MATH001PB

Proof-Based Single Variable Calculus

Module 1: Function

Subsection 1.1: Introduction to Functions

Def^{n} 1.1.1 Function

Let A and B be sets.

A function f from A to B, denoted $f: A \to B$, satisfying the following condition:

- $f \subseteq A \times B$
- $\forall x \in A, \exists_{=1} y \in B$, such that $(x, y) \in f$

Terminologies:

- The set A is called the domain of f.
- The set B is called the codomain of f.
- For each $x \in A$, the unique element $b \in B$ such that $(x, b) \in f$ is denoted by f(x).
  It is called the value of f at x or the image of x under f.
- The set of all images $\{f(x) \in B : x \in A\}$ is called the range or image of f, where $Range(f) \subseteq B$

Def^{n} 1.1.2 Graph of a function

The graph of a function $f: A \to B$ is defined as:

$\text{Graph}(f) = \{\,(a, b) \in A \times B \mid b = f(a)\,\}$

Proposition 1.1.3 Vertical Line Test

A set $G \subset A \times B$ is the graph of a function $f: A \to B$ iff

(1) $\forall a \in A, \exists b \in B, (a, b) \in G$

(2) $\forall a \in A, \forall b_1, b_2 \in B, \big((a, b_1) \in G \wedge (a, b_2) \in G\big) \Longrightarrow \mathrm{b}_1 = \mathrm{b}_2$

Proof.

($\Longrightarrow$) Assume $G \subseteq A \times B$ is a graph of a function $f: A \to B$.

By Def[n] 1.1.1, $\forall a \in A, \exists_{=1} b \in B, (a, b) \in G.$ (3)

Let $a \in A, b_1, b_2 \in B$.

Suppose $(a, b_1) \in G$ and $(a, b_2) \in G$.

Then by uniqueness of b in (3), $b_1 = f(a) = b_2$.

($\Longleftarrow$) Assume (1) and (2).

By (1), $\forall a \in A, \exists b \in B, (a, b) \in G$

Then by (2), b is unique for each a.

Define $f: A \to B$ by $f(a) =$ the unique $b \in B$ such that $(a, b) \in G$.

Then $G = \{(a, f(a)) \mid a \in A\}$.

G is the graph of f. ■

Subsection 1.2 Combination and Composition of Functions

Def[n] 1.2.1 Let f and g be functions with domain A and B respectively.

We define new functions as follows with domains $A \cap B$ as follows.

- $(f+g)(x) = f(x) + g(x)$
- $(f-g)(x) = f(x) - g(x)$
- $(fg)(x) = f(x) \cdot g(x)$
- $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$, with domain $\{x \in A \cap B : g(x) \neq 0\}$

Defn 1.2.2 Composition of Functions

Let $f: B \to C$ and $g: A \to B$ be functions. The composition $f \circ g$ is the function $(f \circ g): A \to C$ defined by:

$(f \circ g)(x) = f\big(g(x)\big)$ for all $x \in A$

The domain of $f \circ g$ is the set of all x in the domain of g such that $g(x)$ is in the domain of f.

Proposition 1.2.3 Non-commutativity of Composition

$\exists x \in \mathbb{R}$ such that $f \circ g(x) \neq g \circ f(x)$.

Proof.

We shall prove by counterexample.

Define $f: \mathbb{R} \to \mathbb{R}$ by $f(x) = x^2$

Define $g: \mathbb{R} \to \mathbb{R}$ by $g(x) = x + 1$

By def$^{\text{n}}$ 1.2.2, $(f \circ g)(x) = x^2 + 2x + 1$ and $(g \circ f)(x) = x^2 + 1$.

By observation, $f \circ g \neq g \circ f$. ■

Subsection 1.3: Inverse Functions

Def$^{\text{n}}$ 1.3.1 Injectivity

A function is injective (or one-to-one) if

$$\forall a_1, a_2 \in A, f(a_1) = f(a_2) \Longrightarrow a_1 = a_2$$

Proposition 1.3.2 Horizontal Line Test

Let $f: A \to B$ be a function. Let G be the graph of the function f.

Then f is injective iff

(1) $\forall b \in B, \forall a_1, a_2 \in A, \big((a_1, b) \in G \wedge (a_2, b) \in G\big) \Longrightarrow a_1 = a_2$

Proof. By definition, $(a, b) \in G \Longrightarrow f(a) = b$.

The logical form of Proposition 1.3.2 is exactly the same as Def^{n} 1.3.1. ■

Def^{n} 1.3.3 Surjectivity

A function $f: A \to B$ is surjective (or onto) if

$\forall b \in B, \exists a \in A$ such that $f(a) = b$.

Def^{n} 1.3.4 Bijection

A function that is both injective and surjective is bijective.

Def^{n} 1.3.5 Inverse Function

Let $f: A \to B$ be an bijective function.

The inverse function f^{-1}:B\to A is defined by:

$$f^{-1} = \{ (b, a) \in B \times A \mid (a, b) \in f \}$$

Equivalently,

$\forall b \in B$, $f^{-1}(b) = a \Leftrightarrow f(a) = b$.

Theorem 1.3.6 Properties of the Inverse Function.

Let $f: A \to B$ be a bijective function and let $f^{-1}: B \to A$ be its inverse.

(1) $Domain(f^{-1}) = Range(f)$
(2) $Range(f^{-1}) = Domain(f)$
(3) $\forall a \in A, (f^{-1} \circ f)(a) = a$
(4) $\forall b \in B, (f \circ f^{-1})(b) = b$

Proof. By def^{n} 1.3.5.

Proposition 1.3.7 Existence of Inverse Function

Let $f: A \to B$, $\exists g: B \to A$ such that $\forall a \in A, (g \circ f)(a) = a \wedge \forall b \in B, (f \circ g)(b) = b$ $\Leftrightarrow$ f is bijective.

Moreover, $\exists_{=1} g = f^{-1}$ from Def^{n} 1.3.5.

Proof.

($\Longrightarrow$) Assume $\exists \mathrm{g}: B \to A$.

Let $f(a_1) = f(a_2)$. Then $\mathrm{g}(f(a_1)) = \mathrm{g}(f(a_2)) \Longrightarrow a_1 = a_2$.

Thus, f is injective. (1)

Let $b \in B$. Then $a = \mathrm{g}(b) \in A$ satisfies $f(a) = f(\mathrm{g}(b)) = b$.

Thus, f is surjective. (2)

Thus, f is bijective.

($\Longleftarrow$) Assume f is bijective. Define $f^{-1}: B \to A$ by:

$\mathrm{g}(b) =$ the unique $a \in A$ such that $f(a) = b$

This function is well-defined because.

(1) Surjectivity ensures such a exists for each $b \in B$.

(2) Injectivity ensures that a is unique.

By construction, $\forall a \in A, (g \circ f)(a) = a \wedge \forall b \in B, (f \circ g)(b) = b$.

(Uniqueness) Suppose $g_1, g_2: B \to A$ both satisfy $\forall a \in A, \forall b \in B, g_1(f(a)) = a \wedge f(g_1(b)) = b$ and similarly for g_2.

Then $\forall b \in B, g_1(b) = g_1\left(f(g_2(b))\right) = (g_1 \circ f)(g_2(b)) = g_2(b)$.

Thus, $g_1 = g_2$.

By definition, $g = f^{-1}$.

(Equivalence with Def^{n} 1.3.5) Since g is unique and f^{-1} from Def^{n} 1.3.5 satisfies the same properties (by Theorem 1.3.6), $g = f^{-1}$.

Corollary 1.3.8 Method for Finding the Inverse

Let $f: A \to B$ be a bijective function. The inverse function f^{-1} can be found by:

1. Let $y = f(x)$.
2. Solve for x in terms of y and obtain $x = g(y)$.
3. Then $\forall y \in B, f^{-1}(y) = g(y)$.

Subsection 1.4 Properties of Functions

Def$^{\text{n}}$ 1.4.1 Boundedness.

(1) A function f is bounded above if

$\exists M \in \mathbb{R}$ such that $\forall x \in Domain(f), f(x) \leq M$.

M is called an upper bound for f.

(2) A function f is bounded below if

$\exists \mathrm{m} \in \mathbb{R}$ such that $\forall x \in Domain(f), f(x) \geq \mathrm{m}$.

m is called a lower bound for f.

(3) A function f is bounded if it is both bounded above and bounded below.
Equivalently, $\exists K > 0$ such that $\forall x \in A, |f(x)| \leq K$.

Def$^{\text{n}}$ 1.4.2 Even and Odd Functions

Let f be a function where $\forall x \in Domain(f), -x \in Domain(f)$.

f is an even function if $\forall x \in Domain(f), f(-x) = f(x)$.

f is an odd function if $\forall x \in Domain(f), f(-x) = -f(x)$.

Def$^{\text{n}}$ 1.4.3 Periodicity

A function f is periodic if

$\exists p \in \mathbb{R}^{+}$ such that $\forall x \in Domain(f), f(x+p) = f(x)$.

Def^{n} 1.4.4 Monotonicity

Let f be a function and let I be an interval contained in the domain of f.

(1) f is monotonically increasing on I if $\forall x_1, x_2 \in I, x_1 < x_2 \Rightarrow \left(f(x_1) \leq f(x_2)\right)$.
(2) f is strictly increasing on I if $\forall x_1, x_2 \in I, x_1 < x_2 \Rightarrow \left(f(x_1) < f(x_2)\right)$.
(3) f is monotonically decreasing on I if $\forall x_1, x_2 \in I, x_1 < x_2 \Rightarrow \left(f(x_1) \ \geq f(x_2)\right)$.
(4) f is strictly decreasing on I if $\forall x_1, x_2 \in I, x_1 < x_2 \Rightarrow \left(f(x_1) > f(x_2)\right)$.

Proposition 1.4.5 Strictly Monotonic Functions are Injective.

If a function f is strictly monotonic on an interval I, then f is injective on I.

Proof. WLOG, Assume that f is strictly increasing on I.

Let $a_1, a_2 \in I$ and suppose $f(a_1) = f(a_2)$.

WLOG, if $a_1 < a_2$, then by definition $f(a_1) < f(a_2)$, which is a contradiction.

Thus, $a_1 = a_2$. f is injective on I.

Definitions.

- Bijective
- Bounded
- Bounded above
- Bounded below
- Codomain
- Composition of Functions
- Domain
- Even Function
- Function
- Graph of a Function
- Horizontal Line Test
- Image (of an element/function)
- Injective Function
- Inverse Function
- Lower Bound
- Monotonic Decreasing Function
- Monotonic Function
- Monotonic Increasing Function
- Odd Function
- One-to-one Function (Injective)
- Onto Function (Surjective)
- Range
- Strictly Decreasing Function
- Strictly Increasing Function
- Strictly Monotonic Function
- Surjective Function
- Upper Bound
- Vertical Line Test