# Heap Sort

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# Heapsort

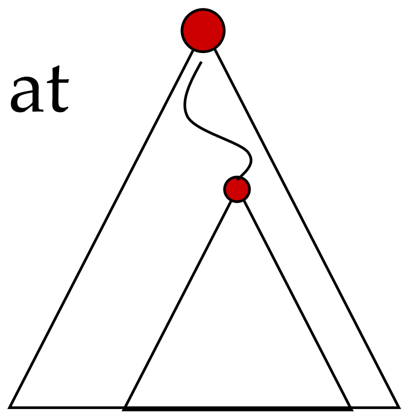
- Combines the better attributes of merge sort and insertion sort.
  - Like merge sort, but unlike insertion sort, running time is $O(n \lg n)$.
  - Like insertion sort, but unlike merge sort, sorts in place.
- Introduces an algorithm design technique
  - Create data structure (*heap*) to manage information during the execution of an algorithm.
- The *heap* has other applications beside sorting.
  - Priority Queues

# Data Structure Binary Heap

- Array viewed as a nearly complete binary tree.
  - Physically – linear array.
  - Logically – binary tree, filled on all levels (except lowest.)
- Map from array elements to tree nodes and vice versa
  - Root – $A[1]$
  - Left[$i$] – $A[2i]$
  - Right[$i$] – $A[2i+1]$
  - Parent[$i$] – $A[\lfloor i/2 \rfloor]$
- length[$A$] – number of elements in array $A$.
- heap-size[$A$] – number of elements in heap stored in $A$.
  - heap-size[$A$] $\leq$ length[$A$]

# Heap Property (Max and Min)

- Max-Heap
  - For every node excluding the root, value is at most that of its parent: $A[parent[i]] \geq A[i]$
- Largest element is stored at the root.
- In any subtree, no values are larger than the value stored at subtree root.



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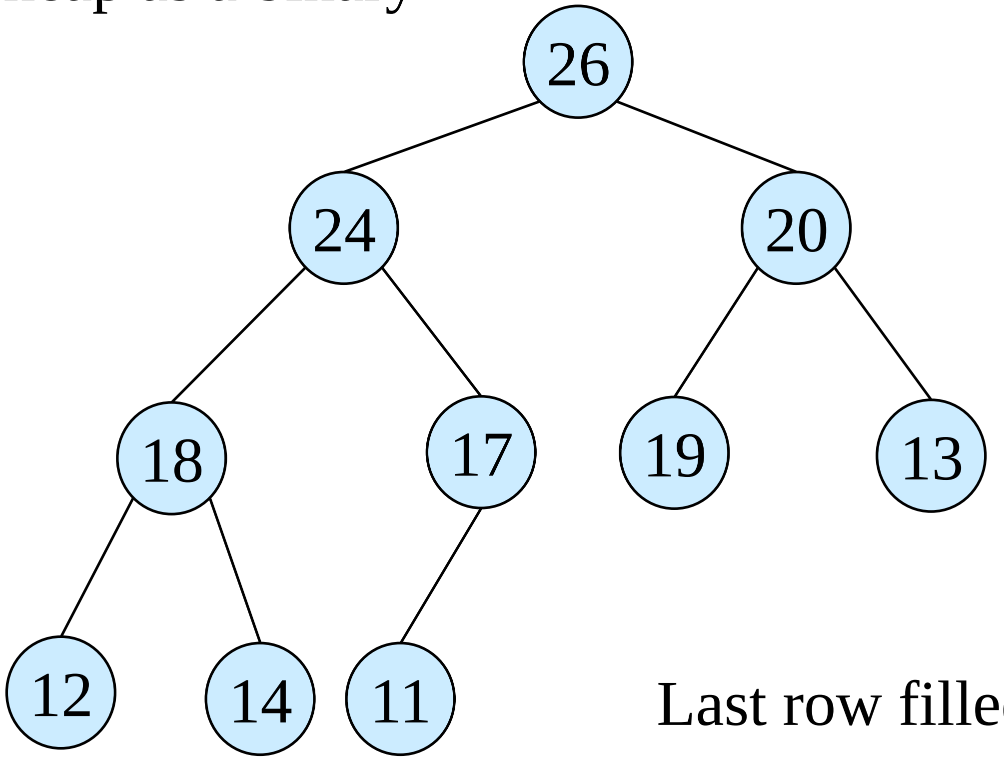
| 26 | 24 | 20 | 18 | 17 | 19 | 13 | 12 | 14 | 11 |
|---|---|---|---|---|---|---|---|---|---|
| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |

Max-heap as an array.

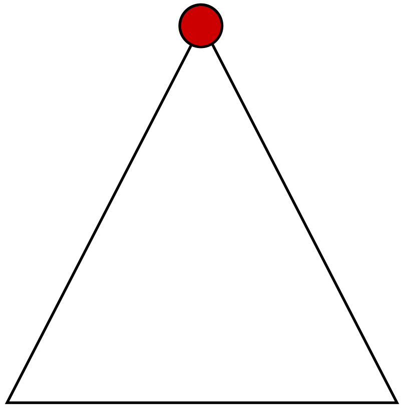
Max-heap as a binary tree.




Last row filled from left to right.

# Height

- *Height of a node in a tree*: the number of edges on the longest simple downward path from the node to a leaf.
- *Height of a tree*: the height of the root.
- Height of a heap: $\lfloor \lg n \rfloor$
  - Basic operations on a heap run in $O(\lg n)$ time

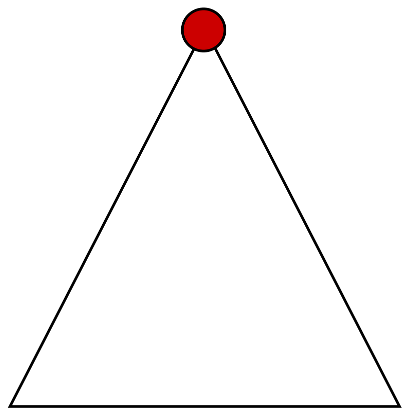


# Heaps in Sorting

- Use max-heaps for sorting.
- The array representation of max-heap is not sorted.
- Steps in sorting
  - Convert the given array of size $n$ to a max-heap (*BuildMaxHeap*)
  - Swap the first and last elements of the array.
    - Now, the largest element is in the last position – where it belongs.
    - That leaves $n - 1$ elements to be placed in their appropriate locations.
    - However, the array of first $n - 1$ elements is no longer a max-heap.
    - Float the element at the root down one of its subtrees so that the array remains a max-heap (MaxHeapify)
    - Repeat step 2 until the array is sorted.

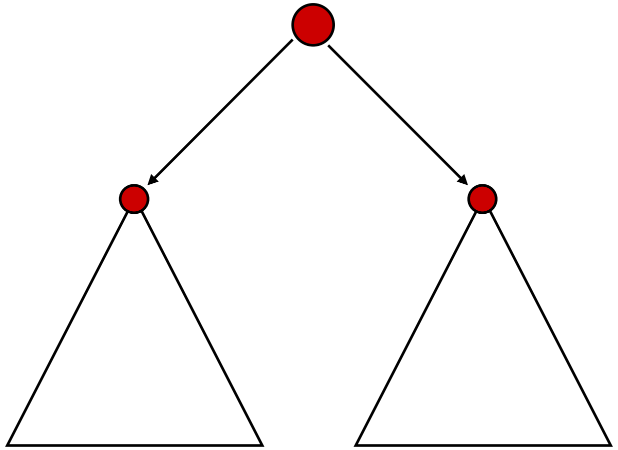
# Heap Characteristics

- *Height* $= \lfloor \lg n \rfloor$
- No. of *leaves* $= \lceil n/2 \rceil$
- No. of nodes of height $h$ $\leq \lceil n/2^{h+1} \rceil$



# Maintaining the heap property

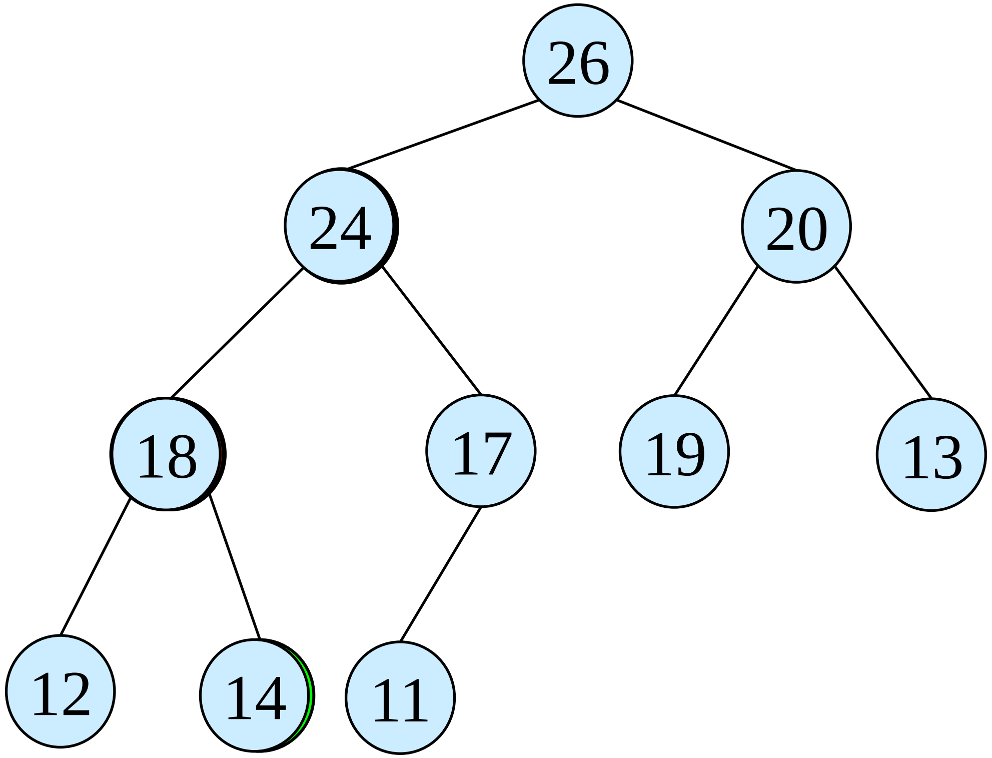
- Suppose two subtrees are max-heaps, but the root violates the max-heap property.



- Fix the offending node by exchanging the value at the node with the larger of the values at its children.
  - May lead to the subtree at the child not being a heap.
- Recursively fix the children until all of them satisfy the max-heap property.

# MaxHeapify – Example

- MaxHeapify($A$, 2)



# Procedure MaxHeapify

<u>*MaxHeapify*(*A*, *i*)</u>

$l \leftarrow$ left($i$)
$r \leftarrow$ right($i$)
**if** $l \leq$ *heap-size*[$A$] and $A[l] > A[i]$
    **then** $largest \leftarrow l$
    **else** $largest \leftarrow i$
**if** $r \leq$ *heap-size*[$A$] **and** $A[r] > A[largest]$
    **then** $largest \leftarrow r$
**if** $largest \neq i$
    **then** exchange $A[i] \leftrightarrow A[largest]$
        *MaxHeapify*($A$, $largest$)

<u>Assumption:</u>

Left($i$) and Right($i$) are max-heaps.

# Running Time for MaxHeapify

<u>*MaxHeapify(A, i)*</u>

1. $l \leftarrow$ left($i$)
2. $r \leftarrow$ right($i$)
3. **if** $l \leq$ *heap-size*[$A$] and $A[l] > A[i]$
4.     **then** *largest* $\leftarrow l$
5.     **else** *largest* $\leftarrow i$
6. **if** $r \leq$ *heap-size*[$A$] **and** $A[r] > A[largest]$
7.     **then** *largest* $\leftarrow r$
8. **if** *largest* $\neq i$
9.     **then** exchange $A[i] \leftrightarrow A[largest]$
10.         *MaxHeapify*($A$, *largest*)

Time to fix node $i$ and its children = $\boldsymbol{\Theta(1)}$

PLUS

Time to fix the subtree rooted at one of $i$'s children = $T$(size of subree at *largest*)

# Running Time for MaxHeapify(A, n)

- $T(n) = T(largest) + \Theta(1)$
- $largest \leq 2n/3$ (worst case occurs when the last row of tree is exactly half full)
- $T(n) \leq T(2n/3) + \Theta(1) \Rightarrow T(n) = O(\lg n)$
- Alternately, MaxHeapify takes $O(h)$ where $h$ is the height of the node where MaxHeapify is applied

# Building a heap

- Use *MaxHeapify* to convert an array $A$ into a max-heap.
- How?
- Call MaxHeapify on each element in a bottom-up manner.

*BuildMaxHeap(A)*

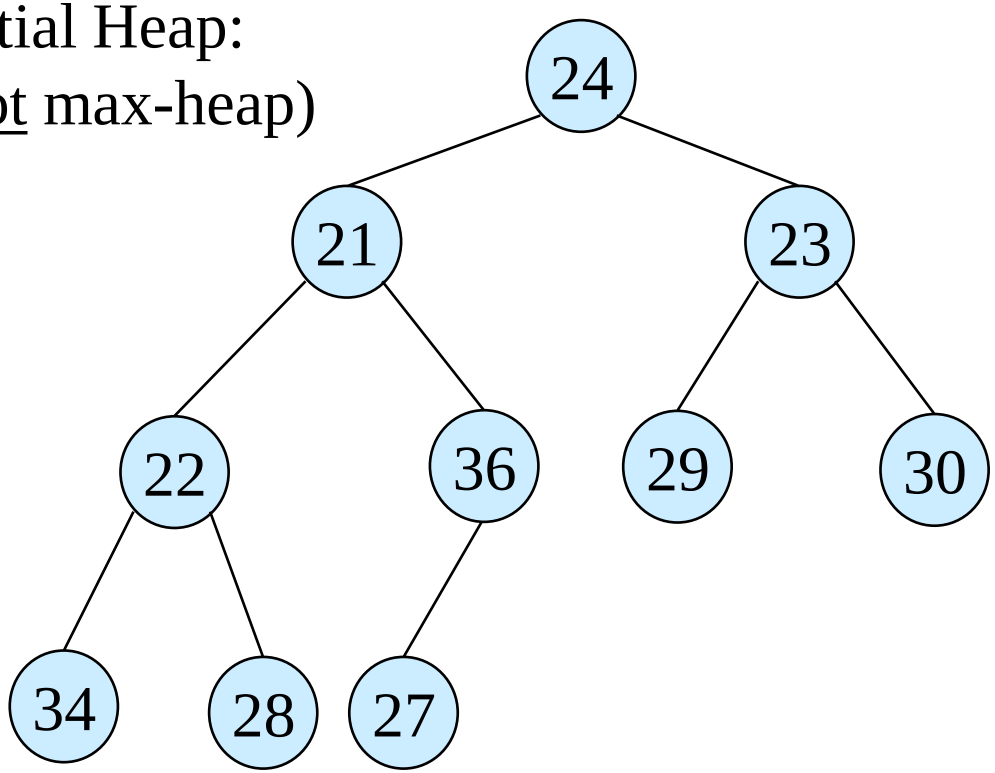
1. $\textit{heap-size}[A] \leftarrow \textit{length}[A]$
2. **for** $i \leftarrow \lfloor \textit{length}[A]/2 \rfloor$ **downto** 1
3. **do** $\textit{MaxHeapify}(A, i)$

# BuildMaxHeap – Example

- Input Array

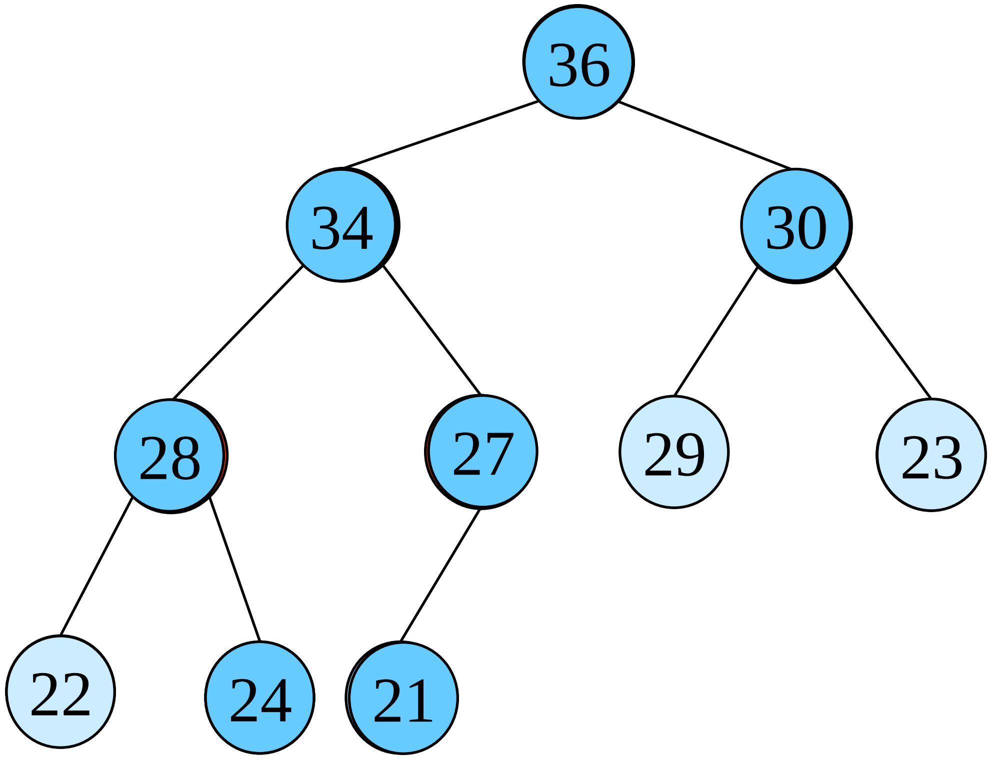
| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 24 | 21 | 23 | 22 | 36 | 29 | 30 | 34 | 28 | 27 |

Initial Heap:
(not max-heap)



# BuildMaxHeap – Example

- MaxHeapify($\lfloor 10/2 \rfloor = 5$)
- MaxHeapify(4)
- MaxHeapify(3)
- MaxHeapify(2)
- MaxHeapify(1)



# Correctness of BuildMaxHeap

- Loop Invariant: At the start of each iteration of the **for** loop, each node $i+1$, $i+2$, …, $n$ is the root of a max-heap.
- Initialization:
  - Before first iteration $i = \lfloor n/2 \rfloor$
  - Nodes $\lfloor n/2 \rfloor+1$, $\lfloor n/2 \rfloor+2$, …, $n$ are leaves and hence roots of max-heaps.
- Maintenance:
  - By LI, subtrees at children of node $i$ are max heaps.
  - Hence, MaxHeapify($i$) renders node $i$ a max heap root (while preserving the max heap root property of higher-numbered nodes).
  - Decrementing $i$ reestablishes the loop invariant for the next iteration.

# Running Time of BuildMaxHeap

- Loose upper bound:
  - Cost of a *MaxHeapify* call × No. of calls to *MaxHeapify*
  - $O(\lg n) \times O(n) = O(n \lg n)$
- Tighter bound:
  - Cost of a call to *MaxHeapify* at a node depends on the height, $h$, of the node – $O(h)$.
  - Height of most nodes smaller than $n$.
  - Height of nodes $h$ ranges from 0 to $\lfloor \lg n \rfloor$.
  - No. of nodes of height $h$ is $\lceil n/2^{h+1} \rceil$

# Running Time of BuildMaxHeap

- Tighter Bound for $T(BuildMaxHeap)$

  $T(BuildMaxHeap)$

$$\sum_{h=0}^{\lfloor \lg n \rfloor} \left\lceil \frac{n}{2^{h+1}} \right\rceil O(h) = O\left( n \sum_{h=0}^{\lfloor \lg n \rfloor} \frac{h}{2^h} \right)$$

$$\sum_{h=0}^{\lfloor \lg n \rfloor} \frac{h}{2^h} \le \sum_{h=0}^{\infty} \frac{h}{2^h} \qquad , x = 1/2 \text{ in (A.8)}$$

$$= \frac{1/2}{(1-1/2)^2} = 2$$

$$O\left( n \sum_{h=0}^{\lfloor \lg n \rfloor} \frac{h}{2^h} \right) = O\left( n \sum_{h=0}^{\infty} \frac{h}{2^h} \right) = O(n)$$

- Can build a heap from an unordered array in linear time

# Heapsort

- Sort by maintaining the as yet unsorted elements as a max-heap.
- Start by building a max-heap on all elements in $A$.
  - Maximum element is in the root, $A[1]$.
- Move the maximum element to its correct final position.
  - Exchange $A[1]$ with $A[n]$.
- Discard $A[n]$ – it is now sorted.
  - Decrement heap-size[$A$].
- Restore the max-heap property on $A[1..n–1]$.
  - Call *MaxHeapify*($A$, 1).
- Repeat until heap-size[$A$] is reduced to 2.

# Heapsort(A)

<u>*HeapSort*(*A*)</u>

1. Build-Max-Heap($A$)
2. **for** $i \leftarrow length[A]$ **downto** 2
3. **do** exchange $A[1] \leftrightarrow A[i]$
4. $heap\text{-}size[A] \leftarrow heap\text{-}size[A] - 1$
5. $MaxHeapify(A, 1)$

# Heapsort – Example

| | | | | | | | | | |
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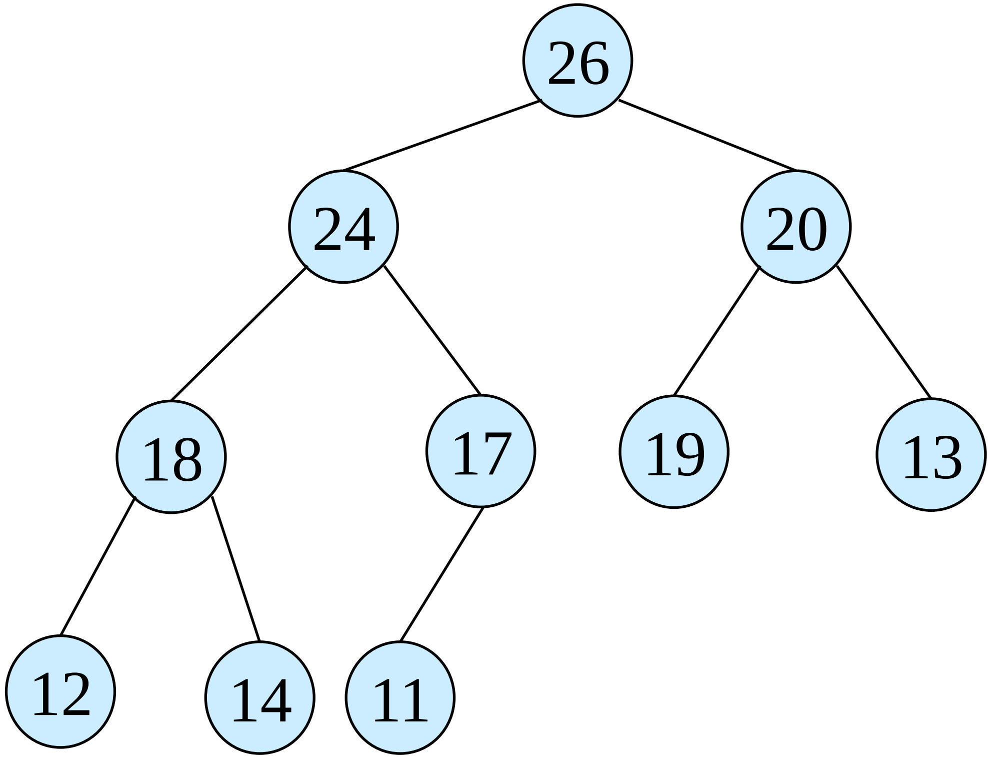

# Algorithm Analysis

*HeapSort*($A$)

1. Build-Max-Heap($A$)
2. **for** $i \leftarrow length[A]$ **downto** 2
3. **do** exchange $A[1] \leftrightarrow A[i]$
4. $heap\text{-}size[A] \leftarrow heap\text{-}size[A] - 1$
5. $MaxHeapify(A, 1)$

- In-place
- Not Stable
- Build-Max-Heap takes $O(n)$ and each of the $n$-$1$ calls to Max-Heapify takes time $O(\lg n)$.
- Therefore, $T(n) = O(n \lg n)$

# Heap Procedures for Sorting

- MaxHeapify $O(\lg n)$
- BuildMaxHeap $O(n)$
- HeapSort $O(n \lg n)$

# Priority Queue

- Popular & important application of heaps.
- Max and min priority queues.
- Maintains a *dynamic* set *S* of elements.
- Each set element has a *key* – an associated value.
- Goal is to support insertion and extraction efficiently.
- Applications:
  - Ready list of processes in operating systems by their priorities – the list is highly dynamic
  - In event-driven simulators to maintain the list of events to be simulated in order of their time of occurrence.

# Basic Operations

- Operations on a max-priority queue:
  - Insert($S$, $x$) - inserts the element $x$ into the set $S$
    - $S \leftarrow S \cup \{x\}$.
  - Maximum($S$) - returns the element of $S$ with the largest key.
  - Extract-Max($S$) - removes and returns the element of $S$ with the largest key.
  - Increase-Key($S$, $x$, $k$) – increases the value of element $x$'s key to the new value $k$.
- Min-priority queue supports Insert, Minimum, Extract-Min, and Decrease-Key.
- Heap gives a good compromise between fast insertion but slow extraction and vice versa.

# Heap Property (Max and Min)

- Max-Heap
  - For every node excluding the root, value is at most that of its parent: $A[parent[i]] \geq A[i]$
- Largest element is stored at the root.
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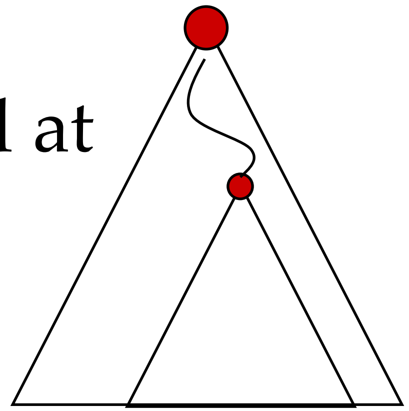

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# Heap-Extract-Max(A)

Implements the Extract-Max operation.

*Heap-Extract-Max*($A$)

1. if *heap-size*[$A$] < 1
2. then error "heap underflow"
3. *max* ← $A$[1]
4. $A$[1] ← $A$[*heap-size*[$A$]]
5. *heap-size*[$A$] ← *heap-size*[$A$] - 1
6. MaxHeapify($A$, 1)
7. return max

Running time : Dominated by the running time of MaxHeapify
$= O(\lg n)$

# Heap-Insert(*A*, *key*)

```
Heap-Insert(A, key)
heap-size[A] ← heap-size[A] + 1
  i ← heap-size[A]
while i > 1 and A[Parent(i)] < key
      do A[i] ← A[Parent(i)]
            i ← Parent(i)
A[i] ← key
```

- Running time is $O(\lg n)$
  - The path traced from the new leaf to the root has length $O(\lg n)$

# Heap-Increase-Key(A, i, key)

- <u>*Heap-Increase-Key*(*A*, *i*, *key*)</u>

**If** *key* < *A*[*i*]
   **then error** "new key is smaller than the current key"
*A*[*i*] ← *key*
**while** *i* > 1 **and** *A*[Parent[*i*]] < *A*[*i*]
      **do** exchange *A*[*i*] ↔ *A*[Parent[*i*]]
         *i* ← Parent[*i*]

- <u>*Heap-Insert*(*A*, *key*)</u>

*heap-size*[*A*] ← *heap-size*[*A*] + 1
*A*[*heap-size*[*A*]] ← –∞
*Heap-Increase-Key*(*A*, *heap-size*[*A*], *key*)

# Examples

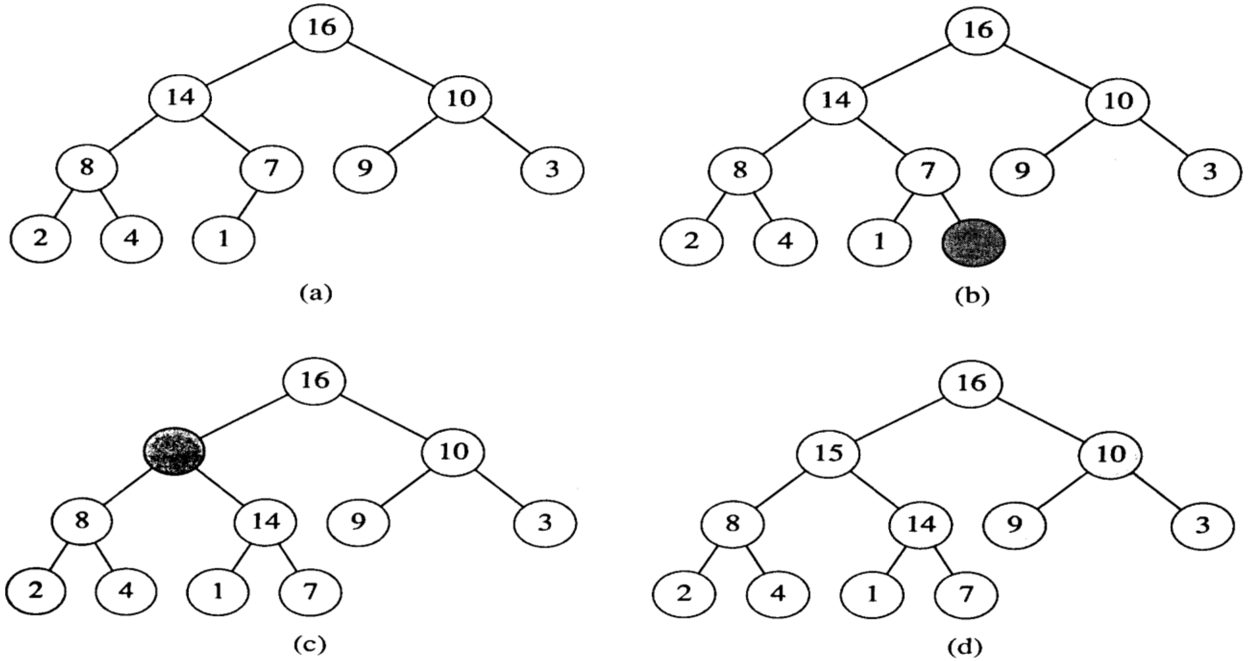


**Figure 7.5** The operation of HEAP-INSERT. **(a)** The heap of Figure 7.4(a) before we insert a node with key 15. **(b)** A new leaf is added to the tree. **(c)** Values on the path from the new leaf to the root are copied down until a place for the key 15 is found. **(d)** The key 15 is inserted.