Chapter 2

Calculus I – Differentiation

Figure 2.1: Gottfried Wilhelm Leibniz, Joseph-Louis Lagrange, Sir Isaac Newton, Leonhard Euler (pictured from left to right). Each introduced their notation for the differential operator acting on a function f: Leibniz – $\frac{df}{dx}$, Lagrange – f', Newton – $\dot{f}$, Euler – $D_x f$

Assignment 3: Calculus I

Fri, 13 Dec 19:00

2.1 Introducing the differential operator

Differentiation is the process of finding the instantaneous rate of change of a function $f(x)$, denoted as $f'(x) \equiv \frac{d}{dx} f(x)$. Below, we can write some foundational rules and definitions:

2.1.1 Power Rule

For the power function $f(x) = x^n$, the derivative is:

Definition 27 (Power Rule).

$$\frac{d}{dx}x^n = nx^{n-1}.$$

2.1.2 Linearity of Differentiation

Differentiation is represented by the operator $\frac{d}{dx}$. It is a linear operator, meaning:

$$\frac{d}{dx}\left[af(x) + bg(x) + ch(x) + \dots\right] = a\frac{d}{dx}f(x) + b\frac{d}{dx}g(x) + c\frac{d}{dx}h(x) + \dots$$

2.1.3 First Principles

The derivative from first principles is defined as:

Definition 28 (Differentiation from first principles).

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

2.2 Derivatives of Sine and Cosine

For trigonometric functions:

$$\frac{d}{dx}\sin x = \cos x, \quad \frac{d}{dx}\cos x = -\sin x.$$

For a constant a:

$$\frac{d}{dx}\sin(ax) = a\cos(ax), \quad \frac{d}{dx}\cos(ax) = -a\sin(ax).$$

Example (Deriving trig derivatives using differentiation from first principles). We may derive the derivatives of sine, or cosine, using differentation from first principles. Let us consider the derivative of $\sin(x)$. By definition, its derivative is:

$$\sin'(x) = \lim_{h \to 0} \frac{\sin(x+h) - \sin(x)}{h}.$$

Using the trigonometric identity $\sin(A+B) = \sin A \cos B + \cos A \sin B$:

$$\sin(x+h) = \sin x \cos h + \cos x \sin h.$$

Substituting this back into the limit:

$$\sin'(x) = \lim_{h \to 0} \frac{\sin x \cos h + \cos x \sin h - \sin x}{h}.$$

Separating terms:

$$\sin'(x) = \lim_{h \to 0} \left(\sin x \frac{\cos h - 1}{h} + \cos x \frac{\sin h}{h} \right).$$

Using the known limits:

$$\lim_{h \to 0} \frac{\sin h}{h} = 1, \quad \lim_{h \to 0} \frac{cosh - 1}{h} = 0,$$

we find:

$$\sin'(x) = \cos x.$$

2.3 Product Rule

The derivative of the product of two functions is:

Definition 29 (Product Rule).

$$\frac{d}{dx}\big[f(x)g(x)\big] = f'(x)g(x) + f(x)g'(x).$$

Proof. To prove the product rule from first principles:

$$\frac{d}{dx}\big[f(x)g(x)\big] = \lim_{h \to 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h}.$$

This may be written as:

$$\frac{d}{dx}\big[f(x)g(x)\big] = \lim_{h \to 0} \frac{f(x+h)g(x+h) - f(x)g(x+h) + f(x)g(x+h) - f(x)g(x)}{h}.$$

Separating terms:

$$\frac{d}{dx}\big[f(x)g(x)\big] = \lim_{h \to 0} \frac{g(x+h)[f(x+h) - f(x)]}{h} + \lim_{h \to 0} \frac{f(x)[g(x+h) - g(x)]}{h}.$$

Taking limits:

$$\frac{d}{dx}\big[f(x)g(x)\big] = g(x)f'(x) + f(x)g'(x).$$

□

Example.

$$(x^2 \sin(x))' = (x^2)' \sin(x) + (x^2)(\sin(x))' = 2x \sin(x) + x^2 \cos(x)$$

2.4 Chain Rule

The chain rule helps differentiate composite functions $f(g(x))$:

Definition 30 (Chain Rule).

$$\frac{d}{dx} f(g(x)) = \frac{df}{dg} \cdot \frac{dg}{dx} = f'_g(g) \cdot g'_x(x).$$

Example.

$$\frac{d}{dx}\sin(x^2) = \frac{d}{d(x^2)}\sin(x^2) \cdot \frac{d}{dx}x^2 = 2x\cos(x^2)$$

2.5 Quotient Rule

The derivative of the quotient of two functions is:

Definition 31 (Quotient Rule)**.**

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{f'(x)g(x) - f(x)g'(x)}{\big[g(x)\big]^2}.$$

Example.

$$\frac{d}{dx}\left(\frac{\sin x}{x^2}\right) = \frac{(\cos x)x^2 - 2(\sin x)x}{x^4} = \frac{x(\cos x) - 2\sin x}{x^3}$$

2.6 Additional Trigonometric Derivatives

For the tangent and secant functions:

$$\frac{d}{dx}\tan x = \sec^2 x, \quad \frac{d}{dx}\sec x = \sec x \tan x.$$

2.7 Maximising/minimising a function

The maxima/minima, i.e. extrema of a function can be found by setting the derivative of the function to 0, granted the function is both differentiable (smooth) and continuous (contains no singularities).

Definition 32 (Extrema of a function)**.** The extrema of a function exist at $f'(x) = 0$, granted the function is both differentiable and continuous.

Example (Snell's Law)**.** Consider a ray of light travelling through 2 different media, each with a refractive index n_1 and n_2. The total travel time is given by:

$$T = \frac{\sqrt{x^2 + h_1^2}}{v_1} + \frac{\sqrt{(d-x)^2 + h_2^2}}{v_2},$$

where $v_1 = \frac{c}{n_1}$ and $v_2 = \frac{c}{n_2}$ are the speeds of light in the two media. Substituting these speeds:

$$T = n_1\sqrt{x^2 + h_1^2} + n_2\sqrt{(d-x)^2 + h_2^2}.$$

To minimize T, differentiate with respect to x:

$$\frac{dT}{dx} = \frac{n_1 x}{\sqrt{x^2 + h_1^2}} - \frac{n_2(d-x)}{\sqrt{(d-x)^2 + h_2^2}}.$$

Setting $\frac{dT}{dx} = 0$:

$$\frac{n_1 x}{\sqrt{x^2 + h_1^2}} = \frac{n_2(d-x)}{\sqrt{(d-x)^2 + h_2^2}}.$$

Rearranging, we find:

$$n_1 \sin\theta_1 = n_2 \sin\theta_2,$$

which is Snell's law. The ray in which the line would trace as a result is shown below.