

$$\frac{d}{dx}\sin(ax) = a\cos(ax), \quad \frac{d}{dx}\cos(ax) = -a\sin(ax).$$

Example (Deriving trig derivatives using differentiation from first principles). We may derive the derivatives of sine, or cosine, using differentation from first principles. Let us consider the derivative of $\sin(x)$. By definition, its derivative is:

$$\sin'(x) = \lim_{h\to 0}\frac{\sin(x+h) - \sin(x)}{h}.$$

Using the trigonometric identity $\sin(A+B) = \sin A\cos B + \cos A\sin B$:

$$\sin(x+h) = \sin x\cos h + \cos x\sin h.$$

Substituting this back into the limit:

$$\sin'(x) = \lim_{h\to 0}\frac{\sin x\cos h + \cos x\sin h - \sin x}{h}.$$

Separating terms:

$$\sin'(x) = \lim_{h\to 0}\left(\sin x\frac{\cos h - 1}{h} + \cos x\frac{\sin h}{h}\right).$$

Using the known limits:

$$\lim_{h\to 0}\frac{\sin h}{h}=1,\quad \lim_{h\to 0}\frac{cosh-1}{h}=0,$$

we find:

$$\sin'(x)=\cos x.$$

2.3 Product Rule

The derivative of the product of two functions is:

Definition 29 (Product Rule).

$$\frac{d}{dx}\big[f(x)g(x)\big]=f'(x)g(x)+f(x)g'(x).$$

Proof. To prove the product rule from first principles:

$$\frac{d}{dx}\big[f(x)g(x)\big]=\lim_{h\to 0}\frac{f(x+h)g(x+h)-f(x)g(x)}{h}.$$

This may be written as:

$$\frac{d}{dx}\big[f(x)g(x)\big]=\lim_{h\to 0}\frac{f(x+h)g(x+h)-f(x)g(x+h)+f(x)g(x+h)-f(x)g(x)}{h}.$$

Separating terms:

$$\frac{d}{dx}\big[f(x)g(x)\big]=\lim_{h\to 0}\frac{g(x+h)[f(x+h)-f(x)]}{h}+\lim_{h\to 0}\frac{f(x)[g(x+h)-g(x)]}{h}.$$

Taking limits:

$$\frac{d}{dx}\big[f(x)g(x)\big]=g(x)f'(x)+f(x)g'(x).$$

□

Example.

$$(x^2\sin(x))'=(x^2)'\sin(x)+(x^2)(\sin(x))'=2x\sin(x)+x^2\cos(x)$$

2.4 Chain Rule

The chain rule helps differentiate composite functions $f(g(x))$:

Definition 30 (Chain Rule).

$$\frac{d}{dx}f(g(x))=\frac{df}{dg}\cdot\frac{dg}{dx}=f'_g(g)\cdot g'_x(x).$$

Example.

$$\frac{d}{dx}\sin(x^2) = \frac{d}{d(x^2)}\sin(x^2)\cdot\frac{d}{dx}x^2 = 2x\cos(x^2)$$

2.5 Quotient Rule

The derivative of the quotient of two functions is:

Definition 31 (Quotient Rule).

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{f'(x)g(x) - f(x)g'(x)}{\big[g(x)\big]^2}.$$

Example.

$$\frac{d}{dx}\left(\frac{\sin x}{x^2}\right) = \frac{(\cos x)x^2 - 2(\sin x)x}{x^4} = \frac{x(\cos x) - 2\sin x}{x^3}$$

2.6 Additional Trigonometric Derivatives

For the tangent and secant functions:

$$\frac{d}{dx}\tan x = \sec^2 x, \quad \frac{d}{dx}\sec x = \sec x\tan x.$$

2.7 Maximising/minimising a function

The maxima/minima, i.e. extrema of a function can be found by setting the derivative of the function to 0, granted the function is both differentiable (smooth) and continuous (contains no singularities).

Definition 32 (Extrema of a function). The extrema of a function exist at $f'(x) = 0$, granted the function is both differentiable and continuous.

Example (Snell's Law). Consider a ray of light travelling through 2 different media, each with a refractive index n_1 and n_2. The total travel time is given by:

$$T = \frac{\sqrt{x^2 + h_1^2}}{v_1} + \frac{\sqrt{(d-x)^2 + h_2^2}}{v_2},$$

where $v_1 = \frac{c}{n_1}$ and $v_2 = \frac{c}{n_2}$ are the speeds of light in the two media. Substituting these speeds:

$$T = n_1\sqrt{x^2 + h_1^2} + n_2\sqrt{(d-x)^2 + h_2^2}.$$

To minimize T, differentiate with respect to x:

$$\frac{dT}{dx} = \frac{n_1 x}{\sqrt{x^2 + h_1^2}} - \frac{n_2(d-x)}{\sqrt{(d-x)^2 + h_2^2}}.$$

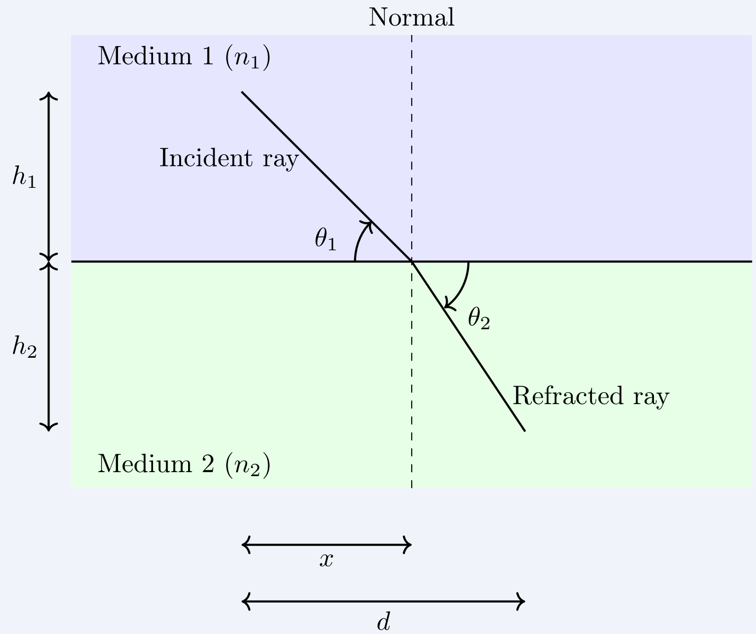
Setting $\frac{dT}{dx} = 0$:

$$\frac{n_1 x}{\sqrt{x^2 + h_1^2}} = \frac{n_2(d - x)}{\sqrt{(d - x)^2 + h_2^2}}.$$

Rearranging, we find:

$$n_1 \sin\theta_1 = n_2 \sin\theta_2,$$

which is Snell's law. The ray in which the line would trace as a result is shown below.



Chapter 3

Calculus II – Integration

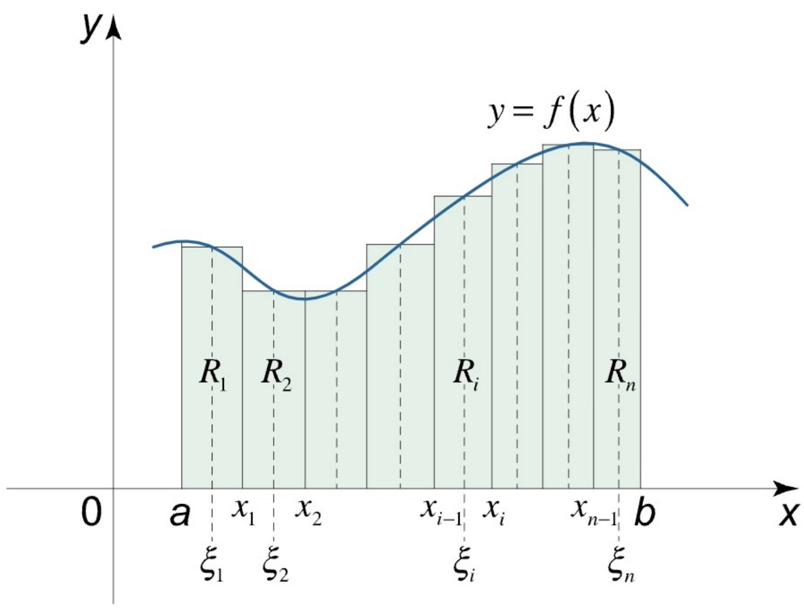


Figure 3.1: Illustration of Riemann Integration. The result of a definite integral is the area between the curve and the x-axis. Riemann integration approximates the integral as a finite sum of rectangles/trapezia, which act to represent the area beneath the curve.

Assignment 4: Calculus II

Fri, 17 Jan 19:00

3.1 Indefinite Integral

3.1.1 The Antiderivative Function

An antiderivative function, $F(x)$, is a function whose derivative is a given function $f(x)$, i.e., $F'(x) = f(x)$. If $F(x)$ is an antiderivative of $f(x)$, then $F(x) + C$, where C is a constant, is also an antiderivative.

> **Example** ($f(x) = 3x^2$)**.** Antiderivatives include $F_1(x) = x^3$, $F_2(x) = x^3 + 4$, and $F_3(x) = x^3 - 7$.

3.1.2 Indefinite Integral

The set of all antiderivatives of a given function $f(x)$ is represented by an indefinite integral:

Definition 33 (Indefinite Integral).

$$\int f(x)\,dx = F(x) + C, \quad C \in \mathbb{R}$$

The notation used above means C belongs to the set of real numbers. The function $f(x)$ is called the integrand, and dx denotes the variable of integration, i.e. the variable you are integrating with respect to.

Corollary. Some properties of indefinite integrals include:

$$\int (f(x) \pm g(x))\,dx = \int f(x)\,dx \pm \int g(x)\,dx, \quad \int kf(x)\,dx = k\int f(x)\,dx.$$

3.2 Definite Integral

The definite integral is the limit of the Riemann sum:

Lemma 1 (Riemann sum).

$$\lim_{\Delta x \to 0} \sum f(x)\Delta x = \int_a^b f(x)\,dx.$$

Theorem 1 (Fundamental theorem of calculus). The fundamental theorem of calculus states that:

$$\int_a^b f(x)\,dx = F(b) - F(a),$$

where $F(x)$ is an antiderivative of $f(x)$.

Example (Calculating the area of curves using definite integration). To find the area between the curves $y = x^2$ and $y = x$ from $x = -1$ to $x = 1$:

$$A = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}.$$

3.3 Integration Techniques

3.3.1 Integration by Parts

The integration by parts formula is derived from the product rule of differentiation. This technique is useful for integrals of the form $u(x)v'(x)$, where $u(x)$ is often easily differentiable, and $v'(x)$ is often easy to integrate.

Definition 34 (Integration by Parts).

$$\int u\,v'\,dx = uv - \int u'\,v\,dx.$$

Example ($\int x \sin x\,dx$). To compute $\int x \sin x\,dx$, set $u = x$ and $v' = \sin x$, giving:

$$\int x \sin x\,dx = -x\cos x + \int \cos x\,dx = -x\cos x + \sin x + C.$$

3.3.2 Integration by Substitution

Integration by substitution is based on the chain rule of differentiation:

$$\frac{d}{dx} f(u(x)) = f'(u(x)) \cdot u'(x).$$

For integrals of composite functions, substitute $u(x) = g(x)$, then $du = g'(x)dx$, and the integral becomes:

Definition 35 (Substitution rule).

$$\int f(g(x))g'(x)\,dx = \int f(u)\,du.$$

Note if one is dealing with a definite integral, one us substitute each of the respective limits for u.

Example ($\int \cos x \sin^3 x\,dx$). To compute $\int \cos x \sin^3 x\,dx$, let $u = \sin x$, so $du = \cos x\,dx$, and the integral becomes:

$$\int u^3\,du = \frac{1}{4}u^4 + C \equiv \frac{1}{4}\sin^4 x + C.$$

3.4 Integrals in Geometry and Physics

Example (Volume of a sphere). The volume of a sphere of radius R can be calculated using integration.

We start by considering the equation of the sphere:

$$x^2 + y^2 + z^2 = R^2.$$

To compute the volume, we use the method of slicing the sphere up into slabs. We slice the sphere into thin slabs along the z-axis. Each disk at a given z has a radius of $r(z) = \sqrt{R^2 - z^2}$, which comes from solving the equation of the sphere for x and y.

The area of a cross-sectional slab in the x-y plane is given by:

$$A(z) = \pi r(z)^2 = \pi(R^2 - z^2).$$

The volume of a thin disk with thickness dz is:

$$dV = A(z)\,dz = \pi(R^2 - z^2)\,dz.$$

To find the total volume, we integrate this expression from the total parameter space of z: $z = -R$ to $z = R$:

$$V = \int_{-R}^{R} \pi(R^2 - z^2)\,dz.$$

$$= \pi \left[R^2 z - \frac{z^3}{3}\right]_{-R}^{R}.$$

Evaluating the integral:

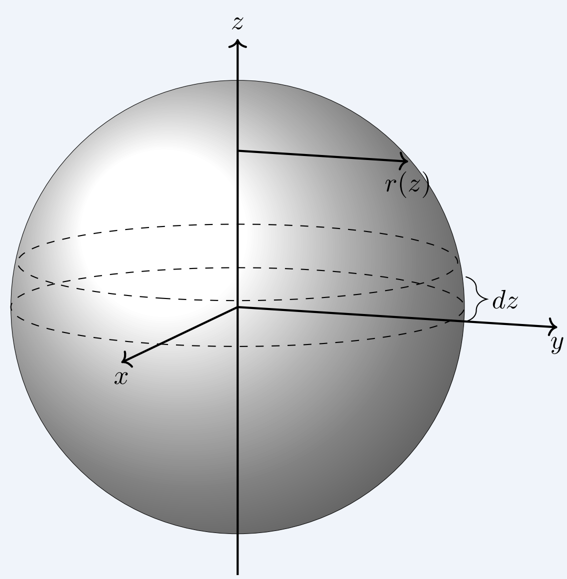
$$V = \pi \left[\left(R^2 R - \frac{R^3}{3}\right) - \left(R^2(-R) - \frac{(-R)^3}{3}\right)\right].$$

In simplifying the expression, we find:

$$V = \pi \left[R^3 - \frac{R^3}{3} + R^3 + \frac{R^3}{3}\right] = \pi \left[2R^3 - \frac{R^3}{3} + \frac{R^3}{3}\right] = \pi \cdot \frac{4R^3}{3}.$$

Thus, the volume of the sphere is:

$$V = \frac{4}{3}\pi R^3.$$



Part III

Resources

Chapter 1

University admissions exam advice

1.0.1 Overview

- PAT is the exam used at Oxford for those who want to apply for Physics and Philosophy.
- It is a two-hour test that examines your ability on maths and physics topics. The topics in which they assess are "core" topics, the topics that are essential building blocks for learning maths and physics. These are very much like the topics we study at COMPOS. The current PAT syllabus currently examines: elementary mathematics, algebra, calculus, mechanics, waves and optics, electricity and magnetism, and the natural world.

1.0.2 Advice

PAT provides the following advice:

Example (PAT Advice).

- "Look over a range of **past papers** to help to familiarise you with the format of the test and the content covered. We also publish reports for each test; reports contain information such as the average mark on the paper and the mark students needed to achieve an interview. Do not expect to get all of it correct – most years the average is 50-60%".

  The main comment to take away here is that past papers are an important part of preparing for the PAT exam. Past papers in general are an important part of the revision process, from GCSE to Uni. They serve as an excellent example of applying the knowledge that you have learned during your revision phase. You first learn the physics through solving problems, and then you have a chance to apply the skills learned in the exam.

- "Familiarise yourself with the syllabus. The material is aimed at AS level maths and physics plus knowledge of material covered at GCSE. However we cannot guarantee when the material will be covered in your school so you might find you need to teach yourself a few topics before the exam."

  Have a read through the ***syllabus*** *in detail. To familiarise yourself with the syllabus, read some of your notes from your A-Level books, COMPOS guidebook, COMPOS assignments, and have a go at redoing some problems for understanding, to ensure the concepts are retained.*

- "Get practice doing some **problem solving**/hard physics questions which are not A level questions. It is advisable to do questions from a range of other sources, not just A level type questions, which can be more structured in nature than the PAT. See our page on useful websites and resources for the PAT."

  COMPOS is a really good resource for this! We will do lots of problems which involve going beyond the spec. Revisiting some of these problems in this guidebook, and practicing some questions using the resources outlined in section 1.0.3 is a great way to go beyond your A-Level spec.

- "Try doing some questions under **timed conditions**. One of the things which students who have taken the test say is hard is the number of questions you need to do in only 2 hours. Practising some questions under timed conditions near the date of the exam will mean you are more likely to get to the end of the paper."

  Sorting out your exam technique is important for PAT and admissions tests. Often when I start off preparing for exams, I use the papers as a tool to carry on learning the content. As I do more, the quicker I get and I start to focus on timing on top of grasping the content. Learning the style of the papers is important, as the questions you do are likely to be similar to the questions seen in the real thing – so getting a good technique together is key.

1.0.3 PAT resources

- COMPOS problems: great resource to have to answer questions in bulk, in relation to core syllabus material.
- Isaac Physics: lots of problems to solve on core Syllabus material, including some problems that bridge the gap between A-Level and University content.
- British Physics Olympiad: I recommend the question bank resource they have, I use it often to prepare questions for tutorials.
- PhysicsLab - "Next Time": contains conceptual physics based questions, quite useful for interview prep!