Chapter 3

Calculus II – Integration

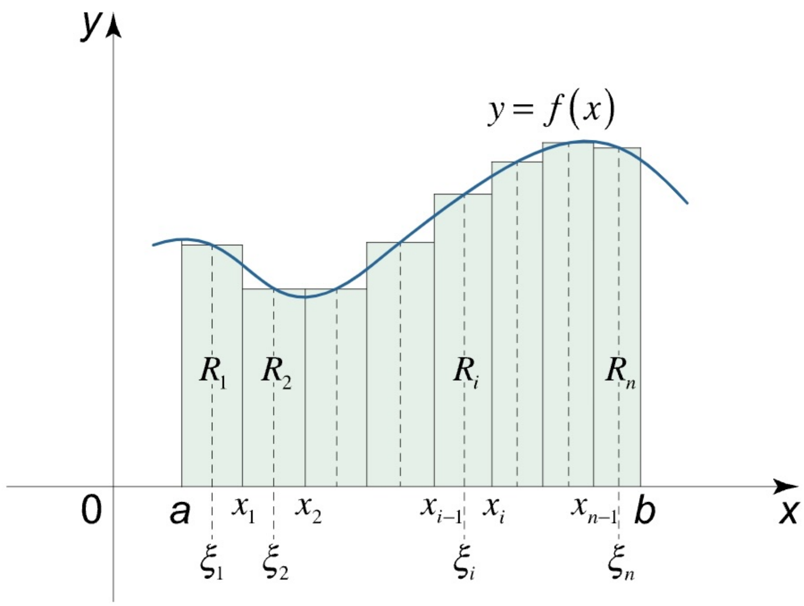

Figure 3.1: Illustration of Riemann Integration. The result of a definite integral is the area between the curve and the x-axis. Riemann integration approximates the integral as a finite sum of rectangles/trapezia, which act to represent the area beneath the curve.

Assignment 4: Calculus II

Fri, 17 Jan 19:00

3.1 Indefinite Integral

3.1.1 The Antiderivative Function

An antiderivative function, $F(x)$, is a function whose derivative is a given function $f(x)$, i.e., $F'(x) = f(x)$. If $F(x)$ is an antiderivative of $f(x)$, then $F(x) + C$, where C is a constant, is also an antiderivative.

> **Example** ($f(x) = 3x^2$). Antiderivatives include $F_1(x) = x^3$, $F_2(x) = x^3 + 4$, and $F_3(x) = x^3 - 7$.

3.1.2 Indefinite Integral

The set of all antiderivatives of a given function $f(x)$ is represented by an indefinite integral:

Definition 33 (Indefinite Integral).

$$\int f(x)\,dx = F(x) + C, \quad C \in \mathbb{R}$$

The notation used above means C belongs to the set of real numbers. The function $f(x)$ is called the integrand, and dx denotes the variable of integration, i.e. the variable you are integrating with respect to.

Corollary. Some properties of indefinite integrals include:

$$\int (f(x) \pm g(x))\,dx = \int f(x)\,dx \pm \int g(x)\,dx, \quad \int kf(x)\,dx = k\int f(x)\,dx.$$

3.2 Definite Integral

The definite integral is the limit of the Riemann sum:

Lemma 1 (Riemann sum).

$$\lim_{\Delta x \to 0} \sum f(x)\Delta x = \int_a^b f(x)\,dx.$$

Theorem 1 (Fundamental theorem of calculus). The fundamental theorem of calculus states that:

$$\int_a^b f(x)\,dx = F(b) - F(a),$$

where $F(x)$ is an antiderivative of $f(x)$.

Example (Calculating the area of curves using definite integration). To find the area between the curves $y = x^2$ and $y = x$ from $x = -1$ to $x = 1$:

$$A = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}.$$

3.3 Integration Techniques

3.3.1 Integration by Parts

The integration by parts formula is derived from the product rule of differentiation. This technique is useful for integrals of the form $u(x)v'(x)$, where $u(x)$ is often easily differentiable, and $v'(x)$ is often easy to integrate.

Definition 34 (Integration by Parts).

$$\int u\,v'\,dx = uv - \int u'\,v\,dx.$$

Example ($\int x \sin x\,dx$). To compute $\int x \sin x\,dx$, set $u = x$ and $v' = \sin x$, giving:

$$\int x \sin x\,dx = -x\cos x + \int \cos x\,dx = -x\cos x + \sin x + C.$$

3.3.2 Integration by Substitution

Integration by substitution is based on the chain rule of differentiation:

$$\frac{d}{dx} f(u(x)) = f'(u(x)) \cdot u'(x).$$

For integrals of composite functions, substitute $u(x) = g(x)$, then $du = g'(x)dx$, and the integral becomes:

Definition 35 (Substitution rule).

$$\int f(g(x))g'(x)\,dx = \int f(u)\,du.$$

Note if one is dealing with a definite integral, one us substitute each of the respective limits for u.

Example ($\int \cos x \sin^3 x\,dx$). To compute $\int \cos x \sin^3 x\,dx$, let $u = \sin x$, so $du = \cos x\,dx$, and the integral becomes:

$$\int u^3\,du = \frac{1}{4}u^4 + C \equiv \frac{1}{4}\sin^4 x + C.$$

3.4 Integrals in Geometry and Physics

Example (Volume of a sphere). The volume of a sphere of radius R can be calculated using integration.

We start by considering the equation of the sphere:

$$x^2 + y^2 + z^2 = R^2.$$

To compute the volume, we use the method of slicing the sphere up into slabs. We slice the sphere into thin slabs along the z-axis. Each disk at a given z has a radius of $r(z) = \sqrt{R^2 - z^2}$, which comes from solving the equation of the sphere for x and y.

The area of a cross-sectional slab in the x-y plane is given by:

$$A(z) = \pi r(z)^2 = \pi(R^2 - z^2).$$

The volume of a thin disk with thickness dz is:

$$dV = A(z)\,dz = \pi(R^2 - z^2)\,dz.$$

To find the total volume, we integrate this expression from the total parameter space of z: $z = -R$ to $z = R$:

$$V = \int_{-R}^{R} \pi(R^2 - z^2)\,dz.$$

$$= \pi\left[R^2 z - \frac{z^3}{3}\right]_{-R}^{R}.$$

Evaluating the integral:

$$V = \pi\left[\left(R^2 R - \frac{R^3}{3}\right) - \left(R^2(-R) - \frac{(-R)^3}{3}\right)\right].$$

In simplifying the expression, we find:

$$V = \pi\left[R^3 - \frac{R^3}{3} + R^3 + \frac{R^3}{3}\right] = \pi\left[2R^3 - \frac{R^3}{3} + \frac{R^3}{3}\right] = \pi \cdot \frac{4R^3}{3}.$$

Thus, the volume of the sphere is:

$$V = \frac{4}{3}\pi R^3.$$