Chapter 1

Circular Motion

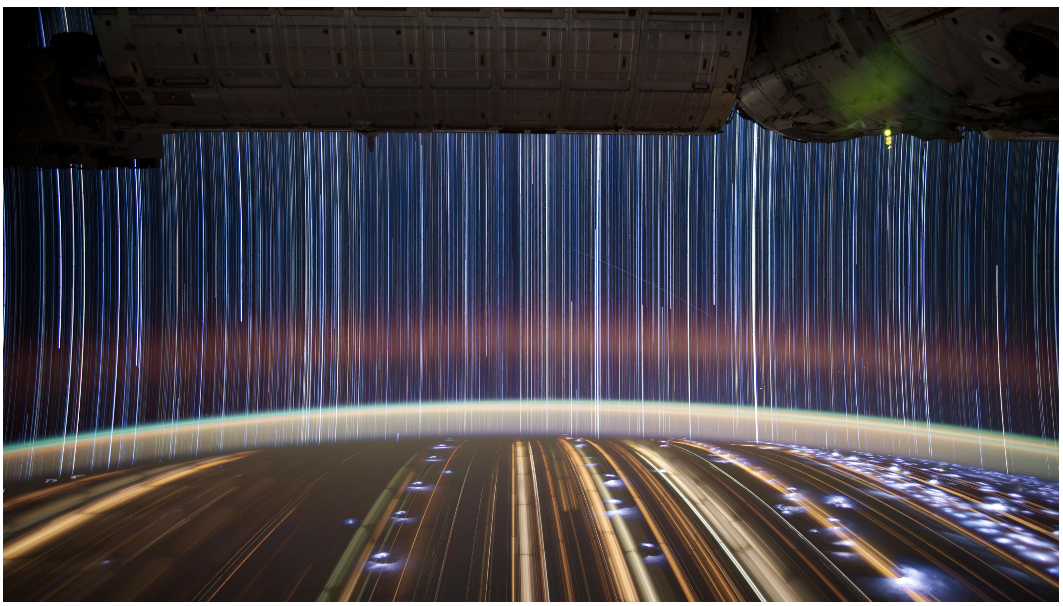

Figure 1.1: Star trails and light-on-Earth trails composite image created by International Space Station Expedition 30 crew member Don Pettit. The ISS is one of 11,000 active satellites as of June 2024 currently in orbit with the Earth.

Assignment 2: Circular Motion

Fri, 8 Nov 19:00

1.1 Angular Motion Kinematics

When an object moves along a circular path, its motion is defined by a few key relationships and variables. This section introduces the core concepts of angular motion.

> **Definition 1** (Angular Displacement). For an object moving in a circle of radius R, the distance it travels along the curve is noted as $x(t)$. The angle swept out by this path,

called the angular displacement, is:

$$\theta(t) = \frac{x(t)}{R}$$

Definition 2 (Angular Velocity and Acceleration)**.** By taking derivatives of angular displacement with respect to time, we get the angular velocity $\omega(t)$ and angular acceleration $\alpha(t)$:

$$\omega(t) = \frac{v(t)}{R}, \quad \alpha(t) = \frac{a_\tau(t)}{R}$$

For **uniform circular motion**, both $v(t)$ and $\omega(t)$ stay constant, giving us $x(t) = vt$ and $\theta(t) = \omega t$.

1.2 Centripetal Acceleration

As an object moves at a constant speed along a circular path, its direction changes continuously, creating a special kind of acceleration toward the circle's center, called centripetal acceleration.

Definition 3 (Centripetal Acceleration)**.** The acceleration that always points toward the center of the circle is known as centripetal acceleration and is given by:

$$a_c = \frac{v^2}{r} = \omega^2 r$$

Proof. Consider velocity vectors $\vec{v}_1$ and $\vec{v}_2$ at points A and B tangent to a circular trajectory and form an angle $\Delta\theta$ with each other. The change in velocity $\Delta\vec{v}$ can be approximated by the vector difference between $\vec{v}_1$ and $\vec{v}_2$. Since $|\vec{v}_1| = |\vec{v}_2| = v$, we find the magnitude of $\Delta\vec{v}$ as follows:

$$|\Delta\vec{v}| = 2v \sin\frac{\Delta\theta}{2}$$

For very small angles, $\sin\frac{\Delta\theta}{2} \approx \frac{\Delta\theta}{2}$, so we can write:

$$|\Delta\vec{v}| \approx v\Delta\theta$$

Now, dividing by the time interval Δt gives the average acceleration:

$$a = \frac{|\Delta\vec{v}|}{\Delta t} = \frac{v\Delta\theta}{\Delta t}$$

Since the angular velocity ω is defined as $\omega = \frac{\Delta\theta}{\Delta t}$, we have:

$$a = v\omega$$

Using the relationship $v = r\omega$, we substitute for ω:

$$a = \frac{v^2}{r}$$

Thus, the centripetal acceleration, directed toward the center of the circular path, is given by:

$$a_c = \frac{v^2}{r}$$

Corollary. Centripetal acceleration works at a right angle to the object's velocity, affecting only its direction and not its speed.

1.3 Uniform Circular Motion Dynamics

When an object moves in a circle, it needs a force to keep it on that path. This force, called centripetal force, keeps it from flying off in a straight line.

Definition 4 (Centripetal Force)**.** The force that pulls an object toward the center of its circular path is the centripetal force:

$$F_c = \frac{mv^2}{r} = m\omega^2 r$$

Centripetal force can be provided by different interactions, such as gravity for a satellite or friction for a car taking a turn. **Period** T and **frequency** f relate to motion around a circle by:

$$T = \frac{1}{f}, \quad \omega = 2\pi f$$

1.4 Non-uniform Circular Motion Dynamics

If an object speeds up or slows down as it moves in a circle, there's more going on. Here, we see both centripetal and tangential accelerations, affecting both direction and speed.

Definition 5 (Total Acceleration)**.** In non-uniform circular motion, total acceleration is the sum of tangential a_τ and centripetal a_c components:

$$a = a_\tau + a_c$$

To analyze this, we typically rely on **Newton's second law** and the principles of energy conservation.

1.5 Centrifugal Force (Apparent Force)

When you're in a rotating reference frame, an outward force, known as centrifugal force, seems to push you away from the center of rotation. This force is an illusion caused by the frame's motion, but it's useful in calculations.

Definition 6 (Centrifugal Force). The centrifugal force appears as an outward pull in a rotating reference frame:

$$F_{\text{centrifugal}} = -ma_c$$

Although it feels like a real force, centrifugal force results from the rotating frame's acceleration and doesn't actually act on the object.