Part II

Maths

Chapter 1

Trigonometry

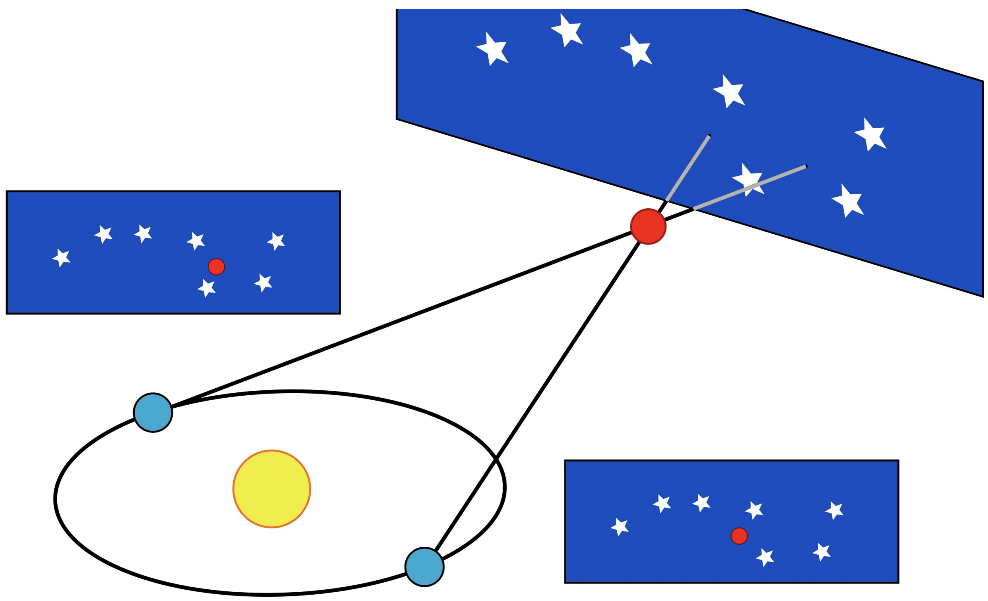


Figure 1.1: Example of a parallax. Shown in the figure is Earth in January, vs Earth in July. There is a relative difference in the apparent position of the start in January versus July, given by the Parallax Angle.

Assignment 2: Trigonometry

Fri, 15 Nov 19:00

1.1 Trigonometric Functions

The coordinates of any point on the unit circle ($r = 1$) are defined by the following trigonometric functions.

Definition 11 (Sine). The sine of an angle θ is the ratio of the length of the opposite side to the length of the hypotenuse:

$$\sin(\theta) = \frac{\text{opposite}}{\text{hypotenuse}}$$

Domain: $\theta \in \mathbb{R}$ (all real numbers)
Range: $\sin(\theta) \in [-1, 1]$

Definition 12 (Cosine)**.** The cosine of an angle θ is the ratio of the length of the adjacent side to the length of the hypotenuse:

$$\cos(\theta) = \frac{\text{adjacent}}{\text{hypotenuse}}$$

Domain: $\theta \in \mathbb{R}$
Range: $\cos(\theta) \in [-1, 1]$

Definition 13 (Tangent)**.** The tangent of an angle θ is the ratio of the length of the opposite side to the length of the adjacent side:

$$\tan(\theta) = \frac{\text{opposite}}{\text{adjacent}} \equiv \frac{\sin(\theta)}{\cos(\theta)}$$

Domain: $\theta \in \mathbb{R} \setminus \left\{ \frac{\pi}{2} + n\pi \mid n \in \mathbb{Z} \right\}$ (all real numbers except odd multiples of $\frac{\pi}{2}$)
Range: $\tan(\theta) \in \mathbb{R}$ (all real numbers)

The Unit Circle

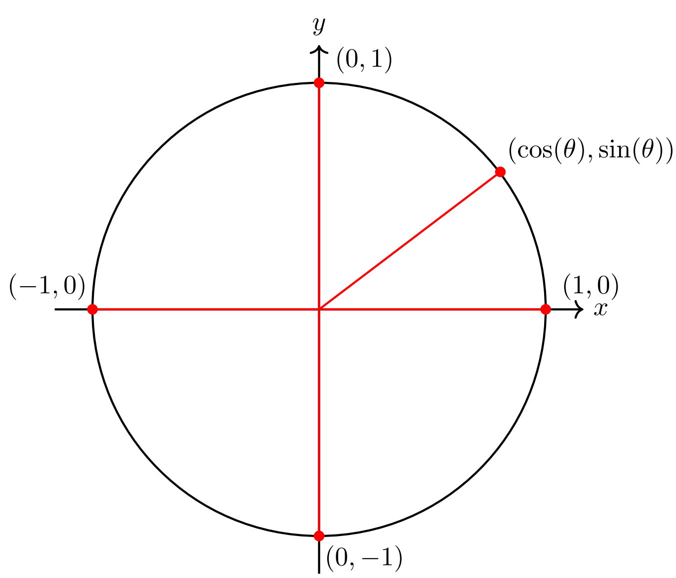


Figure 1.2: The Unit Circle, circle on the xy plane with radius, $r = 1$.

Example (Unit Circle).

- The unit circle is represented by the equation:

$$x^2 + y^2 = 1$$

where x and y are the coordinates of any point on the circle.

- Any point on the unit circle can be described by the coordinates $(\cos(\theta), \sin(\theta))$, where θ is the angle formed by the radius vector and the positive x-axis.
- As θ varies from 0 to 2π, the point on the unit circle traces out a full circle, with $\cos(\theta)$ representing the x-coordinate and $\sin(\theta)$ representing the y-coordinate of the point.
- Key points on the unit circle include:
  - At $\theta = 0°$ (or 0 radians), the point is $(1, 0)$.
  - At $\theta = 90°$ (or $\frac{\pi}{2}$ radians), the point is $(0, 1)$.
  - At $\theta = 180°$ (or π radians), the point is $(-1, 0)$.
  - At $\theta = 270°$ (or $\frac{3\pi}{2}$ radians), the point is $(0, -1)$.

1.2 Reciprocal Trigonometric Functions

The reciprocal trigonometric functions are defined as the inverses of sine, cosine, and tangent.

Definition 14 (Cosecant). The cosecant of θ is the reciprocal of the sine function:

$$\csc(\theta) := \frac{1}{\sin(\theta)}$$

Domain: $\theta \in \mathbb{R} \setminus n\pi$ (all real numbers except integer multiples of π)
Range: $\csc(\theta) \in (-\infty, -1] \cup [1, \infty)$

Definition 15 (Secant). The secant of θ is the reciprocal of the cosine function:

$$\sec(\theta) := \frac{1}{\cos(\theta)}$$

Domain: $\theta \in \mathbb{R} \setminus \frac{\pi}{2} + n\pi$ (all real numbers except odd multiples of $\frac{\pi}{2}$)
Range: $\sec(\theta) \in (-\infty, -1] \cup [1, \infty)$

Definition 16 (Cotangent). The cotangent of θ is the reciprocal of the tangent function:

$$\cot(\theta) := \frac{1}{\tan(\theta)}$$

Domain: $\theta \in \mathbb{R} \setminus n\pi$ (all real numbers except integer multiples of π)
Range: $\cot(\theta) \in \mathbb{R}$ (all real numbers)

1.3 Inverse Trigonometric Functions

Inverse trigonometric functions allow us to determine the angle when given a trigonometric ratio. Here, we discuss arcsine (arcsin) and arccosine (arccos).

Definition 17 (Arcsine (arcsin)). The arcsine function $\arcsin(x)$ is the inverse of the sine function. It returns the angle θ such that:

$$\sin(\theta) = x \Rightarrow \theta = \arcsin(x) \quad \text{and} \quad \theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right].$$

Domain: $x \in [-1, 1]$
Range: $\theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

Definition 18 (Arccosine (arccos)). The arccosine function $\arccos(x)$ is the inverse of the cosine function. It returns the angle θ such that:

$$\cos(\theta) = x \Rightarrow \theta = \arccos(x) \quad \text{and} \quad \theta \in [0, \pi].$$

Domain: $x \in [-1, 1]$
Range: $\theta \in [0, \pi]$

1.3.1 Relationship to the Unit Circle

The arcsine and arccosine functions correspond to angles on the unit circle.

- For $\arcsin(x)$, the angle θ is measured counterclockwise from the positive x-axis and is restricted to the first and fourth quadrants ($-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$).
- For $\arccos(x)$, the angle θ is measured counterclockwise from the positive x-axis and is restricted to the first and second quadrants ($0 \leq \theta \leq \pi$).

1.4 Pythagorean Identities

A fundamental identity in trigonometry.

Definition 19 (Pythagorean Identity). The Pythagorean identity states that for any angle θ:

$$\sin^2(\theta) + \cos^2(\theta) = 1$$

1.5 Sum and Difference Formulas

The sum and difference formulas allow us to compute the trigonometric functions of sums or differences of angles.

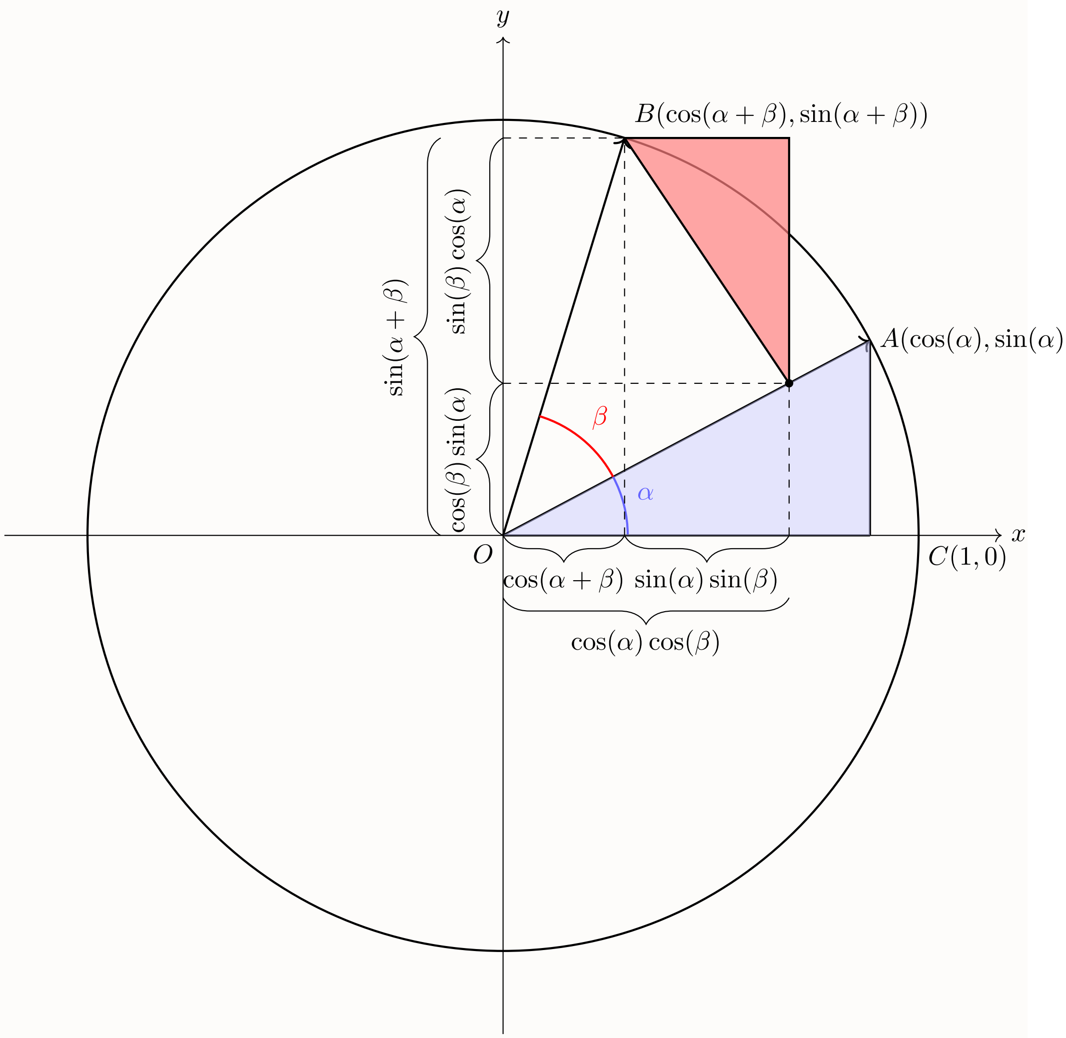
Definition 20 (Sum of Angles for Sine). The sine of the sum of two angles is given by:

$$\sin(A + B) = \sin(A)\cos(B) + \cos(A)\sin(B)$$

Definition 21 (Sum of Angles for Cosine)**.** The cosine of the sum of two angles is given by:

$$\cos(A + B) = \cos(A)\cos(B) - \sin(A)\sin(B)$$

Proof. To prove, we can construct a unit circle, and draw two angles, α and β. Note, at A, this point has an angle of α while B has an angle of $\alpha + \beta$. The key to this problem is the creation of a perpendicular bisector emanating from the point B, and hits the triangle with an angle α. What this creates is a new triangle, with angle β, which is but a triangle with angle β rotated by angle α from the origin. Performing trigonometry on all triangles discussed, we find by matching horizontal and vertical distances highlighted by the underbraces in the figure below: $\sin(\alpha + \beta) = \sin(\alpha)\cos(\beta) + \sin(\beta)\cos(\alpha)$ and $\cos(\alpha)\cos(\beta) = \cos(\alpha + \beta) + \sin(\alpha)\sin(\beta) \Rightarrow \cos(\alpha+\beta) = \cos(\alpha)\cos(\beta)-\sin(\alpha)\sin(\beta)$, completing the proof of definitions 15 and 16.



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Corollary. Noting symmetry of $\sin(-x) = -\sin(x)$ and $\cos(-x) = \cos(x)$, we find as a corollary the following generalised addition formulae:

$$\sin(A \pm B) = \sin(A)\cos(B) \pm \cos(A)\sin(B)$$

$$\cos(A + B) = \cos(A)\cos(B) \mp \sin(A)\sin(B)$$

Definition 22 (Sum of Angles for Tangent)**.** The tangent of the sum of two angles is given by:

$$\tan(A + B) = \frac{\tan(A) + \tan(B)}{1 - \tan(A)\tan(B)}$$

1.6 Trigonometric Equations and Identities

Trigonometric equations and identities are useful tools in simplifying and solving problems in maths.

Definition 23 (Double Angle Formula for Sine)**.** The sine of double an angle is given by:

$$\sin(2\theta) = 2\sin(\theta)\cos(\theta)$$

Definition 24 (Double Angle Formula for Cosine)**.** The cosine of double an angle is given by:

$$\cos(2\theta) = \cos^2(\theta) - \sin^2(\theta)$$

1.7 Applications of Trigonometry

Definition 25 (Law of Sines)**.** In any triangle, the ratio of the length of a side to the sine of its opposite angle is constant:

$$\frac{a}{\sin(A)} = \frac{b}{\sin(B)} = \frac{c}{\sin(C)}$$

Definition 26 (Law of Cosines)**.** In any triangle, the cosine of an angle is related to the lengths of the sides as:

$$c^2 = a^2 + b^2 - 2ab\cos(C)$$

Chapter 2

Calculus I – Differentiation

Figure 2.1: Gottfried Wilhelm Leibniz, Joseph-Louis Lagrange, Sir Isaac Newton, Leonhard Euler (pictured from left to right). Each introduced their notation for the differential operator acting on a function f: Leibniz – $\frac{df}{dx}$, Lagrange – f', Newton – $\dot{f}$, Euler – $D_x f$

Assignment 3: Calculus I

Fri, 13 Dec 19:00

2.1 Introducing the differential operator

Differentiation is the process of finding the instantaneous rate of change of a function $f(x)$, denoted as $f'(x) \equiv \frac{d}{dx} f(x)$. Below, we can write some foundational rules and definitions:

2.1.1 Power Rule

For the power function $f(x) = x^n$, the derivative is:

Definition 27 (Power Rule).

$$\frac{d}{dx}x^n = nx^{n-1}.$$

2.1.2 Linearity of Differentiation

Differentiation is represented by the operator $\frac{d}{dx}$. It is a linear operator, meaning:

$$\frac{d}{dx}\left[af(x) + bg(x) + ch(x) + \dots\right] = a\frac{d}{dx}f(x) + b\frac{d}{dx}g(x) + c\frac{d}{dx}h(x) + \dots$$

2.1.3 First Principles

The derivative from first principles is defined as:

Definition 28 (Differentiation from first principles).

$$f'(x) = \lim_{h\to 0}\frac{f(x+h) - f(x)}{h}.$$

2.2 Derivatives of Sine and Cosine

For trigonometric functions:

$$\frac{d}{dx}\sin x = \cos x, \quad \frac{d}{dx}\cos x = -\sin x.$$

For a constant a:

$$\frac{d}{dx}\sin(ax) = a\cos(ax), \quad \frac{d}{dx}\cos(ax) = -a\sin(ax).$$

Example (Deriving trig derivatives using differentiation from first principles). We may derive the derivatives of sine, or cosine, using differentation from first principles. Let us consider the derivative of $\sin(x)$. By definition, its derivative is:

$$\sin'(x) = \lim_{h\to 0}\frac{\sin(x+h) - \sin(x)}{h}.$$

Using the trigonometric identity $\sin(A+B) = \sin A\cos B + \cos A\sin B$:

$$\sin(x+h) = \sin x\cos h + \cos x\sin h.$$

Substituting this back into the limit:

$$\sin'(x) = \lim_{h\to 0}\frac{\sin x\cos h + \cos x\sin h - \sin x}{h}.$$

Separating terms:

$$\sin'(x) = \lim_{h\to 0}\left(\sin x\frac{\cos h - 1}{h} + \cos x\frac{\sin h}{h}\right).$$

Using the known limits:

$$\lim_{h\to 0} \frac{\sin h}{h} = 1, \quad \lim_{h\to 0} \frac{cosh - 1}{h} = 0,$$

we find:

$$\sin'(x) = \cos x.$$

2.3 Product Rule

The derivative of the product of two functions is:

Definition 29 (Product Rule).

$$\frac{d}{dx}\big[f(x)g(x)\big] = f'(x)g(x) + f(x)g'(x).$$

Proof. To prove the product rule from first principles:

$$\frac{d}{dx}\big[f(x)g(x)\big] = \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h}.$$

This may be written as:

$$\frac{d}{dx}\big[f(x)g(x)\big] = \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x)g(x+h) + f(x)g(x+h) - f(x)g(x)}{h}.$$

Separating terms:

$$\frac{d}{dx}\big[f(x)g(x)\big] = \lim_{h\to 0} \frac{g(x+h)[f(x+h) - f(x)]}{h} + \lim_{h\to 0} \frac{f(x)[g(x+h) - g(x)]}{h}.$$

Taking limits:

$$\frac{d}{dx}\big[f(x)g(x)\big] = g(x)f'(x) + f(x)g'(x).$$

□

Example.

$$(x^2 \sin(x))' = (x^2)' \sin(x) + (x^2)(\sin(x))' = 2x \sin(x) + x^2 \cos(x)$$

2.4 Chain Rule

The chain rule helps differentiate composite functions $f(g(x))$:

Definition 30 (Chain Rule).

$$\frac{d}{dx} f(g(x)) = \frac{df}{dg} \cdot \frac{dg}{dx} = f'_g(g) \cdot g'_x(x).$$

Example.

$$\frac{d}{dx}\sin(x^2) = \frac{d}{d(x^2)}\sin(x^2) \cdot \frac{d}{dx}x^2 = 2x\cos(x^2)$$

2.5 Quotient Rule

The derivative of the quotient of two functions is:

Definition 31 (Quotient Rule)**.**

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{f'(x)g(x) - f(x)g'(x)}{\big[g(x)\big]^2}.$$

Example.

$$\frac{d}{dx}\left(\frac{\sin x}{x^2}\right) = \frac{(\cos x)x^2 - 2(\sin x)x}{x^4} = \frac{x(\cos x) - 2\sin x}{x^3}$$

2.6 Additional Trigonometric Derivatives

For the tangent and secant functions:

$$\frac{d}{dx}\tan x = \sec^2 x, \quad \frac{d}{dx}\sec x = \sec x \tan x.$$

2.7 Maximising/minimising a function

The maxima/minima, i.e. extrema of a function can be found by setting the derivative of the function to 0, granted the function is both differentiable (smooth) and continuous (contains no singularities).

Definition 32 (Extrema of a function)**.** The extrema of a function exist at $f'(x) = 0$, granted the function is both differentiable and continuous.

Example (Snell's Law)**.** Consider a ray of light travelling through 2 different media, each with a refractive index n_1 and n_2. The total travel time is given by:

$$T = \frac{\sqrt{x^2 + h_1^2}}{v_1} + \frac{\sqrt{(d-x)^2 + h_2^2}}{v_2},$$

where $v_1 = \frac{c}{n_1}$ and $v_2 = \frac{c}{n_2}$ are the speeds of light in the two media. Substituting these speeds:

$$T = n_1\sqrt{x^2 + h_1^2} + n_2\sqrt{(d-x)^2 + h_2^2}.$$

To minimize T, differentiate with respect to x:

$$\frac{dT}{dx} = \frac{n_1 x}{\sqrt{x^2 + h_1^2}} - \frac{n_2(d-x)}{\sqrt{(d-x)^2 + h_2^2}}.$$

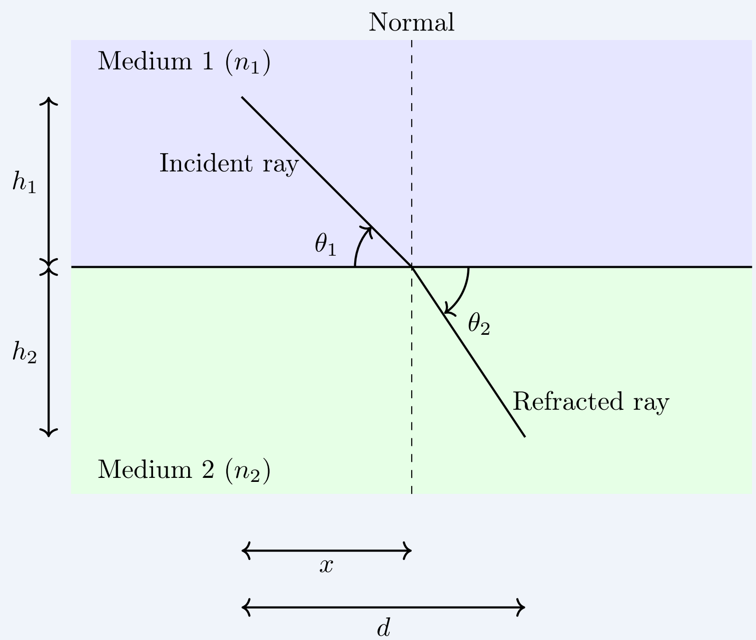
Setting $\frac{dT}{dx} = 0$:

$$\frac{n_1 x}{\sqrt{x^2 + h_1^2}} = \frac{n_2(d - x)}{\sqrt{(d - x)^2 + h_2^2}}.$$

Rearranging, we find:

$$n_1 \sin\theta_1 = n_2 \sin\theta_2,$$

which is Snell's law. The ray in which the line would trace as a result is shown below.



Chapter 3

Calculus II – Integration

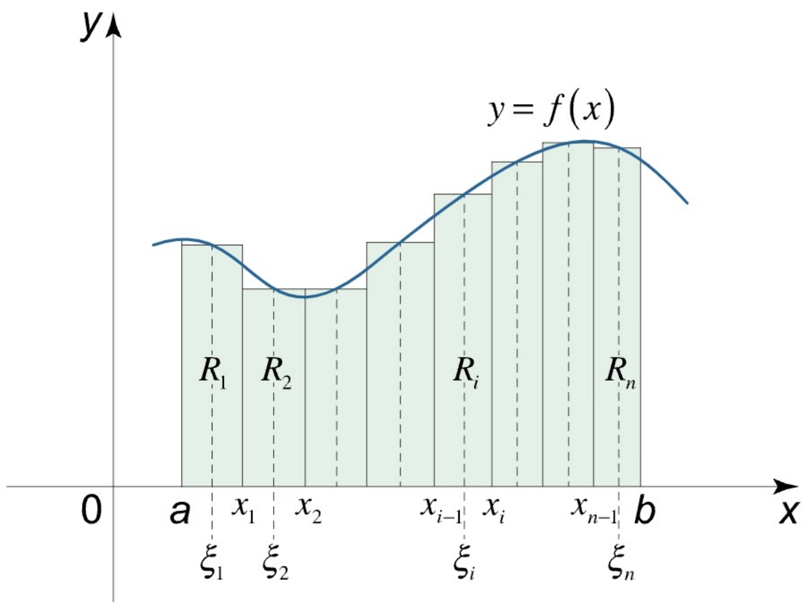


Figure 3.1: Illustration of Riemann Integration. The result of a definite integral is the area between the curve and the x-axis. Riemann integration approximates the integral as a finite sum of rectangles/trapezia, which act to represent the area beneath the curve.

Assignment 4: Calculus II

Fri, 17 Jan 19:00

3.1 Indefinite Integral

3.1.1 The Antiderivative Function

An antiderivative function, $F(x)$, is a function whose derivative is a given function $f(x)$, i.e., $F'(x) = f(x)$. If $F(x)$ is an antiderivative of $f(x)$, then $F(x) + C$, where C is a constant, is also an antiderivative.

> **Example ($f(x) = 3x^2$).** Antiderivatives include $F_1(x) = x^3$, $F_2(x) = x^3 + 4$, and $F_3(x) = x^3 - 7$.

3.1.2 Indefinite Integral

The set of all antiderivatives of a given function $f(x)$ is represented by an indefinite integral:

Definition 33 (Indefinite Integral).

$$\int f(x)\,dx = F(x) + C, \quad C \in \mathbb{R}$$

The notation used above means C belongs to the set of real numbers. The function $f(x)$ is called the integrand, and dx denotes the variable of integration, i.e. the variable you are integrating with respect to.

Corollary. Some properties of indefinite integrals include:

$$\int (f(x) \pm g(x))\,dx = \int f(x)\,dx \pm \int g(x)\,dx, \qquad \int kf(x)\,dx = k\int f(x)\,dx.$$

3.2 Definite Integral

The definite integral is the limit of the Riemann sum:

Lemma 1 (Riemann sum).

$$\lim_{\Delta x \to 0} \sum f(x)\Delta x = \int_a^b f(x)\,dx.$$

Theorem 1 (Fundamental theorem of calculus). The fundamental theorem of calculus states that:

$$\int_a^b f(x)\,dx = F(b) - F(a),$$

where $F(x)$ is an antiderivative of $f(x)$.

Example (Calculating the area of curves using definite integration). To find the area between the curves $y = x^2$ and $y = x$ from $x = -1$ to $x = 1$:

$$A = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}.$$

3.3 Integration Techniques

3.3.1 Integration by Parts

The integration by parts formula is derived from the product rule of differentiation. This technique is useful for integrals of the form $u(x)v'(x)$, where $u(x)$ is often easily differentiable, and $v'(x)$ is often easy to integrate.

Definition 34 (Integration by Parts)**.**

$$\int u\, v'\, dx = uv - \int u'\, v\, dx.$$

Example ($\int x \sin x\, dx$)**.** To compute $\int x \sin x\, dx$, set $u = x$ and $v' = \sin x$, giving:

$$\int x \sin x\, dx = -x \cos x + \int \cos x\, dx = -x \cos x + \sin x + C.$$

3.3.2 Integration by Substitution

Integration by substitution is based on the chain rule of differentiation:

$$\frac{d}{dx} f(u(x)) = f'(u(x)) \cdot u'(x).$$

For integrals of composite functions, substitute $u(x) = g(x)$, then $du = g'(x)dx$, and the integral becomes:

Definition 35 (Substitution rule)**.**

$$\int f(g(x))g'(x)\, dx = \int f(u)\, du.$$

Note if one is dealing with a definite integral, one us substitute each of the respective limits for u.

Example ($\int \cos x \sin^3 x\, dx$)**.** To compute $\int \cos x \sin^3 x\, dx$, let $u = \sin x$, so $du = \cos x\, dx$, and the integral becomes:

$$\int u^3\, du = \frac{1}{4}u^4 + C \equiv \frac{1}{4}\sin^4 x + C.$$

3.4 Integrals in Geometry and Physics

Example (Volume of a sphere)**.** The volume of a sphere of radius R can be calculated using integration.

We start by considering the equation of the sphere:

$$x^2 + y^2 + z^2 = R^2.$$

To compute the volume, we use the method of slicing the sphere up into slabs. We slice the sphere into thin slabs along the z-axis. Each disk at a given z has a radius of $r(z) = \sqrt{R^2 - z^2}$, which comes from solving the equation of the sphere for x and y.

The area of a cross-sectional slab in the x-y plane is given by:

$$A(z) = \pi r(z)^2 = \pi(R^2 - z^2).$$

The volume of a thin disk with thickness dz is:

$$dV = A(z)\,dz = \pi(R^2 - z^2)\,dz.$$

To find the total volume, we integrate this expression from the total parameter space of z: $z = -R$ to $z = R$:

$$V = \int_{-R}^{R} \pi(R^2 - z^2)\,dz.$$
$$= \pi\left[R^2 z - \frac{z^3}{3}\right]_{-R}^{R}.$$

Evaluating the integral:

$$V = \pi\left[\left(R^2R - \frac{R^3}{3}\right) - \left(R^2(-R) - \frac{(-R)^3}{3}\right)\right].$$

In simplifying the expression, we find:

$$V = \pi\left[R^3 - \frac{R^3}{3} + R^3 + \frac{R^3}{3}\right] = \pi\left[2R^3 - \frac{R^3}{3} + \frac{R^3}{3}\right] = \pi \cdot \frac{4R^3}{3}.$$

Thus, the volume of the sphere is:

$$V = \frac{4}{3}\pi R^3.$$